On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line and prove Riemann's hypothesis.

More importantly, the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

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In Section 2, we take the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 3, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$, because the new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and Fourier transformable functions and this condition is satisfied for Riemann's Zeta function, but **not** for Hurwitz zeta function and related functions.

In Appendix A to Appendix F, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

*1.1. **Step 1: Inverse Fourier Transform of** $\xi(\frac{1}{2}+i\omega)$*

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \le \omega \le \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows (link).[3] This is re-derived in Appendix C.

$$E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2n^4\pi^2 e^{\frac{9t}{2}} - 3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2 e^{2t}} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at $s = \frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \le \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

2. Proof of Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Step A: In Section 2.1, we consider $E_p(t) = 2\sum\limits_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}$ and $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{\sigma t}$ and show that $E_q(t) = E_p(-t) = [(-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}]$ and $A(\omega) = \frac{E_{q\omega}(\omega)}{(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))}$. We show that the integral $F(s) = 2A(\omega) = 2\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt$ **converges** for real ω, in the **critical strip** excluding the critical line $0 < |\sigma| < \frac{1}{2}$, where $s = \frac{1}{2} - \sigma + i\omega$.

Step B: In Section 2.2, we evaluate $F(s)$ using the well known theorem $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real and use the substitution $x = e^{2t}$.

Step C: In Section 2.3, we use the substitution $x = e^{2t}$ in the well known equation $\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_1^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}]$.

Step D: In Section 2.4, we prove that $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line using results derived in Steps A, B and C.

2.1. ***Step A***

We start with the function $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{\sigma t}$ and show that the integral $A(\omega) = \int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt$ **converges** for real ω, in the **critical strip** excluding the critical line $0 < |\sigma| < \frac{1}{2}$. Let $s = \frac{1}{2} - \sigma + i\omega$.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} - \sigma + i\omega) = \xi(\frac{1}{2} + \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We consider $E_p(t) = 2\sum\limits_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}$ and $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{\sigma t}$ and show that $E_q(t) = E_p(-t) = (-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$. We use the **fact** that $E_0(t) = E_0(-t)$ and $E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t} = E_0(t)e^{\sigma t}$. The Fourier transform of this equation is given by $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega) = A(\omega)(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))$ which **corresponds** to $\xi(s) = \frac{1}{2}s(s-1)F(s)$ where $F(s) = 2A(s)$ at $s = \frac{1}{2} - \sigma + i\omega$.

$$
\begin{aligned}
A(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} \\
\frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} [\frac{1}{2} + \sigma - 2\pi n^2 e^{2t}] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} [-4\pi n^2 e^{2t} + (\frac{1}{2} + \sigma - 2\pi n^2 e^{2t})^2] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} [\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} - 4\sigma\pi n^2 e^{2t}]
\end{aligned}
\tag{3}
$$

We have arrived at the desired result for $E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t} = E_0(t)e^{\sigma t}$ as follows.

$$
\begin{aligned}
E_q(t) &= (-\frac{1}{4} + \sigma^2)A(t) - 2\sigma \frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_q(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} [(-\frac{1}{4} + \sigma^2) + (-\sigma - 2\sigma^2 + 4\sigma\pi n^2 e^{2t}) + (\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} - 4\sigma\pi n^2 e^{2t})] \\
E_q(t) &= 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}
\end{aligned}
\tag{4}
$$

The Fourier transform of $E_q(t)$ is given by $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ **converges** for real ω, because $\xi(s)$ is an entire function. Using the properties of Fourier transform, we get $E_{q\omega}(\omega) = A(\omega)(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))$ and we see that $A(\omega)$ **converges** for all real ω, **because** $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ and $\frac{1}{(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))}$ converge for all real ω. We can derive $F(s) = \frac{\xi(s)}{\frac{1}{2}s(s-1)}$ by using $s = \frac{1}{2} - \sigma + i\omega$ as follows.

$$
\begin{aligned}
E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega) &= A(\omega)(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma)) \\
F(s) = 2A(s) = 2\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt &= 2\frac{E_{q\omega}(\omega)}{(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))} = \frac{\xi(s)}{\frac{1}{2}s(s-1)}
\end{aligned}
\tag{5}
$$

Hence we have shown that $A(\omega) = \int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt$ **converges** for all real ω, in the region $0 < |\sigma| < \frac{1}{2}$. More arguments for convergence of $A(\omega)$ are presented in Section 2.5.

2.2. ***Step B***

We can write the integral in Eq. 5 as follows and split into two integrals.

$$
F(s) = 2\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt = 2[\int_{0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt + \int_{-\infty}^{0} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt]
$$

$$(6)$$

We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$ in the second integral in Eq. 6, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real and we use $x = e^{2t}$ and derive as follows for $-\infty \leq t \leq \infty$. We use $\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} + \frac{1}{2}e^{-\frac{t}{2}} - \frac{1}{2}e^{\frac{t}{2}}$ derived in Eq C.17 in Appendix C.3 We include $x = 0$ as in textbooks and hence include $t = -\infty$. (Eq.5.5 to Eq.5.6 in link)

$$F(\frac{1}{2} - \sigma + i\omega) = 2[\int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt]$$
$$- \int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{i\omega t} dt \tag{7}$$

2.3. ***Step C***

We use **the well known equation** for $\xi(s)$.(Eq. 5.6 in Ellison's book "Prime Numbers" pages 151-152)

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_1^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{8}$$

We see that $\xi(s)$ is an entire function, for all values of s in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$.

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. 8 and evaluate at $s = \frac{1}{2} - \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2} - \sigma + i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2} - \sigma + i\omega)(-\frac{1}{2} - \sigma + i\omega)\int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{-\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}) dt] \tag{9}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\xi(\frac{1}{2} - \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))[\int_{-\infty}^{0} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt] \tag{10}$$

We can write this as follows using Heaviside step function $u(t)$.

$$\xi(\frac{1}{2} - \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{\sigma t} e^{-i\omega t} dt \tag{11}$$

Statement A: If $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ **has a zero** at $\omega=\omega_0$, **then** $\xi(\frac{1}{2}-\sigma+i\omega) = E_{q\omega}(\omega)$ also **has a zero** at $\omega=\omega_0$, where ω_0 is real and finite, given that $\xi(s)=\xi(1-s)$ and $E_p(t), E_q(t)$ are real and their Fourier transforms have symmetry properties (Appendix F.1). Hence $\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{\sigma t} e^{-i\omega_0 t} dt = -\frac{1}{2}\frac{1}{(-\frac{1}{4}+\sigma^2-\omega_0^2-i\omega_0(2\sigma))}$, from Eq. 11.

The inverse Fourier transform of $\xi(\frac{1}{2}-\sigma+i\omega) = E_{q\omega}(\omega)$ in Eq. 11 is given by $E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$ because $E_0(t) = E_0(-t)$. This is re-derived in Appendix C.1. Hence $F(s)$ in Eq. 5 and Eq. 7 are related by the equation $F(s) = \frac{\xi(s)}{\frac{1}{2}s(s-1)}$.

2.4. *Step D: Final Proof*

If **Statement A** is true, then $F(s) = \frac{\xi(s)}{\frac{1}{2}s(s-1)}$ also **has a zero** at $\omega=\omega_0$, **because** $s(s-1) = (-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))$ **does not** have a zero for real ω, for $s=\frac{1}{2}-\sigma+i\omega$ and $0<\sigma<\frac{1}{2}$. . We can compute Eq. 7 as follows. We use $\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{\sigma t} e^{-i\omega_0 t} dt = -\frac{1}{2}\frac{1}{(-\frac{1}{4}+\sigma^2-\omega_0^2-i\omega_0(2\sigma))}$.

$$F(\frac{1}{2}-\sigma+i\omega_0) = \frac{1}{(\frac{1}{4}-\sigma^2+\omega_0^2+i\omega_0(2\sigma))} - \int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt + \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt = 0 \tag{12}$$

For $0<\sigma<\frac{1}{2}$, we can write

$$\begin{aligned} \int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt &= \frac{1}{\frac{1}{2}+\sigma-i\omega_0} \\ F(\frac{1}{2}-\sigma+i\omega_0) = \frac{1}{(\frac{1}{4}-\sigma^2+\omega_0^2+i\omega_0(2\sigma))} - \frac{1}{\frac{1}{2}+\sigma-i\omega_0} &+ \int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt = 0 \\ F(\frac{1}{2}-\sigma+i\omega_0) = \frac{1}{\frac{1}{2}-\sigma+i\omega_0} &+ \int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt = 0 \end{aligned} \tag{13}$$

We can see that the integral $\int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt$ **diverges** for $0<\sigma<\frac{1}{2}$.

$$\int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega_0 t} dt = \lim_{T\to\infty}[\frac{e^{t(-\frac{1}{2}+\sigma-i\omega_0)}}{(-\frac{1}{2}+\sigma-i\omega_0)}]_{t=-T}^{t=0} = \frac{-1}{\frac{1}{2}-\sigma+i\omega_0} + \frac{1}{\frac{1}{2}-\sigma+i\omega_0}\lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega_0)} \tag{14}$$

Substituting Eq. 14 in Eq. 13, canceling the common term $\frac{1}{\frac{1}{2}-\sigma+i\omega_0}$ we get

$$F(\frac{1}{2}-\sigma+i\omega_0) = \frac{1}{\frac{1}{2}-\sigma+i\omega_0}\lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega_0)} = 0 \tag{15}$$

We can see that $\lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega_0)} \neq 0$ for $0 < \sigma < \frac{1}{2}$ and hence $F(\frac{1}{2}-\sigma+i\omega_0)$ **diverges** for $0 < \sigma < \frac{1}{2}$.

We see that the assumption in **Statement A** that $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ **has a zero** at $\omega = \omega_0$ where ω_0 is real and finite, leads to a **contradiction** for the region $0 < \sigma < \frac{1}{2}$.

We have proved it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2}-\sigma+i\omega) = \xi(\frac{1}{2}+\sigma-i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

2.5. ***Convergence of*** $A(\omega)$

We consider $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$ for $0 < \sigma < \frac{1}{2}$ and show that the integral $\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt$ **converges** for real ω.

$$F(s) = 2\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt = 2\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt \tag{16}$$

Method 1: (1.1) We see that $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} \geq 0$ and finite for all $|t| < \infty$. The **series** in Eq. 16 inside the integral, **converges** for all $t > -\infty$, using Integral test, because $\int_1^{\infty} Ce^{-Bu^2}du$ is finite, where $B = \pi e^{2t} > 0$, $C = e^{\frac{t}{2}} e^{\sigma t}$ and n is replaced by u.

As $t \to \infty$, the integrand in Eq. 16 goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. Hence $A(t)$ is **finite** for all $-\infty < t \leq \infty$.

(1.2) It is well known that the order of Riemann's Xi function at $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant (Titchmarsh). We define $A_0(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and its Fourier transform $A_0(\omega) = \frac{E_{0\omega}(\omega)}{(-\frac{1}{4}-\omega^2)}$ is finite for all $|\omega| \leq \infty$ (Section 2.1) and goes to zero as $|\omega| \to \infty$ with fall-off rate of at least $\omega^A e^{-\frac{|\omega|\pi}{4}}$. Hence $A_0(\omega)$ is absolutely integrable and its inverse Fourier transform $A_0(t)$ goes to zero as $|t| \to \infty$ as per Riemann-Lebesgue Lemma (link). We see that $A(t) = A_0(t)e^{\sigma t}$ and as $t \to -\infty$, the integrand in Eq. 16 goes to zero, with a **fall-off rate** of $e^{\sigma t}$ for $0 < \sigma < \frac{1}{2}$ and with a **faster** fall-off rate as $t \to \infty$, due to the term $e^{-\pi n^2 e^{2t}}$.

Given (1.1) and (1.2), the integrand in Eq. 16 is absolutely integrable and hence $\int_{-\infty}^{\infty} |A(t)|dt$ is finite for $0 < \sigma < \frac{1}{2}$ and the integral $\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt$ **converges** for real ω, in the region $0 < \sigma < \frac{1}{2}$.(Appendix B.4)

Method 2: We can also use the fact that $A(\omega) = \frac{E_{q\omega}(\omega)}{(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))}$ is an **analytic** function for $0 < \sigma < \frac{1}{2}$, which is infinitely differentiable and produces no discontinuities for all $|\omega| \leq \infty$ and has a fall-off rate of at least $O[\omega^A e^{-\frac{|\omega|\pi}{4}}]$, given that $E_{q\omega}(\omega) = E_{0\omega}(\omega + i\sigma)$. Using arguments similar to **Payley-Weiner** theorem in Appendix B.3, it can be shown that the inverse Fourier transform $A(t)$ in Eq. 16 has fall-off rate of **at least** $\frac{1}{t^2}$ as $|t| \to \infty$. We know from (1.1) in above para that $A(t)$ is finite for all $|t| < \infty$. Hence the integrand in Eq. 16 is absolutely **integrable** and the integral **converges** and is finite.(Appendix B.4) More details in Appendix E

It is noted that the second proof in above section suggests that there are no zeros in the critical line corresponding to $\sigma = 0$, which contradicts previously known theorems. There are two possibilities.

• There may be an error in Section 2.1. If this is the case, I request the Referee to point out the error.

• It is possible that there is an error in previously known theorems. For example, in G.H.Hardy's proof on the existence of zeros in the critical line (Titchmarsh[3]), he obtained Eq.10.2.1 (link), by substituting $x = -i * \alpha$ in Eq.2.16.2 (link), which may be incorrect because we cannot find a suitable contour in the complex plane where the integrand vanishes asymptotically, using contour integration method. This is shown in Appendix D.

3. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi\frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (Appendix C). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

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[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ (link). This is re-derived in Appendix C.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned}
\xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\
E_p(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega
\end{aligned} \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line

$y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix B.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$. **Alternate** derivation is in Appendix C.1.

Appendix B. Properties of Fourier Transforms Part 1

Appendix B.1. ***$E_p(t)$ is an absolutely integrable function whose Fourier Transform is finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B.2. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$, as $|\omega| \to \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \to \infty$, where $A, B > 0$ are real.

Appendix B.3. ***Payley-Weiner theorem and Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix B.2) and hence it should have a fall-off rate **at least** $\frac{1}{t^2}$ as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix B.4. ***Fall-off rate and absolutely integrable functions***

It is well known that a Fourier transformable function $f(t)$ which is **finite** at $|t| \leq \infty$ and has a fall-off rate of **at least** $O(\frac{1}{t^2})$ as $|t| \to \infty$ is absolutely integrable because for $0 < T < \infty$, $\int_{-T}^{T} |f(t)|$ is finite, and $\int_{T}^{\infty} O(\frac{1}{t^2}) = [O(-\frac{1}{t})]_{T}^{\infty}$ and $\int_{-\infty}^{-T} O(\frac{1}{t^2}) = [O(-\frac{1}{t})]_{-\infty}^{-T}$ are finite. Hence $\int_{-\infty}^{\infty} |f(t)|$ is finite and hence $f(t)$ is **absolutely integrable**.

Similarly, a Fourier transformable function $f(t)$ which is **finite** at $|t| \leq \infty$ and has a fall-off rate of **at least** $O(e^{-A|t|})$ as $|t| \to \infty$ for $A > 0$, is absolutely integrable because for $0 < T < \infty$, $\int_{-T}^{T} |f(t)|$ is finite, and $\int_{T}^{\infty} O(e^{-A|t|}) = [O(e^{-A|t|})\frac{1}{A}]_{T}^{\infty}$ and $\int_{-\infty}^{-T} O(e^{-A|t|}) = [O(e^{-A|t|}\frac{1}{A})_{-\infty}^{-T}$ are finite. Hence $\int_{-\infty}^{\infty} |f(t)|$ is finite and hence $f(t)$ is **absolutely integrable**.

The references for these well known results are in textbooks on Fourier transforms. (As an example, (Exercise 4.1.4 in link) and (link) and (Example 1 in link)).

Appendix C. Derivation of entire function $\boldsymbol{\xi(s)}$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned}\Gamma(\frac{s}{2}) &= \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx\end{aligned} \tag{C.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{C.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^{\infty} |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{C.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. C.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty}\frac{x^{-(\frac{s}{2}-1)}}{x^2}\frac{(1+2w(x))\sqrt{x}-1)}{2}dx \tag{C.4}$$

Hence we can simplify Eq. C.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}}w(x)dx \tag{C.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_1^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{C.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix C.1. ***Derivation of $E_p(t)$ and $E_0(t)$***

Given that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. C.6 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t} + e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{C.7}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[&\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt \\ &+ \int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{C.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{C.9}$$

We define $A(t) = [\sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t) + \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$
$$A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)]e^{-\sigma t}$$
$$\frac{dA(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}]u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}]u(t)$$
$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-4\pi n^2 e^{-2t} + (-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})^2]u(-t)$$
$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2} - \sigma - 2\pi n^2 e^{2t})^2]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2)] \tag{C.10}$$

We can simplify above equation as follows.

$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2)] \tag{C.11}$$

We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix C.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$
$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t})$$
$$+\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2 + \sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \tag{C.12}$$

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{C.13}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{C.14}$$

Appendix C.2. ***Derivation of*** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}} \tag{C.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2})$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2} \tag{C.16}$$

Appendix C.3. ***Modular Theta functions***

We start with **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ for $0 < x \leq \infty$ (link) and substitute $x = e^{2t}$ and then multiply both sides of the equation by $\frac{1}{2}e^{-\frac{t}{2}}$ as follows, for $-\infty < t \leq \infty$.

$$
\begin{aligned}
\sqrt{x}(1 + 2w(x)) &= (1 + 2w(\frac{1}{x})) \\
e^t(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}) &= 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} \\
\frac{1}{2}e^{\frac{t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \frac{1}{2}e^{-\frac{t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} \\
\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} + \frac{1}{2}e^{-\frac{t}{2}} - \frac{1}{2}e^{\frac{t}{2}}
\end{aligned}
\tag{C.17}
$$

Appendix D. Contour Integration Example

• Let us start with **two sided decaying exponential** function $g(t) = e^{-a|t|}$ whose Fourier transform is given by $G(\omega) = \frac{2a}{a^2+\omega^2}$. We will take inverse Fourier transform of $G(\omega)$ using **Contour Integration** method and show that we get $g(t) = e^{-a|t|}$.

We take a more general function $F(\omega) = G(\omega)A(\omega)$ and substitute $\omega = z = x + iy$ and we get as follows.

$$
\begin{aligned}
F(z) &= \frac{2a}{a^2 + z^2}A(z), \quad F(z) = \frac{2a}{(z + ia)(z - ia)}A(z) \\
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(z)e^{izt}dz &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z)\frac{2a}{(z + ia)(z - ia)}e^{-yt}e^{ixt}dz
\end{aligned}
\tag{D.1}
$$

We can use **Cauchy's Residue** theorem and use a semicircular contour of radius R in upper half plane for $t > 0$ and lower half plane for $t < 0$ and use Jordan's Lemma as $R \to \infty$ and $y \to \pm\infty$ and derive as follows.

• **Case A:** $A(z) = 1$

$$
f(t) = \frac{1}{2\pi}2\pi i[\text{ Residue of } F(z)e^{izt} \text{ at } z = ia \quad + \text{ Residue of } F(z)e^{izt} \text{ at } z = -ia] \tag{D.2}
$$

Case A.1: $t > 0$. Semi circular contour in upper half plane.

$$
f(t) = \frac{1}{2\pi}2\pi i[\text{ Residue of } F(z)e^{izt} \text{ at } z = ia] = i2a[\frac{1}{i2a}e^{-at}] = e^{-at}u(t)
$$

(D.3)

Case A.2: $t < 0$. Semi circular contour in lower half plane.

$$f(t) = \frac{1}{2\pi} 2\pi i[\text{ Residue of } F(z)e^{izt} \text{ at } z = -ia] = -i2a[\frac{1}{-i2a}e^{at}] = e^{at}u(-t)$$

(D.4)

Thus we have derived the inverse Fourier transform of $F(\omega) = \frac{2a}{a^2+\omega^2}$ as $f(t) = e^{-a|t|}$.

In the next section, we investigate **whether** we can substitute $t = i\beta$ in Eq. D.1, where $\beta > 0$ is **real** and get the result $f(i\beta) = e^{-ia\beta}u(t) + e^{ia\beta}u(-t)$. We will show that this is **NOT** possible because we **cannot** find a suitable contour which surrounds the singularities at $z = \pm ia$. We **cannot** get the result $f(i\beta) = e^{-ia\beta}u(t) + e^{ia\beta}u(-t)$.

Appendix D.1. ***Substitution $t = i\beta$ in Eq. D.1***

We want to study **whether** we can substitute $t = i\beta$ in Eq. D.1, where $\beta > 0$ is **real** and write as follows. We will show that this is **NOT** possible.

$$f(i\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(z)e^{iz(i\beta)}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z)\frac{2a}{(z+ia)(z-ia)}e^{-\beta x}e^{-i\beta y}dz \quad \text{(D.5)}$$

.

- **Case B:** $A(z) = 1$

Because the term $e^{-\beta x}$ goes to ∞ as $x \to -\infty$, we **cannot** find a suitable contour which surrounds the singularities at $z = \pm ia$, to evaluate above equation and get the result $f(i\beta) = e^{-ia\beta}u(t) + e^{ia\beta}u(-t)$.

.

- **Case C:** $A(z) = e^{-\pi z^2}$: **Gaussian** function.

$$f(i\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\pi z^2}\frac{2a}{(z+ia)(z-ia)}e^{-\beta x}e^{-i\beta y}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{\pi y^2}e^{-i2\pi xy}\frac{2a}{(z+ia)(z-ia)}e^{-\beta x}e^{-i\beta y}dz$$

(D.6)

Because the term $e^{-\beta x}$ goes to ∞ as $x \to -\infty$ and $e^{\pi y^2}$ goes to ∞ as $y \to \infty$, we **cannot** find a suitable contour which surrounds the singularities at $z = \pm ia$, to evaluate above equation and get the desired result.

.

- **Case D:** $A(z) = E_{0z}(z)$: **Riemann's Xi function.** We know that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ which is the Fourier transform of $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$.

$$A(z) = E_{0z}(z) = \int_{-\infty}^{\infty} E_0(\tau)e^{-iz\tau}d\tau = \int_{-\infty}^{\infty} E_0(\tau)e^{y\tau}e^{-ix\tau}d\tau$$

$$f(i\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z)\frac{2a}{(z+ia)(z-ia)}e^{-\beta x}e^{-i\beta y}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z)\frac{2a}{(z+ia)(z-ia)}e^{-\beta x}e^{-i\beta y}dz$$

(D.7)

Because the term $e^{-\beta x}$ goes to ∞ as $x \to -\infty$, we **cannot** find a suitable contour which surrounds the singularities at $z = \pm ia$, to evaluate above equation and get the desired result.

• Similarly, in G.H.Hardy's proof on the existence of zeros in the critical line (Titchmarsh[3]), he obtained Eq.10.2.1 (link), by substituting $x = -i * \alpha$ in Eq.2.16.2 (link). This may be incorrect because we **cannot** find a suitable contour in the complex plane where the integrand vanishes asymptotically, using contour integration method.

Appendix E. Divergent $\zeta(s)$ for $\sigma' < 1$

Step 1: In this section, we will re-derive Riemann's Xi function $\xi(s)$ and use the steps in Ellison's book "Prime Numbers" pages 151-152[4] and rederive the steps below. (link) We start with the gamma function $\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy$ and substitute $y = \pi n^2 x$ and rederive as follows.

$$\begin{aligned} \Gamma(\frac{s}{2}) &= \int_0^\infty y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \end{aligned} \tag{E.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \tag{E.2}$$

Let $s = \sigma' + i\omega$. For real part of s given by $\sigma' > 1$, we can use theorem of dominated convergence and **interchange** the order of summation and integration as follows. We use the fact that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum\limits_{n=1}^\infty \int_0^\infty |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} w(x) dx \tag{E.3}$$

Given that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. E.3 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$. We define $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$ and write as follows for $\sigma' = \frac{1}{2} + \sigma > 1$.

$$F(\frac{1}{2} + \sigma + i\omega) = 2\int_{-\infty}^\infty \sum_{n=1}^\infty e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} dt = 2\int_{-\infty}^\infty A(t) e^{i\omega t} dt \tag{E.4}$$

Critical Strip: For $0 < \sigma' = \frac{1}{2} + \sigma < 1$, $\zeta(s)$ **diverges** and $F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is said to diverge. There are two possibilities.

• **Case 1:** We **cannot** interchange the order of summation and integration in Eq. E.2 because $F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ diverges in the critical strip. We substitute $x = e^{2t}$ and get $F(s) = 2\sum\limits_{n=1}^{\infty} \int_{-\infty}^\infty e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} dt$.

This is **different** from Eq. 16 where $F(s) = 2\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} dt$ and the results in Section 2.1 **do not conflict** with Eq. E.2.

• **Case 2: If** we want to interchange the order of summation and integration in Eq. E.2 for the critical strip, **then** we must show that $\int_{-\infty}^{\infty} |\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t}| dt$ is finite. We have shown such a result in Section 2.5 and this means that $F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ should converge for the critical strip and the integral $F(s) = 2\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} dt$ represents a **convergent analytic continuation** of $\zeta(s)$ in the critical strip.

This **should not** seem counter-intuitive, we already know that divergent series like $\zeta(0) = -\frac{1}{2}, \quad \zeta(-2m) = 0, \quad \zeta(1-2m) = \frac{(-1)^m B_m}{2m}$ have convergent integral representations (Titchmarsh book pp.18-19).(link)

Appendix F. Other Results

In this section, we start with $\xi(s)$ in Eq. 11 and $F(s)$ in Eq. 7, which are copied below.

$$\xi(s) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma)) \int_{-\infty}^{\infty} [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{\sigma t} e^{-i\omega t} dt$$

$$F(s) = 2[\int_{0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt]$$
$$- \int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt \tag{F.1}$$

We substitute $(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma)) \int_{-\infty}^{\infty} [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{\sigma t} e^{-i\omega t} dt = \xi(s) - \frac{1}{2}$ in $F(s)$ in Eq. F.1 and get

$$\xi(s) = \frac{1}{2}s(s-1)F(s) = \xi(s) - \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))[-\frac{1}{2}\int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \frac{1}{2}\int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt] \tag{F.2}$$

We cancel $\xi(s)$ on both sides of Eq. F.2 and write as follows. For $0 < \sigma < \frac{1}{2}$, we can write

$$\frac{1}{(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))} = [-\int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt]$$
$$\int_{-\infty}^{0} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt = \frac{1}{\frac{1}{2} + \sigma - i\omega}$$
$$\frac{1}{(-\frac{1}{4} + \sigma^2 - \omega^2 - i\omega(2\sigma))} = -\frac{1}{\frac{1}{2} + \sigma - i\omega} - \frac{1}{\frac{1}{2} - \sigma + i\omega} = -\frac{1}{\frac{1}{2} + \sigma - i\omega} + \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt \tag{F.3}$$

Cancelling common terms on both sides of above equation, we get

$$
\begin{aligned}
-\frac{1}{\frac{1}{2}-\sigma+i\omega} &= \int_{-\infty}^{0} e^{-\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt \\
\frac{1}{\frac{1}{2}-\sigma+i\omega} + \int_{-\infty}^{0} & e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt = 0
\end{aligned}
\tag{F.4}
$$

We can see that the integral $\int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt$ diverges for $0 < \sigma < \frac{1}{2}$.

$$
\int_{-\infty}^{0} e^{\frac{-t}{2}} e^{\sigma t} e^{-i\omega t} dt = \lim_{T\to\infty} [\frac{e^{t(-\frac{1}{2}+\sigma-i\omega)}}{(-\frac{1}{2}+\sigma-i\omega)}]_{t=-T}^{t=0} = \frac{-1}{\frac{1}{2}-\sigma+i\omega} + \frac{1}{\frac{1}{2}-\sigma+i\omega} \lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega)}
\tag{F.5}
$$

Substituting Eq. F.5 in Eq. F.4 and canceling common term, we get

$$
\frac{1}{\frac{1}{2}-\sigma+i\omega} \lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega)} = 0
\tag{F.6}
$$

We can see that $\lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma+i\omega)} \neq 0$ and hence above equation diverges for $0 < \sigma < \frac{1}{2}$ and cannot be equal to zero.

This suggests there may be problems in the textbook derivation of $\xi(s)$ in Eq. 8 which uses $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$ which may be approximate.(Ellison's book "Prime Numbers" pages 151-152)

Appendix F.1. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t) dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t) dt = -G_I(-\omega)
\end{aligned}
\tag{F.7}
$$