Exercise 1

a)

Sequence: {cloudy, rainy, sunny, sunny}

$$0.25 \cdot 0.3 \cdot 0.1 \cdot 0.6 = 0.0045$$

b)

Sequence: {sunny, snowy, sunny, rainy}

$$0.25 \cdot 0.0 \cdot 0.0 \cdot 0.1 = 0.0$$

Exercise 2

a)

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{pmatrix} \end{array}$$

b)

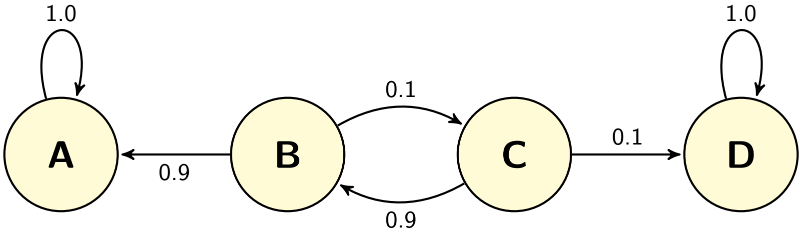
The Markov chain from a) is irreducible. Every state intercommunicates with every other state.

Exercise 3

Given transition state matrix:

$$\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

a)



b)

Since we start in state B, our initial distribution is given by

$$\pi^{(0)} = (0, 1, 0, 0).$$

The probability of being in state A after 8 transitions is

$$\pi^{(0)} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^8$$

$$= (0, 1, 0, 0) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.988946 & 0.00006561 & 0 & 0.0109883 \\ 0.890051 & 0 & 0.00006561 & 0.0109883 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0.988946, 0.00006561, 0, 0.0109883).$$

Therefore the probability is 0.988946.

c)

Since we start in state C, the initial distribution is given by

$$\pi^{(0)} = (0, 0, 1, 0).$$

From the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.98829 & 0.000729 & 0 & 0.010981 \\ 0.889461 & 0 & 0.000729 & 0.10981 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we can see that the probability of ending in state D after 6 transitions is 0.10.981

Exercise 4

Given sequence of observations: O = {coffee,coffee,lemonade}.

Probability matrix:

P	CF	WP
CF	0.7	0.3
WP	0.5	0.5

Emitting probabilities:

B	coffee	water	lemonade
CF	0.6	0.1	0.3
WP	0.1	0.7	0.2

a) Forward algorithm

state \time	t_0(coffee)	t_1(coffee)	t_2(lemonade)	t_3
α_t(CF)	1	$1 \cdot 0.7 \cdot 0.6 = 0.42$	$0.42 \cdot 0.7 \cdot 0.6$ $+ 0.18 \cdot 0.5 \cdot 0.1$ $= 0.1854$	$0.1854 \cdot 0.7 \cdot 0.3$ $+ 0.0846 \cdot 0.5 \cdot 0.2$ $= 0.04739$
α_t(WP)	0	$1 \cdot 0.3 \cdot 0.6 = 0.18$	$0.42 \cdot 0.3 \cdot 0.6$ $+ 0.18 \cdot 0.5 \cdot 0.1$ $= 0.0846$	$0.1854 \cdot 0.3 \cdot 0.3 +$ $0.0846 \cdot 0.5 \cdot 0.2$ $= 0.02515$
P[cof,cof,lem]	1	0.6	0.27	0.07254

b) Backward algorithm

state \time	t_0(coffee)	t_1(coffee)	t_2(lemonade)	t_3
α_t(CF)	$0.162 \cdot 0.7 \cdot 0.6$ $+ 0.025 \cdot 0.3 \cdot 0.6$ $= 0.07254$	$0.3 \cdot 0.7 \cdot 0.6$ $+ 0.2 \cdot 0.3 \cdot 0.6$ $= 0.162$	$1.0 \cdot 0.7 \cdot 0.3$ $+ 1.0 \cdot 0.3 \cdot 0.3$ $= 0.3$	1.0
α_t(WP)	$0.162 \cdot 0.5 \cdot 0.1$ $+ 0.025 \cdot 0.5 \cdot 0.1$ $= 0.00935$	$0.3 \cdot 0.5 \cdot 0.1$ $+ 0.2 \cdot 0.5 \cdot 0.1$ $= 0.025$	$1.0 \cdot 0.5 \cdot 0.2$ $+ 1.0 \cdot 0.5 \cdot 0.2$ $= 0.2$	1.0
P[cof,cof,lem]	0.07254			

c)

state \time	t_0(coffee)	t_1(coffee)	t_2(lemonade)	t_3
δ_t(CF)	1	0.42	$Max[\text{CF} : 0.42 \cdot 0.7 \cdot 0.6 = 0.1764; \text{WP} : 0.18 \cdot 0.5 \cdot 0.1 = 0.009] = 0.1764$	$Max[\text{CF} : 0.1764 \cdot 0.7 \cdot 0.3 = 0.03704; \text{WP} : 0.0756 \cdot 0.5 \cdot 0.2 = 0.00756] = 0.03704$
δ_t(WP)	0	0.18	$Max[\text{CF} : 0.42 \cdot 0.3 \cdot 0.6 = 0.0756; \text{WP} : 0.18 \cdot 0.5 \cdot 0.1 = 0.009] = 0.0756$	$Max[\text{CF} : 0.1764 \cdot 0.3 \cdot 0.3 = 0.01588; \text{WP} : 0.0756 \cdot 0.5 \cdot 0.2 = 0.00756] = 0.01588$
ψ_t(CF)		CF	CF	CF
ψ_t(WP)		CF	CF	CF
x_t	CF	CF	CF	

For the given observations O = {coffee,coffee,lemonade}, the most probable sequence of states is therefore {CF, CF, CF}.