1. Summary of Results: [Author: Akhila Raman]

• It is well known that Shannon's Capacity Limit for a communication channel in the presence of Additive White Gaussian Noise(AWGN) is given by $C = W\log_2(1+\frac{P}{N})$ where C is the Channel Capacity in bits per second, W is the channel bandwidth in Hz, P is the average signal power and N is the average noise power. It uses concept of non-overlapping noise spheres in n-dimensional space, let us call it **System S**.(Shannon's paper pp452-454).

Following results are shown in the context of Orthogonal Frequency Division Multiplexing(OFDM) framework. It could be generalized to M-ary signalling in time domain samples, in a AWGN channel, using suitable pulse shaping for symbols.

• **System A: OFDM with LDPC with symbol by symbol detection**

For the **specific** case of Orthogonal Frequency Division Multiplexing(OFDM) which uses M-ary signalling and **symbol by symbol** detection on each DFT sample in a AWGN channel, followed by (n',k') error correction code on hard decision bits, for example Gallager's Low Density Parity Check(LDPC) Codes(1963), with minimum distance given by d_{min}, which has the property that $\frac{d_{min}}{n'}$ is a non-zero constant as $n' \to \infty$, it can be shown that the channel capacity per unit bandwidth(spectral efficiency) is given by

$$(\frac{C}{W})_{sym} = \log_2(1+\frac{P}{N}*K); \quad K = \tfrac{3}{2} * \frac{1}{(\frac{E_b}{N_0})_{min}} \tag{1}$$

where $(\frac{E_b}{N_0})_{min}$ is the minimum value of $\frac{E_b}{N_0}$ for which the Bit Error Rate(BER) of this system $BER = f(\frac{E_b}{N_0})$ is less than or equal to the Error Correction Rate(ECR) of the associated (n',k') code, as $n' \to \infty$. $BER \leq \frac{(d_{min}-1)}{2n'}$. That would ensure that as $n' \to \infty$, all bit errors are corrected and effective $BER \to 0$. Details in Section **??**.

• **System B: OFDM with LDPC with noise spheres**

We can show that the channel capacity per unit bandwidth, for an OFDM system with (n',k') error correcting code, using Shannon's non-overlapping noise spheres in n-dimensional space, is given by

$$(\frac{C}{W})_{sphere} = \log_2(1+\frac{P}{N}*(3K')) \tag{2}$$

where $K' = \frac{d_{min}}{n'}$ for a certain $\frac{k'}{n'}$, for (n',k') error correcting codes. Details in Section 2 and Section 4.

• We can show that the two expressions for capacity per unit bandwidth in Eq. 1 and Eq. 2 are **different**, for **different** methods of detection and hence must be **studied independently**.

For specific case of $M = 2$(binary signalling on each FFT sample) with $BER = \frac{1}{2}erfc(\sqrt{\frac{E_b}{N_0}})$,and Gallager's (n',k') LDPC code with $\frac{d_{min}}{n'} = K' = 0.11$ and $\frac{k'}{n'} = 0.5$, $(\frac{E_b}{N_0})_{min} = 1.28$ and hence $K = \frac{3}{2*1.28} = 1.1719$ in Eq. 1 and $K' = 0.11$ in Eq. 2.

$$\textbf{System A:}(\frac{C}{W})_{sym} = \log_2(1+\frac{P}{N}*1.1719)$$
$$\textbf{System B:}(\frac{C}{W})_{sphere} = \log_2(1+\frac{P}{N}*0.33))$$

- **Comparison with Shannon's Limit**

It can be shown that **System A** in Eq. 3 is around **4 dB** away from Shannon Limit for a source $\frac{C}{W} = 1$. We can get closer to Shannon's Limit as follows:

A) Soft Decision Iterative MAP Decoders used in LDPC decoding can give **2-3 dB** gain over Hard Decision ML decoder assumed in derivation of Eq. 1.

B) if we can get $\frac{d_{min}}{n'} = 0.175$ for $\frac{k'}{n'} = 0.5$, for a **future** error correcting code [Second MRRW bound allows this. See Table 1 in Page 4], we can show that $(\frac{E_b}{N_0})_{min} = 0.92$ and $K = \frac{3}{2*0.92} = 1.63$. This gives a $10 * \log 10(1.28/0.92) = 1.43$ dB gain over $\frac{d_{min}}{n'} = 0.11$ assumed in Eq. 1.

C) Shannon's Capacity Limit was derived using non-overlapping noise spheres. System A using symbol by symbol detection is different from Shannon's system S and **can have** a different capacity expression.

- **Sanity Check**

We can do a **Sanity Check** of Eq. 1 with Matlab code. If we set $P = 2 * (\frac{E_b}{N_0})_{min} = 2.56$ in Eq. 1, we get channel spectral efficiency $(\frac{C}{W})_{channel} = 2$ and source spectral efficiency $(\frac{C}{W})_{source} = 1$. We can simulate this OFDM system with $M = 2$ and get a $BER = \frac{d_{min}}{2n'} = 0.055$.

If we drop the signal power $P < 2.56$, simulated BER goes higher than 0.055 and Gallager's LDPC code cannot correct all errors as $n' \rightarrow \infty$.

- Modern communication systems which use (n', k') error correcting codes with large n' and k', such as LDPC codes, **do not compare** received signal with transmitted signal corresponding to $2^{k'}$ possible transmitted codewords, due to computational complexity. They do not use the concept of non-overlapping noise spheres in n-dimensional space, as in System S, as in Shannon's theorem, which states that as the number of samples $n = 2WT \rightarrow \infty$, noise samples will become confined to the surface of noise sphere of radius $\sqrt{2WTN}$, which surrounds each signal point.

Instead they may do one of the following:

- They do **symbol by symbol detection** and produce **hard decision bits** followed by (n', k') error correcting codes, as in System A.

- Or they do **soft decision decoding**(System C) such as Iterative Belief Propagation(BP) algorithm, which gives better performance than System A, and even reach performance of System S, but **do not** compare received signal with transmitted signal corresponding to $2^{k'}$ possible transmitted codewords and does not involve non-overlapping noise spheres in n-dimensional space, as in System S and hence System C is distinct from System S and can have **different** capacity expression from System S.

2. System B: OFDM with LDPC with noise spheres

We can derive the capacity expression for the same OFDM system with LDPC codes, but using Shannon's non-overlapping noise spheres in n-dimensional space.

• We can represent transmitted signal points in a single OFDM block of length $N_f = n = 2WT$ samples by geometric representation in a n-dimensional space. Each transmitted signal point in FFT domain is represented by a n-tuple.

• The spectral efficiency $\frac{C}{W} = 2$ can be achieved by **antipodal signalling** by loading one bit represented by a symbol ± 1 on each frequency domain(FFT) sample, with scaling factor $\sqrt{3}$. The number of symbols per FFT sample is given by $M = 2$. We set average noise power $N = 1$ and hence $C = W \log_2(1 + \frac{P}{N})$.

• We use a (n', k') error correcting code with certain $\frac{d_{min}}{n'}$ where $n' = n$ and we see that Shannon's noise sphere of radius $R_s = \sqrt{nN}$ surrounding each transmitted signal point, starts overlapping with more and more adjacent signal points, as $n \to \infty$. We can calculate the signal power required to keep the $2^{k'}$ noise spheres non-overlapping.

• If we use an **additional** scale factor B for each transmitted signal point, for example, $(1, 1, ...1) * \sqrt{3} * B$, noise spheres surrounding each transmitted signal point **will not** overlap, if the radius of Shannon's Noise sphere is less than half the Euclidean distance($d_{eucl_{min}}$) between transmitted signal points corresponding to 2 valid codewords separated by the minimum distance d_{min} of that (n', k') error correcting code. $d_{eucl_{min}} = 2\sqrt{d_{min}}$.

$$\begin{aligned} R_s = \sqrt{nN} &\leq \frac{d_{eucl_{min}}}{2} \\ \sqrt{nN} &\leq B\sqrt{3}\sqrt{d_{min}} \end{aligned} \tag{4}$$

For Gallager's parity-check codes, $\dfrac{d_{min}}{n'} = 0.11 = K'$ for $\frac{k'}{n'} = \frac{1}{2}$. Figure 2.4 in Page 18 in Gallager's book We assumed average noise power $N = 1$ by convention. For binary signalling $M = 2$ and $n' = n$. Hence

$$\begin{aligned} \sqrt{nN} &\leq B\sqrt{3}\sqrt{nK'} \\ \sqrt{nN} &\leq \sqrt{3B^2 nK'} \end{aligned} \tag{5}$$

This implies that $3B^2 \geq \frac{N}{K'} = \frac{1}{0.11}$ and hence we set $3B^2 = \frac{N}{K'}$ (**Result 3.1**) and $B = \frac{1}{\sqrt{0.33}} = 1.74$ for $N = 1$.

• We can show that the modified channel capacity for an OFDM system with (n', k') error correcting code, is given as follows, using Eq. **??**, where $P = P_0 A^2$ and P_0 is the average signal power of the reference system.

$$(\frac{C}{W})_{sphere} = \log_2(1 + 3P_0) = \log_2(1 + \frac{3P}{A^2}) = \log_2(1 + \frac{3PK'}{N}) = \log_2(1 + \frac{P}{N} * (3K')) \tag{6}$$

where A is the overall scale factor applied to the transmitted signal points. For System B, this would be $A = B\sqrt{3}$ and $A^2 = 3B^2 = \frac{N}{K'}$(using Result 3.1). For Gallager's LDPC code with $\frac{k'}{n'} = 0.5$ and $K' = 0.11$, $\frac{C}{W} = \log_2(1 + \frac{P}{N} * (0.33))$.

• We showed that we require an **additional** scale factor of $B = 1.74$ to be applied to the constellation to keep $2^{k'}$ noise spheres non-overlapping when we use (n', k') error correcting code with $\frac{k'}{n'} = \frac{1}{2}$. Effective **source** capacity is reduced by the factor $\frac{k'}{n'} = \frac{1}{2}$. This implies an **additional** average signal power of $B^2 = 3.03$ is required, compared to Shannon's expression, to achieve a **channel** $\frac{C}{W} = 2$ and **source** $\frac{C}{W} = 1$.

3. System A: Capacity expression for M-ary signalling system with symbol by symbol detection followed by an error correction code

Modern OFDM systems with LDPC codes **do not** typically compare received signal with $2^{k'}$ transmitted signals(soft decision decoding) and hence do not use the concept of non-overlapping noise spheres. Instead, they perform symbol by symbol detection, which involves decoding the I-Q symbol in each complex DFT coefficient to obtain the bits, followed by LDPC decoding performed on hard decision bits. We will derive the **Modified Limit** for this system with **symbol by symbol** detection.[In this section, we will decode I and Q separately to obtain bits.]

Step 1

Let us start with a **Reference System** with transmitted symbols per sample as follows: $\pm 1, \pm 3, \pm 5, ... \pm (M-1)$ where M is the number of symbols per sample. The symbols in the Reference System are spaced by 2 units. [We will apply scale factor A later to these symbols, for a given noise power N, to achieve a specified Signal to Noise Ratio(SNR).] This Reference System is plotted in the **Figure**.[Click here for Figure]

The power P_0 of this Reference System is given by

$$P_0 = \frac{2}{M}[1^2 + 3^2 + 5^2 + ... + (M-1)^2] = \frac{2}{M} \sum_{r=0}^{(\frac{M}{2}-1)} (2r+1)^2 \tag{7}$$

- This **Reference system** with M-ary signalling, has a constellation with M symbols per FFT sample.

The number of bits/symbol $= \log_2(M)$.
Each symbol is loaded on a FFT sample in OFDM system and the total number of samples per block of duration T seconds $= n = 2WT$ samples.
The total number of bits per block $= n \log_2(M)$.
The total number of bits per second $C = \frac{n}{T} \log_2(M)$.

Spectral Efficiency of this M-ary signalling OFDM system is given by
$(\frac{C}{W})_{OFDM} = (\frac{C}{W})_{ref} = \frac{n}{WT} \log_2(M) = 2 \log_2(M)$.

Step 2

We can show that spectral efficiency of this Reference system is given by $(\frac{C}{W})_{ref} = 2\log_2(M) = \log_2(1 + 3P_0)$. First we show that $M^2 = 1 + 3P_0$ using Eq. 7.

$$\begin{aligned}
P_0 = \frac{2}{M}[1^2 + 3^2 + ...(M-1)^2] &= \frac{2}{M}[(1^2 + 2^2 + 3^2 + ...(M)^2) - (2^2 + 4^2 + ...(M)^2)] \\
&= \frac{2}{M}[(1^2 + 2^2 + 3^2 + ...(M)^2) - 2^2(1^2 + 2^2 + ...(\frac{M}{2})^2)] \\
&= \frac{2}{M}[(\frac{M*(M+1)*(2M+1)}{6}) - 4(\frac{\frac{M}{2}*(\frac{M}{2}+1)*(M+1)}{6})] \\
&= \frac{2}{M}[(\frac{M*(M+1)}{6})*(2M+1-2*(\frac{M}{2}+1))] \\
P_0 = \frac{2}{M}[(\frac{M*(M+1)}{6})*(M-1)] = \frac{2}{M}(\frac{M*(M^2-1)}{6}) &= (\frac{(M^2-1)}{3}) \\
1 + 3P_0 &= M^2 \\
(\frac{C}{W})_{ref} = 2\log_2(M) &= \log_2(1+3P_0)
\end{aligned}$$

(8)

We can rewrite the Spectral Efficiency of this Reference system as follows.

$$(\frac{C}{W})_{ref} = \log_2(1 + 3P_0)$$

(9)

Step 3: Let us apply the scale factor A to the reference system, to achieve specified SNR for the actual system whose average signal power is given by $P = P_0A^2$.

We can rewrite the spectral efficiency of the actual system which uses symbol by symbol detection as follows.

$$(\frac{C}{W})_{sym} = \log_2(1 + 3P_0) = \log_2(1 + \frac{3P}{A^2})$$

(10)

Step 4: **M=2**: We want to find an expression for constellation scale factor A in terms of noise power N such that, as number of dimensions $n \to \infty$, Bit Error Rate(BER) $\to 0$ and above spectral efficiency can be achieved.

For binary signalling $M = 2$, each FFT sample is loaded with one bit, with symbols ± 1, we know that the Raw BER of this binary signalling system is given by the well known expression

$$BER = P_e = \frac{1}{2}erfc(\sqrt{\frac{E_b}{N_0}}) = \frac{1}{2}erfc(\sqrt{\frac{E_{b(BPSK)}}{N_0}})$$
$$BER = \frac{1}{2}erfc(\sqrt{\frac{P_{BPSK}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}})$$

(11)

where $E_b = E_{b(BPSK)}$ is the Energy Per Bit for a Binary Signalling System and N_0 is the Noise Spectral Density and $erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt$ is the complementary error function. We use $P_{BPSK} = \frac{E_b}{T_b}$, $N = N_0W$ and $WT_b = \frac{1}{2}$ using Nyquist sampling.

Step 5: If we use **Gallager's LDPC codes** (n', k'), for an example code rate $R = \frac{k'}{n'} = \frac{1}{2}$, minimum distance ratio $\delta_0 = \frac{d_{min}}{n'} = 0.11$ as $n' \to \infty$. Given that $d_{min} = 2t + 1$, where t is the number of corrected bit errors, error correction rate ECR is given by $\lim_{n'\to\infty} \frac{t}{n'} = \frac{d_{min}}{2n'} - \frac{1}{n'} = \lim_{n'\to\infty} \frac{d_{min}}{2n'} = 0.055$. Figure 2.4 in Page 18 in Gallager's book

We require the **raw BER** of our OFDM system, which uses this Gallager LDPC codes, to be **less than** error correction rate **ECR**. That would ensure that as $n' \to \infty$, $BER \to 0$.

$$P_e = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}}) \leq \lim_{n'\to\infty} \frac{d_{min}}{2n'}$$

(12)

The minimum value of $\frac{E_b}{N_0}$ which satisfied above inequality is given by $(\frac{E_b}{N_0})_{min} = \frac{A^2}{2N}$ and is a function of M and the $\frac{d_{min}}{n'}$ of the LDPC code. Now we can rewrite Eq. 10 as follows, where $K = \frac{3}{2(\frac{E_b}{N_0})_{min}}$.

$$(\frac{C}{W})_{sym} = \log_2(1 + \frac{3P}{A^2}) = \log_2(1 + \frac{3P}{2N(\frac{E_b}{N_0})_{min}})$$
$$(\frac{C}{W})_{sym} = \log_2(1 + \frac{P}{N} * K) \tag{13}$$

• For Gallager's LDPC code with $\frac{d_{min}}{n'} = 0.11$ and $\frac{k'}{n'} = 0.5$, $(\frac{E_b}{N_0})_{min} = 1.28$ and $K = 1.1719$, using Eq. 11 and Eq. 12.

• Note that the above capacity expression $(\frac{C}{W})_{sym}$ for M-ary signalling system corresponds to the Channel Spectral Efficiency. Because we use a (n', k') error correction code with a code rate of $R = \frac{k'}{n'}$, we can express the Source Spectral Efficiency as follows.

$$(\frac{C}{W})_{sym-channel} = \log_2(1 + \frac{P}{N} * K)$$
$$(\frac{C}{W})_{sym-source} = \frac{k'}{n'} \log_2(1 + \frac{P}{N} * K) \tag{14}$$

• **M=4**:

Each FFT sample in OFDM block has 4 symbols $\pm 1, \pm 3$. See Appendix C.1. The expression for BER for general case of M is derived in Appendix C. For each case of M, we can deduce the value of A which satisfies the expression $BER = 0.055$ in Eq. 10 (If we use Gallager's LDPC codes with $\frac{k'}{n'} = \frac{1}{2}$).

4. Comparison of System A and System B

Channel capacity of System A and System B are as follows, copied from Eq. 14 and Eq. 6.

$$\textbf{System A:}(\frac{C}{W})_{sym} = \log_2(1 + \frac{P}{N} * 1.1719)$$
$$\textbf{System B:}(\frac{C}{W})_{sphere} = \log_2(1 + \frac{P}{N} * 0.33)) \tag{15}$$

We can compare the expressions for **source** spectral efficiency for OFDM (adjusting for $\frac{k}{n} = \frac{1}{2}$) with the spectral efficiency computed by Shannon's original expression in Eq. 3 .

• Table below lists $\frac{E_b}{N_0}$ Vs $\frac{C}{W}$. For Shannon's expression, source $\frac{C}{W}$ and channel $\frac{C}{W}$ are the same. For our OFDM expression, they are different and this explains the first 2 columns.

• we can see that, in order to achieve the same **source** capacity per unit bandwidth, we need **higher** $\frac{E_b}{N_0}$ with our expression for OFDM, compared to Shannon's expression, for the specific case of Gallager's LDPC codes with $\frac{d_{min}}{n} = 0.11$ for $\frac{k}{n} = \frac{1}{2}$.

• The fourth column corresponds to $\frac{E_b}{N_0}$ obtained for OFDM with Gallager's parity-check codes, with symbol by symbol detection as in Eq. 15 $[(\frac{C}{W})_{OFDM_{SYM}(source)} = \frac{k}{n} \log_2(1 + \frac{3P}{2N*(\frac{E_b}{N_0})_{min}})]$ and adjusts $\frac{E_b}{N_0} = \frac{P}{N} \frac{1}{\frac{C}{W} \frac{k}{n}}$ for code rate $\frac{k}{n} = \frac{1}{2}$ and corresponds to source bit rate per unit bandwidth.

• The last column corresponds to $\frac{E_b}{N_0}$ obtained for OFDM with Gallager's parity-check codes, with detection based on noise spheres, as per Eq. 15 [For $M = 2$, $(\frac{C}{W})_{OFDM_{SPHERE}(source)} = log_2(1 + \frac{3P*0.11}{N}) * \frac{k'}{n'}$]. (plot)

$Channel\frac{C}{W}$	$Source\frac{C}{W}$	$\frac{E_b}{N_0}_{(shannon)}$ in dB	$\frac{E_b}{N_0}_{(OFDM_{SYM})}$ (source)	$\frac{E_b}{N_0}_{(OFDM_{SPHERE})}$ (source)
2	1	0	4.08	9.5861
4	2	1.77	7.19	13.576

• It is clear from above table that Symbol by Symbol detection of OFDM system with binary signalling $M = 2$ and Gallager's parity-check codes $\frac{d_{min}}{n'} = 0.11$, **performs better** than a corresponding system which uses soft decision detection based on noise spheres in n-dimensions(choose the nearest neighbour to a received signal point in n-dimensional Euclidean space). The two detection systems are **NOT** equivalent.

• In Section D, we will show that, with modern LDPC codes with higher $\frac{d_{min}}{n}$ or $\frac{d_{min_{average}}}{n}$ where $d_{min_{average}}$ is the average minimum distance between codewords, we can get higher $\frac{C}{W}$ with lower $\frac{E_b}{N_0}$ than the fourth column.

5. Section D

• Modern LDPC codes may have some codewords which are closer in terms of number of bits they differ, but a much higher average minimum distance $d_{min_{average}}$ between codewords, hence we will consider $\frac{d_{min_{average}}}{n'}$. Average minimum distance is given by $d_{min_{average}} = \frac{1}{2^{k'}} \sum_{m=1}^{2^{k'}} d_{min}(m)$ where $d_{min}(m)$ is the minimum distance between codeword m and all other codewords, minimum number of bits by which codeword m differs from other codewords.

For example, consider a code with $n' = 4, k' = 2$ which has $2^{k'} = 4$ codewords. Let us choose the codewords as $(1, 1, 1, 1), (1, 1, -1, 1), (-1, -1, -1, -1), (-1, -1, 1, 1))$. The first 2 codewords have a $d_{min_1} = 1$ but the rest of 2 codewords have a $d_{min_2} = 2$, hence the average inter-codeword minimum distance $d_{min_{average}}$ is closer to 1.5.

$$d_{min_{average}} = \frac{1}{4} * 1 + \frac{1}{4} * 1 + \frac{1}{4} * 2 + \frac{1}{4} * 2 = 1.5 \tag{16}$$

• Radford Neal's (10000,5000) LDPC implementation in C language[4] produces $BER = 0$ for $\frac{E_b}{N_0} = 1.41$ dB (signal amplitude $A = 1$, noise standard deviation $\sqrt{N} = 0.85$) while it produces $BER = 2.438e - 02$ for $\frac{E_b}{N_0} = 0.92$ dB $((A = 1, \sqrt{N} = 0.9))$. [LDPC implementation].

We can use this code with OFDM with $M = 2$ signalling and $\frac{C}{W} = 2$. BER is given by $BER = 0.5 * erfc(\sqrt{\frac{A^2}{2N}}) = 0.1197$ [Note D.1]. We require this $BER < \frac{t}{n'}$ of the code where $\frac{t}{n'}$ is the **average** error correction rate.

This implies that this code has a $\frac{d_{min_{average}}}{n'} = \frac{(2t+1)}{n'} = 0.24$, when $n' = 10000; k' = 5000$, where $d_{min_{average}} = 2t+1$ is the average minimum distance between codewords.

• We can compute the $\frac{d_{min_{average}}}{n'}$ required for a future error correcting code which will produce the **same** $(\frac{C}{W})_{OFDM_{SYM}(source)} = \frac{k}{n} \log_2(1 + \frac{3P}{2N(\frac{E_b}{N_0})_{min}})$ for $\frac{k}{n} = \frac{1}{2}$, as that of $(\frac{C}{W})_{shannon} = \log_2(1 + \frac{P}{N})$, for the same source $\frac{C}{W} = 1$ and same $P = 1$. We require $(\frac{E_b}{N_0})_{min} = 0.5$ and BER is given by $BER = 0.5 * erfc(\sqrt{(\frac{E_b}{N_0})_{min}}) = 0.15866$.

We require this BER to be less than error correction rate $\frac{t}{n'} = \frac{(d_{min_{average}} - 1)}{2n'}$ of the future error correcting code as $n \to \infty$. We require $\frac{d_{min_{average}}}{n'} \geq 0.15866 * 2 = 0.31731$ for the future error correcting code with $\frac{k}{n} = \frac{1}{2}$, which is combined with an OFDM system which uses symbol by symbol detection [System 1], which produces the same source $\frac{C}{W} = 1$ and same average signal power $P = 1$ as compared to Shannon's $\frac{C}{W}$ expression which uses non-overlapping noise spheres in n-dimensional space for detection. [System 2].

- We have already shown that the 2 systems are different and **can produce different results** because they use **different methods of detection**. It may be possible for our System 1 which uses symbol by symbol detection, when combined with a future error correcting code with $0.31731 < \frac{d_{min_{average}}}{n'} < 1$ and with $\frac{k}{n} = \frac{1}{2}$, to exceed the performance of Shannon's expression for $\frac{C}{W} = 1$. This **will not violate** Shannon's Capacity expression because Shannon's Capacity expression is valid for a receiver which uses soft decision detection in n-dimensions using non-overlapping noise spheres and is not valid for our System which uses symbol by symbol detection, followed by error correction decoding on hard bits. This is represented in the Figure below.

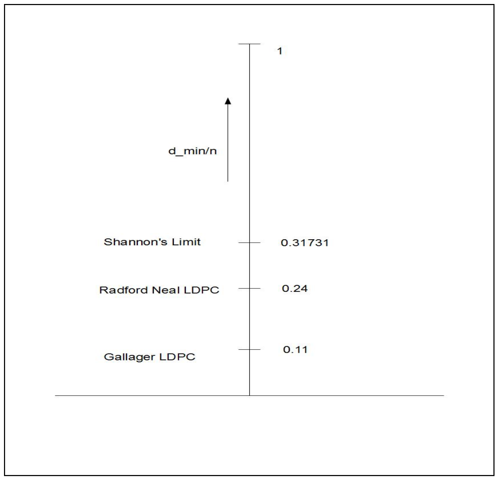

Figure 1:

- **Note D.1**

In numerical simulations in Matlab or C, when we generate a Gaussian Random Variable(RV) of standard deviation σ to simulate zero mean noise[Matlab: w_m=randn(1,N_t)*σ], the average power of noise $w[m]$ is given by $N = \frac{1}{N_t}\sum_{m=1}^{N_t} w[m]^2$ and the standard deviation of noise $w[m]$ is given by $\sigma = \sqrt{\frac{1}{N_t}\sum_{m=1}^{N_t} w[m]^2}$. It is clear that average noise power N always equals square of standard deviation of noise $N = \sigma^2$ in numerical simulations. We know that $N = N_0 W; \sigma^2 = \frac{N_0}{2}$. This implies that $N_0 W = \frac{N_0}{2}$, which implies that $W = \frac{f_s}{2} = \frac{1}{2}$ and $f_s = 1$ in numerical simulations, where W is the signal bandwidth and $f_s = 2W$ is the sampling frequency.

6. Appendix C

Let us derive the expression for Bit Error Rate(BER) for M-ary PAM system used in each FFT sample in OFDM. It is plotted in the Figure below.

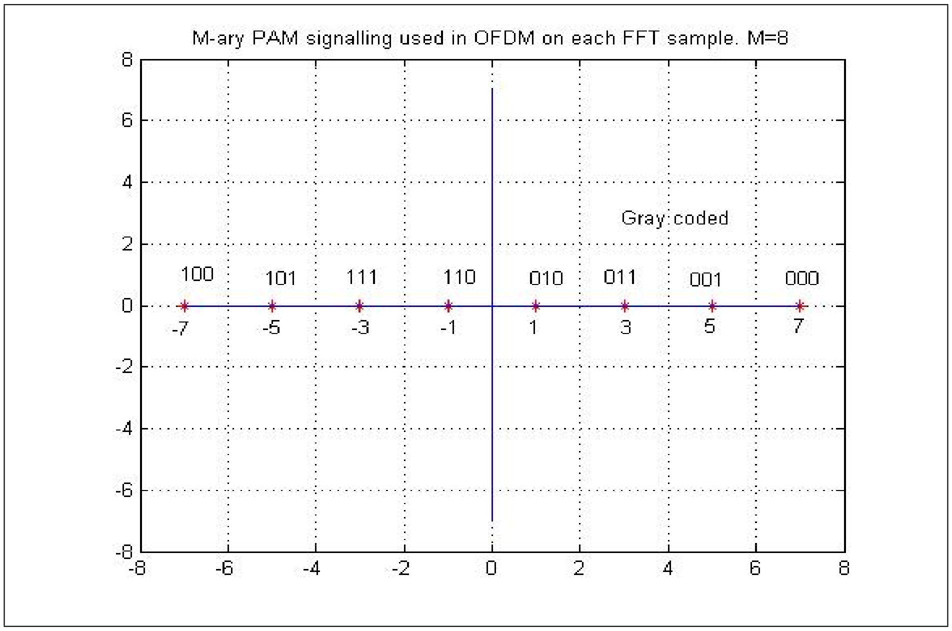

Figure 2:

This reference constellation is scaled by a factor A such that average signal power $P = P_0 A^2$ where P_0 is the average power of the reference constellation. A is chosen to achieve a desired Signal to Noise Ratio(SNR) which yields a desired BER.

Let us start with the outermost symbol numbered 1 to the right, which has a value of $+7A$. Given that the symbol error rate for this symbol is given by the probability that noise takes a value larger than half the distance between adjacent symbols, we can write $P_{ser} = \frac{1}{2}erfc(\sqrt{\frac{E_b}{N_0}}) = \frac{1}{2}erfc(\sqrt{\frac{E_{bpsk}}{N_0}}) = \frac{1}{2}erfc(\sqrt{\frac{P_{bpsk}}{N}WT_b}) = \frac{1}{2}erfc(\sqrt{\frac{P_{bpsk}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}})$. Note that the distance between adjacent symbols is $2A$ and E_b in above expression corresponds to the Energy Per Bit for BPSK constellation and NOT the overall $E_b$for this M-ary signalling system. We can write the BER contributed by this **outermost symbol** as follows.

$$\begin{aligned}BER_1 = \frac{1}{M}[\sum_{m=0}^{M-3}(\frac{1}{2}erfc((2m+1)\sqrt{\frac{A^2}{2N}}) - \frac{1}{2}erfc((2m+3)\sqrt{\frac{A^2}{2N}})] * d_{bits}(s_1, s_{(m+2)})) \\ +\frac{1}{2}erfc((2M-3)\sqrt{\frac{A^2}{2N}}) * d_{bits}(s_1, s_{(M)})]\end{aligned} \tag{17}$$

where $d_{bits}(s_1, s_{(m+2)})$ is the number of bits which differ in Symbol 1 and Symbol $m+2$.

We can write the BER contributed by this **inner symbols** for symbol number $i = 2, 3, ..M/2$ as follows in terms of 2 terms, first term accounts for symbols to the left of this symbol and second term accounts for symbols to the right of this symbol.

$$\begin{aligned}BER_i = \frac{1}{M}[\sum_{m=0}^{M-1-i}(\frac{1}{2}erfc((2m+1)\sqrt{\frac{A^2}{2N}}) - \frac{1}{2}erfc((2m+3)\sqrt{\frac{A^2}{2N}})] * d_{bits}(s_i, s_{(m+1+i)})) \\ +\frac{1}{2}erfc((2M+1-2i)\sqrt{\frac{A^2}{2N}}) * d_{bits}(s_i, s_{(M)})] \\ +[\sum_{m=0}^{i-3}(\frac{1}{2}erfc((2m+1)\sqrt{\frac{A^2}{2N}}) - \frac{1}{2}erfc((2m+3)\sqrt{\frac{A^2}{2N}})] * d_{bits}(s_i, s_{(i-m-1)})) \\ +\frac{1}{2}erfc((2i-3)\sqrt{\frac{A^2}{2N}}) * d_{bits}(s_i, s_{(1)})]\end{aligned}$$

(18)

For symbols numbered $\frac{M}{2}+1,...M$, the BER is the same as the corresponding symbols $\frac{M}{2}, \frac{M}{2}-1,..1$. Hence the total BER of this M-ary signalling system is given by

$$BER = \frac{1}{log_2(M)}\sum_{i=1}^{M} BER_i \tag{19}$$

- **Appendix C.1**

Let us use Gray Coding for this constellation. $+3$ corresponds to bits 00, $+1$ corresponds to bits 01, -1 corresponds to bits 11, -3 corresponds to bits 10.

The Raw BER of this 4-ary signalling system can be expressed as a weighted sum of BER expression for binary signalling case $M = 2$ as follows[as per Appendix C].

$$\begin{aligned}
P_{e_{(M=4)}} &= \frac{1}{2}[0.5*[P_0+(P_0-P_1)+2*P_1]+0.5*[(P_0-P_1)+2(P_1-P_2)+P_2]] \\
P_0 &= \frac{1}{2}erfc(\sqrt{\frac{E_{b_{(BPSK)}}}{N_0}}) = \frac{1}{2}erfc(\sqrt{\frac{P_{BPSK}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}}) \\
P_1 &= \frac{1}{2}erfc(3\sqrt{\frac{E_{b_{(BPSK)}}}{N_0}}) = \frac{1}{2}erfc(3\sqrt{\frac{P_{BPSK}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}}) \\
P_2 &= \frac{1}{2}erfc(5\sqrt{\frac{E_{b_{(BPSK)}}}{N_0}}) = \frac{1}{2}erfc(5\sqrt{\frac{P_{BPSK}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}}) \\
P_3 &= \frac{1}{2}erfc(7\sqrt{\frac{E_{b_{(BPSK)}}}{N_0}}) = \frac{1}{2}erfc(7\sqrt{\frac{P_{BPSK}}{2N}}) = \frac{1}{2}erfc(\sqrt{\frac{A^2}{2N}})
\end{aligned} \tag{20}$$

where $E_{b_{(BPSK)}}$ is the Energy Per Bit for a Binary Signalling System and N_0 is the Noise Spectral Density and the average signal power for Binary signalling system is given by $P_{BPSK} = A^2$ and average noise power is given by $N = N_0W$ and $WT_b = \frac{1}{2}$.

We require this Raw BER $P_{e_{(M=4)}}$ to be less than Error Correction Rate(ECR) of the associated Gallager's LDPC code.
$P_{e_{(M=4)}} \leq \lim_{n'\to\infty} \frac{d_{min}}{2n'}$.

The minimum value of $\frac{E_b}{N_0}$ which satisfied above equality is given by $(\frac{E_b}{N_0})_{min} = \frac{A^2}{2N}$.

For $\frac{d_{min}}{2n'} = 0.055$ and code rate $\frac{k}{n} = 0.5$, $(\frac{E_b}{N_0})_{min}$ is around 1.05 for $M = 4$.

$(\frac{C}{W})_{OFDM} = \log_2(1+\frac{3P}{2N*1.05}) = \log_2(1+\frac{P*1.4285}{N})$.

This seems to perform better than Shannon's Capacity Limit which uses n dimensional noise spheres, $\frac{C}{W} = log_2(1+\frac{P}{N})$, but we should remember that $\frac{C}{W}$ is the source spectral efficiency while $(\frac{C}{W})_{OFDM}$ is the channel spectral efficiency and the **source spectral efficiency** is given by $(\frac{C}{W})_{OFDM(source)} = \frac{k}{n}\log_2(1+\frac{P*1.4285}{N})$.