On the Zeros of Dirichlet Eta Function

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Abstract

Some ideas on Dirichlet Eta Function are derived.

Keywords:

1. Introduction

Let us consider $E_p(t) = \frac{e^{-e^t}}{1+e^{-e^t}} e^{Kt}$ which corresponds to the Eta function $\zeta_a(s) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m^s}$ where $E_p(t)$ is the inverse Fourier Transform of $E(s) = \Gamma(s)\zeta_a(s) = \int_0^\infty [\sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m^s}] e^{-y} y^{s-1} dy$ where $s = \frac{1}{2} + \sigma - i\omega$ and $K = \frac{1}{2} + \sigma$. If we substitute $y = mx$, we have $E_p(s) = \int_0^\infty [\sum_{m=1}^{\infty} (-1)^{m-1} e^{-mx}] x^{s-1} dx = \int_0^\infty \frac{e^{-x}}{1+e^{-x}} x^{s-1} dx$. If we substitute $x = e^t$, we have $E(s) = \int_{-\infty}^{\infty} \frac{e^{-e^t}}{1+e^{-e^t}} e^{st} dt = \int_{-\infty}^{\infty} \frac{e^{-e^t}}{1+e^{-e^t}} e^{Kt} e^{-i\omega t} dt = \int_{\infty}^{\infty} E_p(t) e^{-i\omega t} dt$ where $K = \frac{1}{2} + \sigma$. Let us find the function $P(t)$ which satisfies the equation $E_p(t) = \omega_0^2 P(t) + \frac{d^2 P(t)}{dt^2}$.

Let us use the Taylor series expansion of $E_p(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{k=1}^{\infty} \frac{(-n)^k}{!k} e^{(k+\frac{1}{2}+\sigma)t}$ and use the shorthand notation $E_p(t) = \sum_{n,k} a_{nk} e^{(k+\frac{1}{2}+\sigma)t}$ where $a_{nk} = (-1)^{n-1} \frac{(-n)^k}{!k}$, and we can see that $P(t) = \sum_{n,k} a_{nk} \frac{e^{(k+\frac{1}{2}+\sigma)t}}{((k+\frac{1}{2}+\sigma)^2 + \omega_0^2)}$.

If $E_p(\omega)$ has a zero at $\omega = \omega_0$, we require $P(t)$ to tend to zero as $t \to \infty$ and $t \to -\infty$ because $P(t) = \frac{1}{\omega_0} [\sin(\omega_0 t) \int_{-\infty}^{t} E_p(t) \cos(\omega_0 \tau) d\tau - \cos(\omega_0 t) \int_{-\infty}^{t} E_p(t) \sin(\omega_0 \tau) d\tau]$. We require $P(t) = e^{-\Delta e^t} e^{Kt} g(t)$ where $0 < \Delta << 1$, for above condition to be satisfied, where $g(t)$ is utmost of the order of e^{Rt} where $R > -K$.

2. Section 1

Without loss of generality, let $P(t) = e^{-e^t} e^{Kt} f(t)$ where $f(t) = e^{(1-\Delta)e^t} g(t)$. We can write $f(t) = f_0 + f_1 e^t + f_2 e^{2t} + ...$ given that $P(t) = \sum_{n,k} a_{nk} \frac{e^{(k+\frac{1}{2}+\sigma)t}}{((k+\frac{1}{2}+\sigma)^2 + \omega_0^2)}$.

$$
\begin{aligned}
P(t) &= e^{-e^t} e^{Kt} f(t) \\
\frac{dP}{dt} &= e^{-e^t} e^{Kt} [\frac{df}{dt} + f(t)(K - e^t)] \\
\frac{d^2P}{dt^2} &= e^{-e^t} e^{Kt} [\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(-e^t + (K - e^t)^2)] \\
E(t) = \omega_0^2 P(t) + \frac{d^2P(t)}{dt^2} &= e^{-e^t} e^{Kt} [\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}] = \frac{e^{-e^t}}{1 + e^{-e^t}} e^{Kt} \\
\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) &= \frac{1}{1 + e^{-e^t}}
\end{aligned}
\tag{1}
$$

We see that $f(t) = e^{(1-\Delta)e^t} g(t)$ and that $g(t)$ is utmost of the order of e^{Rt}, given by $O[e^{Rt}]$, where $R > -K$.

$$
\begin{aligned}
f(t) &= e^{(1-\Delta)e^t} g(t) \\
\frac{df}{dt} &= e^{(1-\Delta)e^t} [\frac{dg}{dt} + g(t)(1-\Delta)e^t] \\
\frac{d^2f}{dt^2} &= e^{(1-\Delta)e^t} [\frac{d^2g}{dt^2} + 2\frac{dg}{dt}(1-\Delta)e^t + g(t)[(1-\Delta)e^t + (1-\Delta)^2 e^{2t}]] \\
\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) &= \frac{1}{1 + e^{-e^t}} \\
e^{(1-\Delta)e^t} [\frac{d^2g}{dt^2} + \frac{dg}{dt}[2(1-\Delta)e^t + 2(K - e^t)] & \\
+ g(t)[(1-\Delta)e^t + (1-\Delta)^2 e^{2t} + 2(K - e^t)(1-\Delta)e^t + (\omega_0^2 + K^2 - (2K+1)e^t + e^{2t})]] &= \frac{1}{1 + e^{-e^t}} \\
e^{(1-\Delta)e^t} [\frac{d^2g}{dt^2} + \frac{dg}{dt}[2(K - \Delta e^t] + g(t)[\omega_0^2 + K^2 + e^t(-\Delta(2K+1)) + e^{2t}\Delta^2]] &= \frac{1}{1 + e^{-e^t}}
\end{aligned}
\tag{2}
$$

We have following cases:

Case 1: $\lim_{t\to\infty} g(t) = O[e^{Rt}], R \geq 0$

Without loss of generality, we can write $g(t) = \sum_{r=0}^{R} g_r e^{rt} + h(t)$ where $h(t)$ has terms of lesser order and may include terms of order $O[e^{-Rt}], O[e^{-e^{Rt}}]$ and so on.

Given that $g(t) = \sum_{r=0}^{R} g_r e^{rt} + h(t)$ is of order $O[e^{Rt}]$, $\frac{dg}{dt} = \sum_{r=1}^{R} r g_r e^{rt} + \frac{dh}{dt}$ and $\frac{d^2g}{dt^2} = \sum_{r=1}^{R} r^2 g_r e^{rt} + \frac{d^2h}{dt^2}$, we can see that above equation Eq. 2 is of the order of $O[e^{(1-\Delta)e^t} e^{(R+2)t}]$ and hence we can write the order of the left hand side (LHS) of above equation as $O[e^{(1-\Delta)e^t}e^{(R+2)t}]$ and $\lim_{t\to\infty} LHS = \infty$, while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Hence we see that $g(t) = O[e^{Rt}]$ is **not possible**.

Case 2: $\lim_{t\to\infty} g(t) = e^{-Le^t}O[e^{Rt}], L <= (1-\Delta)$

Without loss of generality, we can write $f(t) = O[e^{\Delta_2 e^t}]e^{Rt}$ where $0 \leq \Delta_2 = 1-\Delta-L << 1$.

Using results in above Cases 1 and 2, we can show that we can write the order of the left hand side (LHS) of above equation Eq. 2 as $O[e^{\Delta_2 e^t}e^{(R+2)t}]$ and $\lim_{t\to\infty} LHS = \infty$, while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Hence we see that $g(t) = e^{-Le^t}O[e^{Rt}], L <= (1-\Delta)$ is **not possible**.

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Case 3: $\lim_{t\to\infty} g(t) = e^{-Le^t}O[e^{Rt}], L > (1-\Delta)$

Without loss of generality, we can write $f(t) = O[e^{-Me^t}]e^{Rt}$ where $M = L-(1-\Delta) > 0$.

We can show that we can write the order of the left hand side (LHS) of above equation Eq. 2 as $O[e^{-Me^t}e^{(R+2)t}]$ and $\lim_{t\to\infty} LHS = 0$, while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Hence we see that $g(t) = e^{-Le^t}O[e^{Rt}], L > (1-\Delta)$ is **not possible**.

Case 4: $\lim_{t\to\infty} g(t) = O[e^{-Rt}], R > 0$

Without loss of generality, we can write $g(t) = \sum_{r=0}^{R} g_r e^{-rt} + h(t)$ where $h(t)$ has terms of lesser order and may include terms of order $O[e^{-e^{Rt}}]$ and so on.

Given that $g(t) = \sum_{r=0}^{R} g_r e^{-rt} + h(t)$ is of order $O[e^{-Rt}]$, $\frac{dg}{dt} = -\sum_{r=1}^{R} r g_r e^{-rt} + \frac{dh}{dt}$ and $\frac{d^2g}{dt^2} = \sum_{r=1}^{R} r^2 g_r e^{-rt} + \frac{d^2h}{dt^2}$, we can see that above equation Eq. 2 is of the order of $O[e^{(1-\Delta)e^t}e^{(-R+2)t}]$ and hence we can write the order of the left hand side (LHS) of above equation as $O[e^{(1-\Delta)e^t}e^{(-R+2)t}]$ and $\lim_{t\to\infty} LHS = \infty$, while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Hence we see that $g(t) = O[e^{-Rt}]$ is **not possible**.

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Case 5: $\lim_{t\to\infty} g(t) = e^{-(1-\Delta)e^t}O[e^{-Rt}], R > 0$

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \frac{1}{1+e^{-e^t}}$$
$$g(t) = e^{-(1-\Delta)e^t} O[e^{-Rt}]$$
$$f(t) = e^{(1-\Delta)e^t} g(t) = O[e^{-Rt}] \tag{3}$$

Without loss of generality, we can write $g(t) = e^{-(1-\Delta)e^t}[\sum_{r=0}^{R} g_r e^{-rt} + h(t)]$ where $h(t)$ has terms of lesser order and may include terms of order $O[e^{-e^{Rt}}]$ and so on. So, we can write $f(t) = e^{(1-\Delta)e^t} g(t) = O[e^{-Rt}]$, hence $\lim_{t\to\infty} \frac{df}{dt} = 0$ and $\lim_{t\to\infty} \frac{d^2f}{dt^2} = 0$ and $\lim_{t\to\infty} f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = 0$ for $R > 2$ and $\lim_{t\to\infty} f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \infty$ for $R < 2$ while the right hand side (RHS) of above equation Eq. 2 $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Special case R=2

Let us consider the case when $\lim_{t\to\infty} f(t) = O[e^{-2t}]$ is a **possible solution** in Eq. 1. We can show that this solution is **NOT** possible as follows.

Let $f(t)$ have a term $\frac{1}{1+e^{2t}}$ which is of order $O[e^{-2t}]$ as $\lim_{t\to\infty}$ and is a **possible solution** in Eq. 1. Given that $P(t), f(t)$ are analytic functions, $P(t) = e^{-e^t} e^{Kt} f(t) = e^{-e^t} O[e^{(K-2)t}], \lim_{t\to-\infty} P(t) \to \infty$ which is **not possible** for $K = \frac{1}{2} + \sigma$ where $0 \le \sigma \le \frac{1}{2}$.

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \frac{1}{1+e^{-e^t}} \tag{4}$$

we can see that the LHS of above equation tends to ∞ as $\lim_{t\to-\infty}$ and RHS tends to 1, thus leading to a contradiction.

In addition, $f(t)$ is also allowed to have additional terms of the order of $O[e^{-e^{At}} e^{Bt}]$ which are of lower order than $O[e^{-2t}]$ as $\lim_{t\to\infty}$. These terms tend to zero as $t \to \infty$ in LHS of equation above. As $t \to -\infty$, these additional terms tend to zero for $B > 0$ and tend to ∞ for $B = 0$ in LHS of equation above, while RHS tends to 1, thus leading to a contradiction.

3. Section 2 : Transcendental f(t) many terms

Let us consider $E_p(t) = \frac{e^{-e^t}}{1+e^{-e^t}} e^{Kt}$. Let $P(t) = e^{-\Delta e^t} e^{Kt} f(t)$ where $f(t) = \cos(g_1(t))h_1(t) + \cos(g_2(t))h_2(t)$ and that $h_1(t), h_2(t)$ is utmost of the order of $O[1]$.

We have **replaced** $f(t), g(t)$ in Section 1 with $f'(t), f(t)$ respectively in this section.

Without loss of generality, let $P(t) = e^{-e^t} e^{Kt} f'(t)$ where $f'(t) = e^{(1-\Delta)e^t} f(t)$.

$$\begin{aligned}
P(t) &= e^{-e^t} e^{Kt} f'(t) \\
\frac{dP}{dt} &= e^{-e^t} e^{Kt}[\frac{df'}{dt} + f'(t)(K - e^t)] \\
\frac{d^2P}{dt^2} &= e^{-e^t} e^{Kt}[\frac{d^2f'}{dt^2} + 2\frac{df'}{dt}(K - e^t) + f'(t)(-e^t + (K - e^t)^2)] \\
E(t) = \omega_0^2 P(t) + \frac{d^2P(t)}{dt^2} &= e^{-e^t} e^{Kt}[\frac{d^2f'}{dt^2} + 2\frac{df'}{dt}(K - e^t) + f'(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}] = \frac{e^{-e^t}}{1+e^{-e^t}} e^{Kt} \\
A(t) &= \frac{d^2f'}{dt^2} + 2\frac{df'}{dt}(K - e^t) + f'(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \frac{1}{1+e^{-e^t}}
\end{aligned} \tag{5}$$

We see that $f'(t) = e^{(1-\Delta)e^t} f(t)$. We can write as follows.

$$\begin{aligned}
f'(t) &= e^{(1-\Delta)e^t} f(t) \\
\frac{df'}{dt} &= e^{(1-\Delta)e^t}[\frac{df}{dt} + f(t)(1-\Delta)e^t] \\
\frac{d^2f'}{dt^2} &= e^{(1-\Delta)e^t}[\frac{d^2f}{dt^2} + 2\frac{df}{dt}(1-\Delta)e^t + f(t)[(1-\Delta)e^t + (1-\Delta)^2 e^{2t}]] \\
A(t) = \frac{d^2f'}{dt^2} &+ 2\frac{df'}{dt}(K - e^t) + f'(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \frac{1}{1+e^{-e^t}} \\
A(t) &= e^{(1-\Delta)e^t}[\frac{d^2f}{dt^2} + \frac{df}{dt}[2(1-\Delta)e^t + 2(K - e^t)] \\
+f(t)[(1-\Delta)e^t + (1-\Delta)^2 e^{2t} &+ 2(K - e^t)(1-\Delta)e^t + (\omega_0^2 + K^2 - (2K+1)e^t + e^{2t})]] = \frac{1}{1+e^{-e^t}} \\
A(t) &= e^{(1-\Delta)e^t}[\frac{d^2f}{dt^2} + \frac{df}{dt}2(K-\Delta)e^t \\
+f(t)[\omega_0^2 + K^2 + e^t(1 - \Delta - 2K - 1 + 2K - 2K\Delta) &+ e^{2t}(1 + \Delta^2 - 2\Delta + 1 - 2 + 2\Delta)]] = \frac{1}{1+e^{-e^t}} \\
A(t) = e^{(1-\Delta)e^t}[\frac{d^2f}{dt^2} + \frac{df}{dt}2(K-\Delta)e^t &+ f(t)[\omega_0^2 + K^2 + e^t(-\Delta(1+2K)) + e^{2t}\Delta^2]] = \frac{1}{1+e^{-e^t}}
\end{aligned} \tag{6}$$

Let us substitute $f(t) = \cos(g_1(t))h_1(t) + \cos(g_2(t))h_2(t)$ and that $h_1(t), h_2(t)$ is **utmost of the order** of $O[1]$. We can write $g_2(t) = g_1(t) + g_{\Delta_1}(t)$.

$$\begin{aligned}
f(t) &= \cos(g_1(t))h_1(t) + \cos(g_2(t))h_2(t) \\
f(t) &= \cos(g_1(t))h_1(t) + \cos(g_1(t) + g_{\Delta_1}(t))h_2(t) \\
f(t) &= \cos(g_1(t))[h_1(t) + \cos(g_{\Delta_1}(t))h_2(t)] - \sin(g_1(t))\sin(g_{\Delta_1}(t))h_2(t) \\
R(t) &= h_1(t) + \cos(g_{\Delta_1}(t))h_2(t) \quad S(t) = \sin(g_{\Delta_1}(t))h_2(t) \\
f(t) &= \cos(g_1(t))R(t) - \sin(g_1(t))S(t)
\end{aligned} \tag{7}$$

We know that $f(t)$ is is utmost of the order of $O[1]$, hence $R(t), S(t)$ are of **utmost of the order** of $O[1]$. Hence we can write as follows.

$$\begin{aligned}
f(t) &= \cos(g_1(t))R(t) - \sin(g_1(t))S(t) \\
\frac{df}{dt} &= \cos(g_1(t))R_1(t) - \sin(g_1(t))S_1(t) \\
R_1(t) &= \frac{dR}{dt} - S(t)\frac{dg_1}{dt}; \quad S_1(t) = R(t)\tfrac{dg_1}{dt} + \tfrac{dS}{dt} \\
\frac{d^2f}{dt^2} &= \cos(g_1(t))R_2(t) - \sin(g_1(t))S_2(t) \\
R_2(t) &= \frac{d^2R}{dt^2} - S(t)\frac{d^2g_1}{dt^2} - R(t)(\frac{dg_1}{dt})^2 - 2\frac{dS}{dt}\frac{dg_1}{dt} \\
S_2(t) &= \frac{d^2S}{dt^2} + R(t)\frac{d^2g_1}{dt^2} - S(t)(\frac{dg_1}{dt})^2 + 2\frac{dR}{dt}\frac{dg_1}{dt}
\end{aligned} \tag{8}$$

Hence we can write Eq. 6 as follows.

$$\begin{aligned}
A(t) = e^{(1-\Delta)e^t}[\frac{d^2f}{dt^2} + \frac{df}{dt}2(K-\Delta)e^t + f(t)[\omega_0^2 + K^2 + e^t(-\Delta(1+2K)) + e^{2t}\Delta^2]] &= \frac{1}{1+e^{-e^t}} \\
\cos(g_1(t))I_1(t) - \sin(g_1(t))I_2(t) &= \frac{1}{1+e^{-e^t}} \\
I_1(t) = e^{(1-\Delta)e^t}[R_2(t) + 2(K-\Delta)e^tR_1(t) + (\omega_0^2 + K^2 - \Delta(2K+1)e^t + \Delta^2e^{2t})R(t)] \\
I_2(t) = e^{(1-\Delta)e^t}[S_2(t) + 2(K-\Delta)e^tS_1(t) + (\omega_0^2 + K^2 - \Delta(2K+1)e^t + \Delta^2e^{2t})S(t)]
\end{aligned} \tag{9}$$

We can rewrite above equations as follows.

$$\begin{aligned}
I_1(t) &= e^{(1-\Delta)e^t}[I_{11}(t) + I_{12}(t)] \\
I_2(t) &= e^{(1-\Delta)e^t}[I_{21}(t) + I_{22}(t)] \\
I_{11}(t) = R_2(t) + 2(K-\Delta)e^tR_1(t), I_{12}(t) &= (\omega_0^2 + K^2 - \Delta(2K+1)e^t + \Delta^2e^{2t})R(t) \\
I_{21}(t) = S_2(t) + 2(K-\Delta)e^tS_1(t), I_{22}(t) &= (\omega_0^2 + K^2 - \Delta(2K+1)e^t + \Delta^2e^{2t})S(t)
\end{aligned} \tag{10}$$

We **require** $\lim_{t\to\infty} I_1(t) = \frac{1}{2}\cos(g_1(t))$ which is of order $O[1]$ and $\lim_{t\to\infty} I_2(t) = -\frac{1}{2}\sin(g_1(t))$ which is of order $O[1]$ for above equation $\cos(g_1(t))I_1(t) - \sin(g_1(t))I_2(t) = \frac{1}{1+e^{-e^t}}$ to be satisfied.

This means that we **require the highest order term** in the equations for $I_1(t)$ and $I_2(t)$ to be of the order of $O[e^{-(1-\Delta)e^t}]$.

We see that the order of the term $I_{12}(t)$ is $O[e^{2t}]$ because $R(t)$ is of **order 1**. Similarly, the order of the term $I_{22}(t)$ is $O[e^{2t}]$ because $S(t)$ is of **order 1**. Hence the order of the terms $I_1(t)$ and $I_2(t)$ are **at least of order** $O[e^{(1-\Delta)e^t}]O[e^{2t}]$.

We can **show that** the terms $I_{12}(t), I_{22}(t)$ are NOT cancelled by $I_{11}(t), I_{21}(t)$ as follows.

Case 1:

$I_{11}(t)$ and $I_{21}(t)$ are of the order of $O[e^{-(1-\Delta)e^t}]\frac{1}{2}\cos(g_1(t))$ as $\lim_{t\to\infty}$. It **cannot** cancel the respective terms $I_{12}(t), I_{22}(t)$ which are of the order of $O[e^{2t}]$. We get the result $I_1(t), I_2(t)$ of the order $O[e^{(1-\Delta)e^t}]O[e^{2t}]$, which is **not** what is required $\lim_{t\to\infty} I_1(t) = \frac{1}{2}\cos(g_1(t))$ and $\lim_{t\to\infty} I_2(t) = -\frac{1}{2}\sin(g_1(t))$.

Case 2:

$I_{11}(t)$ and $I_{21}(t)$ are of the order of $O[e^{2t}]$ as $\lim_{t\to\infty}$. They **can cancel** the respective terms $I_{12}(t), I_{22}(t)$ which are of the order of $O[e^{2t}]$. But we get the result $I_1(t) = 0, I_2(t) = 0$, which is **not** what is required $\lim_{t\to\infty} I_1(t) = \frac{1}{2}\cos(g_1(t))$ and $\lim_{t\to\infty} I_2(t) = -\frac{1}{2}\sin(g_1(t))$.

Case 3:

$I_{11}(t)$ and $I_{21}(t)$ are of order $O[e^{Rt}]$ where $R > 2$, as $\lim_{t\to\infty}$. They **cannot** cancel the respective terms $I_{12}(t), I_{22}(t)$ which are of the order of $O[e^{2t}]$. We get the result $I_1(t), I_2(t)$ of the order $O[e^{(1-\Delta)e^t}]O[e^{Rt}]$, which is **not** what is required $\lim_{t\to\infty} I_1(t) = \frac{1}{2}\cos(g_1(t))$ and $\lim_{t\to\infty} I_2(t) = -\frac{1}{2}\sin(g_1(t))$.

Case 4:

$I_{11}(t)$ and $I_{21}(t)$ are of order $O[e^{Rt}]$ where $R < 2$, as $\lim_{t\to\infty}$. They **cannot** cancel the respective terms $I_{12}(t), I_{22}(t)$ which are of the order of $O[e^{2t}]$. We get the result $I_1(t), I_2(t)$ of the order $O[e^{(1-\Delta)e^t}]O[e^{2t}]$, which is **not** what is required $\lim_{t\to\infty} I_1(t) = \frac{1}{2}\cos(g_1(t))$ and $\lim_{t\to\infty} I_2(t) = -\frac{1}{2}\sin(g_1(t))$.

This means the assumption that $P(t) = e^{-\Delta e^t}e^{Kt}f(t)$ where $f(t) = \cos(g_1(t))h_1(t) + \cos(g_2(t))h_2(t)$ and that $h_1(t), h_2(t)$ is utmost of the order of $O[1]$, leads to a **contradiction**.

Hence the **assumption** that $E_p(\omega)$ has a zero at $\omega = \omega_0$, leads to a **contradiction**.

4. Appendix A

Case 2: $\lim_{t\to\infty} g(t) = O[e^{-Rt}], R > 0$

Without loss of generality, we can write $g(t) = \sum_{r=0}^{R} g_r e^{-rt} + h(t)$ where $h(t)$ has terms of lesser order and may include terms of order $O[e^{-e^{Rt}}]$ and so on.

Given that $g(t) = \sum_{r=0}^{R} g_r e^{-rt} + h(t)$ is of order $O[e^{-Rt}]$, $\frac{dg}{dt} = -\sum_{r=1}^{R} r g_r e^{-rt} + \frac{dh}{dt}$ and $\frac{d^2g}{dt^2} = \sum_{r=1}^{R} r^2 g_r e^{-rt} + \frac{d^2h}{dt^2}$, we can see that above equation is of the order of $O[e^{(1-\Delta)e^t} e^{(-R+2)t}]$ and hence we can write the order of the left hand side (LHS) of above equation as $O[e^{(1-\Delta)e^t} e^{(-R+2)t}]$ and $\lim_{t\to\infty} LHS = \infty$, while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Hence we see that $g(t) = O[e^{-Rt}]$ is **not possible**.

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Case 4: $\lim_{t\to\infty} g(t) = e^{-(1-\Delta)e^t} O[e^{-Rt}], R > 0$

$$\begin{aligned} \frac{d^2 f}{dt^2} + 2\frac{df}{dt}(K - e^t) + f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) &= \frac{1}{1+e^{-e^t}} \\ g(t) &= e^{-(1-\Delta)e^t} O[e^{-Rt}] \\ f(t) = e^{(1-\Delta)e^t} g(t) &= O[e^{-Rt}] \end{aligned} \tag{11}$$

Without loss of generality, we can write $g(t) = e^{-(1-\Delta)e^t}[\sum_{r=0}^{R} g_r e^{-rt} + h(t)]$ where $h(t)$ has terms of lesser order and may include terms of order $O[e^{-e^{Rt}}]$ and so on. So, we can write $f(t) = e^{(1-\Delta)e^t} g(t) = O[e^{-Rt}]$, hence $\lim_{t\to\infty} \frac{df}{dt} = 0$ and $\lim_{t\to\infty} \frac{d^2 f}{dt^2} = 0$ and $\lim_{t\to\infty} f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = 0$ for $R > 2$ and $\lim_{t\to\infty} f(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \infty$ for $R < 2$ while the right hand side (RHS) of above equation $\lim_{t\to\infty} = \frac{1}{1+e^{-e^t}} = 1$ which leads to a **contradiction**.

Special case R=2

We see that $\lim_{t\to\infty} f(t)(\omega_0^2+K^2-(2K+1)e^t+e^{2t}) = 1$ for $R = 2$. Given that $P(t), f(t)$ are holomorphic functions, $P(t) = e^{-e^t} e^{Kt} f(t) = e^{-e^t} O[e^{(K-2)t}]$, $\lim_{t\to-\infty} P(t) \to \infty$ which is **not possible** for $K = \frac{1}{2} + \sigma$ where $0 \le \sigma \le \frac{1}{2}$.

[**Check again**: Given that $f(t)$ is a holomorphic function with order $O[e^{-Rt}]$, this means it may have terms of lower order, for example $O[e^{-(R+S)t}]$ where $S > 0$ and $e^{-O[e^{Rt}]}$ and so on. If we compute $\lim_{t\to-\infty} P(t)$, terms with order lower than $O[e^{-Rt}]$ would be significant at $t \to -\infty$. For example if the lowest order term in $f(t)$ is $O[e^{-(R+S)t}]$, $\lim_{t\to-\infty} P(t) = \lim_{t\to-\infty} e^{-e^t} e^{Kt} O[e^{-(R+S)t}] = O[e^{K-(R+S)t}] \to \infty$, for $R+S > K$. This leads to a contradiction with RHS which tends to 1. Similarly, if the lowest order term in $f(t)$ is $e^{-O[e^{Rt}]}$, $\lim_{t\to-\infty} P(t) = \lim_{t\to-\infty} e^{-e^t} e^{Kt} e^{-O[e^{Rt}]} = 0$. This leads to a contradiction with RHS which tends to 1.

Hence we see that $g(t) = e^{-(1-\Delta)e^t} O[e^{-Rt}], R > 0$ is **not possible**.

Crossover Point: $f(t) = O(e^{-2t})$

The special case of $R = 2$ above, is a cross-over point. For $R > 2$, LHS tends to zero as $\lim_{t\to-\infty}$. For $R < 2$, LHS tends to ∞ as $\lim_{t\to-\infty}$. The exception is the case below.]
Special Case:

Let us consider the case when $\lim_{t\to\infty} f(t) = O[e^{-2t}]$ is a **possible solution** in Eq. 1. We can show that this solution is **NOT** possible as follows. Let us consider $E'_p(t) = E_p(t - t_0)$ where $0 < t_0 < 1$ and we have corresponding $P'(t) = P(t - t_0) = e^{-e^{-t_0}e^t} e^{Kt} e^{-Kt_0} f(t - t_0)$ and we require $f(t - t_0) = O[e^{-2t}]$ to be of the same order as $f(t)$. We can write $P'(t) = e^{-e^t} e^{Kt} e^{-Kt_0} f'(t)$ where $f'(t) = [e^{(1-e^{-t_0})e^t} f(t - t_0)]$ where $(1 - e^{-t_0}) > 0$ and $f'(t)$ is of order $O[e^{(1-e^{-t_0})e^t}]e^{-2t}$ and we can show that this leads to a contradiction in Eq. 1.

$$\frac{d^2 f'}{dt^2} + 2\frac{df'}{dt}(K - e^t) + f'(t)(\omega_0^2 + K^2 - (2K+1)e^t + e^{2t}) = \frac{1}{1 + e^{-e^{-t_0}e^t}} \tag{12}$$

we can see that the LHS of above equation tends to ∞ as $\lim_{t\to\infty}$ and RHS tends to 1, thus leading to a contradiction.