# ON INVOLUTIONS OF MINUSCULE KIRILLOV ALGEBRAS INDUCED BY REAL STRUCTURES

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ABSTRACT. We describe involutions induced by real structures on Kirillov algebras of minuscule representations, viewed as equivariant cohomology algebras of partial flag varieties. The fixed points on spectra of these involutions correspond to coinvariant algebras, which we prove to be modeled by the cohomology of appropriate real partial flag varieties.

## CONTENTS





## 1. INTRODUCTION

The finite-dimensional irreducible representations of a complex semisimple Lie algebra $\mathfrak{g}$ are central to Lie theory. For such a representation $V^\lambda$, labeled by its highest weight $\lambda$, the *Kirillov algebra*

$$\mathcal{C}^\lambda(\mathfrak{g}) \coloneqq (S(\mathfrak{g}) \otimes \operatorname{End}(V^\lambda))^{\mathfrak{g}}$$

was introduced as a new tool in [10].[1] This algebra is commutative if and only if the representation $V^\lambda$ is weight multiplicity free [10, Corollary 1]. However, by recent work of Hausel [7], each $\mathcal{C}^\lambda(\mathfrak{g})$ contains a maximal commutative subalgebra $\mathcal{B}^\lambda(\mathfrak{g})$, called *big algebra*, with intriguing properties. In particular, one can geometrically extract data about the representation $V^\lambda$ from the spectrum $\operatorname{Spec}\mathcal{B}^\lambda(\mathfrak{g})$ [cf. 8]. Suitable natural automorphisms of $\operatorname{Spec}\mathcal{B}^\lambda(\mathfrak{g})$ therefore correspond to symmetries of that data, which motivates the study of big algebra automorphisms as an approach to symmetries in representation theory.

In this paper, we focus on the case where $\lambda$ is *minuscule*, which means that the weights of $V^\lambda$ are all in the Weyl group orbit of $\lambda$. This implies that $V^\lambda$ is weight multiplicity free, so all Kirillov algebras appearing in this paper are commutative and coincide with their big (sub)algebras. We are therefore interested in symmetries of these Kirillov algebras, and how they act on their spectra.

[1] In [10], Kirillov algebras are called *(classical) family algebras*.

Kirillov algebras for minuscule weights have a geometric interpretation as equivariant cohomology algebras. This was first observed in [15], linking $\mathcal{C}^\lambda(\mathfrak{g})$ to a partial flag variety of the connected simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Here we use a slightly different model in terms of the Langlands dual group $G^\vee$ of $G$. The weight $\lambda$ defines a parabolic subgroup $P_\lambda$ of $G^\vee$, and we have a ring isomorphism [7]

$$\mathcal{C}^\lambda(\mathfrak{g}) \cong H^*_{G^\vee}(G^\vee/P_\lambda, \mathbb{C}). \tag{1}$$

Let us mention two useful properties of this isomorphism. Firstly, the Kirillov algebra inherits a grading from $S(\mathfrak{g})$, and (1) becomes a graded isomorphism if the right-hand side is graded by half the cohomological degree. Secondly, $\mathcal{C}^\lambda(\mathfrak{g})$ is a graded algebra over its subring $S(\mathfrak{g})^\mathfrak{g}$ (embedded as $f \mapsto f \otimes \mathrm{id}$). The isomorphism identifies this subring with $H^{2*}_{G^\vee} \coloneqq H^{2*}_{G^\vee}(\mathrm{pt}, \mathbb{C})$, the equivariant cohomology of a point. Altogether, the geometric model is summarised by the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}^\lambda(\mathfrak{g}) \arrow[r, "\cong"] & H^{2*}_{G^\vee}(G^\vee/P_\lambda, \mathbb{C}) \\ S(\mathfrak{g})^\mathfrak{g} \arrow[u, hook] \arrow[r, "\cong"] & H^{2*}_{G^\vee}. \arrow[u, hook] \end{tikzcd} \tag{2}$$

Via this geometric framework, we can obtain Kirillov algebra automorphisms from automorphisms of $G^\vee/P_\lambda$ which respect the $G^\vee$-action. Here we focus on involutions arising in this way from real structures of $G^\vee$. Such a real structure (i. e. an antiholomorphic automorphism) $\sigma$ of $G^\vee$ acts naturally on the weights, and induces a real structure on $G^\vee/P_\lambda$ when $\sigma \cdot \lambda$ is in the same Weyl group orbit as $\lambda$. Taking equivariant cohomology then yields an involution $\sigma^*$ of $H^{2*}_{G^\vee}(G^\vee/P_\lambda) \cong \mathcal{C}^\lambda(\mathfrak{g})$ mapping the subring $H^{2*}_{G^\vee} \cong S(\mathfrak{g})^\mathfrak{g}$ to itself. Due to the aforementioned connection to representation theory, we are interested on the action of $\sigma^*$ on spectra. In particular, we want to describe the fixed point schemes, given by

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}) \arrow[r, "\cong"] \arrow[d] & (\mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g}))^{\sigma^*} \arrow[r, hook] \arrow[d] & \mathrm{Spec}\,\mathcal{C}^\lambda(\mathfrak{g}) \arrow[d] \\ \mathrm{Spec}(S(\mathfrak{g})^\mathfrak{g}_{\sigma^*}) \arrow[r, "\cong"] & (\mathrm{Spec}\,S(\mathfrak{g})^\mathfrak{g})^{\sigma^*} \arrow[r, hook] & \mathrm{Spec}\,S(\mathfrak{g})^\mathfrak{g} = \mathfrak{g}/\!/G. \end{tikzcd} \tag{3}$$

Here the upper $\sigma^*$ denotes sets of fixed points, which form closed affine subschemes with coordinate rings $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ and $S(\mathfrak{g})^\mathfrak{g}_{\sigma^*}$ called *coinvariant rings*. The fixed points of $\sigma^*$ are then encoded by the ring homomorphism

$$S(\mathfrak{g})^\mathfrak{g}_{\sigma^*} \to \mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}. \tag{4}$$

To describe the homomorphism (4) – and with it, the fixed points of $\sigma^*$ – a geometric interpretation like (2) would be very helpful. A natural candidate for such a model is the real form of $G^\vee/P_\lambda$ defined by $\sigma$. However, it turns out that $\sigma^*$ depends only on the *inner class* of $\sigma$, which is not the case for that real form. One therefore has to identify a suitable representative in a given inner class, and this is related to *quasi-compact* real structures. The resulting geometric description – our main result – is as follows:

**Theorem 1.1.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $G$ the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, and $G^\vee$ the Langlands dual group of $G$. Let $\lambda$ be a minuscule weight of $\mathfrak{g}$, viewed as a cocharacter of $G^\vee$. Then any inner class $\mathfrak{S}$ of real structures of $G^\vee$

which fix the Weyl group orbit of $\lambda$ contains a real structure $\sigma$ such that $\sigma_*\lambda = \lambda$ and

$$\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*} \cong H^{2*}_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda, \mathbb{C}) \tag{5}$$

for the parabolic subgroup $P_\lambda \leq G^\vee$ defined by $\lambda$. Moreover, the same inner class contains a quasi-compact real structure $\sigma_0$ with

$$S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \cong H^{2*}_{(G^\vee)^{\sigma_0}}, \tag{6}$$

and an injection

$$\varphi\colon H^{2*}_{(G^\vee)^{\sigma_0}} \hookrightarrow H^{2*}_{(G^\vee)^\sigma}. \tag{7}$$

Combining (5)-(7) yields a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*} \arrow[rr, "\cong"] & & H^{2*}_{(G^\vee)^\sigma}((G^\vee)^\sigma/P^\sigma_\lambda, \mathbb{C}) \\
S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \arrow[u] \arrow[r, "\cong"] & H^{2*}_{(G^\vee)^{\sigma_0}} \arrow[r, hook, "\varphi"] & H^{2*}_{(G^\vee)^\sigma} \arrow[u]
\end{tikzcd} \tag{8}$$

in which the right vertical arrow is the canonical homomorphism for equivariant cohomology.

As an application of Theorem 1.1, we characterise a key algebraic property of the coinvariant homomorphism (4). Namely, the homomorphism $S^{\mathfrak{g}}(\mathfrak{g}) \to \mathcal{C}^\lambda(\mathfrak{g})$ is always finite-free (i. e. defines a finite free module) [cf. 15, p. 277 and Thm. 1.1], which is important in applications to representation theory [7]. This need not be true for $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*} \to \mathcal{C}^\lambda(\mathfrak{g})$:

**Theorem 1.2.** In the setting of Theorem 1.1, $\mathcal{C}^\lambda(\mathfrak{g})_{\sigma^*}$ is finite-free over $S(\mathfrak{g})^{\mathfrak{g}}_{\sigma^*}$ if and only if we can choose $\sigma_0 = \sigma$. This is the case precisely when $\lambda$ is fixed by a quasi-compact real structure in the given inner class.

This paper is structured as follows. In Section 2 we review the relevant background on equivariant cohomology and Kirillov algebras and define the coinvariant rings used here. Section 3 reviews real structures and describes how they act on partial flag varieties and Kirillov algebras. This is then translated to rings of invariant polynomials, which are further discussed in Section 4. Section 4 also contains a discussion of quasi-compactness. The proof of Theorem 1.1 is given in Section 5, and Theorem 1.2 is deduced from it in Section 6. Finally, Section 7 addresses uniqueness questions for the real structures used here, and lists them in tables.



## 2. Equivariant cohomology, Kirillov algebras and coinvariant rings

In this section, we provide further details on relevant background and context. We start with a brief review of equivariant cohomology, a cohomology theory for spaces with group actions.

Let $G$ be a topological group and $X$ a left $G$-space. Let $\mathbb{E}G \to \mathbb{B}G$ be a universal $G$-bundle – in other words, $\mathbb{E}G$ is a contractible space with free right $G$-action, and $\mathbb{B}G = G\backslash\mathbb{E}G$. The *(Borel) $G$-equivariant cohomology* of $X$ is defined as

$$H^*_G(X, R) \coloneqq H^*(X_G, R)$$

where

$$X_G \coloneqq (\mathbb{E}G \times X)/\big((e \cdot g, x) \sim (e, g \cdot x)\big)$$

and $R$ is a ring. From now on, we always take $R = \mathbb{C}$ and drop this from the notation. It is clear from the definition that $H^*_G(X)$ is a graded $\mathbb{C}$-algebra. In fact, the projection $X \to \mathrm{pt}$ to a point induces a map $X_G \to \mathrm{pt}_G$, making $H^*_G(X)$ canonically an algebra over $H^*_G \coloneqq H^*_G(\mathrm{pt})$.

Moreover, equivariant cohomology is functorial for morphisms of spaces with group action. That is, let $H$ be another topological group, $Y$ a left $H$-space, and $f\colon X \to Y$ a map such that

$$f(g \cdot x) = \alpha(g) \cdot f(x) \tag{9}$$

for all $x \in X$, $g \in G$ and a group homomorphism $\alpha\colon G \to H$. Then the pair $(\alpha, f)$ induces a morphism of algebras

$$\begin{tikzcd} H_H^*(Y) \arrow[r] & H_G^*(X) \\ H_H^* \arrow[u] \arrow[r] & H_G^* \arrow[u] \end{tikzcd} \tag{10}$$

making $(G, X) \mapsto H_G^*(X)$ into a functor. As a special case, suppose that $Y = X$, $f = \mathrm{id}_X$, and that $\alpha$ is a homotopy equivalence (in addition to being a group homomorphism). One then readily concludes that the resulting homomorphism $H_H^*(X) \to H_G^*(X)$ is an isomorphism.

Applying this last remark to the case of a point, we see that $H_G^* \cong H_H^*$ whenever $G$ is a subgroup of $H$ whose inclusion is a homotopy equivalence. In particular, this holds when $H$ is a reductive Lie group and $G$ its maximal compact subgroup [see e. g. 11, Prop. 7.19(a)] or when $H$ is a parabolic subgroup of a reductive Lie group and $G$ its Levi factor [e. g. 11, Prop. 7.83(d)].[2] If $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$, we can describe the ring $H_G^*$ via the Chern-Weil [5, p. 116] isomorphism as

$$H_G^* \cong H^*(\mathbb{B}G, \mathbb{C}) \cong S(\mathfrak{g})^G \otimes_{\mathbb{R}} \mathbb{C}. \tag{11}$$

Lastly, let us mention the case where $X = G/H$ is a homogeneous space, for $H \leq G$ a topological subgroup. In this case, any choice of $\mathbb{E}G$ also has a free $H$-action, thus identifying $\mathbb{E}G/H$ with $\mathbb{B}H$. Moreover, one readily checks that $(G/H)_G \cong \mathbb{E}G/H$, so

$$H_G^*(G/H) = H^*((G/H)_G, \mathbb{C}) \cong H^*(\mathbb{B}H, \mathbb{C}) \cong H_H^*. \tag{12}$$

For more details on equivariant cohomology the reader may consult [3].

We now collect the key facts on Kirillov stated already in the introduction.

**Definition 2.1.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\lambda$ a dominant integral weight.[3] Let $V^\lambda$ denote the irreducible $\mathfrak{g}$-representation of highest weight $\lambda$, $\mathrm{End}(V^\lambda)$ is algebra of linear endomorphisms, and $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$. The *Kirillov algebra* with label $\lambda$ is the graded subalgebra

$$\mathcal{C}^\lambda(\mathfrak{g}) := (S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda))^{\mathfrak{g}} \leq S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)$$

consisting of fixed points of the diagonal $\mathfrak{g}$-action.

**Proposition 2.2.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\lambda$ a dominant integral weight. The Kirillov algebra $\mathcal{C}^\lambda(\mathfrak{g})$

(i) has a grading induced from $S(\mathfrak{g})$,
(ii) contains $S(\mathfrak{g})^{\mathfrak{g}}$ as the graded subring $S(\mathfrak{g})^{\mathfrak{g}} \otimes \{\mathrm{id}_{V^\lambda}\}$,
(iii) is finite-free over $S(\mathfrak{g})^{\mathfrak{g}}$, and
(iv) is commutative if and only if $V^\lambda$ is weight-multiplicity free, i. e. if the weight spaces[4] of $V^\lambda$ are one-dimensional.

[2] Here we mean the canonical Levi factor obtained upon fixing a suitable Cartan subalgebra.
[3] Weights, roots etc. can be taken with respect to an arbitrary Cartan subalgebra when none is specified.
[4] Weight spaces are non-zero by convention.

*Proof.* The canonical grading of $S(\mathfrak{g})$ defines a grading of $S(\mathfrak{g}) \otimes \operatorname{End}(V^\lambda)$ (in which the second factor contributes trivially). It is preserved by the diagonal $\mathfrak{g}$-action and therefore restricts to the grading of $\mathcal{C}^\lambda(\mathfrak{g})$ of part (i). Part (ii) is clear. For part (iii), combine [15, Theorem 1.1] with the second paragraph of [loc. cit. p. 277]. Part (iv) is [10, Corollary 1]. □

Recently, Hausel [7] has introduced maximal commutative subalgebras of Kirillov algebras. These are not used in this paper but provide important motivation, so we summarise their key properties. For every $\lambda$, Hausel's *big algebra* $\mathcal{B}^\lambda(\mathfrak{g})$ is a maximal commutative graded $S(\mathfrak{g})^{\mathfrak{g}}$-subalgebra of $\mathcal{C}^\lambda(\mathfrak{g})$. The morphisms

$$\operatorname{Spec} \mathcal{B}^\lambda(\mathfrak{g}) \to \operatorname{Spec} Z(\mathcal{C}^\lambda(\mathfrak{g})) \to \operatorname{Spec} S(\mathfrak{g})^{\mathfrak{g}}$$

(with $Z$ denoting the centre) appear to contain important data of $V^\lambda$ such as its Kashiwara crystal [8, final slide]. In this context, there is the following geometric model:

**Theorem 2.3** (cf. [7], Theorem 3.1)**.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $G$ the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, and $G^\vee$ the Langlands dual group of $G$. For any dominant integral weight $\lambda$ of $\mathfrak{g}$, let $\mathrm{Gr}^\lambda$ denote the affine Schubert variety of $G^\vee$ labeled by $\lambda$. Then we have

(i) $Z(\mathcal{C}^\lambda(\mathfrak{g})) \cong H^{2*}_{G^\vee}(\mathrm{Gr}^\lambda)$
(ii) $\mathcal{B}^\lambda(\mathfrak{g}) \cong IH^{2*}_{G^\vee}(\mathrm{Gr}^\lambda)$, the $G^\vee$-equivariant intersection cohomology of $\mathrm{Gr}^\lambda$, as a module over $H^{2*}_{G^\vee}(\mathrm{Gr}^\lambda)$.

**Corollary 2.4.** Let $\mathfrak{g}$, $G^\vee$ and $\lambda$ be as in Theorem 2.3. Let $P_\lambda \leq G^\vee$ be the parabolic subgroup defined by $\lambda$.[5] If $\lambda$ is minuscule then $\mathcal{C}^\lambda(\mathfrak{g})$ is commutative and isomorphic to $H^{2*}_{G^\vee}(G^\vee/P_\lambda)$.

*Proof.* If $\lambda$ is minuscule, then the Weyl group acts transitively on the weight spaces of $V^\lambda$, so they are all one-dimensional. Thus, $V^\lambda$ is weight multiplicity free and $\mathcal{C}^\lambda(\mathfrak{g})$ is commutative by Proposition 2.2(iv). Theorem 2.3(i) then identifies $\mathcal{C}^\lambda(\mathfrak{g})$ with $H^{2*}_{G^\vee}(\mathrm{Gr}^\lambda)$. But for minuscule $\lambda$, the affine Schubert variety $\mathrm{Gr}^\lambda$ is isomorphic (as a $G^\vee$-space) to $G^\vee/P_\lambda$ [21, Lemma 2.1.13]. □

We end this section with a brief discussion of fixed points and coinvariant rings. In general, if $X$ is any scheme with a self-morphism $f\colon X \to X$, the *fixed-point scheme* $X^f$ is the largest closed subscheme on which $f$ acts trivially. In other words, $X^f$ is the equaliser of $f$ and $\mathrm{id}_X$ in the category of schemes. In the case of affine schemes, there is a particularly easy description:

**Proposition 2.5.** Let $\psi$ be an endomorphism of a ring $A$, and $f$ the induced self-morphism of $\operatorname{Spec} A$. Then the fixed-point scheme $(\operatorname{Spec} A)^f$ is given by $\operatorname{Spec} A_\psi \subseteq \operatorname{Spec} A$, where

$$A_\psi \coloneqq A/(a - \psi(a) \colon a \in A)$$

is the *coinvariant ring.*

*Proof.* As a closed subscheme of the affine scheme $\operatorname{Spec} A$, the fixed-point scheme must be of the form $\operatorname{Spec}(A/I)$ for some ideal $I$ of $A$ [17, Tag 01IF]. By abstract nonsense, the projection $A \to A/I$ must be a coequaliser of $\psi$ and $\mathrm{id}_A$ in the category of commutative rings. As such, its kernel $I$ must be generated as an ideal by $(\mathrm{id}_A - \psi)(A)$, resulting in the claimed description. □

[5] View $\lambda$ as a cocharacter $\mathbb{C}^\times \to G^\vee$, with derivative $\lambda'\colon \mathbb{C} \to \mathfrak{g}$. Then the Lie algebra of $P^\lambda$ is the direct sum of the non-negative eigenspaces of $\lambda'(1) \in \mathfrak{g}$.

## 3. Action of real structures

Here we review real structures of complex Lie groups and describe their action on partial flag varieties. We start by reviewing some definitions, referring to [14, Sections 2–5] or [6, Section 2] for details.

**Definition 3.1.** A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ whose complexification $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathfrak{g}$ (as a complex Lie algebra). A *real structure* of a complex Lie algebra $\mathfrak{g}$ is an antiholomorphic (real) Lie algebra automorphism $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ of order two.

If $\sigma$ is a real structure on a complex Lie algebra $\mathfrak{g}$, then one readily checks that the fixed points $\mathfrak{g}^{\sigma}$ make up a real form of $\mathfrak{g}$. Conversely, for a real form $\mathfrak{g}_0$ the complex conjugation on $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ can be used to define a corresponding real structure on $\mathfrak{g}$ (uniquely up to an automorphism of $\mathfrak{g}$). This way, real forms and real structures are equivalent. On the level of Lie groups, complexification is harder to define[6], but real structures adapt without difficulties:

**Definition 3.2.** A *real structure* of a complex Lie group $G$ is an antiholomorphic smooth automorphism $\sigma\colon G \to G$ of order two. A *real form* of $G$ is a real Lie group isomorphic to the fixed point subgroup $G^{\sigma} \leq G$ for a real structure $\sigma$.

For our purposes, real structures on the group level are equivalent to those on the Lie algebra level:

**Lemma 3.3.** Let $G$ be a complex Lie group with Lie algebra $\mathfrak{g}$. Any real structure on $G$ defines a real structure on $\mathfrak{g}$ by differentiation, and this assignment is injective if $G$ is connected. Conversely, if $G$ is connected and either simply connected or of adjoint type, then any real structure on $\mathfrak{g}$ integrates to a unique real structure on $G$.

*Proof.* The passage from the group level to the Lie algebra level is standard. It is also clear that real structures of $\mathfrak{g}$ lift to $G$ when $G$ is connected and simply connected. For the adjoint type case, it suffices to observe to lift to a simply connected covering group $\tilde{G}$ and to observe that this lift takes the center of $\tilde{G}$ to itself. □

Several equivalence on the set of real structures are commonly used (though their names are not entirely standardised):

**Definition 3.4.** Let $G$ be a connected complex Lie group and $\mathfrak{g}$ its Lie algebra. Let $\sigma_1$, $\sigma_2$ be real structures on $\mathfrak{g}$ or $G$.

(i) $\mathrm{Aut}(\mathfrak{g})$ denotes the group of complex Lie algebra automorphisms of $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ denotes its subgroup generated by elements $\exp(\mathrm{ad}(X))$ for $X \in \mathfrak{g}$.

(ii) $\mathrm{Aut}(G)$ denotes the group of complex Lie group automorphisms of $G$. $\mathrm{Int}(G)$ denotes its subgroup consisting of elements $\mathrm{Conj}_g \coloneqq (h \mapsto ghg^{-1})$ for $g \in G$.

(iii) We say that $\sigma_1$ and $\sigma_2$ are *isomorphic*, denoted $\sigma_1 \approx \sigma_2$, if $\sigma_1 = \psi\sigma_2\psi^{-1}$ for $\psi \in \mathrm{Aut}(\mathfrak{g})$ resp. $\psi \in \mathrm{Aut}(G)$. This is equivalent to the corresponding real forms being (abstractly) isomorphic.

(iv) We say that $\sigma_1$ and $\sigma_2$ are *inner-isomorphic*, denoted $\sigma_1 \approx_i \sigma_2$, if $\sigma_1 = \psi\sigma_2\psi^{-1}$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. This is equivalent to the corresponding real forms being conjugate via the action of $G$.

(v) We say $\sigma_2$ are *inner (to each other)*, denoted $\sigma_1 \sim_i \sigma_2$,, if $\sigma_1 = \psi\sigma_2$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. Equivalently, $\sigma_1 = \sigma_2\psi'$ for $\psi'$ in $\mathrm{Int}(\mathfrak{g})$ resp. $\mathrm{Int}(G)$. The corresponding equivalence classes are called *inner classes*.

[6]It would be easier in the alternative setting of complex algebraic groups.

Note that inner-isomorphic real structures are automatically isomorphic as well as inner to each other (but not vice versa). We now come to a classical result of É. Cartan [4] relating real structures to complex involutions. To state it, we define equivalence relations for such automorphisms analogous to those in Definition 3.4:

**Definition 3.5.** Let $G$ be a connected complex Lie group and $\mathfrak{g}$ its Lie algebra. We write $\mathrm{Aut}_2(\mathfrak{g})$ and $\mathrm{Aut}_2(G)$ for the subgroups of involutions in $\mathrm{Aut}(\mathfrak{g})$ resp. $\mathrm{Aut}(G)$. Let $\theta_1, \theta_2$ be elements of $\mathrm{Aut}_2(\mathfrak{g})$ or $\mathrm{Aut}_2(G)$.

(i) We say that $\theta_1$ and $\theta_2$ are *isomorphic*, denoted $\theta_1 \approx \theta_2$, if $\theta_1 = \psi\theta_2\psi^{-1}$ for $\psi \in \mathrm{Aut}(\mathfrak{g})$ resp. $\psi \in \mathrm{Aut}(G)$.

(ii) We say that $\theta_1$ and $\theta_2$ are *inner-isomorphic*, denoted $\theta_1 \approx_i \theta_2$, if $\theta_1 = \psi\theta_2\psi^{-1}$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$.

(iii) We say $\theta_2$ are *inner (to each other)*, denoted $\theta_1 \sim_i \theta_2$, if $\theta_1 = \psi\theta_2$ for $\psi \in \mathrm{Int}(\mathfrak{g})$ resp. $\psi \in \mathrm{Int}(G)$. Equivalently, $\theta_1 = \theta_2\psi'$ for $\psi'$ in $\mathrm{Int}(\mathfrak{g})$ resp. $\mathrm{Int}(G)$. The corresponding equivalence classes are called *inner classes*.

**Theorem 3.6** (cf. e. g. Section 3 of [14])**.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then $\mathfrak{g}$ has a *compact real structure*, i. e. a real structure $\tau$ such that $\mathrm{Int}(\mathfrak{g}^{\tau})$ is compact. Moreover, every real structure $\sigma$ of $\mathfrak{g}$ is inner-isomorphic to some $\sigma'$ which commutes with $\tau$. Then $\theta \coloneqq \sigma'\tau = \tau\sigma' \in \mathrm{Aut}_2(\mathfrak{g})$. This defines bijections

$$\begin{array}{rcl} \{\text{Real structures of } \mathfrak{g}\}/\approx & \longleftrightarrow & \mathrm{Aut}_2(\mathfrak{g})/\approx \\ \{\text{Real structures of } \mathfrak{g}\}/\approx_i & \longleftrightarrow & \mathrm{Aut}_2(\mathfrak{g})/\approx_i \\ \{\text{Real structures of } \mathfrak{g}\}/\sim_i & \longleftrightarrow & \mathrm{Aut}_2(\mathfrak{g})/\sim_i \\ [\sigma] & \longleftrightarrow & [\theta]. \end{array}$$

which do not depend on the choice of $\tau$. If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, then $\tau$ lifts to $G$ and sets up an analogous correspondence between equivalence classes of real structures on $G$ and $\mathrm{Aut}_2(G)$.

**Definition 3.7.** If a real structure $\sigma$ commutes with a compact real structure $\tau$, the corresponding involutive automorphism $\theta = \sigma\tau$ is called the *Cartan involution (of $\sigma$ with respect to $\tau$)*.

We now come to the action on partial flag varieties. For the remainder of this section, $G$ will denote a connected complex semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h} \le \mathfrak{g}$ be a Cartan subalgebra with corresponding Cartan subgroup $H \coloneqq C_G(\mathfrak{h}) \le G$, and let $\Sigma \le \mathfrak{h}^*$ denote the root system defined by $(\mathfrak{g}, \mathfrak{h})$. Recall that the parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ can be parameterised by elements of $\mathfrak{h}$ as follows: given $v \in \mathfrak{h}$, let

$$\mathfrak{p}_v \coloneqq \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Sigma \\ \alpha(v)\ge 0}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denotes the root space of $\alpha$. Equivalently, $\mathfrak{p}_v$ is the direct sum of non-negative eigenspaces of $\mathrm{ad}(v)$. Each $\mathfrak{p}_v$ is a parabolic subalgebra of $\mathfrak{g}$ and defines a parabolic subgroup $P_v \le G$. Now if $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, H)$ is a cocharacter, we can take the derivative $\lambda' \colon \mathbb{C} \to \mathfrak{h}$ and define

$$P_\lambda \coloneqq P_{\lambda'(1)} = N_G(\mathfrak{p}_{\lambda'(1)}).$$

For the remainder of this section, all real structures of $G$ are assumed to normalise $H$.[7] A real structure $\sigma$ then defines an involution on the cocharacters via

$$\mathrm{Hom}(\mathbb{C}^\times, H) \to \mathrm{Hom}(\mathbb{C}^\times, H), \quad \lambda \mapsto \sigma_*\lambda := \sigma \circ \lambda \circ \overline{(\cdot)},$$

with $\overline{(\cdot)}$ denoting complex conjugation on $\mathbb{C}^\times$. One easily verifies that $P_{\sigma_*\lambda} = \sigma(P_\lambda)$. We say that $\sigma$ *preserves the Weyl group orbit* of a cocharacter $\lambda$ if there exists $w \in W := N_G(H)/H$ such that

$$\sigma_*\lambda = w \cdot \lambda := \mathrm{Conj}_s \circ \lambda, \quad \text{where } w = sH.$$

**Proposition 3.8.** Let $\sigma$ be a real structure of $G$ which normalises the Cartan subgroup $H$ and preserves the Weyl group orbit of a cocharacter $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, H)$. Choose $s \in N_G(H)$ such that $\sigma_*\lambda = \mathrm{Conj}_s \circ \lambda$. Then

$$\underline{\sigma}(gP_\lambda) := \sigma(g)sP_\lambda, \quad g \in G,$$

defines an antiholomorphic involution $\underline{\sigma}$ of $G/P_\lambda$ which does not depend on the choice of $s$.

*Proof.* Firstly, for $g \in G$ and $h \in P_\lambda$ we have

$$\sigma(gh)sP_\lambda = \sigma(g)\sigma(h)sP_\lambda = \sigma(g)s\,\mathrm{Conj}_{s^{-1}}(\sigma(h))P_\lambda = \sigma(g)sP_\lambda$$

because $\mathrm{Conj}_s(P_\lambda) = \sigma(P_\lambda)$, so $\underline{\sigma}(gP_\lambda)$ is well-defined.

Secondly, for another $s' \in N_G(H)$ with $\mathrm{Conj}_{s'} \circ \lambda = \sigma_*\lambda$, we have

$$s'P_\lambda = s(s^{-1}s')P_\lambda = sP_\lambda$$

since $\mathrm{Conj}_{s^{-1}s'}(P_\lambda) = P_\lambda$ and $P_\lambda$ is self-normalising. This shows that $\underline{\sigma}$ does not depend on the choice of $s$.

Lastly, the assumption $\mathrm{Conj}_s \circ \lambda = \sigma_*\lambda$ implies $\mathrm{Conj}_{\sigma(s)s} \circ \lambda = \lambda$. As before, we conclude that $\sigma(s)s \in P_\lambda$. Then

$$\underline{\sigma}^2(gP_\lambda) = g\sigma(s)sP_\lambda = gP_\lambda, \quad g \in G$$

verifies that $\underline{\sigma}$ is indeed an involution. Anti-holomorphicity is easily checked by lifting $\underline{\sigma}$ to $G$. $\square$

*Remark* 3.9. If $\lambda$ is minuscule, the real structure $\underline{\sigma}$ defined in Proposition 3.8 is the restriction to the affine Schubert variety $\mathrm{Gr}^\lambda$ of a real structure on the affine Grassmannian induced by $\sigma$.



As an antiholomorphic involution, $\underline{\sigma}$ should fix a real submanifold of $G/P_\lambda$ whose real dimension is $\dim_{\mathbb{C}} G/P_\lambda$. While this is true, it turns out that this submanifold can be empty:

**Proposition 3.10.** Let $G$, $\sigma$, and $\lambda$ be as in Proposition 3.8. Then

$$(G/P_\lambda)^{\underline{\sigma}} = \begin{cases} G^\sigma/P_\lambda^\sigma & \text{if } \sigma_*\lambda = \lambda \\ \varnothing & \text{otherwise.} \end{cases}$$

*Proof.* The key observation is that $(G/P_\lambda)^{\underline{\sigma}}$ is a closed union of $G^\sigma$-orbits in $G/P_\lambda$. Thus, it is either empty or contains the distinguished orbit through $eP_\lambda$ (where $e \in G$ is the neutral element) [20, Cor. 3.4]. But $\underline{\sigma}(eP_\lambda) = sP_\lambda$ for $s \in N_G(H)$ with $\mathrm{Conj}_s \circ \lambda = \sigma_*\lambda$. This equals $eP_\lambda$ precisely when $s \in P_\lambda$, which is equivalent to $\sigma_*\lambda = \lambda$.

In case $\sigma_*\lambda = \lambda$, we have seen that the fixed submanifold contains the distinguished orbit $G^\sigma \cdot (eP_\lambda) = G^\sigma/P_\lambda^\sigma$. But $\dim_{\mathbb{R}}(G/P_\lambda)^{\underline{\sigma}} = \dim_{\mathbb{C}}(G/P_\lambda)$ is the minimal dimension of $G^\sigma$-orbits [20, Thm. 3.6], which is attained only by the distinguished orbit [loc. cit. Cor. 3.4], so the fixed point set consists of that orbit alone. $\square$

[7] This is a mild assumption: any real structure $\sigma$ of $G$ is inner-isomorphic to one that normalises $H$. Indeed, $\sigma$ is easily seen to normalise some Cartan subgroup $H'$, and $H'$ is conjugate to $H$ via an inner automorphism.

We will be interested in the action of $\underline{\sigma}$ on equivariant cohomology. More precisely, the pair $(\sigma, \underline{\sigma})$ induces an algebra involution

$$\begin{tikzcd} H_G^*(G/P_\lambda) \arrow[r, "\sigma^*"] & H_G^*(G/P_\lambda) \\ H_G^* \arrow[u] \arrow[r, "\sigma^*"] & H_G^* \arrow[u] \end{tikzcd} \tag{13}$$

as in (9)–(10). We finish this section with a discussion of $\sigma^*$.

**Lemma 3.11.** Let $G$, $\sigma$, and $\lambda$ be as in Proposition 3.8. The involution $\sigma^*$ of (13) depends only on the inner class of $\sigma$.

*Proof.* If $\psi = \mathrm{Conj}_u \in \mathrm{Int}(G)$ is such that $\sigma' = \psi\sigma$ is another real structure, one finds that $\underline{\sigma}$ and $\underline{\sigma'}$ differ by the action $\ell_u$ of $u$ on $G/P_\lambda$ by left multiplication. Since $G$ is connected, this action is homotopic to the identity. It follows that $(\sigma')^* = \sigma^*$. □

Using (12), we find that $H_G^*(G/P_\lambda) \cong H_{P_\lambda}^*$. Moreover, $P_\lambda$ has a canonical Levi subgroup $L_\lambda$ whose Lie algebra $\mathfrak{l}_\lambda$ is the centraliser of $\lambda'(1)$ in $\mathfrak{g}$. By the discussion after (10), we then further have $H_G^*(G/P_\lambda) \cong H_{L_\lambda}^*$. Now let $\tau$ be a compact real structure of $G$ which preserves $H$ and commutes with $\sigma$ (which exists e. g.by [14, Prop. II.6]). Then $L_\lambda^\tau$ is a compact real form and hence a maximal compact subgroup of $L_\lambda$, so $H_{L_\lambda}^* \cong H_{L_\lambda^\tau}^*$. At this point, we can apply the isomorphism (11) to conclude

$$H_G^*(G/P_\lambda) \cong H_{P_\lambda}^* \cong H_{L_\lambda}^* \cong H_{L_\lambda^\tau}^* \cong S(\mathfrak{l}_\lambda^\tau)^{L_\lambda^\tau} \otimes_{\mathbb{R}} \mathbb{C} \cong S(\mathfrak{l}_\lambda)^{L_\lambda} \tag{14}$$

(where $\tau$ also denotes the real structure on $\mathfrak{g}$ obtained from $\tau$ by differentiation). Similarly, we have

$$H_G^* \cong H_{G^\tau}^* \cong S(\mathfrak{g}^\tau)^{G^\tau} \otimes_{\mathbb{R}} \mathbb{C} \cong S(\mathfrak{g})^G. \tag{15}$$

Lastly, via the Killing form on $\mathfrak{g}$ we obtain equivariant isomorphisms $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{l}_\lambda \cong \mathfrak{l}_\lambda^*$. We interpret the ring $S(\mathfrak{g}^*)^G$ as the *$G$-invariant polynomials* on $\mathfrak{g}$ and thus denote it as $\mathbb{C}[\mathfrak{g}]^G$ (and extend this notation to other Lie algebras and Lie groups). Then:

$$S(\mathfrak{g})^G \cong \mathbb{C}[\mathfrak{g}]^G, \quad S(\mathfrak{l}_\lambda)^{L_\lambda} \cong \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \tag{16}$$

**Lemma 3.12.** Let $G$, $\sigma$, and $\lambda$ be as in Proposition 3.8 and assume that $\sigma_*\lambda = \lambda$. Under the isomorphisms (14), (15) and (16), the involution $\sigma^*$ of (13) is identified with

$$\begin{tikzcd} \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \arrow[r, "\theta^*"] & \mathbb{C}[\mathfrak{l}_\lambda]^{L_\lambda} \\ \mathbb{C}[\mathfrak{g}]^G \arrow[u, "\mathrm{res}"'] \arrow[r, "\theta^*"] & \mathbb{C}[\mathfrak{g}]^G, \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

where $\theta^*$ denotes precomposition of polynomials with $\theta$ and res their restriction to subalgebras.

*Proof.* Under our assumption, the map $\underline{\sigma}$ has the simple form

$$\underline{\sigma}(gP_\lambda) = \sigma(g)P_\lambda, \quad g \in G.$$

It is then not difficult to trace this through each of the steps in (14)–(16) to arrive at the claimed description. □

## 4. Invariant rings and quasi-compact real structures

Lemma 3.12 translates the involutions (13) we are interested in to the setting of invariant polynomials. After recalling some notions in this context, we come to the key Lemma 4.7. Upon translating back, this result will motivate the use of quasi-compact real structures, as discussed at the end of this section. We begin by reiterating a definition made in passing above:

**Definition 4.1.** Let $G$ be a complex Lie group with Lie algebra $\mathfrak{g}$. Then $\mathbb{C}[\mathfrak{g}]^G$ denotes the subring of $\mathbb{C}[\mathfrak{g}] := S(\mathfrak{g}^*)$ consisting of elements invariant under the canonical $G$-action. If $G = \mathrm{Int}(\mathfrak{g})$, we also denote this ring by $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$.

An important classical result about invariant polynomials is the following theorem of Chevalley:

**Theorem 4.2** (cf. e. g.[19], Thm. 4.9.2)**.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, and $W$ the corresponding Weyl group. Then the canonical restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W := S(\mathfrak{h}^*)^W$ is an isomorphism.

**Corollary 4.3.** Let $G$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, and $N_G(\mathfrak{h})$ its normaliser. Then restriction defines an isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^{N_G(\mathfrak{h})}.$$

*Proof.* Let $G_0$ denote the identity component of $G$. Using the usual decomposition of $\mathfrak{g}$ into its center and derived subalgebra, Theorem 4.2 extends at once to

$$\mathbb{C}[\mathfrak{g}]^{G_0} \cong \mathbb{C}[\mathfrak{h}]^W.$$

To obtain the $G-$invariants, we now have to take into account the action of the component group $G/G_0$. But it is easy to see that $N_G(\mathfrak{h})$ meets all components of $G$, so

$$\mathbb{C}[\mathfrak{g}]^G = (\mathbb{C}[\mathfrak{g}]^{G_0})^{G/G_0} = (\mathbb{C}[\mathfrak{g}]^{G_0})^{N_G(\mathfrak{h})} \cong (\mathbb{C}[\mathfrak{h}]^W)^{N_G(\mathfrak{h})} = \mathbb{C}[\mathfrak{h}]^{N_G(\mathfrak{h})}.$$

□

Having recalled this tool, we now return to the study of involutions on invariant polynomial rings. Directly from the definition, we obtain the following counterpart of Lemma 3.11:

**Proposition 4.4.** Let $\mathfrak{g}$ be a complex Lie algebra and $\theta \in \mathrm{Aut}_2(\mathfrak{g})$. Then the involution $\theta^*$ of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ depends only on the inner class of $\theta$.

This suggests to look for particularly well-behaved involutions in a given inner class. To this end, recall that a *pinning* of a complex reductive Lie algebra $\mathfrak{g}$ consists of a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, a choice of simple roots $\Pi$ for the root system of $(\mathfrak{g}, \mathfrak{h})$, and a nonzero root vector $X_\alpha$ for every $\alpha \in \Pi$.

**Definition 4.5.** Let $\mathfrak{g}$ be a complex reductive Lie algebra. An automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ is called *pinning-preserving* if there exists a pinning $\{\mathfrak{h}, \Pi, \{X_\alpha\}_{\alpha\in\Pi}\}$ of $\mathfrak{g}$ such that $\theta$ maps $\mathfrak{h}$ to itself and permutes the the $X_\alpha$ (hence also the simple roots). If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, then an automorphism of $G$ is, by definition, *pinning-preserving* if its derivative is.

We recall some well-known results in this context:

**Lemma 4.6.** Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta \in \mathrm{Aut}(\mathfrak{g})$.

(i) If $\mathfrak{g}$ is semisimple and $\theta$ preserves a pinning $(\mathfrak{h}, \Pi, \{X_\alpha\}_{\alpha\in\Pi})$, then $\mathfrak{g}^\theta$ is semisimple with Cartan subalgebra $\mathfrak{h}^\theta$. Moreover, if $W$ is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, then its subgroup

$$W_\theta := \{w \in W : w\theta = \theta w\}$$

is identified with the Weyl group of $(\mathfrak{g}^\theta, \mathfrak{h}^\theta)$ by restriction to $\mathfrak{h}^\theta$.

(ii) Without further assumptions on $\mathfrak{g}$ or $\theta$, $\mathfrak{g}^\theta$ is reductive.

(iii) If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ normalised by $\theta$ such that $\dim \mathfrak{h}^\theta$ is maximal among such Cartan subalgebras, then $\mathfrak{h}^\theta$ is a Cartan subalgebra of $\mathfrak{g}^\theta$.

*Proof.* For part (i), we refer to [18, ch. 11].

For part (ii), note that $\theta$ preserves the direct sum decomposition of $\mathfrak{g}$ into its centre $\mathfrak{z}(\mathfrak{g})$ and derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$. The Killing form of $[\mathfrak{g}, \mathfrak{g}]$ restricts to a nondegenerate ad-invariant symmetric bilinear form of $[\mathfrak{g}, \mathfrak{g}]^\theta$, which shows that $[\mathfrak{g}, \mathfrak{g}]^\theta$ is reductive. It follows that $\mathfrak{g}^\theta = \mathfrak{z}(\mathfrak{g})^\theta \oplus [\mathfrak{g}, \mathfrak{g}]^\theta$ is reductive as well. To see that $\mathfrak{h}^\theta$ of part (iii) is a Cartan subalgebra, one can for instance use Gantmacher's normal form [cf. 14, Thm. 4.2], relating $\theta$ to a pinning-preserving automorphism as in part (i). □

**Lemma 4.7.** Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta$ a pinning-preserving automorphism of $\mathfrak{g}$. Then the restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$ is surjective and induces an isomorphism of $\mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$ with the coinvariant ring $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}_{\theta^*}$.

*Proof using reflection group theory.* Let $\mathfrak{g}_{\mathrm{der}} := [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra. The decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{der}}$ is preserved by $\theta$ and yields

$$\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \cong \mathbb{C}[\mathfrak{z}(\mathfrak{g})] \otimes \mathbb{C}[\mathfrak{g}_{\mathrm{der}}]^{\mathfrak{g}_{\mathrm{der}}}.$$

Clearly, it then suffices to prove the lemma for $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}_{\mathrm{der}}$ separately. For the affine space $\mathfrak{z}(\mathfrak{g})$ it follows immediately from Proposition 2.5, so for the remainder we may assume that $\mathfrak{g}$ is semisimple.

Now let $(\mathfrak{h}, \Pi, \{X_\alpha\}_{\alpha\in\Pi})$ be a pinning of $\mathfrak{g}$ preserved by $\theta$. As recalled above, $\mathfrak{h}^\theta$ is then a Cartan subalgebra of $\mathfrak{g}^\theta$, and $W_\theta$ is the corresponding Weyl group. By Corollary 4.3 we then have the commutative diagram of restriction maps

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \arrow[r] \arrow[d, "\cong"'] & \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta} \arrow[d, "\cong"] \\
\mathbb{C}[\mathfrak{h}]^W \arrow[r] & \mathbb{C}[\mathfrak{h}^\theta]^{W_\theta},
\end{tikzcd}$$

allowing us to work in the setting of $\mathfrak{h}$. By [16, Lemma 6.1], $\mathbb{C}[\mathfrak{h}]^W$ admits algebraically independent homogeneous generators $f_i$ ($i = 1, \dots, \operatorname{rank} W$) such that $\theta^* f_i = \varepsilon_i f_i$ for roots of unity $\varepsilon_i$. Moreover, $\theta$ fixes the sum of positive coroots, a regular element of $\mathfrak{h}$, so by [16, Corollary 6.5] the degrees of $W_\theta$ are precisely the $d_i$ with $\varepsilon_i = 1$.

It is clear that the $f_i$ with $\varepsilon_i \neq 1$ vanish on $\mathfrak{h}^\theta$. We have to show that these $f_i$ span the kernel of the restriction $\mathbb{C}[\mathfrak{h}]^W \to \mathbb{C}[\mathfrak{h}^\theta]^{W_\theta}$, and that the restrictions of the $f_i$ with $\varepsilon_i = 1$ generate $\mathbb{C}[\mathfrak{h}^\theta]^{W_\theta}$. To that end, consider the morphism $F : \mathfrak{h}^\theta \to \mathbb{C}^{\dim \mathfrak{h}^\theta}$ whose coordinates are the $f_i$ with $\varepsilon = 1$. Observe [cf. 16, proof of Thm. 3.4] that the fibre of 0 consists only of $0 \in \mathfrak{h}^\theta$ because *all* $f_i$ vanish there. Comparing dimensions, it follows that the coordinates of $F$ are algebraically independent, which completes the proof. □

*Proof using Kostant sections.* Reduce to the semisimple case as before. For a preserved pinning as above, the sum $e \coloneqq \sum_\alpha X_\alpha$ is fixed by $\theta$ and a principal nilpotent element [cf. 13, Section 5] for both $\mathfrak{g}$ and $\mathfrak{g}^\theta$. Upon extending it to an $\mathfrak{sl}_2$-triple $(e, f, h)$ in $\mathfrak{g}^\theta$, we obtain Kostant sections $\mathfrak{s} \coloneqq e + \mathfrak{g}_f$ of $\mathfrak{g}$ and $e + \mathfrak{g}_f^\theta = \mathfrak{s}^\theta$ for $\mathfrak{g}^\theta$. By [12, Thm. 7], the restriction map $\mathcal{I}(\mathfrak{g}) \to \mathcal{I}(\mathfrak{g}^\theta)$ is then equivalent to the restriction $\mathbb{C}[\mathfrak{s}] \to \mathbb{C}[\mathfrak{s}^\theta]$. This is clearly surjective, and Proposition 2.5 identifies $\mathbb{C}[\mathfrak{s}^\theta]$ with the coinvariant ring $\mathbb{C}[\mathfrak{s}]_\theta \cong \mathcal{I}(\mathfrak{g})_{\theta^*}$. □

**Corollary 4.8.** Let $G$ be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$. If $\theta$ is a pinning-preserving automorphism of $G$ (with derivative also denoted by $\theta$), then

$$\mathbb{C}[\mathfrak{g}^\theta]^{G^\theta} = \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}.$$

*Proof.* We have to show that every $g \in G^\theta$ acts trivially on every $f \in \mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$. By Lemma 4.7, $f$ admits an extension $\tilde{f} \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ with $\tilde{f}|_{\mathfrak{g}^\theta} = f$. Moreover, since $G$ is connected, we have $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[\mathfrak{g}]^G$, so $g \cdot \tilde{f} = \tilde{f}$. But this implies $g \cdot f = f$, too. □

The point of the preceding corollary is that it holds despite $G^\theta$ possibly being disconnected. If $\theta$ is not pinning-preserving, the following example shows that $\mathbb{C}[\mathfrak{g}^\theta]^{G^\theta}$ can indeed be strictly smaller than $\mathbb{C}[\mathfrak{g}^\theta]^{\mathfrak{g}^\theta}$.

*Example* 4.9. Let $G = \mathrm{GL}_{2n}(\mathbb{C})$ for some $n \in \mathbb{N}$, and let $\theta$ be inverse-transpose, $\theta(A) = (A^t)^{-1}$. Then $G^\theta = O_{2n}(\mathbb{C})$ with Lie algebra $\mathfrak{g}^\theta = \mathfrak{so}_{2n}(\mathbb{C})$. E. g. using the Chevalley restriction theorem, one finds that

$$\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathfrak{so}_{2n}(\mathbb{C})} \cong \mathbb{C}[x_1, \dots, x_n]^{S_n \ltimes \mathbb{Z}_2^{n-1}}$$

where $S_n \ltimes \mathbb{Z}_2^{n-1}$ acts on the variables by signed permutations with an even number of sign changes. From this ring, we obtain $\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{O_{2n}(\mathbb{C})}$ by taking into account the component group of $O_{2n}(\mathbb{C})$, which has order 2. Its nontrivial element acts on $\mathbb{C}[x_1, \dots, x_n]^{S_n \ltimes \mathbb{Z}_2^{n-1}}$ by a single sign change, so that



$$\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{O_{2n}} \cong \mathbb{C}[x_1, \dots, x_n]^{S_n \ltimes \mathbb{Z}_2^n},$$

with $S_n \ltimes \mathbb{Z}_2^n$ acting by arbitrary signed permutations. This is a strict subring of $\mathbb{C}[\mathfrak{so}_{2n}(\mathbb{C})]^{\mathfrak{so}_{2n}(\mathbb{C})}$ – for instance, $x_1 x_2 \cdots x_n \notin \mathbb{C}[x_1, \dots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$.

We can now translate back to the setting of real structures:

**Corollary 4.10.** Let $\sigma$ be a real structure of a connected complex reductive group $G$ with Cartan involution $\theta$. If $\theta$ is pinning-preserving then the restriction $H_G^* \to H_{G^\sigma}^*$ is surjective and identifies $H_{G^\sigma}^*$ with the coinvariant ring $(H_G^*)_{\sigma^*}$

*Proof.* As in (15)–(16) we obtain compatible isomorphisms

$$H_G^* \cong \mathbb{C}[\mathfrak{g}]^G, \quad H_{G^\sigma}^* \cong \mathbb{C}[\mathfrak{g}^\theta]^{G^\theta},$$

where $\theta$ is the Cartan involution of $\sigma$ (with respect to a suitable compact real structure). The result then follows from Lemma 4.7 in combination with Corollary 4.8. □

**Definition 4.11.** A real structure with pinning-preserving Cartan involution (as in Corollary 4.10) is called *quasi-compact*.

Quasi-compact real structures play a key role in this paper due to Corollary 4.10. The next Proposition establishes basic facts about them, including the reason for their name.

**Proposition 4.12.** Let $G$ be a connected complex reductive group. Every inner class of real structures on $G$ contains a quasi-compact real structure and this real structure is unique up to inner-isomorphism. If $G_0$ is the corresponding real form, then the dimension of its maximal compact subgroup is maximal among all real forms in the given inner class.

*Proof.* Existence and uniqueness of the quasi-compact real structure are equivalent via Theorem 3.6 to such statements about pinning-preserving involutions. In turn, these follow from well-known descriptions of $\mathrm{Aut}(\mathfrak{g})$, see e. g.[14, ch. 4]. The second statement can be checked using Gantmacher's normal form for automorphisms [cf. 14, Thm. 4.2]. ☐

We will also need the following Lemma, which lets us compare the invariant ring of a quasi-compact real form with that of any real form inner to it:

**Lemma 4.13.** Let $\sigma$ be a real structure of a connected complex reductive Lie group $G$ with Lie algebra $\mathfrak{g}$. There exist a quasi-compact real structure $\sigma_0$ and a compact real structure $\tau$ of $G$ such that

- $\sigma_0$ is inner to $\sigma$,
- $\sigma$ and $\sigma_0$ both commute with $\tau$, and
- $\sigma$, $\sigma_0$ and $\tau$ all normalise a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, and
- $\mathfrak{h}^\sigma = \mathfrak{h}^{\sigma_0}$.

Moreover, let $\theta \coloneqq \sigma\tau$ and $\theta_0 \coloneqq \sigma_0\tau$ denote the Cartan involutions, and $W_\theta$, $W_{\theta_0}$ the Weyl groups for $(\mathfrak{g}^\theta, \mathfrak{h}^\theta)$ and $(\mathfrak{g}^{\theta_0}, \mathfrak{h}^{\theta_0})$. Then, as subgroups of $\mathrm{GL}(\mathfrak{h}^\theta) = \mathrm{GL}(\mathfrak{h}^{\theta_0})$, we have

$$W_\theta \leq W_{\theta_0},$$

and it follows that $H^*_{G^{\sigma_0}}$ canonically injects into $H^*_{G^\sigma}$.

*Proof.* The existence of $\sigma_0$ is rather standard. The first requirement can be achieved via Proposition 4.12, and the second via a variant of [14, Prop.3.7]. The third and fourth conditions can be incorporated by conjugation with an appropriate inner automorphism. For the inclusion of Weyl groups, it suffices to check that the each element of the root system of $(\mathfrak{g}^\theta, \mathfrak{h}^\theta)$ is – up to nonzero rescaling – contained in that of $(\mathfrak{g}^{\theta_0}, \mathfrak{h}^{\theta_0})$; this can be done using Gantmacher normal forms [cf. 14, Thm. 4.2].

Finally, via (15)–(16) and Corollary 4.3 we obtain a diagram

$$\begin{tikzcd}
H^*_{G_0^\sigma} \arrow[r, dotted] \arrow[d, "\cong"] & H^*_{G^\sigma} \arrow[d, "\cong"] \\
\mathbb{C}[\mathfrak{g}^{\theta_0}]^{G^{\theta_0}} \arrow[r, dotted] \arrow[d, "\cong"] & \mathbb{C}[\mathfrak{g}^\theta]^{G^\theta} \arrow[d, "\cong"] \\
\mathbb{C}[\mathfrak{h}^\theta]^{N_{G^{\theta_0}}(\mathfrak{h}^\theta)} \arrow[r, dotted] & \mathbb{C}[\mathfrak{h}^\theta]^{N_{G^\theta}(\mathfrak{h}^\theta)}
\end{tikzcd}$$

in which the dotted arrows are to be defined. Equivalently, these rings in the bottom row are the subrings of $\mathbb{C}[\mathfrak{h}^\theta]^{W_{\theta_0}}$ resp. $\mathbb{C}[\mathfrak{h}^\theta]^{W_\theta}$ invariant under the actions of the relevant component groups. But, by the same argument as in the proof of Corollary 4.8, we see that these component groups both act trivially on $\mathbb{C}[\mathfrak{h}^\theta]^{W_{\theta_0}}$. Together with the containment $W_\theta \leq W_{\theta_0}$, this lets us put a canonical injection in the bottom row of the diagram above, finishing the proof. ☐

## 5. Proof of main theorem

We now come to the proof of Theorem 1.1. It is structured into three lemmas, followed by a main body combining everything. The first two lemmas reduce from the semisimple to the simple case, and the third establishes a key fact for most cases.

**Lemma 5.1.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ be its decomposition into simple complex Lie algebras $\mathfrak{g}_i$. Under this isomorphism, a minuscule weight $\lambda$ of $\mathfrak{g}$ decomposes as a sum $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$ of minuscule weights of the $\mathfrak{g}_i$. Moreover, the Kirillov algebra $\mathcal{C}^\lambda(\mathfrak{g})$ decomposes as

$$\mathcal{C}^\lambda(\mathfrak{g}) \cong \mathcal{C}^{\lambda_1}(\mathfrak{g}_1) \otimes \cdots \otimes \mathcal{C}^{\lambda_\ell}(\mathfrak{g}_\ell).$$

*Proof.* The decomposition of $\lambda$ is a simple consequence of highest weight theory, which also yields $V^\lambda \cong V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_\ell}$ (with $\mathfrak{g}_i$ acting on the $i$-th tensor factor). It follows that both factors in $S(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)$ decompose as tensor products compatibly with the decomposition of $\mathfrak{g}$; hence, so does the Kirillov algebra. □

**Lemma 5.2.** Let $G$ be a complex semisimple Lie group of adjoint type[8], and let $G \cong G_1 \times \cdots \times G_\ell$ be its decomposition into simple factors $G_i$. Let $\sigma$ be a real structure on $G$. Then $\sigma$ permutes the $G_i$ with orbits of one or two elements. For each $i$, there are two possibilities:

(a) If $\sigma(G_i) = G_i$, then $\sigma$ restricts to a real structure of $G_i$.
(b) If $\sigma(G_i) = G_j$ with $i \neq j$, then $G_j \cong G_i$ and the Cartan involution of $\sigma|_{G_i \times G_j}$ is isomorphic to the *swap involution*

$$G_i \times G_i \to G_i \times G_i, \quad (g, h) \mapsto (h, g).$$

*Proof.* The permutation property follows from simplicity of the factors, and part (a) is clear. For part (b), we can extend suitable compact real structures of the factors to obtain a compact real structure $\tau$ of $G_i \times G_j$ which preserves the product structure and commutes with $\sigma$. Then, the Cartan involution $\theta \in \mathrm{Aut}_2(G_i \times G_j)$ must map $G_i$ to $G_j$. The resulting $G_i \cong G_j$ can be used to identify $\theta$ with the swap involution up to isomorphism. □

**Lemma 5.3.** Let $\mathfrak{g}$ be a simple complex Lie algebra with a minuscule coweight $\lambda$, and let $\mathfrak{l}_\lambda$ be the corresponding Levi subalgebra (i. e. the centraliser of $\lambda$). Let $\mathfrak{S}$ be an inner class of real structures on $\mathfrak{g}$ which preserve the Weyl group orbit of $\lambda$. If $\mathfrak{g}$ is not of type $A_{2n}$ ($n \in \mathbb{N}$) or if $\mathfrak{S}$ does not contain a split real structure[9], then $\mathfrak{S}$ contains a real structure $\sigma$, unique up to inner-isomorphism, for which

(i) $\sigma_* \lambda = \lambda$, and
(ii) $\sigma|_{\mathfrak{l}_\lambda}$ is quasi-compact.

*Proof.* One can verify this using Satake diagrams. We can realise $\lambda$ as an element of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and since $\lambda$ is minuscule, we can arrange it to be the fundamental coweight of a simple root. We may then restrict to real structures $\sigma \in \mathfrak{S}$ which preserve $\mathfrak{h}$ and for which $\mathfrak{h}^\sigma$ is maximally split.[10] These are then up to inner-isomorphism classified by Satake diagrams [1], which consist of the Dynkin diagram of $\mathfrak{g}$ together with a 2-coloring into black and white of the nodes and an involutive permutation of the white nodes, indicated by arrows.

The assumption that the elements of $\mathfrak{S}$ preserve the Weyl group orbit of $\lambda$ implies that the node representing $\lambda$ has now arrow attached. One then verifies that condition (i) is equivalent to that node being white. Moreover, deleting a white node from a Satake diagram yields the

[8]That is, $G$ is assumed to have trivial centre, which implies that $G$ indeed decomposes into simple factors.
[9]i. e.a real structure whose corresponding real form is split over $\mathbb{R}$
[10]i. e. $\dim(\mathfrak{h}^\theta)$ is minimal for $\theta$ denoting the Cartan involution.

Satake diagram for the restricted real structure on the corresponding Levi subalgebra. Thus, it suffices to check that the Satake diagram of the (unique class of) quasi-compact real structure on $\mathfrak{l}_\lambda$ can be obtained from a Satake diagram of $\mathfrak{g}$ by deleting a white node corresponding to $\lambda$. This is indeed the case whenever $\mathfrak{g}$ is not of type $A_{2n}$, as the reader can verify using the tables in Section 7. □

*Proof of Theorem 1.1.* Since $G$ is simply connected, its Langlands dual $G^\vee$ is of adjoint type and Lemmas 5.1 and 5.2 apply. Clearly, the permutation of simple factors of $G^\vee$ is the same for all $\sigma \in \mathfrak{S}$. By restricting to its orbits, the theorem is reduced to three cases:

(a) $\mathfrak{g}$ is simple and either not of type $A_{2n}$ or $\mathfrak{S}$ does not contain a split real structure.
(b) $\mathfrak{g} \cong \mathfrak{sl}_{2n}$ for some $n \in \mathbb{N}$ and $\mathfrak{S}$ contains a split real structure.
(c) $\mathfrak{g} \cong \mathfrak{g}_s \oplus \mathfrak{g}_s$ for $\mathfrak{g}_s$ simple, and the real structures in $\mathfrak{S}$ swap the two copies of $\mathfrak{g}_s$.

In each case, we now give a real structure $\sigma \in \mathfrak{S}$ for which $\sigma_* \lambda = \lambda$ and such that the identity (5) holds. The latter is equivalent to

$$(H^*_{L_\lambda})_{\sigma^*} \cong H^*_{L^\sigma_\lambda} \tag{17}$$

by Corollary 2.4 and (12).

- In case (a), we can apply Lemma 5.3 to obtain $\sigma \in \mathfrak{S}$ fixing $\lambda$ with quasi-compact restriction to $L_\lambda$. Then (17) follows from Corollary 4.10.
- In case (b), we have $G^\vee = PGL_{2n+1}(\mathbb{C})$. The inner class of its split real structure(s) contains one inner-isomorphism class. Thus, up to isomorphism we can only choose complex conjugation, i. e. $\sigma(A) = \overline{A}$. The minuscule coweights of $PGL_{2n+1}$ are exactly the fundamental coweights (and zero), and are (for standard choices) all fixed by $\sigma$. The corresponding Levi subgroups are $L_k := P(GL_k \times GL_{2n+1-l})$ for $k \in \mathbb{N}$, and using (15) and Corollary 4.3 one finds that

  $$H^*_{L_k} \cong \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{2n+1-1}]^{S_k \times S_{2n+1-k}} / (x_1 + \cdots + y_{2n+1-k}).$$

  The Cartan involution $A \mapsto (A^t)^{-1}$ acts as $-1$ on a Cartan subalgebra, so Lemma 3.12 and Theorem 4.2 imply that $\sigma^*$ acts on homogeneous elements by multiplication with $(-1)^{\deg}$. The coinvariant ring is then isomorphic to

  $$\mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{2n+1-1}]^{S_k \times S_{2n+1-k} \times \mathbb{Z}_2^{2n+1}},$$

  with $\mathbb{Z}_2^{2n+1}$ acting by sign changes on the variables. But a computation similar to that in Example 4.9 identifies that ring with $H^{L^\sigma_k}$, proving (17) for this case.
- In case (c), all elements of $\mathfrak{S}$ are quasi-compact, and can be conjugated by an inner automorphism to fix $\lambda$. The restriction to $\mathfrak{l}_\lambda$ is then also quasi-compact, so we can proceed as in case (a).

The rest of the proof can again be treated uniformly. Firstly, (6) holds for any quasi-compact $\sigma_0 \in \mathfrak{S}$ by Lemma 3.11, (15), and Corollary 4.10. The injection (7) is achieved by Lemma 4.13. The maps in diagram (8) are all derived from restriction maps of invariant polynomial rings, so the diagram commutes. □



## 6. Characterisation of freeness

In this section, Theorem 1.1 is used to characterise freeness of the coinvariant homomorphism (4), resulting in a proof of Theorem 1.2. According to Theorem 1.1, we have to analyse the composition

$$H^*_{(G^\vee)^{\sigma_0}} \xrightarrow{\varphi} H^*_{(G^\vee)^\sigma} \to H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma / P^\sigma_\lambda) \tag{18}$$

where $\varphi$ is the canonical injection constructed in Lemma 4.13.

**Lemma 6.1.** $\varphi$ is finite.

*Proof.* Indeed, it is an injection between finitely generated $\mathbb{C}$-algebras of equal transcendence degree (namely $\operatorname{rank}(G^\vee)^\sigma = \operatorname{rank}(G^\vee)^{\sigma_0}$). ☐

The behaviour of the second map in (18) is related to the geometry of the homogeneous space $X \coloneqq (G^\vee)^\sigma / P_\lambda^\sigma$. Here it is more convenient to work with compact Lie groups, so we fix a compact real structure $\tau$ of $G^\vee$ that commutes with $\sigma$. It will be convenient to choose $\tau$ such that $\tau_* \lambda = -\lambda$, which can be achieved by a standard construction of compact real structures [cf. e. g. 11, Thm. 6.11]. Now, $K \coloneqq ((G^\vee)^\sigma)^\tau$ is a maximal compact subgroup of $(G^\vee)^\sigma$. Since $(G^\vee)^\sigma \cong KP_\lambda^\sigma$ [11, Prop. 7.83f], $K$ acts transitively on $X$, so $X \cong K/L$ where $L \coloneqq P_\lambda^\sigma \cap K = L_\lambda^\sigma \cap K$. The cohomology of such compact homogeneous spaces is particularly well-behaved in the so-called *equal rank* case:

**Theorem 6.2.** If $K$ is a connected compact Lie group and $L \leq K$ a closed subgroup, then the odd singular cohomology of $X = K/L$ vanishes if and only if $K$ and $L$ have the same rank. In this case, we further have an isomorphism $H_K^*(X) \cong H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} H_K^*$ of $H_K^*$-modules.

**Note:** In our convention, *rank* means dimension of maximal torus, which need not coincide with the rank of the root system (due to a possibly positive-dimensional centre).

*Proof.* The first statement is classical: the "only if" part follows from vanishing of the Euler characteristic $\chi(X)$ in the nonequal rank case first shown by Hopf and Samelson [9]. The "if" part is due to Borel [2]. For a more detailed discussion, see [3, ch. 5]. The statement about equivariant cohomology (whose conclusion is known as *equivariant formality*) follows from degeneracy of the Serre spectral sequence for the fibration $X_K \to \mathbb{B}K$ with fibre $X$ [cf. e. g. 3, ch. 9]. ☐

Returning to the analysis before Theorem 6.2, we are lead to compare the ranks of $K = ((G^\vee)^\sigma)^\tau$ and $L = L_\lambda^\sigma \cap K$. We translate the question to the complex setting via the Cartan involution $\theta \coloneqq \sigma\tau$. The theorem then implies:

**Lemma 6.3.** If $(G^\vee)^\theta$ and $L_\lambda^\theta$ have the same rank, then the canonical map

$$H^*_{(G^\vee)^\sigma} \to H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma / P_\lambda^\sigma)$$

is free. Otherwise, $H^*_{(G^\vee)^\sigma}$ has strictly larger transcendence degree than $H^*_{(G^\vee)^\sigma}((G^\vee)^\sigma / P_\lambda^\sigma)$, so the map cannot be injective, and in particular cannot be free.

*Proof.* The first assertion indeed follows immediately from Theorem 6.2. For the statement about transcendence degrees, let $\mathfrak{h} \leq (\mathfrak{g}^\vee)^\theta$ and $\mathfrak{h}' \leq \mathfrak{l}_\lambda^\theta$ be Cartan subalgebras. By Corollary 4.3, the rings involved are invariant subrings of $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}]$ by finite groups.[11] Their transcendence degrees are then the dimensions of $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. ☐

We are thus lead to compare Cartan subalgebras of $(\mathfrak{g}^\vee)^\theta$ and $\mathfrak{l}_\lambda^\theta$, with the following result:

**Lemma 6.4.** Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\theta \in \operatorname{Aut}_2(\mathfrak{g})$. Among Cartan subalgebras of $\mathfrak{g}$ normalised by $\theta$, choose $\mathfrak{h}$ such that $\dim \mathfrak{h}^\theta$ is minimal. Let $\lambda$ be a minuscule coweight of $\mathfrak{g}$ contained in $\mathfrak{h}^{-\theta}$ (i. e. $\theta(\lambda) = -\lambda$) and $\mathfrak{l}_\lambda$ the corresponding Levi subalgebra of $\mathfrak{g}$. If $\theta$ preserves a pinning of $\mathfrak{l}_\lambda$, then $\mathfrak{g}^\theta$ and $\mathfrak{l}_\lambda^\theta$ have equal rank if and only if $\theta$ is also pinning-preserving for $\mathfrak{g}$.

[11] Finiteness follows from the well-known finiteness of Weyl groups and the fact that $G^\theta$ and $L_\lambda^\theta$ have finitely many connected components. One way to see this is that they are homotopic to their intersections with the compact group $K$.

*Proof.* We may assume that $\mathfrak{g}$ is semisimple. The ranks of $\mathfrak{g}^\theta$ and $\mathfrak{l}_\lambda^\theta$ are the maximal dimensions of $\mathfrak{t}^\theta$ for $\theta$-stable Cartan subalgebras $\mathfrak{t}$ of $\mathfrak{g}$ or of $\mathfrak{l}_\lambda$, respectively (cf. Lemma 4.6). In particular, both ranks admit the lower bound $\dim \mathfrak{h}^\theta$. For convenience, we now translate to the parallel setting of real structures, which is more easily found in the literature. That is, we fix a compatible compact real structure $\tau$ of $\mathfrak{g}$ and define $\sigma \coloneqq \tau\theta = \theta\tau$. Then $\mathfrak{t}$ as above correspond to *maximally compact* Cartan subalgebras of $\mathfrak{g}^\sigma$ or $\mathfrak{l}_\lambda^\sigma$.

It is well-known that the conjugacy classes of Cartan subalgebras of a real reductive Lie algebra can be related by *Cayley transforms*, which are defined using *real or noncompact imaginary roots* [cf. 11, p. 390]. In particular, if there are no such roots with respect to $\sigma$, then there is only one such conjugacy class. In the case of a pinning-preserving Cartan involution $\theta$, one can always arrange that there are no real roots and no noncompact imaginary simple roots. As long as $\theta^*\alpha$ is orthogonal to $\alpha$ for every simple $\alpha$ this implies (using [11, Prop. 6.104]) that there are no such roots at all, and all Cartan subalgebras of of the real form are conjugate.

Thus, if there is no simple summand of $\mathfrak{l}_\lambda$ of type $A_{2n}$ on which $\theta$ restricts to a nontrivial involution, then $\mathfrak{l}_\lambda^\sigma$ has only one conjugacy class of ($\theta$-stable) Cartan subalgebras.[12] It then follows that all $\theta$-stable Cartan subalgebras of $\mathfrak{l}_\lambda$ are conjugate via $\mathrm{Int}(\mathfrak{l}_\lambda^\theta)$, so that $\mathfrak{h}^\theta$ must be a Cartan subalgebra of $\mathfrak{l}_\lambda^\theta$, and $\mathrm{rank}\, \mathfrak{l}_\lambda^\theta = \dim \mathfrak{h}^\theta$. A similar analysis shows that if $\theta$ is pinning-preserving on $\mathfrak{g}$ and there are no interfering type $A_{2n}$ summands, $\mathfrak{g}^\theta$ has rank $\dim \mathfrak{h}^\theta$ as well. To finish the proof we thus have to do the following:

(1) Show that a non-quasi-compact real form of a complex reductive Lie algebra always admits more than one isomorphism class of Cartan subalgebras. Since only one of them is maximally compact [11, Prop. 6.61], it will follow that $\mathrm{rank}\, \mathfrak{g}^\theta > \dim \mathfrak{h}^\theta$ in this case.
(2) Check some cases involving type $A_{2n}$ summands separately.

For task (1), it suffices to show that such a real form always has noncompact imaginary roots with respect to a maximally compact Cartan subalgebra. This follows from the classification using *Vogan diagrams* [cf. 11, p. VI.8], since the absence of noncompact imaginary (simple) roots would imply quasi-compactness (by uniqueness of the classification).

For task (2), we may assume that $\mathfrak{g}$ is simple. The cases not yet covered are then up to isomorphism as follows:

- $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$ with $\mathfrak{l}_\lambda = \mathfrak{s}(\mathfrak{gl}_{2k+1} \oplus \mathfrak{gl}_{2n-2k-1})(\mathbb{C})$ for $k, n \in \mathbb{N}$; $\mathfrak{g}^\theta = \mathfrak{so}_{2n}(\mathbb{C})$, and $\mathfrak{l}_\lambda^\theta = \mathfrak{so}_{2k+1}(\mathbb{C}) \oplus \mathfrak{so}_{2n-2k-1}(\mathbb{C})$. Here $\theta$ is not pinning-preserving on $\mathfrak{g}$ and the fixed subalgebras have ranks $n$ and $n-1$, respectively.
- $\mathfrak{g} = \mathfrak{sp}_{4n+2}(\mathbb{C})$ for $n \in \mathbb{N}$ with $\mathfrak{l}_\lambda = \mathfrak{sl}_{2n+1}(\mathbb{C})$; $\mathfrak{g}^\theta = \mathfrak{gl}_{2n+1}(\mathbb{C})$ and $\mathfrak{l}_\lambda^\theta = \mathfrak{so}_{2n+1}(\mathbb{C})$. Here, $\theta$ is again not pinning-preserving on $\mathfrak{g}$, and the fixed subalgebras have ranks $2n+1$ and $n$, respectively.

□

*Remark* 6.5. It seems likely that Lemma 6.4 can be proven more systematically and with fewer assumptions, perhaps using qualitative properties of Cayley transforms [cf. 11, p. VI.7].

We can now collect the auxiliary results of this section into a proof of Theorem 1.2, which characterises freeness of the coinvariant homomorphism (4).

*Proof of Theorem 1.2.* For $\mathfrak{g} = \mathfrak{sl}_{2n+1}(\mathbb{C})$, $n \in \mathbb{N}$, with $\mathfrak{S}$ containing a split real structure, the Theorem can be checked directly. Here all elements of the inner class are quasi-compact, and a simple computation (as in the proof of Theorem 1.1) shows that the coinvariant homomorphism

[12] Indeed, $A_{2n}$ is the only simple type in which a Dynkin diagram automorphism can map a node to an adjacent node.

4 is indeed free in this case. For the remainder of the proof, we assume that no element of $\mathfrak{S}$ restricts to a split real structure on a type $A_{2n}$ summand of $G^\vee$.

It is then clear from the proof of Theorem 1.1 that we can choose $\sigma = \sigma_0$ whenever a quasi-compact real structure in the given inner class fixes $\lambda$. If this is the case, the first map, called $\varphi$, in (18) is an identity. Moreover, the second map is then free by Lemmas 6.4 and 6.3.

On the other hand, if $\sigma$ is not quasi-compact, then Lemmas 6.4 and 6.3 imply that the transcendence degree drops in the second step of (18). By Lemma 6.1, it stays the same in the first step, so overall the coinvariant homomorphism decreases transcendence degree. In particular, it is not injective, hence not free. □

## 7. Remarks on uniqueness and tables

In the proof of Theorem 1.1, three cases were distinguished. In most cases, the real structure guaranteed by the Theorem can be chosen to restrict quasi-compactly to the relevant Levi subgroup; it is then essentially unique (see Lemma 5.3). This restriction property can fail for simple factors of type $A_{2n}$ with the inner class of the split real structure, or on factors of type $G_s \times G_s$, with $G_s$ simple, where the two copies are swapped by the real structures. For both types of factors, the relevant inner classes contain only one inner-isomorphism class. Thus, we could obtain a unique choice of $\sigma$ up to (inner) isomorphism by demanding quasi-compact restriction on all factors where this is possible.

The resulting $\sigma$ are precisely those used in the proof of Theorem 1.1. They are listed, in terms of the real forms, for all minuscule weights in tables 1 and 2, grouped by the inner classes. We also list the Levi subalgebras $\mathfrak{l}_\lambda$ via their (more conveniently notated) derived subalgebras $\mathfrak{l}'_\lambda$.

However, in the way it is stated, Theorem 1.1 could allow for several essentially different choices of $\sigma$. For the sake of completeness, we remark that this can indeed happen. Namely, consider the case of $G^\vee = Sp_{4n}(\mathbb{C})$ with $L_\lambda \cong GL_{2n}(\mathbb{C})$ and the inner class containing a split real form of $G^\vee$. The real structure listed in Table 1 results in the real form $U^*(2n)$ of $GL_{2n}(\mathbb{C})$. However, a calculation as in Example 4.9 shows that we could also have chosen the split real structure of $G^\vee$, resulting in the real form $GL_{2n}(\mathbb{R})$ of $GL_{2n}(\mathbb{C})$.

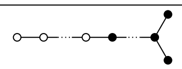
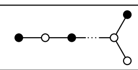
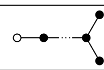
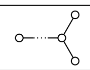
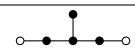
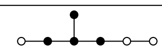
| $\mathfrak{g}^\vee$ | $\mathfrak{l}'_\lambda$ | $(\mathfrak{g}^\vee)^\sigma$ | $(\mathfrak{l}^\sigma_\lambda)'$ | Satake diagram |
|---|---|---|---|---|
| $\mathfrak{sl}_{2n}(\mathbb{C})$ | $\mathfrak{sl}_{2k}(\mathbb{C}) \oplus \mathfrak{sl}_{2n-2k}(\mathbb{C})$ | $\mathfrak{su}^*(2n)$ | $\mathfrak{su}^*(2k) \oplus \mathfrak{su}^*(2n-2k)$ |  |
| $\mathfrak{sl}_{2n}(\mathbb{C})$ | $\mathfrak{sl}_{2k+1}(\mathbb{C}) \oplus \mathfrak{sl}_{2n-2k-1}(\mathbb{C})$ | $\mathfrak{sl}_{2n}(\mathbb{R})$ | $\mathfrak{sl}_{2k+1}(\mathbb{R}) \oplus \mathfrak{sl}_{2n-2k-1}(\mathbb{R})$ |  |
| $\mathfrak{sl}_{2n+1}(\mathbb{C})$ | $\mathfrak{sl}_{k}(\mathbb{C}) \oplus \mathfrak{sl}_{2n+1-k}(\mathbb{C})$ | $\mathfrak{sl}_{2n+1}(\mathbb{R})$ | $\mathfrak{sl}_{k}(\mathbb{R}) \oplus \mathfrak{sl}_{2n+1-k}(\mathbb{R})$ |  |
| $\mathfrak{so}_{2n+1}(\mathbb{C})$ | $\mathfrak{so}_{2n-1}(\mathbb{C})$ | $\mathfrak{so}_{1,2n}(\mathbb{R})$ | $\mathfrak{so}_{2n-1}(\mathbb{R})$ |  |
| $\mathfrak{sp}_{4n}(\mathbb{C})$ | $\mathfrak{sl}_{2n}(\mathbb{C})$ | $\mathfrak{sp}_{2n,2n}(\mathbb{R})$ | $\mathfrak{su}^*(2n)$ |  |
| $\mathfrak{sp}_{4n+2}(\mathbb{C})$ | $\mathfrak{sl}_{2n+1}(\mathbb{C})$ | $\mathfrak{sp}_{4n+2}(\mathbb{R})$ | $\mathfrak{sl}_{2n+1}(\mathbb{R})$ |  |
| $\mathfrak{so}_{4n}(\mathbb{C})$ | $\mathfrak{so}_{4n-2}(\mathbb{C})$ | $\mathfrak{so}_{2,4n-2}(\mathbb{R})$ | $\mathfrak{so}_{1,4n-3}(\mathbb{R})$ |  |
| $\mathfrak{so}_{4n}(\mathbb{C})$ | $\mathfrak{sl}_{2n}(\mathbb{C})$ | $\mathfrak{so}^*(4n)$ | $\mathfrak{su}^*(2n)$ |  |
| $\mathfrak{so}_{4n+2}(\mathbb{C})$ | $\mathfrak{so}_{4n-2}(\mathbb{C})$ | $\mathfrak{so}_{1,4n+1}(\mathbb{R})$ | $\mathfrak{so}_{4n-2}(\mathbb{R})$ |  |
| $\mathfrak{so}_{4n+2}(\mathbb{C})$ | $\mathfrak{sl}_{2n+1}(\mathbb{C})$ | $\mathfrak{so}_{2n+1,2n+1}(\mathbb{R})$ | $\mathfrak{sl}_{2n+1}(\mathbb{R})$ |  |
| $\mathfrak{e}_6(\mathbb{C})$ | $\mathfrak{so}_{10}(\mathbb{C})$ | $\mathfrak{e}_{6,-26}$ | $\mathfrak{so}_{1,9}(\mathbb{R})$ |  |
| $\mathfrak{e}_7(\mathbb{C})$ | $\mathfrak{e}_6(\mathbb{C})$ | $\mathfrak{e}_{7,-25}$ | $\mathfrak{e}_{6,-26}$ |  |



Table 1. Real forms *inner to a split real form* adapted to minuscule coweights and their Satake diagrams

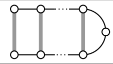
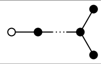
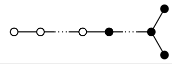
| $\mathfrak{g}^\vee$ | $\mathfrak{l}_\lambda'$ | $(\mathfrak{g}^\vee)^\sigma$ | $(\mathfrak{l}_\lambda^\sigma)'$ | Satake diagram |
|---|---|---|---|---|
| $\mathfrak{sl}_{2n}(\mathbb{C})$ | $\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{su}_{n,n}$ | $\mathfrak{sl}_n(\mathbb{C})_\mathbb{R}$ |  |
| $\mathfrak{so}_{4n}(\mathbb{C})$ | $\mathfrak{so}_{4n-2}(\mathbb{C})$ | $\mathfrak{so}_{1,4n-1}(\mathbb{R})$ | $\mathfrak{so}_{4n-2}(\mathbb{R})$ |  |
| $\mathfrak{so}_{4n+2}(\mathbb{C})$ | $\mathfrak{so}_{4n}(\mathbb{C})$ | $\mathfrak{so}_{2,4n}(\mathbb{R})$ | $\mathfrak{so}_{1,4n-1}(\mathbb{R})$ |  |

Table 2. Real forms *not inner to a split real form* adapted to invariant minuscule coweights and their Satake diagrams. A subscript $\mathbb{R}$ denotes that a complex Lie algebra is considered as its underlying real Lie algebra.

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