1. $E = mc^2$ does not obey Conservation of Energy Example 1

Consider a system of 2 stars S_1, S_2 of rest masses M, located at $x_1 = x_0, y_1 = 0$ and $x_2 = -x_0, y_2 = 0$ respectively with initial velocities $v_{x_1} = -v_0, v_{y_1} = 0$ and $v_{x_2} = v_0, v_{y_2} = 0$ respectively. They move towards each other with increasing velocities.

We know that the **energy law** $\frac{1}{2}M_1v_1^2(t) + \frac{1}{2}M_2v_2^2(t) - \frac{GM_1M_2}{r(t)} = C_0$ where C_0 is a constant, is satisfied at all points in the trajectory of the stars, for classical Newtonian mechanics. [(Richard Feynman Section 13-3) and (this link)] .

We want to **restate** this Energy Law with the assumption $E = mc^2 = \frac{m_0c^2}{\sqrt{1-(\frac{v}{c})^2}}$. Note that m_0c^2 is the **rest energy** of the particle, $\frac{1}{2}m_0v^2$ is the classical **kinetic energy** term, which appears in Newtonian two-body energy conservation law. Let us assume that Higher order terms $\frac{v^4}{c^2}$ and higher, are part of kinetic energy and let us assume that $E = mc^2$ **does not** capture **potential energy** due to outside masses.

In **Section 2**, we derive the energy law for the general case $M_1 \neq M_2, v_1 \neq v_2$ and show that it results in a **contradiction**. In the subsection **1.1** and **Section 3**, we show that general relativity **cannot rescue** this contradiction.

Let us assume that $M_1 = \frac{M_0}{K_1}, M_2 = \frac{M_0}{K_2}, K_1 = \sqrt{1 - \frac{v_1(t)^2(t)}{c^2}}, K_2 = \sqrt{1 - \frac{v_2(t)^2(t)}{c^2}}$ we can write as follows, with kinetic energy term $\frac{1}{2}Mv^2$ replaced by the **whole energy term** $\frac{Mc^2}{K}$ where A_0, A_1, A_2 are constants and $r(t) = |x_2(t) - x_1(t)|$. $\frac{1}{2}C_2$ is the Combination "Choose" operator.

$$
\begin{aligned}
E = \frac{Mc^2}{K}, \quad K = \sqrt{1 - \frac{v^2(t)}{c^2}} \\
\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r(t)} = A_1 \\
\frac{1}{K_1} + \frac{M_2}{K_2M_1} - \frac{GM_2}{K_1K_2r(t)c^2} = \frac{A_1}{M_1c^2} = A_2
\end{aligned}
\tag{1}
$$

For example, let us consider 2 stars of masses $M_1 = 2e30, M_2 = 2e31$ Kgs, starting with initial velocities $v_1 = 1209, v_2 = 382$ km/sec, $c = 3e8$ m/sec, $\frac{v_1}{c} = 4e{-}3$, $\frac{v_2}{c} = 1e{-}3, G = 6.67e{-}11$ and separated by a distance $r(0) = 1e8$ meters. We get $K_1^2 = 1 - 1.6e - 5$ and $K_2^2 = 1 - 1e - 6$.

Matlab simulation of $A_2 - 1$ in above Eq. 1 in (link) produces a plot which shows that the relativistic **total energy** term is **NOT a constant**, even when we reduce the simulation time step further. This **contradicts** the assumption that the total energy of the system is a constant!

Hence the **assumption** that $E = mc^2$leads to a result that the total energy in the system is NOT a constant, which is a **contradiction**!

1.1. ***General Relativity does not explain this anomaly!***

Note that **general relativity** adds terms $\frac{B^2}{r^2(t)}$ to the potential energy in Eq. 1 **much smaller** than Newtonian terms, for the example of 2 ordinary stars moving towards each other, where B is a constant. (See Eq. 7c in Einstein mercury perihelion paper:).

$$
\begin{aligned}
\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r(t)}[1 + \frac{B^2}{r^2(t)}] = A_1 \\
\frac{1}{K_1} + \frac{M_2}{K_2M_1} - \frac{GM_2}{K_1K_2r(t)c^2}[1 + \frac{B^2}{r^2(t)}] = \frac{A_1}{M_1c^2} = A_2
\end{aligned}
\tag{2}
$$

If we choose $r(t)$ **much larger** than B, the last term in above equation $T_5 = \frac{GM}{r(t)K_1K_2c^2}\frac{B^2}{r^2(t)}$ can be **ignored** and the first 2 terms being the same as Eq. 1 , for the same example values in previous subsection, we can arrive at a similar conclusion that the left hand side of above equation is **NOT a constant**!

Hence the **assumption** that $E = mc^2$ and that it **does not** capture **potential energy** due to outside masses. leads to a result that the total energy in the system is NOT a constant, which is a contradiction! Hence general relativity **cannot rescue** above contradiction.

Regarding General relativity(GR) energy conservation law, it is **curious** that Einstein's equations when applied to two body or N-body system, **did not** have **special relativistic** mass expansion terms at all! (Eq. 7b in Einstein mercury perihelion paper:)

NASA JPL scientists use this **Parametric Post-Newtonian** (PPN) model which is Newtonian + GR, for N-body system and it also **does not** have **special relativistic mass expansion terms** at all! [Eq.27 in page 12 link] It seems no one has managed to capture this special relativistic mass expansion terms in GR equations for motion !

This means, GR equations for motion, are likely to have **same problems** with special relativistic mass expansion terms and Energy Conservation Law, similar to what is shown in Section 4.

2. Law of Conservation of Energy Not obeyed in $E = mc^2$ two body system with relativistic mass expansion

In this section, we will use $E = mc^2$ and **add** special-relativistic mass expansion and see whether Law of Conservation of Energy is obeyed, if the energy law $\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r} = A_k$ is satisfied, where A_k is a constant, $K_1 = \sqrt{1-\frac{v_1^2}{c^2}}$, $K_2 = \sqrt{1-\frac{v_2^2}{c^2}}$. We see that $K_1 = \sqrt{1-\frac{(v_{x_1}^2+v_{y_1}^2)}{c^2}}$, $\frac{dK_1}{dt} = \frac{1}{2K_1}\frac{-2}{c^2}(v_{x_1}a_{x_1}+v_{y_1}a_{y_1}) = \frac{-1}{K_1c^2}\vec{a_1}.\vec{v_1}$ and $K_2 = \sqrt{1-\frac{(v_{x_2}^2+v_{y_2}^2)}{c^2}}$, $\frac{dK_2}{dt} = \frac{1}{2K_2}\frac{-2}{c^2}(v_{x_2}a_{x_2}+v_{y_2}a_{y_2}) = \frac{-1}{K_2c^2}\vec{a_2}.\vec{v_2}$, and $v_1\frac{dv_1}{dt} = \vec{a_1}.\vec{v_1}$, $\frac{v_2dv_2}{dt} = \vec{a_2}.\vec{v_2}$.

Let us define gravitational potentials $\Phi_1 = \frac{-GM_1}{K_1r}, \Phi_2 = \frac{-GM_2}{K_2r}$. We can write $\Phi = -\frac{GM_1M_2}{K_1K_2r} = \frac{1}{2}[\frac{M_1}{K_1}\Phi_2 + \frac{M_2}{K_2}\Phi_1]$ and $\frac{d\Phi}{dt} = \frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt} + M_1\Phi_2\frac{-1}{K_1^2}\frac{dK_1}{dt} + \frac{M_2}{K_2}\frac{d\Phi_1}{dt} + M_2\Phi_1\frac{-1}{K_2^2}\frac{dK_2}{dt}]$.

We can differentiate the energy law $\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 + \Phi = \frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r} = A_k$ with respect to **variable** t, we have as follows.

$$
\begin{aligned}
M_1c^2\frac{(\frac{1}{K_1c^2}\vec{a_1}.\vec{v_1})}{K_1^2} + M_2c^2\frac{(\frac{1}{K_2c^2}\vec{a_2}.\vec{v_2})}{K_2^2} + \frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt} + \frac{M_2}{K_2}\frac{d\Phi_1}{dt} + M_1\Phi_2\frac{1}{K_1^3c^2}\vec{a_1}.\vec{v_1} + M_2\Phi_1\frac{1}{K_2^3c^2}\vec{a_2}.\vec{v_2}] = 0 \\
(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} = -\frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt} + \frac{M_2}{K_2}\frac{d\Phi_1}{dt}] \\
(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} = -\frac{M_1}{2K_1}[\frac{d\Phi_2}{dx}\frac{dx}{dt} + \frac{d\Phi_2}{dy}\frac{dy}{dt}] - \frac{M_2}{2K_2}[\frac{d\Phi_1}{dx}\frac{dx}{dt} + \frac{d\Phi_1}{dy}\frac{dy}{dt}] \\
(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} = \\
-\frac{M_1}{2K_1}[(\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y}).(\frac{dx}{dt}\widehat{x} + \frac{dy}{dt}\widehat{y})] - \frac{M_2}{2K_2}[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(\frac{dx}{dt}\widehat{x} + \frac{dy}{dt}\widehat{y})]
\end{aligned}
\tag{3}
$$

Given that $x = x_2 - x_1$ and $y = y_2 - y_1$, and $v_x = \frac{dx}{dt} = v_{x_2} - v_{x_1}$, $v_y = \frac{dy}{dt} = v_{y_2} - v_{y_1}$, $\Phi_1 = \frac{-GM_1}{rK_1}$ and defining $\vec{\Phi_1'} = (\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}), \vec{\Phi_2'} = (\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y})$,and given that $\vec{v_2} = (v_{x_2}\widehat{x} + v_{y_2}\widehat{y}), \vec{v_1} = (v_{x_1}\widehat{x} + v_{y_1}\widehat{y})$ we can write

$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}$$
$$= -\frac{M_2}{2K_2}[\Phi_1^{'}.((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})] - \frac{M_1}{2K_1}[\Phi_2^{'}.((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})]$$
$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}$$
$$= -\frac{M_2}{2K_2}\Phi_1^{'}.(v_{x_2}\widehat{x} + v_{y_2}\widehat{y}) - \frac{M_1}{2K_1}\Phi_2^{'}.(v_{x_2}\widehat{x} + v_{y_2}\widehat{y}) + \frac{M_2}{2K_2}\Phi_1^{'}.(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})] + \frac{M_1}{2K_1}\Phi_2^{'}.(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})$$
$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}$$
$$= -[(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}).\vec{v_2})] + [(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}).\vec{v_1}]$$
$$((\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} - (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}})).\vec{v_1} + ((\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2} + (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}})).\vec{v_2} = 0$$
$$\vec{F_1}.\vec{v_1} + \vec{F_2}.\vec{v_2} = 0 \tag{4}$$

Given that both the vectors $\vec{v_1}, \vec{v_2}$ and $\vec{F_1}, \vec{F_2}$ are in the same XY plane, $\vec{F_1}.\vec{v_1} \neq 0, \vec{F_2}.\vec{v_2} \neq 0$ for all t as the 2 bodies move along the orbit, unless $\vec{F_1} = \vec{F_2} = 0$. So we require the following conditions be satisfied for the energy law to be true for the general case.

$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} = (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}) = A(r);$$
$$(\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2} = -(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}) = -A(r);$$
$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} = -(\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2} \tag{5}$$

We require following to be satisfied, from above equations.

$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})a_{x_1} = -(\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})a_{x_2}$$
$$(\frac{M_1}{K_1^3} + M_1\Phi_2\frac{1}{2K_1^3c^2})a_{y_1} = -(\frac{M_2}{K_2^3} + M_2\Phi_1\frac{1}{2K_2^3c^2})a_{y_2}$$
$$\frac{a_{x_1}}{a_{x_2}} = -\frac{M_2K_1^3(1 + \frac{\Phi_1}{2c^2})}{M_1K_2^3(1 + \frac{\Phi_2}{2c^2})}) \tag{6}$$

But we know that

$$
\begin{aligned}
\vec{a_1} = a_{x_1}\widehat{x} + a_{y_1}\widehat{y} &= \frac{GM_2 x}{K_2 r^3}\widehat{x} + \frac{GM_2 y}{K_2 r^3}\widehat{y} \\
\vec{a_2} = a_{x_2}\widehat{x} + a_{y_2}\widehat{y} &= \frac{-GM_1 x}{K_1 r^3}\widehat{x} + \frac{-GM_1 y}{K_1 r^3}\widehat{y} \\
\frac{a_{x_1}}{a_{x_2}} &= \frac{-M_2 K_1}{M_1 K_2}
\end{aligned}
\tag{7}
$$

Equating above 2 sets of equations, we get

$$
\begin{aligned}
\frac{a_{x_1}}{a_{x_2}} = \frac{-M_2 K_1}{M_1 K_2} &= -\frac{M_2 K_1^3 (1 + \frac{\Phi_1}{2c^2})}{M_1 K_2^3 (1 + \frac{\Phi_2}{2c^2}})\\
K_2^2 (1 + \frac{\Phi_2}{2c^2}) &= K_1^2 (1 + \frac{\Phi_1}{2c^2}) \\
\Phi_1 = \frac{-GM_1}{K_1 r}, \Phi_2 &= \frac{-GM_2}{K_2 r}
\end{aligned}
\tag{8}
$$

In general, above equation is NOT satisfied for the general case of objects with different masses and velocities and hence **energy law is NOT obeyed**.

For example, let us consider 2 stars of masses $M_1 = 2e30, M_2 = 2e31$ Kgs, starting with initial velocities $v_1 = 1209, v_2 = 382$ km/sec, $c = 3e8$ m/sec, $\frac{v_1}{c} = 4e{-}3$, $\frac{v_2}{c} = 1e{-}3$, $G = 6.67e{-}11$ and separated by a distance $r(0) = 1e8$ meters.

We get $\frac{\Phi_1}{2c^2} = -7.41e - 6, K_1^2 = 1 - 1.6e - 5$ and $\frac{\Phi_2}{2c^2} = -7.41e - 5, K_2^2 = 1 - 1e - 6$. We can see that left hand side of the above equation is $1 - 7.5e - 5$ and the right hand side is $1 - 2.3e - 5$, thus leading to a conclusion that **energy law is NOT obeyed**.

Matlab simulation of above Eq. 8 in (link) produces a plot (link) shows that left hand side and right hand side of above equation are **not equal**, even when we reduce the simulation time step further. This **contradicts** the assumption that the total energy of the system is a constant!

We can do a sanity check of Eq. 4 in (plot) which shows **Red** plot is actual simulation of **first derivative** of relativistic total energy and is compared with theoretical equation which is in **blue**, we can see that both match. We can see that left hand side of Eq. 4, which is first derivative of relativistic total energy $\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 + \Phi = \frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1 M_2}{K_1 K_2 r} = A_k$, is NOT zero! We can also see that the relativistic **total energy** term in (plot) is **NOT a constant**.

Thus we have shown that Law of Conservation of Energy is **not obeyed** for the general case of two objects starting at arbitrary velocities, if we assume that $E = mc^2$ and special relativistic mass expansion term $\frac{m_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}}$ is included in Newtonian two body system.

3. Law of Conservation of Energy Not obeyed in Einstein's General Relativity two body system with $E = mc^2$ and relativistic mass expansion

Einstein mercury perihelion paper is here. We take Eq. 7b, 8a, 10 and get as follows. $\alpha = \frac{2GM}{Kc^2}$ where $K = \sqrt{1 - \frac{v^2}{c^2}}$

$$
\begin{aligned}
\frac{d^2 x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1 + \frac{\alpha}{r} + 2u^2 - 3(\frac{dr}{ds})^2] = \frac{-\alpha x_\nu}{2r^3}[1 + \frac{\alpha}{r} - u^2 + 3(u^2 - (\frac{dr}{ds})^2)] \\
r^2 \frac{d\phi}{ds} = B \\
u^2 = (\frac{dr}{ds})^2 + r^2(\frac{d\phi}{ds})^2 \\
u^2 - (\frac{dr}{ds})^2 = r^2(\frac{d\phi}{ds})^2 = \frac{B^2}{r^2} \\
\frac{d^2 x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]
\end{aligned}
\tag{9}
$$

We see that $x = [x_\nu]_{\nu=1}, y = [x_\nu]_{\nu=2}$ Let us investigate if $a_x = \frac{d^2 x}{ds^2} = \frac{-\alpha x}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}], a_y = \frac{d^2 y}{ds^2} = \frac{-\alpha y}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$ and a general $\Phi = \frac{-\alpha}{2r}[\sum_{n=0}^{N} \frac{f_n}{r^n}]$ where $f_N \neq 0$ ($\Phi = \frac{-\alpha}{2r}[1 + \frac{B^2}{r^2}]$ is a special case) satisfy the general case energy law $\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r}[\sum_{n=0}^{N} \frac{f_n}{r^n}] = A_k$.

We can use the **same method** in previous section 2 to show that Law of Conservation of Energy is **Not obeyed**. Because the particular form of a_x, a_y, Φ **do not affect** the result derived in previous section,

Thus we have proved that $\frac{d^2 x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$ and a general $\Phi = \frac{-\alpha}{2r}[\sum_{n=0}^{N} \frac{f_n}{r^n}]$ where $f_N \neq 0$ does not satisfy the general case energy law, for the general case of two objects starting at arbitrary velocities, if we assume that $E = mc^2$ and special relativistic mass expansion term $\frac{m_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}}$ is included.

4. Law of Conservation of Energy Not obeyed in Newtonian two body system with relativistic mass expansion

Richard Feynman showed that Law of Conservation of Energy is Obeyed in Standard Newtonian two body system (Section 13-3 in link 2).[See **Appendix 1.**]

In this section, we will **add** special-relativistic mass expansion and see whether Law of Conservation of Energy is obeyed, if the energy law $\frac{1}{2}\frac{M_1}{K_1}{v_1}^2+\frac{1}{2}\frac{M_2}{K_2}{v_2}^2-\frac{GM_1M_2}{K_1K_2r}=A_k$ is satisfied, where A_k is a constant, $K_1=\sqrt{1-\frac{v_1^2}{c^2}}$, $K_2=\sqrt{1-\frac{v_2^2}{c^2}}$. Let us define gravitational potentials $\Phi_1=\frac{-GM_1}{K_1r}, \Phi_2=\frac{-GM_2}{K_2r}$. We see that $K_1=\sqrt{1-\frac{(v_{x_1}^2+v_{y_1}^2)}{c^2}}$, $\frac{dK_1}{dt}=\frac{1}{2K_1}\frac{-2}{c^2}(v_{x_1}a_{x_1}+v_{y_1}a_{y_1})=\frac{-1}{K_1c^2}\vec{a_1}.\vec{v_1}$ and $K_2=\sqrt{1-\frac{(v_{x_2}^2+v_{y_2}^2)}{c^2}}$, $\frac{dK_2}{dt}=\frac{1}{2K_2}\frac{-2}{c^2}(v_{x_2}a_{x_2}+v_{y_2}a_{y_2})=\frac{-1}{K_2c^2}\vec{a_2}.\vec{v_2}$, and $v_1\frac{dv_1}{dt}=\vec{a_1}.\vec{v_1}$, $\frac{v_2dv_2}{dt}=\vec{a_2}.\vec{v_2}$.

Let us define gravitational potentials $\Phi_1=\frac{-GM_1}{K_1r}, \Phi_2=\frac{-GM_2}{K_2r}$. We can write $\Phi=-\frac{GM_1M_2}{K_1K_2r}=\frac{1}{2}[\frac{M_1}{K_1}\Phi_2+\frac{M_2}{K_2}\Phi_1]$ and $\frac{d\Phi}{dt}=\frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt}+M_1\Phi_2\frac{-1}{K_1^2}\frac{dK_1}{dt}+\frac{M_2}{K_2}\frac{d\Phi_1}{dt}+M_2\Phi_1\frac{-1}{K_2^2}\frac{dK_2}{dt}]$.

We can differentiate the energy law $\frac{1}{2}\frac{M_1}{K_1}{v_1}^2+\frac{1}{2}\frac{M_2}{K_2}{v_2}^2+\Phi=\frac{1}{2}\frac{M_1}{K_1}{v_1}^2+\frac{1}{2}\frac{M_2}{K_2}{v_2}^2-\frac{GM_1M_2}{K_1K_2r}=A_k$ with respect to **variable** t, we have as follows. Define $A_1=\frac{1-\frac{v_1^2}{2c^2}}{K_1^2}, A_2=\frac{1-\frac{v_2^2}{2c^2}}{K_2^2}$

$$
\begin{aligned}
&\frac{M_1}{2}\frac{(2K_1\vec{a_1}.\vec{v_1}+{v_1}^2\frac{1}{K_1c^2}\vec{a_1}.\vec{v_1})}{K_1^2}+\frac{M_2}{2}\frac{(2K_2\vec{a_2}.\vec{v_2}+{v_2}^2\frac{1}{K_2c^2}\vec{a_2}.\vec{v_2})}{K_2^2}\\
&+\frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt}+\frac{M_2}{K_2}\frac{d\Phi_1}{dt}+M_1\Phi_2\frac{1}{K_1^3c^2}\vec{a_1}.\vec{v_1}+M_2\Phi_1\frac{1}{K_2^3c^2}\vec{a_2}.\vec{v_2}]=0\\
&(\frac{M_1}{2K_1^3c^2}(2c^2-{v_1}^2)+M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1}+(\frac{M_2}{2K_2^3c^2}(2c^2-{v_2}^2)+M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}\\
&=-\frac{1}{2}[\frac{M_1}{K_1}\frac{d\Phi_2}{dt}+\frac{M_2}{K_2}\frac{d\Phi_1}{dt}]\\
&(\frac{M_1}{K_1}A_1+M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1}+(\frac{M_2}{K_2}A_2+M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}=\\
&-\frac{M_1}{2K_1}[\frac{d\Phi_2}{dx}\frac{dx}{dt}+\frac{d\Phi_2}{dy}\frac{dy}{dt}]-\frac{M_2}{2K_2}[\frac{d\Phi_1}{dx}\frac{dx}{dt}+\frac{d\Phi_1}{dy}\frac{dy}{dt}]\\
&(\frac{M_1}{K_1}A_1+M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1}+(\frac{M_2}{K_2}A_2+M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2}=\\
&-\frac{M_1}{2K_1}[(\frac{d\Phi_2}{dx}\widehat{x}+\frac{d\Phi_2}{dy}\widehat{y}).(\frac{dx}{dt}\widehat{x}+\frac{dy}{dt}\widehat{y})]-\frac{M_2}{2K_2}[(\frac{d\Phi_1}{dx}\widehat{x}+\frac{d\Phi_1}{dy}\widehat{y}).(\frac{dx}{dt}\widehat{x}+\frac{dy}{dt}\widehat{y})]
\end{aligned}
\tag{10}
$$

Given that $x=x_2-x_1$ and $y=y_2-y_1$, and $v_x=\frac{dx}{dt}=v_{x_2}-v_{x_1}$, $v_y=\frac{dy}{dt}=v_{y_2}-v_{y_1}$, $\Phi_1=\frac{-GM_1}{rK_1}$ and defining $\vec{\Phi_1'}=(\frac{d\Phi_1}{dx}\widehat{x}+\frac{d\Phi_1}{dy}\widehat{y}), \vec{\Phi_2'}=(\frac{d\Phi_2}{dx}\widehat{x}+\frac{d\Phi_2}{dy}\widehat{y})$,and given that $\vec{v_2}=(v_{x_2}\widehat{x}+v_{y_2}\widehat{y}), \vec{v_1}=(v_{x_1}\widehat{x}+v_{y_1}\widehat{y})$ we can write

$$
\begin{aligned}
&(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} \\
&= -\frac{M_2}{2K_2}[\Phi_1^{'}.((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})] - \frac{M_1}{2K_1}[\Phi_2^{'}.((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})] \\
&(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} \\
&= -\frac{M_2}{2K_2}\Phi_1^{'}.(v_{x_2}\widehat{x} + v_{y_2}\widehat{y}) - \frac{M_1}{2K_1}\Phi_2^{'}.(v_{x_2}\widehat{x} + v_{y_2}\widehat{y}) + \frac{M_2}{2K_2}\Phi_1^{'}.(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})] + \frac{M_1}{2K_1}\Phi_2^{'}.(v_{x_1}\widehat{x} + v_{y_1}\widehat{y}) \\
&(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1}.\vec{v_1} + (\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}.\vec{v_2} \\
&= -[(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}).\vec{v_2})] + [(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}).\vec{v_1}] \\
&((\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} - (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}})).\vec{v_1} + ((\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2} + (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}})).\vec{v_2} = 0 \\
&\vec{F_1}.\vec{v_1} + \vec{F_2}.\vec{v_2} = 0
\end{aligned}
\tag{11}
$$

Given that both the vectors $\vec{v_1}, \vec{v_2}$ and $\vec{F_1}, \vec{F_2}$ are in the same XY plane, $\vec{F_1}.\vec{v_1} \neq 0, \vec{F_2}.\vec{v_2} \neq 0$ for all t as the 2 bodies move along the orbit, unless $\vec{F_1} = \vec{F_2} = 0$. So we require the following conditions be satisfied for the energy law to be true for the general case.

$$
\begin{aligned}
(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} &= (\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}) = A(r); \\
(\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2} &= -(\frac{M_2}{2K_2}\vec{\Phi_1^{'}} + \frac{M_1}{2K_1}\vec{\Phi_2^{'}}) = -A(r); \\
(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})\vec{a_1} &= -(\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})\vec{a_2}
\end{aligned}
\tag{12}
$$

We require following to be satisfied, from above equations.

$$
\begin{aligned}
(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})a_{x_1} &= -(\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})a_{x_2} \\
(\frac{M_1}{K_1}A_1 + M_1\Phi_2\frac{1}{2K_1^3c^2})a_{y_1} &= -(\frac{M_2}{K_2}A_2 + M_2\Phi_1\frac{1}{2K_2^3c^2})a_{y_2} \\
\frac{a_{x_1}}{a_{x_2}} &= -\frac{M_2A_2K_1(1 + \frac{\Phi_1}{2c^2})}{M_1A_1K_2(1 + \frac{\Phi_2}{2c^2})})
\end{aligned}
\tag{13}
$$

But we know that

$$
\begin{aligned}
\vec{a_1} = a_{x_1}\widehat{x} + a_{y_1}\widehat{y} &= \frac{GM_2x}{K_2r^3}\widehat{x} + \frac{GM_2y}{K_2r^3}\widehat{y} \\
\vec{a_2} = a_{x_2}\widehat{x} + a_{y_2}\widehat{y} &= \frac{-GM_1x}{K_1r^3}\widehat{x} + \frac{-GM_1y}{K_1r^3}\widehat{y} \\
\frac{a_{x_1}}{a_{x_2}} &= \frac{-M_2K_1}{M_1K_2}
\end{aligned}
\tag{14}
$$

Equating above 2 sets of equations, we get as follows, given that $A_1 = \frac{1-\frac{v_1^2}{2c^2}}{K_1^2}, A_2 = \frac{1-\frac{v_2^2}{2c^2}}{K_2^2}$.

$$
\begin{aligned}
\frac{a_{x_1}}{a_{x_2}} = \frac{-M_2K_1}{M_1K_2} &= -\frac{M_2A_2K_1(1+\frac{\Phi_1}{2c^2})}{M_1A_1K_2(1+\frac{\Phi_2}{2c^2}})\\
A_1(1+\frac{\Phi_2}{2c^2}) &= A_2(1+\frac{\Phi_1}{2c^2}) \\
\Phi_1 = \frac{-GM_1}{K_1r}, \Phi_2 &= \frac{-GM_2}{K_2r}
\end{aligned}
\tag{15}
$$

In general, above equation is NOT satisfied for the general case of objects with different masses and velocities and hence **energy law is NOT obeyed**.

For example, let us consider 2 stars of masses $M_1 = 2e30, M_2 = 2e31$ Kgs, starting with initial velocities $v_1 = 1209, v_2 = 382$ km/sec, $c = 3e8$ m/sec, $\frac{v_1}{c} = 4e-3$, $\frac{v_2}{c} = 1e-3$, $G = 6.67e-11$ and separated by a distance $r(0) = 1e8$ meters.

We get $\frac{\Phi_1}{2c^2} = -7.41e-6, A_2 = 1 + 8.1e-7$ and $\frac{\Phi_2}{2c^2} = -7.41e-5, A_1 = 1 + 8.1e-6$. We can see that left hand side of the above equation is $1 - 6.6e - 5$ and the right hand side is $1 - 6.6e - 6$, thus leading to a conclusion that **energy law is NOT obeyed**.

Matlab simulation of above Eq. 15 in (link) produces a plot (link) shows that left hand side and right hand side of above equation are **not equal**, even when we reduce the simulation time step further. This **contradicts** the assumption that the total energy of the system is a constant!

We can do a sanity check of Eq. 11 in (plot) which shows **Red** plot is actual simulation of **first derivative** of relativistic total energy and is compared with theoretical equation which is in **blue**, we can see that both match. We can see that left hand side of Eq. 11, which is first derivative of relativistic total energy $\frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 + \Phi = \frac{M_1}{K_1}c^2 + \frac{M_2}{K_2}c^2 - \frac{GM_1M_2}{K_1K_2r} = A_k$, is NOT zero! We can also see that the relativistic **total energy** term in (plot) is **NOT a constant**.

Thus we have shown that Law of Conservation of Energy is **not obeyed** for the general case of two objects with arbitrary velocities and masses, if special relativistic mass expansion term $\frac{m_0}{\sqrt{1-\frac{v^2(t)}{c^2}}}$ is included in Newtonian two body system.

5. Appendix 1: Law of Conservation of Energy Obeyed in Standard Newtonian two body system

Richard Feynman showed that Law of Conservation of Energy is Obeyed in Standard Newtonian two body system (Section 13-3 in link 2).

Let us investigate if $\frac{d^2\vec{r}}{dt^2} = \frac{-GM_1\vec{r}}{r^3}$ and Newtonian gravitational potential $\Phi_1 = \frac{-GM_1}{r}$ satisfy the general case energy law $\frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 - \frac{GM_1M_2}{r} = \frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 + M_2\Phi_1 = A_k$ where A_k is a constant.

Differentiating it with respect to variable t , we have as follows, given that $a_1 = \frac{dv_1}{dt}, a_2 = \frac{dv_2}{dt}$, we get

$$\begin{aligned}
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} + M_2\frac{d\Phi_1}{dt} &= 0 \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[\frac{d\Phi_1}{dx}\frac{dx}{dt} + \frac{d\Phi_1}{dy}\frac{dy}{dt}] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(\frac{dx}{dt}\widehat{x} + \frac{dy}{dt}\widehat{y})]
\end{aligned} \tag{16}$$

Given that $x = x_2 - x_1$ and $y = y_2 - y_1$, and $v_x = \frac{dx}{dt} = v_{x_2} - v_{x_1}$, $v_y = \frac{dy}{dt} = v_{y_2} - v_{y_1}$, and defining $\Phi_2 = \frac{-GM_2}{r} = \Phi_1\frac{M_2}{M_1}$, and $\vec{\Phi_1'} = (\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y})$, $\vec{\Phi_2'} = (\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y})$,and given that $\vec{v_2} = (v_{x_2}\widehat{x} + v_{y_2}\widehat{y})$, $\vec{v_1} = (v_{x_1}\widehat{x} + v_{y_1}\widehat{y})$ we can write

$$\begin{aligned}
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(v_{x_2}\widehat{x} + v_{y_2}\widehat{y})] + M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(v_{x_2}\widehat{x} + v_{y_2}\widehat{y})] + M_1[(\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y}).(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[\vec{\Phi_1'}.\vec{v_2})] + M_1[\vec{\Phi_2'}.\vec{v_1}] \\
M_1[\vec{a_1} - \vec{\Phi_2'}].\vec{v_1} + M_2[\vec{a_2} + \vec{\Phi_1'}].\vec{v_2} &= 0 \\
\vec{F_1}.\vec{v_1} + \vec{F_2}.\vec{v_2} &= 0
\end{aligned} \tag{17}$$

Given that both the vectors $\vec{v_1}, \vec{v_2}$ and $\vec{F_1}, \vec{F_2}$ are in the same XY plane, in general, $\vec{F_1}.\vec{v_1} \neq 0, \vec{F_2}.\vec{v_2} \neq 0$ for all t as the 2 bodies move along the orbit, unless $\vec{F_1} = \vec{F_2} = 0$. So we require the following conditions be satisfied for the energy law to be true for the general case.

$$\vec{a_1} = \vec{\Phi_2'}; \quad a_{x_1} = \frac{d\Phi_2}{dx}; \quad a_{y_1} = \frac{d\Phi_2}{dy};$$
$$\vec{a_2} = -\vec{\Phi_1'}; \quad a_{x_2} = -\frac{d\Phi_1}{dx}; \quad a_{y_2} = -\frac{d\Phi_1}{dy};$$

$$(18)$$

Let us investigate if above conditions are satisfied. Given that $a_{x_2} = \frac{-GM_1x}{r^3}, a_{y_2} = \frac{-GM_1y}{r^3}$ and $-\frac{d\Phi_1}{dx} = \frac{-GM_1x}{r^3}, -\frac{d\Phi_1}{dy} = \frac{-GM_1y}{r^3}$ and hence $a_{x_2} = -\frac{d\Phi_1}{dx}, a_{y_2} = -\frac{d\Phi_1}{dy}$ is indeed satisfied. Similarly, it can be verified that $a_{x_1} = \frac{GM_2x}{r^3} = \frac{d\Phi_2}{dx}, a_{y_1} = \frac{GM_2y}{r^3} = \frac{d\Phi_2}{dy}$ are also satisfied.

We have shown that $\frac{d^2\vec{r}}{dt^2} = \frac{-GM_1\vec{r}}{r^3}$ and gravitational potential $\Phi = \frac{-GM_1}{r}$ do indeed satisfy energy law $\frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 - \frac{GM_1M_2}{r} = A_k$ and the Law of Areas.

6. Law of Conservation of Energy Not obeyed in Einstein's General Relativity two body system

Einstein mercury perihelion paper is here. We take Eq. 7b, 8a, 10 and get as follows.

$$\begin{aligned}
\frac{d^2x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1+\frac{\alpha}{r}+2u^2-3(\frac{dr}{ds})^2] = \frac{-\alpha x_\nu}{2r^3}[1+\frac{\alpha}{r}-u^2+3(u^2-(\frac{dr}{ds})^2)] \\
r^2\frac{d\phi}{ds} = B \\
u^2 = (\frac{dr}{ds})^2 + r^2(\frac{d\phi}{ds})^2 \\
u^2-(\frac{dr}{ds})^2 = r^2(\frac{d\phi}{ds})^2 = \frac{B^2}{r^2} \\
\frac{d^2x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1+\frac{\alpha}{r}-u^2+\frac{3B^2}{r^2}]
\end{aligned} \tag{19}$$

Let us investigate if $\frac{d^2x_\nu}{ds^2} = \frac{-\alpha x_\nu}{2r^3}[1+\frac{\alpha}{r}-u^2+\frac{3B^2}{r^2}]$ and a general $\Phi = \frac{-\alpha}{2r}[\sum_{n=0}^{N}\frac{f_n}{r^n}]$ where $f_N \neq 0$ [Assumption 1].($\Phi = \frac{-\alpha}{2r}[1+\frac{B^2}{r^2}]$ is a special case) satisfy the general case energy law $\frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 + M_2\Phi = A_k$.

Differentiating it with respect to variable s (we could also differentiate w.r.to the time variable t and similar arguments would apply) , we have as follows, given that $a_1 = \frac{dv_1}{ds}$, we get

$$\begin{aligned}
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} + M_2\frac{d\Phi}{ds} = 0 \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} = -M_2[\frac{d\Phi}{dx}\frac{dx}{ds} + \frac{d\Phi}{dy}\frac{dy}{ds}] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} = -M_2[(\frac{d\Phi}{dx}\widehat{x} + \frac{d\Phi}{dy}\widehat{y}).(\frac{dx}{ds}\widehat{x} + \frac{dy}{ds}\widehat{y})]
\end{aligned} \tag{20}$$

Given that $x = x_2 - x_1$ and $y = y_2 - y_1$, and $v_x = \frac{dx}{ds} = v_{x_2} - v_{x_1}$, $v_y = \frac{dy}{ds} = v_{y_2} - v_{y_1}$, $\Phi = \frac{-GM_1}{rc^2}[\sum_{n=0}^{N}\frac{f_n}{r^n}]$ and defining $\Phi_1 = \frac{-GM_1}{rc^2}[\sum_{n=0}^{N}\frac{f_n}{r^n}]$, $\Phi_2 = \frac{-GM_2}{rc^2}[\sum_{n=0}^{N}\frac{f_n}{r^n}]$, and $\vec{\Phi_1'} = (\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y})$, $\vec{\Phi_2'} = (\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y})$,and given that $\vec{v_2} = (v_{x_2}\widehat{x} + v_{y_2}\widehat{y})$, $\vec{v_1} = (v_{x_1}\widehat{x} + v_{y_1}\widehat{y})$ we can write

$$
\begin{aligned}
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi}{dx}\widehat{x} + \frac{d\Phi}{dy}\widehat{y}).((v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y})] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[(\frac{d\Phi_1}{dx}\widehat{x} + \frac{d\Phi_1}{dy}\widehat{y}).(v_{x_2}\widehat{x} + v_{y_2}\widehat{y})] + M_1[(\frac{d\Phi_2}{dx}\widehat{x} + \frac{d\Phi_2}{dy}\widehat{y}).(v_{x_1}\widehat{x} + v_{y_1}\widehat{y})] \\
M_1\vec{a_1}.\vec{v_1} + M_2\vec{a_2}.\vec{v_2} &= -M_2[\vec{\Phi_1'}.\vec{v_2})] + M_1[\vec{\Phi_2'}.\vec{v_1}] \\
M_1[\vec{a_1} - \vec{\Phi_2'}].\vec{v_1} + M_2[\vec{a_2} + \vec{\Phi_1'}].\vec{v_2} &= 0 \\
\vec{F_1}.\vec{v_1} + \vec{F_2}.\vec{v_2} &= 0
\end{aligned}
\tag{21}
$$

Given that both the vectors $\vec{v_1}, \vec{v_2}$ and $\vec{F_1}, \vec{F_2}$ are in the same XY plane, $\vec{F_1}.\vec{v_1} \neq 0, \vec{F_2}.\vec{v_2} \neq 0$ for all s as the 2 bodies move along the orbit, unless $\vec{F_1} = \vec{F_2} = 0$ so we require the following conditions be satisfied for the energy law to be true for the general case.

$$
\begin{aligned}
\vec{a_1} = \vec{\Phi_2'}; &\quad a_{x_1} = \tfrac{d\Phi_2}{dx}; \quad a_{y_1} = \frac{d\Phi_2}{dy}; \\
\vec{a_2} = -\vec{\Phi_1'}; &\quad a_{x_2} = -\tfrac{d\Phi_1}{dx}; \quad a_{y_2} = -\frac{d\Phi_1}{dy};
\end{aligned}
\tag{22}
$$

Let us investigate if above conditions are satisfied. Let us first consider $a_{x_2} = -\frac{d\Phi_1}{dx}; a_{y_2} = -\frac{d\Phi_1}{dy}$;. Given that $\Phi_1 = \frac{-GM_1}{rc^2}[\sum_{n=0}^{N} \frac{f_n}{r^n}], a_{x_2} = \frac{-\alpha x}{2r^3}[1+\frac{\alpha}{r}-u^2+\frac{3B^2}{r^2}], a_{y_2} = \frac{-\alpha y}{2r^3}[1+\frac{\alpha}{r}-u^2+\frac{3B^2}{r^2}]$ and the motion of the 2 bodies are confined to XY plane. Given that $\frac{dr}{dx} = \frac{x}{r}$ we have

$$
\begin{aligned}
\Phi_1 &= \frac{-GM_1}{c^2}[\sum_{n=0}^{N} \frac{f_n}{r^{n+1}}] \\
\frac{d\Phi_1}{dx} &= \frac{GM_1}{c^2}[\sum_{n=0}^{N} \frac{f_n(n+1)r^{n-1}x]}{r^{2(n+1)}}] = \frac{GM_1}{c^2}[\sum_{n=0}^{N} \frac{f_n(n+1)x]}{r^{n+3}}] = -a_{x_2} \\
\frac{d\Phi_1}{dy} &= \frac{GM_1}{c^2}[\sum_{n=0}^{N} \frac{f_n(n+1)r^{n-1}y]}{r^{2(n+1)}}] = \frac{GM_1}{c^2}[\sum_{n=0}^{N} \frac{f_n(n+1)y]}{r^{n+3}}] = -a_{y_2}
\end{aligned}
\tag{23}
$$

Substituting $a_{x_2} = \frac{-\alpha x}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}], a_{y_2} = \frac{-\alpha y}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$ and comparing the 2 sides, we get

$$[\sum_{n=0}^{N} \frac{f_n(n+1)]}{r^n}] = [1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$$

$$\begin{aligned} u^2 &= \sum_{n=0}^{N} \frac{g_n}{r^n} \\ g_n &= -f_n(n+1) \qquad N > 2 \\ g_0 &= 1 - f_0; \quad g_1 = \alpha - 2f_1; \quad g_2 = 3B^2 - 3f_2 \end{aligned} \tag{24}$$

Differentiating both sides w.r.to s, given that $\vec{u} = (v_{x_2} - v_{x_1})\widehat{x} + (v_{y_2} - v_{y_1})\widehat{y}$, $\frac{dr}{ds} = \frac{\vec{r}.\vec{u}}{r}$, $\frac{d\vec{u}}{ds} = \vec{a} = (a_{x_2} - a_{x_1})\widehat{x} + (a_{y_2} - a_{y_1})\widehat{y}$, given that $a_{x_1} = -a_{x_2}\frac{M_2}{M_1}$, $a_{y_1} = -a_{y_2}\frac{M_1}{M_2}$ we get

$$\begin{aligned} 2\vec{u}.\frac{d\vec{u}}{ds} &= \sum_{n=0}^{N} \frac{-g_n}{r^{n+2}}\vec{r}.\vec{u} \\ \vec{a} &= \sum_{n=0}^{N} \frac{-g_n}{2r^{n+2}}\vec{r} \\ a_{x_2} - a_{x_1} &= x\sum_{n=0}^{N} \frac{-g_n}{2r^{n+2}} \\ a_{x_2} &= \frac{x}{(1+\frac{M_2}{M_1})}\sum_{n=0}^{N} \frac{-g_n}{2r^{n+2}} \end{aligned} \tag{25}$$

Substituting $a_{x_2} = \frac{-\alpha x}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$ once again, where $x = x_2 - x_1$, we get

$$\begin{aligned} [1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}] &= \frac{2}{\alpha(1+\frac{M_2}{M_1})}\sum_{n=0}^{N} \frac{g_n}{2r^{n-1}} \\ \sum_{n=0}^{N} \frac{f_n(n+1)}{r^n} &= \frac{2}{\alpha(1+\frac{M_2}{M_1})}\sum_{n=0}^{N} \frac{g_n}{2r^{n-1}} \end{aligned} \tag{26}$$

Comparing $\frac{1}{r^N}$ terms on both sides, we get $(N+1)f_N = 0$ which implies $f_N = 0$ which contradicts Assumption 1 that $f_N \neq 0$.

Thus we have proved that $\frac{d^2x_v}{ds^2} = \frac{-\alpha x_v}{2r^3}[1 + \frac{\alpha}{r} - u^2 + \frac{3B^2}{r^2}]$ and a general $\Phi = \frac{-\alpha}{2r}[\sum_{n=0}^{N} \frac{f_n}{r^n}]$ where $f_N \neq 0$ does not satisfy the general case energy law $\frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 + M_2\Phi = A_k$.

6.1. *Law of Areas and general relativity*

Let us investigate if the Law of Areas and Law of Conservation of Energy are satisfied for Einstein's General Relativity Second Order Approximation of Gravitation, say $\frac{d^2\vec{r}}{ds^2} = \frac{-\alpha\vec{r}}{2r^3}[1 + \frac{\alpha}{r} + 2u^2 - 3(\frac{dr}{ds})^2] = f(r)\vec{r}$ and if we consider corresponding gravitational potential $\Phi = \frac{-\alpha}{2r}[1 + \frac{B^2}{r^2}]$ where $\alpha = \frac{2GM_1}{c^2}$ where M_1 is the mass of the first body and c is the speed of light in vaccum, $r^2\frac{d\theta}{ds} = B$. Here, x, y, z, r, u are all functions of s instead of t and $\frac{dx_4}{ds} = 1 + \frac{\alpha}{r}, x_4 = ct$.

Similar to Section 1, we can show that $\frac{d}{ds}(\vec{r}\times\vec{v}) = \frac{d\vec{r}}{ds}\times\vec{v}+\vec{r}\times\frac{d\vec{v}}{ds} = 0$ using the fact that,$\vec{v} = \frac{d\vec{r}}{ds}$, $\frac{d\vec{r}}{ds}\times\vec{v} = \vec{v}\times\vec{v} = 0$ and $\vec{r}\times\frac{d\vec{v}}{ds} = \vec{r}\times[f(r)\vec{r}] = f(r)[\vec{r}\times\vec{r}] = 0$, hence $\frac{d}{ds}(\vec{r}\times\vec{v}) = 0$.Which implies that $\vec{r}\times\vec{v} = \vec{h}$ is a constant as a function of time variable s. This implies **conservation of angular momentum**.

From Section 1, we can see that $\vec{h}.\vec{r} = 0$. Hence the motion of the particle is restricted to the plane orthogonal to $\vec{h}$, provided $\vec{h} \neq 0$. This implies that $\vec{r} = x\widehat{x} + y\widehat{y}$ and $\vec{h} = h_z\widehat{z}$ if we choose the Z-axis along the direction of $\vec{h}$.

Similar to Section 1, let $\vec{h} = (0, 0, h)$ and $\vec{r} = (x, y, 0) = (r\cos\theta, r\sin\theta, 0)$. Hence $\vec{v} = \frac{d\vec{r}}{ds} = (\frac{dr}{ds}\cos\theta - r\sin\theta\frac{d\theta}{ds}, \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{ds}, 0)$. Now we can compute $\vec{r}\times\vec{v} = [r\cos\theta[\frac{dr}{ds}\sin\theta + r\cos\theta\frac{d\theta}{ds}] - r\sin\theta[\frac{dr}{ds}\cos\theta - r\sin\theta\frac{d\theta}{ds}]]\widehat{z} = r^2\frac{d\theta}{ds}\widehat{z}$. Hence $\vec{r}\times\vec{v} = \vec{h} = h\widehat{z} = r^2\frac{d\theta}{ds}\widehat{z}$ and hence $r^2\frac{d\theta}{ds} = h$ is a constant. If A denotes the area swept by the radius vector in the orbital plane, $\frac{dA}{ds} = \frac{1}{2}r^2\frac{d\theta}{ds} = \frac{h}{2}$, we can see that **Kepler's Law of Areas** is also satisfied.

We have shown that Einstein's General Relativity Second Order Approximation $\frac{d^2\vec{r}}{ds^2} = \frac{-\alpha\vec{r}}{2r^3}[1+\frac{\alpha}{r} + 2u^2 - 3(\frac{dr}{ds})^2]$ and $\Phi = \frac{-\alpha}{2r}[1 + \frac{B^2}{r^2}]$ satisfies the Law of Areas but not the general case energy law $\frac{1}{2}M_1{v_1}^2 + \frac{1}{2}M_2{v_2}^2 + M_2\Phi = A_k$.