

On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s -plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.^{[4] [5]} We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.^[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.^[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < \text{Re}[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In ?? to ??, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$, where ω, t are real, as follows (link).^[3]

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t , given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in link using contour integration.

$$E_p(t) = E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \leq \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. Step 2: On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t) = f(t) e^{-\sigma t} u(-t) + f(t) e^{\sigma t} u(t)$, where $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ and $f_1(t) = e^{\sigma t_0} E'_p(t + t_0)$ and $f_2(t) = e^{-\sigma t_0} E'_p(t - t_0)$ and $E'_p(t) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$ and t_0, t_2 are real and $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, for every value of t_0 , to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

1.3. Step 3: On the zeros of the function $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g(t)$ and compute its real part given by $G_R(\omega) = G_R(\omega, t_2, t_0)$ and we can write as follows.

$$\begin{aligned} G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0)] \cos(\omega \tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0) e^{-2\sigma \tau} + E'_{0n}(\tau + t_0)] \cos(\omega \tau) d\tau \end{aligned} \quad (3)$$

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **each fixed value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$.

1.4. Step 4: Zero Crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

In Section 2.3, we show the result in Eq. 4 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ is an **odd** function of t_0 , for all t_0 , for a given value of t_2 as follows.

$$\begin{aligned} P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau] \end{aligned} \quad (4)$$

1.5. Step 5: Final Step

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2 . In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2, t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use $E'_0(t) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_{0n}(t) = E'_0(-t)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \quad (5)$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $\tau = [0, t_{0c}]$ where $t_{0c} > 0$. For $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, we see that $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi}{2t_{0c}}\tau$ lies in the range $[0, \frac{\pi}{2}]$ and hence $\sin(\omega_{c1}\tau) > 0$ in that interval $\tau = [0, t_{0c}]$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t , where $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix A.1.

2.1. New function $g(t)$

Let us consider the function $E_p'(t) = E_p'(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t)e^{-\sigma t}$, where t_2 is finite and real, and $E_0'(t) = E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E_{p\omega}'(\omega) = E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0)$ and $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega) = F(\omega, t_2, t_0) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$.

Let us consider a new function $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We note that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), F(\omega)$ and $G(\omega)$ which are also functions of variables t_2, t_0 . Similarly we use the shorthand notation for the functions $E_p'(t), E_0'(t)$ and $E_{p\omega}'(\omega)$ which are also functions of variable t_2 .

We can show that $E_p(t), E_p'(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \rightarrow \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}'(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \rightarrow \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix A.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t .

If Statement 1 is true, then we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0 , for a given fixed value of t_2 . This implies that the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite, where $G_R(\omega)$ crosses the zero line to the opposite sign, and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega)$ crosses the zero line to the opposite sign, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**. This is shown

in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0 , for a given fixed value of t_2 , where $G_R(\omega)$ crosses the zero line to the opposite sign and $\omega_z(t_2, t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t)$ where $f_1(t) = e^{\sigma t_0}E'_p(t + t_0)$ and $f_2(t) = e^{-\sigma t_0}E'_p(t - t_0)$, $E'_p(t) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2)$, and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E'_{p\omega}(\omega)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_2, t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \quad (6)$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$. (Appendix A.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 6 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (7)$$

We can split the integral in Eq. 7 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \left[\int_{-\infty}^0 G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (8)$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (link). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 8 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_0^{\infty} G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (9)$$

In Appendix A.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \rightarrow \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 9 is zero. For finite $\omega \geq 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$. We see that $G_R(\omega)$ is **not** an all zero function and that $G_R(\omega, t_2, t_0)$ is a continuous function of ω , for a fixed t_0 and t_2 (Section 5.1).

• **Case 1:** $G_R(\omega') \geq 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (link). Hence $F_R(\omega) > 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at

$\omega' = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line and becomes negative, where $\omega_z(t_2, t_0)$ is real and finite.

• **Case 2:** $G_R(\omega') \leq 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (link). Hence $F_R(\omega) < 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where it crosses the zero line and becomes positive, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**. We call this **Statement 2**. In the rest of the sections, we consider only the **first** zero crossing away from origin, where $G_R(\omega)$ crosses the zero line to the opposite sign.

2.2. On the zeros of a related function $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0 , for a given value of t_2 , to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the Fourier transform of the function $g_1(t)$ given by $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$. We use $g_1(t) = f_1(t)e^{-\sigma t}u(-t) + f_1(t)e^{\sigma t}u(t) = e^{\sigma t_0}E_p'(t+t_0)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p'(t+t_0)e^{\sigma t}u(t)$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$ and $G_1(\omega)$ which are also functions of variables t_2, t_0 . Similarly we use the shorthand notation for the functions $E_p'(t), E_0'(t)$ and $E_{0n}'(t)$ which are also functions of variable t_2 .

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^0 g_1(t)e^{-i\omega t}dt + \int_0^{\infty} g_1(t)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 e^{\sigma t_0}E_p'(t+t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_0^{\infty} e^{\sigma t_0}E_p'(t+t_0)e^{\sigma t}e^{-i\omega t}dt \end{aligned} \tag{10}$$

We use $E_p'(t) = E_0'(t)e^{-\sigma t}$ where $E_0'(t) = E_0(t-t_2) - E_0(t+t_2)$ and $E_p'(t+t_0) = E_0'(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 10, we have

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^0 E_0'(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_0^{\infty} E_0'(t+t_0)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 E_0'(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E_0'(-t+t_0)e^{i\omega t}dt \end{aligned} \tag{11}$$

We define $E_{0n}'(t) = E_0'(-t)$ and get $E_0'(-t+t_0) = E_{0n}'(t-t_0)$ and write Eq. 11 as follows.

$$G_1(\omega) = \int_{-\infty}^0 E_0'(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E_{0n}'(t-t_0)e^{i\omega t}dt = G_R(\omega) + iG_I(\omega) \tag{12}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t} \cos(\omega t) dt + \int_{-\infty}^0 E'_{0n}(t-t_0) \cos(\omega t) dt \quad (13)$$

2.3. Zero crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

Now we consider the function $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t) = e^{-\sigma t_0} E'_p(t+t_0) + e^{\sigma t_0} E'_p(t-t_0)$ where $f_1(t) = e^{\sigma t_0} E'_p(t+t_0)$ and $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0} E'_p(t-t_0)$ and $g(t)h(t) = f(t)$ where $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g(t)$ and compute its real part using the procedure in above section, similar to Eq. 13 and we can write as follows. We substitute $t = \tau$.

$$\begin{aligned} G_R(\omega, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_0) + e^{2\sigma t_0} G_{1R}(\omega, -t_0) \\ G_{1R}(\omega, t_0) &= \int_{-\infty}^0 [E'_0(\tau+t_0)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0)] \cos(\omega\tau) d\tau \\ G_R(\omega, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau+t_0)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0)] \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau-t_0)e^{-2\sigma\tau} + E'_{0n}(\tau+t_0)] \cos(\omega\tau) d\tau \end{aligned} \quad (14)$$

We require $G_R(\omega, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **every given fixed value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ and we can rearrange the terms as follows.

$$\begin{aligned} P(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau+t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau+t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \int_{-\infty}^0 [e^{2\sigma t_0} E'_0(\tau-t_0)e^{-2\sigma\tau} + e^{-2\sigma t_0} E'_{0n}(\tau-t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \end{aligned} \quad (15)$$

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0 .

$$\begin{aligned} P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\ P_{odd}(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau+t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau+t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \end{aligned} \quad (16)$$

We see that $f(t, t_0) = e^{-\sigma t_0} E'_p(t+t_0) + e^{\sigma t_0} E'_p(t-t_0) = f(t, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0 , for **every fixed value** of t_2 .

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 16 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E'_{0n}(\tau) = E'_0(-\tau)$ and $E'_0(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

We **note** that we use the **shorthand** notation for the functions $E'_0(t)$ and $E'_{0n}(t)$ which are also functions of variable t_2 .

$$\begin{aligned}
P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned} \tag{17}$$

In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for a given value of t_2 . This means there are **no** Dirac delta functions present in $\omega_z(t_2, t_0)$.

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$.

In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2 , we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 , for every value of t_2 , and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, we see that $\omega_z(t_2, t_0)t_0$ will **certainly pass through** π , as t_0 is increased from zero to ∞ .

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 17 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\begin{aligned}
&\int_{-\infty}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&- \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned} \tag{18}$$

We split the integral in the left hand side of Eq. 18 and write as follows.

$$\begin{aligned}
&[\int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&+ e^{2\sigma t_{0c}} [\int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&- \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned}$$

(19)

We combine the terms with common integrals and cancel common terms in Eq. 19 as follows.

$$\begin{aligned}
& \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned} \tag{20}$$

We can rearrange the terms in Eq. 20 as follows.

$$\begin{aligned}
& \int_{-t_{0c}}^{t_{0c}} [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned} \tag{21}$$

We denote the right hand side of Eq. 21 as RHS . We can split the integral in Eq. 21 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^0 + \int_0^{t_{0c}}$ as follows.

$$\begin{aligned}
& \int_{-t_{0c}}^0 [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& + \int_0^{t_{0c}} [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS
\end{aligned} \tag{22}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 22 as follows. We use $E'_0(-\tau) = E'_{0n}(\tau)$ and $E'_{0n}(-\tau) = E'_0(\tau)$.

$$\begin{aligned}
& \int_{t_{0c}}^0 [E'_{0n}(\tau) e^{2\sigma\tau} + E'_0(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& + \int_0^{t_{0c}} [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS
\end{aligned} \tag{23}$$

Given that $\int_{t_{0c}}^0 = -\int_0^{t_{0c}}$, we can simplify as follows.

$$\int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \tag{24}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 21 as follows. We use $E'_{0n}(-\tau) = E'_0(\tau)$.

$$RHS = 2 \sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{25}$$

We split the integral on the right hand side in Eq. 25 as follows.

$$RHS = 2 \sinh(2\sigma t_{0c}) \left[\int_0^\infty E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \quad (26)$$

We consolidate the integrals with the term $\int_0^{t_{0c}} E'_0(\tau)$ in Eq. 24 and Eq. 26 as follows. We use $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (27)$$

We cancel common terms in Eq. 27 as follows.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (28)$$

We substitute $E'_0(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E'_{0n}(\tau) = E'_0(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$. Hence we see that $E'_{0n}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E'_0(\tau)$. We can write Eq. 28 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (29)$$

We substitute $2 \cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2 \cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 29 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (30)$$

Next Step:

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 30 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$\begin{aligned}
RHS = & \sinh(2\sigma t_{0c}) [\cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& - \cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau]
\end{aligned} \tag{31}$$

In Eq. 31, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$. Hence we cancel common terms and write Eq. 31 and Eq. 30 as follows.

$$\begin{aligned}
& \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)) d\tau \\
& = -\sinh(2\sigma t_{0c}) \left[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right]
\end{aligned} \tag{32}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 32 as follows. Given that $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ , we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)) d\tau = 0 \tag{33}$$

We can multiply Eq. 33 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})] (\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)) d\tau = 0 \tag{34}$$

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and hence the sinusoidal function varies over a **half cycle** and is > 0 , in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 34, we see that in the interval $0 < \tau < t_{0c}$, the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0 , in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t \geq \frac{1}{8}$, we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 4.7). The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 34. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 34 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

4. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. It is stated in Conrey's article that $\Phi(t)$ is positive for $t > 0$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t > 0$ (link). This is derived independently below.

$$\begin{aligned} E_0(t) = \Phi(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\ E_0(t) &= \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}] \end{aligned} \quad (35)$$

We show that $X(t) = \frac{E_0(t)}{2\pi}$ is a **strictly decreasing** function for $t \geq 0$ as follows.

- In Section 4.1, it is shown that the second derivative of $X(t)$, given by $X_2(t) = \frac{d^2 X(t)}{dt^2} < 0$ for $t = 0$.
- In Section 4.2, it is shown that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. (example plot) Hence $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_{min}$.
- In Section 4.3, it is shown that $\frac{dX(t)}{dt} < 0$ for $t \geq t'_{min}$, where t'_{min} corresponds to the first minima of $X_{11}(t)$ which is the partial term in $\frac{dX(t)}{dt}$ corresponding to $n = 1$.
- In Section 4.4, it is shown that $\frac{dX(t)}{dt} < 0$ for $t_{min} < t < t'_{min}$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is strictly decreasing for all $t > 0$ and $E_0(t) = 2\pi X(t)$ is **strictly decreasing** for all $t > 0$.

4.1. $\frac{d^2 X(t)}{dt^2} < 0$ **for** $t = 0$

We consider $X(t) = \frac{E_0(t)}{2\pi} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$ and take the first derivative of $X(t)$ as follows. We note that $E_0(t)$ is an analytic function for $|t| \leq \infty$ and is infinitely differentiable in that interval.

$$\begin{aligned} \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t}) (\frac{1}{2} - 2\pi n^2 e^{2t})] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}] \end{aligned} \quad (36)$$

We take the second derivative of $X(t)$ as follows.

$$\begin{aligned}
\frac{d^2 X(t)}{dt^2} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t} + (-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2})(\frac{5}{2} - 2\pi n^2 e^{2t})] \\
\frac{d^2 X(t)}{dt^2} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t} \\
&\quad + (-10\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t} - \frac{75}{4} + 8\pi^3 n^6 e^{6t} - 30\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t})] \\
\frac{d^2 X(t)}{dt^2} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t, n), \quad Z(t, n) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4}
\end{aligned} \tag{37}$$

• **Case 1:** $n = 1, t = 0$: The partial term for $n = 1$ and $t = 0$ in $\frac{d^2 X(t)}{dt^2}$ in Eq. 37, given by $A_1 < 0$ shown as follows.

$$A_1 = e^{-\pi}(8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4}) < 0 \tag{38}$$

At $n = 1, t = 0$, we see that $Z(t, n) = 8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4} = \pi(8\pi^2 - 56\pi + \frac{165}{2}) - \frac{75}{4} \leq 3 * (8 * 10 - 56 * 3 + 83) - 18 = -33 < 0$ (**Result 1**), because $3 < \pi < \frac{22}{7} = 3.1429$ (link) and $\pi^2 < 10$ and $(8 * 10 - 56 * 3 + 83) < 0$. We note that the term in the brackets is negative and hence we multiply it by the minimum value of π . Hence we see that the partial term for $n = 1$ and $t = 0$ in $\frac{d^2 X(t)}{dt^2}$ given by $A_1 < 0$.

• **Case 2:** $n > 1, t = 0$: the partial terms for $n > 1$ and $t = 0$ in $\frac{d^2 X(t)}{dt^2}$ in Eq. 37, given by $A_2 > 0$ and $A_2 < |A_1|$ shown as follows.

$$A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} (8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4}) > 0 \tag{39}$$

At $n = 2, t = 0$, we see that $Z(t, n) = 8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4} = \pi n^2(8\pi^2 n^4 - 56\pi n^2 + \frac{165}{2}) - \frac{75}{4} > 3 * 2^2(8 * 3^2 * 2^4 - 56 * 4 * 2^2 + 82) - 19 = 4037 > 0$, because $3 < \pi < 4$ and $(8 * 3^2 * 2^4 - 56 * 4 * 2^2 + 82) > 0$. At $n > 2, t = 0$, we see that $Z(t, n) > 0$ due to the dominant term $8\pi^2 n^4$. Hence the partial terms for $n > 1$ and $t = 0$ in $\frac{d^2 X(t)}{dt^2}$ in Eq. 37, given by $A_2 > 0$.

• We can show that at $t = 0$, $A_1 \leq -0.4283$ and $A_2 < |A_1|$ and hence $\frac{d^2 X(t)}{dt^2} < 0$ in Eq. 37.

At $n = 1, t = 0$, we see that, given $3 < \pi < \frac{22}{7} = 3.1429$ (link) and $e^{-\pi} \geq e^{-0.1429}(e^{-\frac{1}{2}})^6$ and $e^{-0.1429} \geq 1 - 0.1429 = 0.8571$ and $e^{-\frac{1}{2}} \geq 1 - \frac{1}{2} = \frac{1}{2}$, we can write $e^{-\pi} \geq \frac{0.8571}{2^6}$ and $|A_1| = e^{-\pi}|Z(t, n)| \geq 33 * \frac{0.8571}{2^6} \geq 32 * \frac{0.8571}{64} = 0.4283$ (using Result 1). Hence we see that the partial term for $n = 1, t = 0$ in $\frac{d^2 X(t)}{dt^2}$ in Eq. 37, given by $A_1 \leq -0.4283$ (**Result 2**).

At $n > 1, t = 0$, we see that $A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} Z(0, n) \leq \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} * (8\pi^3 n^6 + \frac{165}{2}\pi n^2)$ and we **want to show** $A_2 < |A_1|$ where $A_1 \leq -0.4283$. We use the fact that $(n^2)^4 e^{-\pi n^2} < e^{-(\pi-1)n^2}$ and $(n^2)^2 e^{-\pi n^2} < e^{-(\pi-1)n^2}$.

$$A_2 \leq \sum_{n=2}^{\infty} e^{-(\pi-1)n^2} (8\pi^3 + \frac{165}{2}\pi)$$

(40)

We use $3 < \pi < 3.1429$ and $\sum_{n=2}^{\infty} f(n) \leq \int_{u=2}^{\infty} f(u) du$.

$$A_2 \leq (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-(\pi-1)u^2} du \leq (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-2u^2} du \quad (41)$$

We substitute $\sqrt{2}u = t$ as follows. We use $8\pi^3 + \frac{165}{2}\pi = \pi(8\pi^2 + \frac{165}{2}) \leq 3.1429(8 * 10 + 83) \leq 514$. Given that $\pi < 4$, we see that $\frac{\sqrt{\pi}}{2\sqrt{2}} < 1$ as follows. We use the complementary error function $erfc(z)$. (link)

$$\begin{aligned} erfc(z) &= \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt \\ A_2 &\leq \frac{514}{\sqrt{2}} \int_{t=2\sqrt{2}}^{\infty} e^{-t^2} dt \leq \frac{514}{\sqrt{2}} \left(\frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2}) \right) \leq \frac{514}{\sqrt{2}} \frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2}) \\ A_2 &\leq 514 * erfc(2\sqrt{2}) = 0.0326 \end{aligned} \quad (42)$$

We can see that $A_1 + A_2 < 0$ given that $A_1 \leq -0.4283$ (using Result 2).

Alternatively, we can use the fact that $erfc(z) \leq \frac{e^{-z^2}}{z\sqrt{\pi}}$ (link) . Hence $A_2 \leq 514 \frac{e^{-8}}{2\sqrt{2}\pi} \leq 514 \frac{e^{-8}}{4} \leq 129e^{-8}$, given that $2 < \pi < 4$. We can show that $A_1 + A_2 < 0$ as follows, using Eq. 38 and $e > 2$.

$$\begin{aligned} A_2 &\leq 129e^{-8}, \quad A_1 \leq -33e^{-\pi} \leq -33e^{-4} \\ A_1 + A_2 &\leq -33e^{-4} + 129e^{-8} = e^{-4}(-33 + 129e^{-4}) \\ A_1 + A_2 &\leq e^{-4}(-33 + 129 * 2^{-4}) = e^{-4}(-33 + \frac{129}{16}) < 0 \\ A_1 + A_2 &< 0 \\ (\frac{d^2 X(t)}{dt^2})_{t=0} &= A_1 + A_2 < 0 \end{aligned} \quad (43)$$

We have shown that the second derivative of $X(t)$, given by $X_2(t) = \frac{d^2 X(t)}{dt^2} < 0$ for $t = 0$. (**Result 3**)

4.2. $\frac{dX(t)}{dt} < 0$ **for** $0 < t \leq t_{min}$

In the sections below, we will show that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a negative minimum value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. Thus we will show that $X(t)$ is a **strictly decreasing** function of t . We will also show that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. (example plot)

We see that $\frac{d^2 X(t)}{dt^2} < 0$ at $t = 0$ (using Result 3) . Hence, as t increases from zero (point A), $\frac{dX(t)}{dt}$ reaches a **negative minimum** value at $t = t_{min}$ (point B) and then starts increasing towards zero, in the neighborhood of $t > t_{min}$. We see that $\frac{d^2 X(t)}{dt^2}$ remains negative in $0 \leq t < t_{min}$ and then reaches a **zero** at $t = t_{min}$ and then becomes **positive** in the neighborhood of $t > t_{min}$. (example plot) Hence $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_{min}$. (**Result 4**)

4.3. $\frac{dX(t)}{dt} < 0$ **for** $t \geq t'_{min}$

We note that $\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t)$ in Eq. 36, where $X_{11}(t)$ has partial term corresponding to $n = 1$ and $X_{12}(t)$ has partial terms corresponding to $n > 1$.

$$\begin{aligned}\frac{dX(t)}{dt} &= X_{11}(t) + X_{12}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}] \\ X_{11}(t) &= e^{-\pi e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 e^{4t} + 15\pi e^{2t} - \frac{15}{2}] \\ X_{12}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]\end{aligned}\tag{44}$$

For $t > t_{min}$, $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ in Eq. 37 reaches a **positive maximum** at $t = t_{max2}$ (point C) and then starts decreasing to zero, in the neighborhood $t > t_{max2}$. $X_{21}(t)$ has partial term corresponding to $n = 1$ and $X_{22}(t)$ has partial terms corresponding to $n > 1$. This means that $\frac{dX(t)}{dt}$ **increases** towards zero, in the neighborhood $t > t_{min}$. (example plot)

$$\begin{aligned}\frac{d^2 X(t)}{dt^2} &= X_{21}(t) + X_{22}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t, n), \quad Z(t, n) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4} \\ X_{21}(t) &= e^{-\pi e^{2t}} e^{\frac{5t}{2}} (8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2}\pi e^{2t} - \frac{75}{4}) \\ X_{22}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})\end{aligned}\tag{45}$$

We see that $X_{11}(t)$ in Eq. 44 is positive at $t = 0$. As t increases from zero (point A), $X_{11}(t)$ starts from a positive value and decreases to a negative value due to the dominant term $-4\pi^2 e^{4t}$, passing through **zero** at $t = t_z$. Given that $\lim_{t \rightarrow \infty} X_{11}(t) = 0$, it reaches a **negative minimum** value at $t = t'_{min}$ (point B') and then starts **increasing** towards zero, for $t > t'_{min}$ (**Result 5**). (example plot) We note that $X_{11}(t)$ **remains negative** for $t > t_z$ due to the dominant term $-4\pi^2 e^{4t}$ in Eq. 44.

This means that the derivative of $X_{11}(t)$ given by $X_{21}(t)$ in Eq. 45 is **negative** at $0 \leq t < t'_{min}$ and $X_{21}(t)$ reaches a **zero** at $t = t'_{min}$ (**Result 6**) and then becomes **positive** for $t > t'_{min}$ (using Result 5) and falls to zero as $t \rightarrow \infty$, given that $\lim_{t \rightarrow \infty} X_{21}(t) = 0$. We note that, when $X_{21}(t)$ in Eq. 45 becomes positive after crossing $t = t'_{min}$, it **remains positive** as $t \rightarrow \infty$ due to the dominant term $8\pi^3 e^{6t}$ in Eq. 45 (**Result 7**).

Hence $X_{11}(t)$ is a **strictly decreasing** function for $t < t'_{min}$ and then it starts **increasing** towards zero as $t \rightarrow \infty$. We note that $X_{11}(t)$ **cannot** become positive at some $t > t'_{min}$, because if it did become positive again, then it would have to decrease to zero as $t \rightarrow \infty$, which would **require** $X_{21}(t)$ to become negative again, which is **not** the case as shown in Result 7.

We will show that $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) = 0$ **only for** $t = t_{min}$, and it becomes positive for $t > t_{min}$ and then starts decreasing towards zero, as $t \rightarrow \infty$.

In Section 4.5, it is shown that $X_{22}(t)$ is **strictly decreasing** for $t > 0$ (Result 13) and hence $X_{22}(t) > 0$ for $t > 0$, given that $\lim_{t \rightarrow \infty} X_{22}(t) = 0$. (**Result 8**).

We **note that** $t_{min} < t'_{min}$ given that $X_{21}(t) > 0$ for $t > t'_{min}$ (using Result 7) and $X_{22}(t) > 0$ for $t \geq t'_{min}$ (using Result 8) and hence $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ for $t \geq t'_{min}$ (**Result 9**) and given that $\frac{d^2 X(t)}{dt^2} = 0$ at $t = t_{min}$ and $\frac{d^2 X(t)}{dt^2} < 0$ at $0 \leq t < t_{min}$.

We see that $X_{12}(t) < 0$ for all $t \geq 0$ in Eq. 44 due to the dominant term $-4\pi^2 n^4 e^{4t}$. Because $\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t)$ is negative at $t = t'_{min}$, we see that $\frac{dX(t)}{dt}$ starts **increasing** from a negative value towards zero. (using Result 9) Hence $\frac{dX(t)}{dt} < 0$ for $t \geq t'_{min}$. (**Result 10**)

We note that $\frac{dX(t)}{dt}$ **cannot** become positive at some $t > t'_{min}$, because if it did become positive again, then it would have to decrease to zero as $t \rightarrow \infty$, which would **require** $\frac{d^2 X(t)}{dt^2}$ to become negative again, which is **not** the case as shown in Result 9.

4.4. $\frac{dX(t)}{dt} < 0$ **for** $t_{min} < t < t'_{min}$

We can show that the second derivative $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ becomes zero **only once** at $t = t_{min}$, in the interval $0 < t < t'_{min}$. At $t = t_{min}$, we see that $X_{22}(t_{min}) = X_0 > 0$ (using Result 8) and hence $X_{21}(t_{min}) = -X_0 < 0$. In the interval $t_{min} < t < t'_{min}$, we see that $X_{22}(t)$ is **strictly decreasing** and remains positive (using Result 13 in Section 4.5) and decreases from X_0 further towards zero at a **slower** rate, while $X_{21}(t)$ increases from the negative value $-X_0$ towards zero **faster**, to make $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ and this is explained below.

We see that $X_{21}(t)$ is **strictly increasing** for $t_{min} < t < t'_{min}$ (using Result 14 in Section 4.6) and **reaches zero** at $t = t'_{min}$ at a **faster** rate than $X_{22}(t)$ because $X_{22}(t)$ is **strictly decreasing** in the interval $t_{min} < t < t'_{min}$ and $X_{22}(t) > 0$ at $t = t'_{min}$ and has not yet reached zero (using Result 8). Hence $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ in the interval $t_{min} < t < t'_{min}$ (**Result 11**). Hence $\frac{dX(t)}{dt}$ is **increasing** from a negative value towards zero, in the interval $t_{min} < t < t'_{min}$.

We can **rule out** the Case A that $\frac{dX(t)}{dt} > 0$ somewhere in the interval $t > t_{min}$ and reaches a maximum at $t = t_{max}$, as follows. We see that $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ becomes zero **only once** at $t = t_{min}$, in the interval $0 < t < t'_{min}$ and is positive for $t_{min} < t < t'_{min}$ (using Result 11) and is positive for $t \geq t'_{min}$ (using Result 9) and **does not** become zero again at $t = t_{max}$, which is required for Case A. Hence $\frac{dX(t)}{dt} < 0$ in the interval $t_{min} < t < t'_{min}$. (**Result 12**)

We have shown in earlier sections that $\frac{dX(t)}{dt} < 0$ in the interval $0 < t \leq t_{min}$ (using Result 4) and $t \geq t'_{min}$ (using Result 10). We see that $\frac{dX(t)}{dt} < 0$ in the interval $t_{min} < t < t'_{min}$ (using Result 12) and hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is **strictly decreasing** for all $t > 0$. Hence $E_0(t) = 2\pi X(t)$ is **strictly decreasing** for all $t > 0$.

4.5. **Second derivative given by X_{22} is a strictly decreasing function for $t > 0$**

We consider $X_{22}(t)$ in Eq. 45 as follows.

$$X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4}) \quad (46)$$

We see that $X_{22}(t)$ in Eq. 46, is **positive** for each $n = 2, 3, \dots$, for **all** $t \geq 0$, as follows. We see that $X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (\pi^2 n^4 e^{4t} (8\pi n^2 e^{2t} - 56) + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})$. For $n = 2, t = 0$, we see that the partial term in $X_{22}(t)$ given by $2^2 e^{-\pi 2^2} (8\pi^3 2^6 - 56\pi^2 2^4 + \frac{165}{2}\pi 2^2 - \frac{75}{4}) = 4e^{-4\pi} (\pi^2 2^4 (32\pi - 56) + 165 * 2\pi - \frac{75}{4}) > 0$.

For $n \geq 2$ and $t \geq 0$, $X_{22}(t) > 0$ due to the **dominant term** $8\pi^3 n^6 e^{6t}$.

We compute the **derivative** of $X_{22}(t)$ as follows.

$$\begin{aligned}
X_{32}(t) &= \frac{dX_{22}(t)}{dt} = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t} \\
&\quad + (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})(\frac{5}{2} - 2\pi n^2 e^{2t})] \\
X_{32}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t} \\
&\quad + (20\pi^3 n^6 e^{6t} - 28 * 5\pi^2 n^4 e^{4t} + \frac{165 * 5}{4}\pi n^2 e^{2t} - \frac{75 * 5}{8} - 16\pi^4 n^8 e^{8t} + 56 * 2\pi^3 n^6 e^{6t} - 165\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t})] \\
X_{32}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Y'(t, n), \quad Y'(t, n) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75 * 5}{8}
\end{aligned} \tag{47}$$

We examine the term $Y'(t, n) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75 * 5}{8}$ in Eq. 47. We see that, for $n = 2$ and $t = 0$, $Y'(0, 2) = -\pi^3 2^6 (16\pi * 4 - 180) - \pi^2 (529\pi * 2^2 - 408.75) - \frac{75 * 5}{8} < 0$. We see that, **for all** $n > 1$ and $t \geq 0$, $Y'(t, n) < 0$ due to the dominant term $-16\pi^4 n^8 e^{8t}$. Hence $X_{32}(t) < 0$ for all $t \geq 0$.

Hence $X_{22}(t)$ is a **strictly decreasing function** for $t > 0$. (**Result 13**) (example plot)

4.6. Second derivative given by X_{21} is a strictly increasing function for $t_{min} < t < t'_{min}$

We can show that $X_{21}(t)$ is a **strictly increasing function** for $t_{min} < t < t'_{min}$. We take the derivative of $X_{21}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} (8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2}\pi e^{2t} - \frac{75}{4})$ in Eq. 45, given by $X_{31}(t)$ and set $n = 1$ in the summand in Eq. 47 as follows.

$$X_{31}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} Z'(t), \quad Z'(t) = -16\pi^4 e^{8t} + 180\pi^3 e^{6t} - 529\pi^2 e^{4t} + 408.75\pi e^{2t} - \frac{75 * 5}{8} \tag{48}$$

We see that $X_{21}(t)$ in Eq. 45 is **negative** at $0 \leq t < t'_{min}$ (using Result 6). In the interval $t_{min} < t < t'_{min}$, we consider 3 cases for $X_{21}(t)$ as follows.

Case 1: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and increases to zero at $t = t'_{min}$. This is **possible** because we know that $X_{21}(t)$ in Eq. 45 is **negative** at $0 \leq t < t'_{min}$ and $X_{21}(t) = 0$ at $t = t'_{min}$ (using Result 6).

Case 2: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and decreases to a more negative value and reaches a negative minimum at $t = t_c < t'_{min}$ and then starts increasing towards zero. This requires $X_{31}(t) < 0$ in the interval $t_{min} < t < t_c$ and $X_{31}(t) > 0$ in the interval $t_c < t < t'_{min}$, which is **NOT** possible because of the dominant term $-16\pi^4 e^{8t}$ in Eq. 48.

Case 3: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and increases to a less negative value and has a point of inflection at $t = t_d < t'_{min}$ and then starts decreasing towards a more negative value and has a second point of inflection at $t = t_e < t'_{min}$ and then starts increasing towards zero. This requires $X_{31}(t) < 0$ in the interval $t_d < t < t_e$ and $X_{31}(t) > 0$ in the interval $t_e < t < t'_{min}$, which is **NOT** possible because of the

dominant term $-16\pi^4 e^{8t}$ in Eq. 48.

Similarly, we can extend the argument to the case where $X_{21}(t)$ has many points of inflection in the interval $t_{min} < t < t'_{min}$.

Hence we see that only **Case 1** is possible and hence $X_{21}(t)$ is a **strictly increasing function** for $t_{min} < t < t'_{min}$. (**Result 14**) (example plot)

4.7. **Result** $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 4 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 34 .

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t , and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c})$ to $E_0(3t_{0c})$, which is **less than** $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c})$ to $E_0(-t_{0c})$ respectively. Hence we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 34 , for $t_{0c} > 0$.

5. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2

We see from Section 2.1 that $\omega_z(t_2, t_0)$ is shown to be **finite and non-zero** for all $|t_0| < \infty$ and that $\omega_z(t_2, t_0)$ is an even function of variable t_0 , for a given value of t_2 . For a given t_2 and t_0 , $\omega_z(t_2, t_0)$ can have more than one value, but we consider only the first zero crossing away from origin in the section below, where $G_R(\omega)$ crosses the zero line to the opposite sign, as detailed in **Lemma 1** in Section 2.1 and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$. (example plot)

We consider the Fourier transform of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0 , the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 , for all $|t_0| < \infty$, for **each** fixed value of t_2 . This is shown in 3 steps below. $G_R(\omega, t_2, t_0)$ is a function of two variables ω and t_0 , for a given **fixed** value of t_2 and we use Implicit Function Theorem in R^2 .

- It is shown in Section 5.1 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω , as shown in Eq. 49.

- It is shown in Section 5.2 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_0 , as shown in Eq. 50.

- It is shown in Section 5.3 that the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **continuous** function of t_0 , for a given t_2 , using **Implicit Function Theorem** in R^2 .

5.1. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of ω

$G_R(\omega) = G_R(\omega, t_2, t_0)$ in Eq. 14 is copied below and is partially differentiable **at least** twice with respect to ω and the integrals converge in Eq. 49 for $0 < \sigma < \frac{1}{2}$, because the term $\tau^r E'_0(\tau - t_0, t_2) e^{-2\sigma\tau}$ has asymptotic fall-off rate of $\tau^r e^{(\frac{5}{2}-2\sigma)}$ and the integrands are absolutely integrable for $r = 0, 1, 2$ and the integrands are continuous functions of variables ω . We can interchange the order of partial differentiation and integration in Eq. 49.

$$\begin{aligned}
G_R(\omega) &= G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \\
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial \omega^2} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau]
\end{aligned} \tag{49}$$

5.2. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_0

$G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals converge in Eq. 50 and Eq. 51 shown as follows. The integrands are absolutely integrable and the integrands are continuous functions of variables t_0 and we can interchange the order of partial differentiation and integration in Eq. 50 and Eq. 51.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
&\quad + 2\sigma e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau
\end{aligned} \tag{50}$$

The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_0 is given by

$$\begin{aligned}
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad - 4\sigma e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0^2} \cos(\omega\tau) d\tau \\
&\quad + 4\sigma^2 e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + 4\sigma e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau - t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0^2} \cos(\omega\tau) d\tau
\end{aligned} \tag{51}$$

We consider the term $E'_0(\tau + t_0, t_2)$ first. We see that $E'_0(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$. We consider the term $E_0(\tau + t_2)$ first and can show that the integrals converge in Eq. 50 and Eq. 51, as follows.

$$\begin{aligned}
E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned} \tag{52}$$

We can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows.

$$\begin{aligned}
\frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\
&\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}) \\
\frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\
&\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})
\end{aligned} \tag{53}$$

Similarly we can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 - t_0) = -\frac{\partial}{\partial \tau} E_0(\tau + t_2 - t_0)$ and we can write one of the terms corresponding to the term $E_0(\tau + t_2)$ in the second integral in Eq. 51 as follows.

$$\begin{aligned}
&\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau = \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0, t_2))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\
&= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\
&= [E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\
&\quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau) d\tau
\end{aligned}$$

(54)

We see that the integrals in Eq. 54 converge and hence the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2+t_0,t_2)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau$ in Eq. 54 also converges. Similarly, we can use above procedure for the term $E_0(\tau-t_2)$ in the second integral in Eq. 51 and show that $\int_{-\infty}^0 \frac{\partial(E_0(\tau-t_2+t_0,t_2)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau$ also converges. Similarly, we can use above procedure for the term $E'_{0n}(\tau-t_0,t_2)$ in Eq. 51 and show that the corresponding integral converges.

We can use the above procedure in Eq. 52 to Eq. 54 for the term $\frac{\partial^2(E'_0(\tau+t_0,t_2)e^{-2\sigma\tau}+E'_{0n}(\tau-t_0,t_2))}{\partial t_0^2}$ and show that all the terms and integrals in Eq. 50 and Eq. 51 converge.

5.3. Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 , for a given t_2 .

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 , for a given fixed value of t_2 , with continuous partial derivatives (Section 5.1 and Section 5.2) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0 , for all $|t_0| < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for all $|t_0| < \infty$, for each fixed value of t_2 .

5.4. 3-dimensional case

- It is shown in Section 5.5 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2 . We can use the procedure in previous subsections and Implicit Function Theorem and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 , for all $|t_2| < \infty$, for **each** fixed value of t_0 .

- We can use the procedure in previous subsections and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 and t_0 , for all $|t_0| < \infty$ and $|t_2| < \infty$, using Implicit Function Theorem in R^3 .

We use **Implicit Function Theorem** for the three dimensional case (link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2 , with continuous partial derivatives (Section 5.1, Section 5.2 and Section 5.5) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$.

5.5. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_2

$G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals converge in Eq. 55 and Eq. 56 shown as follows. The integrands are absolutely integrable and the integrands are continuous functions of variables t_2 and we can interchange the order of partial differentiation and integration in Eq. 55 and Eq. 56.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau+t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau-t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau+t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau+t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau-t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau+t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau
\end{aligned}$$

(55)

The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_2 is given by

$$\begin{aligned} \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2 (E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2 (E'_0(\tau - t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau \end{aligned} \quad (56)$$

We consider the term $E'_0(\tau + t_0, t_2)$ first. We see that $E'_0(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$. We consider the term $E_0(\tau + t_2)$ first and can show that the integrals converge in Eq. 55 and Eq. 56, as follows. We consider Eq. 52 and show that $\frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows.

$$\begin{aligned} \frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \end{aligned} \quad (57)$$

Similarly we can show that $\frac{\partial}{\partial t_2} E_0(\tau + t_2 - t_0) = -\frac{\partial}{\partial \tau} E_0(\tau + t_2 - t_0)$ and we can write one of the terms corresponding to the term $E_0(\tau + t_2)$ in the first integral in Eq. 55 as follows.

$$\begin{aligned} \int_{-\infty}^0 \frac{\partial (E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial (E_0(\tau + t_2 + t_0, t_2))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\ &= \int_{-\infty}^0 \frac{\partial (E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) \frac{\partial (e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\ &= [E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\ &\quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau} \cos(\omega\tau) d\tau \end{aligned} \quad (58)$$

We see that the integrals in Eq. 58 converge and hence the integral $\int_{-\infty}^0 \frac{\partial (E_0(\tau + t_2 + t_0, t_2) e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 58 also converges. Similarly, we can use above procedure for the term $E_0(\tau - t_2)$ in the first integral in Eq. 55 and show that $\int_{-\infty}^0 \frac{\partial (E_0(\tau - t_2 + t_0, t_2) e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau$ also converges. Similarly, we can use above procedure for the term $E'_{0n}(\tau - t_0, t_2)$ in Eq. 55 and show that the corresponding integral converges.

We can use the above procedure in Eq. 56 to Eq. 58 for the term $\frac{\partial^2 (E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2^2}$ and show that all the terms and integrals in Eq. 55 and Eq. 56 converge.

6. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied^[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi) \xi(1-s, \bar{\chi})$ ^[7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real ^[4] and then derive $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t) e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function which is **strictly decreasing** function for $t > 0$ (Section 4). These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

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- [4] Fern Ellison and William J. Ellison, Prime Numbers (1985). pp.147 to 152
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[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Properties of Fourier Transforms Part 2

Appendix A.1. $E_p(t), h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite.

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t} d\omega$. We see that $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \rightarrow \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \rightarrow -\infty$, $E_p(t)$ goes to zero, because for every value of n , the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t \rightarrow \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t} dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t) dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$ and $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = e^{\sigma t_0} E'_p(t + t_0)$ and $f_2(t) = e^{-\sigma t_0} E'_p(t - t_0)$ and $E'_p(t) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$. We can see that $g(t)h(t) = f(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix A.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$ and hence are absolutely **integrable functions** and the integrals $\int_{-\infty}^{\infty} |E_0(t)| dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}| dt < \infty$. Hence $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)| dt = \int_{-\infty}^{\infty} g(t) dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix A.2. Convolution integral convergence

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ and $\int_{-\infty}^{\infty} H(\omega) d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $[\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where

$$h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t).$$

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \quad (\text{A.1})$$

Appendix A.3. **Fall off rate of Fourier Transform of functions**

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

Appendix A.4. **Payley-Weiner theorem and Exponential Fall off rate of analytic functions.**

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant^[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s -plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \rightarrow \infty$, as $|t| \rightarrow \infty$ (Appendix A.3) and hence it should have **exponential fall-off** rates as $|t| \rightarrow \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$.