1. Fourier Transform Properties: Shifted function asymptotic result

Let us consider the Fourier transform of the shifted function $f(x - x_0)$ where x_0 is real and finite, below. (link and Eq. 55 in link)

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$
$$\int_{-\infty}^{\infty} f(x - x_0) e^{-ikx} dx = e^{-ikx_0} F(k) \tag{1}$$

We derive the asymptotic result for the Fourier transform of $\lim_{x_0 \to \infty} f(x - x_0)$ using **principle of mathematical induction**, as follows.

Set $x_0 = Nx_1$. For $N = 1$, the Fourier transform of $f_1(x) = f(x - x_1)$ is given by $F_1(k) = F(k)e^{-ikx_1}$.

For $N = 2$, the Fourier transform of $f_2(x) = f_1(x - x_1) = f(x - 2x_1)$ is given by $F_2(k) = F_1(k)e^{-ikx_1} = F(k)e^{-i2kx_1}$.

Inductive Hypothesis: Let $f_N(x) = f_{N-1}(x - x_1) = f(x - Nx_1)$. Its Fourier transform is given by $F_N(k) = F_{N-1}(k)e^{-ikx_1} = F(k)e^{-iNkx_1}$.

Inductive Result: Set $N = N + 1$. We get $f_{N+1}(x) = f_N(x - x_1) = f(x - (N + 1)x_1)$. Its Fourier transform is given by $F_{N+1}(k) = F_N(k)e^{-ikx_1} = F(k)e^{-i(N+1)kx_1}$.

Hence we can use the principle of mathematical induction and get the inductive result that the Fourier transform of $\lim_{x_0 \to \infty} f(x - x_0) = \lim_{N \to \infty} f_N(x)$ is given by $\lim_{N \to \infty} e^{-iNkx_1} F(k)$.

$$\int_{-\infty}^{\infty} \lim_{x_0 \to \infty} f(x - x_0) e^{-ikx} dx = \lim_{N \to \infty} e^{-iNkx_1} F(k) \tag{2}$$

It is noted that $\lim_{N \to \infty} e^{-iNkx_1}$ is indeterminate and finite and $|e^{-iNkx_1}| \leq 1$ as $N \to \infty$. If $f(x)$ is an absolutely integrable function, then $F(k)$ is finite for all k. Hence $\lim_{N \to \infty} e^{-iNkx_1} F(k)$ is finite and indeterminate for all k.