1. Riemann Zeta Function and two-sided decaying exponential functions. (Akhila Raman)

- **Riemann's Zeta** function is given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$. Its analytic continuation to the whole s-plane is derived from **Riemann's Xi** Function $\xi(s)$ which is evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $\Phi(t) = E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real. (Brian Conrey's 2003 article). (Derived in link)

$$\Phi(t) = E_0(t) = E_0(-t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{1}$$

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- The **Inverse Fourier Transform** of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We will show that $E_p(t) = E_0(t)e^{-\sigma t}$ and $0 \leq |\sigma| < \frac{1}{2}$ corresponds to the critical strip. We can write $E_p(t)$ as an **infinite summation** of **two-sided decaying exponential** functions using Taylor series. (Details in link)

$$E_p(t) = E_0(t)e^{-\sigma t}, \quad E_0(t) = E_0(-t) = \sum_{n,k,r,p} c_{nkrp}e^{b_{krp}t}$$

$$E_p(t) = [\sum_{n,k,r,p} c_{nkrp}e^{b_{krp}t}u(-t) + \sum_{n,k,r,p} c_{nkrp}e^{-b_{krp}t}u(t)]e^{-\sigma t}$$

$$b_{krp} = (2k + \frac{5}{2} + 2r - 2p), \quad \sum_{n,k,r,p} c_{nkrp} = \sum_{r=0}^{1}\sum_{n=1}^{\infty} e_{nr}\sum_{k=0}^{\infty} d_{nk}\sum_{p=0}^{k}\binom{k}{p}(-1)^p$$

$$e_{n1} = a_n, \quad e_{n0} = -b_n, \quad a_n = 4\pi^2 n^4 e^{-\pi n^2}, \quad b_n = 6\pi n^2 e^{-\pi n^2}, \quad d_{nk} = \frac{(-\pi n^2)^k}{!(k)} \tag{2}$$

- We know that a real **two-sided decaying exponential function** $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, where $u(t)$ is Heaviside unit step function and $a, b > 0$ are real, has Fourier Transform given by $G_0(\omega)$ as follows. (Page 6)

$$G_0(\omega) = \int_{-\infty}^{\infty} g_0(t)e^{-i\omega t}dt = [\frac{b}{b^2 + \omega^2} + \frac{a}{a^2 + \omega^2}] + i\omega[\frac{1}{b^2 + \omega^2} - \frac{1}{a^2 + \omega^2}] \tag{3}$$

We can see that the real part of $G_0(\omega)$ given by $\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}$ **does not have zeros** for any finite and real value of ω and hence $G_0(\omega)$ does not have zeros for any finite real value of ω.

Given that $E_p(t)$ is expressed as an **infinite summation of two-sided decaying exponential functions**, we could investigate if its Fourier transform $E_{p\omega}(\omega)$ also does not have zeros for any finite real value of ω.

• **Step 4: Statement 1**: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider **a toy example** with a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We will show that the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t)+g(-t)]$ given by $G_{even}(\omega) = G_R(\omega)$ must have **at least one zero** at $\omega=\omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite. (link)

As an **example**, consider $E_p(t) = e^{bt}u(t) + e^{-at}u(-t)$ where $a,b>\sigma>0$ are real and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$. We see that $g(t) = e^{(b-\sigma)t}u(-t) + e^{-(a-\sigma)t}u(t)$. The real part of Fourier transform of $g(t)$ is given by $G_R(\omega) = \frac{(b-\sigma)}{(b-\sigma)^2+\omega^2} + \frac{(a-\sigma)}{(a-\sigma)^2+\omega^2}$ **does not** have any zeros for real and finite ω. The Fourier transform of $h(t)$ is given by $H(\omega) = \frac{2\sigma}{(\sigma)^2+\omega^2}$ also **does not** have any zeros for real and finite ω.

Because $g(t)h(t) = E_p(t)$ corresponds to **convolution** of the respective Fourier transforms (plot), therefore real part of Fourier transform of $E_p(t)$ given by $Re[E_{p\omega}(\omega)]$ **cannot** have zeros for real and finite ω, which **contradicts** Statement 1. Therefore $G_R(\omega)$ must have **at least one zero** at $\omega=\omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite.

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• **Step 4.1:** Similarly, in Section 2.1 (link), we consider a **modified even symmetric** function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{3\sigma t}u(t)$ for $|t_0| \leq \infty$ where $f(t) = e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$ where $g(t)h(t) = f(t)$ and show that Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$, for **every value** of t_0, to satisfy Statement 1, where $\omega_2(t_0)$ is real and finite. (link)

If there is more than one solution for $\omega_2(t_0)$, these different solutions can remain distinct. This is shown by an example video simulation in link. It is shown that $\omega_2(t_0)$ is a well defined continuous function, which is **at least** differentiable twice. (link)

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• **Step 5:** In Section 2.1 (link), we compute the fourier transform of the even function $g(t)$ given by $G_R(\omega)$. We require $G_R(\omega)=0$ for $\omega=\omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**.

It is shown that $R(t_0) = G_R(\omega_2(t_0), t_0)$ is an **odd** function of variable t_0 as follows.

$$R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \tag{4}$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r,p} c_{nkrp}e^{b_{krp}t}e^{-\sigma t}$, we use the fact that $E_0(t) = E_0(-t)$, we can write as follows.

$$R(t_0) = \sum_{n,k,r,p} c_{nkrp} \frac{(b_{krp} - 2\sigma)e^{(b_{krp})t_0}}{(\omega_2^2(t_0) + (b_{krp} - 2\sigma)^2)} \tag{5}$$

We see that there is a **one to one correspondence** between the integral representation in Eq. 4 and Taylor series representation in Eq. 5. Given that it is easier to show integral convergence, we use only the integral representation in subsequent steps.

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- **Step 6:** In Section 2.2 (link), we derive the first 2 derivatives of $R(t_0)$ at $t_0 = 0$ as follows, where $e_0 = E_0(0), \omega_{20} = [\omega_2(t_0)]_{t_0=0}$. $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$, $m_2 = -\omega_{22}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$.

$$\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= e_0 + n_0\omega_{20} + 2\sigma m_0 \\
(\frac{d^2R(t_0)}{dt_0^2})_{t_0=0} &= m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - m_0\frac{\omega_{20}^2}{2}
\end{aligned} \tag{6}$$

Given that $R(t_0) = G_R(\omega_2(t_0), t_0)$ is an **odd** function of variable t_0, we get $m_0 = 0$ and $m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} = 0$.

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- **Step 7** In Section 2.3 (link), we replace $E_p(t)$ by $E_{pp}(t) = e^{\sigma t_2}E_p(t + t_2) + e^{-\sigma t_2}E_p(t - t_2)$, for $|t_2| \leq \infty$ and derive as follows.

$$\begin{aligned}
m_0^{'}(t_2) &= R'(t_2) + R'(-t_2) = 0 \\
R'(t_2) &= e^{2\sigma t_2}[\cos(\omega_{20}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}(t_2)\tau)d\tau + \sin(\omega_{20}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}(t_2)\tau)d\tau] \\
A(t_2) &= m_2^{'}(t_2) + \sigma e_0^{'}(t_2) + 2\sigma n_0^{'}(t_2)\omega_2(t_2) = 0 \\
e_0^{'}(t_2) &= E_0(t_2) + E_0(-t_2) \\
n_0^{'}(t_2) &= n_{0p}(t_2) + n_{0p}(-t_2) \\
n_{0p}(t_2) &= e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
m_2^{'}(t_2) &= m_{2p}(t_2) + m_{2p}(-t_2) \\
m_{2p}(t_2) &= -\frac{1}{2}\frac{d^2\omega_2(t_2)}{dt_2^2}e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau \\
&\quad - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau]
\end{aligned} \tag{7}$$

• **Step 8:** In Section 2.4 (link), we consider the asymptotic case and show that $\lim_{t_2\to\infty}\omega_2(t_2) = \omega_z$ (link) and derive as follows.

$$\begin{aligned}
\lim_{t_2\to\infty} A(t_2) = \lim_{t_2\to\infty} 2\sigma\omega_z n_0^{'}(t_2) = 0 \\
\lim_{t_2\to\infty} n_0^{'}(t_2) = 0 \\
\lim_{t_2\to\infty} m_0^{'}(t_2) = 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{i(\omega_z t)}dt = 0
\end{aligned} \tag{8}$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We repeat above steps N times till $2^N\sigma > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-2^N\sigma\tau}e^{-i\omega_{zN}\tau}d\tau = 0$. In each iteration n, we use $h(t) = e^{2^n\sigma t}u(-t) + e^{-3*2^n\sigma t}u(t)$. We know that the Fourier Transform of $E_0(t)e^{-2^N\sigma t}$ **does not** have a real zero for $2^N\sigma > \frac{1}{2}$, corresponding to $Re[s] > 1$ and we show a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$.