# On the Zeros of Riemann's Zeta Function

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**Abstract**

Some ideas on Riemann's Zeta Function are derived.

*Keywords:*

### *0.1.* ***Introduction***

Let us start with this analytic continuation of Riemann's Zeta Function $\xi(\frac{1}{2}+i\omega) = E_0(\omega)$. Its Inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_0(\omega)e^{i\omega t}d\omega$

$$E_0(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{1}$$

We know that $E_0(t) = E_0(-t)$ is an even function of $t$. This is shown in Appendix D. Let us consider the Fourier Transform of $\xi(\frac{1}{2}-\sigma+i\omega)$, where $0 < \sigma < \frac{1}{2}$. It is given by

$$E_p(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} = E_0(t)e^{-\sigma t} \tag{2}$$

**Step 1:**

Let us use the Taylor series expansion of $E_p(t) = [\sum_{n,k}(a_{nk}e^{(2k+\frac{9}{2})t} - b_{nk}e^{(2k+\frac{5}{2})t})]e^{-\sigma t}$ and use the short-hand notation $\sum_{n,k,r} c_{nkr}e^{b_{kr}t}e^{-\sigma t}$ where a new symbol $\sum_{n,k,r} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{1}$, where $b_{kr} = (2k+\frac{5}{2}+2r), c_{nk0} = a_{nk}, c_{nk1} = -b_{nk}$, $a_{nk} = 2\pi^2 n^4 d_{nk}; b_{nk} = 3\pi n^2 d_{nk}$ and $d_{nk} = \frac{(-\pi n^2)^k}{!(k)}$. [**In the Section 2**, we will show similar results for a general $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_p(\omega)e^{i\omega t}d\omega$, without using Taylor series expansion.]

**Step 2**

Lets us assume that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$, given by $E_p(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, has a zero at $\omega = \omega_0$. Let us call it **Assumption 1**. We will prove that this assumption leads to a contradiction for $\sigma \neq 0$.

We know that a two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, where $u(t)$ is Heaviside unit step function and it has the Fourier Transform given by $G_0(\omega) = \int_{-\infty}^{\infty} g_0(t)e^{-i\omega t}dt = \frac{1}{b+i\omega} + \frac{1}{a-i\omega} = \frac{b-i\omega}{b^2+\omega^2} + \frac{a+i\omega}{a^2+\omega^2} = [\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}] + i\omega[\frac{1}{a^2+\omega^2} - \frac{1}{b^2+\omega^2}]$. We can see that the real part of $G_0(\omega)$ given by $\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}$ **does not have zeros** for any real $\omega$.

Given that the Inverse Fourier Transform of Riemann Zeta function $E_p(t)$ is expressed as an **infinite summation of two-sided decaying exponential functions** in Step 1, we will show that $E_p(t)$ does not have real zeros in its Fourier Transform.

## 1. Section 1

**Theorem 1**: Riemann's Zeta Function $\xi(\frac{1}{2} - \sigma + i\omega) = E_p(\omega)$ does not have zeros for any real value of $-\infty \le \omega \le \infty$, for $\sigma \ne 0$, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_p(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ , $E_0(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$, given that **given that** the Fourier Transform of $E_0(t)$ given by $\xi(\frac{1}{2} + i\omega)$, has a known real zero at some $\omega = \omega_z$ **and if** the Fourier Transform of $E_0(t)e^{-(2^N\sigma)t}$ is known to NOT have a real zero for $(2^N\sigma) > K_0$ where $K_0$ is some finite constant and $N$ is a positive integer.

**Proof**: The proof of this theorem is shown in subsections below. We will prove it for $\sigma > 0$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $\sigma \ne 0$.

## 2. Section 1.1 Two-sided function

Let us form a new function $f(t) = e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)$ where $t_0$ is a constant delay for $-\infty \le t_0 \le \infty$. Given that the Fourier Transform of $E_p(t)$ has a zero at $\omega = \omega_0$, we can see that the Fourier Transform of this new function $f(t)$ also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t)+g_+(t)u(t)$ where $g(t)$ is a real and asymmetric function of variable $t$ and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$ . We can see that $g(t)[e^{\sigma t}u(-t) + e^{-\sigma t}u(t)] = f(t)$.

As shown in Appendix A.1, we take the fourier transform of $g(t)$ and get $G(\omega)$ as follows.

$$\begin{aligned} G(\omega) = e^{-\sigma t_0}e^{-i\omega t_0}e^{-\sigma t_0}\int_{-\infty}^{-t_0} E_p(\tau)e^{-\sigma\tau}e^{-i\omega\tau}d\tau + e^{\sigma t_0}e^{i\omega t_0}e^{\sigma t_0}\int_{-\infty}^{t_0} E_p(\tau)e^{-\sigma\tau}e^{-i\omega\tau}d\tau \\ +e^{-\sigma t_0}e^{-i\omega t_0}e^{\sigma t_0}\int_{-\infty}^{t_0} E_q(\tau)e^{-\sigma\tau}e^{i\omega\tau}d\tau + e^{\sigma t_0}e^{i\omega t_0}e^{-\sigma t_0}\int_{-\infty}^{-t_0} E_q(\tau)e^{-\sigma\tau}e^{i\omega\tau}d\tau \\ G(\omega) = G_R(\omega) + iG_I(\omega) \end{aligned} \tag{3}$$

We wish to compute the fourier transform of the function $g_{even}(t) = g(t) + g(-t)$ given by $G_{even}(\omega) = G_R(\omega)$. We require $G_{even}(\omega) = 0$ for $\omega = \omega_0(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**.

Comparing the **real parts** of $G(\omega)$, given $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_0(-t)e^{\sigma t}$ and we require $G_R(\omega) = 0$ for $\omega = \omega_0(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**, we have

$$\begin{aligned} G_R(\omega) = G_1(\omega, t_0) + G_1(\omega, -t_0) \\ G_1(\omega, t_0) = e^{2\sigma t_0}[\cos(\omega t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega\tau)d\tau + \sin(\omega t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega\tau)d\tau] \\ +[\cos(\omega t_0)\int_{-\infty}^{t_0} E_0(-\tau)\cos(\omega\tau)d\tau + \sin(\omega t_0)\int_{-\infty}^{t_0} E_0(-\tau)\sin(\omega\tau)d\tau] \end{aligned}$$

$$(4)$$

Given that $E_p(t) = E_0(t)e^{-\sigma t}, E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t}$, Hence we can see that $P(t_0) = G_1(\omega_0(t_0), t_0)$ is an odd function of variable $t_0$.

$$G_R(\omega_0(t_0), t_0) = G_1(\omega_0(t_0), t_0) + G_1(\omega_0(t_0), -t_0) = 0$$
$$P(t_0) = G_1(\omega_0(t_0), t_0)$$
$$P(t_0) = e^{2\sigma t_0}[\cos(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_0(t_0)\tau)d\tau + \sin(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_0(t_0)\tau)d\tau]$$
$$+[\cos(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_0(-\tau)\cos(\omega_0(t_0)\tau)d\tau + \sin(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_0(-\tau)\sin(\omega_0(t_0)\tau)d\tau] \tag{5}$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r} c_{nkr}e^{b_{kr}t}e^{-\sigma t}$, we use the fact that $E_0(t) = E_0(-t)$, we can write as follows.

$$P(t_0) = \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)\frac{e^{(b_{kr})t_0}}{(\omega_0^2(t_0) + (b_{kr} - 2\sigma)^2)} + (b_{kr})\frac{e^{b_{kr}t_0}}{(\omega_0^2(t_0) + b_{kr}^2)}] \tag{6}$$

Now we evaluate $P(t_0)$ at $t_0 = 0$. Given that $P(t_0)$ is an **odd function** of variable $t_0$ , we can equate it to zero, $P(t_0)$ evaluated at $t_0 = 0$ as follows.

We define $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{00}\tau)d\tau$, $m_{0p} = \int_{-\infty}^{0} E_0(-\tau)\cos(\omega_{00}\tau)d\tau$.

$$[P(t_0)]_{t_0=0} = m_0 + m_{0p} = 0$$
$$m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{00}\tau)d\tau$$
$$m_{0p} = \int_{-\infty}^{0} E_0(-\tau)\cos(\omega_{00}\tau)d\tau \tag{7}$$

If we replace $E_p(t)$ in above section by $E_{pp}(t) = e^{\sigma t_2}E_p(t + t_2) + e^{-\sigma t_2}E_p(t - t_2) = [E_0(t + t_2) + E_0(t - t_2)]e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, where $E_0^{'}(t) = E_0(t + t_2) + E_0(t - t_2)$, this location of the zeros in Fourier transform of $g(t)$ are represented by $\omega_0^{'}(t_2, t_0)$ and using method in section, we can get similar results as in Eq. 5.

$$P^{'}(t_0) = e^{2\sigma t_0}[\cos(\omega_0^{'}(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_0^{'}(t_2, t_0)\tau)d\tau + \sin(\omega_0^{'}(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_0^{'}(t_2, t_0)\tau)d\tau]$$
$$+[\cos(\omega_0^{'}(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(-\tau)\cos(\omega_0^{'}(t_2, t_0)\tau)d\tau + \sin(\omega_0^{'}(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(-\tau)\sin(\omega_0^{'}(t_2, t_0)\tau)d\tau] \tag{8}$$

Similar to $m_0 + m_{0p} = 0$ in Eq. 7, Now we evaluate $P^{'}(t_0)$ at $t_0 = 0$. Given that $P^{'}(t_0)$ is an **odd function** of variable $t_0$ , we can equate it to zero, $P^{'}(t_0)$ evaluated at $t_0 = 0$ as follows.

$$\begin{aligned} [P^{'}(t_0)]_{t_0=0} &= m_0^{'} + m_{0p}^{'} = 0 \\ m_0^{'} &= \int_{-\infty}^{0} E_0^{'}(\tau) e^{-2\sigma\tau} \cos(\omega_0^{'}(t_2,0)\tau) d\tau \\ m_{0p}^{'} &= \int_{-\infty}^{0} E_0^{'}(-\tau) \cos(\omega_0^{'}(t_2,0)\tau) d\tau \end{aligned} \tag{9}$$

We can expand above integrals as follows using $E_0^{'}(t) = E_0(t+t_2) + E_0(t-t_2)$. We can show that $m_0^{'}(t_2) + m_{0p}^{'}(t_2) = x_{odd}(t_2)$ is an **odd function** of variable $t_2$. Let $\omega_0^{'}(t_2, 0) = \omega_0^{'}(t_2)$.

$$\begin{aligned} m_0^{'}(t_2) + m_{0p}^{'}(t_2) &= x_{odd}(t_2) \\ m_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_0^{'}(t_2)t_2) \int_{-\infty}^{t_2} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_0^{'}(t_2)\tau) d\tau &+ \sin(\omega_0^{'}(t_2)t_2) \int_{-\infty}^{t_2} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_0^{'}(t_2)\tau) d\tau] \\ m_{0p}^{'}(t_2) = [\cos(\omega_0^{'}(t_2)t_2) \int_{-\infty}^{t_2} E_0(\tau) \cos(\omega_0^{'}(t_2)\tau) d\tau &+ \sin(\omega_0^{'}(t_2)t_2) \int_{-\infty}^{t_2} E_0(\tau) \sin(\omega_0^{'}(t_2)\tau) d\tau] \end{aligned} \tag{10}$$

Next we compare Eq. 5 and Eq. 10 and see that $\omega_0^{'}(t_2) = \omega_0(t_2)$.

### 2.1. ***Section 1.2 Single-sided function***

Next, we repeat the procedure in section 1.1 for the single sided function $f(t) = e^{\sigma t_0} E_p(t + t_0)$ where $t_0$ is a constant delay and we can see that the Fourier Transform of this new function also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real and asymmetric function of variable $t$ and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$ . We can see that $g_1(t)h(t) = f_1(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

Similarly, we can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**. In general, $\omega_2(t_0) \neq \omega_0(t_0)$.

It can be shown that $G_R(\omega_3(t_0), t_0) = G_1(\omega_2(t_0), t_0) = 0$ and $R(t_0) = G_1(\omega_3(t_0), t_0)$ is an odd function of variable $t_0$ and is given as follows. This result is shown in **Appendix A.2**.

$$
\begin{aligned}
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau)d\tau] \\
+[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega_2(t_0)\tau)d\tau] = 0
\end{aligned}
\tag{11}
$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r} c_{nkr} e^{b_{kr}t} e^{-\sigma t}$, we use the fact that $E_0(t) = E_0(-t)$, we can write as follows.

$$
R(t_0) = \sum_{n,k,r} c_{nkr} [\frac{(b_{kr} - 2\sigma)e^{(b_{kr})t_0}}{(\omega_2^2(t_0) + (b_{kr} - 2\sigma)^2)} + \frac{b_{kr}e^{-(b_{kr})t_0}}{(\omega_2^2(t_0) + b_{kr}^2)}] = 0
\tag{12}
$$

Now we evaluate $R(t_0)$ at $t_0 = 0$. Given that $R(t_0)$ is an **odd function** of variable $t_0$ , we can equate it to zero, $R(t_0)$ evaluated at $t_0 = 0$ as follows.

We define $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_{00}\tau)d\tau$, $m_{0p} = \int_{-\infty}^{0} E_0(-\tau) \cos(\omega_{00}\tau)d\tau$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 + m_{0p} = 0 \\
m_0 &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_{00}\tau)d\tau \\
m_{0p} &= \int_{-\infty}^{0} E_0(-\tau) \cos(\omega_{00}\tau)d\tau
\end{aligned}
\tag{13}
$$

If we replace $E_p(t)$ in above section by $E_{pp}(t) = e^{\sigma t_2} E_p(t + t_2) + e^{-\sigma t_2} E_p(t - t_2) = [E_0(t + t_2) + E_0(t - t_2)]e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, where $E_0^{'}(t) = E_0(t + t_2) + E_0(t - t_2)$, this location of the zeros in Fourier transform of $g(t)$ are represented by $\omega_0^{'}(t_2, t_0)$ and using method in section, we can get similar results as in Eq. 11.

$$R^{'}(t_0) = e^{2\sigma t_0}[\cos(\omega_2^{'}(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_2^{'}(t_2,t_0)\tau)d\tau + \sin(\omega_2^{'}(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_2^{'}(t_2,t_0)\tau)d\tau]$$
$$+[\cos(\omega_2^{'}(t_2,t_0)t_0)\int_{-\infty}^{-t_0} E_0^{'}(-\tau)\cos(\omega_2^{'}(t_2,t_0)\tau)d\tau - \sin(\omega_2^{'}(t_2,t_0)t_0)\int_{-\infty}^{-t_0} E_0^{'}(-\tau)\sin(\omega_2^{'}(t_2,t_0)\tau)d\tau] = 0 \tag{14}$$

Similar to $m_0 + m_{0p} = 0$ in Eq. 13, Now we evaluate $R^{'}(t_0)$ at $t_0 = 0$ and equate it to zero, $R^{'}(t_0)$ evaluated at $t_0 = 0$ as follows.

$$\begin{aligned}
[R^{'}(t_0)]_{t_0=0} &= m_0^{'} + m_{0p}^{'} = 0 \\
m_0^{'} &= \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_2^{'}(t_2,0)\tau)d\tau \\
m_{0p}^{'} &= \int_{-\infty}^{0} E_0^{'}(-\tau)\cos(\omega_2^{'}(t_2,0)\tau)d\tau
\end{aligned} \tag{15}$$

We can expand above integrals as follows using $E_0^{'}(t) = E_0(t+t_2) + E_0(t-t_2)$. We can show that $m_0^{'}(t_2) + m_{0p}^{'}(t_2) = y_{odd}(t_2)$ is an **odd function** of variable $t_2$. Let $\omega_2^{'}(t_2,0) = \omega_2^{'}(t_2)$.

$$\begin{aligned}
m_0^{'}(t_2) + m_{0p}^{'}(t_2) &= y_{odd}(t_2) \\
m_0^{'}(t_2) &= e^{2\sigma t_2}[\cos(\omega_2^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2^{'}(t_2)\tau)d\tau + \sin(\omega_2^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2^{'}(t_2)\tau)d\tau] \\
m_{0p}^{'}(t_2) &= [\cos(\omega_2^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)\cos(\omega_2^{'}(t_2)\tau)d\tau + \sin(\omega_2^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)\sin(\omega_2^{'}(t_2)\tau)d\tau]
\end{aligned} \tag{16}$$

Next we compare Eq. 10 and Eq. 16 and see that $\omega_2^{'}(t_2) = \omega_0^{'}(t_2) = \omega_0(t_2)$ which is an even function of variable $t_2$.

### 2.2. ***Section 1.3***

Next, take Eq. 14 and evaluate it at $t_2 = 0$. We can see that $E_0^{'}(t) = E_0(t + t_2) + E_0(t - t_2) = 2E_0(t)$. Let $\omega_2^{'}(t_2, 0) = \omega_2^{'}(t_2)$.

$$\begin{aligned} R^{'}(t_0) = 2e^{2\sigma t_0}[\cos(\omega_2^{'}(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_2^{'}(t_0)\tau)d\tau + \sin(\omega_2^{'}(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_2^{'}(t_0)\tau)d\tau] \\ +[\cos(\omega_2^{'}(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega_2^{'}(t_0)\tau)d\tau - \sin(\omega_2^{'}(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega_2^{'}(t_0)\tau)d\tau] = 0 \end{aligned} \tag{17}$$

We have shown in Section 1.2 that $\omega_2^{'}(t_0) = \omega_0(t_0)$ which is an even function of variable $t_0$. Hence we can write as follows.

$$\begin{aligned} R^{'}(t_0) = 2e^{2\sigma t_0}[\cos(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_0(t_0)\tau)d\tau + \sin(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_0(t_0)\tau)d\tau] \\ +2[\cos(\omega_0(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega_0(t_0)\tau)d\tau - \sin(\omega_0(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega_0(t_0)\tau)d\tau] = 0 \end{aligned} \tag{18}$$

We wish to evaluate $\lim_{t_0 \to -\infty} R^{'}(t_0)$. Let $\lim_{t_0 \to \pm\infty} \omega_0(t_0) = \omega_z'$.
Let us define $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_0(t_0)\tau)d\tau, Q_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau)d\tau$ and $I_0(t_0) = \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega_0(t_0)\tau)d\tau, Q_0(t_0) = \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega_2(t_0)\tau)d\tau$ . We can see that as $t_0 \to -\infty$, $I_1(t_0) = 0, Q_1(t_0) = 0$.

Let $I_{00} = \lim_{t_0 \to -\infty} I_0(t_0) = \int_{-\infty}^{\infty} E_0(-\tau) \cos(\omega_z'\tau)d\tau, Q_{00} = \lim_{t_0 \to -\infty} Q_0(t_0), \int_{-\infty}^{\infty} E_0(-\tau) \sin(\omega_z'\tau)d\tau$. We can write as follows.

$$\begin{aligned} \lim_{t_0 \to -\infty} R^{'}(t_0) = 0 \\ \lim_{t_0 \to -\infty} e^{2\sigma t_0}[I_1(t_0) \cos(\omega_z' t_0) + Q_1(t_0) \sin(\omega_z' t_0)] + I_0(t_0) \cos(\omega_z' t_0) - Q_0(t_0) \sin(\omega_z' t_0) = 0 \\ \lim_{t_0 \to -\infty} I_1(t_0) = Q_1(t_0) = 0 \\ \lim_{t_0 \to +\infty} I_{00} \cos(\omega_z' t_0) + Q_{00} \sin(\omega_z' t_0) = 0 \end{aligned} \tag{19}$$

We wish to evaluate $\lim_{t_0 \to +\infty} R^{'}(t_0)$. We can see that as $t_0 \to \infty$, $I_0(t_0) = 0, Q_0(t_0) = 0$. Let $I_{10} = \lim_{t_0 \to \infty} I_1(t_0) = \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_z' t_0)d\tau, Q_{10} = \lim_{t_0 \to \infty} Q_1(t_0) = \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_z' t_0)d\tau$. We can write as follows.

$$\begin{aligned} \lim_{t_0 \to \infty} R^{'}(t_0) = 0 \\ \lim_{t_0 \to \infty} e^{2\sigma t_0}[I_1(t_0) \cos(\omega_z' t_0) + Q_1(t_0) \sin(\omega_z' t_0)] + I_0(t_0) \cos(\omega_z' t_0) - Q_0(t_0) \sin(\omega_z' t_0) = 0 \\ \lim_{t_0 \to \infty} I_0(t_0) = Q_0(t_0) = 0 \\ \lim_{t_0 \to \infty} e^{2\sigma t_0}[I_1(t_0) \cos(\omega_z' t_0) + Q_1(t_0) \sin(\omega_z' t_0)] = 0 \\ \lim_{t_0 \to +\infty} I_{10} \cos(\omega_z' t_0) + Q_{10} \sin(\omega_z' t_0) = 0 \end{aligned}$$

$$(20)$$

We see that $\lim_{t_0 \to \pm\infty} \omega_0(t_0) = \omega_z'$ where $\omega_z$ is an **isolated zero** of the underlying analytic function $\lim_{t_0 \to \infty} g(t)$ and $\lim_{t_0 \to \infty} \omega_2(t_0) = \frac{d^2\omega_0(t_0)}{dt_0^2} = 0$.[ **Show this in more detail**, using $g_{even}(t)$ ]. So we have as follows.

$$\lim_{t_0 \to +\infty} I_{00} \cos(\omega_z' t_0) + Q_{00} \sin(\omega_z' t_0) = 0$$
$$\lim_{t_0 \to +\infty} I_{10} \cos(\omega_z' t_0) + Q_{10} \sin(\omega_z' t_0) = 0$$
$$(21)$$

From Eq. 16, this means $\lim_{t_0 \to \infty} m_0(t_0) = 0, \lim_{t_0 \to \infty} m_{0p}(t_0) = 0$. Now we can use these in next subsection and show the final result.

### 2.3. ***Section 1.4***

Now we find the first derivative of $R(t_0)$ in Eq. 11, as shown in Appendix E and F. Given that $R(t_0) = 0$ for all $t_0$ , we can equate to zero, zeroth and first derivative evaluated at $t_0 = 0$ as follows. We use the fact that $\omega_2(t_0) = \omega_0(t_0)$.

We define $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{00}\tau)d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{00}\tau)d\tau$, $m_{0p} = \int_{-\infty}^{0} E_0(-\tau)\cos(\omega_{00}\tau)d\tau$, $n_{0p} = \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_{00}\tau)d\tau$. Given that $E_0(t) = E_0(-t), e_1 = 0$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 + m_{0p} = 0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= \omega_{00}[n_0 - n_{0p}] + 2\sigma m_0 = 0
\end{aligned} \tag{22}
$$

If we replace $E_p(t)$ in above section by $E_{pp}(t) = e^{\sigma t_2}E_p(t+t_2) = E_0(t+t_2)e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, where $E_0^{'}(t) = E_0(t+t_2)$, the location of the zeros in Fourier transform of $g(t)$ are represented by $\omega_0^{'}(t_0, t_2)$ and using method in Section 1.1, we can get similar results.

Similarly, if $\omega_{00}[n_0 - n_{0p}] + 2\sigma m_0 = 0$ as derived in Section 1.1, we can show that $\omega_0^{'}(t_2)[n_0^{'}(t_2) - n_{0p}^{'}(-t_2)] + 2\sigma m_0^{'}(t_2) = y_{odd}(t_2)$ is an **odd function** of variable $t_2$ as follows.

$$
\begin{aligned}
\omega_0^{'}(t_2)[n_0^{'}(t_2) - n_{0p}^{'}(-t_2)] + 2\sigma m_0^{'}(t_2) &= y_{odd}(t_2) \\
m_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_0^{'}(t_2)\tau)d\tau &+ \sin(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_0^{'}(t_2)\tau)d\tau] \\
n_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(-\tau)e^{-2\sigma\tau}\sin(\omega_0^{'}(t_2)\tau)d\tau &- \sin(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(-\tau)e^{-2\sigma\tau}\cos(\omega_0^{'}(t_2)\tau)d\tau] \\
n_{0p}^{'}(-t_2) = [\cos(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{-t_2} E_0(-\tau)\sin(\omega_0^{'}(t_2)\tau)d\tau &+ \sin(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{-t_2} E_0(-\tau)\cos(\omega_0^{'}(t_2)\tau)d\tau]
\end{aligned} \tag{23}
$$

We can see that as $t_2 \to -\infty$,$\lim_{t_2\to-\infty}\omega_0^{'}(t_2)[n_0^{'}(t_2) - n_{0p}^{'}(-t_2)] + 2\sigma m_0^{'}(t_2) = 0$. This means $\lim_{t_2\to-\infty} n_{0p}^{'}(-t_2) = \lim_{t_2\to-\infty}[\cos(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{\infty} E_0(-\tau)\sin(\omega_0^{'}(t_2)\tau)d\tau - \sin(\omega_0^{'}(t_2)t_2)\int_{-\infty}^{\infty} E_0(-\tau)\cos(\omega_0^{'}(t_2)\tau)d\tau]$. We can **show that** $\omega_0^{'}(t_2) = \omega_0(t_2)$ and $\lim_{t_2\to-\infty}\omega_0(t_2) = \omega_z'$. Hence we can write as follows.

If we write $I_{00} = \int_{-\infty}^{\infty} E_0(-t)\cos(\omega_0(t_2)t)dt$ and $Q_{00} = \int_{-\infty}^{\infty} E_0(-t)\sin(\omega_0(t_2)t)dt$, and $\lim_{t_2\to\infty}(\omega_0(t_2) = \omega_z'$ we can write as $\lim_{t_2\to\infty}$ as follows.

$$
\begin{aligned}
Q_{00}\cos(\omega_z' t_0) - I_{00}\sin(\omega_z' t_0) &= 0 \\
I_{00}\cos(\omega_z' t_0) + Q_{00}\sin(\omega_z' t_0) &= 0 \\
\frac{I_{00}}{Q_{00}} = \frac{\cos(\omega_z' t_2)}{\sin(\omega_z' t_2)} &= -\frac{Q_{00}}{I_{00}}
\end{aligned} \tag{24}
$$

For the general case of $\lim_{t_2\to\infty}\frac{\sin(\omega_z' t_2)}{\cos(\omega_z' t_2)} \neq 0, \pm\infty$, we get $I_{00}^2 + Q_{00}^2 = 0$. This implies that $I_{00} = Q_{00} = 0$ and $\int_{-\infty}^{\infty} E_0(-t)e^{i(\omega_z' t)}dt = 0$. We know that $E_0(t) = E_0(-t)$ and that the Fourier Transform of $E_0(t)$ has **at least one isolated zero** at $\omega = \omega_z$. Hence we see that $\omega_z' = \omega_z$.

### 2.4. ***Section 1.5 Final result***

We can see that as $t_2 \to +\infty$,$\lim_{t_2\to\infty} \omega_0^{'}(t_2)[n_0^{'}(t_2) - n_{0p}^{'}(-t_2)] + 2\sigma m_0^{'}(t_2) = 0$.

We showed in the last subsection that $\lim_{t_2\to\infty} m_0(t_2) = 0$. Given that $\omega_0^{'}(t_2) = \omega_0(t_2)$, we know that $\lim_{t_2\to\infty} n_{0p}^{'}(-t_2) = 0$ and we see that $\lim_{t_2\to\infty} m_0(t_2) = m_0^{'}(t_2) = 0$. Hence we can see that $\lim_{t_2\to\infty} n_0^{'}(t_2) = 0$. These two equations can be expanded as follows.

$$\lim_{t_2\to\infty} n_0^{'}(t_2) = 0$$
$$n_0^{'}(t_2) = \lim_{t_2\to\infty} e^{2\sigma t_2}[\cos(\omega_0(t_2)t_2) \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_0(t_2)\tau)d\tau - \sin(\omega_0(t_2)t_2) \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_0(t_2)\tau)d\tau]$$
$$\lim_{t_2\to\infty} m_0^{'}(t_2) = 0$$
$$m_0^{'}(t_2) = \lim_{t_2\to\infty} e^{2\sigma t_2}[\cos(\omega_0(t_2)t_2) \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_0(t_2)\tau)d\tau + \sin(\omega_0(t_2)t_2) \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_0(t_2)\tau)d\tau] = 0 \tag{25}$$

If we write $I_{10} = \int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t} \cos(\omega_0 t)dt$ and $Q_{10} = E_0(t)e^{-2\sigma t} \sin(\omega_0 t)dt$, and $\lim_{t_2\to\infty}(\omega_0(t_2) = \omega_z$ we can write

$$\begin{aligned}
\lim_{t_2\to\infty} \cos(\omega_z t_2)Q_{10} - \lim_{t_1\to\infty} \sin(\omega_z t_2)I_{10} &= 0\\
\lim_{t_2\to\infty} \cos(\omega_z t_2)I_{10} + \lim_{t_1\to\infty} \sin(\omega_z t_2)Q_{10} &= 0\\
\frac{Q_{10}}{I_{10}} = \lim_{t_2\to\infty} \frac{\sin(\omega_z t_2)}{\cos(\omega_z t_2)} &= -\frac{I_{10}}{Q_{10}}
\end{aligned} \tag{26}$$

For the general case of $\lim_{t_2\to\infty} \frac{\sin(\omega_z t_2)}{\cos(\omega_z t_2)} \neq 0, \pm\infty$, we get $I_{10}^2 + Q_{10}^2 = 0$. This implies that $I_{10} = Q_{10} = 0$ and $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{i(\omega_z t)}dt = 0$.

We started with **Assumption 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

Now we can repeat the steps in Section 2, starting with the new result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$ and modified $h(t) = e^{2\sigma t}u(-t) + e^{-2\sigma t}u(t)$ and derive the next result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-4\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We can repeat above steps N times till $2^N\sigma > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-2^N\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$. In each iteration $n$, we use $h(t) = e^{2^n\sigma t}u(-t) + e^{-2^n\sigma t}u(t)$. We know that the Fourier Transform of $E_0(t)e^{-2^N\sigma t} = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-2^N\sigma t}$ **does not** have a real zero for $2^N\sigma > \frac{1}{2}$. Here we use the well known fact that $E_0(t) = E_0(-t)$.

Hence we have produced a **contradiction** of **Assumption 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$.

**Method 2:**

In the last step, where we have derived the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-2^N\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$, we replace $\tau = -t$ and get $\int_{-\infty}^{\infty} E_0(-t)e^{2^N\sigma t}e^{i\omega_z t}dt = 0$. We define $E_{0p}(t) = E_0(-t)$, hence $\int_{-\infty}^{\infty} E_{0p}(t)e^{2^N\sigma t}e^{i\omega_z t}dt = 0$.

We repeat the procedure in this section **one more time**, we use $h(t) = e^{2^N\sigma t}u(-t) + e^{-2^N\sigma t}u(t)$ and we can derive the result $\int_{-\infty}^{\infty} E_{0p}(\tau)e^{-2^{(N+1)}\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$. Now we replace $\tau = -t$ and get $\int_{-\infty}^{\infty} E_{0p}(-t)e^{2^{(N+1)}\sigma t}e^{i\omega_z t}dt = 0$. Given that $E_{0p}(t) = E_0(-t)$, we get $\int_{-\infty}^{\infty} E_0(t)e^{2^{(N+1)}\sigma t}e^{i\omega_z t}dt = 0$.

We know that the Fourier Transform of $E_0(t)e^{2^{(N+1)}\sigma t} = \sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{2^{(N+1)}\sigma t}$ **does not** have a real zero for $2^{(N+1)}\sigma > \frac{1}{2}$.

Hence we have produced a **contradiction** of **Assumption 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$.

## 3. Section 2

**Theorem 2**: Any Fourier Transformable real function of the form $E_p(t) = E_0(t)e^{-\sigma t}$ does not have zeros in its Fourier Transform given by $E_p(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, for any real value of $-\infty \leq \omega \leq \infty$, for $\sigma \neq 0$, **ONLY IF** the Fourier Transform of $E_0(t)$ has a known real zero at some $\omega = \omega_z$ **and if** the Fourier Transform of $E_0(t)e^{-(2^N\sigma)t}$ is known to NOT have a real zero for $(2^N\sigma) > K_0$ where $K_0$ is some finite constant and $N$ is a positive integer.

For example, this theorem holds for **Dirichlet L-functions** given by $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ where $\chi(n)$ is a Dirichlet character. Hence this theorem proves **Generalized Riemann Hypothesis**.

**Proof**: The proof of this theorem is done as follows. We replace $E_p(t)$ in section 1 by the generalized riemann zeta function and repeat the procedure in section 1 and we geta similar result.

## 4. Section 3

**Theorem 1.4**: **Dirichlet Eta Function** $\zeta_a(s) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^s}$ does not have zeros for any real value of $-\infty \leq \omega \leq \infty$, for $\sigma \neq 0$, where $s = \frac{1}{2} - \sigma + i\omega$, **given that** the Fourier Transform of $E_0(t)$ given by $\xi(\frac{1}{2}+i\omega)(1-2^{(1-s)})$, has a known real zero at some $\omega = \omega_z$ **and if** the Fourier Transform of $E_0(t)e^{-(2^N\sigma)t}$ is known to NOT have a real zero for $(2^N\sigma) > K_0$ where $K_0$ is some finite constant and $N$ is a positive integer.

**Proof**: The proof of this theorem is shown below.

Let us consider $E_p(t) = E_0(t)e^{-\sigma t}, E_0(t) = \frac{e^{-e^t}}{1+e^{-e^t}}e^{\frac{1}{2}t}$ which corresponds to the Eta function $\zeta_a(s) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^s}$ where $E_p(t)$ is the inverse Fourier Transform of $E(s) = \Gamma(s)\zeta_a(s) = \int_0^{\infty}[\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^s}]e^{-y}y^{s-1}dy$ where $s = \frac{1}{2} - \sigma + i\omega$.

If we substitute $y = mx$, we have $E_p(s) = \int_0^{\infty}[\sum_{m=1}^{\infty}(-1)^{m-1}e^{-mx}]x^{s-1}dx = \int_0^{\infty}\frac{e^{-x}}{1+e^{-x}}x^{s-1}dx$. If we substitute $x = e^t$, we have $E(s) = \int_{-\infty}^{\infty}\frac{e^{-e^t}}{1+e^{-e^t}}e^{st}dt = \int_{-\infty}^{\infty}\frac{e^{-e^t}}{1+e^{-e^t}}e^{\frac{1}{2}t}e^{-\sigma t}e^{-i\omega t}dt = \int_{\infty}^{\infty}E_p(t)e^{-i\omega t}dt.$

Let us use the Taylor series expansion of $E_p(t) = \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{k=1}^{\infty}\frac{(-n)^k}{!k}e^{(k+\frac{1}{2}-\sigma)t}$ and use the shorthand notation $E_p(t) = \sum_{n,k}a_{nk}e^{(k+\frac{1}{2}-\sigma)t}$ where $a_{nk} = (-1)^{n-1}\frac{(-n)^k}{!k}$. We define $r = 0, c_{nkr} = a_{nk}, b_{kr} = k + \frac{1}{2}$ and express $E_p(t) = \sum_{n,k,r}c_{nkr}e^{b_{kr}t}e^{-\sigma t}$ similar to previous sections.

We can see that $E_0(t) \neq E_0(-t)$ and is **not** an even function of variable $t$. We will show that the **Assumption 1** which assumes that the Fourier Transform of the function $E_p(t) = \frac{e^{-e^t}}{1+e^{-e^t}}e^{\frac{1}{2}t}e^{-\sigma t}$, given by Dirichlet eta function $\zeta_a(s) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^{\frac{1}{2}-\sigma+i\omega}}$ has a zero at $\omega = \omega_0$, leads to a **contradiction** for $\sigma \neq 0$.

**Next, we repeat** the steps in Section 1.1, 1.2 and 1.3 and we can produce a contradiction of **Assumption 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty}E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$.

## 5. Appendix A.1 Two sided f(t)

### Step 1

Lets us assume that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$, given by $E_p(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, has a zero at $\omega = \omega_0$. Let us call it **Assumption 1**.

Let us form a new function $f(t) = e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)$ where $t_0$ is a constant delay and we can see that the Fourier Transform of this new function also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t)+g_+(t)u(t)$ where $g(t)$ is a real and asymmetric function of variable $t$ and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$ . We can see that $g(t)[e^{\sigma t}u(-t) + e^{-\sigma t}u(t)] = f(t)$.

We can see that $g(t)$ is a real $L^1$ integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[ Riemann Lebesgue Lemma]. This is explained in detail in **Appendix C**. We can write $g(t) = g_{even}(t)+g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable $t$.

If we take the fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $G(\omega) * H(\omega) = F(\omega)$ where $*$ denotes convolution operation given by $F(\omega) = \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+w_0^2)}$ is the fourier transform of the function $h(t)$.

For every value of $t_0$, we require the fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the fourier transform of the even function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_0(t_0)$ for every value of $t_0$. [ Because $H(\omega) = \frac{2\sigma}{(\sigma^2+w_0^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H(\omega)$ obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Assumption 1**.]

Similarly, the fourier transform of the even function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one real zero** at $\omega = \omega_1(t_0)$ for every value of $t_0$. In general, $\omega_1(t_0) \neq \omega_0(t_0)$. Because $H(\omega) = \frac{2\sigma}{(\sigma^2+w_0^2)}$ does not have real zeros, if $G_I(\omega)$ does not have real zeros, then $F_I(\omega) = G_I(\omega) * H(\omega)$ obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, cannot possibly have real zeros, which goes against **Assumption 1**. This is shown in **Appendix A.2**.

### Step 2

Let us compute the fourier transform of the function $g(t)$ given by $G(\omega) = G_R(\omega) + iG_I(\omega)$. Let us define $E_q(t) = E_p(-t)$. We can see that $E_p(t-t_0) = E_q(-t+t_0)$ and $E_p(t+t_0) = E_q(-t-t_0)$. Substituting $t = -t$ in the second integral below, we have

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} g_-(t)e^{-i\omega t}dt + \int_{0}^{\infty} g_+(t)e^{-i\omega t}dt]$$

$$G(\omega) = \int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)]e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} [e^{-\sigma t_0}E_q(-t + t_0) + e^{\sigma t_0}E_q(-t - t_0)]e^{\sigma t}e^{-i\omega t}dt$$

$$G(\omega) = \int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)]e^{-\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} [e^{-\sigma t_0}E_q(t + t_0) + e^{\sigma t_0}E_q(t - t_0)]e^{-\sigma t}e^{i\omega t}dt \tag{27}$$

Using the substitutions $t - t_0 = \tau, dt = d\tau$ and $t + t_0 = \lambda, dt = d\lambda$, we can write the above equation as follows.

$$\begin{aligned} G(\omega) = e^{-\sigma t_0} e^{-i\omega t_0} e^{-\sigma t_0} \int_{-\infty}^{-t_0} E_p(\tau) e^{-\sigma\tau} e^{-i\omega\tau} d\tau + e^{\sigma t_0} e^{i\omega t_0} e^{\sigma t_0} \int_{-\infty}^{t_0} E_p(\tau) e^{-\sigma\tau} e^{-i\omega\tau} d\tau \\ + e^{-\sigma t_0} e^{-i\omega t_0} e^{\sigma t_0} \int_{-\infty}^{t_0} E_q(\tau) e^{-\sigma\tau} e^{i\omega\tau} d\tau + e^{\sigma t_0} e^{i\omega t_0} e^{-\sigma t_0} \int_{-\infty}^{-t_0} E_q(\tau) e^{-\sigma\tau} e^{i\omega\tau} d\tau \end{aligned} \tag{28}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and simplified by cancelling common terms. Comparing the **real parts** of $G(\omega)$, given $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_0(-t)e^{\sigma t}$, we have

$$\begin{aligned} G_R(\omega) = G_1(\omega, t_0) + G_1(\omega, -t_0) \\ G_1(\omega, t_0) = e^{2\sigma t_0}[\cos(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega\tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega\tau) d\tau] \\ + [\cos(\omega t_0) \int_{-\infty}^{t_0} E_0(-\tau) \cos(\omega\tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E_0(-\tau) \sin(\omega\tau) d\tau] \\ + \end{aligned} \tag{29}$$

We require $G_R(\omega) = 0$ for $\omega = \omega_0(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**.

Given that $E_p(t) = E_0(t)e^{-\sigma t}, E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t}$, Hence we can see that $P(t_0) = G_1(\omega_0(t_0), t_0)$ is an odd function of variable $t_0$.

$$\begin{aligned} G_R(\omega_0(t_0), t_0) = G_1(\omega_0(t_0), t_0) + G_1(\omega_0(t_0), -t_0) = 0 \\ P(t_0) = e^{2\sigma t_0}[\cos(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_0(t_0)\tau) d\tau + \sin(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_0(t_0)\tau) d\tau] \\ + [\cos(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(-\tau) \cos(\omega_0(t_0)\tau) d\tau + \sin(\omega_0(t_0)t_0) \int_{-\infty}^{t_0} E_0(-\tau) \sin(\omega_0(t_0)\tau) d\tau] \end{aligned} \tag{30}$$

Comparing the **imaginary parts** of $G(\omega)$, given $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_0(-t)e^{\sigma t}$, we have

$$\begin{aligned} G_I(\omega) = G_2(\omega, t_0) + G_2(\omega, -t_0) \\ G_2(\omega, t_0) = e^{2\sigma t_0}[\sin(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega\tau) d\tau - \cos(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega\tau) d\tau] \\ - [\sin(\omega t_0) \int_{-\infty}^{t_0} E_0(-\tau) \cos(\omega\tau) d\tau - \cos(\omega t_0) \int_{-\infty}^{t_0} E_0(-\tau) \sin(\omega\tau) d\tau] \\ + \end{aligned} \tag{31}$$

We require $G_I(\omega) = 0$ for $\omega = \omega_1(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**.

Given that $E_p(t) = E_0(t)e^{-\sigma t}, E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t}$, Hence we can see that $Q(t_0) = G_2(\omega_1(t_0), t_0)$ is an odd function of variable $t_0$.

$$
\begin{aligned}
G_I(\omega_1(t_0), t_0) &= G_2(\omega_1(t_0), t_0) + G_2(\omega_1(t_0), -t_0) = 0 \\
Q(t_0) = e^{2\sigma t_0}[\sin(\omega_1(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_1(t_0)\tau) d\tau &- \cos(\omega_1(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_1(t_0)\tau) d\tau] \\
-[\sin(\omega_1(t_0)t_0) \int_{-\infty}^{t_0} E_0(-\tau) \cos(\omega_1(t_0)\tau) d\tau &- \cos(\omega_1(t_0)t_0) \int_{-\infty}^{t_0} E_0(-\tau) \sin(\omega_1(t_0)\tau) d\tau]
\end{aligned} \tag{32}
$$

## 6. Appendix A.2 Single sided f(t)

### Step 1

Lets us assume that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$, given by $E_p(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, has a zero at $\omega = \omega_0$. Let us call it **Assumption 1**.

Let us form a new function $f(t) = e^{\sigma t_0} E_p(t + t_0)$ where $t_0$ is a constant delay and we can see that the Fourier Transform of this new function also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real and asymmetric function of variable $t$ and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$ . We can see that $g(t)[e^{\sigma t}u(-t) + e^{-\sigma t}u(t)] = f(t)$.

We can see that $g(t)$ is a real $L^1$ integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[ Riemann Lebesgue Lemma]. This is explained in detail in **Appendix C**. We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable $t$.

If we take the fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $G(\omega) * H(\omega) = F(\omega)$ where $*$ denotes convolution operation given by $F(\omega) = \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + w_0^2)}$ is the fourier transform of the function $h(t)$.

For every value of $t_0$, we require the fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the fourier transform of the even function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_0(t_0)$ for every value of $t_0$. [ Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + w_0^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H(\omega)$ obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Assumption 1**.]

Similarly, the fourier transform of the even function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one real zero** at $\omega = \omega_1(t_0)$ for every value of $t_0$. In general, $\omega_1(t_0) \neq \omega_0(t_0)$. Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + w_0^2)}$ does not have real zeros, if $G_I(\omega)$ does not have real zeros, then $F_I(\omega) = G_I(\omega) * H(\omega)$ obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, cannot possibly have real zeros, which goes against **Assumption 1**. This is shown in **Appendix 0.3**.

### Step 2

Let us compute the fourier transform of the function $g(t)$ given by $G(\omega) = G_R(\omega) + iG_I(\omega)$. Let us define $E_q(t) = E_p(-t)$. We can see that $E_p(t - t_0) = E_q(-t + t_0)$ and $E_p(t + t_0) = E_q(-t - t_0)$. Substituting $t = -t$ in the second integral below, we have

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} g_-(t)e^{-i\omega t}dt + \int_{0}^{\infty} g_+(t)e^{-i\omega t}dt]$$
$$G(\omega) = \int_{-\infty}^{0} e^{\sigma t_0} E_p(t + t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0} E_q(-t - t_0)e^{\sigma t}e^{-i\omega t}dt$$
$$G(\omega) = \int_{-\infty}^{0} e^{\sigma t_0} E_p(t + t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} e^{\sigma t_0} E_q(t - t_0)e^{-\sigma t}e^{i\omega t}dt \tag{33}$$

Using the substitutions $t + t_0 = \tau, dt = d\tau$ we can write the above equation as follows.

$$G(\omega) = e^{\sigma t_0} e^{i\omega t_0} e^{\sigma t_0} \int_{-\infty}^{t_0} E_p(\tau) e^{-\sigma\tau} e^{-i\omega\tau} d\tau + e^{\sigma t_0} e^{i\omega t_0} e^{-\sigma t_0} \int_{-\infty}^{-t_0} E_q(\tau) e^{-\sigma\tau} e^{i\omega\tau} d\tau \tag{34}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and simplified by cancelling common terms. Comparing the **real parts** of $G(\omega)$, given $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t}$, we have

$$\begin{aligned} G_R(\omega) &= G_1(\omega, t_0) \\ G_1(\omega, t_0) = e^{2\sigma t_0}[\cos(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega\tau) d\tau &+ \sin(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega\tau) d\tau] \\ +[\cos(\omega t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega\tau) d\tau &- \sin(\omega t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega\tau) d\tau] \\ &+ \end{aligned} \tag{35}$$

We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**.
Given that $E_p(t) = E_0(t)e^{-\sigma t}, E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t}$, Hence we can see that $R(t_0) = G_1(\omega_2(t_0), t_0) = 0$.

$$\begin{aligned} G_R(\omega_2(t_0), t_0) &= R(t_0) = G_1(\omega_2(t_0), t_0) = 0 \\ R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau &+ \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] \\ +[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega_2(t_0)\tau) d\tau &- \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega_2(t_0)\tau) d\tau] = 0 \end{aligned} \tag{36}$$

Comparing the **imaginary parts** of $G(\omega)$, given $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_0(t)e^{\sigma t}$, we have

$$\begin{aligned} G_I(\omega) &= G_2(\omega, t_0) \\ G_2(\omega, t_0) = e^{2\sigma t_0}[\sin(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega\tau) d\tau &- \cos(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega\tau) d\tau] \\ +[\sin(\omega t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \cos(\omega\tau) d\tau &+ \cos(\omega t_0) \int_{-\infty}^{-t_0} E_0(-\tau) \sin(\omega\tau) d\tau] \\ &+ \end{aligned} \tag{37}$$

We require $G_I(\omega) = 0$ for $\omega = \omega_3(t_0)$ for every value of $t_0$, to satisfy **Assumption 1**.
Given that $E_p(t) = E_0(t)e^{-\sigma t}, E_q(t) = E_p(-t) = E_0(t)e^{\sigma t}$, Hence we can see that $S(t_0) = G_2(\omega_2(t_0), t_0) = 0$ .

$$
\begin{aligned}
G_I(\omega_2(t_0), t_0) = S(t_0) = G_2(\omega_3(t_0), t_0) = 0 \\
S(t_0) = e^{2\sigma t_0}[\sin(\omega_3(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_3(t_0)\tau)d\tau - \cos(\omega_3(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_3(t_0)\tau)d\tau] \\
+[\sin(\omega_3(t_0)t_0)\int_{-\infty}^{-t_0} E_0(-\tau)\cos(\omega_3(t_0)\tau)d\tau + \cos(\omega_3(t_0)t_0)\int_{-\infty}^{-t_0} E_0(-\tau)\sin(\omega_3(t_0)\tau)d\tau] = 0 \\
S(t_0) = e^{2\sigma t_0}[\sin(\omega_3(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_3(t_0)\tau)d\tau - \cos(\omega_3(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_3(t_0)\tau)d\tau] \\
-[-\sin(\omega_3(t_0)t_0)\int_{-\infty}^{-t_0} E_0(-\tau)\cos(\omega_3(t_0)\tau)d\tau - \cos(\omega_3(t_0)t_0)\int_{-\infty}^{-t_0} E_0(-\tau)\sin(\omega_3(t_0)\tau)d\tau] = 0
\end{aligned} \tag{38}
$$

## 7. Appendix C

In this section, we will rederive the Inverse Fourier Transform of Riemann Zeta function $E_0(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us start with this analytic continuation of Riemann's Zeta Function $\xi(\frac{1}{2} + i\omega) = E_0(\omega)$. Its Inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_0(\omega)e^{i\omega t}d\omega$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and rederive the steps below. We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and rederive as follows. We define $s = \frac{1}{2} + \sigma + i\omega$.

$$
\begin{aligned}
\Gamma(\frac{s}{2}) &= \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy \\
\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx
\end{aligned} \tag{39}
$$

For $\sigma > 1$, we can do a summation of both sides of above equation for all positive integers $n$ and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

$$
\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{40}
$$

For $Re(s) > 1$, we can use theorem of doominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$.

$$
\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{41}
$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}s(s-1)\int_{0}^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{42}$$

In the next subsection, we show that $\xi(s) = \xi(1-s)$ by doing an analytic continuation of $\xi(s)$ for all values of $Re[s]$ in the complex plane.

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in above equation and get

$$\xi(s) = \frac{1}{2}s(s-1)\int_{-\infty}^{\infty}[e^{st} + e^{(1-s)t}]\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}dt \tag{43}$$

We evaluate above equation at $s = \frac{1}{2} + i\omega$ as follows.

$$\begin{aligned} \xi(\frac{1}{2}+i\omega) &= \frac{1}{2}(\frac{1}{2}+i\omega)(-\frac{1}{2}+i\omega)\int_{-\infty}^{\infty}[e^{\frac{t}{2}}e^{i\omega t} + e^{\frac{t}{2}}e^{-i\omega t}]\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}dt \\ \xi(\frac{1}{2}+i\omega) &= \frac{1}{2}[-(\frac{1}{4}+\omega^2)[\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-i\omega t}dt + \int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{i\omega t}dt]] \end{aligned} \tag{44}$$

We define $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ and get the inverse Fourier transform of $\xi(\frac{1}{2}+i\omega)$ given by $E_0(t)$ as follows.

$$\begin{aligned} E_0(t) &= \frac{1}{2}[-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\ A(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}] \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[-4\pi n^2 e^{2t} + (\frac{1}{2} - 2\pi n^2 e^{2t})^2] \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[\frac{1}{4} + 4\pi^2 n^4 e^{4t} - 2\pi n^2 e^{2t} - 4\pi n^2 e^{2t}] \end{aligned} \tag{45}$$

We have arrived at the desired result for $E_0(t)$ as follows.

$$\begin{aligned} E_0(t) &= \frac{1}{2}[-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\ E_0(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] \end{aligned}$$

$$ \tag{46} $$

We can evaluate the inverse fourier transform of $\xi(\frac{1}{2}+\sigma+i\omega)$ as $E_p(t) = E_0(t)e^{-\sigma t}$. The Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ is given by $E_p(\omega) = E_0(\omega - i\sigma)$. We know that $E_0(\omega) = \xi(\frac{1}{2}+i\omega)$ and hence $E_p(\omega) = E_0(\omega - i\sigma) = \xi(\frac{1}{2}+\sigma+i\omega)$.

### *7.1. Appendix C.1*

We divide the range of integration in the right hand side of Eq. 41 in two intervals $[0,1]$ and $[1,\infty]$ and substitute $x->\frac{1}{x}$ in the interval $[0,1]$ as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{-\frac{s}{2}-1}\frac{\sqrt{x}}{2}[1-\frac{1}{\sqrt{x}}+w(x)]dx \tag{47}$$

Now we use the fact that for $x>0$, $w(\frac{1}{x}) = w(x)x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}$ and we get

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{-(s+1)}{2}}w(x)dx \tag{48}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and use the fact that $\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ and we get for $Re[s]>1$

$$\xi(s) = \frac{1}{2}[1+s(s-1)\int_1^\infty [x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}}]\frac{w(x)}{x}dx] \tag{49}$$

Now we do an analytic continuation of $\xi(s)$ for all values of $Re[s]$ in the complex plane and we see that $\xi(s) = \xi(1-s)$.

## 8. Appendix D

In Section 1, we mentioned that $E_0(t) = E_0(-t)$ where $E_0(t) = \sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$. This is derived from the well known result $\displaystyle\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x}} = \sum_{n=1}^{\infty} e^{-\pi n^2 x}x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}$ for $x>0$ **[Result A]**. This is rederived here.

For $x = e^{-2t}$ where $-\infty \le t \le \infty$,

$$\sum_{n=1}^{\infty} e^{-\pi n^2e^{2t}} = \sum_{n=1}^{\infty} e^{-\pi n^2e^{-2t}}e^{-t} + \frac{1}{2}e^{-t} - \frac{1}{2} \tag{50}$$

Multiplying above equation by $e^{\frac{t}{2}}$,

$$
\begin{aligned}
f(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} + \frac{1}{2}e^{-\frac{t}{2}} - \frac{1}{2}e^{\frac{t}{2}} \\
\frac{df(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}] = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}}[-\frac{1}{2} + 2\pi n^2 e^{-2t}] - \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{4}e^{\frac{t}{2}} \\
\frac{d^2 f(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[(\frac{1}{2} - 2\pi n^2 e^{2t})^2 - 4\pi n^2 e^{2t}] \\
&= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{4} + 4\pi^2 n^4 e^{4t} - 2\pi n^2 e^{2t} - 4\pi n^2 e^{2t}] \\
&= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}}[(-\frac{1}{2} + 2\pi n^2 e^{-2t})^2 - 4\pi n^2 e^{-2t}] + \frac{1}{8}e^{-\frac{t}{2}} - \frac{1}{8}e^{\frac{t}{2}}
\end{aligned}
\tag{51}
$$

We wish to compute $E_0(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \frac{1}{2}\frac{d^2 f(t)}{dt^2} - \frac{1}{8}f(t)$. Comparing the right hand side of above equations, we have

$$
\begin{aligned}
E_0(t) &= \frac{1}{2}\frac{d^2 f(t)}{dt^2} - \frac{1}{8}f(t) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}}[\frac{1}{4} + 4\pi^2 n^4 e^{-4t} - 2\pi n^2 e^{-2t} - 4\pi n^2 e^{-2t}] + \frac{1}{16}e^{-\frac{t}{2}} - \frac{1}{16}e^{\frac{t}{2}} \\
&\quad -\frac{1}{8}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} - \frac{1}{16}e^{-\frac{t}{2}} + \frac{1}{16}e^{\frac{t}{2}} \\
E_0(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}}[2\pi^2 n^4 e^{-4t} - 3\pi n^2 e^{-2t}] = E_0(-t)
\end{aligned}
\tag{52}
$$

Thus we have shown the result $E_0(t) = E_0(-t)$ which is derived from the assumption of the Result A $\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x}} = \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}$ for $x > 0$ .

## 9. Appendix D.1

It is shown that $\omega_{00} \neq 0$ in previous sections.

We see that $E_p(t) = E_0(te^{-\sigma t}$ where $E_0(t) = \sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us form a new function $f(t) = e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)$ where $t_0$ is a constant delay for $-\infty \leq t_0 \leq \infty$. Given that the Fourier Transform of $E_p(t)$ has a zero at $\omega = \omega_0$, we can see that the Fourier Transform of this new function $f(t)$ also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real and asymmetric function of variable $t$ and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$ . We can see that $g(t)[e^{\sigma t}u(-t) + e^{-\sigma t}u(t)] = f(t)$.

We can see that $E_p(t), f(t), g(t)$ are all positive and are $> 0$ for all values of $t$. Hence $g_{even}(t) = g(t) + g(-t) > 0$ for all values of $t$ and hence its Fourier transform evaluated at $\omega = 0$ cannot be zero and hence $\omega_{00} = [\omega_0(t_0)]_{t_0=0}$ cannot be zero.

## 10. Appendix D.2

If we consider any real $L^1$ integrable function $g(t)$, its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \rightarrow \pm\infty$, as per **Riemann-Lebesgue Lemma**.

We can see that $h(t) = e^{-\sigma t_0}[e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$ is a real $L^1$ integrable function and its Fourier transform given by $H(\omega) = e^{-\sigma t_0}[(\frac{\sigma}{\sigma^2+\omega^2} + \frac{3\sigma}{9\sigma^2+\omega^2}) + i\omega(\frac{1}{\sigma^2+\omega^2} - \frac{1}{9\sigma^2+\omega^2})]$, is finite for $|\omega| < \infty$ and goes to zero as $\omega \rightarrow \pm\infty$, as per **Riemann-Lebesgue Lemma**.

If we take the fourier transform of the equation $g(t)h(t) = f(t)$, we can see that $f(t)$ is also a real $L^1$ integrable function and its Fourier transform is given by $G(\omega) * H(\omega) = F(\omega)$ where $*$ denotes convolution operation given by $F(\omega) = \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$. We can see that $F(\omega)$ is also finite for $|\omega| < \infty$ for $L^1$ integrable function and goes to zero as $\omega \rightarrow \pm\infty$, as per **Riemann-Lebesgue Lemma**.

## 11. Method 3: First derivative of P(t)

In this section, we will consider the function $B(t) = e^{\Delta t}[\cos(\omega_0(t)t) \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau} \cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t) \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau} \sin(\omega_0(t)\tau)d\tau]$ and compute the value at $t = 0$ of its zeroth and first derivatives.

$$B(t) = e^{\Delta t}[\cos(\omega_0(t)t) \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau} \cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t) \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau} \sin(\omega_0(t)\tau)d\tau] \tag{53}$$

Then we will compute the value at $t = 0$ of zeroth and first derivatives of the functions $m_0(t), m_{0p}(t)$ as follows, by substituting $\Delta = 2\sigma$ and $\Delta = 0$ respectively.

$$m_0(t) = e^{2\sigma t}[\cos(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_0(t)\tau)d\tau]$$
$$m_{0p}(t) = \cos(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)\cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)\sin(\omega_0(t)\tau)d\tau \tag{54}$$

**Step S2.1a**

We can see that the zeroth derivative of the functions $m_0(t), m_{0p}(t)$ are as follows.

$$[m_0(t)]_{t=0} = m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{00}\tau)d\tau$$
$$(m_{0p}(t))_{t=0} = m_{0p} = \int_{-\infty}^{0} E_0(\tau)\cos(\omega_{00}\tau)d\tau \tag{55}$$

**Step S2.2a**

We wish to compute the first derivative of the function $A(t)$ in Eq. 53 as follows. Let us define $I_1(t) = \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\cos(\omega_0(t)\tau)d\tau$ and $I_2(t) = \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\sin(\omega_0(t)\tau)d\tau$. Let us define $\theta(t) = \omega_0(t)t$, so we have $\frac{d\theta(t)}{dt} = \omega_0(t) + t\frac{d\omega_0(t)}{dt}$, $\frac{d^2\theta(t)}{dt^2} = 2\frac{d\omega_0(t)}{dt} + t\frac{d^2\omega_0(t)}{dt^2}$. Given that $\omega_0(t)$ is an even function of variable $t$ we can see that $\omega_0(0) = \omega_{00}$, $[\frac{d\omega_0(t)}{dt}]_{t=0} = 0$, $[\frac{d^2\omega_0(t)}{dt_0^2}]_{t=0} = 2\omega_{02}$, $\theta(0) = 0$, $[\frac{d\theta(t)}{dt}]_{t=0} = \omega_{01}$, $[\frac{d^2\theta(t)}{dt_0^2}]_{t=0} = 0$ and write

$$B(t) = e^{\Delta t}[\cos(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t)\int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\sin(\omega_0(t)\tau)d\tau]$$
$$B(t) = e^{\Delta t}[\cos(\omega_0(t)t)I_1(t) + \sin(\omega_0(t)t)I_2(t)]$$
$$\frac{dB(t)}{dt} = \Delta e^{\Delta t}[I_1(t)\cos(\omega_0(t)t) + I_2(t)\sin(\omega_0(t)t)]$$
$$+e^{\Delta t}[\cos(\omega_0(t)t)\frac{dI_1(t)}{dt} + \sin(\omega_0(t)t)\frac{dI_2(t)}{dt} - I_1(t)\sin(\omega_0(t)t)\frac{d\theta(t)}{dt} + I_2(t)\cos(\omega_0(t)t)\frac{d\theta(t)}{dt}$$
$$\frac{dB(t)}{dt} = e^{\Delta t}[\cos(\omega_0(t)t)[\Delta I_1(t) + \frac{dI_1(t)}{dt} + I_2(t)\frac{d\theta(t)}{dt}] + \sin(\omega_0(t)t)[\Delta I_2(t) + \frac{dI_2(t)}{dt} - I_1(t)\frac{d\theta(t)}{dt}]] \tag{56}$$

We wish to calculate the terms $\frac{dI_2(t)}{dt}, \frac{dI_1(t)}{dt}$. Let us use the Taylor series expansion of $E_0(t) = [\sum_{n,k}(a_{nk}e^{(2k+\frac{9}{2})t} - b_{nk}e^{(2k+\frac{5}{2})t})]$ and use the shorthand notation $\sum_{n,k,r} c_{nkr}e^{b_{kr}t}$ for $r = 0, 1$, where $b_{kr} = (2k + \frac{5}{2} + 2r), c_{nk0} = a_{nk}, c_{nk1} = -b_{nk}$. We define $b_{kr\Delta} = b_{kr} - \Delta$ where $\Delta = 2\sigma$ and $\Delta = 0$ for a general $E_0(t)e^{-\Delta t} = \sum_{n,k,r} c_{nkr}e^{b_{kr\Delta}t}$ [**In the next section**, we will show similar results for a general $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_0(\omega)e^{i\omega t}d\omega$, without using Taylor series expansion.]

We will use the well known result $\int_{-\infty}^{t} e^{b_{kr\Delta}\tau}\cos(\omega_0(t)\tau)d\tau = \frac{e^{b_{kr\Delta}t}}{(b_{kr\Delta}^2+\omega_0^2(t))}[b_{kr\Delta}\cos(\omega_0(t)t)+\omega_0(t)\sin(\omega_0(t)t)]$ and $\int_{-\infty}^{t} e^{b_{kr\Delta}\tau}\sin(\omega_0(t)\tau)d\tau = \frac{e^{b_{kr\Delta}t}}{(b_{kr\Delta}^2+\omega_0^2(t))}[b_{kr\Delta}\sin(\omega_0(t)t) - \omega_0(t)\cos(\omega_0(t)t)]$. Given the fact that every term in taylor series expansion of $E_p(t)$ converges inside the integral, we can interchange the order of summation and integration as follows, using the theorem of dominated convergence.

$$
\begin{aligned}
I_1(t) = \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\cos(\omega_0(t)\tau)d\tau &= \sum_{n,k,r} c_{nkr}[\frac{e^{b_{kr\Delta}t}}{(b_{kr\Delta}^2+\omega_0^2(t))}[b_{kr\Delta}\cos(\omega_0(t)t)+\omega_0(t)\sin(\omega_0(t)t)] \\
\frac{dI_1(t)}{dt} &= \sum_{n,k,r} c_{nkr}[\frac{1}{(b_{kr\Delta}^2+\omega_0^2(t))^2}[(b_{kr\Delta}^2+\omega_0^2(t))e^{b_{kr\Delta}t}[\cos(\omega_0(t)t)(b_{kr\Delta}^2+\omega_0(t)\frac{d\theta(t)}{dt}) \\
&\quad + \sin(\omega_0(t)t)(b_{kr\Delta}\omega_0(t) - b_{kr\Delta}\frac{d\theta(t)}{dt} + \frac{d\omega_0(t)}{dt})] \\
&\quad - \frac{1}{(b_{kr\Delta}^2+\omega_0^2(t))^2}e^{b_{kr\Delta}t}[b_{kr\Delta}\cos(\omega_0(t)t)+\omega_0(t)\sin(\omega_0(t)t)](2\omega_0(t)\frac{d\omega_0(t)}{dt})] \\
(\frac{dI_1(t)}{dt})_{t=0} &= \sum_{n,k,r} c_{nkr} = E_0(0) = e_0
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
I_2(t) = \int_{-\infty}^{t} E_0(\tau)e^{-\Delta\tau}\sin(\omega_0(t)\tau)d\tau &= \sum_{n,k,r} c_{nkr}[\frac{e^{b_{kr\Delta}t}}{(b_{kr\Delta}^2+\omega_0^2(t))}[b_{kr\Delta}\sin(\omega_0(t)t)-\omega_0(t)\cos(\omega_0(t)t)] \\
\frac{dI_2(t)}{dt} &= \sum_{n,k,r} c_{nkr}[\frac{1}{(b_{kr\Delta}^2+\omega_0^2(t))^2}[(b_{kr\Delta}^2+\omega_0^2(t))e^{b_{kr\Delta}t}[\sin(\omega_0(t)t)(b_{kr\Delta}^2+\omega_0(t)\frac{d\theta(t)}{dt}) \\
&\quad - \cos(\omega_0(t)t)(b_{kr\Delta}\omega_0(t) - b_{kr\Delta}\frac{d\theta(t)}{dt} + \frac{d\omega_0(t)}{dt})] \\
&\quad - \frac{1}{(b_{kr\Delta}^2+\omega_0^2(t))^2}e^{b_{kr\Delta}t}[b_{kr\Delta}sin(\omega_0(t)t)-\omega_0(t)\cos(\omega_0(t)t)](2\omega_0(t)\frac{d\omega_0(t)}{dt})] \\
(\frac{dI_2(t)}{dt})_{t=0} &= 0
\end{aligned}
\tag{58}
$$

We will use results in Eq. 57 and Eq. 58 in above equation as below. Given that at $t = 0$, $\frac{dI_1(t)}{dt} = e_0$ and $\frac{d\theta(t)}{dt} = \omega_{00}$, $I_1(0) = m_0, I_2(0) = n_0$ we can write

$$
\begin{aligned}
(\frac{dB(t)}{dt})_{t=0} &= \Delta I_1(0) + e_0 + I_2(0)\omega_{00} \\
(\frac{dB(t)}{dt})_{t=0} &= \Delta m_0 + e_0 + n_0\omega_{00}
\end{aligned}
\tag{59}
$$

Now we can substitute $\Delta = 2\sigma$ and $\Delta = 0$ respectively and derive results below.

$$
\begin{aligned}
(\frac{dm_0(t)}{dt})_{t=0} &= 2\sigma m_0 + e_0 + n_0\omega_{00} \\
(\frac{dm_{0p}(t)}{dt})_{t=0} &= e_0 + n_{0p}\omega_{00}
\end{aligned}
$$

$$(60)$$

So we can write second derivatives of $B(t)$ as below. **Check Again**

$$\begin{aligned} b_0 = [B(t)]_{t=0} = \int_{-\infty}^{0} E_0(\tau) e^{-\Delta\tau} \cos{(\omega_0(t)\tau)} d\tau = m_0 \\ b_1 = [\frac{dB(t)}{dt}]_{t=0} = \Delta I_1(0) + [\frac{dI_1(t)}{dt}]_{t=0} + I_2(0)\omega_{01} = \Delta m_0 + e_0 + n_0\omega_{01} \end{aligned} \tag{61}$$

We can use this procedure to obtain desired result as follows.

$$\begin{aligned} R(t) = e^{2\sigma t}[\cos{(\omega_0(t)t)} \int_{-\infty}^{t} E_0(\tau) e^{-2\sigma\tau} \cos{(\omega_0(t)\tau)} d\tau + \sin{(\omega_0(t)t)} \int_{-\infty}^{t} E_0(\tau) e^{-2\sigma\tau} \sin{(\omega_0(t)\tau)} d\tau] \\ + \cos{(\omega_0(t)t)} \int_{-\infty}^{-t} E_0(-\tau) \cos{(\omega_0(t)\tau)} d\tau - \sin{(\omega_0(t)t)} \int_{-\infty}^{-t} E_0(-\tau) \sin{(\omega_0(t)\tau)} d\tau \\ r_0 = [R(t)]_{t=0} = \int_{-\infty}^{0} E_0(\tau) e^{-2\sigma\tau} \cos{(\omega_0(t)\tau)} d\tau + \int_{-\infty}^{0} E_0(-\tau) \cos{(\omega_0(t)\tau)} d\tau = m_0 + m_{0p} \\ r_1 = [\frac{dR(t)}{dt}]_{t=0} = 2\sigma m_0 + e_0 + n_0\omega_{00} - (e_0 + n_{0p}\omega_{00}) = \omega_{00}(n_0 - n_{0p}) + 2\sigma m_0 \end{aligned} \tag{62}$$

## 12. $\omega_0(t)$ is at least 2 times differentiable around $t = 0$

In Section 2 Eq. 63, we derived
$P(t_0) = 2e^{\sigma t_0}[\cos(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_p(\tau)\cos(\omega_0(t_0)\tau)d\tau + \sin(\omega_0(t_0)t_0)\int_{-\infty}^{t_0} E_p(\tau)\sin(\omega_0(t_0)\tau)d\tau]$ and observed that $P(t_0) + P(t_0) = 0$ for all $-\infty \le t_0 \le \infty$. Replacing $t_0$ by $t$, we have

$$P(t) = 2e^{\sigma t}[\cos(\omega_0(t)t)\int_{-\infty}^{t} E_p(\tau)\cos(\omega_0(t)\tau)d\tau + \sin(\omega_0(t)t)\int_{-\infty}^{t} E_p(\tau)\sin(\omega_0(t)\tau)d\tau]$$
$$f(t) = P(t) + P(t) = 0 \tag{63}$$

In this section, we will show that $\omega_0(t)$ is at least 2 times differentiable around $t = 0$, so that $\frac{d\omega_0(t)}{dt}$ and $\frac{d^2\omega_0(t)}{dt^2}$ evaluated at $t = 0$ remain finite and continuous around $t = 0$.

Let us consider an interval $[-dt, dt]$ around $t = 0$ and we see that $f(t) = 0$ in that interval and is continuous in that interval and hence $\omega_0(t)$ cannot have **dirac delta function** $\delta(t)$ in that interval. If it did have dirac delta function in that interval, integrals in above equation will produce a **jump discontinuity** around $t = 0$, which is clearly **not** the case.

Now we consider the first derivative of $f(t)$ and we know that $\frac{df(t)}{dt} = 0$ in the interval $[-dt, dt]$ around $t = 0$ and is continuous in that interval. We know from Eq. 53 and other equations in previous section, that $\frac{df(t)}{dt}$ has the term $\frac{d\omega_0(t)}{dt}$. Hence $\frac{d\omega_0(t)}{dt}$ is **continuous** in that interval, which means $\omega_0(t)$ must have terms of the order of $t^2$ or higher.

We can also see that $\omega_0(t)$ cannot have **jump discontinuity** like a heaviside unit step function in that interval. If it did have jump discontinuity in that interval, integrals in above equation will produce a **triangular** ramp around $t = 0$, and when we take first derivative of $f(t)$, this triangular ramp will produce a **jump discontinuity** in $\frac{df(t)}{dt}$, which is clearly **not** the case.

Now we consider the **second derivative** of $f(t)$ and we know that $\frac{d^2f(t)}{dt^2} = 0$ in the interval $[-dt, dt]$ around $t = 0$ and is continuous in that interval. We know from Eq. 53 and other equations in previous section, that $\frac{d^2f(t)}{dt^2}$ has the term $\frac{d^2\omega_0(t)}{dt^2}$. Hence $\frac{d^2\omega_0(t)}{dt^2}$ is **continuous** in that interval, which means $\omega_0(t)$ must have terms of the **order** of $t^3$ or higher.

We can also see that $\omega_0(t)$ **cannot** have terms of the order of $t$ like a triangular ramp function in that interval. If it did have such a term in that interval, integrals in above equation will produce terms of order $t^2$ around $t = 0$, and when we take second derivative of $f(t)$, this term will produce a **jump discontinuity** in $\frac{df(t)}{dt}$, which is clearly **not** the case.

Hence we have shown that the terms $\frac{d\omega_0(t)}{dt}$ and $\frac{d^2\omega_0(t)}{dt^2}$ iare **continuous** in that interval $[-dt, dt]$ around $t = 0$ and hence $\omega_0(t)$ is at least 2 times differentiable.

**Method 2:**

By assumption, we know that $\omega_0(t)$ is finite at $t = 0$ and $\omega_0(t) = \omega_0(-t)$. Hence $\frac{d\omega_0(t)}{dt}$ is an odd function of variable $t$ and hence is zero at $t = 0$. This holds **even if** $\omega_0(t)$ is nowhere differentiable. Hence these 2 results are sufficient for our procedure outlined in previous sections.