On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

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1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω

when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 3, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$, because the new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and Fourier transformable functions and this condition is satisfied for Riemann's Zeta function, but **not** for Hurwitz zeta function and related functions.

In Appendix A to Appendix I, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \le \omega \le \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows (link).[3] This is re-derived in Appendix F.

$$E_0(t) = \Phi(t) = \Xi 2\sum_{n=1}^{\infty}[2n^4\pi^2 e^{\frac{9t}{2}} - 3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2 e^{2t}} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \le \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

In **Section 2.1**, we will show that the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \ne 0$ to satisfy Statement 1, where ω_1 is real and finite.

1.3. ***Step 3: On the zeros of the function*** $G_I(\omega)$

In **Section 2.2**, we compute the Fourier transform of the function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$. We **require** $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, to satisfy Statement 1. Hence $S_0 = G_I(\omega_1) = 0$ and we will derive as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{3}$$

1.4. ***Step 4: Even order Derivatives of*** $g(t)$

In **Section 2.3** and **Section 2.4**, we consider the **even order derivative** of the function $g(t)$ given by $g_{2r}(t) = \frac{1}{!(2r)}\frac{d^{2r}g(t)}{dt^{2r}}$ and compute the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and show results as follows. We will also show that **dirac delta functions vanish** in the computation of $g_{2r_{odd}}(t)$.

$$S_{2r} = \frac{1}{!(2r)}[-\int_{-\infty}^{0}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\sin(\omega_1\tau)d\tau] = 0 \tag{4}$$

1.5. ***Step 5: New Function*** $A(t_1)$

In **Section 2.4**, we consider a new function $g_{a_{odd}}(t, t_1) = \sum_{r=0}^{\infty} g_{2r_{odd}}(t)t_1^{2r}$, for real $-\infty < t_1 < \infty$ and compute its Fourier transform $G_{a_I}(\omega, t_1)$, evaluate it at $\omega = \omega_1$ and set it to zero, using the procedure above. We get $A(t_1) = [G_{a_I}(\omega, t_1)]_{\omega=\omega_1} = 0$. We will show that it can be written as follows, where $x(\tau) = E_0(\tau)e^{-\sigma\tau}$.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{5}$$

We can write $A(t_1) = \frac{1}{2}[y(t_1) + y(-t_1)] = 0$ as follows. Given that $\omega_1 \neq 0$, we will show that

$$y(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)\int_{-\infty}^{t_1}(E_0(t) - x(t))\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int_{-\infty}^{t_1}(E_0(t) - x(t))\cos(\omega_1 t)dt] = y_{odd}(t_1) \tag{6}$$

We can see that $y(t_1)$ is an **odd function** of variable t_1.

1.6. ***Step 6: Final Step in the proof of theorem.***

In **Section 2.5**, we will evaluate the **odd** symmetry function $z_{odd}(t_1)$ as follows.

$$\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = z_{odd}(t_1)$$
$$\frac{\omega_1}{2}[x(t_1) - E_0(t_1)] = z_{odd}(t_1)$$
$$\frac{\omega_1}{2}[E_0(t_1)e^{-2\sigma t_1} - E_0(t_1)] = z_{odd}(t_1)$$
$$\frac{\omega_1}{2}E_0(t_1)[e^{-2\sigma t_1} - 1] = z_{odd}(t_1)$$

$$(7)$$

We will show that $\omega_1 \neq 0$ (Section 2.1). We know that $E_0(t_1) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t_1} - 3\pi n^2 e^{2t_1}]e^{-\pi n^2 e^{2t_1}} e^{\frac{t_1}{2}}$ is an **even function** of variable t_1 and $E_0(t_1) \neq 0$, hence we require $(e^{-2\sigma t_1} - 1)$ to be an **odd function** of variable t_1, to satisfy Eq. 7, which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence this proves Riemann hypothesis.

2. Proof of Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

The inverse Fourier Transform of the function $E_{p\omega}(\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

Given that $E_{0\omega}(\omega)$ is an entire function and finite for all ω, we see that $E_0(t) = 0$ at $t = \pm\infty$, because if $E_0(t) \neq 0$ at $t = \pm\infty$, then its Fourier transform $E_{0\omega}(\omega)$ will not be finite. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

2.1. *On the zeros of a related function* $G(\omega)$

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can show that $E_p(t), h(t), g(t)$ are real L^1 integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix B.1.

If we take the Fourier transform of the equation $g(t)h(t) = E_p(t)$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = E_{p\omega}(\omega)$ as per convolution theorem (link), where $*$ denotes **convolution** operation given by $E_{p\omega}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix G.1.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ where ω_1 is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_I(\omega)$ does not have at least one zero for some $\omega = \omega_1 \neq 0$, **then** the **imaginary part** of $E_{p\omega}(\omega)$ given by $E_I(\omega) = \frac{1}{2\pi}[G_I(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, where $g(t)h(t) = E_p(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its imaginary part given by $E_I(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_I(\omega)$ **does not** have at least one zero for finite $\omega = \omega_1 \neq 0$ and show that $E_I(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the imaginary parts as follows.

$$E_I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(\omega')H(\omega - \omega')d\omega' \tag{8}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix B.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 8 and we get

$$E_I(\omega) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} G_I(\omega')\frac{1}{(\sigma^2 + (\omega - \omega')^2)}d\omega' \tag{9}$$

We can split the integral in Eq. 9 as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_I(\omega')\frac{1}{(\sigma^2 + (\omega - \omega')^2)}d\omega' + \int_{0}^{\infty} G_I(\omega')\frac{1}{(\sigma^2 + (\omega - \omega')^2)}d\omega'] \tag{10}$$

We see that $G_I(-\omega) = -G_I(\omega)$ because $g(t)$ is a real function (Appendix G.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 10 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}\int_{0}^{\infty} G_I(\omega')[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} - \frac{1}{(\sigma^2 + (\omega + \omega')^2)}]d\omega' \tag{11}$$

In Appendix B.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 11 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} - \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_I(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) > 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix G.2). Hence $E_I(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 6. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

- **Case 2:** $G_I(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) < 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix G.2). Hence $E_I(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

We have shown that, $G_I(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**. We will investigate if Statement 2 leads to a contradiction for $0 < \sigma < \frac{1}{2}$.

2.2. *On the zeros of the function* $G_I(\omega)$

We take the Fourier transform of $g(t)$ and get $G(\omega)$ as follows. In Section 2.1 second paragraph, it is shown that the Fourier transform of $g(t)$ is finite for all $|\omega| \leq \infty$.

$$g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} E_p(t)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_p(t)e^{\sigma t}e^{-i\omega t}dt \tag{12}$$

We can substitute $t = -\tau$ in the second integral in Eq. 12 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows.

$$G(\omega) = \int_{-\infty}^{0} E_p(\tau)e^{-\sigma\tau}e^{-i\omega\tau}d\tau + \int_{-\infty}^{0} E_q(\tau)e^{-\sigma\tau}e^{i\omega\tau}d\tau = G_R(\omega) + iG_I(\omega) \tag{13}$$

Eq. 13 can be expanded as follows using Euler's formula $e^{i\omega\tau} = \cos(\omega\tau) + i\sin(\omega\tau)$ and comparing the **imaginary parts** of $G(\omega)$, we can write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_I(\omega) = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega\tau)d\tau$$

(14)

We require $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, to satisfy **Statement 1** as shown in Section 2.1.

We can set $S_0 = G_I(\omega_1) = 0$ and write as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{15}$$

The integrals in Eq. 15 converge because they are derived from the Fourier transform of $g(t)$ which is finite for all $|\omega| \leq \infty$ as shown in second paragraph in Section 2.1.

2.3. ***Even order Derivatives of*** $g(t)$

In Section 1.1, we showed that $E_p(t)$ is a real **analytic** function in the interval $-\infty \leq t \leq \infty$ which is infinitely differentiable in that interval. Let us consider the $(2r)^{th}$ derivative of the function $g(t)$ given by $g_{2r}(t) = \frac{1}{!(2r)}\frac{d^{2r}g(t)}{dt^{2r}}$ where $r = 0, 1, ..., \infty$. Its Fourier transform is given by $G_{2r}(\omega) = \int_{-\infty}^{\infty} g_{2r}(t)e^{-i\omega t}dt$.

We can see that $g_2(t) = \frac{1}{!(2)}\frac{d^2g(t)}{dt^2}$ produces a **Dirac delta function**, which is an **even function** of variable t (link). Hence, when we take the **odd part** of $g_2(t)$ given by $g_{2_{odd}}(t) = \frac{1}{2}[g_2(t) - g_2(-t)]$, the dirac delta impulse function **vanishes** (Appendix C).

We take the **odd part** of $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and the dirac delta impulse function related terms **vanish** because dirac delta function $\delta(t)$ has even symmetry (link) and its even derivatives $\delta^{2r}(t)$ are **even functions** of variable t, given the well known relation $t^{2r}\delta^{2r}(t) = (-1)^{2r}(!(2r))\delta(t) = (!(2r))\delta(t)$ and we see that t^{2r} has even symmetry for $r = 0, 1, ..., \infty$ (Eq. 17 in link). This is shown in detail in **Appendix C**.

We take the Fourier transform of $g_{2r_{odd}}(t)$ and we see that $G_{2r_I}(\omega) = 0$ for the **same** $\omega = \omega_1$ because $G_{2r}(\omega) = \frac{1}{!(2r)}(-\omega^2)^r G(\omega) = \frac{1}{!(2r)}(-\omega^2)^r[G_R(\omega) + iG_I(\omega)]$ and hence $G_{2r_I}(\omega) = \frac{1}{!(2r)}(-\omega^2)^r G_I(\omega)$ (link)

First we compute the Fourier transform of $g_{2r}(t)$ given by $G_{2r}(\omega)$ as follows.

$$G_{2r}(\omega) = \frac{1}{!(2r)}[\int_{-\infty}^{0} \frac{d^{2r}(E_p(t)e^{-\sigma t})}{dt^{2r}}e^{-i\omega t}dt + \int_{0}^{\infty} \frac{d^{2r}(E_p(t)e^{\sigma t})}{dt^{2r}}e^{-i\omega t}dt] \tag{16}$$

We can substitute $t = -\tau$ in the second integral in Eq. 16 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_{2r}(\omega) = \frac{1}{!(2r)}[\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}e^{-i\omega\tau}d\tau + \int_{-\infty}^{0} \frac{d^{2r}E_0(-\tau)}{d\tau^{2r}}e^{i\omega\tau}d\tau] \tag{17}$$

Eq. 17 can be expanded as follows using Euler's formula $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and comparing the **imaginary parts** of $G_{2r}(\omega) = G_{2r_R}(\omega) + iG_{2r_I}(\omega)$, we can write as follows.

$$G_{2r_I}(\omega) = \frac{1}{!(2r)}[-\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega\tau)d\tau + \int_{-\infty}^{0} \frac{d^{2r}E_0(-\tau)}{d\tau^{2r}} \sin(\omega\tau)d\tau] \tag{18}$$

We require $G_{2r_I}(\omega) = 0$ for the **same** $\omega = \omega_1$, to satisfy **Statement 1**, because we derived the result that $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$ in Section 2.1 and $G_{2r_I}(\omega) = \frac{1}{!(2r)}(-\omega^2)^r G_I(\omega)$. Hence $S_{2r} = G_{2r_I}(\omega_1) = 0$ and is given as follows. (Integral convergence shown in Appendix I.1)

$$S_{2r} = G_{2r_I}(\omega_1) = \frac{1}{!(2r)}[-\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau] = 0 \tag{19}$$

2.4. ***New Function*** $A(t_1)$

We form a new function $g_{a_{odd}}(t, t_1) = \sum_{r=0}^{\infty} g_{2r_{odd}}(t) t_1^{2r}$, for real $-\infty < t_1 < \infty$ and compute its Fourier transform $G_{a_I}(\omega, t_1)$, evaluate it at $\omega = \omega_1$ and set it to zero, using the procedure above. We get $A(t_1) = [G_{a_I}(\omega, t_1)]_{\omega=\omega_1} = 0$. We will show that $A(t_1)$ in Eq. 20 equals Eq. 21 and the integrals in Eq. 21 converge. (Integral convergence shown in Appendix I.2 and Appendix I.4)

$$A(t_1) = [G_{a_I}(\omega, t_1)]_{\omega=\omega_1} = -\int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1\tau)d\tau = 0 \tag{20}$$

For the specific case of **complex exponential** function $C(\tau) = e^{i\omega\tau}$, we define a new function $D(\tau, t_1) = \sum_{r=0}^{\infty} \frac{d^{2r}C(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$ which can be written as $D(\tau, t_1) = \frac{1}{2}[C(\tau + t_1) + C(\tau - t_1)]$. We can show similar results for the summation terms in Eq. 20 as follows.

Let $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$. In Eq. 20 we have $f_1(\tau, t_1) = \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$. In **Appendix E**, we show that $f_1(\tau, t_1) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ using the inverse Fourier transform representation of $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, given that $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ is an analytic function and is Fourier transformable. Similarly, we can show that $f_2(\tau, t_1) = \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)]$. Hence we can write Eq. 20 as follows. (Integral convergence shown in Appendix I.2)

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0} [x(\tau + t_1) + x(\tau - t_1)] \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [E_0(\tau + t_1) + E_0(\tau - t_1)] \sin(\omega_1\tau)d\tau] = 0 \tag{21}$$

We can write $A(t_1) = y(t_1) + y(-t_1) = 0$ in Eq. 21 and substitute $\tau + t_1 = t$ as follows. We can see that $y(t_1)$ is an **odd function** of variable t_1.

$$y(t_1) = -\frac{1}{2}[\cos(\omega_1 t_1)\int_{-\infty}^{t_1} x(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int_{-\infty}^{t_1} x(t)\cos(\omega_1 t)dt]$$
$$+\frac{1}{2}[\cos(\omega_1 t_1)\int_{-\infty}^{t_1} E_0(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int_{-\infty}^{t_1} E_0(t)\cos(\omega_1 t)dt] = y_{odd}(t_1) \tag{22}$$

2.5. ***Final Step in the proof of theorem.***

In Eq. 22, we evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = z_{odd}(t_1)$ as follows, where $z_{odd}(t_1)$ is an **odd function** of variable t_1. In **Appendix D**, we show that if $f(t) = [\int x(\tau)d\tau]_{\tau=t}$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is an analytic function and we also derive in detail the equation $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$. We use $x(t_1) = E_0(t_1)e^{-2\sigma t_1}$ below.

$$\begin{aligned} \frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\ \frac{\omega_1}{2}[x(t_1) - E_0(t_1)] &= z_{odd}(t_1) \\ \frac{\omega_1}{2}[E_0(t_1)e^{-2\sigma t_1} - E_0(t_1)] &= z_{odd}(t_1) \\ \frac{\omega_1}{2}E_0(t_1)[e^{-2\sigma t_1} - 1] &= z_{odd}(t_1) \end{aligned} \tag{23}$$

We use the fact that $\omega_1 \neq 0$ (Section 2.1). We know that $E_0(t_1) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t_1} - 3\pi n^2 e^{2t_1}]e^{-\pi n^2 e^{2t_1}} e^{\frac{t_1}{2}}$ is an **even function** of variable t_1 and $E_0(t_1) \neq 0$, hence we require $(e^{-2\sigma t_1} - 1)$ to be an **odd function** of variable t_1, which is possible **only** for $\sigma = 0$ corresponding to the critical line.(Appendix H)

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does**

not contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \dfrac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (Appendix F). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$ and requires specific conditions to be satisfied to ensure convergence of integrals as explained in Appendix I. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

- **Declarations of interest: none**. This research is self funded by the author.

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$

(link). This is re-derived in Appendix F.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \le |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned} \xi(\frac{1}{2}+\sigma+i\omega) &= \xi(\frac{1}{2}+i(\omega-i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega-i\sigma) \\ E_p(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega-i\sigma)e^{i\omega t}d\omega \end{aligned} \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x+iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2}+iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2}+iz)$ is an entire function in the Z-plane.

Given that $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \le \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. In **Section 2** paragraph 6, we showed that $E_p(t) \ge 0$ for all $|t| \le \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 integrable function, for $0 \le |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2}+iz) = \xi(\frac{1}{2}-y+ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \le y \le 0$, given that $E_0(t)e^{yt}$ is a L^1 integrable function in the interval $-\infty \le t \le \infty$ as per Riemann-Lebesgue Lemma (link). Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned} E_p(t) &= e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \\ E_p(t) = E_0(t)e^{-\sigma t} &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \end{aligned} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$. **Alternate** derivation is in Appendix F.1.

Appendix B. Properties of Fourier Transforms Part 1

Appendix B.1. $E_p(t), h(t), g(t)$ ***are*** L^1 ***integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t}$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$. We see that $E_p(t) = 0$ at $t = \pm\infty$ and hence $E_p(t) \geq 0$ for all $|t| \leq \infty$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an L^1 **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

We can see that $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t) \geq 0$ for all $|t| \leq \infty$ because $E_p(t) \geq 0$ for all $|t| \leq \infty$. Given that $E_p(t) = E_0(t)e^{-\sigma t} = [E_0(t)u(-t) + E_0(-t)u(t)]e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$, we see that $g(t)$ goes to zero as $t \to -\infty$ with its order of decay greater than $e^{\frac{3t}{2}}$ and $g(t)$ goes to zero as $t \to \infty$ with its order of decay greater than $e^{-\frac{5t}{2}}$, for $0 < \sigma < \frac{1}{2}$. Hence $g(t)$ is an L^1 **integrable function** because $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B.2. ***Convolution integral convergence***

Let us consider a function whose **first derivative is discontinuous** at $t = 0$, for example $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are L^1 integrable functions and their Fourier Transforms are finite as shown in Appendix B.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{B.1}$$

Appendix C. Dirac delta derivatives vanish when we consider even derivatives of $g(t)$ and take their odd part $g_{2r_{odd}}(t)$

Let us consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2g(t)}{dt^2}$ where $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $g(t)h(t) = E_p(t)$. In Section 1.1, we showed that $E_p(t)$ is an analytic function in the interval $-\infty \leq t \leq \infty$. Even derivatives of $g(t)$ have dirac delta functions at $t = 0$.

We can show that **dirac delta function** $d_0(t) = \delta(t)$ and its **even derivatives** $d_{2r-2}(t)$, which are present in $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ **vanish**, when we take the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ for positive integer r, because **dirac delta function and its even derivatives have even symmetry**, while $g_{2r_{odd}}(t)$ has **odd symmetry**.

The dirac delta function $\delta(t)$ has even symmetry (link) and its even derivatives $\delta^{2r}(t)$ are **even functions** of variable t, given the well known relation $t^{2r}\delta^{2r}(t) = (-1)^{2r}(!(2r))\delta(t) = (!(2r))\delta(t)$ and we see that t^{2r} has even symmetry for $r = 0, 1, ..., \infty$ (Eq. 17 in link).

$$\begin{aligned}
g(t) &= g_-(t)u(-t) + g_+(t)u(t) \\
g_-(t) &= E_p(t)e^{-\sigma t}, \quad g_+(t) = E_p(t)e^{\sigma t} \\
g_2(t) = \frac{d^2g(t)}{dt^2} &= \frac{d^2g_-(t)}{dt^2}u(-t) + \frac{d^2g_+(t)}{dt^2}u(t) + A_0d_0(t), \quad A_0 = [\frac{dg_+(t)}{dt} - \frac{dg_-(t)}{dt}]_{t=0} \\
g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}} &= \frac{d^{2r}g_-(t)}{dt^{2r}}u(-t) + \frac{d^{2r}g_+(t)}{dt^{2r}}u(t) + A_{2r-2}d_0(t) + \sum_{k=0}^{r-2} A_{2k}\frac{d^{2r-2-2k}(d_0(t))}{dt^{2r-2-2k}} \\
A_{2r-2} = [\frac{d^{2r-1}g_+(t)}{dt^{2r-1}} &- \frac{d^{2r-1}g_-(t)}{dt^{2r-1}}]_{t=0}, \quad A_{2k} = [\frac{d^{2k+1}g_+(t)}{dt^{2k+1}} - \frac{d^{2k+1}g_-(t)}{dt^{2k+1}}]_{t=0}
\end{aligned} \tag{C.1}$$

Then we take the **odd part** of the functions $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}(g_{2r}(t) - g_{2r}(-t))$ and take their Fourier transforms given by $iG_{2r_I}(\omega) = i(-\omega^2)^rG_I(\omega)$. We can see that the Fourier transform of the delta function and its even derivatives **vanish** given that **dirac delta function and its even derivatives have even symmetry** in Eq. C.1 and **do not interfere** with the results.

Appendix D. Derivation of Result 1

- First we show that if $f(t) = [\int x(\tau)d\tau]_{\tau=t}$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is a real analytic function in the interval $-\infty \leq t \leq \infty$.

If $x(\tau)$ is an analytic function, then we can express it using taylor series expansion around $\tau = 0$ as follows, where $x_n = \frac{1}{!n}[\frac{d^n(x(\tau))}{d\tau^n}]_{\tau=0}$ and K_0 is an integration constant in the indefinite integral $f(\tau) = \int x(\tau)d\tau$.

$$\begin{aligned}
x(\tau) &= \sum_{n=0}^{\infty} x_n\tau^n = x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... \\
f(\tau) = \int x(\tau)d\tau &= K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ... \\
\frac{df(\tau)}{d\tau} &= x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... = x(\tau)
\end{aligned} \tag{D.1}$$

Now we can repeat the steps above for $f(t) = [\int x(\tau)d\tau]_{\tau=t}$ as follows.

$$f(t) = [\int x(\tau)d\tau]_{\tau=t} = [K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ...]_{\tau=t} = K_0 + x_0t + x_1\frac{t^2}{2} + x_2\frac{t^3}{3} + x_3\frac{t^4}{4} + ...$$
$$\frac{df(t)}{dt} = x_0 + x_1t + x_2t^2 + x_3t^3 + ... = x(t) \tag{D.2}$$

We have shown that if $f(t) = [\int x(\tau)d\tau]_{\tau=t}$, then $\frac{df(t)}{dt} = x(t)$.

• Now, we start with $y(t_1)$ in Eq. 22 and derive in detail $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$ in Eq. 23 as follows, where $x(t_1) = E_0(t_1)e^{-2\sigma t_1}$. We use the fact that both $x(t_1)$ and $E_0(t_1)$ are analytic functions in the interval $-\infty \le t \le \infty$. (Section 1.1)

We define $\int (E_0(t) - x(t)) \sin(\omega_1 t)dt = I_1(t) = J_1(t) + K_1$ and $\int (E_0(t) - x(t)) \cos(\omega_1 t)dt = I_2(t) = J_2(t) + K_2$ where K_1, K_2 are integration constants and $J_1(t), J_2(t)$ do not have constant terms. We can simplify $y(t_1)$ in Eq. 22 and evaluate the indefinite integrals at upper limit and lower limit **separately** as follows.

$$y(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)\int_{-\infty}^{t_1}(E_0(t) - x(t))\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int_{-\infty}^{t_1}(E_0(t) - x(t))\cos(\omega_1 t)dt]$$
$$y(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)[\int (E_0(t) - x(t))\sin(\omega_1 t)dt]_{t=t_1} - \sin(\omega_1 t_1)[\int (E_0(t) - x(t))\cos(\omega_1 t)dt]_{t=t_1}]]$$
$$-\frac{1}{2}[\cos(\omega_1 t_1)[\int (E_0(t) - x(t))\sin(\omega_1 t)dt]_{t=-\infty} - \sin(\omega_1 t_1)[\int (E_0(t) - x(t))\cos(\omega_1 t)dt]_{t=-\infty}]]$$
$$y(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)[J_1(t) + K_1]_{t=t_1} - \sin(\omega_1 t_1)[J_2(t) + K_2]_{t=t_1}]$$
$$-\frac{1}{2}[\cos(\omega_1 t_1)[J_1(t) + K_1]_{t=-\infty} - \sin(\omega_1 t_1)[J_2(t) + K_2]_{t=-\infty}] \tag{D.3}$$

Integration constants K_1, K_2 get cancelled at the upper limit and lower limit. Let $K_3 = [J_1(t)]_{t=-\infty}, K_4 = [J_2(t)]_{t=-\infty}$. We can simplify as follows.

$$y(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)[J_1(t)]_{t=t_1} - \sin(\omega_1 t_1)[J_2(t)]_{t=t_1}] - \frac{1}{2}[K_3\cos(\omega_1 t_1) - K_4\sin(\omega_1 t_1)]$$
$$y(t_1) = Z(t_1) + Z_2(t_1)$$
$$Z_2(t_1) = -\frac{1}{2}[K_3\cos(\omega_1 t_1) - K_4\sin(\omega_1 t_1)]$$
$$Z(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)[J_1(t)]_{t=t_1} - \sin(\omega_1 t_1)[J_2(t)]_{t=t_1}]$$
$$Z(t_1) = \frac{1}{2}[\cos(\omega_1 t_1)[\int (E_0(t) - x(t))\sin(\omega_1 t)dt]_{t=t_1} - \sin(\omega_1 t_1)[\int (E_0(t) - x(t))\cos(\omega_1 t)dt]_{t=t_1}] \tag{D.4}$$

We take the first derivative of $Z(t_1)$ as follows. We use the fact that $\frac{d}{dt_1}([\int x(t)\sin(\omega_1 t)dt]_{t=t_1}) = x(t_1)\sin(\omega_1 t_1)$ and $\frac{d}{dt_1}([\int x(t)\cos(\omega_1 t)dt]_{t=t_1}) = x(t_1)\cos(\omega_1 t_1)$, as per Eq. D.2 and **cancel** common terms.

$$\frac{dZ(t_1)}{dt_1} = \frac{\omega_1}{2}[-\sin(\omega_1 t_1)[\int (E_0(t) - x(t))\sin(\omega_1 t)dt]_{t=t_1} - \cos(\omega_1 t_1)[\int (E_0(t) - x(t))\cos(\omega_1 t)dt]_{t=t_1}]$$

(D.5)

We take the second derivative of $Z(t_1)$ and simplify using $\cos^2(\omega_1 t_1) + \sin^2(\omega_1 t_1) = 1$, as follows.

$$\frac{d^2Z(t_1)}{dt_1^2} = \frac{\omega_1^2}{2}[-\cos(\omega_1 t_1)[\int (E_0(t) - x(t))\sin(\omega_1 t)dt]_{t=t_1} + \sin(\omega_1 t_1)[\int (E_0(t) - x(t))\cos(\omega_1 t)dt]_{t=t_1}] + \frac{\omega_1}{2}(x(t_1) - E_0(t_1)) \tag{D.6}$$

Now we evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1)$ from Eq. D.6 and Eq. D.4 and cancel common terms and get Eq. D.7. We use the fact that $\frac{d^2Z_2(t_1)}{dt_1^2} + \omega_1^2 Z_2(t_1) = 0$ where $Z_2(t_1) = -\frac{1}{2}[K_3\cos(\omega_1 t_1) - K_4\sin(\omega_1 t_1)]$ in Eq. D.4

$$\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{d^2Z(t_1)}{dt_1^2} + \omega_1^2 Z(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)] \tag{D.7}$$

Appendix E. Derivation of Result 2

We start with Eq. 20 as follows.

$$A(t_1) = -\int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau = 0 \tag{E.1}$$

In Eq. E.1 we have $f(\tau, t_1) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}$ inside the first integral, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ and we will show that $f(\tau, t_1) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$, given that $E_0(\tau)e^{-2\sigma\tau}$ is an analytic function in the interval $-\infty \leq \tau \leq \infty$ and hence infinitely differentiable (Section 1.1) and it is also Fourier transformable.

Similarly, we can show that $d(\tau, t_1) = \sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)]$ inside the second integral in Eq. E.1 .

We substitute $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ in the equation for $f(\tau, t_1)$ and we write as follows.

$$f(\tau, t_1) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{d^{2r}([\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega]e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} \tag{E.2}$$

It is well known that the order of Riemann's Xi function at $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (Page 257 in Titchmarsh book).

We can use **Fubini**'s theorem and we can interchange the order of integration and summation in Eq. E.2 and write Eq. E.3, because $\int_{-\infty}^{\infty} |E_{0\omega}(\omega) \quad \sum_{r=0}^{\infty} \frac{t_1^{2r}}{!(2r)}(i\omega - 2\sigma)^{2r} e^{(i\omega-2\sigma)\tau} \quad |d\omega$ is finite, because for every value of r, **the integral converges**, because $\frac{1}{!(2r)}(i\omega - 2\sigma)^{2r} E_{0\omega}(\omega)$ is finite for all $|\omega| \le \infty$ and has a fall-off rate of the order of $\omega^A e^{-\frac{|\omega|\pi}{4}}(i\omega - 2\sigma)^{2r}\frac{1}{!(2r)}$ and also the series sum from $r = 0, ...\infty$ converges due to the factorial term $!(2r)$, using Series Ratio Test. The series converges absolutely with limit $L = \lim_{r\to\infty} |\frac{S_{r+1}}{S_r}| = \lim_{r\to\infty} |\frac{a_{r+1}(!(2r))}{a_r(!(2r+2))}| = \lim_{r\to\infty} |\frac{((i\omega-2\sigma)t_1)^2}{(2r+2)(2r+1)}| < 1$, where $a_r = ((i\omega - 2\sigma)t_1)^{2r}$ and S_r is the $(r)^{th}$ term in the series.

After interchanging the order of integration and summation in $f(\tau, t_1)$ in Eq. E.3, we show that it equals $f(\tau, t_1) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ in Eq. E.4 and in Eq. E.5, which is **finite** for all $|\tau| \le \infty$.

$$f(\tau, t_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty} \frac{d^{2r}e^{(i\omega-2\sigma)\tau}}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty}(i\omega - 2\sigma)^{2r}e^{(i\omega-2\sigma)\tau}\frac{t_1^{2r}}{!(2r)}]d\omega \quad (E.3)$$

We can simplify this equation as follows.

$$\begin{aligned} f(\tau, t_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)t_1} + e^{-(i\omega-2\sigma)t_1}]e^{(i\omega-2\sigma)\tau}d\omega \\ f(\tau, t_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)(\tau+t_1)} + e^{(i\omega-2\sigma)(\tau-t_1)}]d\omega \end{aligned} \quad (E.4)$$

We can simplify this equation as follows, using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ and $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$f(\tau, t_1) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)] \quad (E.5)$$

Comparing Eq. E.2 and Eq. E.5, we can see that $f(\tau, t_1) = \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$.

Using similar arguments, we see that $d(\tau, t_1) = \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)]$.

$$\begin{aligned} f(\tau, t_1) &= \sum_{r=0}^{\infty} \frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)] \\ d(\tau, t_1) &= \sum_{r=0}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)] \end{aligned} \quad (E.6)$$

Hence we can write Eq. E.1 as follows.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0} [x(\tau + t_1) + x(\tau - t_1)] \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [E_0(\tau + t_1) + E_0(\tau - t_1)] \sin(\omega_1\tau)d\tau] = 0 \quad (E.7)$$

Appendix F. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and rederive the steps below[4]. We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned}\Gamma(\frac{s}{2}) &= \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx\end{aligned} \tag{F.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{F.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^{\infty} |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{F.3}$$

For real part of s less than 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. F.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty}\frac{x^{-(\frac{s}{2}-1)}}{x^2}\frac{(1+2w(x))\sqrt{x}-1)}{2}dx \tag{F.4}$$

Hence we can simplify Eq. F.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}}w(x)dx \tag{F.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s)=\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)=\frac{1}{2}[1+s(s-1)\int_1^{\infty}(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{F.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s)=\xi(1-s)$ [4].

Appendix F.1. ***Derivation of $E_p(t)$ and $E_0(t)$***

Given that $w(x)=\sum_{n=1}^{\infty}e^{-\pi n^2 x}$, we substitute $x=e^{2t}$, $\frac{dx}{x}=2dt$ in Eq. F.6 and evaluate at $s=\frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{F.7}$$

We can substitute $t=-t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[&\int_{-\infty}^{0}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt\\&+\int_0^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{F.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{F.9}$$

We define $A(t)=[\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$E_p(t)=\frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2}$$

$$A(t)=[\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$$

$$\frac{dA(t)}{dt}=\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t)$$

$$\begin{aligned}\frac{d^2A(t)}{dt^2}&=\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t)\\&+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty}e^{-\pi n^2}(1-4\pi n^2)]\end{aligned}$$

$$(F.10)$$

We can simplify above equation as follows.

$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2)]$$

$$(F.11)$$

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix F.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$
$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}) + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$

$$(F.12)$$

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

$$(F.13)$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$

$$(F.14)$$

Appendix F.2. ***Derivation of*** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$
\begin{aligned}
F(x) = 1 + 2w(x) &= \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}}
\end{aligned}
\tag{F.15}
$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$
\begin{aligned}
[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} &= \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2}) \\
\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) &= -\frac{1}{2}
\end{aligned}
\tag{F.16}
$$

Appendix G. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix G.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{G.1}
$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$
\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'
\end{aligned}
$$

$$ \tag{G.2} $$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. G.2 and arrive at the convolution theorem.

$$ F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{G.3} $$

Appendix G.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$ \begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{G.4} $$

Appendix G.3. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$ \begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{G.5} $$

Appendix H. Derivation of Result 4

In this section, we show that, if $f(t_1) = (e^{-2\sigma t_1} - 1)$ is an **odd function** of variable t_1 for real σ, this is possible **only** for $\sigma = 0$. We can see that $e^{-2\sigma t_1}$ is an analytic function in the interval $|t_1| \leq \infty$ and can be represented by its Taylor series expansion. We can equate the even part of $f(t_1)$ to zero, as follows.

$$ \begin{aligned} f(t_1) = (e^{-2\sigma t_1} - 1) = -2\sigma t_1 + \frac{(-2\sigma t_1)^2}{!2} + \frac{(-2\sigma t_1)^3}{!3} + \frac{(-2\sigma t_1)^4}{!4} + ... &= f_{odd}(t_1) \\ f_{even}(t_1) = \frac{(-2\sigma t_1)^2}{!2} + \frac{(-2\sigma t_1)^4}{!4} + \frac{(-2\sigma t_1)^6}{!6} + ... &= 0 \\ \frac{d^2 f_{even}(t_1)}{dt_1^2} = 4\sigma^2 + \frac{16\sigma^4(t_1)^2}{!2} + \frac{64\sigma^6(t_1)^4}{!4} + ... &= 0 \end{aligned} \tag{H.1} $$

We take the second derivative of above equation for $f_{even}(t_1)$ and evaluate it at $t_1 = 0$. We get $[\frac{d^2 f_{even}(t_1)}{dt_1^2}]_{t_1=0} = 4\sigma^2 = 0$. This is possible only for $\sigma = 0$ and hence we have shown that if $f(t_1) = (e^{-2\sigma t_1} - 1)$ is an **odd function** of variable t_1 for real σ, this is possible **only** for $\sigma = 0$.

Appendix I. Integral Convergence

Appendix I.1. ***Integral convergence for*** S_{2r}

In this section, we will show that the integrals in Eq. 19 copied in Eq. I.1 are finite. We use $E_0(t) = E_0(-t)$.

$$S_{2r} = G_{2r_I}(\omega_1) = \frac{1}{!(2r)}[-\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau] = 0 \qquad \text{(I.1)}$$

It is well known that the order of Riemann's Xi function at $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (Page 257 in Titchmarsh book).

Hence the Fourier transform of $C(\tau) = \frac{1}{!(2r)}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}$ is given by $C(\omega) = \frac{1}{!(2r)}(-\omega^2)^r E_{0\omega}(\omega)$ has a fall-off rate of the order of $\omega^A e^{-\frac{|\omega|\pi}{4}}(-\omega^2)^r \frac{1}{!(2r)}$ as $|\omega| \to \infty$, which is finite for $|\omega| < \infty$ and goes to zero as $|\omega| \to \infty$ for $r = 0, 1, ..\infty$. Hence $\frac{1}{!(2r)}\int_{-\infty}^{\infty} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau$ is finite.

Given fall-off rate conditions in previous paragraph, $\int_{-\infty}^{\infty}|C(\omega)|d\omega < \infty$. Hence its inverse Fourier transform $C(\tau) < \infty$ for $|\tau| \le \infty$ and $C(\tau)$ goes to zero as $|\tau| \to \infty$ as per Riemann-Lebesgue Lemma. As shown in Appendix I.3, $C(\tau)$ goes to zero as $\tau \to -\infty$ with its **order of decay** greater than $e^{\frac{5\tau}{2}}$. Hence $\int_{-\infty}^{0}|C(\tau)|d\tau < \infty$ and the second integral $\frac{1}{!(2r)}\int_{-\infty}^{0} \frac{d^{2r}E_0(\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau$ in Eq. I.1 **converges** to a finite value.

Similarly, $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ in Eq. I.1, is an **analytic** function which is infinitely differentiable which produces no discontinuities in $|\tau| \le \infty$. Hence its Fourier transform $X(\omega)$ has a fall-off rate of the order of $\omega^B e^{-|\omega|B_2}$ as $|\omega| \to \infty$, where $B > 0, B_2 > 0$ are constants, similar to $E_{0\omega}(\omega)$ in paragraph 2 in this section.

Hence the Fourier transform of $D(\tau) = \frac{1}{!(2r)}\frac{d^{2r}x(\tau)}{d\tau^{2r}}$ is given by $D(\omega) = \frac{1}{!(2r)}(-\omega^2)^r X(\omega)$ has a fall-off rate of the order of $\omega^B e^{-|\omega|B_2}(-\omega^2)^r \frac{1}{!(2r)}$ as $|\omega| \to \infty$, which is finite for $|\omega| < \infty$ and goes to zero as $|\omega| \to \infty$ for $r = 0, 1, ..\infty$. Hence $\frac{1}{!(2r)}\int_{-\infty}^{\infty} \frac{d^{2r}E_0(\tau)e^{-2\sigma\tau}}{d\tau^{2r}} \sin(\omega_1\tau)d\tau$ is finite.

Given fall-off rate conditions in previous paragraph, $\int_{-\infty}^{\infty}|D(\omega)|d\omega < \infty$. Its inverse Fourier transform $D(\tau) < \infty$ for $|\tau| \le \infty$ and $D(\tau)$ goes to zero as $|\tau| \to \infty$ as per Riemann-Lebesgue Lemma. As shown in Appendix I.3, $D(\tau)$ goes to zero as $\tau \to -\infty$ with **order of decay** greater than $e^{\frac{3\tau}{2}}$. Hence $\int_{-\infty}^{0}|D(\tau)|d\tau < \infty$ and the second integral $\frac{1}{!(2r)}\int_{-\infty}^{0} \frac{d^{2r}E_0(\tau)e^{-2\sigma\tau}}{d\tau^{2r}} \sin(\omega_1\tau)d\tau$ in Eq. I.1 **converges** to a finite value.

Appendix I.2. ***Integral convergence for*** $A(t)$

In this section, we show that the integrals in Eq. 21 copied in Eq. I.2 are finite. We use $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \qquad \text{(I.2)}$$

In Section 1.1, we showed that $E_0(\tau)$ is a real **analytic** function in the interval $-\infty \le \tau \le \infty$. Hence we see that $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ is also a real **analytic** function in the same interval. Hence $x(\tau) = E_0(\tau)e^{-2\sigma\tau} = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau} - 3\pi n^2e^{2\tau}]e^{-\pi n^2e^{2\tau}}e^{\frac{\tau}{2}}e^{-2\sigma\tau}$ in the first integral in Eq. I.2 goes to zero as $\tau \to -\infty$ with its **order of decay** greater than $e^{\frac{3\tau}{2}}$ for $0 < \sigma < \frac{1}{2}$ and $E_0(\tau)$ in the second integral goes to zero as $t \to -\infty$ with its order of decay greater than $e^{\frac{5\tau}{2}}$. Both $x(\tau)$ and $E_0(\tau)$ are finite in the interval $|\tau| \le \infty$.

Hence $\int_{-\infty}^{\infty}|x(\tau)|d\tau$ and $\int_{-\infty}^{0}|x(\tau)|d\tau$ is finite and hence $\int_{-\infty}^{0}x(\tau)\sin(\omega_1\tau)d\tau$ is finite. Because $x(\tau+t_1)+x(\tau-t_1)$ are shifted versions of $x(\tau)$, for $-\infty < t_1 < \infty$, we see that $\int_{-\infty}^{0}[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau$ is also finite.

Similarly, we can see that $\int_{-\infty}^{0}[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau$ is also finite, using arguments in previous two paragraphs for the function $E_0(\tau)$.

Appendix I.3. ***Integral convergence for*** S_{2r}***: Method 2***

In the first integral in Eq. I.1, we consider $f(t) = E_0(t)e^{-2\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-2\sigma t}$. We see that $f(t)$ goes to zero as $t \to -\infty$ with its **order of decay** greater than $e^{\frac{3t}{2}}$ for $0 < \sigma < \frac{1}{2}$ and $E_0(t)$ in the second integral goes to zero as $t \to -\infty$ with its order of decay greater than $e^{\frac{5t}{2}}$.

Now we consider the $(2r)^{th}$ derivative of $f(t)$ given by $f_{2r}(t) = \frac{1}{!(2r)}\frac{d^{2r}f(t)}{dt^{2r}}$ for $r = 0, 1, ...$ We can derive as follows, where C_r, D_r are real constants.

$$f(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-2\sigma t}$$

$$\frac{df(t)}{dt} = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{-\pi n^2e^{2t}}e^{t(\frac{9}{2}-2\sigma)}[(\frac{9}{2}-2\sigma) - 2\pi n^2e^{2t}] - [3\pi n^2e^{-\pi n^2e^{2t}}e^{t(\frac{5}{2}-2\sigma)}[(\frac{5}{2}-2\sigma) - 2\pi n^2e^{2t}]$$

$$\frac{1}{!(2)}\frac{d^2f(t)}{dt^2} = \sum_{n=1}^{\infty}[2\pi^2n^4e^{-\pi n^2e^{2t}}e^{t(\frac{9}{2}-2\sigma)}[-4\pi n^2e^{2t} + ((\frac{9}{2}-2\sigma) - 2\pi n^2e^{2t})^2]$$
$$-[3\pi n^2e^{-\pi n^2e^{2t}}e^{t(\frac{5}{2}-2\sigma)}[-4\pi n^2e^{2t} + ((\frac{5}{2}-2\sigma) - 2\pi n^2e^{2t})^2]$$

$$\frac{1}{!(2)}\frac{d^2f(t)}{dt^2} = \frac{1}{!(2)}\sum_{n=1}^{\infty}(-6\pi n^2)e^{-\pi n^2e^{2t}}e^{t(\frac{5}{2}-2\sigma)}[C_0 + C_1n^2e^{2t} + C_2n^4e^{4t} + C_3n^6e^{6t}]$$

$$f_{2r}(t) = \frac{1}{!(2r)}\frac{d^{2r}f(t)}{dt^{2r}} = \frac{1}{!(2r)}\sum_{n=1}^{\infty}(-6\pi n^2)e^{-\pi n^2e^{2t}}e^{t(\frac{5}{2}-2\sigma)}[D_0 + D_1n^2e^{2t} + D_2n^4e^{4t} + ... + D_{2r+1}n^{4r+2}e^{(4r+2)t}] \tag{I.3}$$

We see that $f_{2r}(t) = \frac{1}{!(2r)}\frac{d^{2r}f(t)}{dt^{2r}}$ is **finite** in the interval $-\infty < t \leq 0$ because of the terms $e^{-\pi n^2e^{2t}}$ and $!(2r)$. We see that $f_{2r}(t)$ goes to zero as $t \to -\infty$ with its **order of decay** greater than $e^{\frac{3t}{2}}$ for $0 < \sigma < \frac{1}{2}$, and the term $\frac{1}{!(2r)}\frac{d^{2r}E_0(t)}{dt^{2r}}$ goes to zero as $t \to -\infty$ with its order of decay greater than $e^{\frac{5t}{2}}$. Hence $\int_{-\infty}^{0}|f_{2r}(\tau)|d\tau = \int_{-\infty}^{0}|\frac{1}{!(2r)}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}|d\tau$ in the first integral in Eq. I.1 is finite.

We can set $\sigma = 0$ and show that $\int_{-\infty}^{0}|\frac{1}{!(2r)}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}|d\tau$ in the second integral in Eq. I.1 is finite. Hence the first and second integrals in Eq. I.1 are finite.

Appendix I.4. ***Integral convergence for*** $A(t)$***: Method 2***

Let us consider Eq. 20 copied below in Eq. I.4 and show that the integrals converge. We use the results in the above section for Eq. I.1 and then we take the series summation from $r = 0$ to $r = \infty$.

$$A(t_1) = -\int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau = 0$$

(I.4)

We can show that the series in Eq. I.4, summed from $r = 0, ...\infty$ converges due to the factorial term $!(2r)$, using Series Ratio Test.

In Eq. I.3, for each r, consider the term $b_r = \frac{1}{!(2r)} \frac{d^{2r} f(\tau)}{dt^{2r}} = \frac{1}{!(2r)} \sum_{n=1}^{\infty} (-6\pi n^2) e^{-\pi n^2 e^{2\tau}} e^{\tau(\frac{5}{2}-2\sigma)} [D_0 + D_1 n^2 e^{2\tau} + D_2 n^4 e^{4\tau} + ... + D_{2r+1} n^{4r+2} e^{(4r+2)\tau}]$ and the summation over n is finite, because of the term $e^{-\pi n^2 e^{2\tau}}$, in the interval $-\infty < \tau \leq 0$.

For $\tau = 0$ in Eq. I.4, the integrands equal zero due to the term $\sin(\omega_1 \tau)$. For $\tau = -\infty$, it is shown in Appendix I.3 paragraph 3 that the integrands in Eq. I.4 equal zero. For $-\infty < \tau < 0$, as $r \to \infty$, the terms $D_{2r+1} n^{4r+2} e^{(4r+2)\tau}$ **go to zero**, for $r = 0, 1, ..\infty$. We write $b_r = \frac{a_r}{!(2r)}$ and get the ratio of $\frac{a_{r+1}}{a_r}$ for the series in the first integral in Eq. I.4.

$$\frac{a_{r+1}}{a_r} = \frac{\sum_{n=1}^{\infty} (-6\pi n^2) e^{-\pi n^2 e^{2\tau}} e^{\tau(\frac{5}{2}-2\sigma)} D_0'}{\sum_{n=1}^{\infty} (-6\pi n^2) e^{-\pi n^2 e^{2\tau}} e^{\tau(\frac{5}{2}-2\sigma)} D_0} = \frac{D_0'}{D_0}$$

$$L = \lim_{r\to\infty} |\frac{S_{r+1}}{S_r}| = \lim_{r\to\infty} |\frac{a_{r+1} t_1^2 (!(2r))}{a_r (!(2r+2))}| = \lim_{r\to\infty} |\frac{D_0' t_1^2}{D_0 (2r+2)(2r+1)}| < 1 \tag{I.5}$$

In the first integral in Eq. I.4, the series converges absolutely with limit L, where S_r is the $(r)^{th}$ term in the series and $a_r = \frac{d^{2r}(E_0(\tau) e^{-2\sigma\tau})}{d\tau^{2r}}$ and the first integral in Eq. I.4 converges.

We set $\sigma = 0$ and use above procedure for the second integral in Eq. I.4 and define $a_r = \frac{d^{2r} E_0(\tau)}{d\tau^{2r}}$ and show that the second integral in Eq. I.4 converges.

Hence the integrals in Eq. I.4 are finite.