

On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s -plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.^{[4] [5]} We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.^[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.^[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < \text{Re}[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new

method discussed in this paper will lead to a correct solution by other researchers.

In Section 2 to Section 7, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 8, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$.

We present an **outline** of the new method below.

1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$, where ω, t are real, as follows (link).^[3]

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t , given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t) = E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \leq \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. Step 2: On the zeros of a related function $G(\omega, t_2, t_0)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t, t_2, t_0) = f(t, t_2, t_0) e^{-\sigma t} u(-t) + f(t, t_2, t_0) e^{\sigma t} u(t)$, where $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ and $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ and $E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$ and t_0, t_2 are real and $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can

73 see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

74
75 In Section 2.1, we will show that the Fourier transform of the **even function** $g_{\text{even}}(t, t_2, t_0) =$
76 $\frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_0, t_2)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$,
77 for every value of t_0 , for a given value of t_2 , to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

78 1.3. Step 3: On the zeros of the function $G_R(\omega, t_0, t_2)$

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80 In Section 2.2, we compute the Fourier transform of the function $g(t, t_2, t_0)$ and compute its real
81 part given by $G_R(\omega, t_2, t_0)$ and we can write as follows.

$$\begin{aligned} G_R(\omega, t_2, t_0) = & e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ & + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \end{aligned}$$

(3)

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83 We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **each fixed value**
84 of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) =$
85 $G_R(\omega_z(t_2, t_0), t_2, t_0) = 0$.

86 1.4. Step 4: Zero Crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

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88 In Section 2.3, we show the result in Eq. 4 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that
89 $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = P_{\text{odd}}(t_2, t_0) + P_{\text{odd}}(t_2, -t_0) = 0$ and that $P_{\text{odd}}(t_2, t_0)$ is an **odd**
90 function of t_0 , for a given value of t_2 as follows.

$$\begin{aligned} P_{\text{odd}}(t_2, t_0) = & [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ & + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ & + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau] \end{aligned}$$

(4)

92 1.5. Step 5: Final Step

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94 In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all
95 $|t_0| \leq \infty$ and $|t_2| \leq \infty$. In Section 7, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

96
97 In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute
98 in the equation for $P_{\text{odd}}(t_2, t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use
99 $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_{0n}(t, t_2) = E'_0(-t, t_2)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$$

(5)

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$, where $t_{0c} > 0$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t , where $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix B.1.

2.1. New function $g(t, t_2, t_0)$

Let us consider the function $E_p'(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t, t_2)e^{-\sigma t}$, where t_2 is finite and real, and $E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0, t_2)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega, t_2, t_0) = E_{p\omega}'(\omega, t_2)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero

at the **same** $\omega = \omega_0$.

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Let us consider a new function $g(t, t_2, t_0) = g_-(t, t_2, t_0)u(-t) + g_+(t, t_2, t_0)u(t)$ where $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}$ and $g_+(t, t_2, t_0) = f(t, t_2, t_0)e^{\sigma t}$. We can see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

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We can show that $E_p(t), E'_p(t, t_2), h(t), g(t, t_2, t_0)$ are real absolutely integrable functions and go to zero as $t \rightarrow \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E'_{p\omega}(\omega, t_2), H(\omega), G(\omega, t_2, t_0)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \rightarrow \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix B.1.

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If we take the Fourier transform of the equation $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega, t_2, t_0)*H(\omega)] = F(\omega, t_2, t_0) = E'_{p\omega}(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega, t_2, t_0) + iF_I(\omega, t_2, t_0)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega', t_2, t_0)H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega, t_2, t_0) = G_R(\omega, t_0, t_2) + iG_I(\omega, t_2, t_0)$ is the Fourier transform of the function $g(t, t_2, t_0)$. We can write $g(t, t_2, t_0) = g_{\text{even}}(t, t_2, t_0) + g_{\text{odd}}(t, t_2, t_0)$ where $g_{\text{even}}(t, t_2, t_0)$ is an even function and $g_{\text{odd}}(t, t_2, t_0)$ is an odd function of variable t .

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If Statement 1 is true, then we require the Fourier transform of the function $f(t, t_2, t_0)$ given by $F(\omega, t_2, t_0)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0 , for a given fixed value of t_2 . This implies that the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_0, t_2)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign, and can be different from ω_0 in general. We call this **Statement 2**.

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Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega, t_0, t_2)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign, **then the real part** of $F(\omega, t_2, t_0)$ given by $F_R(\omega, t_2, t_0) = \frac{1}{2\pi}[G_R(\omega, t_0, t_2) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega, t_0, t_2)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**. This is shown in detail in Lemma 1.

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Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_0, t_2)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0 , for a given finite value of t_2 , where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign and $\omega_z(t_2, t_0)$ is real and finite, where $g(t, t_2, t_0)h(t) = f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0}E'_p(t - t_0, t_2)$, $E'_p(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2)$, and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

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Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega, t_2, t_0) = E'_{p\omega}(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega, t_2, t_0)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

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Let us consider the case where $G_R(\omega, t_2, t_0)$ **does not** have at least one zero for finite $\omega =$

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186 $\omega_z(t_2, t_0) \neq 0$ and show that $F_R(\omega, t_2, t_0)$ does not have at least one zero at finite $\omega \neq 0$ for this case,
 187 which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only
 188 for the real parts as follows.

$$F_R(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) H(\omega - \omega') d\omega' \quad (6)$$

189 We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega, t_2, t_0)$ and $H(\omega)$
 190 have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t, t_2, t_0)$ and $h(t)$ are discontinuous
 191 at $t = 0$.(Appendix B.2)

192 We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 6 and we get

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (7)$$

194 We can split the integral in Eq. 7 as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \left[\int_{-\infty}^0 G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (8)$$

196 We see that $G_R(-\omega, t_2, t_0) = G_R(\omega, t_2, t_0)$ because $g(t, t_2, t_0)$ is a real function (link). We can
 197 substitute $\omega' = -\omega''$ in the first integral in Eq. 8 and substituting $\omega'' = \omega'$ in the result, we can write
 198 as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_0^{\infty} G_R(\omega', t_2, t_0) \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (9)$$

200 In Appendix B.1 last paragraph, it is shown that $G(\omega, t_2, t_0)$ is finite for $|\omega| \leq \infty$ and goes to
 201 zero as $|\omega| \rightarrow \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 9 is zero. For
 202 finite $\omega \geq 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$. We see that
 203 $G_R(\omega', t_0, t_2)$ is **not** an all zero function and that $G_R(\omega', t_2, t_0)$ is a continuous function of ω' , for a
 204 fixed t_0 and t_2 (Section 4.1).

205
 206 • **Case 1:** $G_R(\omega', t_2, t_0) \geq 0$ for all finite $\omega' > 0$

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 208 We see that $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because
 209 $f(t, t_2, t_0)$ is a real function (link). Hence $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \leq 0$.

210
 211 This **contradicts** Statement 1 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$
 212 because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega', t_2, t_0)$ must have **at**
 213 **least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line and becomes negative, where
 214 $\omega_z(t_2, t_0)$ is real and finite.

215
 216 • **Case 2:** $G_R(\omega', t_2, t_0) \leq 0$ for all finite $\omega' > 0$

217

218 We see that $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because
 219 $f(t, t_2, t_0)$ is a real function (link). Hence $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \leq 0$.

220
 221 This **contradicts** Statement 1 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$.
 222 Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where it crosses the
 223 zero line and becomes positive, where $\omega_z(t_2, t_0)$ is real and finite.

224
 225 We have shown that, $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where
 226 it crosses the zero line to the opposite sign, to satisfy **Statement 1**. We call this **Statement 2**. In
 227 the rest of the sections, we consider only the **first** zero crossing away from origin, where $G_R(\omega, t_0, t_2)$
 228 crosses the zero line to the opposite sign. Hence $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for every finite
 229 value of t_2 .

2.2. On the zeros of a related function $G(\omega, t_2, t_0)$

230
 231
 232 We can compute the fourier transform of the function $g_{even}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$
 233 given by $G_R(\omega, t_2, t_0)$. We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0 , for a
 234 given value of t_2 , to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

235
 236 First we compute the Fourier transform of the function $g_1(t, t_2, t_0)$ given by $G_1(\omega, t_2, t_0) = G_{1R}(\omega, t_2, t_0) +$
 237 $iG_{1I}(\omega, t_2, t_0)$. We use $g_1(t, t_2, t_0) = f_1(t, t_2, t_0)e^{-\sigma t}u(-t) + f_1(t, t_2, t_0)e^{\sigma t}u(t) = e^{\sigma t_0}E'_p(t+t_0, t_2)e^{-\sigma t}u(-t) +$
 238 $e^{\sigma t_0}E'_p(t+t_0, t_2)e^{\sigma t}u(t)$.

239

$$\begin{aligned} G_1(\omega, t_2, t_0) &= \int_{-\infty}^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt = \int_{-\infty}^0 g_1(t, t_2, t_0)e^{-i\omega t}dt + \int_0^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt \\ G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 e^{\sigma t_0}E'_p(t+t_0, t_2)e^{-\sigma t}e^{-i\omega t}dt + \int_0^{\infty} e^{\sigma t_0}E'_p(t+t_0, t_2)e^{\sigma t}e^{-i\omega t}dt \end{aligned}$$

240

(10)

241 We use $E'_p(t, t_2) = E'_0(t, t_2)e^{-\sigma t}$ where $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_p(t + t_0, t_2) =$
 242 $E'_0(t + t_0, t_2)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 10, we have

$$\begin{aligned} G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 E'_0(t+t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_0^{\infty} E'_0(t+t_0, t_2)e^{-i\omega t}dt \\ G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 E'_0(t+t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_0(-t+t_0, t_2)e^{i\omega t}dt \end{aligned}$$

243

(11)

244 We define $E'_{0n}(t, t_2) = E'_0(-t, t_2)$ and get $E'_0(-t+t_0, t_2) = E'_{0n}(t-t_0, t_2)$ and write Eq. 11 as
 245 follows.

$$G_1(\omega, t_2, t_0) = \int_{-\infty}^0 E'_0(t+t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_{0n}(t-t_0, t_2)e^{i\omega t}dt = G_{1R}(\omega, t_2, t_0) + iG_{1I}(\omega, t_2, t_0)$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$.
Comparing the **real parts** of $G_1(\omega)$, we have

$$G_{1R}(\omega, t_2, t_0) = \int_{-\infty}^0 E'_0(t + t_0, t_2) e^{-2\sigma t} \cos(\omega t) dt + \int_{-\infty}^0 E'_{0n}(t - t_0, t_2) \cos(\omega t) dt$$

249

2.3. Zero crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0 , for a fixed t_2

251

Now we consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t + t_0, t_2) + e^{\sigma t_0} E'_p(t - t_0, t_2)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ and $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g(t, t_2, t_0)$ and compute its real part $G_R(\omega, t_2, t_0)$ using the procedure in above section, similar to Eq. 13 and we can write as follows. We substitute $t = \tau$.

$$\begin{aligned} G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_2, t_0) + e^{2\sigma t_0} G_{1R}(\omega, t_2, -t_0) \\ G_{1R}(\omega, t_2, t_0) &= \int_{-\infty}^0 [E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega \tau) d\tau \\ G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega \tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega \tau) d\tau \end{aligned}$$

258

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **every finite value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = 0$ and we can rearrange the terms as follows.

$$\begin{aligned} P(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0) \tau) d\tau \\ &\quad + \int_{-\infty}^0 [e^{2\sigma t_0} E'_0(\tau - t_0, t_2) e^{-2\sigma \tau} + e^{-2\sigma t_0} E'_{0n}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0) \tau) d\tau = 0 \end{aligned}$$

262

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0 .

263

$$\begin{aligned} P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\ P_{odd}(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0) \tau) d\tau \end{aligned}$$

264

We see that $f(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t + t_0, t_2) + e^{\sigma t_0} E'_p(t - t_0, t_2) = f(t, t_2, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0 , for **every finite value** of t_2 .

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 16 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E'_{0n}(\tau, t_2) = E'_0(-\tau, t_2)$ and $E'_0(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

$$\begin{aligned}
P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}
\tag{17}$$

In Section 2.1, it is shown that $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for a given value of t_2 .

In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$.

In Section 7, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given that $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2 , we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2 and given that t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function and is positive for $t_0 > 0$ because $0 < \omega_z(t_2, t_0) < \infty$.

We see that $\omega_z(t_2, t_0) > 0$ as $t_0 \rightarrow \infty$ and is a **continuous** function of variable t_0 and t_2 , as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$ and that the order of $\omega_z(t_2, t_0)$ is greater than or equal to 1 (Section 5 and Section 6) and the order of $\omega_z(t_2, t_0)t_0$ is greater than or equal to $O[t_0]$. As t_0 is increased from zero towards ∞ , the continuous function $\omega_z(t_2, t_0)t_0$ starts from zero and increases towards ∞ with order greater than or equal to $O[t_0]$ and will pass through $\frac{\pi}{2}$.

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 17 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\begin{aligned}
&\int_{-\infty}^{t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&- \int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned}$$

We split the first two integrals in the left hand side of Eq. 18 and write as follows.

We combine the terms with common integrals and cancel common terms in Eq. 19 as follows.

We can rearrange the terms in Eq. 20 as follows.

We denote the right hand side of Eq. 21 as *RHS*. We can split the integral in Eq. 21 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^0 + \int_0^{t_{0c}}$ as follows.

We substitute $\tau = -\tau$ in the first integral in Eq. 22 as follows. We use $E'_0(-\tau, t_{2c}) = E'_{0n}(\tau, t_{2c})$ and $E'_{0n}(-\tau, t_{2c}) = E'_0(\tau, t_{2c})$.

$$\begin{aligned} & \int_{-t_{0c}}^0 [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \int_0^{t_{0c}} [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned}$$

308 Given that $\int_{t_{0c}}^0 = -\int_0^{t_{0c}}$, we can simplify as follows.

$$\int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS$$

309

(24)

310 We substitute $\tau = -\tau$ in the right hand side of Eq. 21 as follows. We use $E'_{0n}(-\tau, t_{2c}) = E'_0(\tau, t_{2c})$.

$$RHS = 2 \sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$

311

(25)

312 We split the integral on the right hand side in Eq. 25 as follows.

$$RHS = 2 \sinh(2\sigma t_{0c}) \left[\int_0^{\infty} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right]$$

313

(26)

314 We consolidate the integrals with the term $\int_0^{t_{0c}} E'_0(\tau, t_{2c})$ in Eq. 24 and Eq. 26 as follows. We use
 315 $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^{\infty} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

316

(27)

317 We cancel common terms in Eq. 27 as follows.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^{\infty} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

318

(28)

319 We substitute $E'_0(\tau, t_{2c}) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E'_{0n}(\tau, t_{2c}) = E'_0(-\tau, t_{2c}) = E_0(-\tau - t_{2c}) -$
 320 $E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that
 321 $E_0(\tau) = E_0(-\tau)$. Hence we see that $E'_{0n}(\tau, t_{2c}) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E'_0(\tau, t_{2c})$. We can
 322 write Eq. 28 as follows.

$$\begin{aligned}
& \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& = 2 \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned}$$

(29)

We substitute $2 \cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2 \cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 29 as follows.

$$\begin{aligned}
& \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& = \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned}$$

(30)

Next Step:

We denote the right hand side of Eq. 30 as *RHS*. We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 30 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$\begin{aligned}
RHS = & \sinh(2\sigma t_{0c}) [\cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& + \sin(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{-t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& - \cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau]
\end{aligned}$$

(31)

In Eq. 31, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$. Hence we cancel common terms and write Eq. 31 and Eq. 30 as follows.

$$\begin{aligned}
& \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
& = -\sinh(2\sigma t_{0c}) \left[\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right]
\end{aligned}$$

(32)

We use $\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 32 as follows. Given that $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ , we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)) d\tau = 0$$

(33)

We can multiply Eq. 33 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)) d\tau = 0$$

(34)

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and the sinusoidal function is > 0 , in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 34, we see that in the interval $0 < \tau < t_{0c}$, the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0 , in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ (Section 7), we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 7.3) in the interval $0 < \tau < t_{0c}$. The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 34. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 34 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

4. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2

We see from Section 2.1 that $\omega_z(t_2, t_0)$ is shown to be **finite and non-zero** for all $|t_0| < \infty$ and $|t_2| < \infty$ and that $\omega_z(t_2, t_0)$ is an even function of variable t_0 , for a given value of t_2 . For a given t_2 and t_0 , $\omega_z(t_2, t_0)$ can have more than one value, but we consider only the first zero crossing away from origin in the section below, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, as detailed in **Lemma 1** in Section 2.1 and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$. (example plot)

We consider the Fourier transform of the even part of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0 , the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 , for all $|t_0| < \infty$, for **each** finite value of t_2 . This is shown in the steps below. For a given **finite** value of t_2 , $G_R(\omega, t_2, t_0)$ is a function of two

variables ω and t_0 , and we use Implicit Function Theorem in R^2 .

• It is shown in Section 4.1 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω , as shown in Eq. 35.

• It is shown in Section 4.2 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_0 , as shown in Eq. 37 and Eq. 42.

• It is shown in Section 4.3 that the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **continuous** function of t_0 , for a given t_2 , using **Implicit Function Theorem** in R^2 .

• It is shown in Section 4.4 that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$, using **Implicit Function Theorem** in R^3 .

4.1. $G_R(\omega, t_2, t_0)$ is *partially differentiable twice as a function of ω*

$G_R(\omega, t_2, t_0)$ in Eq. 14 is copied below and we can expand $G_R(\omega, t_2, t_0)$ in Eq. 35 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 17.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau = G'_{1R}(\omega, t_2, t_0) + G'_{1R}(\omega, t_2, -t_0) \\
G'_{1R}(\omega, t_2, t_0) &= [\cos(\omega t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \cos(\omega\tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \sin(\omega\tau) d\tau] \\
&+ e^{-2\sigma t_0} [\cos(\omega t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \cos(\omega\tau) d\tau - \sin(\omega t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \sin(\omega\tau) d\tau]
\end{aligned} \tag{35}$$

We could then use $E'_0(t, t_2) = (E_0(t - t_2) - E_0(t + t_2))$ and $E'_{0n}(t, t_2) = E'_0(-t, t_2)$ and substitute $t + t_2 = t$ and $t - t_2 = t'$ and expanding it using the procedure used in Eq. 35. The integrands are absolutely integrable and we could then use theorem of dominated convergence as follows.

$G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω and the integrals converge in Eq. 35 for $0 < \sigma < \frac{1}{2}$, because the term $\tau^r E'_0(\tau - t_0, t_2)e^{-2\sigma\tau}$ has exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$, for $r = 0, 1, 2$ (Appendix B.4). The integrands are absolutely integrable and the integrands are analytic functions of variables ω and t_0 , for a given t_2 . We can interchange the order of partial differentiation and integration in Eq. 36 using theorem of dominated convergence, recursively as follows.(link) (We could also use theorem 3 in link and link.)

$$\begin{aligned}
\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \\
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial \omega^2} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau]
\end{aligned}$$

(36)

403

404 *4.2. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_0*

405

406 $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals converge
407 in Eq. 37 and Eq. 42 shown as follows. The integrands are absolutely integrable because the term
408 $E'_0(\tau - t_0, t_2)e^{-2\sigma\tau}$ has exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Appendix B.4). The
409 integrands are analytic functions of variables ω and t_0 , for a given t_2 and we can expand $G_R(\omega, t_2, t_0)$
410 in Eq. 37 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 35. We can interchange the
411 order of partial differentiation and integration in Eq. 37 and Eq. 42 using theorem of dominated
412 convergence as follows. (link) (We could also use theorem 3 in link and link)

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
&\quad + 2\sigma e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau
\end{aligned}$$

(37)

413

414 We can show that the integrals in Eq. 37 converge, as follows. We see that $E'_0(\tau + t_0, t_2) =$
415 $E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$.

416

417 We see that $E'_0(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$ and $E'_{0n}(\tau, t_2) = E'_0(-\tau, t_2) = -E'_0(\tau, t_2)$ because
418 $E_0(-\tau) = E_0(\tau)$. We get $E'_{0n}(\tau - t_0, t_2) = -E'_0(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$
419 given that $E_0(\tau) = E_0(-\tau)$. We see that the third integral in Eq. 37 converges because the term
420 $E'_0(\tau - t_0, t_2)e^{-2\sigma\tau}$ has exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Appendix B.4).

421

422 We consider the integrand in the fourth integral in Eq. 37 first and use the results in the above
 423 paragraph.

$$\begin{aligned} \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_0} \\ &\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_0} \end{aligned} \quad (38)$$

425 We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 38 and can show that the integrals converge in
 426 Eq. 37, as follows.

$$\begin{aligned} E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\ E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \end{aligned} \quad (39)$$

428 We can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows. (**Result A**)

$$\begin{aligned} \frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \end{aligned} \quad (40)$$

430 We can replace t_0 by $-t_0$ in Eq. 40 and show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 - t_0) = -\frac{\partial}{\partial \tau} E_0(\tau + t_2 - t_0)$. (**Result B**)

431
 432 We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 38, corresponding to the term in the fourth
 433 integral in Eq. 37, using Result A, as follows.

$$\begin{aligned} \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\ &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0) \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\ &= [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\ &\quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \cos(\omega\tau) d\tau \end{aligned}$$

We see that the integrals in Eq. 41 converge and hence the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau$ in Eq. 41 also converges. We set $\sigma = 0$ and $t_0 = -t_0$ and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_0} \cos(\omega\tau) d\tau$ in Eq. 38 also converges, using Result B.

We set $t_2 = -t_2$ in Eq. 39 to Eq. 41 and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau$ in Eq. 38 also converges. We set $\sigma = 0$ and $t_0 = -t_0$ and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau-t_0-t_2))}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 38 also converges, using Result B. Hence the fourth integral in Eq. 37 corresponding to the terms in Eq. 38, also converges.

We can see that the last two integrals in Eq. 37 converge, by setting $t_0 = -t_0$ and using Result B. Hence all the integrals in Eq. 37 converge.

The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_0 is given by $\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} = \frac{\partial}{\partial t_0} \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ as follows. We use the result in Eq. 41 and we can interchange the order of partial differentiation and integration in Eq. 42 using theorem of dominated convergence as follows.

$$\begin{aligned} \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ &\quad - 4\sigma e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\ &\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0^2} \cos(\omega\tau) d\tau \\ &\quad + 4\sigma^2 e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\ &\quad + 4\sigma e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0^2} \cos(\omega\tau) d\tau \end{aligned}$$

450

We can use the above procedure in Eq. 39 to Eq. 41 for the term $\frac{\partial^2(E'_0(\tau+t_0,t_2)e^{-2\sigma\tau}+E'_{0n}(\tau-t_0,t_2))}{\partial t_0^2} = \frac{\partial I(\tau, t_0, t_2)}{\partial t_0}$ where $I(\tau, t_0, t_2) = \frac{\partial(E'_0(\tau+t_0,t_2)e^{-2\sigma\tau}+E'_{0n}(\tau-t_0,t_2))}{\partial t_0}$ in the third integral in Eq. 42 and we can show that it converges, using the procedure used in Eq. 41 twice.

454

We can see that the last three integrals in Eq. 42 converge, by setting $t_0 = -t_0$ and using Result B. Hence all the integrals in Eq. 37 and Eq. 42 converge.

457

4.3. Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 , for a given t_2 .

458

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 , for a given fixed value of t_2 ,

461

with continuous partial derivatives (Section 4.1 and Section 4.2) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0 , for all $|t_0| < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for all $|t_0| < \infty$, for each finite value of t_2 .

• It is shown in Section 4.5 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2 . We can use the procedure in previous subsections and Implicit Function Theorem and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 , for all $|t_2| < \infty$, for **each** finite value of t_0 .

4.4. *Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 and t_2*

We can use the procedure in previous subsections and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 and t_0 , for all $|t_0| < \infty$ and $|t_2| < \infty$, using Implicit Function Theorem in R^3 .

We use **Implicit Function Theorem** for the three dimensional case (link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2 , with continuous partial derivatives (Section 4.1, Section 4.2 and Section 4.5) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$ (**Result E**). We see that t_0, t_2 are real and as they increase to larger and larger values without bounds, for each such finite value of t_0 and t_2 , we see that $\omega_z(t_2, t_0)$ should remain a **continuous** function, as $|t_0| \rightarrow \infty$ and $|t_2| \rightarrow \infty$ (Section 6.1).

4.5. *$G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_2*

$G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals converge in Eq. 43 and Eq. 47 shown as follows. The integrands are analytic functions of variables ω and t_2 , for a given t_0 and we can expand $G_R(\omega, t_2, t_0)$ in Eq. 43 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 35. We can interchange the order of partial differentiation and integration in Eq. 43 using theorem of dominated convergence as follows. (link) (We could also use theorem 3 in link and link)

$$\begin{aligned} G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\ \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau \end{aligned}$$

(43)

495 We use the procedure outlined in Eq. 38 to Eq. 41, with t_0 replaced by t_2 and show that all the
 496 integrals in Eq. 43 converge, as follows.

497
 498 We see that $E'_0(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E'_{0n}(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) -$
 499 $E_0(\tau - t_0 - t_2)$ (the paras above Eq. 38). We consider the integrand in the third integral in Eq. 43
 500 first.

$$\begin{aligned} \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2} \\ &\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2} \end{aligned} \quad (44)$$

502 We consider the term $E_0(\tau + t_0 + t_2)$ first and can show that the integrals converge in Eq. 43, as
 503 follows. We consider Eq. 39 and show that $\frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows. (**Result**
 504 **C**)

$$\begin{aligned} \frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] \\ &\quad + \left(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}\right) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \end{aligned} \quad (45)$$

506 We can replace t_2 by $-t_2$ in Eq. 45 and show that $\frac{\partial}{\partial t_2} E_0(\tau + t_0 - t_2) = -\frac{\partial}{\partial \tau} E_0(\tau + t_0 - t_2)$ (**Result**
 507 **D**). We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 44, corresponding to the term in the third
 508 integral in Eq. 43 as follows.

$$\begin{aligned} \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\ &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0) \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\ &= [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\ &\quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \cos(\omega\tau) d\tau \end{aligned} \quad (46)$$

510 We see that the integrals in Eq. 46 converge and hence the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau$
 511 in Eq. 46 also converges. We set $\sigma = 0$ and $t_0 = -t_0$ and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 - t_0))}{\partial t_2} \cos(\omega\tau) d\tau$

512 in Eq. 44 also converges.

513

514 We set $t_2 = -t_2$ in Eq. 45 to Eq. 46 and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau)d\tau$
 515 in Eq. 44 also converges, using Result D. We set $\sigma = 0$ and $t_0 = -t_0$ and see that the integral
 516 $\int_{-\infty}^0 \frac{\partial(E_0(\tau-t_2-t_0))}{\partial t_2} \cos(\omega\tau)d\tau$ in Eq. 44 also converges. Hence the third integral in Eq. 43 correspond-
 517 ing to the terms in Eq. 44, also converges.

518

519 The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_2 is given by $\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} = \frac{\partial}{\partial t_2} \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$
 520 as follows. We use the result in Eq. 46 and we can interchange the order of partial differentiation and
 521 integration in Eq. 47 using theorem of dominated convergence as follows.

$$\begin{aligned} \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau+t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0, t_2))}{\partial t_2^2} \cos(\omega\tau)d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau-t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau+t_0, t_2))}{\partial t_2^2} \cos(\omega\tau)d\tau \end{aligned} \quad (47)$$

522

523 We can use the above procedure in Eq. 45 to Eq. 46 for the term $\frac{\partial^2(E'_0(\tau+t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0, t_2))}{\partial t_2^2} =$
 524 $\frac{\partial I(\tau, t_0, t_2)}{\partial t_2}$ where $I(\tau, t_0, t_2) = \frac{\partial(E'_0(\tau+t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau-t_0, t_2))}{\partial t_2}$ in the first integral in Eq. 47 and we can
 525 show that it converges, using the procedure used in Eq. 46 twice.

526

527 We can see that the second integral in Eq. 47 converge, by setting $t_0 = -t_0$ and using the proce-
 528 dure in this section. Hence all the integrals in Eq. 43 and Eq. 47 converge.

529

530 **5. $\omega_z(t_2, t_0)t_0$ does not decrease towards zero, as finite t_0 increases without bounds**

531

532 It is noted that we **do not** use $\lim_{t_0 \rightarrow \infty}$ in this section. Instead we consider real t_0 increases to a
533 larger and larger finite value without bounds.

534

535 We copy $P_{odd}(t_2, t_0)$ in Eq. 48 as follows. We use $E'_{0n}(\tau, t_2) = E'_0(-\tau, t_2)$ and $E'_0(\tau, t_2) = E_0(\tau -$
536 $t_2) - E_0(\tau + t_2)$.

$$\begin{aligned} P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &\quad P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \end{aligned}$$

537

(48)

538 Let us assume that $\omega_z(t_2, t_0)t_0$ decreases to a very small finite value close to zero, as real t_0 in-
539 creases to a larger and larger finite value without bounds. (**Statement B**) We see that t_0 is a real
540 number and as it increases to a larger and larger finite value without bounds, we can use the approx-
541 imations $\cos(\omega_z(t_2, t_0)t_0) \approx 1$, $\sin(\omega_z(t_2, t_0)t_0) \approx \omega_z(t_2, t_0)t_0 \approx 0$. We see that $\cos(\omega_z(t_2, t_0)\tau)$ and
542 $\sin(\omega_z(t_2, t_0)\tau)$ are finite and the integrals in the expression for $P_{odd}(t_2, t_0)$ in Eq. 48 converge to a
543 finite value.

544

545 We note that $E'_{0n}(\tau, t_2) = -E'_0(\tau, t_2) = E_0(\tau + t_2) - E_0(\tau - t_2)$. We choose $t_2 = 2t_0$ and in the
546 interval $[-\infty, t_0]$, we see that $E'_0(\tau, t_2) \approx -E_0(\tau + t_2)$ and $E'_{0n}(\tau, t_2) \approx E_0(\tau + t_2)$, for sufficiently
547 large t_0 . We choose t_3 such that $E_0(\tau)$ approximates zero for $\tau > t_3$ and $E_0(\tau + t_2)$ approximates
548 zero for $\tau > -t_2 + t_3$. As t_0, t_2 increase without bounds, we see that $t_3 \ll t_0$ and in the interval
549 $[-\infty, -t_2 + t_3]$, we see that the term $\cos(\omega_z(t_2, t_0)\tau) \approx 1$.

550

551 We see that the term $P_{odd}(t_2, -t_0)$ approaches a value very close to zero, as real t_0 increases to a
552 larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals $\int_{-\infty}^{-t_0}$. Hence
553 we can write as follows.

$$\begin{aligned} P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) &\approx \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \\ -e^{4\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma(\tau + 2t_0)} \cos(\omega_z(t_2, t_0)\tau) d\tau &+ e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \\ -e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma(\tau + 2t_0)} \cos(\omega_z(t_2, t_0)\tau) d\tau &+ \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \end{aligned}$$

554

(49)

Hence for sufficiently large t_0 , the integral $\int_{-\infty}^{t_0} E_0(\tau+2t_0)e^{-2\sigma(\tau+2t_0)} \cos(\omega_z(t_2, t_0)\tau) d\tau \approx \int_{-\infty}^{\infty} E_0(\tau+2t_0)e^{-2\sigma(\tau+2t_0)} d\tau = \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau} d\tau$ which remains finite and non-zero and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds.

The term $e^{2\sigma t_0}$ in $P_{odd}(t_2, t_0)$ in Eq. 49 increases to a larger and larger finite value **exponentially** and hence the term $P_{odd}(t_2, t_0)$ approaches a larger and larger finite value exponentially and hence $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ **cannot** equal zero in this case.

Hence **Statement B** is **false** and $\omega_z(t_2, t_0)t_0$ **does not** decrease towards zero, as finite t_0 increases without bounds. Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$ (Section 4), we see that the the order of $\omega_z(t_2, t_0)t_0$ is greater than or equal to 1, as finite t_0 increases without bounds.

5.1. *Order of $\omega_z(t_2, t_0)t_0$ is greater than $O[1]$*

We can show that the order of $\omega_z(t_2, t_0)t_0$ is greater than 1, using similar arguments in this section. We see that the term $e^{2\sigma t_0}$ in $P_{odd}(t_2, t_0)$ in Eq. 48 increases to a larger and larger finite value **exponentially**, while the integrals remain finite and **do not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds and hence the term $P_{odd}(t_2, t_0)$ approaches a larger and larger finite value exponentially and hence $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ **cannot** equal zero in this case.

Let us assume that the order of $\omega_z(t_2, t_0)t_0$ is 1, as real t_0 increases to a larger and larger finite value without bounds. (**Statement B**).

We see that the term $P_{odd}(t_2, -t_0)$ approaches a value very close to zero, as real t_0 increases to a larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals $\int_{-\infty}^{-t_0}$.

We note that $E'_{0n}(\tau, t_2) = -E'_0(\tau, t_2) = E_0(\tau + t_2) - E_0(\tau - t_2)$. We choose $t_2 = 2t_0$ and in the interval $[-\infty, t_0]$, we see that $E'_0(\tau, t_2) \approx -E_0(\tau + t_2)$ and $E'_{0n}(\tau, t_2) \approx E_0(\tau + t_2)$, for sufficiently large t_0 . We choose t_3 such that $E_0(\tau)$ approximates zero for $\tau > t_3$ and $E_0(\tau + t_2)$ approximates zero for $\tau > -t_2 + t_3$. As t_0, t_2 increase without bounds, we see that $t_3 < t_0$.

$$\begin{aligned} P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) &\approx -e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau + 2t_0)e^{-2\sigma(\tau+2t_0)} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau + 2t_0)e^{-2\sigma(\tau+2t_0)} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &\quad + [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \sin(\omega_z(t_2, t_0)\tau) d\tau] = 0 \end{aligned}$$

(50)

Hence for sufficiently large t_0 , the integrals in in Eq. 50 remain finite. We multiply above equation by $e^{-\sigma t_0}$ and the last two integrals can be ignored for sufficiently large t_0 and we approximate as follows.

$$P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) \approx -e^{\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma(\tau + 2t_0)} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau = 0$$

(51)

We substitute $\tau + 2t_0 = t$ and write as follows.

$$I(t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) \approx -e^{\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau = -e^{\sigma t_0} Q(t_0) = 0$$

(52)

For sufficiently large t_0 , the term $E_0(t)e^{-2\sigma t}$ and the integrand in Eq. 52 fall off with an order $e^{-\pi e^{2t}} e^{(\frac{5}{2}-2\sigma)t}$, given that the term $\cos(\omega_z(t_2, t_0)(t - 3t_0))$ has an order of 1. The integrand is very close to zero for sufficiently large t_0 and the integral $\int_{3t_0}^{6t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau$ is vanishingly small compared to the integral $\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau$. We increase t_0 to $2t_0$ and we can compute $I(2t_0)$ as follows.

$$\begin{aligned} I(2t_0) &\approx -e^{2\sigma t_0} \int_{-\infty}^{6t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau \\ &= -e^{2\sigma t_0} \left[\int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau + \int_{3t_0}^{6t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau \right] = 0 \end{aligned}$$

(53)

For sufficiently large t_0 , the last integral falls off with an order $e^{-\pi e^{6t_0}} e^{(\frac{5}{2}-2\sigma)3t_0}$ and when multiplied by the term $e^{2\sigma t_0}$ has a fall-off rate $e^{-\pi e^{6t_0}} e^{(\frac{5}{2}-2\sigma)3t_0} e^{2\sigma t_0}$. But the first term in Eq. 53 given by $-e^{2\sigma t_0} Q(t_0) = e^{\sigma t_0} I(t_0)$ has increased exponentially by a factor $e^{\sigma t_0}$, as we increase t_0 to $2t_0$.

Hence the integral $\int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) d\tau$ in Eq. 52 **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds.

The term $e^{\sigma t_0}$ in $P_{odd}(t_2, t_0)$ in Eq. 53 increases to a larger and larger finite value **exponentially** and hence the term $P_{odd}(t_2, t_0)$ approaches a larger and larger finite value exponentially and hence $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ **cannot** equal zero in this case.

Hence **Statement C** is **false** and the order of $\omega_z(t_2, t_0)t_0$ is **not** 1, as finite t_0 increases without bounds. We see that $\omega_z(t_2, t_0)t_0$ **does not** decrease towards zero, as finite t_0 increases without bounds (Section 5). Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $|t_0| < \infty$ and $|t_2| < \infty$ (Section 4), we see that the the order of $\omega_z(t_2, t_0)t_0$ is greater than 1, as finite t_0 increases without bounds.

617 **6. $\omega_z(t_2, t_0)$ is non-zero as $t_0 \rightarrow \infty$**

618

619 In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for every finite value of t_2 . In this
620 section, it is shown that $\omega_z(t_2, t_0)$ remains non-zero, as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$.

621

622 **Method 1:** We will show that the location of zeros in the Fourier transform of $\lim_{t_0 \rightarrow \infty} g(t, t_2, t_0)$
623 and $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = e^{-2\sigma t_0} g(t, t_2, t_0)$ in Eq. 57, given by $\lim_{t_0 \rightarrow \infty} \omega_z(t_2, t_0)$ approaches a non-zero
624 value.

625

626 • It is shown that $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = \lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ and its Fourier transform has a zero
627 at the location $\omega = \omega_c$ on the critical line, which is finite and non-zero.

628

629 • In Section 6.2, it is shown that $\lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ **does not** vanish to an all-zero function and
630 remains finite and non-zero.

631

632 • We know that the Fourier transform of the function $E_0(t - 3t_0)$ given by $E_{0\omega}(\omega)e^{-i\omega 3t_0}$ does **not**
633 have a zero at $\omega = 0$, given that $E_0(t) \geq 0$ for all $|t| \leq \infty$. It is shown that the Fourier transform of
634 the function $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = \lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ also does not have a zero at $\omega = 0$ and hence
635 $\lim_{t_0 \rightarrow \infty} \omega_z(t_2, t_0)$ **cannot** equal zero.

636

637 We note that $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ and $f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) +$
638 $e^{2\sigma t_0}f_2(t, t_2, t_0) = e^{-\sigma t_0}E_p'(t + t_0, t_2) + e^{\sigma t_0}E_p'(t - t_0, t_2)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p'(t + t_0, t_2)$ and
639 $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0}E_p'(t - t_0, t_2)$, $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) =$
640 $E_0'(t, t_2)e^{-\sigma t}$, $E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ and $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) =$
641 $[e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$. We start with finite values of t_0 and t_2 .

$$\begin{aligned} g(t, t_2, t_0) &= u(-t)[e^{-\sigma t_0}E_p'(t + t_0, t_2) + e^{\sigma t_0}E_p'(t - t_0, t_2)]e^{-\sigma t} + u(t)[e^{-\sigma t_0}E_p'(t + t_0, t_2) + e^{\sigma t_0}E_p'(t - t_0, t_2)]e^{\sigma t} \\ &= u(-t)[E_p'(t + t_0, t_2)e^{-\sigma(t+t_0)} + E_p'(t - t_0, t_2)e^{-\sigma(t-t_0)}] \\ &\quad + u(t)[e^{-2\sigma t_0}E_p'(t + t_0, t_2)e^{\sigma(t+t_0)} + e^{2\sigma t_0}E_p'(t - t_0, t_2)e^{\sigma(t-t_0)}] \end{aligned}$$

642 (54)

643 We define $g'(t, t_2, t_0) = e^{-2\sigma t_0}g(t, t_2, t_0)$ whose Fourier transform is given by $G'(\omega, t_2, t_0)$. The
644 scaling factor $e^{-2\sigma t_0}$ does not affect the location of zeros in $G'(\omega, t_2, t_0)$ whose **zeros** are the **same** as
645 that of $G(\omega, t_2, t_0)$, given by $\omega_z(t_2, t_0)$, for every value of t_0 .

$$\begin{aligned} g'(t, t_2, t_0) &= u(-t)e^{-2\sigma t_0}[E_p'(t + t_0, t_2)e^{-\sigma(t+t_0)} + E_p'(t - t_0, t_2)e^{-\sigma(t-t_0)}] \\ &\quad + u(t)[e^{-4\sigma t_0}E_p'(t + t_0, t_2)e^{\sigma(t+t_0)} + E_p'(t - t_0, t_2)e^{\sigma(t-t_0)}] \end{aligned}$$

646

(55)

647 As we increase t_0 towards ∞ , the first 3 terms in Eq. 55 go to zero, due to the terms $e^{-2\sigma t_0}$ and
648 $e^{-4\sigma t_0}$, given that $E_p'(t, t_2)e^{-\sigma t}$ and its t-shifted versions are finite. We use $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) -$
649 $e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t, t_2)e^{-\sigma t}$, where $E_0'(t, t_2) = (E_0(t - t_2) - E_0(t + t_2))$.
650 For the choice of $t_2 = 2t_0$ used in Section 3, we can write $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0)$ as follows.

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) &= \lim_{t_0 \rightarrow \infty} u(t) [E_p'(t - t_0, t_2) e^{\sigma(t-t_0)}] = \lim_{t_0 \rightarrow \infty} u(t) [E_0'(t - t_0, t_2)] \\ &= \lim_{t_0 \rightarrow \infty} u(t) [E_0(t - t_2 - t_0) - E_0(t + t_2 - t_0)] = \lim_{t_0 \rightarrow \infty} u(t) [E_0(t - 3t_0) - E_0(t + t_0)] \end{aligned}$$

(56)

We see that the discontinuity at $t = 0$ represented by $u(t)$ in Eq. 56, has a value given by $\lim_{t_0 \rightarrow \infty} E_0(-3t_0) - E_0(t_0)$ goes to zero, as t_0 goes towards ∞ and the term $E_0(t + t_0)u(t)$ goes to zero and we can write as follows.

$$\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = \lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$$

(57)

We want to find out the location of zeros in the Fourier transform of $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0)$ in Eq. 57, given by $\lim_{t_0 \rightarrow \infty} \omega_z(t_2, t_0)$.

We know that the Fourier transform of $E_0(t) = \Phi(t)$ given by $\xi(\frac{1}{2} + i\omega)$ has an infinity of zeros on the critical line.^[2] We consider one of the zeros to the right of origin given by $\omega = \omega_c$ which is finite and non-zero. We see that t_0 is a real number and as it increases to a larger and larger value without bounds, for each such finite value of t_0 , we see that the Fourier transform of $E_0(t - 3t_0)$ given by $E_{0\omega}(\omega)e^{-i\omega 3t_0}$, has a zero at the **same** location $\omega = \omega_c$. (**Result H**) This statement should hold, as $t_0 \rightarrow \infty$. More details in Section 6.1.

Secondly, we see that $\int_{-\infty}^{\infty} E_0(t - 3t_0)dt$ is non-zero, given that $E_0(t) > 0$ for $|t| < \infty$ and that $E_0(t)$ goes to zero as $|t| \rightarrow \infty$ (Appendix B.1). We see that $\int_{-\infty}^{\infty} E_0(t - 3t_0)dt$ remains non-zero, as t_0 increases to a larger and larger value without bounds (**Result I**). This statement should hold, as $t_0 \rightarrow \infty$ and hence $\lim_{t_0 \rightarrow \infty} \omega_z(t_2, t_0)$ is non-zero. More details in Section 6.1.

We see that the Fourier transform of $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = \lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ has a zero at the **same** location $\omega = \omega_c$.

We see that $\omega_z(t_2, t_0)$ is non-zero, as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$ and hence the order of $\omega_z(t_2, t_0)$ is greater than or equal to 1.

We see that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$, given that the Fourier transform of $\lim_{t_0 \rightarrow \infty} g'(t, t_2, t_0) = \lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ has a zero at the **same** location $\omega = \omega_c$.

• **Method 2:** We see that t_0 is a real number and as it increases to a larger and larger value without bounds, for each such finite value of t_0 , the integral in Eq. ?? converges because $E_0(t - 3t_0)$ is an absolutely integrable function (Appendix B.1). Hence we can use Lemma 1 in Section 2.1 and show that $\omega_z(t_2, t_0)$ is non-zero and finite. (**Result G**) This statement should hold, as $t_0 \rightarrow \infty$. More details in Section 6.1.

Hence we can use Lemma 1 in Section 2.1 and show that $\omega_z(t_2, t_0)$ is non-zero and finite, as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$ and hence the order of $\omega_z(t_2, t_0)$ is greater than or equal to 1.

If we assume the case of $\omega_z(t_2, t_0)$ tending to zero asymptotically, we can show that this leads to a contradiction, similar to Lemma 1. We see that $F(\omega, t_2, t_0)$ has a zero at $\omega = \omega_0$ as $t_0 \rightarrow \infty$ and $t_2 \rightarrow \infty$, and $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω . If $G_R(\omega, t_0, t_2)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign, **then the real part** of $F(\omega, t_2, t_0)$ given by $F_R(\omega, t_2, t_0) = \frac{1}{2\pi}[G_R(\omega, t_0, t_2) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega, t_0, t_2)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**.

6.1. Discussion of $\lim_{t_0 \rightarrow \infty}$

We see that t_0 is a real number and as it increases to a larger and larger value without bounds, given that there is no largest real number, for each such finite value of t_0 , Result E, F, G, H and I are valid. As $\lim_{t_0 \rightarrow \infty}$, Result E, F, G, H and I should remain valid, as explained below.

Infinity is defined as an unbounded quantity that is greater than every real number. (**Statement A**) This is based on definitions like $\frac{1}{0} = \infty$ and $\infty + 1 = \infty$, which **do not** have proof. (link) These definitions are stated **without** proof, presumably because the rules obeyed by the set of real numbers, are not obeyed by the defined quantity ∞ and hence Statement A is assumed and as $\lim_{t_0 \rightarrow \infty}$, the rules which dictate the interchanging of order of limit and integration are deemed necessary.

It is argued here that, as $\lim_{t_0 \rightarrow \infty}$, Result E, F, G, H and I should remain valid, given that Statement A is based on assumed definitions without proof.

6.2. $\lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ **does not vanish and approach an all-zero function**

In this section, it is shown that $\lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ **does not** vanish and approach an all-zero function and remains finite and non-zero for finite t .

We consider 3 points in the function $E_0(t)$ namely Point A at $t = 0$ given by $E_0(0)$, Point B at $t = 1$ given by $E_0(1)$ and Point C at $t = -1$ given by $E_0(-1) = E_0(1)$. In the shifted function $E_0(t - 3t_0)$, Points A, B and C move to $t = 3t_0$, $t = 3t_0 + 1$ and $t = 3t_0 - 1$ respectively, with the **same non-zero** values $E_0(0)$, $E_0(1)$ and $E_0(-1)$ respectively, without any decrease in values or change in the shape of $E_0(t)$. (**Result J**)

We see that t_0 is a real number and as it increases to a larger and larger value without bounds, given that there is no largest real number, for each such finite value of t_0 , Result J is **valid**.

As $\lim_{t_0 \rightarrow \infty}$, Result J should continue to hold, given that Points A, B and C move to $t = \infty$, $t = \infty + 1 = \infty$ and $t = \infty - 1 = \infty$ respectively, with the **same non-zero** values $E_0(0)$, $E_0(1)$ and $E_0(-1)$ respectively, without any decrease in values or change in the shape of $E_0(t)$. We note that $\lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ **cannot** suddenly become an all-zero function.

The notion that $\lim_{t_0 \rightarrow \infty} E_0(t - 3t_0)$ becomes an all-zero function is based on arbitrary definition of infinity **without** proof, as explained in Section 6.1.

7. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. It is known that $\Phi(t)$ is positive for $|t| < \infty$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t > 0$. (link). This is shown below.

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

$$E_0(t) = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$$

(58)

We show that $X(t) = \frac{E_0(t)}{2}$ is a **strictly decreasing** function for $t > 0$ as follows.

- In Section 7.1, it is shown that the first derivative of $X(t)$, given by $\frac{dX(t)}{dt} < 0$ for $t > t_z$ where $t_z = \frac{1}{2} \log \frac{y_z}{\pi}$ and $y_z = 3.16$.

- In Section 7.2, it is shown that, $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$.

Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is strictly decreasing for all $t > 0$ and $E_0(t) = 2X(t)$ is **strictly decreasing** for all $t > 0$.

7.1. $\frac{dX(t)}{dt} < 0$ **for** $t > t_z$

We consider $X(t) = \frac{E_0(t)}{2} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$ and take the first derivative of $X(t)$ as follows. We note that $E_0(t)$ is an analytic function for $|t| \leq \infty$ and is infinitely differentiable in that interval.

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t}) (\frac{1}{2} - 2\pi n^2 e^{2t})]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]$$

(59)

We substitute $y = \pi e^{2t}$ in Eq. 59 and define $A(y)$ such that $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$. [8]

$$A(y) = \sum_{n=1}^{\infty} n^2 e^{-n^2 y} [-4n^4 y^2 + 15n^2 y - \frac{15}{2}]$$

We see that $A(y) = 0$ at $y = \pi$, given that $\frac{dX(t)}{dt} = 0$ at $t = 0$, because $X(t)$ is an even function of variable t . The quadratic expression $B(y, n) = (-4n^4y^2 + 15n^2y - \frac{15}{2})$ in Eq. 60 has roots at $y = \frac{-15n^2 \pm \sqrt{225n^4 - 120n^4}}{-8n^4} = \frac{(15 \pm \sqrt{105})}{8n^2}$. We see that the second derivative of $B(y, n)$ given by $-8n^4$, is negative for all y and n and hence $B(y, n)$ is a concave down function for each n , which reaches a maximum at $y = \frac{15}{8n^2}$ and given the dominant term $-4n^4y^2$ in Eq. 60, we see that $B(y, n) < 0$, for $y > \frac{(15 + \sqrt{105})}{8} > 3.16 = y_z$, for $n \geq 1$ and hence $A(y) < 0$ for $y > y_z$. Hence $\frac{dX(t)}{dt} < 0$ for $t > \frac{1}{2} \log \frac{y_z}{\pi} = t_z$ (**Result 1**).

We want to show that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. It suffices to show that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq 3.16$ and hence $A(y) < 0$ for $\pi < y \leq 3.16$.

7.2. $\frac{dX(t)}{dt} < 0$ **for** $0 < t \leq t_z$

It is shown in this section that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq 3.16$ and hence $A(y) < 0$ for $\pi < y \leq 3.16$ [8]. We take the derivative of $A(y)$ in Eq. 60 and take the factor n^2 out of the brackets, as follows.

$$\begin{aligned} \frac{dA(y)}{dy} &= \sum_{n=1}^{\infty} n^2 e^{-n^2 y} [-8n^4 y + 15n^2 + (-4n^4 y^2 + 15n^2 y - \frac{15}{2})(-n^2)] \\ \frac{dA(y)}{dy} &= \sum_{n=1}^{\infty} n^4 e^{-n^2 y} [-8n^2 y + 15 + 4n^4 y^2 - 15n^2 y + \frac{15}{2}] = \sum_{n=1}^{\infty} n^4 e^{-n^2 y} [4n^4 y^2 - 23n^2 y + \frac{45}{2}] \end{aligned}$$

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We examine the term $C(y, n) = n^4 e^{-n^2 y} (4n^4 y^2 - 23n^2 y + \frac{45}{2})$ in Eq. 61 in the interval $\pi \leq y \leq 3.16$ and show that $\frac{dA(y)}{dy} = C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < 0$, as follows.

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For $n = 1$, we see that $C(y, 1) = e^{-y} (4y^2 - 23y + \frac{45}{2}) < 0$ in the interval $\pi \leq y \leq 3.16$. Given that $3.16 < 4$ and $3.16^2 < 10$ and $\pi > 3$ and $(4y^2 - 23y + \frac{45}{2}) < 0$ in the interval $\pi \leq y \leq 3.16$, we see that $C(y, 1) < e^{-4} (4 * 10 - 23 * 3 + \frac{45}{2}) < e^{-4} (40 - 69 + 23) = -6e^{-4} = C_{max}(1)$ where $C_{max}(1)$ is the maximum value of $C(y, 1)$ in the interval $\pi \leq y \leq 3.16$.

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$$C(y, 1) = e^{-y} (4y^2 - 23y + \frac{45}{2}) < -6e^{-4}, \quad \pi \leq y \leq 3.16$$

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For $n > 1$, in the interval $\pi \leq y \leq 3.16$, we can write $C(y, n)$ as follows, given that $-23n^2 y + \frac{45}{2} < 0$ and $\pi > 3$ and $3.16^2 < 10$.

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$$C(y, n) = n^4 e^{-n^2 y} (4n^4 y^2 - 23n^2 y + \frac{45}{2}) < n^4 e^{-\pi n^2} (4n^4 (3.16)^2) < 40n^8 e^{-\pi n^2} < 40n^8 e^{-3n^2}$$

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783 We want to show that $\frac{dA(y)}{dy} = C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < 0$ in the interval $\pi \leq y \leq 3.16$. Using
 784 Eq. 62 and Eq. 63, we write

$$\begin{aligned} \frac{dA(y)}{dy} &= C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < -6e^{-4} + \sum_{n=2}^{\infty} 40n^8 e^{-3n^2} \\ e^4 \frac{dA(y)}{dy} &< -6 + \sum_{n=2}^{\infty} 40n^8 e^{4-3n^2} \end{aligned} \quad (64)$$

786 We want to show that $e^4 \frac{dA(y)}{dy} < 0$ in the interval $\pi \leq y \leq 3.16$. We compute $\log(n^8 e^{4-3n^2})$ as
 787 follows. We note that $f(x) = \log x$ is a concave down function whose second derivative given by
 788 $-\frac{1}{x^2} < 0$ for $|x| < \infty$ and we can write $f(x) = \log x \leq f(x_0) + f'(x_0)(x - x_0)$ using its tangent line
 789 equation. We set $x = n$ and $x_0 = 2$ below.

$$\begin{aligned} \log(n^8 e^{4-3n^2}) &= 8 \log n + (4 - 3n^2) \leq 8(\log 2 + \frac{1}{2}(n - 2)) + (4 - 3n^2) \\ \log(n^8 e^{4-3n^2}) &\leq 8 \log 2 + 4n - 4 - 3n^2 \end{aligned} \quad (65)$$

791 We note that $g(x) = 4x - 4 - 3x^2$ in Eq. 65 is a concave down function whose second derivative
 792 given by $-6 < 0$ for all x and we can write $g(x) \leq g(x_0) + g'(x_0)(x - x_0)$ using its tangent line
 793 equation. We set $x = n$ and $x_0 = 2$ and write Eq. 65 as follows.

$$\begin{aligned} \log(n^8 e^{4-3n^2}) &\leq 8 \log 2 - 8 - 8(n - 2) \leq 8 \log 2 + 8(1 - n) \\ n^8 e^{4-3n^2} &\leq 2^8 e^{8(1-n)} \end{aligned} \quad (66)$$

795 We substitute the result in Eq. 66 in Eq. 64 as follows.

$$\begin{aligned} e^4 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 \sum_{n=2}^{\infty} e^{8(1-n)} \\ e^4 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 * e^8 \sum_{n=2}^{\infty} e^{-8n} \\ e^4 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 * e^8 \frac{e^{-8*2}}{1 - e^{-8}} \\ e^4 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 * \frac{e^{-8}}{1 - e^{-8}} \\ e^4 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 * \frac{1}{e^8 - 1} \end{aligned} \quad (67)$$

797 We multiply Eq. 67 by $\frac{(e^8 - 1)}{6}$ and write as follows.

$$e^4 \frac{dA(y)}{dy} \frac{(e^8 - 1)}{6} < -e^8 + 1 + 40 * \frac{256}{6} = -1273.3 \quad (68)$$

We see that $e^4 \frac{dA(y)}{dy} \frac{(e^8 - 1)}{6} < 0$ in Eq. 68 and hence $\frac{dA(y)}{dy} < 0$, in the interval $\pi \leq y \leq 3.16$. Given that $A(y) = 0$ at $y = \pi$, we see that $A(y) < 0$ in Eq. 60, for $\pi < y \leq 3.16$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y) < 0$ in the interval $0 < t \leq t_z$. (**Result 2**)

In Section 7.1, it is shown that $\frac{dX(t)}{dt} < 0$ for $t > t_z$ (from Result 1). In this section, we have shown that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$.

Hence $E_0(t) = 2X(t)$ is a **strictly decreasing function** for $t > 0$.

7.3. Result $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 7 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 34.

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t , and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c})$ to $E_0(3t_{0c})$, which is **less than** $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c})$ to $E_0(-t_{0c})$ respectively. Hence we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 34, for $t_{0c} > 0$ and $t_{2c} = 2t_{0c}$.

8. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1 - s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1 - s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied^[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi) \xi(1 - s, \bar{\chi})$ ^[7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and

entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$,

where $x > 0$ is real ^[4] and then derive $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function which is **strictly decreasing** function for $t > 0$ (Section 7). These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} -$

873 $3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). This is re-derived in link.

874

875 We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) =$
 876 $E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned}\xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega\end{aligned}$$

877

(A.1)

878 We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \quad (A.2)$$

879 We can evaluate the above integral in the complex plane using contour integration, substituting
 880 $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2
 881 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line
 882 $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded
 883 by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

884

885 In **Appendix B.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely
 886 integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

887

888 We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$,
 889 **goes to zero** as $x \rightarrow \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because
 890 $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in
 891 Eq. A.2 **vanishes** along the contours C_2 and C_4 . Using Cauchy's Integral theroem, we can write
 892 Eq. A.2 as follows.

$$\begin{aligned}E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}\end{aligned}$$

893

(A.3)

894 Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

895 Appendix B. Properties of Fourier Transforms

896

897 *Appendix B.1. $E_p(t), h(t), g(t, t_2, t_0)$ are absolutely integrable functions and their Fourier*
 898 *Transforms are finite.*

899

900 The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) =$
 901 $E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$. We see that $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for

all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \rightarrow \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \rightarrow -\infty$, $E_p(t)$ goes to zero, because for every value of n , the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t}$ goes to zero, at $t \rightarrow \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t} dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t) dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ where $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$ and $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ and $E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$. We can see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)h(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix B.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^3}$ as $|t| \rightarrow \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)| dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}| dt < \infty$. Hence $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t, t_2, t_0)| dt = \int_{-\infty}^{\infty} g(t, t_2, t_0) dt$ is finite and its Fourier transform $G(\omega, t_2, t_0)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B.2. Convolution integral convergence

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ and $\int_{-\infty}^{\infty} H(\omega) d\omega$ converges.

Let us consider the function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$. We can see that $G(\omega, t_2, t_0), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the **first derivatives** of $g(t, t_2, t_0), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t, t_2, t_0)$ are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega', t_2, t_0) H(\omega - \omega') d\omega' = \frac{1}{2\pi} [G(\omega, t_2, t_0) * H(\omega)] \quad (\text{B.1})$$

944 *Appendix B.3. Fall off rate of Fourier Transform of functions*

945

946 Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose
 947 $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$
 948 has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform
 949 $P_N(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

950

951 This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a
 952 term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

953

954 We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$
 955 then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

956 *Appendix B.4. Payley-Weiner theorem and Exponential Fall off rate of analytic func-*
 957 *tions.*

958

959 We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of
 960 their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$
 961 and $x(t) = E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^3}$ as $|t| \rightarrow \infty$ for $0 < \sigma < \frac{1}{2}$.

962

963 We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by
 964 $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant^[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) =$
 965 $E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$ and they are absolutely inte-
 966 grable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

967

968 Given that $\xi(s)$ is an entire function in the s -plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic**
 969 functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and
 970 $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster
 971 than $\frac{1}{t^M}$ as $M \rightarrow \infty$, as $|t| \rightarrow \infty$ (Appendix B.3) and hence it should have **exponential fall-off**
 972 rates as $|t| \rightarrow \infty$.

973

974 We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ has a fall-off rate of **at least** $\frac{1}{t^3}$ as
 975 $|t| \rightarrow \infty$, because its Fourier transform is an **analytic** function for all $|\omega| \leq \infty$ with **exponential**
 976 **fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$.