On a new method towards proof of Riemann's Hypothesis

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Abstract

It is well known that a real two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, does not have zeros in its Fourier Transform, where $u(t)$ is Heaviside unit step function and $a, b > 0$ are real. We consider the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$, which is expressed as an **infinite summation of two-sided decaying exponential functions** using Taylor series expansion.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line and prove Riemann's hypothesis. We also use the new method **without** using Taylor series expansion and prove Riemann's hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function.[5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function $\xi(s)$ also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]
Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ has no zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, critical strip $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$, which is expressed as an **infinite summation of two-sided decaying exponential functions** using Taylor series expansion and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

In Section 3, we prove Riemann's hypothesis **without** using Taylor series representation of $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$.

In Appendix A to Appendix H, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=E_{0\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows[3]. (Page 5 in Brian Conrey's 2003 article) This is re-derived in Appendix H.

$$E_0(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2 e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2 e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t}-3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and even function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t}-3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}=f(e^{2t})e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

1.2. *Step 2: Taylor's series representation of* $E_p(t)$

We can substitute $z=e^{2t}$ in Eq. 2 and we can expand real analytic function $f(z)$ using Taylor series expansion around $z=0$ as follows.

$$\begin{gathered}
f(z)=2\sum_{n=1}^{\infty}[2\pi^2 n^4 z^2-3\pi n^2 z]e^{-\pi n^2 z}=\sum_{n,k}(a_{nk}z^{(k+2)}-b_{nk}z^{k+1})\\
a_{nk}=4\pi^2 n^4 d_{nk},\quad b_{nk}=6\pi n^2 d_{nk},\quad d_{nk}=\frac{(-\pi n^2)^k}{!(k)},\quad \sum_{n,k}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}
\end{gathered} \tag{3}$$

Now we can substitute $z=e^{2t}$ in Eq. 3 and write the Taylor series expansion of $E_p(t)$ in Eq. 2 and use the shorthand notation as follows.

$$\begin{gathered}
E_p(t)=[\sum_{n,k}(a_{nk}e^{(2k+\frac{9}{2})t}-b_{nk}e^{(2k+\frac{5}{2})t})]e^{-\sigma t}=\sum_{n,k,r}c_{nkr}e^{b_{kr}t}e^{-\sigma t}\\
\sum_{n,k,r}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{1},\quad b_{kr}=(2k+\frac{5}{2}+2r),\quad c_{nk1}=a_{nk},\quad c_{nk0}=-b_{nk}
\end{gathered} \tag{4}$$

Given that $E_0(t)=E_0(-t)$, we can write $E_p(t)=E_0(t)e^{-\sigma t}$ as an **infinite summation** of two-sided decaying exponential functions as follows.

$$E_p(t)=[\sum_{n,k,r}c_{nkr}e^{b_{kr}t}u(-t)+\sum_{n,k,r}c_{nkr}e^{-b_{kr}t}u(t)]e^{-\sigma t} \tag{5}$$

In Appendix B, we show that we can also expand $f(z)$ using an alternate Taylor series expansion around $z=1$.

1.3. *Step 3: Two-sided decaying exponentials and zeros in their Fourier transform*

We know that a real two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, where $u(t)$ is Heaviside unit step function and $a, b > 0$ and t are real, has Fourier Transform given by $G_0(\omega)$, where ω is real, as follows. (link)

$$\begin{aligned} G_0(\omega) &= \int_{-\infty}^{\infty} g_0(t)e^{-i\omega t}dt = \frac{1}{b - i\omega} + \frac{1}{a + i\omega} = \frac{b + i\omega}{b^2 + \omega^2} + \frac{a - i\omega}{a^2 + \omega^2} \\ &= [\frac{b}{b^2 + \omega^2} + \frac{a}{a^2 + \omega^2}] + i\omega[\frac{1}{b^2 + \omega^2} - \frac{1}{a^2 + \omega^2}] \end{aligned} \tag{6}$$

We can see that the real part of $G_0(\omega)$ given by $\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}$ **does not have zeros** for any finite value of ω and hence $G_0(\omega)$ does not have zeros for any finite value of ω.

Given that the inverse Fourier Transform of Riemann Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ given by $E_p(t)$ is expressed as an **infinite summation of two-sided decaying exponential functions** in previous subsection, we will investigate if $E_{p\omega}(\omega)$ also does not have zeros for any finite real value of ω.

1.4. *Step 4: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

In **Section 2.1**, we will show that the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite.

1.5. *Step 5: On the zeros of the function* $G_I(\omega)$

In **Section 2.2**, we compute the Fourier transform of the function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$. We **require** $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, to satisfy Statement 1. Hence $S_0 = G_I(\omega_1) = 0$ and we will derive as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{7}$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r} c_{nkr}e^{b_{kr}t}e^{-\sigma t}$, and we use the fact that $E_0(t) = E_0(-t)$ and we will derive as follows.

$$S_0 = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{8}$$

1.6. ***Step 6: Even order Derivatives of*** $g(t)$

In **Section 2.3**, we consider the **even order derivative** of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ and compute the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and show results as follows. We will also show that **dirac delta functions vanish** in the computation of $g_{2r_{odd}}(t)$.

$$S_{2r} = \omega_1 \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)^{2r} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^{2r} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{9}$$

1.7. ***Step 7: New Function*** $A(t_1)$

Next, we will form a new function $A(t_1) = \sum_{r=0}^{\infty} S_{2r} \frac{t_1^{2r}}{!(2r)} = 0$ for $-\infty \leq t_1 \leq \infty$ where t_1 is real and we can write

$$A(t_1) = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1} + e^{-(b_{kr}-2\sigma)t_1})}{2} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)} + e^{-(b_{kr}t_1)})}{2} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{10}$$

We can write $A(t_1) = \frac{\omega_1}{2}[y(t_1) + y(-t_1)] = 0$ as follows. We know that $\omega_1 \neq 0$ and we can write

$$y(t_1) = \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1}}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)}}{(\omega_1^2 + b_{kr}^2)}] = -y(-t_1) = y_{odd}(t_1) \tag{11}$$

We can see that $y(t_1)$ is an **odd function** of variable t_1.

1.8. ***Step 8: Final Step in the proof of theorem.***

We can evaluate the **odd** symmetry function $z_{odd}(t_1)$ as follows.

$$\begin{aligned} \frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\ \sum_{n,k,r} c_{nkr}[e^{(b_{kr}-2\sigma)t_1} - e^{(b_{kr}t_1)}] &= z_{odd}(t_1) \\ \sum_{n,k,r} c_{nkr} e^{(b_{kr}t_1)}(e^{-2\sigma t_1} - 1) &= z_{odd}(t_1) \end{aligned} \tag{12}$$

We know that $\sum_{n,k,r} c_{nkr} e^{(b_{kr}t_1)} = E_0(t_1)$ is an **even function** of variable t_1, hence we require $(e^{-2\sigma t_1} - 1)$ to be an **odd function** of variable t_1, to satisfy Eq. 12, which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence this proves Riemann hypothesis.

In **Section 3**, we will prove the same result, **without** using Taylor series expansion for $E_p(t)$.

2. Proof of Riemann's Hypothesis using Taylor Series Expansion of $E_p(t)$

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$, $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$, given that $E_0(t) = E_0(-t)$ is an even function of variable t.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2}+\sigma+i\omega) = \xi(\frac{1}{2}-\sigma-i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

The inverse Fourier Transform of the function $E_{p\omega}(\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$. We see that $E_p(t) = 0$ at $t = \pm\infty$ and its Fourier transform given by $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ does not have a zero at $\omega = 0$ and hence $\omega_0 \neq 0$.

2.1. ***On the zeros of a related function*** $G(\omega)$

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can show that $E_p(t), h(t), g(t)$ are real L^1 integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \le \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma. This is shown in detail in Appendix C.5.

If we take the Fourier transform of the equation $g(t)h(t) = E_p(t)$, we get $\frac{1}{2\pi}[G(\omega)*H(\omega)] = E_{p\omega}(\omega)$ as per convolution theorem, where $*$ denotes **convolution** operation given by $E_{p\omega}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma-i\omega}+\frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega)+iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix C.1.

We can write $g(t) = g_{even}(t)+g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t)-g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ where ω_1 is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_I(\omega)$ does not have at least one zero for some $\omega = \omega_1 \neq 0$, **then** the **imaginary part** of $E_{p\omega}(\omega)$ given by $E_I(\omega) = \frac{1}{2\pi}[G_I(\omega)*H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t)-g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, where $g(t)h(t) = E_p(t)$

and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its imaginary part given by $E_I(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_I(\omega)$ **does not** have at least one zero for finite $\omega = \omega_1 \neq 0$ and show that $E_I(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the imaginary parts as follows.

$$E_I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(\omega')H(\omega-\omega')d\omega' \tag{13}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.6)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 13 and we get

$$E_I(\omega) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' \tag{14}$$

We can write Eq. 14 as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' + \int_{0}^{\infty} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega'] \tag{15}$$

We see that $G_I(-\omega) = -G_I(\omega)$ because $g(t)$ is a real function (Appendix C.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 15 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}\int_{0}^{\infty} G_I(\omega')[\frac{1}{(\sigma^2+(\omega-\omega')^2)} - \frac{1}{(\sigma^2+(\omega+\omega')^2)}]d\omega' \tag{16}$$

We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 16 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} - \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

Case 1: $G_I(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) > 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_I(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

Case 2: $G_I(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) < 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_I(\omega) > 0$ for all finite $\omega < 0$.
This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

We have shown that, $G_I(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**. We will investigate if Statement 2 leads to a contradiction for $0 < \sigma < \frac{1}{2}$.

2.2. *On the zeros of the function* $G_I(\omega)$

We take the Fourier transform of $g(t)$ and get $G(\omega)$ as follows.

$$g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$$
$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} E_p(t)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_p(t)e^{\sigma t}e^{-i\omega t}dt \tag{17}$$

We can substitute $t = -\tau$ in the second integral in Eq. 17 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows.

$$G(\omega) = \int_{-\infty}^{0} E_p(\tau)e^{-\sigma\tau}e^{-i\omega\tau}d\tau + \int_{-\infty}^{0} E_q(\tau)e^{-\sigma\tau}e^{i\omega\tau}d\tau = G_R(\omega) + iG_I(\omega) \tag{18}$$

Eq. 18 can be expanded as follows using Euler's formula $e^{i\omega\tau} = \cos(\omega\tau) + i\sin(\omega\tau)$ and comparing the **imaginary parts** of $G(\omega)$, we can write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_I(\omega) = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega\tau)d\tau \tag{19}$$

We require $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, to satisfy **Statement 1** as shown in Section 2.1.

We can set $S_0 = G_I(\omega_1) = 0$ and write as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{20}$$

We use Taylor series representation of $E_p(\tau) = \sum_{n,k,r} c_{nkr}e^{b_{kr}\tau}e^{-\sigma\tau}$, and we use the fact that $E_0(\tau) = E_0(-\tau)$. We can see that $b_{kr} = (2k + \frac{5}{2} + 2r) > 2\sigma$ for all k, r and $0 < \sigma < \frac{1}{2}$. We can interchange the order of integration and summation in Eq. 20 because for each term in Taylor series, integral in Eq. 20 converges.

We use the well known result $\int e^{a\tau}\sin(\omega_1\tau)d\tau = \frac{e^{a\tau}}{(\omega_1^2+a^2)}[a\sin(\omega_1\tau) - \omega_1\cos(\omega_1\tau)]$ in Eq. 20 and then evaluate the integral at $\tau = 0$ for $a = (b_{kr} - 2\sigma)$ in the first integral and $a = b_{kr}$ in the second integral.

We can see that the two integrals in Eq. 20 equal zero when evaluated at the lower limit of $\tau = -\infty$ because $b_{kr} - 2\sigma > 0$ for all k, r and $0 < \sigma < \frac{1}{2}$. Hence we can write as follows.

$$S_0 = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{21}$$

2.3. ***Second Derivative of*** $g(t)$

We know that $E_p(t)$ is an **analytic** function in the interval $-\infty \leq t \leq \infty$ which is infinitely differentiable in that interval. Let us consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2g(t)}{dt^2}$ where $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$.

We can see that $g_2(t) = \frac{d^2g(t)}{dt^2}$ produces a **Dirac delta function**, which is an **even function** of variable t. Hence, when we take the **odd part** of $g_2(t)$ given by $g_{2_{odd}}(t) = \frac{1}{2}[g_2(t) - g_2(-t)]$, the dirac delta impulse function **vanishes** (Appendix D). We will compute the Fourier transform of $g_{2_{odd}}(t)$ given by $G_{2_I}(\omega)$ shortly.

First we compute the Fourier transform of $g_2(t)$ given by $G_2(\omega)$ as follows.

$$G_2(\omega) = \int_{-\infty}^{0} \frac{d^2(E_p(t)e^{-\sigma t})}{dt^2} e^{-i\omega t} dt + \int_{0}^{\infty} \frac{d^2(E_p(t)e^{\sigma t})}{dt^2} e^{-i\omega t} dt \tag{22}$$

We can substitute $t = -\tau$ in the second integral in Eq. 22 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_2(\omega) = \int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} e^{-i\omega\tau} d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2} e^{i\omega\tau} d\tau \tag{23}$$

Eq. 23 can be expanded as follows using Euler's formula $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and comparing the **imaginary parts** of $G_2(\omega) = G_{2_R}(\omega) + iG_{2_I}(\omega)$, we can write as follows.

$$G_{2_I}(\omega) = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} \sin(\omega\tau) d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2} \sin(\omega\tau) d\tau \tag{24}$$

We see that the Fourier transform of $g_{2_{odd}}(t)$ is given by $iG_{2_I}(\omega)$ where $G_2(\omega) = G_{2_R}(\omega) + iG_{2_I}(\omega)$ and $G_2(\omega)$ is the Fourier transform of $g_2(t)$. We see that $G_2(\omega) = -\omega^2 G(\omega) = -\omega^2[G_R(\omega) + iG_I(\omega)]$ and hence $G_{2_I}(\omega) = -\omega^2 G_I(\omega)$.

We require $G_{2_I}(\omega) = 0$ for the **same** $\omega = \omega_1$, to satisfy **Statement 1**, because we derived the result that $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$ in Section 2.1 and $G_{2_I}(\omega) = -\omega^2 G_I(\omega)$. Hence $S_2 = G_{2_I}(\omega_1) = 0$ and is given as follows.

$$S_2 = G_{2_I}(\omega_1) = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} \sin(\omega_1\tau) d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2} \sin(\omega_1\tau) d\tau = 0 \tag{25}$$

Using Taylor series representation of $E_0(\tau) = \sum_{n,k,r} c_{nkr} e^{b_{kr}\tau}$ and we use the fact that $E_0(\tau) = E_0(-\tau)$, we can write as follows.

$$S_2 = \omega_1 \sum_{n,k,r} c_{nkr} [(b_{kr} - 2\sigma)^2 \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^2 \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{26}$$

2.4. ***Even order Derivatives of*** $g(t)$

We know that $E_p(t)$ is an **analytic** function in the interval $-\infty \leq t \leq \infty$ which is infinitely differentiable in that interval. Let us consider the $(2r)^{th}$ derivative of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ where $r = 0, 1, ..., \infty$. Its Fourier transform is given by $G_{2r}(\omega) = \int_{-\infty}^{\infty} g_{2r}(t)e^{-i\omega t}dt$. We take the **odd part** of $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and the dirac delta impulse function related terms **vanish** because dirac delta and its even derivatives are **even functions** of variable t. This is shown in detail in **Appendix D**.

We take the Fourier transform of $g_{2r_{odd}}(t)$ and we see that $G_{2r_I}(\omega) = 0$ for the **same** $\omega = \omega_1$ because $G_{2r}(\omega) = (-\omega^2)^r G(\omega) = (-\omega^2)^r [G_R(\omega) + iG_I(\omega)]$ and hence $G_{2r_I}(\omega) = (-\omega^2)^r G_I(\omega)$ and we derived the result that $G_I(\omega) = 0$ for $\omega = \omega_1$ in Section 2.1 . We can derive results similar to Eq. 25, Eq. 26 as follows.

$$S_{2r} = G_{2r_I}(\omega_1) = -\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1\tau)d\tau] + \int_{-\infty}^{0} \frac{d^{2r}E_0(-\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau = 0 \tag{27}$$

Using Taylor series representation of $E_0(\tau) = \sum_{n,k,r} c_{nkr}e^{b_{kr}\tau}$ and we use the fact that $E_0(\tau) = E_0(-\tau)$, we can write as follows.

$$S_{2r} = \omega_1 \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)^{2r} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^{2r} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{28}$$

Now, we can form a new function $A(t_1)$ as follows, for real $-\infty \leq t_1 \leq \infty$.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r} \frac{t_1^{2r}}{!(2r)} = 0$$

$$A(t_1) = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1} + e^{-(b_{kr}-2\sigma)t_1})}{2} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)} + e^{-(b_{kr}t_1)})}{2} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{29}$$

We see that $A(t_1) = \frac{\omega_1}{2}[y(t_1) + y(-t_1)] = 0$ where $y(t_1)$ is an **odd** function of variable t_1, because there is **at least one non-zero** value of $\omega_1 \neq 0$ as explained in Section 2.1, we write as follows.

$$y(t_1) = \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1}}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)}}{(\omega_1^2 + b_{kr}^2)}] = -y(-t_1) = y_{odd}(t_1) \tag{30}$$

We can evaluate the **odd** symmetry function $z_{odd}(t_1)$ as follows.

$$\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = z_{odd}(t_1)$$

$$\sum_{n,k,r} c_{nkr}[e^{(b_{kr}-2\sigma)t_1} - e^{(b_{kr}t_1)}] = z_{odd}(t_1)$$

$$\sum_{n,k,r} c_{nkr}e^{(b_{kr}t_1)}(e^{-2\sigma t_1} - 1) = z_{odd}(t_1)$$

(31)

We know that $\sum_{n,k,r} c_{nkr} e^{(b_{kr} t_1)} = E_0(t_1)$ is an **even function** of variable t_1 and $E_0(t_1) \neq 0$, hence we require $(e^{-2\sigma t_1} - 1)$ to be an **odd function** of variable t_1 to satisfy Eq. 31, which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3. Proof of Riemann's Hypothesis without using Taylor series expansion of $E_p(t)$

In this section, we re-derive the results in Section 2.3 and Section 2.4 **without** using Taylor series expansion of $E_p(t)$. Results in Section 2.1 and Section 2.2 hold for the case **without** using Taylor series expansion of $E_p(t)$ as well.

We consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2 g(t)}{dt^2}$ and using procedure in Section 2.3, we can write as follows.

$$S_2 = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau) e^{-2\sigma\tau})}{d\tau^2} \sin(\omega_1 \tau) d\tau + \int_{-\infty}^{0} \frac{d^2 E_0(-\tau)}{d\tau^2} \sin(\omega_1 \tau) d\tau = 0 \tag{32}$$

Let us consider the $(2r)^{th}$ **derivative** of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r} g(t)}{dt^{2r}}$ and using procedure discussed in Section 2.4, we can write as follows. We use the fact that $E_0(\tau) = E_0(-\tau)$.

$$S_{2r} = -\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau) e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1 \tau) d\tau + \int_{-\infty}^{0} \frac{d^{2r} E_0(\tau)}{d\tau^{2r}} \sin(\omega_1 \tau) d\tau = 0 \tag{33}$$

We can form a new function $A(t_1)$ as follows, for real $-\infty \leq t_1 \leq \infty$. We can see that for every value of r, the integrals in the equation below converge and we can interchange the order of integration and summation as follows.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r} \frac{t_1^{2r}}{!(2r)} = -\int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r}(E_0(\tau) e^{-2\sigma\tau})}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1 \tau) d\tau + \int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r} E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1 \tau) d\tau = 0 \tag{34}$$

For the specific case of **complex exponential** function $C(\tau) = e^{i\omega\tau}$, we define a new function $D(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r} C(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$ which can be written as $D(\tau) = \frac{1}{2}[C(\tau + t_1) + C(\tau - t_1)]$. We can show similar results for the summation terms in Eq. 34 as follows.

Let $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$. In Eq. 34 we have $f_1(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$. In **Appendix F**, we show that $f_1(\tau) = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)]$ using the inverse Fourier transform representation of $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, given that $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ is an analytic function and is Fourier transformable. Similarly, we can show that $f_2(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)]$. Hence we can write Eq. 34 as follows.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0} [x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{35}$$

We define $B(t_1) = -\int_{-\infty}[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau$ and evaluate the integral at the lower limit of $\tau = -\infty$. We can evaluate the integrals in Eq. 35 separately at the upper limit and lower limit as follows.

$$A(t_1) = \frac{1}{2}[-\int^{0} [x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int^{0} [E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau - B(t_1)] = 0 \tag{36}$$

We see that $B(t_1)$ equals **integration constant** K_I, added to an extra term which is **non-zero** in the **general** case.

In **Appendix G**, for the **specific case** of our function $E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} = E_0(t)e^{-\sigma t}$ and for $0 < |\sigma| < \frac{1}{2}$, we show that $B(t_1) = 0 + K_I$ by using integration by parts method and evaluating the integrals in Eq. 36 at the lower limit of $\tau = -\infty$.

Integration constant K_I gets **cancelled** at the upper and lower limit of the indefinite integrals in Eq. 36 given by $-\int[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau$ and $\int[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau$.

3.1. ***Final Step in the proof of theorem.***

We can write $A(t_1) = y(t_1) + y(-t_1) = 0$ as follows, with integrals evaluated **only** at the upper limit and integration constant K_I **omitted** in equations below because it gets **cancelled** at the upper and lower limit of the indefinite integrals in Eq. 36.

$$y(t_1) = -\frac{1}{2}\int^{0} x(\tau+t_1)\sin(\omega_1\tau)d\tau + \frac{1}{2}\int^{0} E_0(\tau+t_1)\sin(\omega_1\tau)d\tau = -y(-t_1) = y_{odd}(t_1) \tag{37}$$

We can see that $y(t_1)$ is an **odd function** of variable t_1.

We can substitute $\tau + t_1 = t$ and write as follows.

$$y(t_1) = -\frac{1}{2}[\cos(\omega_1t_1)\int^{t_1} x(t)\sin(\omega_1t)dt - \sin(\omega_1t_1)\int^{t_1} x(t)\cos(\omega_1t)dt]$$
$$+\frac{1}{2}[\cos(\omega_1t_1)\int^{t_1} E_0(t)\sin(\omega_1t)dt - \sin(\omega_1t_1)\int^{t_1} E_0(t)\cos(\omega_1t)dt] = y_{odd}(t_1)$$

(38)

We can evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = z_{odd}(t_1)$ as follows, where $z_{odd}(t_1)$ is an **odd function** of variable t_1. In **Appendix E**, we show that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is an analytic function and the indefinite integral is evaluated only at the upper limit and we also derive in detail the equation $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$. We use $x(t_1) = E_0(t_1)e^{-2\sigma t_1}$ below.

$$\begin{aligned}
\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\
\frac{\omega_1}{2}[x(t_1) - E_0(t_1)] &= z_{odd}(t_1) \\
\frac{\omega_1}{2}[E_0(t_1)e^{-2\sigma t_1} - E_0(t_1)] &= z_{odd}(t_1) \\
\frac{\omega_1}{2}E_0(t_1)[e^{-2\sigma t_1} - 1] &= z_{odd}(t_1)
\end{aligned} \tag{39}$$

We use the fact that $\omega_1 \neq 0$. We know that $E_0(t_1) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t_1} - 3\pi n^2 e^{2t_1}]e^{-\pi n^2 e^{2t_1}} e^{\frac{t_1}{2}}$ is an **even function** of variable t_1 and $E_0(t_1) \neq 0$, hence we require $e^{-2\sigma t_1} - 1$ to be an **odd function** of variable t_1 which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

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Appendix A.

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$. This is re-derived in Appendix H.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$, is given by $E_p(t)=E_0(t)e^{-\sigma t}$ where $0<|\sigma|<\frac{1}{2}$ is real.

$$
\begin{aligned}
\xi(\frac{1}{2}+\sigma+i\omega)=\xi(\frac{1}{2}+i(\omega-i\sigma))=E_{p\omega}(\omega)=E_{0\omega}(\omega-i\sigma)\\
E_p(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{p\omega}(\omega)e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega-i\sigma)e^{i\omega t}d\omega
\end{aligned}
\tag{A.1}
$$

We substitute $\omega'=\omega-i\sigma$ in Eq. A.1 as follows.

$$
E_p(t)=e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma}E_{0\omega}(\omega')e^{i\omega' t}d\omega'
\tag{A.2}
$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega'=z=x+iy$ and we use a rectangular contour comprised of C_1 along the line $x=[-\infty,\infty]$, C_2 along the line $z=[\infty,\infty-i\sigma]$, C_3 along the line $z=[\infty-i\sigma,-\infty-i\sigma]$ and then C_4 along the line $z=[-\infty-i\sigma,-\infty]$. We can see that $E_{0\omega}(z)=\xi(\frac{1}{2}+iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2}+iz)$ is an entire function in the Z-plane.

Given that $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega|\leq\infty$ and also for $\omega=0$. Hence $\int_{-\infty}^{\infty}E_p(t)dt$ is finite. We see that $E_p(t)\geq 0$ for all $|t|\leq\infty$. Hence we can write $\int_{-\infty}^{\infty}|E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 integrable function .

We use the fact that $E_{0\omega}(z)=\xi(\frac{1}{2}+iz)=\xi(\frac{1}{2}-y+ix)=\int_{-\infty}^{\infty}E_0(t)e^{-izt}dt=\int_{-\infty}^{\infty}E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x\to\pm\infty$ when $-\sigma\leq y\leq 0$, given that $E_0(t)e^{yt}$ is a L^1 integrable function in the interval $-\infty\leq t\leq\infty$ as per (Riemann-Lebesgue Lemma). Hence the integral in above equation **vanishes** along the contours C_2 and C_4. We can write Eq. A.2 as follows.

$$
\begin{aligned}
E_p(t)=e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega')e^{i\omega' t}d\omega'\\
E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}
\end{aligned}
\tag{A.3}
$$

Thus we have arrived at the desired result $E_p(t)=E_0(t)e^{-\sigma t}$.

Appendix B. Alternate Taylor's series representation of $E_p(t)$

We can substitute $z = e^{2t}$ in Eq. 2 for $E_p(t)$ reproduced below.

$$E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} = E_0(t)e^{-\sigma t} = f(e^{2t})e^{\frac{t}{2}} e^{-\sigma t}$$
$$f(z) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 z^2 - 3\pi n^2 z]e^{-\pi n^2 z} \tag{B.1}$$

We can expand the real analytic function $f(z)$ using Taylor series expansion **around** $z = 1$ as follows.

$$f(z) = [\sum_{n,k}(a_{nk}(z-1)^{(k+2)} - b_{nk}(z-1)^{(k+1)})]e^{-\pi n^2}$$
$$\sum_{n,k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}, \quad a_{nk} = 4\pi^2 n^4 d_{nk}, \quad b_{nk} = 6\pi n^2 d_{nk}, \quad d_{nk} = \frac{(-\pi n^2)^k}{!(k)} \tag{B.2}$$

Now we substitute $z = e^{2t}$ in Eq. B.2 and we can write the Taylor series expansion of $E_p(t)$ as follows and we use binomial series expansion for $(e^{2t}-1)^v = \sum_{p=0}^{v}\binom{v}{p}(-1)^p e^{2t(v-p)}$ for v is a positive integer.

$$E_p(t) = [\sum_{n,k}(a_{nk}(e^{2t}-1)^{(k+2)} - b_{nk}(e^{2t}-1)^{(k+1)})]e^{-\pi n^2} e^{\frac{t}{2}} e^{-\sigma t}$$
$$E_p(t) = [\sum_{n,k}(a_{nk}[\sum_{p=0}^{k+2}\binom{k+2}{p}(-1)^p e^{2t(k+2-p)}] - b_{nk}[\sum_{p=0}^{k+1}\binom{k+1}{p}(-1)^p e^{2t(k+1-p)}]]e^{-\pi n^2} e^{\frac{t}{2}} e^{-\sigma t} \tag{B.3}$$

This equation can be simplified as follows.

$$E_p(t) = \sum_{n,k}[\sum_{p=0}^{k+2} a'_{nkp} e^{(2k+\frac{9}{2}-2p)t} - \sum_{p=0}^{k+1} b'_{nkp} e^{(2k+\frac{5}{2}-2p)t}]e^{-\sigma t} = E_0(t)e^{-\sigma t}$$
$$a'_{nkp} = a_{nk}e^{-\pi n^2}\binom{k+2}{p}(-1)^p, b'_{nkp} = b_{nk}e^{-\pi n^2}\binom{k+1}{p}(-1)^p \tag{B.4}$$

Given that $E_0(t) = E_0(-t)$, we can write $E_p(t)$ as an **infinite summation** of two-sided decaying exponential functions as follows.

$$E_p(t) = E_0(t)e^{-\sigma t}, E_0(t) = \sum_{n,k}[\sum_{p=0}^{k+2} a'_{nkp} e^{(2k+\frac{9}{2}-2p)t} - \sum_{p=0}^{k+1} b'_{nkp} e^{(2k+\frac{5}{2}-2p)t}]$$
$$E_p(t) = [E_0(t)u(-t) + E_0(-t)u(t)]e^{-\sigma t} \tag{B.5}$$

Appendix C. Properties of Fourier Transforms

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix C.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{C.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and interchange the order of integration in equations below.

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'
\end{aligned} \tag{C.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. C.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{C.3}$$

Appendix C.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned} \tag{C.4}$$

Appendix C.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \tag{C.5}$$

Appendix C.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \tag{C.6}$$

Appendix C.5. $E_p(t), h(t), g(t)$ ***are*** L^1 ***integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t}$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$. We see that $E_p(t) = 0$ at $t = \pm\infty$ and hence $E_p(t) \ge 0$ for all $|t| \le \infty$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \le \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \ge 0$ for all $|t| \le \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an L^1 **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

We can see that $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t) \ge 0$ for all $|t| \le \infty$ because $E_p(t) \ge 0$ for all $|t| \le \infty$. Given that $E_p(t) = E_0(t)e^{-\sigma t} = [E_0(t)u(-t) + E_0(-t)u(t)]e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} -$

$3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$, we see that $g(t)$ goes to zero as $t \to -\infty$ with its order of decay greater than e^{2t} and $g(t)$ goes to zero as $t \to \infty$ with its order of decay greater than $e^{\frac{-5t}{2}}$, for $0 < \sigma < \frac{1}{2}$. Hence $g(t)$ is an L^1 **integrable function** and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

Appendix C.6. ***Convolution integral convergence***

Let us consider a function whose **first derivative is discontinuous** at $t = 0$, for example $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $E_p(t), h(t), g(t)$ are L^1 integrable functions and their Fourier Transforms are finite as shown in Appendix C.5. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.7}$$

Appendix D. Dirac delta derivatives vanish when we consider even derivatives of $g(t)$ and take their odd part $g_{2r_{odd}}(t)$

Let us consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2g(t)}{dt^2}$ where $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $g(t)h(t) = E_p(t)$. We see that $E_p(t)$ is an analytic function in the interval $-\infty \leq t \leq \infty$ and even derivatives of $g(t)$ have dirac delta functions at $t = 0$.

We can show that **dirac delta function** $d_0(t) = \delta(t)$ which is present in $g_2(t)$ and its **even derivatives** $d_{2r-2}(t)$ which are present in $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ **vanish**, when we take the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ for positive integer r, because **dirac delta function and its even derivatives have even symmetry**, while $g_{2r_{odd}}(t)$ has **odd symmetry**.

$$g(t) = g_-(t)u(-t) + g_+(t)u(t)$$
$$g_-(t) = E_p(t)e^{-\sigma t}, \quad g_+(t) = E_p(t)e^{\sigma t}$$
$$g_2(t) = \frac{d^2g(t)}{dt^2} = \frac{d^2g_-(t)}{dt^2}u(-t) + \frac{d^2g_+(t)}{dt^2}u(t) + A_0d_0(t), \quad A_0 = [\frac{dg_+(t)}{dt} - \frac{dg_-(t)}{dt}]_{t=0}$$
$$g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}} = \frac{d^{2r}g_-(t)}{dt^{2r}}u(-t) + \frac{d^{2r}g_+(t)}{dt^{2r}}u(t) + A_{2r-2}d_0(t) + \sum_{k=0}^{r-2} A_{2k}\frac{d^{2r-2-2k}(d_{2k}(t))}{dt^{2r-2-2k}}$$
$$A_{2r-2} = [\frac{d^{2r-1}g_+(t)}{dt^{2r-1}} - \frac{d^{2r-1}g_-(t)}{dt^{2r-1}}]_{t=0}, \quad A_{2k} = [\frac{d^{2k+1}g_+(t)}{dt^{2k+1}} - \frac{d^{2k+1}g_-(t)}{dt^{2k+1}}]_{t=0}$$

(D.1)

Then we take the **odd part** of the functions $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}(g_{2r}(t) - g_{2r}(-t))$ and take their Fourier transforms given by $iG_{2r_I}(\omega) = i(-\omega^2)^r G_I(\omega)$. We can see that the Fourier transform of the delta function and its even derivatives **vanish** given that **dirac delta function and its even derivatives have even symmetry** in Eq. D.1 and **do not interfere** with the results. This is shown below.

Let us consider the Fourier transform of Dirac delta function $d_0(t) = \delta(t)$ and its derivatives for $r = 0, 1, ...\infty$. We use notation $F[d_0(t)]$ to represent Fourier transform of $d_0(t)$. We use the well known property that $F[\frac{d^{2r}g(t)}{dt^{2r}}] = (-\omega^2)^r G(\omega)$ for a general Fourier transformable function $g(t)$.

$$
\begin{aligned}
F[d_0(t)] &= \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = 1 \\
F[d_2(t)] &= \int_{-\infty}^{\infty} \frac{d^2 d_0(t)}{dt^2} e^{-i\omega t}dt = -\omega^2 \\
F[d_{2r}(t)] &= \int_{-\infty}^{\infty} \frac{d^{2r} d_0(t)}{dt^{2r}} e^{-i\omega t}dt = (-\omega^2)^r
\end{aligned}
\tag{D.2}
$$

We can see that $(-\omega^2)^r$ is a real and even function of ω and hence its inverse Fourier transform given by $\frac{d^{2r}d_0(t)}{dt^{2r}}$ is also a real and even function of t.

Appendix E.

In this section, we show that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is an analytic function in the interval $-\infty \leq t \leq \infty$ and the indefinite integral is evaluated only at the upper limit.

If $x(\tau)$ is an analytic function, then we can express it using taylor series expansion around $\tau = 0$ as follows, where $x_n = \frac{1}{!n}[\frac{d^n(x(\tau))}{d\tau^n}]_{\tau=0}$ and K_0 is an integration constant in the indefinite integral $f(\tau) = \int x(\tau)d\tau$.

$$
\begin{aligned}
x(\tau) &= x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... \\
f(\tau) = \int x(\tau)d\tau &= K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ... \\
\frac{df(\tau)}{d\tau} &= x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... = x(\tau)
\end{aligned}
\tag{E.1}
$$

Now we can repeat the steps above for $f(t) = \int^t x(\tau)d\tau$ as follows.

$$
\begin{aligned}
f(t) = \int^t x(\tau)d\tau = [K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ...]^t &= K_0 + x_0t + x_1\frac{t^2}{2} + x_2\frac{t^3}{3} + x_3\frac{t^4}{4} + ... \\
\frac{df(t)}{dt} &= x_0 + x_1t + x_2t^2 + x_3t^3 + ... = x(t)
\end{aligned}
\tag{E.2}
$$

We have shown that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$.

Now, we start with $y(t_1)$ in Eq. 38 and derive in detail $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$ in Eq. 39 as follows.

$$\begin{aligned} y(t_1) = &-\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ &+\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] \end{aligned} \tag{E.3}$$

We take the first derivative of $y(t_1)$ as follows.

$$\begin{aligned} \frac{dy(t_1)}{dt_1} = &-\frac{\omega_1}{2}[-\sin(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt - \cos(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ &+\frac{\omega_1}{2}[-\sin(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt - \cos(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] \end{aligned} \tag{E.4}$$

We take the second derivative of $y(t_1)$ as follows.

$$\begin{aligned} \frac{d^2y(t_1)}{dt_1^2} = &-\frac{\omega_1^2}{2}[-\cos(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt + \sin(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ &+\frac{\omega_1^2}{2}[-\cos(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt + \sin(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] + \frac{\omega_1}{2}[x(t_1) - E_0(t_1)] \end{aligned} \tag{E.5}$$

Now we evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1)$ as follows and get Eq. 39 .

$$\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)] \tag{E.6}$$

Appendix F.

We start with Eq. 34 as follows.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r}\frac{t_1^{2r}}{!(2r)} = -\int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau] + \int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau] = 0 \tag{F.1}$$

In Eq. F.1 we have $f(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}$ inside the first integral, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ and we can show that $f(\tau) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$, given that $E_0(\tau)e^{-2\sigma\tau}$ is an analytic function in the interval $-\infty \leq \tau \leq \infty$

and hence infinitely differentiable and it is also Fourier transformable.

Similarly, we can show that $d(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)]$ inside the second integral.

We substitute $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ in the equation for $f(\tau)$ and we write as follows.

$$f(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2\pi}\sum_{r=0}^{\infty} \frac{d^{2r}([\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega]e^{-2\sigma\tau})}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} \tag{F.2}$$

In Appendix C.6, we have shown that if the $(N-1)^{th}$ **derivative** of a function $P(t)$ is **discontinuous** at $t=0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$. We know that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$ and it should have a fall-off rate **at least** of the order of $e^{-A|\omega|}$ where $A > 0$.

We can interchange the order of integration and summation as follows, because for every value of r in equation below, **the integral converges**.

$$f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty} \frac{d^{2r}e^{(i\omega-2\sigma)\tau}}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty}(i\omega-2\sigma)^{2r}e^{(i\omega-2\sigma)\tau}\frac{t_1^{2r}}{!(2r)}]d\omega \tag{F.3}$$

We can simplify this equation as follows.

$$\begin{aligned} f(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)t_1} + e^{-(i\omega-2\sigma)t_1}]e^{(i\omega-2\sigma)\tau}d\omega \\ f(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)(\tau+t_1)} + e^{(i\omega-2\sigma)(\tau-t_1)}]d\omega \end{aligned} \tag{F.4}$$

We can simplify this equation as follows, using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ and $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$f(\tau) = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)] \tag{F.5}$$

Comparing Eq. F.2 and Eq. F.5, we can see that $f(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)]$.

Similarly, we see that $d(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)]$.

$$\begin{aligned} f(\tau) &= \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)] \\ d(\tau) &= \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)] \end{aligned}$$

(F.6)

Hence we can write Eq. F.1 as follows.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{F.7}$$

Appendix G.

In this section, we want to show that the inner indefinite integral in Eq. F.7 reproduced below, can be expressed as $I_0(\tau,t_1) = J_0(\tau,t_1)+K_I$ where K_I is the integration constant and we will show that $J_0(\tau,t_1)=0$ when evaluated at the lower limit of $\tau=-\infty$, for the **specific case** of our function $E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} = E_0(t)e^{-\sigma t}$. The **integration constant** K_I gets cancelled when evaluating $I_0(\tau,t_1)$ at the upper and lower limits of the integral.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{G.1}$$

The inner indefinite integral in Eq. G.1 can be written as follows, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$I_0(\tau,t_1) = \frac{1}{2}[-\int[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{G.2}$$

We can write $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ as follows using the shorthand notation $E_0(t) = \sum_{n,r} c_{nr}e^{-\pi n^2e^{2t}}e^{K_rt}$ where $\sum_{n,r} = \sum_{n=1}^{\infty}\sum_{r=0}^{1}$, where $c_{n1} = a_n, c_{n0} = -b_n$, $a_n = 4\pi^2n^4; b_n = 6\pi n^2$ and $K_r = \frac{5}{2}+2r > 1$ for $r = 0,1$.

Appendix G.1.

In Eq. G.2, let us evaluate the indefinite integral term $I_1(\tau) = \int x(\tau)\cos(\omega_1\tau)d\tau$ as follows, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$. We show that the indefinite integral can be expressed as $I_1(\tau) = J_1(\tau)+K_{I_1}$ where K_{I_1} is the integration constant and we will show that $J_1(\tau)=0$ when evaluated at the lower limit of $\tau=-\infty$,

$$I_1(\tau) = \int x(\tau)\cos(\omega_1\tau)d\tau = \int\sum_{n,r} c_{nr}e^{-\pi n^2e^{2\tau}}e^{K_r\tau}e^{-2\sigma\tau}\cos(\omega_1\tau)d\tau \tag{G.3}$$

Using theorem of dominated convergence, we can interchange the order of integration and summation as follows, given that for every value of n and r, the integral converges.

$$I_1(\tau) = \sum_{n,r} c_{nr} \int e^{-\pi n^2 e^{2\tau}} e^{K_r \tau} e^{-2\sigma\tau} \cos(\omega_1 \tau) d\tau \tag{G.4}$$

We substitute $e^{2\tau} = x, dx = 2xd\tau, \tau = \frac{\log_e(x)}{2}$ and write as follows. we use $K_2 = \frac{(K_r - 2\sigma)}{2} - 1$.

$$I_1(x) = \frac{1}{2} \sum_{n,r} c_{nr} \int e^{-\pi n^2 x} x^{K_2} \cos(\omega_1 \frac{\log_e(x)}{2}) dx \tag{G.5}$$

Using **integration by parts** method $\int udv = uv - \int vdu$, we substitute $u = e^{-\pi n^2 x} \cos(\omega_1 \frac{\log_e(x)}{2}), dv = x^{K_2} dx$ and hence we get $v = \frac{x^{(K_2+1)}}{(K_2+1)}$ and $du = e^{-\pi n^2 x}[\cos(\omega_1 \frac{\log_e(x)}{2})(-\pi n^2) - \sin(\omega_1 \frac{\log_e(x)}{2})(\frac{\omega_1}{2x})]dx$ and we can write as follows using this method repeatedly.

$$\begin{aligned} I_1(x) = \frac{1}{2} \sum_{n,r} c_{nr} e^{-\pi n^2 x} [\frac{x^{(K_2+1)}}{(K_2+1)} \cos(\omega_1 \frac{\log_e(x)}{2}) \\ - \frac{x^{(K_2+2)}}{(K_2+1)(K_2+2)} [\cos(\omega_1 \frac{\log_e(x)}{2})[(-\pi n^2)] + \sin(\omega_1 \frac{\log_e(x)}{2})[\frac{-\omega_1}{2x}]] \\ + \frac{x^{(K_2+3)}}{(K_2+1)(K_2+2)(K_2+3)} [\cos(\omega_1 \frac{\log_e(x)}{2})[(-\pi n^2)^2 - (\frac{\omega_1}{2x})^2] + \sin(\omega_1 \frac{\log_e(x)}{2})[2(-\pi n^2)\frac{-\omega_1}{2x}]] - ...] + K_{I_1} \end{aligned} \tag{G.6}$$

We can simplify this as follows.

$$\begin{aligned} I_1(x) = \frac{1}{2} \sum_{n,r} c_{nr} e^{-\pi n^2 x} [\frac{x^{(K_2+1)}}{(K_2+1)} \cos(\omega_1 \frac{\log_e(x)}{2}) \\ - \frac{x^{(K_2+1)}}{(K_2+1)(K_2+2)} [\cos(\omega_1 \frac{\log_e(x)}{2})[(-\pi n^2)x] + \sin(\omega_1 \frac{\log_e(x)}{2})[\frac{-\omega_1}{2}]] \\ + \frac{x^{(K_2+1)}}{(K_2+1)(K_2+2)(K_2+3)} [\cos(\omega_1 \frac{\log_e(x)}{2})[(-\pi n^2)^2 x^2 - (\frac{\omega_1}{2})^2] + \sin(\omega_1 \frac{\log_e(x)}{2})[2(-\pi n^2)\frac{-\omega_1 x}{2}]] - ...] + K_{I_1} \end{aligned} \tag{G.7}$$

We want to evaluate the above indefinite integral $I_1(x)$ at the lower limit of $\tau = -\infty$ which corresponds to $x = 0$ under the substitution $e^{2\tau} = x$. We can see that $I_1(x) = 0$ at $x = 0$ plus an **integration constant** K_{I_1} which gets cancelled when evaluating the indefinite integral at the upper and lower limits. We can see that $K_2 + 1 > 0$ given that $K_2 = \frac{(K_r - 2\sigma)}{2} - 1$ and $K_r = \frac{5}{2} + 2r > 1$ for $r = 0, 1$ and $0 < |\sigma| < \frac{1}{2}$.

Similar to above method, we can evaluate the indefinite integral term $I_2(\tau) = \int x(\tau) \sin(\omega_1 \tau) d\tau$ in Eq. G.2 and we can show that the indefinite integral **equals zero**, when evaluated at the lower limit of $\tau = -\infty$, plus an **integration constant** which gets cancelled when evaluating the indefinite integral at the upper and lower limits.

We can use integration by parts method for the terms $x(\tau + t_1), x(\tau - t_1), E_0(\tau + t_1), E_0(\tau - t_1)$ in Eq. G.2 and show that the indefinite integral **equals zero**, when evaluated at the lower limit of $\tau = -\infty$, plus an

integration constant which gets cancelled when evaluating the indefinite integral at the upper and lower limits.

Hence $I_0(\tau, t_1)$ in Eq. G.2, when evaluated at the lower limit of $\tau = -\infty$, equals zero plus the **integration constant** K_I.

We can see that the indefinite integral $I_1(x)$ in Eq. G.7 evaluated at the upper limit of $\tau = t$ which corresponds to $x = e^{2t}$ is a finite value plus **integration constant** K_{I_1}.

Hence $I_0(\tau, t_1)$ in Eq. G.2 is finite.

Appendix H.

Let us start with Riemann's Xi Function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$.

In this section, we will re-derive the inverse Fourier Transform of Riemann's Xi function as $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. [4]

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and rederive the steps below. We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and rederive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{H.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{H.2}$$

For real part of s greater than 1, we can use theorem of doominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{H.3}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}s(s-1)\int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx$$

$$\tag{H.4}$$

$\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane. We see that $\xi(s) = \xi(1-s)$.

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in above equation and get

$$\xi(s) = 2\frac{1}{2}s(s-1)\int_{-\infty}^{\infty} e^{st}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} dt \tag{H.5}$$

We evaluate above equation at $s = \frac{1}{2} + i\omega$ as follows.

$$\begin{aligned}\xi(\frac{1}{2}+i\omega) &= 2\frac{1}{2}(\frac{1}{2}+i\omega)(-\frac{1}{2}+i\omega)\int_{-\infty}^{\infty} e^{\frac{t}{2}}e^{i\omega t}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} dt \\ \xi(\frac{1}{2}+i\omega) &= 2\frac{1}{2}[-(\frac{1}{4}+\omega^2)\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{i\omega t} dt\end{aligned} \tag{H.6}$$

We define $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and get the inverse Fourier transform of $\xi(\frac{1}{2}+i\omega)$ given by $E_0(t)$ as follows.

$$\begin{aligned}E_0(t) &= 2\frac{1}{2}[-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\ A(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}] \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi n^2 e^{2t} + (\frac{1}{2} - 2\pi n^2 e^{2t})^2] \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{4} + 4\pi^2 n^4 e^{4t} - 2\pi n^2 e^{2t} - 4\pi n^2 e^{2t}]\end{aligned} \tag{H.7}$$

We have arrived at the desired result for $E_0(t)$ as follows.

$$\begin{aligned}E_0(t) &= [-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\ E_0(t) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]\end{aligned} \tag{H.8}$$