ON INVOLUTIONS OF MINUSCULE KIRILLOV ALGEBRAS INDUCED BY REAL STRUCTURES

MISCHA ELKNER

ABSTRACT. We study Kirillov algebras attached to minuscule highest weight representations of semisimple Lie algebras. They can be viewed as equivariant cohomology algebras of partial flag varieties. Real structures on the varieties then induce involutions of these algebras. We describe how these involutions act on the spectra of minuscule Kirillov algebras, and model the fixed points via the equivariant cohomology of real partial flag varieties. We then use this model to characterise freeness of the fixed point coordinate ring over the appropriate base. As an application, we recover a $q = -1$ phenomenon of Stembridge in the minuscule case by geometric means.

CONTENTS

1. INTRODUCTION

The finite-dimensional irreducible representations of a complex semisimple Lie algebra $\mathfrak{g}$ are central to Lie theory. For such a representation V^λ, labelled by its highest weight λ,[1] the *Kirillov algebra*

$$\mathcal{C}^\lambda(\mathfrak{g}) := (S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda))^{\mathfrak{g}}$$

was introduced in [18].[2] This algebra is commutative if and only if the representation V^λ is weight multiplicity free [18, Cor. 1]. However, by recent work of Hausel [12], each $\mathcal{C}^\lambda(\mathfrak{g})$ contains a maximal commutative subalgebra $\mathcal{B}^\lambda(\mathfrak{g})$, called *big algebra*, with intriguing properties. In particular, one can geometrically extract data about the representation V^λ from the spectrum $\mathrm{Spec}\,\mathcal{B}^\lambda(\mathfrak{g})$ (cf. [14]). Suitable automorphisms of $\mathrm{Spec}\,\mathcal{B}^\lambda(\mathfrak{g})$ therefore correspond to symmetries of that data, which motivates the study of big algebra automorphisms as an approach to symmetries in representation theory.

This paper realises a first step in that direction: we study the case where λ is *minuscule*, which entails that $\mathcal{B}^\lambda(\mathfrak{g})$ is the full Kirillov algebra. Among the automorphisms of $\mathcal{B}^\lambda(\mathfrak{g}) = \mathcal{C}^\lambda(\mathfrak{g})$, we focus on involutions induced by real structures of the Langlands dual of $\mathfrak{g}$. More precisely, it has been observed in [24] that minuscule Kirillov algebras of $\mathfrak{g}$ are isomorphic to equivariant cohomology algebras of

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[1]With respect to some Borel and Cartan subalgebra. When this choice does not matter, we do not specify it, and simply speak of weights, roots, Weyl group etc. of $\mathfrak{g}$.

[2]In [18], Kirillov algebras are called *(classical) family algebras*.

partial flag varieties of the connected simply connected Lie group G with Lie algebra $\mathfrak{g}$. Here we use a slightly different model in terms of the Langlands dual group $G^\vee$ of G. The weight λ defines a parabolic subgroup P_λ of $G^\vee$, and we have a ring isomorphism [12]

$$(1) \qquad \mathcal{C}^\lambda(\mathfrak{g}) \cong H^*_{G^\vee}(G^\vee/P_\lambda, \mathbb{C}).$$

Let us mention two useful properties of this isomorphism. Firstly, the Kirillov algebra inherits a grading from $S(\mathfrak{g})$, and (1) becomes a graded isomorphism if the right-hand side is graded by half the cohomological degree. Secondly, $\mathcal{C}^\lambda(\mathfrak{g})$ is a graded algebra over its subring $S(\mathfrak{g})^{\mathfrak{g}}$ (embedded as $f \mapsto f \otimes \mathrm{id}$). The isomorphism identifies this subring with $H^{2*}_{G^\vee} := H^{2*}_{G^\vee}(\mathrm{pt}, \mathbb{C})$, the equivariant cohomology of a point. Altogether, the geometric model is summarised by the following commutative diagram:

$$(2) \qquad \begin{array}{ccc} \mathcal{C}^\lambda(\mathfrak{g}) & \overset{\cong}{\longrightarrow} & H^{2*}_{G^\vee}(G^\vee/P_\lambda, \mathbb{C}) \\ \big\uparrow & & \big\uparrow \\ S(\mathfrak{g})^{\mathfrak{g}} & \overset{\cong}{\longrightarrow} & H^{2*}_{G^\vee}. \end{array}$$

Through this framework, we obtain Kirillov algebra automorphisms from automorphisms of the $G^\vee$-space $G^\vee/P_\lambda$. Here, we focus on involutions arising in this way from real structures of $G^\vee$. Such a real structure (i.e. an antiholomorphic automorphism) σ of $G^\vee$ acts naturally on the coweights, and induces a real structure on $G^\vee/P_\lambda$ if $\sigma_*\lambda$ is in the same $G^\vee$-orbit as λ. Taking equivariant cohomology then yields an involution σ^* of $H^{2*}_{G^\vee}(G^\vee/P_\lambda) \cong \mathcal{C}^\lambda(\mathfrak{g})$ mapping the subring $H^{2*}_{G^\vee} \cong S(\mathfrak{g})^{\mathfrak{g}}$ to itself. We are interested on the action of σ^* on spectra. In particular, we want to describe the fixed point schemes, given by

$$(3) \qquad \begin{array}{ccccc} \mathrm{Spec}(\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}) & \overset{\cong}{\longrightarrow} & (\mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g}))^{\sigma^*} & \lhook\joinrel\longrightarrow & \mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*}) & \overset{\cong}{\longrightarrow} & (\mathrm{Spec}\,S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*} & \lhook\joinrel\longrightarrow & \mathrm{Spec}\,S(\mathfrak{g})^{\mathfrak{g}} \cong \mathfrak{g}/\!/G. \end{array}$$

Here the upper σ^* denotes sets of fixed points, which form closed affine subschemes whose coordinate rings $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ and $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*}$ we call *coinvariant rings*. Dually to the left vertical map, we have a homomorphism

$$(4) \qquad S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \to \mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}.$$

To describe the homomorphism (4) – and with it, the fixed points of σ^* – a geometric model similar to (2) is desirable. A simple candidate for such a model is to use the real form of $G^\vee/P_\lambda$ defined by σ. However, it turns out that σ^* depends only on the *inner class* of σ, which is not the case for the real form of $G^\vee/P_\lambda$. One therefore has to identify a suitable representative in a given inner class. Roughly speaking, σ should be chosen as compact as possible while preserving the parabolic P_λ. When made precise, being "as compact as possible" means that σ should induce a *quasi-compact* real structure (see Definition 4.11) on a Levi subgroup of P_λ. We show that such a choice of representative σ exists (see §5) and use this to obtain the desired model. Our main result is as follows:

Theorem 1.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, and $G^\vee$ the Langlands dual group of G. Let λ be a minuscule weight of G, viewed as a cocharacter of $G^\vee$. Then any inner class $\mathfrak{S}$ of real structures of $G^\vee$ which fix the $G^\vee$-orbit of λ contains a real structure σ such that $\sigma_*\lambda = \lambda$ and*

$$(5) \qquad \mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*} \cong H^{2*}_{(G^\vee)^\sigma}((G^\vee)^\sigma/P_\lambda^\sigma, \mathbb{C})$$

for the parabolic subgroup $P_\lambda \leq G^\vee$ defined by λ. Moreover, the same inner class contains a quasi-compact real structure σ_0 with

$$(6) \qquad S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \cong H^{2*}_{(G^\vee)^{\sigma_0}},$$

and there is a canonical injection

$$(7) \qquad \varphi \colon H^{2*}_{(G^\vee)^{\sigma_0}} \hookrightarrow H^{2*}_{(G^\vee)^\sigma}.$$

Combining (5)-(7) *yields a commutative diagram*

$$(8)$$

$$
\begin{array}{ccc}
\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*} & \xrightarrow{\ \cong\ } & H^{2*}_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda, \mathbb{C}) \\
\big\uparrow & & \big\uparrow \\
S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} & \xrightarrow{\ \cong\ } H^{2*}_{(G^\vee)^{\sigma_0}} \xhookrightarrow{\ \varphi\ } & H^{2*}_{(G^\vee)^\sigma}
\end{array}
$$

in which the right vertical arrow is the structure map for equivariant cohomology.

For a discussion of uniqueness of the real structure σ see Appendix A. We use Theorem 1.1 to describe a key algebraic property of the coinvariant homomorphism (4):

Theorem 1.2. *In the setting of Theorem 1.1, either $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ is a free module of finite rank over $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*}$ or the homomorphism* (4) *is non-injective. The former case holds precisely when we can choose $\sigma_0 = \sigma$ in Theorem 1.1, which is equivalent to λ being fixed by a quasi-compact real structure in the given inner class $\mathfrak{S}$.*

Structural results like Theorem 1.2 are of inherent interest, but are also related to combinatorial applications, one of which we now describe. Namely, for a generic $x \in \operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}}$ in the base, the fibre in $\operatorname{Spec} \mathcal{C}^\lambda(\mathfrak{g})$ is finite and can be identified with the set $\operatorname{wt}(\lambda)$ of weights of V^λ. By choosing $x \in (\operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$, we obtain an action of σ^* on that fibre, hence an involution on $\operatorname{wt}(\lambda)$. Now if σ is (inner to) a split real structure, it is not hard to identify this action on $\operatorname{wt}(\lambda)$ with that of the longest element w_0 of the Weyl group. This action has been considered by Stembridge [30] and shown to fulfil a $q = -1$ phenomenon. Another part of this phenomenon is played by the *Dynkin polynomial* $\mathcal{D}^\lambda$ (see (22)). In the setting of [30], this is the rank generating function for the ranked poset $\operatorname{wt}(\lambda)$, but we can also interpret it as the Poincaré polynomial of $\mathcal{C}^\lambda(\mathfrak{g})$ over $S(\mathfrak{g})^{\mathfrak{g}}$ [24]. Stembridge shows that

$$(9) \qquad \# \operatorname{wt}(\lambda)^{w_0} = \mathcal{D}^\lambda(-1).$$

Both sides of this identity have natural interpretations in our setup. Indeed, we have just indicated that $\# \operatorname{wt}(\lambda)^{w_0}$ is the number of σ^*-fixed points in a generic fibre over a σ^*-fixed base point. Moreover, still assuming that σ is (inner to) a split real structure, we will show that σ^* acts on $\mathcal{C}^\lambda(\mathfrak{g})$ as $(-1)^{\deg}$, that is, it acts like $-1 \in \mathbb{C}^\times$ through the $\mathbb{C}^\times$-action corresponding to the natural grading on $\mathcal{C}^\lambda(\mathfrak{g})$. The same is then true for the restriction of σ^* to the functions on a fibre as above. Moreover, that function ring, denoted $\mathcal{C}^\lambda_x(\mathfrak{g})$, inherits the Poincaré polynomial $\mathcal{D}^\lambda$ from $\mathcal{C}^\lambda(\mathfrak{g})$ – now as a graded $\mathbb{C}$-algebra. Thus, $\mathcal{D}^\lambda(-1)$ is simply the trace of σ^* on $\mathcal{C}^\lambda_x(\mathfrak{g})$. Altogether, we have the following Theorem, which recovers (9):

Theorem 1.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and λ a minuscule weight of $\mathfrak{g}$. Further, let G be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, $G^\vee$ its Langlands dual, and σ a split real structure of $G^\vee$. Denoting the canonical map $\operatorname{Spec} \mathcal{C}^\lambda(\mathfrak{g}) \to \operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}}$ by π, the fixed-point scheme $(\operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$ contains a dense open subset U such that the fibre $\pi^{-1}(x)$ is reduced for each $x \in U$. For such x, we then have*

$$\mathcal{D}^\lambda(-1) = \#(\pi^{-1}(x))^{\sigma^*} = \# \operatorname{wt}(\lambda)^{w_0},$$

where w_0 is the longest element of the Weyl group. Moreover, this quantity is nonzero if and only if λ is fixed by a quasi-compact real structure inner to σ.

An analogous analysis is possible for involutions induced by non-split inner classes, although the action on $\mathcal{C}^\lambda(\mathfrak{g})$ can be more complicated than $(-1)^{\deg}$. Moreover, as indicated above, the work in this paper should generalise substantially, with arbitrary big algebras in place of minuscule Kirillov algebras. In particular, we expect Theorem 1.3 to generalise to arbitrary dominant integral weights,

thereby recovering results in [29]. A similar description in this generality has recently been obtained for twining in [35].

This paper is structured as follows. In §2 we review the relevant background on equivariant cohomology and Kirillov algebras and define the coinvariant rings used here. §3 reviews real structures and describes the induced actions on partial flag varieties and Kirillov algebras. This is then translated to rings of invariant polynomials, which are further discussed in §4. That section also contains a discussion of quasi-compactness. The proof of Theorem 1.1 is given in Section 5, and Theorem 1.2 is deduced from it in §6. In §7 we describe how our involutions relate to involutions on weights and derive Theorem 1.3. Finally, §8 briefly discusses the aforementioned generalisations to big algebras and arbitrary weights, as well as a connection to the theory of Higgs bundles. Appendix A discusses in which sense the real structure appearing in Theorem 1.1 is unique, and lists the real structures in tables.

Conventions

In this paper, all cohomology is taken with complex coefficients.

Given an action of a group G on a space or algebraic structure X, the fixed points are denoted by X^G. Similarly, if f is an automorphism of X, the fixed points under f are denoted X^f. However, if $\mathfrak{g}$ is a Lie algebra acting linearly on a vector space V, then $V^{\mathfrak{g}}$ denotes the subspace of elements annihilated by $\mathfrak{g}$ (so $V^{\mathfrak{g}} = V^G$ for the corresponding action of the connected simply connected Lie group with Lie algebra $\mathfrak{g}$).

Although our main results are for semisimple complex Lie algebras/groups, we will also need to incorporate the reductive case, for which we use the following standard conventions. A Cartan subalgebra $\mathfrak{h}$ of a reductive Lie algebra $\mathfrak{g}$ is the direct sum of the centre $\mathfrak{z}(\mathfrak{g})$ and a Cartan subalgebra $\mathfrak{h}^{ss}$ of the semisimple part $\mathfrak{g}^{ss} = [\mathfrak{g}, \mathfrak{g}]$. The roots and Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are defined in terms of $(\mathfrak{g}^{ss}, \mathfrak{h}^{ss})$ and extended in the obvious way. Thus, if G is reductive with Lie algebra $\mathfrak{g}$ and $H \leq G$ is the Cartan subgroup corresponding to $\mathfrak{h}$, then the Weyl group is isomorphic to $N_G(H)/H$ if G is connected but can be strictly smaller otherwise.

For semisimple $\mathfrak{g}$, we freely use the canonical $\mathfrak{g}$-equivariant isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ afforded by the Killing form. In particular, we freely view the Weyl group with respect to a Cartan subalgebra $\mathfrak{h}$ as a subgroup of $\mathrm{GL}(\mathfrak{h})$.

We freely identify (co)weights of a complex Lie group G with their induced (co)weights of the Lie algebra $\mathfrak{g}$ (and vice versa when lifts exist). As mentioned previously, we do not specify a choice of Cartan (or Borel) subalgebra when talking about weights, roots, etc., unless necessary. The *weight spaces* of a representation are by convention at least one-dimensional.

A homomorphism of algebras $A \to B$ is called *free* if it equips B with the structure of a free A-module.

2. Equivariant cohomology, Kirillov algebras and coinvariant rings

In this section, we provide further details on relevant background and context. We start with a brief review of equivariant cohomology, a cohomology theory for spaces with group actions.

Let G be a topological group and X a left G-space. Let $\mathbb{E}G \to \mathbb{B}G$ be a universal G-bundle – in other words, $\mathbb{E}G$ is a contractible space with free right G-action, and $\mathbb{B}G = \mathbb{E}G/G$. ($\mathbb{E}G$ is not

unique, but unique up to G-homotopy equivalence.) The *(Borel) G-equivariant cohomology* of X with coefficients in a ring R is defined as

$$H_G^*(X, R) := H^*(X_G, R)$$

where

$$X_G := (\mathbb{E}G \times X)/\big((e \cdot g, x) \sim (e, g \cdot x)\big)$$

From now on, we always take $R = \mathbb{C}$ and drop this from the notation. It is clear from the definition that $H_G^*(X)$ is a graded $\mathbb{C}$-algebra. In fact, the projection $X \to \mathrm{pt}$ to a point induces a map $X_G \to \mathrm{pt}_G$, making $H_G^*(X)$ canonically an algebra over $H_G^* := H_G^*(\mathrm{pt})$.

Moreover, equivariant cohomology is functorial for morphisms of spaces with group action. That is, let H be another topological group, Y a left H-space, $\alpha: G \to H$ a group homomorphism, and $f: X \to Y$ a map such that

(10) $$f(g \cdot x) = \alpha(g) \cdot f(x)$$

for all $x \in X$, $g \in G$. Then the pair (α, f) induces a morphism of algebras

(11)
$$\begin{array}{ccc} H_H^*(Y) & \longrightarrow & H_G^*(X) \\ \uparrow & & \uparrow \\ H_H^* & \longrightarrow & H_G^* \end{array}$$

making $(G, X) \mapsto H_G^*(X)$ into a functor. As a special case, suppose that $Y = X$, $f = \mathrm{id}_X$, and that α is a homotopy equivalence (in addition to being a group homomorphism). One then readily concludes that the resulting homomorphism $H_H^*(X) \to H_G^*(X)$ is an isomorphism.

Applying this last remark to the case of a point, we see that $H_G^* \cong H_H^*$ whenever there is a group homomorphism $G \to H$ which is also a homotopy equivalence. In particular, this holds when H is a reductive Lie group and G its maximal compact subgroup (see e. g. [19, Prop. 7.19(a)]) or when G is a parabolic subgroup of a reductive Lie group and H its Levi factor (e. g. [19, Prop. 7.83(d)]). If G is a compact Lie group with Lie algebra $\mathfrak{g}$, we can describe the ring H_G^* via the Chern-Weil isomorphism [8, p. 116] as

(12) $$H_G^* \cong H^*(\mathbb{B}G) \cong S(\mathfrak{g})^G \otimes_{\mathbb{R}} \mathbb{C}.$$

Lastly, let us mention the case where $X = G/H$ is a homogeneous space, for $H \leq G$ a topological subgroup. In this case, any choice of $\mathbb{E}G$ also has a free H-action, so it can be viewed as $\mathbb{E}H$ as well. Moreover, one readily checks that $(G/H)_G \cong \mathbb{E}G/H$, so

(13) $$H_G^*(G/H) = H^*((G/H)_G) \cong H^*(\mathbb{B}H) \cong H_H^*.$$

For more details on equivariant cohomology we refer to [6].

We now collect the key facts on Kirillov algebras stated already in the introduction.

Definition 2.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and λ a dominant integral weight. Let V^λ denote the irreducible $\mathfrak{g}$-representation of highest weight λ, $\mathrm{End}(V^\lambda)$ is algebra of linear endomorphisms[3], and $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$. The *Kirillov algebra* with label λ is the graded subalgebra

$$\mathcal{C}^\lambda(\mathfrak{g}) := (S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda))^\mathfrak{g} \leq S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)$$

consisting of fixed points of the diagonal $\mathfrak{g}$-action.

If $\mathfrak{h} \leq \mathfrak{g}$ is a Cartan subalgebra and W the Weyl group, Panyushev [24] defines

$$\mathbb{C}^\lambda(\mathfrak{h}) := (S(\mathfrak{h}) \otimes \mathrm{End}_\mathfrak{h}(V^\lambda))^W$$

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[3]not to be confused with the algebra of $\mathfrak{g}$-equivariant endomorphisms (which is isomorphic to $\mathbb{C}$ by Schur's lemma)

where $\mathrm{End}_{\mathfrak{h}}(V^{\lambda})$ denotes the $\mathfrak{h}$-equivariant endomorphisms. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{h}$ with $\mathfrak{h}^*$ through the Killing form yields an injective restriction map

$$(14) \qquad\qquad\qquad r_{\lambda} \colon \mathcal{C}^{\lambda}(\mathfrak{g}) \hookrightarrow \mathcal{C}^{\lambda}(\mathfrak{h}).$$

Proposition 2.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and λ a dominant integral weight. The Kirillov algebra $\mathcal{C}^{\lambda}(\mathfrak{g})$*

  (i) *has a grading induced from $S(\mathfrak{g})$,*
  (ii) *contains $S(\mathfrak{g})^{\mathfrak{g}}$ as the graded subring $S(\mathfrak{g})^{\mathfrak{g}} \otimes \{\mathrm{id}_{V^{\lambda}}\}$,*
  (iii) *is finite-free over $S(\mathfrak{g})^{\mathfrak{g}}$, and*
  (iv) *is commutative if and only if V^{λ} is weight-multiplicity free, i. e. if the weight spaces[4] of V^{λ} are one-dimensional.*
  (v) *Moreover, r_{λ} is an isomorphism if and only if λ is minuscule.*

Proof. The canonical grading of $S(\mathfrak{g})$ defines a grading of $S(\mathfrak{g}) \otimes \mathrm{End}(V^{\lambda})$ (in which the second factor contributes trivially). It is preserved by the diagonal $\mathfrak{g}$-action and therefore restricts to the grading of $\mathcal{C}^{\lambda}(\mathfrak{g})$ of part (i). Part (ii) is clear. For part (iii), combine [24, Thm. 1.1] with the second paragraph of [loc. cit. p. 277]. Part (iv) is [18, Cor. 1], and part (v) is [24, Cor. 2.9]. $\qquad\square$

Recently, Hausel [12], building on previous work including [25, 9], has introduced certain maximal commutative subalgebras of Kirillov algebras, called *big algebras*. The minuscule Kirillov algebras studied in this paper are the simplest examples of big algebras, and general big algebras provide important motivation. We now summarise some of their key properties. For every λ, Hausel's big algebra $\mathcal{B}^{\lambda}(\mathfrak{g})$ is a maximal commutative graded $S(\mathfrak{g})^{\mathfrak{g}}$-subalgebra of $\mathcal{C}^{\lambda}(\mathfrak{g})$. The morphisms

$$\mathrm{Spec}\,\mathcal{B}^{\lambda}(\mathfrak{g}) \to \mathrm{Spec}\,Z(\mathcal{C}^{\lambda}(\mathfrak{g})) \to \mathrm{Spec}\,S(\mathfrak{g})^{\mathfrak{g}}$$

(with Z denoting the centre) appear to encode important invariants of V^{λ} such as its Kashiwara crystal [14, final slide]. These algebras admit the following geometric model:

Theorem 2.3 (cf. [12], Theorem 3.1). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, and $G^{\vee}$ the Langlands dual group of G. For any dominant integral weight λ of $\mathfrak{g}$, let $\mathrm{Gr}^{\leq \lambda}$ denote the affine Schubert variety of $G^{\vee}$ labelled by λ. Then we have*

  (i) $Z(\mathcal{C}^{\lambda}(\mathfrak{g})) \cong H^{2*}_{G^{\vee}}(\mathrm{Gr}^{\leq \lambda})$
  (ii) $\mathcal{B}^{\lambda}(\mathfrak{g}) \cong IH^{2*}_{G^{\vee}}(\mathrm{Gr}^{\leq \lambda})$, *the $G^{\vee}$-equivariant intersection cohomology of $\mathrm{Gr}^{\leq \lambda}$, as a module over $H^{2*}_{G^{\vee}}(\mathrm{Gr}^{\leq \lambda})$.*

In this paper, we focus on the minuscule case. Recall that a dominant integral weight λ is called *minuscule* if it fulfils one of the following equivalent conditions:

  • there does not exist a positive root α for which $\lambda - \alpha$ is dominant,
  • the weights of V^{λ} are all contained in the Weyl group orbit of λ.

Since the highest weight space of V^{λ} is one-dimensional, the second condition shows that V^{λ} is then weight multiplicity free. The minuscule weights form a subset of the fundamental weights; in particular there are finitely many.

Corollary 2.4. *Let $\mathfrak{g}$, $G^{\vee}$ and λ be as in Theorem 2.3. Let $P_{\lambda} \leq G^{\vee}$ be the parabolic subgroup defined by λ.[5] If λ is minuscule then $\mathcal{C}^{\lambda}(\mathfrak{g})$ is commutative and isomorphic to $H^{2*}_{G^{\vee}}(G^{\vee}/P_{\lambda})$.*

Proof. Since λ is minuscule, the affine Schubert variety $\mathrm{Gr}^{\leq \lambda}$ is smooth, and in fact isomorphic (as a $G^{\vee}$-space) to $G^{\vee}/P_{\lambda}$ [33, Lemma 2.1.13]. $\qquad\square$

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[4]Weight spaces are non-zero by convention.
[5]View λ as a cocharacter $\mathbb{C}^{\times} \to G^{\vee}$, with derivative $d\lambda \colon \mathbb{C} \to \mathfrak{g}$. Then the Lie algebra of P^{λ} is the direct sum of the non-negative eigenspaces of $d\lambda(1) \in \mathfrak{g}$, and P_{λ} itself is the normaliser in $G^{\vee}$ of that Lie algebra.

Remark 2.5. We appeal to Theorem 2.3 mostly to contextualise the result. Without appealing to the theorem, an isomorphism $\mathcal{C}^\lambda(\mathfrak{g}) \cong H^{2*}_{G^\vee}(G^\vee/P_\lambda)$ for minuscule λ can be obtained more concretely as follows. For any Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, [24, Thm. 2.6] gives an explicit isomorphism $\mathcal{C}^\lambda(\mathfrak{g}) \cong S(\mathfrak{h})^{W_\lambda}$, where W_λ is the stabiliser of λ in the Weyl group. The Weyl group is preserved by Langlands duality, and $\mathfrak{h}^* \cong \mathfrak{h}$ (as a W-space) can be viewed as an abstract Cartan subalgebra of the Langlands dual $\mathfrak{g}^\vee$. Via the Chevalley restriction theorem (see Theorem 4.3 below) and the Chern-Weil isomorphism (see (12) below), we then get

$$\mathcal{C}^\lambda(\mathfrak{g}) \cong S(\mathfrak{h})^{W_\lambda} \cong S(\mathfrak{h}^*)^{W_\lambda} \cong S(\mathfrak{l}_\lambda)^{\mathfrak{l}_\lambda} \cong H^{2*}_{G^\vee}(G^\vee/P_\lambda)$$

where $\mathfrak{l}_\lambda$ is a canonical Levi subalgebra defined by λ (see also the discussion at the end of §3).

We end this section with a brief discussion of fixed points and coinvariant rings. In general, if X is any scheme with a self-morphism $f\colon X \to X$, the *fixed-point scheme* X^f is the largest closed subscheme on which f acts trivially. In other words, X^f is the equaliser of f and id_X in the category of schemes. In the case of affine schemes, there is a particularly easy description:

Proposition 2.6. *Let ψ be an endomorphism of a ring A, and f the induced self-morphism of* $\mathrm{Spec}\, A$. *Then the fixed-point scheme* $(\mathrm{Spec}\, A)^f$ *is given by* $\mathrm{Spec}\, A_\psi \subseteq \mathrm{Spec}\, A$, *where*

$$A_\psi := A/(a - \psi(a)\colon a \in A)$$

is the coinvariant ring.

Proof. As a closed subscheme of the affine scheme $\mathrm{Spec}\, A$, the fixed-point scheme must be of the form $\mathrm{Spec}(A/I)$ for some ideal I of A [27, Tag 01IF]. The projection $A \to A/I$ is then a coequaliser of ψ and id_A in the category of commutative rings, so I is the ideal $(a - \psi(a)\colon a \in A)$ as claimed. $\qquad\square$

3. Action of real structures

Here we review real structures of complex Lie groups and describe their action on partial flag varieties. We then study the resulting involutions on equivariant cohomology. We start by setting up definitions, referring to [23, §2–5] or [10, §2] for details.

Definition 3.1. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ whose complexification $\mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{C}$ is isomorphic to $\mathfrak{g}$ (as a complex Lie algebra). A *real structure* of a complex Lie algebra $\mathfrak{g}$ is a conjugate-linear Lie algebra automorphism $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ of order two.

If σ is a real structure on a complex Lie algebra $\mathfrak{g}$, then one readily checks that the fixed points $\mathfrak{g}^\sigma$ make up a real form of $\mathfrak{g}$. Conversely, for a real form $\mathfrak{g}_0$ the complex conjugation on $\mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{C}$ can be used to define a corresponding real structure on $\mathfrak{g}$ (uniquely up to an automorphism of $\mathfrak{g}$). This way, real forms and real structures are equivalent. The latter concept generalises more conveniently to complex Lie groups:

Definition 3.2. A *real structure* of a complex Lie group G is an antiholomorphic automorphism of order two.

For our purposes, real structures on the group level are equivalent to those on the Lie algebra level:

Lemma 3.3. *Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Any real structure on G defines a real structure on $\mathfrak{g}$ by differentiation, and this assignment is injective if G is connected. Conversely, if G is connected and either simply connected or of adjoint type, then any real structure on $\mathfrak{g}$ integrates to a unique real structure on G.*

Proof. The passage from the group level to the Lie algebra level is standard. It is also clear that real structures of $\mathfrak{g}$ lift to G when G is connected and simply connected. For the adjoint type case, it suffices to lift to a simply connected covering group $\tilde{G}$ and to observe that this lift preserves the centre of $\tilde{G}$. $\qquad\square$

Several equivalence relations on the set of real structures are commonly used (though their names are not entirely standardised):

Definition 3.4 (Equivalence relations for real structures)**.** Let G be a connected complex Lie group and $\mathfrak{g}$ its Lie algebra. Let σ_1, σ_2 be real structures on $\mathfrak{g}$ or G.

   (i) $\mathrm{Aut}(\mathfrak{g})$ denotes the group of complex Lie algebra automorphisms of $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ denotes its subgroup generated by elements $\exp(\mathrm{ad}(X))$ for $X \in \mathfrak{g}$.

   (ii) $\mathrm{Aut}(G)$ denotes the group of complex Lie group automorphisms of G. $\mathrm{Int}(G)$ denotes its subgroup consisting of elements $\mathrm{Conj}_g := (h \mapsto ghg^{-1})$ for $g \in G$.

   (iii) We say that σ_1 and σ_2 are *isomorphic*, denoted $\sigma_1 \approx \sigma_2$, if $\sigma_1 = \psi\sigma_2\psi^{-1}$ for $\psi \in \mathrm{Aut}(\mathfrak{g})$ resp. $\psi \in \mathrm{Aut}(G)$. This is equivalent to the corresponding real forms being (abstractly) isomorphic.

   (iv) We say that σ_1 and σ_2 are *inner-isomorphic*, denoted $\sigma_1 \approx_i \sigma_2$, if $\sigma_1 = \psi\sigma_2\psi^{-1}$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. This is equivalent to the corresponding real forms being isomorphic via conjugation by some $g \in G$.

   (v) We say that σ_1 and σ_2 are *inner (to each other)*, denoted $\sigma_1 \sim_i \sigma_2$, if $\sigma_1 = \psi\sigma_2$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. Equivalently, $\sigma_1 = \sigma_2\chi$ for χ in $\mathrm{Int}(\mathfrak{g})$ resp. $\mathrm{Int}(G)$. The corresponding equivalence classes are called *inner classes*.

Note that inner-isomorphic real structures are automatically isomorphic as well as inner to each other (the latter because $\mathrm{Int}(\mathfrak{g})$ is normal in $\mathrm{Aut}(\mathfrak{g})$). We now come to a classical result of É. Cartan [7] relating real structures to complex involutions. To state it, we define equivalence relations for such automorphisms analogous to those in Definition 3.4:

Definition 3.5 (Equivalence relations for holomorphic involutions)**.** Let G be a connected complex Lie group and $\mathfrak{g}$ its Lie algebra. We write $\mathrm{Aut}_2(\mathfrak{g})$ and $\mathrm{Aut}_2(G)$ for the subgroups of involutions in $\mathrm{Aut}(\mathfrak{g})$ resp. $\mathrm{Aut}(G)$. Let θ_1, θ_2 be elements of $\mathrm{Aut}_2(\mathfrak{g})$ or $\mathrm{Aut}_2(G)$.

   (i) We say that θ_1 and θ_2 are *isomorphic*, denoted $\theta_1 \approx \theta_2$, if $\theta_1 = \psi\theta_2\psi^{-1}$ for $\psi \in \mathrm{Aut}(\mathfrak{g})$ resp. $\psi \in \mathrm{Aut}(G)$.

   (ii) We say that θ_1 and θ_2 are *inner-isomorphic*, denoted $\theta_1 \approx_i \theta_2$, if $\theta_1 = \psi\theta_2\psi^{-1}$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$.

   (iii) We say that θ_1 and θ_2 are *inner (to each other)*, denoted $\theta_1 \sim_i \theta_2$, if $\theta_1 = \psi\theta_2$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. Equivalently, $\theta_1 = \theta_2\chi$ for χ in $\mathrm{Int}(\mathfrak{g})$ resp. $\mathrm{Int}(G)$. The corresponding equivalence classes are called *inner classes*.

Theorem 3.6 (cf. e. g. §3 of [23])**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then $\mathfrak{g}$ has a compact real structure, i. e. a real structure τ such that $\mathrm{Int}(\mathfrak{g}^\tau)$ is compact. Moreover, every real structure σ of $\mathfrak{g}$ is inner-isomorphic to some σ' which commutes with τ. Then $\theta := \sigma'\tau = \tau\sigma' \in \mathrm{Aut}_2(\mathfrak{g})$. This defines bijections*

$$\begin{aligned}
\{\text{Real structures of } \mathfrak{g}\}/\approx \quad &\longleftrightarrow \quad \mathrm{Aut}_2(\mathfrak{g})/\approx \\
\{\text{Real structures of } \mathfrak{g}\}/\approx_i \quad &\longleftrightarrow \quad \mathrm{Aut}_2(\mathfrak{g})/\approx_i \\
\{\text{Real structures of } \mathfrak{g}\}/\sim_i \quad &\longleftrightarrow \quad \mathrm{Aut}_2(\mathfrak{g})/\sim_i \\
[\sigma] \quad &\longleftrightarrow \quad [\theta].
\end{aligned}$$

which do not depend on the choice of τ. If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then τ lifts to G and sets up an analogous correspondence between equivalence classes of real structures on G and equivalence classes in $\mathrm{Aut}_2(G)$.

Definition 3.7. If a real structure σ commutes with a compact real structure τ, the corresponding involutive automorphism $\theta = \sigma\tau$ is called the *Cartan involution (of σ with respect to τ)*.

For later use, we also recall the notion of a pinning-preserving automorphism. A *pinning* of a complex reductive Lie algebra $\mathfrak{g}$ consists of a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, a choice of simple roots Π for the root system of $(\mathfrak{g}, \mathfrak{h})$, and a nonzero root vector X_α for every $\alpha \in \Pi$.

Definition 3.8. Let $\mathfrak{g}$ be a complex reductive Lie algebra. An automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ is called *pinning-preserving* if there exists a pinning $\{\mathfrak{h}, \Pi, \{X_\alpha\}_{\alpha \in \Pi}\}$ of $\mathfrak{g}$ such that θ maps $\mathfrak{h}$ to itself and permutes the X_α (hence also the simple roots). If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then an automorphism of G is, by definition, *pinning-preserving* if its derivative is.[6]

We now come to the action on partial flag varieties. For the remainder of this section, G will denote a connected complex semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra with corresponding Cartan subgroup $H := C_G(\mathfrak{h}) \leq G$, and let $\Sigma \leq \mathfrak{h}^*$ denote the root system defined by $(\mathfrak{g}, \mathfrak{h})$. Recall that the parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ can be parameterised by elements of $\mathfrak{h}$ as follows: given $v \in \mathfrak{h}$, let

$$\mathfrak{p}_v := \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Sigma \\ \alpha(v) \geq 0}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denotes the root space of α. Equivalently, $\mathfrak{p}_v$ is the direct sum of non-negative eigenspaces of $\mathrm{ad}(v)$. Each $\mathfrak{p}_v$ is a parabolic subalgebra of $\mathfrak{g}$ and defines a parabolic subgroup $P_v := N_G(\mathfrak{p}_v) \leq G$. Now if $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, H)$ is a cocharacter, we can take the derivative $d\lambda \colon \mathbb{C} \to \mathfrak{h}$ and define $P_\lambda := P_{d\lambda(1)}$.

Now let σ be a real structure of G which preserves H. This defines an action on the cocharacters given by

$$\mathrm{Hom}(\mathbb{C}^\times, H) \to \mathrm{Hom}(\mathbb{C}^\times, H), \quad \lambda \mapsto \sigma_* \lambda := \sigma \circ \lambda \circ \overline{(\cdot)},$$

with $\overline{(\cdot)}$ denoting complex conjugation on $\mathbb{C}^\times$. One easily verifies that $P_{\sigma_* \lambda} = \sigma(P_\lambda)$. We will be interested in the case where $P_{\sigma_* \lambda}$ is conjugate to P_λ, which can be expressed using the Weyl group. Recall that the Weyl group $W = N_G(H)/H$ has a natural action on the cocharacters induced by the action of $N_G(H)$. Explicitly, $g \in N_G(H)$ acts as $\lambda \mapsto \mathrm{Conj}_g \circ \lambda$ where $\mathrm{Conj}_g \colon H \to H$ denotes conjugation by g. Thus, it makes sense to speak of the *Weyl group orbit* of a cocharacter λ, and we will focus on the case where σ preserves this orbit (see also Remark 3.11 below).

Remark 3.9. (Notions of preserved orbits)

(1) The explicit choice of Cartan subalgebra $\mathfrak{h}$ (and Cartan subgroup H) is merely for convenience. If no choice is made, a coweight should be viewed as a homomorphism $\lambda \colon \mathbb{C}^\times \to G$ valued in semisimple elements. The action of a real structure σ on coweights is defined as above, and now G acts on these by $g \cdot \lambda := \mathrm{Conj}_g \circ \lambda$. Given any Cartan subgroup H, we can arrange $\sigma(H) = H$ by replacing σ with an inner-isomorphic real structure. If λ takes values in a Cartan subgroup H preserved by σ, then σ preserves the G-orbit of λ if and only if it preserves its Weyl group orbit.

(2) Note further that these notions of preserving orbits depend only on the inner class of σ. More precisely, if $\phi \in \mathrm{Int}(G)$, it is clear that $\sigma\phi$ preserves the G-orbit of λ if and only if σ does. If both σ and $\sigma\phi$ preserve a Cartan subgroup H such that λ takes values in H, then ϕ normalises H so that $(\sigma\phi)_* \lambda \in W \cdot \lambda$ if and only if $\sigma_* \lambda \in W \cdot \lambda$. Now, if $B \leq G$ is a Borel subgroup containing H such that λ is dominant with respect to B, there exists a quasi-split real structure σ^{qs} inner to σ which preserves both H and B (see e.g. [19, Problem VI.31]). Moreover, one can check that there exist a split real structure σ^s preserving H and B and a pinning-preserving holomorphic involution η such that $\sigma^{qs} = \eta \sigma^s$. We then have $\sigma^s_* \lambda = \lambda$ (irrespective of $\sigma_* \lambda$), so altogether

$$\sigma_* \lambda \in W \cdot \lambda \quad \Longleftrightarrow \quad \sigma^{qs}_* \lambda = \lambda \quad \Longleftrightarrow \quad \eta \circ \lambda = \lambda.$$

Proposition 3.10. *Let σ be a real structure of G which preserves the Cartan subgroup H as well as the Weyl group orbit of a cocharacter $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, H)$. Choose $s \in N_G(H)$ such that $\sigma_* \lambda = \mathrm{Conj}_s \circ \lambda$. Then*

$$\underline{\sigma}(gP_\lambda) := \sigma(g)sP_\lambda, \quad g \in G,$$

defines an antiholomorphic involution $\underline{\sigma}$ of G/P_λ which does not depend on the choice of s.

---

[6]This is equivalent to the more common definition involving a pinning of G.

Proof. Firstly, for $g \in G$ and $h \in P_\lambda$ we have

$$\sigma(gh)sP_\lambda = \sigma(g)\sigma(h)sP_\lambda = \sigma(g)s\,\mathrm{Conj}_{s^{-1}}(\sigma(h))P_\lambda = \sigma(g)sP_\lambda$$

because $\mathrm{Conj}_s(P_\lambda) = \sigma(P_\lambda)$, so $\underline{\sigma}(gP_\lambda)$ is well-defined.

Secondly, for another $s' \in N_G(H)$ with $\mathrm{Conj}_{s'} \circ \lambda = \sigma_* \lambda$, we have

$$s'P_\lambda = s(s^{-1}s')P_\lambda = sP_\lambda$$

since $\mathrm{Conj}_{s^{-1}s'}(P_\lambda) = P_\lambda$ and P_λ is self-normalising (forcing $s^{-1}s' \in P_\lambda$). This shows that $\underline{\sigma}$ does not depend on the choice of s.

Lastly, the assumption $\mathrm{Conj}_s \circ \lambda = \sigma_* \lambda$ implies $\mathrm{Conj}_{\sigma(s)s} \circ \lambda = \lambda$. As before, we conclude that $\sigma(s)s \in P_\lambda$. Then

$$\underline{\sigma}^2(gP_\lambda) = g\sigma(s)sP_\lambda = gP_\lambda, \quad g \in G$$

verifies that $\underline{\sigma}$ is indeed an involution. Antiholomorphicity is easily checked by lifting $\underline{\sigma}$ to G. $\square$

Remark 3.11. The real structure σ induces a real structure $\tilde{\sigma}$ on the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_G$ of G. The affine Schubert varieties of Gr can be labelled by W-orbits of coweights, and $\tilde{\sigma}$ permutes the affine Schubert varieties as σ_* permutes these W-orbits. In particular, $\tilde{\sigma}$ preserves the affine Schubert variety $\mathrm{Gr}^{\leq \lambda}$ if and only if $\sigma_* \lambda \in W \cdot \lambda$. If this is the case, $\tilde{\sigma}$ restricts to a real structure of $\mathrm{Gr}^{\leq \lambda}$. Now, if λ is in addition minuscule, then $\mathrm{Gr}^{\leq \lambda}$ is canonically isomorphic to G/P_λ, and the isomorphism identifies $\tilde{\sigma}$ with the real structure $\underline{\sigma}$ of Proposition 3.10.

As an antiholomorphic involution, $\underline{\sigma}$ fixes a real submanifold of G/P_λ whose real dimension is $\dim_{\mathbb{C}} G/P_\lambda$. While this is true, it turns out that this submanifold can be empty:

Proposition 3.12. *Let G, σ, and λ be as in Proposition 3.10. Then*

$$(G/P_\lambda)^{\underline{\sigma}} = \begin{cases} G^\sigma/P_\lambda^\sigma & \text{if } \sigma_* \lambda = \lambda \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. The key observation is that $(G/P_\lambda)^{\underline{\sigma}}$ is a closed union of G^σ-orbits in G/P_λ. Thus, it is either empty or contains the distinguished orbit through eP_λ (where $e \in G$ is the neutral element) [32, Cor. 3.4]. But $\underline{\sigma}(eP_\lambda) = sP_\lambda$ for $s \in N_G(H)$ with $\mathrm{Conj}_s \circ \lambda = \sigma_* \lambda$. This equals eP_λ precisely when $s \in P_\lambda$, which is equivalent to $\sigma_* \lambda = \lambda$.

In case $\sigma_* \lambda = \lambda$, we have seen that the fixed submanifold contains the distinguished orbit $G^\sigma \cdot (eP_\lambda) = G^\sigma/P_\lambda^\sigma$. But $\dim_{\mathbb{R}}(G/P_\lambda)^{\underline{\sigma}} = \dim_{\mathbb{C}}(G/P_\lambda)$ is the minimal dimension of G^σ-orbits [32, Thm. 3.6], which is attained only by the distinguished orbit [loc. cit. Cor. 3.4], so the fixed point set consists of that orbit alone. $\square$

We will be interested in the action of $\underline{\sigma}$ on equivariant cohomology. More precisely, the pair $(\sigma, \underline{\sigma})$ induces an algebra involution

(15)
$$\begin{array}{ccc} H_G^*(G/P_\lambda) & \xrightarrow{\;\sigma^*\;} & H_G^*(G/P_\lambda) \\ \uparrow & & \uparrow \\ H_G^* & \xrightarrow{\;\sigma^*\;} & H_G^* \end{array}$$

as in (10)–(11). We finish this section with a description of σ^*.

Lemma 3.13. *Let G, σ, and λ be as in Proposition 3.10. The involution σ^* of (15) depends only on the inner class of σ.*[7]

Proof. If $\psi = \mathrm{Conj}_u \in \mathrm{Int}(G)$ is such that $\sigma' = \psi\sigma$ is another real structure, one finds that $\underline{\sigma}$ and $\underline{\sigma}'$ differ by the action ℓ_u of u on G/P_λ by left multiplication. Since G is connected, this action is homotopic to the identity. It follows that $(\sigma')^* = \sigma^*$. $\square$

---

[7]This lemma is a special case of Remark A.7(i) in [34].

Using (13), we find that $H_G^*(G/P_\lambda) \cong H_{P_\lambda}^*$. Moreover, P_λ has a canonical Levi subgroup L_λ whose Lie algebra $\mathfrak{l}_\lambda$ is the centraliser of $d\lambda(1)$ in $\mathfrak{g}$. By the discussion after (11), we then further have $H_G^*(G/P_\lambda) \cong H_{L_\lambda}^*$. Now let τ be a compact real structure of G which preserves H and commutes with σ (which exists e. g. by [23, Prop. II.6]). Then L_λ^τ is a compact real form and hence a maximal compact subgroup of L_λ, so $H_{L_\lambda}^* \cong H_{L_\lambda^\tau}^*$. At this point, we can apply the isomorphism (12) to conclude

$$(16) \qquad H_G^*(G/P_\lambda) \cong H_{P_\lambda}^* \cong H_{L_\lambda}^* \cong H_{L_\lambda^\tau}^* \cong S(\mathfrak{l}_\lambda^\tau)^{L_\lambda^\tau} \otimes_{\mathbb{R}} \mathbb{C} \cong S(\mathfrak{l}_\lambda)^{L_\lambda}$$

(where τ also denotes the real structure on $\mathfrak{g}$ obtained from τ by differentiation). Similarly, we have

$$(17) \qquad H_G^* \cong H_{G^\tau}^* \cong S(\mathfrak{g}^\tau)^{G^\tau} \otimes_{\mathbb{R}} \mathbb{C} \cong S(\mathfrak{g})^G.$$

Lastly, via the Killing form on $\mathfrak{g}$ we obtain equivariant isomorphisms $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{l}_\lambda \cong \mathfrak{l}_\lambda^*$. We interpret the ring $S(\mathfrak{g}^*)^G$ as the *G-invariant polynomials* on $\mathfrak{g}$ and thus denote it as $\mathbb{C}[\mathfrak{g}]^G$ (and extend this notation to other Lie algebras and Lie groups). Then:

$$(18) \qquad S(\mathfrak{g})^G \cong \mathbb{C}[\mathfrak{g}]^G, \quad S(\mathfrak{l}_\lambda)^{L_\lambda} \cong \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda}$$

Lemma 3.14. *Let G, σ, and λ be as in Proposition 3.10 and assume that $\sigma_*\lambda = \lambda$. Under the isomorphisms (16), (17) and (18), the involution σ^* of (15) is identified with*

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} & \xrightarrow{\theta^*} & \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \\
{\scriptstyle\text{res}}\uparrow & & {\scriptstyle\text{res}}\uparrow \\
\mathbb{C}[\mathfrak{g}]^G & \xrightarrow{\theta^*} & \mathbb{C}[\mathfrak{g}]^G,
\end{array}
$$

where θ^* denotes precomposition of polynomials with θ and res *their restriction to subalgebras.*

Proof. Under our assumption, the map $\underline{\sigma}$ has the simple form

$$\underline{\sigma}(gP_\lambda) = \sigma(g)P_\lambda, \quad g \in G.$$

It is then not difficult to trace this through each of the steps in (16)–(18) to arrive at the claimed description. $\qquad\square$

The description in Lemma 3.14 will be used for the proof of Theorem 1.1. However, translating this further through the isomorphism $\mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \cong \mathcal{C}^\lambda(\mathfrak{g})$, the involutions σ^* turn out to fit into a more general class of finite order automorphisms of $\mathcal{C}^\lambda(\mathfrak{g})$, which we now introduce.[8] The main idea is that two natural sources of automorphisms are (suitable) automorphisms of $\mathfrak{g}$ and "rescaling the argument" on $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$.

Proposition 3.15. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and λ be a dominant integral weight (with respect to some choice of positivity) with corresponding irreducible representation $\rho^\lambda \colon \mathfrak{g} \to \mathrm{End}(V^\lambda)$. Suppose given*

- *an automorphism $\eta \in \mathrm{Aut}(\mathfrak{g})$ such that $\eta(\mathfrak{h}) = \mathfrak{h}$, and*
- *a root of unity $\zeta \in \mathbb{C}^\times$.[9]*

(i) *The map $\rho^\lambda \circ \eta$ defines another irreducible representation on V^λ whose highest weight $\eta^*\lambda$ is conjugate to $\lambda \circ \eta$ through the Weyl group.*

(ii) *Up to rescaling, there is a unique linear isomorphism $A = A_{\eta,\lambda} \colon V^\lambda \to V^{\eta^*\lambda}$ such that*

$$A(x \cdot v) = \eta(x) \cdot A(v) \quad \forall x \in \mathfrak{g}, v \in V^\lambda.$$

---

[8]We note that their construction, in Proposition 3.15 below, closely resembles that in [10, Thm. 6.3].

[9]The importance of this parameter was explained to us by Miguel González.

(iii) *If $\eta^*\lambda = \lambda$, then*

$$\widetilde{\iota_{\eta,\zeta}} := S(\zeta\eta) \otimes \mathrm{Conj}_A \colon S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda) \to S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)$$

   restricts to an automorphism $\iota_{\eta,\zeta}$ of $\mathcal{C}^\lambda(\mathfrak{g})$. (Here $S(\zeta\eta)$ denotes the automorphism of $S(\mathfrak{g})$ induced by the linear isomorphism $x \mapsto \zeta \cdot \eta(x)$, and Conj_A is conjugation by $A = A_{\eta,\lambda}$.)

(iv) *Let $\psi \in \mathrm{Int}(\mathfrak{g})$ normalise $\mathfrak{h}$ and put $\eta' := \eta\psi$. Then $\eta^*\lambda = (\eta')^*\lambda$, and if this equals λ we have $\iota_{\eta',\zeta} = \iota_{\eta,\zeta}$.*

(v) *If $\eta \in \mathrm{Int}(\mathfrak{g})$, then $\eta^*\lambda = \lambda$ for all dominant integral weights λ, and $\iota_{\eta,\zeta} = \iota_{\mathrm{id},\zeta} = \zeta^{\deg}$. That is, $\iota_{\eta,\zeta}$ acts on homogeneous elements of degree k by multiplication with ζ^k.*

Proof. The first item is straightforward, and the second an easy consequence of Schur's Lemma. For (iii), it suffices to observe that $\widetilde{\iota_{\eta,\zeta}}$ intertwines the usual $\mathfrak{g}$-action on both factors with the action twisted by η; in particular it sends invariant elements to invariant elements. For item (iv) we use that $\lambda \circ \psi$ is in the same Weyl group orbit as λ, and then the definition of $\mathcal{C}^\lambda(\mathfrak{g})$ to see that the contribution of ψ cancels out. Lastly, (v) is an application of (iv) followed by a straightforward computation of $\iota_{\mathrm{id},\zeta}$. $\qquad\square$

Proposition 3.16. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, $G^\vee$ the Langlands dual group of G and $\mathfrak{g}^\vee$ its Lie algebra. Let λ be a minuscule weight of $\mathfrak{g}$, which we view also as a cocharacter of $G^\vee$, and σ a real structure of $G^\vee$ preserving the $G^\vee$-orbit of λ. Then there exists $\eta \in \mathrm{Aut}_2(\mathfrak{g})$ such that the involution σ^* of $H_{G^\vee}^{2*}(G^\vee/P_\lambda)$ defined as in (15) fits into a commutative diagram*

$$
\begin{array}{ccc}
H_{G^\vee}^{2*}(G^\vee/P_\lambda) & \xrightarrow{\ \sigma^*\ } & H_{G^\vee}^{2*}(G^\vee/P_\lambda) \\[4pt]
{\scriptstyle\cong}\big\uparrow & & {\scriptstyle\cong}\big\uparrow \\[4pt]
\mathcal{C}^\lambda(\mathfrak{g}) & \xrightarrow{\ \iota_{\eta,-1}\ } & \mathcal{C}^\lambda(\mathfrak{g}).
\end{array}
$$

An explicit construction of η goes as follows: let σ^s be any split real structure of $G^\vee$, and let κ denote the image of $\sigma\sigma^s$ in $\mathrm{Out}(\mathfrak{g}^\vee) := \mathrm{Aut}(\mathfrak{g}^\vee)/\mathrm{Int}(\mathfrak{g}^\vee)$. Use the canonical isomorphism $\mathrm{Out}(\mathfrak{g}) \cong \mathrm{Out}(\mathfrak{g}^\vee)$, to identify κ with an element of $\mathrm{Out}(\mathfrak{g})$, and define $\eta \in \mathrm{Aut}(\mathfrak{g})$ to be any lift (of order two) of that element.

Proof. Let $\mathfrak{b} \leq \mathfrak{g}$ be a Borel subalgebra and $\mathfrak{h} \leq \mathfrak{b}$ a Cartan subalgebra; we may realise λ as an element of $\mathfrak{h}^*$ dominant with respect to $\mathfrak{b}$ and identify $\mathfrak{h}^*$ with a Cartan subalgebra of $\mathfrak{g}^\vee$ preserved by σ. To keep track of identifications, we divide the isomorphism $H_{G^\vee}^{2*}(G^\vee/P_\lambda) \cong \mathcal{C}^\lambda(\mathfrak{g})$ sketched in Remark 2.5 into numbered steps (recalled below):

$$H_{G^\vee}^{2*}(G^\vee/P_\lambda) \overset{\mathbf{1}}{\cong} \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \overset{\mathbf{2}}{\cong} \mathbb{C}[\mathfrak{h}^*]^{W_\lambda} \overset{\mathbf{3}}{\cong} (S(\mathfrak{h}) \otimes \mathrm{End}_\mathfrak{h}(V^\lambda))^W \overset{\mathbf{4}}{\cong} \mathcal{C}^\lambda(\mathfrak{g}).$$

By Remark 3.9(2), there exists a quasi-split real structure σ^{qs} inner to σ such that $\sigma^{qs}(\mathfrak{h}^*) = \mathfrak{h}^*$ and $\sigma_*^{qs}\lambda = \lambda$. Let θ be the Cartan involution of σ^{qs}. By Lemma 3.14, step **1** (cf. (16)–(18)) then identifies σ^* with θ^*. Step **2** is the Chevalley restriction theorem for the connected reductive group L_λ and the Cartan subalgebra $\mathfrak{h}^* \leq \mathfrak{l}_\lambda$ (see Corollary 4.4 below). In particular, this step identifies θ^* with $(\theta|_{\mathfrak{h}^*})^*$, which we still denote θ^* by abuse of notation.

Now, if σ^s is split then θ acts on $\mathfrak{h}^*$ by -1. Since all isomorphisms above preserve the gradings, they identify σ^* with $(-1)^{\deg} = \iota_{\mathrm{id},-1}$ in this case. More generally, there exists a pinning-preserving involution $\widetilde{\eta}$ of $\mathfrak{g}^\vee$ such that $\widetilde{\eta}(\mathfrak{h}^*) = \mathfrak{h}^*$ and $\sigma^{qs}\widetilde{\eta}$ is a split real structure.[10] It therefore remains to translate the involution $\widetilde{\eta}^*$ through steps **3** and **4**.

---

[10]To obtain $\eta^\vee$, use the action of σ^{qs} on simple roots with respect to a Borel subalgebra fixed by σ^{qs}.

Step **3** consists of noticing that the ring $\mathrm{End}_\mathfrak{h}(V^\lambda)$ (the $\mathfrak{h}$-equivariant endomorphisms of V^λ) is isomorphic to $\bigoplus_\mu \mathrm{End}(V_\mu^\lambda)$ where V_μ^λ are the weight spaces. Hence,

$$\mathcal{C}^\lambda(\mathfrak{h}) \overset{\mathrm{def}}{=} (S(\mathfrak{h}) \otimes \mathrm{End}_\mathfrak{h}(V^\lambda))^W \cong (S(\mathfrak{h}) \otimes \bigoplus \mathrm{End}(V_\mu^\lambda))^W,$$

and since W acts transitively on the one-dimensional summands $\mathrm{End}(V_\mu^\lambda)$, any W-invariant element of $S(\mathfrak{h}) \otimes \bigoplus \mathrm{End}(V_\mu^\lambda)$ is determined by its component in $S(\mathfrak{h}) \otimes \mathrm{End}(V_\lambda^\lambda)$. Restriction to this component yields an isomorphism $\mathcal{C}^\lambda(\mathfrak{h}) \to S(\mathfrak{h})^{W_\lambda} = \mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$, the inverse of **3** (cf. [24, Thm. 2.6]). The isomorphism **4** is simply the inverse of the restriction map r_λ from (14) (cf. Proposition 2.2(v)).

We claim that steps **3** and **4** identify $\tilde{\eta}^*$ with $\iota_{\eta,1}$, where $\eta \in \mathrm{Aut}(\mathfrak{g})$ preserves $\mathfrak{h}$ and lifts the class of $\tilde{\eta}$ in $\mathrm{Out}(\mathfrak{g}^\vee) \cong \mathrm{Out}(\mathfrak{g})$. To see this, let $f \in \mathcal{C}^\lambda(\mathfrak{g})$ be arbitrary. We may assume η to be pinning-preserving, so that $\lambda \circ \eta = \eta$ and Conj_A acts trivially on $\mathrm{End}(V_\lambda^\lambda)$. If the $\mathrm{End}(V_\lambda^\lambda)$ component of $r_\lambda(f)$ is f_λ, then that of $r_\lambda(\iota_{\eta,1}(f))$ is $S(\eta)(f_\lambda)$. But, viewing $S(\mathfrak{h})$ as $\mathbb{C}[\mathfrak{h}^*]$, this is precisely $\tilde{\eta}^* f_\lambda$, which proves our claim.

Altogether, the isomorphisms **1** through **4** identify $\sigma^* = (\sigma^{qs}\tilde{\eta})^*$ with $\iota_{\mathrm{id},-1} \circ \iota_{\eta,1} = \iota_{\eta,-1}$ as we wanted to show. $\qquad\square$

4. Invariant rings and quasi-compact real structures

Lemma 3.14 translates the involutions (15) we are interested in to the setting of invariant polynomials. In this section, we collect important facts about invariant polynomials, most importantly the key Lemma 4.7. This result motivates the use of quasi-compact real structures, as discussed at the end of the section. We begin by reiterating a definition from the previous section:

Definition 4.1. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Then $\mathbb{C}[\mathfrak{g}]^G$ denotes the subring of $\mathbb{C}[\mathfrak{g}] := S(\mathfrak{g}^*)$ consisting of elements invariant under the canonical G-action. Analogously, $\mathbb{C}[\mathfrak{g}]^\mathfrak{g}$ denotes the subring of $\mathbb{C}[\mathfrak{g}]$ consisting of elements annihilated by $\mathfrak{g}$.

Remark 4.2. By definition of $\mathrm{Int}(\mathfrak{g})$, we have $\mathbb{C}[\mathfrak{g}]^\mathfrak{g} = \mathbb{C}[\mathfrak{g}]^{\mathrm{Int}(\mathfrak{g})}$. If G is connected, the adjoint action $G \to \mathrm{Aut}(\mathfrak{g})$ has values in $\mathrm{Int}(\mathfrak{g})$, so $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{g}]^\mathfrak{g}$. For disconnected G, $\mathbb{C}[\mathfrak{g}]^G$ can be strictly smaller than $\mathbb{C}[\mathfrak{g}]^\mathfrak{g}$, see Example 4.9 below.

An important classical result about invariant polynomials is the following theorem of Chevalley:

Theorem 4.3 (cf. [31], Thm. 4.9.2). *Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, and W the corresponding Weyl group. Then the canonical restriction map $\mathbb{C}[\mathfrak{g}]^\mathfrak{g} \to \mathbb{C}[\mathfrak{h}]^W := S(\mathfrak{h}^*)^W$ is an isomorphism.*

Corollary 4.4. *Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, and $N_G(\mathfrak{h})$ its normaliser. Then restriction defines an isomorphism*

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^{N_G(\mathfrak{h})}.$$

Proof. Let G_0 denote the identity component of G. Using the usual decomposition of $\mathfrak{g}$ into its centre and derived subalgebra, Theorem 4.3 extends at once to

$$\mathbb{C}[\mathfrak{g}]^{G_0} \cong \mathbb{C}[\mathfrak{h}]^W.$$

To obtain the G−invariants, we now have to take into account the action of the component group G/G_0. But it is easy to see that $N_G(\mathfrak{h})$ meets all components of G, so

$$\mathbb{C}[\mathfrak{g}]^G = (\mathbb{C}[\mathfrak{g}]^{G_0})^{G/G_0} = (\mathbb{C}[\mathfrak{g}]^{G_0})^{N_G(\mathfrak{h})} \cong (\mathbb{C}[\mathfrak{h}]^W)^{N_G(\mathfrak{h})} = \mathbb{C}[\mathfrak{h}]^{N_G(\mathfrak{h})}.$$

$\qquad\square$

Having recalled this tool, we now return to the study of involutions on invariant polynomial rings. Directly from the definition, we obtain the following counterpart of Lemma 3.13:

Proposition 4.5. *Let $\mathfrak{g}$ be a complex Lie algebra and $\theta \in \mathrm{Aut}_2(\mathfrak{g})$. Then the involution θ^* of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ depends only on the inner class of θ.*

This suggests to look for particularly well-behaved involutions in a given inner class, and a natural candidate are those that preserve a pinning (see Definition 3.8). We recall some well-known results in this context:

Lemma 4.6. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta \in \mathrm{Aut}(\mathfrak{g})$.*

 (i) *The fixed subalgebra $\mathfrak{g}^{\theta}$ is reductive.*
 (ii) *If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ preserved by θ such that $\dim \mathfrak{h}^{\theta}$ is maximal among such Cartan subalgebras, then $\mathfrak{h}^{\theta}$ is a Cartan subalgebra of $\mathfrak{g}^{\theta}$.*
 (iii) *If $\mathfrak{g}$ is semisimple and θ preserves a pinning $(\mathfrak{h}, \Pi, \{X_{\alpha}\}_{\alpha \in \Pi})$, then $\mathfrak{g}^{\theta}$ is semisimple with Cartan subalgebra $\mathfrak{h}^{\theta}$. Moreover, if W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, then its subgroup*

$$W^{\theta} := \{w \in W : w\theta = \theta w\}$$

 is identified with the Weyl group of $(\mathfrak{g}^{\theta}, \mathfrak{h}^{\theta})$ by restriction to $\mathfrak{h}^{\theta}$.

Proof. For part (i), note that θ preserves the direct sum decomposition of $\mathfrak{g}$ into its centre $\mathfrak{z}(\mathfrak{g})$ and derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$. The Killing form of $[\mathfrak{g}, \mathfrak{g}]$ restricts to a nondegenerate ad-invariant symmetric bilinear form of $[\mathfrak{g}, \mathfrak{g}]^{\theta}$, which shows that $[\mathfrak{g}, \mathfrak{g}]^{\theta}$ is reductive. It follows that $\mathfrak{g}^{\theta} = \mathfrak{z}(\mathfrak{g})^{\theta} \oplus [\mathfrak{g}, \mathfrak{g}]^{\theta}$ is reductive as well.

To see that $\mathfrak{h}^{\theta}$ of part (ii) is a Cartan subalgebra, one can for instance use Gantmacher's normal form, cf. [23, Thm. 4.2], relating θ to a pinning-preserving automorphism.

For part (iii), we refer to [28, ch. 11]. $\qquad\square$

Lemma 4.7. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and θ a pinning-preserving automorphism of $\mathfrak{g}$. Then the restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}^{\theta}]^{\mathfrak{g}^{\theta}}$ is surjective and induces an isomorphism of $\mathbb{C}[\mathfrak{g}^{\theta}]^{\mathfrak{g}^{\theta}}$ with the coinvariant ring $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}_{\theta^*}$.*

Proof using reflection group theory. Let $\mathfrak{g}_{\mathrm{der}} := [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra. The decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{der}}$ is preserved by θ and yields

$$\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \cong \mathbb{C}[\mathfrak{z}(\mathfrak{g})] \otimes \mathbb{C}[\mathfrak{g}_{\mathrm{der}}]^{\mathfrak{g}_{\mathrm{der}}}.$$

Clearly, it then suffices to prove the lemma for $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}_{\mathrm{der}}$ separately. For the affine space $\mathfrak{z}(\mathfrak{g})$ it follows immediately from Proposition 2.6, so for the remainder we may assume that $\mathfrak{g}$ is semisimple.

Now let $(\mathfrak{h}, \Pi, \{X_{\alpha}\}_{\alpha \in \Pi})$ be a pinning of $\mathfrak{g}$ preserved by θ. As recalled above, $\mathfrak{h}^{\theta}$ is then a Cartan subalgebra of $\mathfrak{g}^{\theta}$, and W^{θ} is the corresponding Weyl group. By Corollary 4.4 we then have the commutative diagram of restriction maps

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} & \longrightarrow & \mathbb{C}[\mathfrak{g}^{\theta}]^{\mathfrak{g}^{\theta}} \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
\mathbb{C}[\mathfrak{h}]^{W} & \longrightarrow & \mathbb{C}[\mathfrak{h}^{\theta}]^{W^{\theta}},
\end{array}
$$

allowing us to pass to the setting of $\mathfrak{h}$. By [26, Lemma 6.1], $\mathbb{C}[\mathfrak{h}]^{W}$ admits algebraically independent homogeneous generators f_i $(i = 1, \ldots, \mathrm{rank}\, W)$ such that $\theta^* f_i = \varepsilon_i f_i$ for roots of unity ε_i. Moreover, θ fixes the sum of positive coroots, a regular element of $\mathfrak{h}$, so by [26, Corollary 6.5] the degrees of W^{θ} are precisely the d_i with $\varepsilon_i = 1$.

It is clear that the f_i with $\varepsilon_i \neq 1$ vanish on $\mathfrak{h}^{\theta}$. We have to show that these f_i span the kernel of the restriction $\mathbb{C}[\mathfrak{h}]^{W} \to \mathbb{C}[\mathfrak{h}^{\theta}]^{W^{\theta}}$, and that the restrictions of the f_i with $\varepsilon_i = 1$ generate $\mathbb{C}[\mathfrak{h}^{\theta}]^{W^{\theta}}$. To that end, consider the morphism $F : \mathfrak{h}^{\theta} \to \mathbb{C}^{\dim \mathfrak{h}^{\theta}}$ whose coordinates are the f_i with $\varepsilon = 1$. Observe (cf. [26, proof of Thm. 3.4]) that the fibre of 0 consists only of $0 \in \mathfrak{h}^{\theta}$ because *all* f_i vanish there. Comparing

dimensions, it follows that the coordinates of F are algebraically independent, which completes the proof. $\qquad\square$

Proof using Kostant sections. Reduce to the semisimple case as before. For a preserved pinning as above, the sum $e := \sum_\alpha X_\alpha$ is fixed by θ and a principal nilpotent element (cf. [21, §5]) for both $\mathfrak{g}$ and $\mathfrak{g}^\theta$. Upon extending it to an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{g}^\theta$, we obtain Kostant sections $\mathfrak{s} := e + \mathfrak{g}_f$ of $\mathfrak{g}$ and $e + \mathfrak{g}_f^\theta = \mathfrak{s}^\theta$ for $\mathfrak{g}^\theta$. By [20, Thm. 7], the restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$ is then equivalent to the restriction $\mathbb{C}[\mathfrak{s}] \to \mathbb{C}[\mathfrak{s}^\theta]$. This is clearly surjective, and Proposition 2.6 identifies $\mathbb{C}[\mathfrak{s}^\theta]$ with the coinvariant ring $\mathbb{C}[\mathfrak{s}]_\theta \cong \mathbb{C}[\mathfrak{g}]_{\theta^*}^{\mathfrak{g}}$. $\qquad\square$

Corollary 4.8. *Let G be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$. If θ is a pinning-preserving automorphism of G (with derivative also denoted by θ), then*

$$\mathbb{C}[\mathfrak{g}^\theta]^{G^\theta} = \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}.$$

Proof. We have to show that every $g \in G^\theta$ acts trivially on every $f \in \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$. By Lemma 4.7, f admits an extension $\tilde{f} \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ with $\tilde{f}|_{\mathfrak{g}^\theta} = f$. Moreover, since G is connected, we have $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[\mathfrak{g}]^G$, so $g \cdot \tilde{f} = \tilde{f}$. But this implies $g \cdot f = f$, too. $\qquad\square$

The point of the preceding corollary is that it holds even though G^θ can be disconnected. If θ is not pinning-preserving, the following example shows that $\mathbb{C}[\mathfrak{g}^\theta]^{G^\theta}$ can indeed be strictly smaller than $\mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$.

Example 4.9. Let $G = \mathrm{GL}_{2n}(\mathbb{C})$ for some $n \in \mathbb{N}$, and let θ be inverse-transpose, $\theta(A) = (A^t)^{-1}$. Then $G^\theta = \mathrm{O}_{2n}(\mathbb{C})$ with Lie algebra $\mathfrak{g}^\theta = \mathfrak{so}_{2n}(\mathbb{C})$. Using the Chevalley restriction theorem, one finds that

$$\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathfrak{so}_{2n}(\mathbb{C})} \cong \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^{n-1}}$$

where $S_n \ltimes \mathbb{Z}_2^{n-1}$ acts on the variables by signed permutations with an even number of sign changes. From this ring, we obtain $\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathrm{O}_{2n}(\mathbb{C})}$ by taking into account the component group of $\mathrm{O}_{2n}(\mathbb{C})$, which has order 2. Its nontrivial element acts on $\mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^{n-1}}$ by a single sign change, so that

$$\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathrm{O}_{2n}} \cong \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n},$$

with $S_n \ltimes \mathbb{Z}_2^n$ acting by arbitrary signed permutations. This is a strict subring of $\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathfrak{so}_{2n}(\mathbb{C})}$ — for instance, $x_1 x_2 \cdots x_n \notin \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$.

We can now translate back to the setting of real structures:

Corollary 4.10. *Let σ be a real structure of a connected complex reductive group G with Cartan involution θ. If θ is pinning-preserving then the restriction $H_G^* \to H_{G^\sigma}^*$ is surjective and identifies $H_{G^\sigma}^*$ with the coinvariant ring $(H_G^*)_{\sigma^*}$*

Proof. As in (17)–(18) we obtain compatible isomorphisms

$$H_G^* \cong \mathbb{C}[\mathfrak{g}]^G, \quad H_{G^\sigma}^* \cong \mathbb{C}[\mathfrak{g}^\theta]^{G^\theta},$$

where θ is the Cartan involution of σ (with respect to a suitable compact real structure). The result then follows from Lemma 4.7 in combination with Corollary 4.8. $\qquad\square$

Definition 4.11. A real structure with pinning-preserving Cartan involution (as in Corollary 4.10) is called *quasi-compact*.

Quasi-compact real structures play a key role in this paper due to Corollary 4.10. The next Proposition establishes basic facts about them, including the reason for their name.[11]

---

[11]While uncommon, the name *quasi-compact* is not new, appearing for example in [1, Rmk. 8.3].

Proposition 4.12. *Let G be a connected complex reductive group. Every inner class of real structures on G contains a quasi-compact real structure and this real structure is unique up to inner-isomorphism. If G_0 is the corresponding real form, then the dimension of its maximal compact subgroup is maximal among all real forms in the given inner class.*

Proof. Existence and uniqueness of the quasi-compact real structure are equivalent via Theorem 3.6 to corresponding statements about pinning-preserving involutions. In turn, these follow from well-known descriptions of $\mathrm{Aut}(\mathfrak{g})$, see e.g. [23, ch. 4]. The second statement can be checked using Gantmacher's normal form for automorphisms, cf. [23, Thm. 4.2]. $\qquad\square$

We will also need the following Lemma, which lets us compare the invariant ring of a quasi-compact real form with that of any real form inner to it:

Lemma 4.13. *Let σ be a real structure of a connected complex reductive Lie group G with Lie algebra $\mathfrak{g}$. There exist a quasi-compact real structure σ_0 and a compact real structure τ of G such that*

- *σ_0 is inner to σ,*
- *σ and σ_0 both commute with τ,*
- *there exists a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ preserved by σ, σ_0 and τ, and*
- *$\mathfrak{h}^\sigma = \mathfrak{h}^{\sigma_0}$.*

Moreover, let $\theta := \sigma\tau$ and $\theta_0 := \sigma_0\tau$ denote the Cartan involutions, and $W(\theta)$, $W(\theta_0)$ the Weyl groups for $(\mathfrak{g}^\theta, \mathfrak{h}^\theta)$ and $(\mathfrak{g}^{\theta_0}, \mathfrak{h}^{\theta_0})$.[12] Then, as subgroups of $\mathrm{GL}(\mathfrak{h}^\theta) = \mathrm{GL}(\mathfrak{h}^{\theta_0})$, we have

$$W(\theta) \leq W(\theta_0),$$

and it follows that $H^*_{G^{\sigma_0}}$ canonically injects into $H^*_{G^\sigma}$.

Proof. The existence of σ_0 is rather standard: the first requirement can be achieved via Proposition 4.12, and the second via a variant of [23, Prop. 3.7]. The third and fourth conditions can be incorporated by conjugation with an appropriate inner automorphism. For the inclusion of Weyl groups, it suffices to check that each element of the root system of $(\mathfrak{g}^\theta, \mathfrak{h}^\theta)$ is – up to nonzero rescaling – contained in that of $(\mathfrak{g}^{\theta_0}, \mathfrak{h}^{\theta_0})$; this can be done using Gantmacher normal forms, cf. [23, Thm. 4.2].

Finally, via (17)–(18) and Corollary 4.4 we obtain a diagram

$$
\begin{array}{ccc}
H^*_{G_0^\sigma} & \dashrightarrow & H^*_{G^\sigma} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathbb{C}[\mathfrak{g}^{\theta_0}]^{G^{\theta_0}} & \dashrightarrow & \mathbb{C}[\mathfrak{g}^{\theta}]^{G^{\theta}} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathbb{C}[\mathfrak{h}^{\theta}]^{N_{G^{\theta_0}}(\mathfrak{h}^{\theta})} & \dashrightarrow & \mathbb{C}[\mathfrak{h}^{\theta}]^{N_{G^{\theta}}(\mathfrak{h}^{\theta})}
\end{array}
$$

in which the dashed arrows are yet to be constructed – of course it suffices to construct one of them. The rings in the bottom row are the subrings of $\mathbb{C}[\mathfrak{h}^\theta]^{W(\theta_0)}$ resp. $\mathbb{C}[\mathfrak{h}^\theta]^{W(\theta)}$ invariant under the actions of the relevant component groups. But, by the same argument as in the proof of Corollary 4.8, we see that these component groups both act trivially on $\mathbb{C}[\mathfrak{h}^\theta]^{W(\theta_0)}$. Together with the containment $W(\theta) \leq W(\theta_0)$, this lets us put a canonical injection in the bottom row of the diagram above, finishing the proof. $\qquad\square$

---

[12]Here we are using Lemma 4.6 to see that $\mathfrak{h}^\theta$ and $\mathfrak{h}^{\theta_0}$ are Cartan subalgebras. Also, $W(\theta_0) = W^{\theta_0}$ is the centraliser of θ_0 in W by Lemma 4.6(iii), but such a description need not hold for $W(\theta)$.

5. Proof of main theorem

We now come to the proof of Theorem 1.1. It is structured into three lemmas, followed by a main body combining everything. The first two lemmas reduce from the semisimple to the simple case; the third gives a construction of the appropriate real structure in almost all simple types. (For the one remaining type, A_{2n}, we supply ad hoc arguments.)

Lemma 5.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ be its decomposition into simple complex Lie algebras $\mathfrak{g}_i$. Under this isomorphism, a minuscule weight λ of $\mathfrak{g}$ decomposes as a sum $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$ of minuscule weights of the $\mathfrak{g}_i$. Moreover, the Kirillov algebra $\mathcal{C}^\lambda(\mathfrak{g})$ decomposes as*

$$\mathcal{C}^\lambda(\mathfrak{g}) \cong \mathcal{C}^{\lambda_1}(\mathfrak{g}_1) \otimes \cdots \otimes \mathcal{C}^{\lambda_\ell}(\mathfrak{g}_\ell).$$

Proof. The decomposition of λ is a simple consequence of highest weight theory, which also yields $V^\lambda \cong V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_\ell}$ (with $\mathfrak{g}_i$ acting on the i-th tensor factor). It follows that both factors in $S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)$ decompose as tensor products compatibly with the decomposition of $\mathfrak{g}$; hence, so does the Kirillov algebra. $\qquad\square$

Lemma 5.2. *Let G be a connected complex semisimple Lie group of adjoint type[13], and let $G \cong G_1 \times \cdots \times G_\ell$ be its decomposition into simple factors G_i. Let σ be a real structure on G. Then σ permutes the G_i with orbits of one or two elements. For each i, there are two possibilities:*

(a) *If $\sigma(G_i) = G_i$, then σ restricts to a real structure of G_i.*
(b) *If $\sigma(G_i) = G_j$ with $i \neq j$, then $G_j \cong G_i$ and the Cartan involution of $\sigma|_{G_i \times G_j}$ is isomorphic to the* swap involution

$$G_i \times G_i \to G_i \times G_i, \quad (g, h) \mapsto (h, g).$$

Proof. For all i, j, the intersection $\sigma(G_i) \cap G_j$ is either trivial or all of G_j by simplicity, so σ indeed permutes the factors. The statement about orbit sizes and part (a) are clear. For part (b), choose any compact real structure τ_i of G_i and observe that $\sigma \tau_i \sigma$ is a compact real structure of G_j – indeed, its fixed point subgroup is isomorphic via σ to the compact Lie group $G_i^{\tau_i}$. The product $\tau := \tau_i \times (\sigma \tau_i \sigma)$ is then a compact real structure of $G_i \times G_j$ which commutes with σ. By construction, the Cartan involution $\theta := \sigma \tau \in \mathrm{Aut}_2(G_i \times G_j)$ maps G_i to G_j, and the resulting $G_i \cong G_j$ can be used to identify θ with the swap involution up to isomorphism. $\qquad\square$

Lemma 5.3. *Let $\mathfrak{g}$ be a simple complex Lie algebra with a minuscule coweight λ, and let $\mathfrak{l}_\lambda$ be the corresponding Levi subalgebra (i. e. the centraliser of λ). Let $\mathfrak{S}$ be an inner class of real structures on $\mathfrak{g}$ which preserve the $\mathrm{Ad}(\mathfrak{g})$-orbit of λ. If $\mathfrak{g}$ is not of type A_{2n} ($n \in \mathbb{N}$), then $\mathfrak{S}$ contains a real structure σ, unique up to inner-isomorphism, for which*

(i) *$\sigma_* \lambda = \lambda$, and*
(ii) *$\sigma|_{\mathfrak{l}_\lambda}$ is quasi-compact.*

Proof. We may identify λ with the coroot of a simple root α_1 with respect to a Borel subalgebra $\mathfrak{b} \leq \mathfrak{g}$ and Cartan subalgebra $\mathfrak{h} \leq \mathfrak{b}$. Denote the remaining simple roots by $\alpha_2, \ldots, \alpha_r$ and extend this data to a pinning by choosing root vectors $e_i \in \mathfrak{g}_{\alpha_i}$. These choices yield a set of standard generators e_i, f_i, h_i, which define a canonical Chevalley involution (also known as Weyl involution) $\omega \colon \mathfrak{g} \to \mathfrak{g}$ with

$$\omega(h_i) = -h_i, \quad \omega(e_i) = -f_i, \quad \omega(f_i) = -e_i.$$

as well as a canonical split real structure σ_s which fixes the generators. Their composition $\tau := \sigma \omega$ is a canonical compact real structure. (For details see [23, p. 18].)

With this setup, $\mathfrak{l}_\lambda$ inherits the standard generators e_i, f_i, h_i, $i \geq 2$; in particular ω restricts to $\mathfrak{l}_\lambda$ as its canonical Chevalley involution. Following [23, ch. 4], we explicitly relate $\omega|_{\mathfrak{l}_\lambda}$ to a pinning-preserving involution in its inner class. Let $\mathbf{h} = \sum_{i=2}^r r_i h_i$ be the dual of the sum of positive roots of $\mathfrak{l}_\lambda$ and extend it to an $\mathfrak{sl}_2$-triple by $\mathbf{e} := \sum_{i=2}^r \sqrt{r_i} e_i$ and $\mathbf{f} := \sum_{i=2}^r \sqrt{r_i} f_i$. It is not hard to show that

---

[13]This ensures that G decomposes into simple factors.

$\varphi \coloneqq \exp(\mathrm{ad}(\frac{\pi}{2}(\mathbf{e}-\mathbf{f}))) \in \mathrm{Int}(\mathfrak{l}_\lambda)$ commutes with $\omega|_{\mathfrak{l}_\lambda}$ and that $\omega|_{\mathfrak{l}_\lambda}\varphi$ is a pinning-preserving involution of $\mathfrak{l}_\lambda$. Being an inner automorphism, φ has an obvious extension $\tilde{\varphi} \in \mathrm{Int}(\mathfrak{g})$ which commutes with ω and τ, and fixes λ. We claim that $\tilde{\varphi}^2 = \mathrm{id}$.

Since $\varphi^2 = \mathrm{id}_{\mathfrak{l}_\lambda}$, it suffices to show that $\tilde{\varphi}^2(e_1) = e_1$. (Indeed, any automorphism of $\mathfrak{g}$ is uniquely determined by its action on $\mathfrak{h}$ and the e_i, see e.g. [17, Thm. 14.2].) To this end, observe that e_1 is a lowest weight vector for an $\mathfrak{sl}_2$-invariant subspace V of $\mathfrak{g}$ with respect to the $\mathfrak{sl}_2$-triple $(\mathbf{e}, \mathbf{f}, \mathbf{h})$. Lifting the representation to SL_2 identifies $\tilde{\varphi}^2$ with the action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so we have to show that the lowest weight of V is even, or equivalently that $\dim V$ is odd. Since λ is minuscule, the highest root α_{max} of $\mathfrak{g}$ is of the form $\alpha_1 + \sum_{i=2}^r n_i \alpha_i$ for $n_i \in \mathbb{N}$. Moreover, it is clear that $\mathrm{ad}(\mathbf{e})^n(e_1) \in \mathfrak{g}_{\alpha_{max}}$ for $n \coloneqq \sum_{i=2}^r n_i$. On the other hand, it is also clear that $\mathrm{ad}(\mathbf{f})^n$ maps $\mathfrak{g}_{\alpha_{max}}$ to $\mathfrak{g}_{\alpha_1}$. These observations combine to show that $\mathfrak{g}_{\alpha_{max}}$ is contained in V as its highest weight space, and that $\dim V = n + 1$ – the height of α_{max}. This number is known to equal $c - 1$ where c is the Coxeter number (see e.g. [5, Prop. VI.31]). Altogether, we have $\tilde{\varphi}^2 = \mathrm{id}$ provided that the Coxeter number of $\mathfrak{g}$ is even, which is guaranteed by the exclusion of type A_{2n}.

If $\sigma_s \in \mathfrak{S}$, we may now take $\sigma \coloneqq \sigma_s \tilde{\varphi}$, which indeed fixes λ and whose Cartan involution $\omega\varphi$ preserves a pinning of $\mathfrak{l}_\lambda$. In general, the assumption on $\mathfrak{S}$ implies that there exists an involution $\eta \in \mathrm{Aut}(\mathfrak{g})$, preserving the chosen pinning, fixing λ, and commuting with ω and σ_s, such that $\eta\sigma_s \in \mathfrak{S}$ (cf. proof of Proposition 3.16). Then η commutes also with $\tilde{\varphi}$ and we may take $\sigma \coloneqq \eta\sigma_s\tilde{\varphi}$. This proves existence.

As for uniqueness, suppose now that σ' is any (other) real structure with the required properties. Up to conjugation by inner automorphisms, we may assume that σ' preserves the previously chosen Cartan subalgebra $\mathfrak{h}$ and commutes with τ, and that the Cartan involution $\theta' \coloneqq \sigma'\tau$ preserves the chosen pinning of $\mathfrak{l}_\lambda$. But since θ' is inner to $\theta \coloneqq \sigma\tau$, the same holds for their restrictions to $\mathfrak{l}_\lambda$. Then, since both preserve the same pinning, we find $\theta'|_{\mathfrak{l}_\lambda} = \theta|_{\mathfrak{l}_\lambda}$. The inner automorphism $\psi \coloneqq \theta\theta'$ then normalises $\mathfrak{h}$ and fixes all simple roots, whence $\psi = \exp(\mathrm{ad}\, t)$ for some $t \in \mathfrak{h}$. Since ψ also fixes $\mathfrak{l}_\lambda$ pointwise, t must be a scalar multiple of $\lambda \in \mathfrak{h}$. But this yields

$$\theta' = \theta\psi = \theta\exp(\mathrm{ad}\, t) = \exp(\mathrm{ad}(-t/2))\,\theta\,\exp(\mathrm{ad}(t/2)),$$

so θ' is inner-isomorphic to θ as we needed to show.   □

Remark 5.4 (Satake diagrams). One can also verify the lemma using Satake diagrams. As before, we can identify λ with the fundamental coweight of a simple root. We may then restrict to real structures $\sigma \in \mathfrak{S}$ which preserve $\mathfrak{h}$ and for which $\mathfrak{h}^\sigma$ is maximally split.[14] These are classified up to inner-isomorphism by Satake diagrams [3], which consist of the Dynkin diagram of $\mathfrak{g}$ together with a choice of painted nodes and an involutive permutation of the unpainted nodes, indicated by arrows.

The assumption that the elements of $\mathfrak{S}$ preserve the $\mathrm{Ad}(\mathfrak{g})$-orbit of λ implies that the node representing λ has no arrow attached (in any of the Satake diagrams resulting from $\mathfrak{S}$). One then verifies that condition (i) is equivalent to that node being unpainted. Moreover, deleting an unpainted node with no arrow from a Satake diagram yields the Satake diagram for the restricted real structure on the corresponding Levi subalgebra. Thus, it suffices to check that there is exactly one Satake diagram corresponding to a real structure in $\mathfrak{S}$ such that the node for λ is unpainted and such that deleting that node yields the Satake diagram of a quasi-compact real structure of $\mathfrak{l}_\lambda$. This is indeed the case whenever $\mathfrak{g}$ is not of type A_{2n}, as can be verified using the tables in Appendix A. The advantage of this approach is that it provides the inner-isomorphism type of σ via its Satake diagram.

Proof of Theorem 1.1. Since G is simply connected, its Langlands dual $G^\vee$ is of adjoint type and Lemmas 5.1 and 5.2 apply. Clearly, the permutation of simple factors of $G^\vee$ is the same for all $\sigma \in \mathfrak{S}$. By restricting to its orbits, the theorem is reduced to three cases:

(a) $\mathfrak{g}$ is simple and not of type A_{2n}
(b) $\mathfrak{g} \cong \mathfrak{sl}_{2n+1}$ for some $n \in \mathbb{N}$.
(c) $\mathfrak{g} \cong \mathfrak{g}_s \oplus \mathfrak{g}_s$ for $\mathfrak{g}_s$ simple, and the real structures in $\mathfrak{S}$ swap the two copies of $\mathfrak{g}_s$.

---

[14]i.e. $\dim(\mathfrak{h}^\theta)$ is minimal possible where θ denotes the Cartan involution.

In each case, we now give a real structure $\sigma \in \mathfrak{S}$ for which $\sigma_* \lambda = \lambda$ and such that the identity (5) holds. The latter is equivalent to

$$(19) \qquad (H^*_{L_\lambda})_{\sigma^*} \cong H^*_{L^\sigma_\lambda}$$

by Corollary 2.4 and (13).

- In case (a), we apply Lemma 5.3 to obtain $\sigma \in \mathfrak{S}$ fixing λ with quasi-compact restriction to L_λ. Then (19) follows from Corollary 4.10.
- In case (b), we have $G^\vee = \mathrm{PGL}_{2n+1}(\mathbb{C})$. Only the inner class of the split real structure fixes any minuscule coweights, and this inner class consists of a single inner-isomorphism class. We therefore only have to analyse the real structure $\sigma \colon A \mapsto \overline{A}$. The minuscule coweights of PGL_{2n+1} are exactly the fundamental coweights (and zero), and are (for standard choices) all fixed by σ. The corresponding Levi subgroups are $L_k := \mathrm{P}(\mathrm{GL}_k \times \mathrm{GL}_{2n+1-l})$ for $k \in \mathbb{N}$, and using (17) and Corollary 4.4 one finds that

$$H^*_{L_k} \cong \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{2n+1-1}]^{S_k \times S_{2n+1-k}} / (x_1 + \dots + y_{2n+1-k}).$$

  The Cartan involution $A \mapsto (A^t)^{-1}$ acts as -1 on a Cartan subalgebra, so Lemma 3.14 and Theorem 4.3 imply that σ^* acts on homogeneous elements by multiplication with $(-1)^{\deg}$. The coinvariant ring is then isomorphic to

$$\mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{2n+1-1}]^{S_k \times S_{2n+1-k} \times \mathbb{Z}_2^{2n+1}},$$

  with $\mathbb{Z}_2^{2n+1}$ acting by sign changes on the variables. But a computation similar to that in Example 4.9 identifies that ring with $H^*_{L^\sigma_k}$, proving (19) for this case.
- In case (c), all elements of $\mathfrak{S}$ are quasi-compact, and can be conjugated by an inner automorphism to fix λ. The restriction to $\mathfrak{l}_\lambda$ is then also quasi-compact, so we can proceed as in case (a).

This concludes the case distinction; the rest of the proof is again uniform. Firstly, (6) holds for any quasi-compact $\sigma_0 \in \mathfrak{S}$ by Lemma 3.13, (17), and Corollary 4.10. The injection (7) is achieved by Lemma 4.13. The maps in diagram (8) are all derived from restriction maps of invariant polynomial rings, so the diagram commutes. $\qquad \square$

6. Characterisation of freeness

In this section, Theorem 1.1 is used to characterise freeness of the coinvariant homomorphism (4), resulting in a proof of Theorem 1.2. According to Theorem 1.1, we have to analyse the composition

$$(20) \qquad H^*_{(G^\vee)^{\sigma_0}} \xrightarrow{\varphi} H^*_{(G^\vee)^\sigma} \to H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma / P^\sigma_\lambda)$$

where φ is the canonical injection constructed in Lemma 4.13.

Lemma 6.1. φ is finite.

Proof. Indeed, it is an injection between finitely generated $\mathbb{C}$-algebras of equal transcendence degree (namely $\mathrm{rank}(G^\vee)^\sigma = \mathrm{rank}(G^\vee)^{\sigma_0}$). $\qquad \square$

The behaviour of the second map in (20) is related to the geometry of the homogeneous space $X := (G^\vee)^\sigma / P^\sigma_\lambda$. Here it is more convenient to work with compact Lie groups, so we fix a compact real structure τ of $G^\vee$ that commutes with σ. It will be convenient to choose τ such that $\tau_* \lambda = -\lambda$, which can be achieved by a standard construction of compact real structures, cf. [19, Thm. 6.11]. Now, $K := ((G^\vee)^\sigma)^\tau$ is a maximal compact subgroup of $(G^\vee)^\sigma$. Since $(G^\vee)^\sigma \cong KP^\sigma_\lambda$ [19, Prop. 7.83f], K acts transitively on X, so $X \cong K/L$ where $L := P^\sigma_\lambda \cap K = L^\sigma_\lambda \cap K$. The cohomology of such compact homogeneous spaces is particularly well-behaved in the so-called *equal rank* case:

Theorem 6.2. *If K is a connected compact Lie group and $L \leq K$ a closed subgroup, then the odd singular cohomology of $X = K/L$ vanishes if and only if K and L have the same rank. In this case, we further have an isomorphism $H^*_K(X) \cong H^*(X) \otimes_\mathbb{C} H^*_K$ of H^*_K-modules.*

Note: In our convention, *rank* means dimension of maximal torus. This need not coincide with the rank of the root system (due to a possibly positive-dimensional centre).

Proof. The first statement is classical: the "only if" part follows from vanishing of the Euler characteristic $\chi(X)$ in the nonequal rank case first shown by Hopf and Samelson [16]. The "if" part is due to Borel [4]. For a more detailed discussion, see [6, ch. 5]. The statement about equivariant cohomology (known as *equivariant formality*) follows from degeneracy of the Serre spectral sequence for the fibration $X_K \to \mathbb{B}K$ with fibre X, cf. [6, ch. 9].   □

Returning to the analysis before Theorem 6.2, we are led to compare the ranks of $K = ((G^\vee)^\sigma)^\tau$ and $L = L^\sigma_\lambda \cap K$. We translate the question to the complex setting via the Cartan involution $\theta := \sigma\tau$, noting that K and L are compact real forms of $(G^\vee)^\theta$ and L^θ_λ, respectively. The theorem then implies:

Lemma 6.3. *If $(G^\vee)^\theta$ and L^θ_λ have the same rank, then $H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda)$ is a free module over $H^*_{(G^\vee)^\sigma}$. Otherwise, $H^*_{(G^\vee)^\sigma}$ has strictly larger transcendence degree than $H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda)$, so the module structure cannot be free.*

Proof. The first assertion indeed follows immediately from Theorem 6.2. For the statement about transcendence degrees, let $\mathfrak{h} \leq (\mathfrak{g}^\vee)^\theta$ and $\mathfrak{h}' \leq \mathfrak{l}^\theta_\lambda$ be Cartan subalgebras. By Corollary 4.4, the rings involved are invariant subrings of $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}']$ by finite groups.[15] Their transcendence degrees are then the dimensions of $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. If $H^*_{(G^\vee)^\sigma}$ has strictly larger transcendence degree than $H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda)$, the structure map $H^*_{(G^\vee)^\sigma} \to H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda)$ is not injective, so it does not define a free module structure.   □

We are thus led to compare Cartan subalgebras of $(\mathfrak{g}^\vee)^\theta$ and $\mathfrak{l}^\theta_\lambda$, with the following result:

Lemma 6.4. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta \in \mathrm{Aut}_2(\mathfrak{g})$. Among Cartan subalgebras of $\mathfrak{g}$ preserved by θ, choose $\mathfrak{h}$ such that $\dim \mathfrak{h}^\theta$ is minimal. Let λ be a minuscule coweight of $\mathfrak{g}$ contained in $\mathfrak{h}^{-\theta}$ (i. e. $\theta(\lambda) = -\lambda$) and $\mathfrak{l}_\lambda$ the corresponding Levi subalgebra of $\mathfrak{g}$. If θ preserves a pinning of $\mathfrak{l}_\lambda$, then $\mathfrak{g}^\theta$ and $\mathfrak{l}^\theta_\lambda$ have equal rank if and only if θ is also pinning-preserving for $\mathfrak{g}$.*

Proof. Since $\mathfrak{l}_\lambda$ contains the centre of $\mathfrak{g}$, we may assume that $\mathfrak{g}$ is semisimple. The ranks of $\mathfrak{g}^\theta$ and $\mathfrak{l}^\theta_\lambda$ are the maximal dimensions of $\mathfrak{t}^\theta$ for θ-stable Cartan subalgebras $\mathfrak{t}$ of $\mathfrak{g}$ or of $\mathfrak{l}_\lambda$, respectively (cf. Lemma 4.6). In particular, both ranks admit the lower bound $\dim \mathfrak{h}^\theta$.

For convenience, we now translate to the parallel setting of real structures, which is more easily found in the literature. That is, we fix a compatible compact real structure τ of $\mathfrak{g}$ and define $\sigma := \tau\theta = \theta\tau$. Then $\mathfrak{t}$ as above correspond to *maximally compact* Cartan subalgebras of $\mathfrak{g}^\sigma$ or $\mathfrak{l}^\sigma_\lambda$.

The conjugacy classes of Cartan subalgebras of a real reductive Lie algebra can be related by *Cayley transforms*, which are defined using *real or noncompact imaginary roots*, cf. [19, p. 390]. In particular, if there are no such roots with respect to σ, then there is only one such conjugacy class. If the Cartan involution θ preserves a pinning $(\mathfrak{t}, \Pi, \{X_\alpha\})$, there are no real roots with respect to $\mathfrak{t}$ and no noncompact imaginary simple roots in Π. Now suppose that, additionally, $\theta^*\alpha$ is either equal to or orthogonal to α for every simple α. Using [19, Prop. 6.104], this implies that there are no real or noncompact imaginary roots at all, so all Cartan subalgebras of of the real form are conjugate. Since θ acts on the simple roots according to an automorphism of the Dynkin diagram, this assumption holds in all simple types except for the nontrivial involution of type A_{2n}.

Thus, suppose that $\mathfrak{l}_\lambda$ has no simple summand of type A_{2n} on which θ restricts to a nontrivial involution. Then the above analysis implies that $\mathfrak{l}^\sigma_\lambda$ has only one conjugacy class of θ-stable Cartan subalgebras. In particular $\mathfrak{h}^\theta$ must be a Cartan subalgebra of $\mathfrak{l}^\theta_\lambda$, and so $\mathrm{rank}\,\mathfrak{l}^\theta_\lambda = \dim \mathfrak{h}^\theta$. A similar analysis shows that if θ is pinning-preserving on $\mathfrak{g}$ and there are no interfering type A_{2n} summands, $\mathfrak{g}^\theta$ has rank $\dim \mathfrak{h}^\theta$ as well. To finish the proof we thus have to do the following:

---

[15]Finiteness follows from the well-known finiteness of Weyl groups and the fact that G^θ and L^θ_λ have finitely many connected components. One way to see this is that they are homotopic to their intersections with the compact group K.

(1) Show that a non-quasi-compact real form of a complex reductive Lie algebra always admits more than one isomorphism class of Cartan subalgebras. Since only one of them is maximally compact [19, Prop. 6.61], it will follow that rank $\mathfrak{g}^\theta >$ dim $\mathfrak{h}^\theta$ in this case.

(2) Check the cases involving type A_{2n} summands with non-inner Cartan involutions separately.

For task (1), it suffices to show that such a real form always has noncompact imaginary roots with respect to a maximally compact Cartan subalgebra. This follows from the classification using *Vogan diagrams*, cf. [19, p. VI.8], since the absence of noncompact imaginary roots would imply quasi-compactness (by uniqueness of the classification).

For task (2), we may assume that $\mathfrak{g}$ is simple. For $\mathfrak{g} = \mathfrak{sl}_{2n+1}$, one finds that no non-inner involution restricts to a pinning preserving involution on any $\mathfrak{l}_\lambda$ defined as above. Thus, we only have to treat the cases where $\mathfrak{l}_\lambda$ has some type A_{2n} summand on which the Cartan involution acts nontrivially. Up to isomorphism, they are as follows:

- $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$ with $\mathfrak{l}_\lambda = \mathfrak{s}(\mathfrak{gl}_{2k+1} \oplus \mathfrak{gl}_{2n-2k-1})(\mathbb{C})$ for $k, n \in \mathbb{N}$; $\mathfrak{g}^\theta = \mathfrak{so}_{2n}(\mathbb{C})$, and $\mathfrak{l}_\lambda^\theta = \mathfrak{so}_{2k+1}(\mathbb{C}) \oplus \mathfrak{so}_{2n-2k-1}(\mathbb{C})$. Here θ is not pinning-preserving on $\mathfrak{g}$ and the fixed subalgebras have ranks n and $n - 1$, respectively.

- $\mathfrak{g} = \mathfrak{sp}_{4n+2}(\mathbb{C})$ for $n \in \mathbb{N}$ with $\mathfrak{l}_\lambda = \mathfrak{sl}_{2n+1}(\mathbb{C})$; $\mathfrak{g}^\theta = \mathfrak{gl}_{2n+1}(\mathbb{C})$ and $\mathfrak{l}_\lambda^\theta = \mathfrak{so}_{2n+1}(\mathbb{C})$. Here, θ is again not pinning-preserving on $\mathfrak{g}$, and the fixed subalgebras have ranks $2n + 1$ and n, respectively.

$\square$

Remark 6.5. It seems likely that Lemma 6.4 can be proven more systematically and with fewer assumptions, perhaps using qualitative properties of Cayley transforms, cf. [19, p. VI.7].

We now collect the auxiliary results of this section into a proof of Theorem 1.2, which characterises freeness of the coinvariant homomorphism (4).

Proof of Theorem 1.2. For $\mathfrak{g} = \mathfrak{sl}_{2n+1}(\mathbb{C})$, $n \in \mathbb{N}$, with $\mathfrak{S}$ containing a split real structure, the Theorem can be checked directly. Here all elements of the inner class are quasi-compact, and a simple computation (as in the proof of Theorem 1.1) shows that the coinvariant homomorphism (4) is indeed free in this case. For the remainder of the proof, we assume that no element of $\mathfrak{S}$ restricts to a split real structure on a type A_{2n} factor of $G^\vee$.

It is then clear from the proof of Theorem 1.1 that we may choose $\sigma = \sigma_0$ whenever a quasi-compact real structure in the given inner class fixes λ. If this is the case, the first map φ in (20) is an identity. Moreover, the second map is then free (i.e. defines a free module structure) by Lemmas 6.4 and 6.3.

On the other hand, if σ is not quasi-compact, then Lemmas 6.4 and 6.3 imply that the transcendence degree drops in the second step of (20). By Lemma 6.1, it stays the same in the first step, so overall the coinvariant homomorphism decreases transcendence degree. In particular, it is not injective, hence cannot give rise to a free module structure. $\square$

7. Action on fibres and weights

In this section, we relate the involutions discussed above to combinatorial information about the representation V^λ. In the general case, where λ is any dominant integral weight, one should work with the big algebra [12], whose fibres over suitable points of Spec $S(\mathfrak{g})^\mathfrak{g}$ can be identified with the canonical basis of V^λ (cf. [14, final slide]). For simplicity, we instead (continue to) restrict ourselves to the case where $\mathcal{C}^\lambda(\mathfrak{g})$ is commutative and thus agrees with the big algebra (cf. [12, Thm. 2.1]). By Proposition 2.2, this means that V^λ is weight multiplicity free, so instead of the canonical basis we can work with just the set of weights wt(λ). We begin this section by describing how to identify wt(λ) with fibres of the map

$$\text{(21)} \qquad\qquad \pi \colon \text{Spec}\, \mathcal{C}^\lambda(\mathfrak{g}) \to \text{Spec}\, S(\mathfrak{g})^\mathfrak{g}.$$

For involutions induced by real structures, we then describe how the fixed points of the action on weights are encoded by the coinvariant homomorphism (4). This is applied to the special case of

split real structures, where we recover (the weight multiplicity free case of) a $q = -1$ phenomenon of Stembridge [30, 29]. Along the way, we obtain a combinatorial obstruction to freeness of the coinvariant homomorphism (4), see Proposition 7.6. In all of this section, we **assume that V^λ is weight multiplicity free** (although this is not required in Lemma 7.1). For Theorem 1.3, proved below, λ is assumed minuscule.

The starting point is to consider, for $x \in \mathfrak{g}$, the evaluation homomorphism $S(\mathfrak{g})^\mathfrak{g} \cong S(\mathfrak{g}^*)^\mathfrak{g} \to \mathbb{C}$, in which we first use the Killing form to view $S(\mathfrak{g})^\mathfrak{g}$ as $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}$ and then evaluate these on $\mathfrak{g}$. Applying this to the first factor of $\mathcal{C}^\lambda(\mathfrak{g})$, we obtain a map

$$\mathrm{ev}_x \colon \mathcal{C}^\lambda = (S(\mathfrak{g})^\mathfrak{g} \otimes \mathrm{End}(V^\lambda))^\mathfrak{g} \to \mathrm{End}(V^\lambda).$$

The image of ev_x is a subalgebra of $\mathrm{End}(V^\lambda)$ which we denote by

$$\mathcal{C}_x^\lambda(\mathfrak{g}) := \mathrm{ev}_x(\mathcal{C}^\lambda(\mathfrak{g})).$$

Now assume that x is a semisimple element, contained in some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. It is then convenient to restrict the Kirillov algebra to $\mathfrak{h}$ via the map r_λ of (14). This map turns out to be generically an isomorphism. Namely, let $D \in S(\mathfrak{h}^*)^W$ be the *discriminant*, defined as the symmetric product of all positive roots. Via the Killing form and Chevalley's restriction theorem (Theorem 4.3), we can view D as an element of $S(\mathfrak{g})^\mathfrak{g}$ as well as of $\mathcal{C}^\lambda(\mathfrak{g})$ and $\mathcal{C}^\lambda(\mathfrak{h})$. The generic behaviour of (14) is then as follows:

Lemma 7.1 ([24], Lemma 2.1)**.** *The injection* (14) *induces an isomorphism* $\mathcal{C}^\lambda(\mathfrak{g})_D \cong \mathcal{C}^\lambda(\mathfrak{h})_D$ *of* $S(\mathfrak{g})_D^\mathfrak{g}$*-algebras, where the index D denotes localisation at D.*

Using this lemma, we can now describe the fibres of π over (points represented by) regular semisimple elements. Recall that the centraliser $\mathfrak{c}_\mathfrak{g}(x)$ of a regular semisimple element is a Cartan subalgebra of $\mathfrak{g}$. We have:

Proposition 7.2. *If $x \in \mathfrak{g}$ is a regular semisimple element with corresponding Cartan subalgebra $\mathfrak{h} := \mathfrak{c}_\mathfrak{g}(x)$, then $\mathcal{C}_x^\lambda(\mathfrak{g}) = \mathrm{End}_\mathfrak{h}(V^\lambda)$. Since V^λ is weight multiplicity free, it follows that $\mathcal{C}_x^\lambda(\mathfrak{g}) \cong \mathbb{C}^{\mathrm{wt}(\lambda)}$. Moreover, viewing x as a closed point of $\mathrm{Spec}\, S(\mathfrak{g})^\mathfrak{g} \cong \mathrm{Spec}\, \mathbb{C}[\mathfrak{g}]^\mathfrak{g} \cong \mathrm{Spec}\, \mathbb{C}[\mathfrak{h}]^W$, we further have $\pi^{-1}(x) \cong \mathrm{Spec}(\mathcal{C}_x^\lambda(\mathfrak{g}))$. Thus, $\pi^{-1}(x)$ is a zero-dimensional reduced scheme with underlying set $\mathrm{wt}(\lambda)$.*

Proof. Evaluation in x also defines a homomorphism $\widetilde{\mathrm{ev}}_x \colon \mathcal{C}^\lambda(\mathfrak{h}) \to \mathrm{End}_\mathfrak{h}(V^\lambda)$. This variant $\widetilde{\mathrm{ev}}_x$ is easily seen to be surjective: indeed, $A \in \mathrm{End}_\mathfrak{h}(V^\lambda)$ has preimage $\sum_{w \in W} \delta_{w \cdot x} \otimes (w \cdot A)$ where $\delta_x \in S(\mathfrak{h}) \cong S(\mathfrak{h}^*)$ is one on x and zero on $(W \cdot x) \setminus \{x\}$. Moreover, it is clear that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}^\lambda(\mathfrak{g}) & \xrightarrow{\ \mathrm{ev}_x\ } & \mathrm{End}(V^\lambda) \\
\downarrow & \nearrow_{\widetilde{\mathrm{ev}}_x} & \\
\mathcal{C}^\lambda(\mathfrak{h}). & &
\end{array}
$$

Since x is regular, the discriminant D does not vanish at x and so both ev_x and $\widetilde{\mathrm{ev}}_x$ factor through the localisations at D. But upon localisation, the vertical map becomes an isomorphism (by Lemma 7.1), so the images coincide. Thus,

$$\mathcal{C}_x^\lambda(\mathfrak{g}) = \mathrm{ev}_x(\mathcal{C}_x^\lambda(\mathfrak{g})) = \widetilde{\mathrm{ev}}_x(\mathcal{C}^\lambda(\mathfrak{h})) = \mathrm{End}_\mathfrak{h}(V^\lambda).$$

For the second assertion, note that endomorphisms in $\mathrm{End}_\mathfrak{h}(V^\lambda)$ have to send each weight space to itself. By weight multiplicity freeness, each weight space is one-dimensional, so such endomorphisms are diagonal with respect to the weight space decomposition.

It remains to show that $\pi^{-1}(x)$ is isomorphic to the spectrum of $\mathcal{C}_x^\lambda(\mathfrak{g})$. Since $D(x) \neq 0$, Lemma 7.1 allows to compute $\pi^{-1}(x)$ as a fibre of $\tilde{\pi} \colon \mathrm{Spec}\, \mathcal{C}^\lambda(\mathfrak{h}) \to \mathrm{Spec}\, S(\mathfrak{h})^W \cong \mathrm{Spec}\, S(\mathfrak{g})^\mathfrak{g}$. By definition, this fibre is given by the spectrum of

$$\mathcal{C}^\lambda(\mathfrak{h}) \otimes_{S(\mathfrak{h})^W} S(\mathfrak{h})^W / (\ker(\widetilde{\mathrm{ev}}_x|_{S(\mathfrak{h})^W})) \cong \mathcal{C}^\lambda(\mathfrak{h}) / (\mathcal{C}^\lambda(\mathfrak{h}) \cdot \ker(\widetilde{\mathrm{ev}}_x|_{S(\mathfrak{h})^W})).$$

It is easy to see that the ideal $\mathcal{C}^\lambda(\mathfrak{h}) \cdot \ker(\widetilde{\mathrm{ev}}_x|_{S(\mathfrak{h})^W})$ equals the kernel of $\widetilde{\mathrm{ev}}_x$, so we conclude that

$$\pi^{-1}(x) \cong \widetilde{\pi}^{-1}(x) \cong \mathrm{Spec}(\mathcal{C}^\lambda(\mathfrak{h})/\ker(\widetilde{\mathrm{ev}}_x)) \cong \mathrm{Spec}\, \mathcal{C}_x^\lambda(\mathfrak{g})$$

as claimed.   □

Thus, the map π of (21) has reduced fibres over all regular semisimple points in the base. We will also require the base points to be fixed by the involutions under consideration, which can be arranged without breaking the good fibre behaviour. Indeed, if σ^* is an involution of the algebra $S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{C}^\lambda(\mathfrak{g})$ induced by a real structure (see §3), then the action of σ^* on the base ring $S(\mathfrak{g})^{\mathfrak{g}}$ is by a complex linear involution θ of $\mathfrak{g}$ (cf. Lemma 3.14). Moreover, only the inner class of θ matters, so we may assume that θ preserves a pinning. As observed in the proofs of Lemma 4.7, a pinning-preserving automorphism fixes a regular semisimple element of $\mathfrak{g}$ and we conclude:

Lemma 7.3. *For an involution σ^* induced by a real structure, there is a dense open subset U of the fixed point scheme $(\mathrm{Spec}\, S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$ such that $\pi^{-1}(x) \cong \mathrm{Spec}\, \mathbb{C}^{\mathrm{wt}(\lambda)}$ for every $x \in U$.*

Proof. We may take U to be the intersection of $(\mathrm{Spec}\, S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$ with the open subset of $\mathrm{Spec}\, S(\mathfrak{g}) \cong \mathfrak{g}$ of regular semisimple elements. Then U is open in $(\mathrm{Spec}\, S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$, hence dense open (by irreducibility) since it is nonempty by the discussion above.   □

For x as in the lemma, the involution σ^* restricts to an involution of $\pi^{-1}(x)$, given by an involutive permutation of the underlying set $\mathrm{wt}(\lambda)$. By Proposition 3.16, σ^* is of the form $\iota_{\eta,-1}$ for a certain $\eta \in \mathrm{Aut}_2(\mathfrak{g})$, allowing for a convenient description of the induced permutation:

Proposition 7.4. *Let $\eta \in \mathrm{Aut}_2(\mathfrak{g})$ such that $\eta^*\lambda = \lambda$ (with notation as in Proposition 3.15(i)). By replacing η with an involution in the same inner class if necessary, we may assume that there exists a regular semisimple $x \in \mathfrak{g}$ such that $\eta(x) = -x$. The restriction of $\iota := \iota_{\eta,-1}$ to the fibre $\pi^{-1}(x) \cong \mathrm{wt}(\lambda)$ is then given by*

$$\mathrm{wt}(\lambda) \to \mathrm{wt}(\lambda), \quad \mu \mapsto \mu \circ \eta.$$

Proof. To arrange $\eta(x) = -x$ for some regular semisimple x, take η to be a *principal involution* in the given inner class – see [2, Def. 6.13 and Thm. 6.14]. The involution of $\mathcal{C}_x^\lambda(\mathfrak{g})$ induced by ι is then simply $\mathrm{Conj}_{A_\eta,\lambda}$, which is easily seen to act on the weights as claimed.   □

Since these fibres are zero-dimensional and reduced, it is easy to count the fixed points:

Lemma 7.5. *Let A be a finite-dimensional reduced commutative $\mathbb{C}$-algebra and ι an algebra automorphism of A, corresponding to an automorphism of $\mathrm{Spec}\, A$ also denoted ι. Then*

$$\#(\mathrm{Spec}\, A)^\iota = \mathrm{tr}\,\iota = \dim_\mathbb{C} A_\iota,$$

where A_ι denotes the coinvariant ring (see Proposition 2.6).

Proof. The assumptions imply that $X := \mathrm{Spec}\, A$ is a finite and reduced, so $A \cong \mathbb{C}^X$. In particular, A has a $\mathbb{C}$-basis consisting of functions δ_x which are 1 on $x \in X$ and zero on $X \setminus \{x\}$. Moreover, ι automatically has finite order, so that its trace coincides with the dimension of its eigenspace for eigenvalue 1. Thus,

$$\mathrm{tr}\,\iota = \#\{x \in X \mid \delta_x \circ \iota = \delta_x\} = \#(\mathrm{Spec}\, A)^\iota$$

That eigenspace can also be identified with the underlying vector space of

$$A_\iota \cong \mathbb{C}^X/(\{\delta_x \mid \delta_x \circ \iota \neq \delta_x\}) = \mathbb{C}^X/\mathbb{C} \cdot \{\delta_x \mid \delta_x \circ \iota \neq \delta_x\},$$

which finishes the proof.   □

Thus, the number of fixed points in a suitable fibre $\pi^{-1}(x)$ is given by $\dim_\mathbb{C} \mathcal{C}_x^\lambda(\mathfrak{g})$. If the full coinvariant ring $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ is already free over $(S(\mathfrak{g})^{\mathfrak{g}})_{\sigma^*}$ then that dimension is equal to its rank. Note that in this case the corresponding morphism on spectra is surjective, so in particular $\pi^{-1}(x)^{\sigma^*} \neq \varnothing$. Conversely, if $\pi^{-1}(x)^{\sigma^*} = \varnothing$, then the same argument shows that the coinvariant homomorphism (4) is

not injective. We can phrase this as a simple obstruction to freeness, shedding some light on Theorem 1.2:

Proposition 7.6. *Let $\iota = \iota_{\eta,-1}$ be an involution of $\mathcal{C}^\lambda(\mathfrak{g})$ of the form given in Proposition 3.15. Assume that η has a regular semisimple eigenvector in $\mathfrak{g}$ with eigenvalue -1. If the set*

$$\{\mu \in \mathrm{wt}(\lambda) \mid \mu = \mu \circ \eta\}$$

is empty, then the coinvariant homomorphism $S(\mathfrak{g})_\iota^\mathfrak{g} \to \mathcal{C}^\lambda(\mathfrak{g})_\iota$ is not injective, hence not free.

Proof. Follows directly from the discussion above and Proposition 7.4.   □

Remark 7.7. When $\mathfrak{g}$ is simple, it is not hard describe the action of η on weights explicitly (as is done for inner η below). For instance, if η is a Chevalley involution (meaning that η acts as -1 on a Cartan subalgebra), then this action is simply $\mu \mapsto -\mu$. If 0 is not a weight of V^λ, this corresponds to a non-free coinvariant homomorphism. Along these lines, one could prove combinatorially that the freeness condition in Theorem 1.2 is necessary.

For the rest of this section, we **specialise to the case where σ is a split real structure**.

In this case, $\sigma^* = (-1)^{\deg}$ according to Proposition 3.16. To extract information on fixed points in fibres from this fact, let us recall (from Proposition 2.2) that $\mathcal{C}^\lambda(\mathfrak{g})$ is finite-free over its graded subring $S(\mathfrak{g})^\mathfrak{g}$. Thus, there is a graded $\mathbb{C}$-vector space V such that $\mathcal{C}^\lambda(\mathfrak{g}) \cong S(\mathfrak{g})^\mathfrak{g} \otimes_\mathbb{C} V$ as a graded $S(\mathfrak{g})^\mathfrak{g}$-module. Using the description by invariant polynomial rings in §4 one can show [24, p. 11] that $V = V^\lambda$ with principal grading.[16] This grading is defined in terms of the weight spaces V_μ^λ, $\mu \in \mathrm{wt}\,\lambda$. Namely, all weights of V^λ are obtained from the lowest weight $-\lambda^*$ by successively adding simple roots; if k such additions are needed to obtain μ we place V_μ^λ in degree k. If Δ denotes the root system and $\rho^\vee := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha^\vee$ the half sum of positive coroots, then $\langle\rho^\vee,\alpha\rangle = 1$ for all simple roots α. Thus, the degree of V_μ^λ can be computed as $k = \langle\rho^\vee,\mu+\lambda^*\rangle = \langle\rho^\vee,\mu+\lambda\rangle$ and the Poincaré polynomial for this grading is the *Dynkin polynomial*, cf. [24, §3]

$$(22) \qquad \mathcal{D}^\lambda(q) = \sum_{\mu\in\mathrm{wt}\,\lambda}(\dim V_\mu^\lambda)q^{\langle\mu+\lambda,\rho^\vee\rangle} = \prod_{\alpha\in\Delta^+}\frac{1-q^{\langle\rho+\lambda,\alpha^\vee\rangle}}{1-q^{\langle\rho,\alpha^\vee\rangle}}.$$

From the graded module isomorphism $\mathcal{C}^\lambda(\mathfrak{g}) \cong S(\mathfrak{g})^\mathfrak{g} \otimes_\mathbb{C} V^\lambda$, it follows that $\mathcal{C}_x^\lambda(\mathfrak{g}) \cong \mathcal{C}^\lambda(\mathfrak{g}) \otimes_{S(\mathfrak{g})^\mathfrak{g}}\mathbb{C} \cong V^\lambda$ as a graded $\mathbb{C}$-vector space, for any $x \in \mathfrak{g}$. If x is moreover fixed by σ^*, it follows that σ^* restricts to $\mathcal{C}_x^\lambda(\mathfrak{g})$ as $(-1)^{\deg}$. In particular, its trace is given by the Poincaré polynomial evaluated at -1. Via Lemma 7.5 we conclude:

Proposition 7.8. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, λ a minuscule weight, G the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, $G^\vee$ its Langlands dual, and σ a split real structure of $G^\vee$. Then, given $x \in (\operatorname{Spec} S(\mathfrak{g})^\mathfrak{g})^{\sigma^*}$ with reduced fibre as in Lemma 7.3, we have*

$$\#(\pi^{-1}(x))^{\sigma^*} = \mathcal{D}^\lambda(-1).$$

To finish the analysis, we now want to compute the restriction of σ^* to a good fibre $\operatorname{Spec}\mathcal{C}_x^\lambda(\mathfrak{g})$ again, but from the point of view of $\mathcal{C}_x^\lambda(\mathfrak{g}) \cong \mathbb{C}^{\mathrm{wt}\,\lambda}$. In this picture, the grading appears less natural and the explicit permutation on the set of weights from Proposition 7.4 is preferable. Here, the involution η appearing in the proposition will be inner and map a regular semisimple element x to its negative. Modifying η if necessary, we may assume that x lies in the dominant Weyl chamber, so that the action of η on weights is given by the longest element w_0 of W. We are now in position to synthesise a proof of Theorem 1.3 from the discussion in this section.

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[16]Alternatively, one can use the geometric model in Theorem 2.3 together with the geometric Satake equivalence [22] to conclude this for all big algebras, in particular for minuscule Kirillov algebras.

Proof of Theorem 1.3. The first claim of the Theorem is that there is a dense open subset U of the fixed point scheme $(\operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$ over which the fibres of (21) are reduced; this is shown in Lemma 7.3.

For such a base point x, Proposition 7.8 shows that the number of fixed points in its fibre is $\mathcal{D}^\lambda(-1)$. By the discussion immediately above this proof, this is also the number of weights of V^λ fixed by the longest Weyl group element.[17]

Finally, it is claimed that the fibre is non-empty if and only if λ is fixed by a quasi-compact real structure inner to a split real structure. According to Theorem 1.2, the latter condition is equivalent to injectivity of the coinvariant homomorphism $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \to \mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$. In turn, this is equivalent to surjectivity of the map $(\operatorname{Spec} \mathcal{C}^\lambda(\mathfrak{g}))^{\sigma^*} \to (\operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}})^{\sigma^*}$ (using Proposition 2.6 and the fact that the map is finite, hence closed). But it is clear that this map is surjective if and only if its fibres over the dense open subset U are nonempty. $\qquad\square$

Remark 7.9. Instead of the route taken here, one can also observe that the principal grading on V^λ corresponds to weight spaces for a suitable element of the adjoint group $\operatorname{Ad}(\mathfrak{g})$, and that this element is conjugate to a representative of w_0. This is the original strategy used in [30]. The inner automorphism η, arising in the discussion above, which maps a regular semisimple element to its negative is indeed a special involution in the sense of [30, p. 15]. However, the approach we have taken here seems more natural in the context of Kirillov algebras.

Finally, we present a slight extension of Theorem 1.3, which makes some progress towards proving conjectures of Hausel stated in [15, slide 7]. Following Hausel, we introduce the following variant of (22), a priori as a rational function:

$$(23) \qquad \mathcal{D}^\lambda_{\mathrm{ev}}(q) := \prod_{\substack{\alpha \in \Delta^+ \\ \langle \rho+\lambda, \alpha^\vee \rangle \in 2\mathbb{Z}}} \left(1 - q^{\langle \rho+\lambda, \alpha^\vee \rangle}\right) \prod_{\substack{\alpha \in \Delta^+ \\ \langle \rho, \alpha^\vee \rangle \in 2\mathbb{Z}}} \left(1 - q^{\langle \rho, \alpha^\vee \rangle}\right)^{-1}.$$

Two simple facts about this are as follows:

Proposition 7.10.
  (i) $\mathcal{D}^\lambda_{\mathrm{ev}}$ *is a polynomial.*
  (ii) $\mathcal{D}^\lambda_{\mathrm{ev}}(1) = \mathcal{D}^\lambda(-1)$.

Proof. Since $\mathcal{D}^\lambda$ is a fraction of products of cyclotomic polynomials by (22), it admits a factorisation

$$\mathcal{D}^\lambda(q) = \frac{\prod_{z \in Z_1}(q-z)}{\prod_{z \in Z_2}(q-z)}$$

where Z_1, Z_2 are multisets of roots of unity different from 1. The multiplicity of each z in Z_1 is at least the multiplicity in Z_2 because $\mathcal{D}^\lambda$ is a polynomial. By removing all odd roots of unity from both Z_1 and Z_2, we arrive at a factorisation of $\mathcal{D}^\lambda_{\mathrm{ev}}$, which is thus also a polynomial, proving (i). The definition of $\mathcal{D}^\lambda_{\mathrm{ev}}$ immediately yields that it is an even polynomial, so in particular $\mathcal{D}^\lambda_{\mathrm{ev}}(1) = \mathcal{D}^\lambda_{\mathrm{ev}}(-1)$. Part (ii) then follows from (i) and the fact that

$$(1 - q^{2k+1})/(1-q) = 1 + q + \cdots q^{2k}$$

evaluates to 1 at $q = -1$ for any $k \in \mathbb{N}$. $\qquad\square$

Theorem 7.11. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and λ a minuscule weight of $\mathfrak{g}$. Further, let G be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, $G^\vee$ its Langlands dual, and σ a split real structure of $G^\vee$. Then the coinvariant homomorphism $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \to \mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ is free if and only if $\mathcal{D}^\lambda_{\mathrm{ev}}(1) \neq 0$. If this is the case, $\mathcal{D}^\lambda_{\mathrm{ev}}$ is the Poincaré polynomial of $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ over $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*}$.*

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[17]There, the base point x is further restricted to be an eigenvector of the involution η, and it is not obvious whether this works for all $x \in U$. However, the fixed point count in terms of $\mathcal{D}^\lambda$ is independent of that restriction and gives the same result in all fibres over points in U.

Proof. The characterisation of freeness follows from Theorems 1.2 and 1.3 and Proposition 7.10. In the free case, we see from Theorems 1.2 and 1.1 that the coinvariant homomorphism is the canonical map

$$H^{2*}_{(G^\vee)^\sigma} \to H^{2*}_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda)$$

for a quasi-compact real structure inner to σ. Using the identification $\sigma^* = (-1)^{\deg}$ and unravelling the proof of Theorem 1.1, we may identify this map with

$$\mathbb{C}[\mathfrak{g}]^\mathfrak{g}_{(-1)^{\deg}} \to \mathbb{C}[\mathfrak{l}_\lambda]^{\mathfrak{l}_\lambda}_{(-1)^{\deg}}.$$

These are polynomial rings with generators graded in the even degrees of the Weyl groups W and W_λ. Therefore, the Poincaré polynomial is given by

$$P(q) \coloneqq \prod_{d \in D(W) \cap 2\mathbb{Z}} (1 - q^d) \prod_{d \in D(W_\lambda) \cap 2\mathbb{Z}} (1 - q^d)^{-1}$$

where $D(W), D(W_\lambda)$ are the respective multisets of degrees. But the analogous expression

$$\prod_{d \in D(W)} (1 - q^d) \prod_{d \in D(W_\lambda)} (1 - q^d)^{-1}$$

equals $\mathcal{D}^\lambda$ (see [24, Prop. 3.7]), so a similar argument as in the proof of Proposition 7.10 shows that $P = \mathcal{D}^\lambda_{\mathrm{ev}}$. □

8. Outlook

As remarked in the introduction, minuscule Kirillov algebras are comparatively simple special cases of Hausel's *big algebras* [12]. As such, they provide a valuable testing ground for the study of general big algebras, and we expect that the main results of this paper can be extended to that context. However, the relatively elementary methods used here need to be adapted to treat the general case.

In some more detail, big algebras are modelled by the equivariant intersection cohomology of affine Schubert varieties (cf. Theorem 2.3), though this does not account for the ring structure in general. Nevertheless, there still is a well-defined action of real structures, and we do expect the coinvariant ring to be modelled by a real affine Schubert variety similarly to Theorem 1.1. Such a geometric model would likely also lead to a characterisation of freeness as in Theorem 1.2.

In the special case of a split real structure, Theorem 1.3 of this paper recovers a minuscule $q = -1$ phenomenon due to Stembridge [30]. That work was generalised to arbitrary representations, where one counts fixed elements of the canonical basis instead of fixed weights, see [29]. We expect the action of split real structures on general big algebras to recover this via a similar analysis as in §7.

Another future direction is to describe the case of non-free coinvariant homomorphisms $S(\mathfrak{g})^\mathfrak{g}_{\sigma*} \to \mathcal{C}^\lambda(\mathfrak{g})_{\sigma*}$ in more detail. It follows from Lemma 7.3 and the proof of Theorem 1.3 that the image of $\mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g})_{\sigma*}$ in $S(\mathfrak{g})^\mathfrak{g}_{\sigma*}$ must in this case be contained in the vanishing locus $V(D)$ of the discriminant. One can then ask for a maximal closed subscheme Z of $S(\mathfrak{g})^\mathfrak{g}_{\sigma*}$ (contained in $V(D)$) such that $\mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g})_{\sigma*}$ is finite-free over Z. The analysis in §6 suggests that such a Z can be obtained as the spectrum of $\mathbb{C}[\mathfrak{k}]^K$ for a suitable (compact) subgroup K of $(G^\vee)^{\sigma_0}$ with Lie algebra $\mathfrak{k}$, but it is not yet clear to us how to define K in a uniform way.

Let us also remark that there is a connection to moduli spaces of Higgs bundles. When $\mathfrak{g}$ is of type A, the spectra of minuscule Kirillov algebras are isomorphic to *very stable type* $(1, \ldots, 1)$ *upward flows* in the moduli space of Higgs bundles[18] [13] (see also [11, §4]). Involutions (and more general automorphisms) on the whole moduli space, resembling the automorphisms $\iota_{\eta,\zeta}$ of Proposition 3.15, have been studied in [10]. The Kirillov algebra involutions studied here should correspond to such global involutions of the moduli space restricted to suitable upward flows. The case of $\iota_{\mathrm{id},-1}$ is discussed in [11].

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[18]of fixed rank and degree, over a smooth projective curve of genus at least two

Finally, one can ask how natural our focus on real structures is for the material covered in this paper. Indeed, significant parts of our arguments proceed via the (holomorphic) Cartan involutions corresponding to the real structures. However, the geometric model in Theorem 1.1 does seem most natural when stated through real structures (as opposed to invariant polynomial rings for complex Lie groups). Currently, this provides only heuristic input towards describing the coinvariant homomorphism: namely, it is free precisely when the modelling real partial flag variety is an equal rank homogeneous space. It would be interesting to further incorporate the real geometry into the description.

Appendix A. Remarks on uniqueness and tables

In the proof of Theorem 1.1 (see §5), three cases were distinguished. In most cases, the real structure guaranteed by the Theorem can be chosen to restrict quasi-compactly to the relevant Levi subgroup; it is then essentially unique (see Lemma 5.3). This restriction property can fail for simple factors of type A_{2n} with the inner class of the split real structure. This inner class consists of only one inner-isomorphism class, Thus, we could obtain a unique choice of σ up to isomorphism by demanding quasi-compact restriction on all factors where this is possible.

The resulting σ are precisely those used in the proof of Theorem 1.1. However, in the way it is stated, Theorem 1.1 could allow for several essentially different choices of σ. For the sake of completeness, we remark that this can indeed happen. Namely, consider the case of $G^\vee = Sp_{4n}(\mathbb{C})$ with $L_\lambda \cong \mathrm{GL}_{2n}(\mathbb{C})$ and the inner class containing a split real form of $G^\vee$. The real structure provided by Lemma 5.3 results in the real form $U^*(2n)$ of $\mathrm{GL}_{2n}(\mathbb{C})$. However, a calculation as in Example 4.9 shows that we could also have chosen the split real structure of $G^\vee$, resulting in the real form $\mathrm{GL}_{2n}(\mathbb{R})$ of $\mathrm{GL}_{2n}(\mathbb{C})$.

The real structures used in the proof of Theorem 1.1 are listed below, in terms of the corresponding real forms, for all minuscule weights of simple complex Lie algebras. This leaves (up to direct sums) the case of $\mathfrak{g}^\vee = \mathfrak{g}_s \oplus \mathfrak{g}_s$ where $\mathfrak{g}_s$ is simple and the real structure permutes the summands. However, all such real structures are isomorphic; up to isomorphism the Satake diagram consists of two copies of the Dynkin diagram of $\mathfrak{g}_s$ with corresponding nodes connected by arrows.

Except when $\mathfrak{g}$ is of type D_4, there are then one or two inner classes of real structures for each $\mathfrak{g}$, so we can group them by whether they are inner to a split real structure. If we identify isomorphic (though not necessarily inner-isomorphic) real structures, then even for D_4 this results in only two cases. Thus, Table 1 contains inner classes of split real structures, and Table 2 the remaining ones. We also list the Levi subalgebras $\mathfrak{l}_\lambda$ via their (more conveniently notated) derived subalgebras $\mathfrak{l}'_\lambda$ and include Satake diagrams. For the role of Satake diagrams in this context, see Remark 5.4.

$\mathfrak{g}^\vee$	$\mathfrak{l}^I_\lambda$	$(\mathfrak{g}^\vee)^\sigma$	$(\mathfrak{l}^\sigma_\lambda)'$	Satake diagram
$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_{2k}(\mathbb{C}) \oplus$ $\mathfrak{sl}_{2n-2k}(\mathbb{C})$	$\mathfrak{su}^*(2n)$ (quasi-compact)	$\mathfrak{su}^*(2k) \oplus$ $\mathfrak{su}^*(2n-2k)$	(1 2 3 … 2k … 2n-1)
$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_{2k+1}(\mathbb{C}) \oplus$ $\mathfrak{sl}_{2n-2k-1}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sl}_{2k+1}(\mathbb{R}) \oplus$ $\mathfrak{sl}_{2n-2k-1}(\mathbb{R})$	(1 2 3 2k+1 … 2n-1)
$\mathfrak{sl}_{2n+1}(\mathbb{C})$	$\mathfrak{sl}_{k}(\mathbb{C}) \oplus$ $\mathfrak{sl}_{2n+1-k}(\mathbb{C})$	$\mathfrak{sl}_{2n+1}(\mathbb{R})$ (quasi-compact)	$\mathfrak{sl}_{k}(\mathbb{R}) \oplus$ $\mathfrak{sl}_{2n+1-k}(\mathbb{R})$	(1 2 3 k … 2n)
$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{so}_{2n-1}(\mathbb{C})$	$\mathfrak{so}_{1,2n}(\mathbb{R})$	$\mathfrak{so}_{2n-1}(\mathbb{R})$	(1 2 3 … n)
$\mathfrak{sp}_{4n}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sp}_{2n,2n}(\mathbb{R})$	$\mathfrak{su}^*(2n)$	(1 2 3 … 2n)
$\mathfrak{sp}_{4n+2}(\mathbb{C})$	$\mathfrak{sl}_{2n+1}(\mathbb{C})$	$\mathfrak{sp}_{4n+2}(\mathbb{R})$	$\mathfrak{sl}_{2n+1}(\mathbb{R})$	(1 2 3 … 2n+1)
$\mathfrak{so}_{4n}(\mathbb{C})$	$\mathfrak{so}_{4n-2}(\mathbb{C})$	$\mathfrak{so}_{2,4n-2}(\mathbb{R})$	$\mathfrak{so}_{1,4n-3}(\mathbb{R})$	(1 2 3 … 2n)
$\mathfrak{so}_{4n}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n)$	(1 2 3 … 2n)
$\mathfrak{so}_{4n+2}(\mathbb{C})$	$\mathfrak{so}_{4n-2}(\mathbb{C})$	$\mathfrak{so}_{1,4n+1}(\mathbb{R})$ (quasi-compact)	$\mathfrak{so}_{4n-2}(\mathbb{R})$	(1 2 3 … 2n+1)
$\mathfrak{so}_{4n+2}(\mathbb{C})$	$\mathfrak{sl}_{2n+1}(\mathbb{C})$	$\mathfrak{so}_{2n+1,2n+1}(\mathbb{R})$	$\mathfrak{sl}_{2n+1}(\mathbb{R})$	(1 2 3 … 2n+1)
$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{so}_{10}(\mathbb{C})$	$\mathfrak{e}_{6,-26}$ (quasi-compact)	$\mathfrak{so}_{1,9}(\mathbb{R})$	(2; 1 3 4 5 6)
$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_{7,-25}$	$\mathfrak{e}_{6,-26}$	(2; 1 3 4 5 6 7)

TABLE 1. Real forms *inner to a split real form* adapted to minuscule coweights λ, and their Satake diagrams with the (unpainted) node corresponding to λ crossed out. If the real form $(\mathfrak{g}^\vee)^\sigma$ is quasi-compact, this is remarked in the third column.

$\mathfrak{g}^\vee$	$\mathfrak{l}^I_\lambda$	$(\mathfrak{g}^\vee)^\sigma$	$(\mathfrak{l}^\sigma_\lambda)'$	Satake diagram
$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{su}_{n,n}$	$\mathfrak{sl}_n(\mathbb{C})_\mathbb{R}$	(1 2 3 … n; 2n-1)
$\mathfrak{so}_{4n}(\mathbb{C})$	$\mathfrak{so}_{4n-2}(\mathbb{C})$	$\mathfrak{so}_{1,4n-1}(\mathbb{R})$ (quasi-compact)	$\mathfrak{so}_{4n-2}(\mathbb{R})$	(1 2 3 … 2n)
$\mathfrak{so}_{4n+2}(\mathbb{C})$	$\mathfrak{so}_{4n}(\mathbb{C})$	$\mathfrak{so}_{2,4n}(\mathbb{R})$	$\mathfrak{so}_{1,4n-1}(\mathbb{R})$	(1 2 3 … 2n+1)

TABLE 2. Real forms *not inner to a split real form* adapted to minuscule coweights λ, and their Satake diagrams with the (unpainted) node corresponding to λ crossed out. If the real form $(\mathfrak{g}^\vee)^\sigma$ is quasi-compact, this is remarked in the third column. A subscript $\mathbb{R}$ denotes that a complex Lie algebra is considered as its underlying real Lie algebra.

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Mischa Elkner,
Institute of Science and Technology Austria (ISTA),
Am Campus 1,
3400 Klosterneuburg,
Austria
mischa.elkner@ist.ac.at