$SL(2, \mathbb{R})$ as anti-de Sitter Space

Introduction: As we've seen in the lectures, de Sitter and anti-de Sitter space aren't geodesically connected. More precisely, we looked at the geodesics starting at one fixed point p and saw that they missed parts of the space. Another way to put this is that the exponential map $\exp_p$ isn't surjective.

There is also a (related but different) concept of exponential maps for Lie groups. A standard example of a Lie group whose exponential map isn't surjective is $SL(2, \mathbb{R})$ – this can be shown directly with some linear algebra[1].

Our goal here is to show that these two examples of non-surjective exponential maps are in fact one and the same: $SL(2, \mathbb{R})$ admits a Lorentzian metric making it isometric to 3-dimensional anti-de Sitter space, and the geodesics starting at the unit matrix $I \in SL(2, \mathbb{R})$ are precisely the one-parameter subgroups.

A Lorentzian metric on $SL(2, \mathbb{R})$: As in the construction of anti-de Sitter space, we start with a surrounding vector space, the space $M_2(\mathbb{R})$ of 2×2 real matrices. We want to provide $M_2(\mathbb{R})$ with a semi-Riemannian metric, and since it's a vector space, a scalar product will do. Even simpler than that, we can look for a quadratic form on $M_2(\mathbb{R})$, and soon find a very natural one: the determinant $\det: M_2(\mathbb{R}) \to \mathbb{R}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto ad - bc$.

Like any quadratic form, det is represented by a symmetric matrix (with respect to the standard basis), which we can diagonalize. Here we get

$$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = S \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} S^t,$$

where

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in O(4).$$

This shows that det corresponds to an index 2 scalar product b (i. e. $\det(X) = b(X, X)$) such that S yields an isometry $(M_2(\mathbb{R}), b) \cong \mathbb{R}^4_2$.

Recall that we defined anti-de Sitter space as the Lorentzian submanifold of $\mathbb{R}^4_2$ consisting of all vectors $x \in \mathbb{R}^4_2$ with $\|x\|^2 = 1$. Under our isometry,

[1] See e. g. the question part in this thread.

these correspond to matrices of determinant 1. We conclude that $SL(2,\mathbb{R})$ is a Lorentzian submanifold of $M_2(\mathbb{R})$ isometric to 3-dimensional anti-de Sitter space.

One-parameter subgroups are geodesics: Now that we have a Lie group with a Lorentzian structure, it's very tempting to compare the exponential maps[2]. Both are maps $\mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$, and both are defined via certain curves: geodesics for the semi-Riemannian exponential, and one-parameter subgroups for the Lie group one. Therefore, we can restrict our attention to these curves.

Before we get started, we observe that the metric g on $SL(2,\mathbb{R})$ induced by b is left-invariant. That is, we have

$$g(L_X, L_Y) = b(X,Y), \quad \forall X, Y \in \mathfrak{sl}(2,\mathbb{R}),$$

where L_X denotes the left-invariant vector field induced by X. An easy way to see this is to observe that multiplication by $A \in SL(2,\mathbb{R})$ preserves the determinant:

$$\det(AB) = \det(A)\det(B) = \det(B), \quad \forall B \in M_2(\mathbb{R}).$$

By polarisation, it also preserves b, so we have

$$g(L_X, L_Y)(A) = b(AX, AY) = b(X,Y), \quad \forall A \in SL(2,\mathbb{R}).$$

Now let $X \in \mathfrak{sl}(2,\mathbb{R})$, and write $c\colon \mathbb{R} \to SL(2,\mathbb{R})$ for the induced one-parameter subgroup: $c(t) = \mathrm{Fl}_t^{L_X}(I)$. We want to show that $c' = L_X \circ c \in \mathfrak{X}(c)$ is parallel. We have

$$(L_X \circ c)'(t) = \nabla_{c'(t)} L_X = (\nabla_{L_X} L_X)(c(t)),$$

so it's sufficient to prove $\nabla_{L_X} L_X = 0$.

Clearly, it's enough to show $g(\nabla_{L_X} L_X, L_Y) = 0$ for all $Y \in \mathfrak{sl}(2,\mathbb{R})$. By the Koszul formula, and the fact that $g(L_X, L_Y)$ is constant, we get

$$\begin{aligned}
2g(\nabla_{L_X} L_X, L_Y) &= 0 + 0 - 0 - g(L_X, [L_X, L_Y]) + g(L_X, [L_Y, L_X]) + 0 \\
&= 2g(L_X, [L_Y, L_X]) \\
&= 2g(L_X, L_{[Y,X]}) \\
&= 2b(X, [Y,X]).
\end{aligned}$$

Hence, c is a geodesic iff $b(X, [Y,X]) = 0$ for all $Y \in \mathfrak{sl}(2,\mathbb{R})$. This can probably be checked by a brute-force computation, but let's do it more conceptually: First, we rewrite $[Y,X]$ as $\mathrm{ad}_Y(X)$. This of course is just a change of notation, but in the form $b(X, \mathrm{ad}_Y(X)) = 0$, we see that the identity would follow from ad_Y being skew-adjoint with respect to b. But that means that ad should map into $\mathfrak{so}(\mathfrak{sl}(2,\mathbb{R}), b)$, which is equivalent to Ad mapping into $SO(\mathfrak{sl}(2,\mathbb{R}), b)$. This

[2] based at I in the semi-Riemannian case

is the case precisely if Ad_A respects the scalar product – or equivalently the quadratic form – on $\mathfrak{sl}(2,\mathbb{R})$, for all $A \in SL(2,\mathbb{R})$, i. e.

$$\det(\mathrm{Ad}_A(Z)) \stackrel{!}{=} \det(Z), \quad \forall Z \in \mathfrak{sl}(2,\mathbb{R}).$$

But in this matrix group setting we have the simple formula $\mathrm{Ad}_A(Z) = AZA^{-1}$. Conjugation clearly preserves the determinant[3], so the above identity is true.

We conclude that $\nabla_{L_X} L_X = 0$ for arbitrary $X \in \mathfrak{sl}(2,\mathbb{R})$, and from this that all one-parameter subgroups are geodesics. This already exhausts all possible geodesics at I, since they are uniquely determined by their initial tangent vector. Thus, the geodesics based at I are precisely the one-parameter subgroups. In particular, the two exponential maps are equal.

Personally, I think of this as a nice explanation for why $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ isn't surjective – the non-surjectivity seems much clearer (certainly easier to visualize) in the Lorentzian setting.

[3]Conjugation with a determinant 1 matrix preserves it even more. ☺