Computational Photography

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Project 3

- Topics
  - Linear filtering
  - Convolution
  - Discrete Fourier transform
  - Frequency domain filtering
  - Inverse filtering/deconvolution

Linear Filter

$$\mathcal{F}(f(x)) = e^{f(x)}$$

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$\Rightarrow$ **shift invariant**

Fourier Transform of Gaussian

$$F(w) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi w x} dx$$

Fourier Transform of Gaussian

$$
\begin{aligned}
F(w) &= \int_{-\infty}^{\infty} f(x) e^{-i2\pi w x} dx \\
&= \int_{-\infty}^{\infty} A e^{-ax^2} e^{-i2\pi w x} dx
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completing the square:

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&= A e^{-w^2\pi^2/a} \sqrt{\frac{\pi}{a}}
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Integral of a Gaussian

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Trick No 1: Squaring and taking the root doesn't change anything!

Integral of a Gaussian

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&= \sqrt{\frac{\pi}{a}}
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Naive DFT implementation

$$F_m = \sum_{n=0}^{M-1} f_n e^{-i2\pi \frac{mn}{M}}$$

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$$\begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{M-1} \end{pmatrix} = \begin{pmatrix} e^{-i2\pi \frac{0*0}{M}} & e^{-i2\pi \frac{0*1}{M}} & ... & e^{-i2\pi \frac{0(M-1)}{M}} \\ e^{-i2\pi \frac{1*0}{M}} & e^{-i2\pi \frac{1*1}{M}} & ... & e^{-i2\pi \frac{1(M-1)}{M}} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-i2\pi \frac{(M-1)0}{M}} & e^{-i2\pi \frac{(M-1)1}{M}} & ... & e^{-i2\pi \frac{(M-1)(M-1)}{M}} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{M-1} \end{pmatrix}$$

DFT in Matlab

- Commands `fft2`, `ifft2`, `fftshift`, `ifftshift`
- Result of `fft2` has center of frequency domain (i.e., zero-frequency, or DC component) at top left
- `fftshift` shifts center of frequency domain to center of matrix
  - `ifftshift` performs inverse shift
- Use `fftshift` for visualization

DFT filtering

1. DFT with zero-padding using `fft2`
   - DFT filtering corresponds to circular convolution
   - Padding to avoid boundary artifacts
   - Padding to twice the image size is always safe
2. Generate filter function
3. Multiply Fourier transforms
4. Apply inverse DFT using `ifft2`
5. Crop to original size

Linear filtering

- Matlab functions
  - `fspecial` creates a 2-d filter of a specified type and size.
  - `imfilter` filters an image with a filter created using `fspecial`.

Generating filter functions

- Generate a frequency domain filter from a spatial filter using `freqz2`
- Directly construct filter in frequency domain

Meshgrid

- Use `meshgrid` to generate distance matrices

```
>> [x y] = meshgrid(-1:1,-1:1)

x =

    -1     0     1
    -1     0     1
    -1     0     1


y =

    -1    -1    -1
     0     0     0
     1     1     1
```

```
>> d = x.^2+y.^2

d =

     2     1     2
     1     0     1
     2     1     2

>> 1./(1+d)

ans =

    0.3333    0.5000    0.3333
    0.5000    1.0000    0.5000
    0.3333    0.5000    0.3333
```