

On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.^{[4] [5]} We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.^[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.^[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix F, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$, where ω, t are real, as follows (link).^[3]

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t , given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t) = E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \leq \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. Step 2: On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. In Section 2.1, we consider a **modified even symmetric** function $g(t) = f(t) e^{-\sigma t} u(-t) + f(t) e^{3\sigma t} u(t)$ for $|t_0| \leq \infty$ where $f(t) = e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)$ and $h(t) = [e^{\sigma t} u(-t) + e^{-3\sigma t} u(t)]$ where $g(t)h(t) = f(t)$ and show that Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$, for **every value** of t_0 , to satisfy Statement 1, where $\omega_2(t_0)$ is real and finite. (Appendix F).

1.3. Step 3: On the zeros of the function $G_R(\omega)$

In Section 2.1, we compute the Fourier transform of the even function $g(t)$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of t_0 , to satisfy **Statement 1**. In general, $\omega_2(t_0) \neq \omega_0$.

It is shown that $R(t_0) = G_R(\omega_2(t_0), t_0)$ is an **odd** function of variable t_0 as follows.

$$R(t_0) = e^{2\sigma t_0} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] \quad (3)$$

1.4. Step 4: Asymptotic fall off rate argument

In Section 2.2, we consider the asymptotic case and show that $\lim_{t_2 \rightarrow \infty} \omega_2(t_2) = \omega_z$ and derive as follows.

$$\begin{aligned}
\lim_{t_0 \rightarrow \infty} \cos(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_z \tau) d\tau + \sin(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_z \tau) d\tau &= 0 \\
\lim_{t_0 \rightarrow \infty} \cos(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_z \tau) d\tau - \sin(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_z \tau) d\tau &= 0 \\
\int_{-\infty}^{\infty} E_0(t) e^{-2\sigma t} e^{i(\omega_z t)} dt &= 0
\end{aligned} \tag{4}$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau) e^{-\sigma\tau} e^{-i\omega_0\tau} d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau) e^{-2\sigma\tau} e^{-i\omega_z\tau} d\tau = 0$.

We repeat above steps N times till $2^N \sigma > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau) e^{-2^N \sigma\tau} e^{-i\omega_z N\tau} d\tau = 0$. In each iteration n , we use $h(t) = e^{2^n \sigma t} u(-t) + e^{-3 \cdot 2^n \sigma t} u(t)$. We know that the Fourier Transform of $E_0(t) e^{-2^N \sigma t}$ **does not** have a real zero for $2^N \sigma > \frac{1}{2}$, corresponding to $Re[s] > 1$ and we show a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t) e^{-\sigma t}$ has a zero at $\omega = \omega_0$.

Details are in the next page.

2. An Approach towards Riemann's Hypothesis: Method 1

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t , where $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$, $E_p(t) = E_0(t) e^{-\sigma t}$ and $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix D.1.

2.1. On a related function $G(\omega)$

Let us form a new function $f(t) = e^{-\sigma t_0} E_p(t-t_0) + e^{\sigma t_0} E_p(t+t_0) = [E_0(t+t_0) + E_0(t-t_0)] e^{-\sigma t} = E_{0n}(t) e^{-\sigma t}$, where $|t_0| \leq \infty$, $E_{0n}(t) = E_{0n}(-t) = E_0(t+t_0) + E_0(t-t_0)$. Its Fourier Transform given by $F(\omega) = E_{p\omega}(\omega) [e^{-\sigma t_0} e^{-i\omega t_0} + e^{\sigma t_0} e^{i\omega t_0}]$ also has a zero at $\omega = \omega_0$.

Let us consider a real and **even symmetric** function $g(t) = g(-t) = g_-(t)u(-t) + g_+(t)u(t)$ where $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = g_-(-t) = f(-t)e^{\sigma t} = f(t)e^{3\sigma t}$, because $f(t) = E_{0n}(t)e^{-\sigma t}$, $f(-t)e^{\sigma t} = E_{0n}(t)e^{2\sigma t}$, $f(t)e^{3\sigma t} = E_{0n}(t)e^{2\sigma t}$ and $E_{0n}(t) = E_{0n}(-t)$. We see that $g(t) = E_{0n}(t)e^{-2\sigma t}u(-t) + E_{0n}(t)e^{2\sigma t}u(t)$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$.

We can see that $g(t)$ is a real L^1 integrable function, its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \rightarrow \pm\infty$, as per **Riemann-Lebesgue Lemma** [Riemann Lebesgue Lemma]. This is explained in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$, we get $\frac{1}{2\pi} [G(\omega) * H(\omega)] = F(\omega)$ where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega'$ and $H(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{3\sigma + i\omega}] = [\frac{\sigma}{(\sigma^2 + \omega^2)} + \frac{3\sigma}{(9\sigma^2 + \omega^2)}] + i\omega [\frac{1}{(\sigma^2 - \omega^2)} - \frac{1}{(9\sigma^2 + \omega^2)}]$ is the Fourier transform of the function $h(t)$.

For **every value** of t_0 , we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_2(t_0)$ for **every value** of t_0 . Because the real part of $H(\omega)$ given by $H_R(\omega) = \frac{\sigma}{(\sigma^2 + \omega^2)} + \frac{3\sigma}{(9\sigma^2 + \omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

This is explained in detail in Appendix F.

Next Step

Let us compute the Fourier transform of the function $g(t)$ given by $G(\omega)$.

$$\begin{aligned}
g(t) &= g_-(t)u(-t) + g_+(t)u(t) = g_-(t)u(-t) + g_-(-t)u(t) \\
g(t) &= [e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)]e^{-\sigma t}u(-t) + [e^{-\sigma t_0} E_p(-t - t_0) + e^{\sigma t_0} E_p(-t + t_0)]e^{\sigma t}u(t) \\
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = \int_{-\infty}^0 [e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)]e^{-\sigma t}e^{-i\omega t} dt \\
&\quad + \int_0^{\infty} [e^{-\sigma t_0} E_p(-t - t_0) + e^{\sigma t_0} E_p(-t + t_0)]e^{\sigma t}e^{-i\omega t} dt
\end{aligned}$$

(5)

In the second integral in above equation ,we can substitute $t = -t$ and we get

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^0 [e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)]e^{-\sigma t}e^{-i\omega t} dt + \int_{-\infty}^0 [e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)]e^{-\sigma t}e^{i\omega t} dt \\
G(\omega) &= 2 \int_{-\infty}^0 [e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)]e^{-\sigma t} \cos \omega t dt = G_R(\omega) + iG_I(\omega) = G_R(\omega)
\end{aligned}$$

(6)

Using the substitutions $t - t_0 = \tau, dt = d\tau$ and $t + t_0 = \tau, dt = d\tau$, we can write the above equation as follows. We use $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$.

$$\begin{aligned}
G_R(\omega) &= G_R(\omega, t_0) = G_2(\omega, t_0) + G_2(\omega, -t_0) \\
G_2(\omega, t_0) &= 2e^{\sigma t_0}e^{\sigma t_0}[\cos(\omega t_0) \int_{-\infty}^{t_0} E_p(\tau)e^{-\sigma\tau} \cos(\omega\tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E_p(\tau)e^{-\sigma\tau} \sin(\omega\tau) d\tau] \\
G_2(\omega, t_0) &= 2e^{2\sigma t_0}[\cos(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega\tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega\tau) d\tau]
\end{aligned}$$

(7)

We require $G(\omega) = G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for **every value** of t_0 , to satisfy **Statement 1**. Hence we can see that $R(t_0) = \frac{1}{2}G_2(\omega_2(t_0), t_0)$ is an **odd function** of variable t_0 .

$$\begin{aligned}
G(\omega_2(t_0), t_0) &= G_2(\omega_2(t_0), t_0) + G_2(\omega_2(t_0), -t_0) = 0 \\
R(t_0) &= \frac{1}{2}G_2(\omega_2(t_0), t_0) \\
R(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] \\
S(t_0) &= R(t_0) + R(-t_0) = 0
\end{aligned}$$

(8)

We see that $f(t) = e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_2(t_0)$ is an **even** function of variable t_0 .

2.2. Final Proof: Asymptotic Fall off rate argument.

In Section 2.3, we show that $\lim_{t_0 \rightarrow \infty} g(t)$ is an **analytic** function, with the **magnitude** of the step discontinuity at $t = 0$ **decreasing to zero**, and its Fourier transform is an analytic function with **isolated zeros** and each isolated zero has a single value, as $\lim_{t_0 \rightarrow \infty}$.

In Section 2.4, we show that $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z$ is a constant and we **rule out** the pathological case of $\omega_2(t_0)$ which is discontinuous everywhere and/or ill-defined. It is shown that the integrals $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau$ and $I_2(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau$ in Eq. 9 **converge** as $\lim_{t_0 \rightarrow \infty}$.

As $\lim_{t_0 \rightarrow \infty}$, we can compute $S(t_0)$ in Eq. 8 as follows. The expression for $R(-t_0)$ goes to zero as $\lim_{t_0 \rightarrow \infty}$, due to the term $e^{-2\sigma t_0}$. In the equation for $R(t_0)$, the term $\lim_{t_0 \rightarrow \infty} e^{2\sigma t_0} = \infty$. Hence we require $\lim_{t_0 \rightarrow \infty} \cos(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_z \tau) d\tau + \sin(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_z \tau) d\tau = 0$. We use $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z$ and write as follows.

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} S(t_0) &= \lim_{t_0 \rightarrow \infty} e^{2\sigma t_0} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] = 0 \\ &= \lim_{t_0 \rightarrow \infty} \cos(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_z \tau) d\tau + \sin(\omega_z t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_z \tau) d\tau = 0 \end{aligned} \quad (9)$$

We define $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau$ and $I_2(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau$ in Eq. 9 and note that $\lim_{t_0 \rightarrow \infty} I_1(t_0)$ and $\lim_{t_0 \rightarrow \infty} I_2(t_0)$ tend to a constant, which is finite and determinate, given that $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z$ (**Statement A**). We require $\lim_{t_0 \rightarrow \infty} U(t_0) = 0$ and we consider following 2 cases.

• **Case 1:** We consider the general case and require the terms $I_1(t_0)$ and $I_2(t_0)$ have an **asymptotic fall-off** rate of e^{-Kt_0} , as $\lim_{t_0 \rightarrow \infty}$, where $K > 2\sigma$, to satisfy the equation $U(t_0) = 0$.

Given the **asymptotic fall-off** rate of e^{-Kt_0} , where $K > 2\sigma$, we see that $\lim_{t_0 \rightarrow \infty} I_1(t_0) = 0$ and $\lim_{t_0 \rightarrow \infty} I_2(t_0) = 0$.

• **Case 2:** We consider the specific case $\lim_{t_0 \rightarrow \infty} I_1(t_0) = V(t_0) \sin(\omega_z t_0)$ and $\lim_{t_0 \rightarrow \infty} I_2(t_0) = -V(t_0) \cos(\omega_z t_0)$, which will satisfy the equation $\lim_{t_0 \rightarrow \infty} U(t_0) = 0$. Given Statement A, we require $\lim_{t_0 \rightarrow \infty} V(t_0)$ to have an **asymptotic fall-off** rate of e^{-Kt_0} , as $\lim_{t_0 \rightarrow \infty}$, where $K > 0$, to satisfy the equation $U(t_0) = 0$.

If $K \leq 0$, we see that $\lim_{t_0 \rightarrow \infty} I_1(t_0) = V(t_0) \sin(\omega_z t_0)$ and $\lim_{t_0 \rightarrow \infty} I_2(t_0) = -V(t_0) \cos(\omega_z t_0)$ will be oscillatory due to the terms $\sin(\omega_z t_0)$ and $\cos(\omega_z t_0)$ which is **not** valid, given Statement A.

Given the **asymptotic fall-off** rate of e^{-Kt_0} , where $K > 0$, we see that $\lim_{t_0 \rightarrow \infty} I_1(t_0) = 0$ and $\lim_{t_0 \rightarrow \infty} I_2(t_0) = 0$.

In both cases, we arrive at the **result** that $\int_{-\infty}^{\infty} E_0(\tau) e^{-2\sigma\tau} e^{-i\omega_z \tau} d\tau = 0$.

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t) e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau) e^{-\sigma\tau} e^{-i\omega_0 \tau} d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau) e^{-2\sigma\tau} e^{-i\omega_z \tau} d\tau = 0$.

Now we can repeat the steps in Section 2, starting with the new result that $\int_{-\infty}^{\infty} E_0(\tau) e^{-2\sigma\tau} e^{-i\omega_z \tau} d\tau = 0$ and σ replaced by 2σ and derive the next result that $\int_{-\infty}^{\infty} E_0(\tau) e^{-4\sigma\tau} e^{-i\omega_{(z1)} \tau} d\tau = 0$.

We can repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau) e^{-(2^{N+1}\sigma)\tau} e^{-i\omega_{(zN)} \tau} d\tau = 0$. In each iteration n , we use $h(t) = e^{(2^{N+1}\sigma)t} u(-t) + e^{-3*(2^{N+1}\sigma)t} u(t)$, $\omega_2(t_0)$ replaced by $\omega_{2n}(t_0)$ and ω_z replaced by $\omega_{(zn)}$. We know that the Fourier Transform of $E_0(t) e^{-(2^{N+1}\sigma)t} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-(2^{N+1}\sigma)t}$ given by $E_{pN}(\omega) = \xi(\frac{1}{2} + 2^N \sigma + i\omega)$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

2.3. Analytic Functions and Isolated Zeros.

In this section, we show that $\lim_{t_0 \rightarrow \infty} g(t)$ is an analytic function, with the magnitude of the step discontinuity at $t = 0$ decreasing to zero, and its Fourier transform is an analytic function with isolated zeros, as $\lim_{t_0 \rightarrow \infty}$. Hence $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z \neq 0$ which is a constant and $\lim_{t_0 \rightarrow \infty} \frac{d^2 \omega_2(t_0)}{dt_0^2} = 0$.

We see that $g(t) = E'_0(t)e^{-2\sigma t}u(-t) + E'_0(t)e^{2\sigma t}u(t)$ where $E'_0(t) = E'_0(-t) = E_0(t + t_0) + E_0(t - t_0)$ and its first derivative has a **step** discontinuity at $t = 0$ with magnitude $\Delta_d = 4\sigma E'_0(0) = 4\sigma(E_0(t_0) + E_0(-t_0))$. As $\lim_{t_0 \rightarrow \infty}$, $\Delta_d \rightarrow 0$ because $E_0(t_0)$ and $E_0(-t_0)$ decrease to zero as $\lim_{t_0 \rightarrow \infty}$ and hence $\lim_{t_0 \rightarrow \infty} g(t) = \lim_{t_0 \rightarrow \infty} [E'_0(t)e^{-2\sigma t} + E'_0(t)e^{2\sigma t}]$ is an **analytic** function.

We use a **scale factor** and get $g_s(t) = g(t)e^{-2\sigma t_0}$, so that $\lim_{t_0 \rightarrow \infty} g_s(t)$ remains **finite** for all $|t| \leq \infty$. This scale factor **does not** affect the location of zeros in the Fourier transform of $g(t)$ and $g_s(t)$. Hence $\lim_{t_0 \rightarrow \infty} g_s(t) = \lim_{t_0 \rightarrow \infty} E'_0(t)[e^{-2\sigma t} + e^{2\sigma t}]e^{-2\sigma t_0} = E_0(t + t_0)e^{-2\sigma(t+t_0)} + E_0(t - t_0)e^{2\sigma(t-t_0)} + E_0(t - t_0)e^{-2\sigma(t+t_0)} + E_0(t + t_0)e^{2\sigma(t-t_0)}$.

The Fourier transform of $g_s(t)$ is given by $G_s(\omega)$ and $\lim_{t_0 \rightarrow \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0} + E_{0\omega}(\omega - i2\sigma)e^{-i\omega t_0}e^{-4\sigma t_0} + E_{0\omega}(\omega + i2\sigma)e^{i\omega t_0}e^{-4\sigma t_0}$. As $\lim_{t_0 \rightarrow \infty}$, the last two terms in $\lim_{t_0 \rightarrow \infty} G_s(\omega)$ go to zero.

Hence $\lim_{t_0 \rightarrow \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0}$ is an **analytic function** for all $|\omega| \leq \infty$ because it is derived from the **entire function** $\xi(s)$ and we know that $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. The same statement holds for $\lim_{t_0 \rightarrow \infty} G(\omega)$ which differs only by a scale factor $e^{-2\sigma t_0}$.

We use the well known result that **analytic functions have isolated zeros**.(link) Hence $\lim_{t_0 \rightarrow \infty} G_s(\omega)$ and $\lim_{t_0 \rightarrow \infty} G(\omega)$ have **isolated zeros** at $\omega = \omega_2(t_0) = \omega_z$ and the **second derivative** given by $\lim_{t_0 \rightarrow \infty} \frac{d^2 \omega_2(t_0)}{dt_0^2} = 0$.

2.3.1. Isolated Zeros are single valued.

We consider $g_s(t) = g(t)e^{-2\sigma t_0}$ and see that $\lim_{t_0 \rightarrow \infty} g_s(t) = E_0(t + t_0)e^{-2\sigma(t+t_0)} + E_0(t - t_0)e^{2\sigma(t-t_0)} + E_0(t - t_0)e^{-2\sigma(t+t_0)} + E_0(t + t_0)e^{2\sigma(t-t_0)}$ is an **analytic** function, whose Fourier transform is given by $\lim_{t_0 \rightarrow \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0}$ which is derived from the **entire function** $\xi(s)$ where $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$.

We know that **analytic functions have isolated zeros** (link) and each isolated zero has a **single value**. For example, the analytic function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$ corresponding to the **critical line**, is well known to have isolated zeros and each isolated zero has a **single value**. Hence we can expect the analytic function $\lim_{t_0 \rightarrow \infty} G_s(\omega)$ to have **isolated zeros** and each isolated zero to have a **single value**. Hence $\lim_{t_0 \rightarrow \infty} \omega_2(t_0)$ is **not** a pathological function with multiple values or ill-defined, but $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z$ is a well defined constant.

2.4. Integral convergence

In this section, we show that $\lim_{t_0 \rightarrow \infty} \omega_2(t_0)$ equals a well defined constant and that the integrals $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau$ and $I_2(t_0) = \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau$ in Eq. 9 **converge** as $\lim_{t_0 \rightarrow \infty}$.

We take Eq. 8 and write it as follows using $E'_0(t) = E_0(t + t_0) + E_0(t - t_0)$. If we substitute $t + t_0 = \tau$ and $t - t_0 = \tau$, we get Eq. 8 copied below.

$$\begin{aligned} S(t_0) &= R(t_0) + R(-t_0) = \int_{-\infty}^0 E'_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau = 0 \\ R(t_0) &= e^{2\sigma t_0} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] \end{aligned} \quad (10)$$

We consider the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined (**Statement 2**). Then $S(t_0) = \int_{-\infty}^0 E'_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau$ is ill-defined everywhere as a result (**Statement 3**) and we can show that this pathological case **does not** apply to $\omega_2(t_0)$.

We see that the integral $S(t_0) = 0$ in Eq. 10 **converges** for $|t_0| \leq \infty$ and this result is derived from **Statement 1** (Riemann's Xi function has a zero in the critical strip excluding the critical line). This contradicts Statement 2 and 3.

Given that $\lim_{t_0 \rightarrow \infty} R(-t_0) = 0$, we have $\lim_{t_0 \rightarrow \infty} R(t_0) = \lim_{t_0 \rightarrow \infty} S(t_0) = 0$.

- We can **rule out** the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined, as follows. **If Statements 1, 2 and 3** were true, **then** the result that the integral in Eq. 10 converges, suggests one of the following:

- a) Statement 1 is true and above result **contradicts** Statement 2 and 3 and hence we can **rule out** pathological case for $\omega_2(t_0)$ **or**

- b) Statements 2 and 3 are true and **Statement 1 is false** and we complete the proof of theorem 1 at this point. We **do not** require to show that $\omega_2(t_0)$ is **not** pathological, for this case.

- Let us consider the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined. In Eq. 9 copied below, we see that $\cos(\omega_2(t_0)t_0)$ and $\sin(\omega_2(t_0)t_0)$ are ill-defined and the integrals are also ill-defined functions.

$$\lim_{t_0 \rightarrow \infty} \cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau = 0 \quad (11)$$

Hence we see that (ill-defined function) * (ill-defined function) + (ill-defined function) * (ill-defined function) = 0, as $\lim_{t_0 \rightarrow \infty}$, and this **does not** make sense. Therefore the assumption that $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined is **false**, if **Statement 1** is true.

- Let us consider the case where $\omega_2(t_0)$ is well defined but **first derivative** $\frac{d\omega_2(t_0)}{dt_0}$ is **discontinuous everywhere** and/or ill-defined. We take the first derivative of Eq. 10.

$$\frac{dS(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0} \int_{-\infty}^0 \tau E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau + \int_{-\infty}^0 \frac{dE'_0(\tau)}{dt_0} e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau = 0 \quad (12)$$

Hence we see that (ill-defined function) * (ill-defined function) + (ill-defined function) = 0, for all $|t_0| \leq \infty$, and this **does not** make sense. Therefore the assumption that $\frac{d\omega_2(t_0)}{dt_0}$ is **discontinuous everywhere** and/or ill-defined is **false**, if **Statement 1** is true.

• If Statement 1 is true, **then** we have shown that **Statement 2 is false** and hence $\omega_2(t_0)$ is a well-defined function and $\lim_{t_0 \rightarrow \infty} \omega_2(t_0) = \omega_z$ is a well defined constant (also shown in Section 2.3). Hence the results derived in Section 2.2 are **valid**, with constant ω_z . Hence the integrals $I_1(t_0), I_2(t_0)$ in Eq. 9 converge.

References

- [1] Bernhard Riemann, On the Number of Prime Numbers less than a Given Quantity.(Ueber die Anzahl der Primzahlen untereiner gegebenen Grosse.) Monatsberichte der Berliner Akademie, November 1859. (Link to Riemann's 1859 paper)
- [2] Hardy, G.H., Littlewood, J.E. The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift volume 10, pp.283 to 317 (1921).
- [3] E. C. Titchmarsh, The Theory of the Riemann Zeta Function. (1986) pp.254 to 255
- [4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152
- [5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)
- [6] Mathworld article on Hurwitz Zeta functions. (Link)
- [7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). This is re-derived in Appendix B.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t) e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega \end{aligned} \quad (\text{A.1})$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \quad (\text{A.2})$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix D.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t) = E_0(t) e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$, **goes to zero** as $x \rightarrow \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t) e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4 . Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned} E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \end{aligned} \quad (\text{A.3})$$

Thus we have arrived at the desired result $E_p(t) = E_0(t) e^{-\sigma t}$. **Alternate** derivation is in Appendix B.1.

Appendix B. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below^[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma\left(\frac{s}{2}\right)(\pi n^2)^{-\frac{s}{2}} &= \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\end{aligned}\tag{B.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$.

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\tag{B.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ and

$$\sum_{n=1}^\infty \int_0^\infty |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma\left(\frac{\sigma'}{2}\right)\pi^{-\frac{\sigma'}{2}}\zeta(\sigma').$$

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} w(x) dx\tag{B.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. B.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0, 1]$ and $[1, \infty)$ and make the substitution $x \rightarrow \frac{1}{x}$ in the first interval $(0, 1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $x > 0$ is real.^[4]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_1^\infty \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1}{2} dx\tag{B.4}$$

Hence we can simplify Eq. B.4 as follows.

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_1^\infty x^{\frac{-(s+1)}{2}} w(x) dx\tag{B.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}]\tag{B.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ ^[4].

Appendix B.1. **Derivation of $E_p(t)$ and $E_0(t)$**

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. B.6 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} [1 + 2(\frac{1}{2} + \sigma + i\omega)(-\frac{1}{2} + \sigma + i\omega) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t}) dt] \quad (\text{B.7})$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) & [\int_{-\infty}^0 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \\ & + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt] \end{aligned} \quad (\text{B.8})$$

We can write this as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) \int_{-\infty}^{\infty} [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} e^{-i\omega t} dt \quad (\text{B.9})$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2} + \sigma + i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned} E_p(t) &= \frac{1}{2} \delta(t) + (-\frac{1}{4} + \sigma^2) A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2 A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}] u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}] u(t) \\ \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-4\pi n^2 e^{-2t} + (-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})^2] u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [-4\pi n^2 e^{2t} + (\frac{1}{2} - \sigma - 2\pi n^2 e^{2t})^2] u(t) + \delta(t) [\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2)] \end{aligned} \quad (\text{B.10})$$

We can simplify above equation as follows.

$$\begin{aligned} \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma \pi n^2 e^{-2t}] u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma \pi n^2 e^{2t}] u(t) + \delta(t) [\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2)] \end{aligned} \quad (\text{B.11})$$

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real^[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix B.2) and hence **dirac delta terms cancel each other** in equation below.

$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + \left(-\frac{1}{4} + \sigma^2\right)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}\right) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}\right] u(t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}\right) + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}\right] u(-t)
\end{aligned} \tag{B.12}$$

We can simplify above equation as follows.

$$\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned} \tag{B.13}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$\begin{aligned}
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) &= E_0(t)e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned} \tag{B.14}$$

Appendix B.2. Derivation of $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real^[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$\begin{aligned}
F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2 \frac{1}{x}} \left(\frac{1}{x^2}\right) + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \left(\frac{-1}{2}\right) \frac{1}{x^{\frac{3}{2}}}
\end{aligned} \tag{B.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$\begin{aligned}
\left[\frac{dF(x)}{dx}\right]_{x=1} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2} = \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2} + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}) \left(\frac{-1}{2}\right) \\
&\quad \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}
\end{aligned}
\tag{B.16}$$

Appendix C. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix C.1. *Convolution Theorem: Multiplication of $g(t)$ and $h(t)$ corresponds to convolution in Fourier transform domain*

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi} [G(\omega) * H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} g(t) h(t) e^{-i\omega t} dt \tag{C.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') e^{i\omega' t} d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(\omega') e^{i\omega' t} d\omega' \right] h(t) e^{-i\omega t} dt \\
F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[\int_{-\infty}^{\infty} e^{i\omega' t} h(t) e^{-i\omega t} dt \right] d\omega' \\
F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[\int_{-\infty}^{\infty} h(t) e^{-i(\omega - \omega') t} dt \right] d\omega'
\end{aligned}
\tag{C.2}$$

We substitute $\int_{-\infty}^{\infty} h(t) e^{-i(\omega - \omega') t} dt = H(\omega - \omega')$ in Eq. C.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega' = \int_{-\infty}^{\infty} g(t) h(t) e^{-i\omega t} dt \tag{C.3}$$

Appendix C.2. *Fourier transform of Real $g(t)$*

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t) \cos(\omega t) dt = G_R(-\omega) \\
G_I(\omega) &= - \int_{-\infty}^{\infty} g(t) \sin(\omega t) dt = -G_I(-\omega)
\end{aligned}
\tag{C.4}$$

Appendix C.3. Even part of $g(t)$ corresponds to real part of Fourier transform $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t} dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \quad (\text{C.5})$$

Appendix C.4. Odd part of $g(t)$ corresponds to imaginary part of Fourier transform $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{\text{odd}}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{odd}}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t} dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \quad (\text{C.6})$$

Appendix D. Properties of Fourier Transforms Part 2

Appendix D.1. $E_p(t), h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite.

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t} d\omega$. We see that $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \rightarrow \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \rightarrow -\infty$, $E_p(t)$ goes to zero, because for every value of n , the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t} dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t) dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix D.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix D.2. Convolution integral convergence

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix D.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi} [G(\omega) * H(\omega)] \quad (D.1)$$

Appendix D.3. Fall off rate of Fourier Transform of functions

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \rightarrow \infty$, as $|\omega| \rightarrow \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \rightarrow \infty$, where $A, B > 0$ are real.

Appendix D.4. Payley-Weiner theorem and Fall off rate of analytic functions.

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant^[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off rate** $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived

from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \rightarrow \infty$, as $|t| \rightarrow \infty$ (Appendix D.3) and hence it should have a fall-off rate **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \rightarrow \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$.

Appendix E. On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \rightarrow \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \rightarrow \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = E_p(t)$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = E_{p\omega}(\omega)$ as per convolution theorem (link), where $*$ denotes **convolution** operation given by $E_{p\omega}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix C.1.

We can write $g(t) = g_{\text{even}}(t) + g_{\text{odd}}(t)$ where $g_{\text{even}}(t)$ is an even function and $g_{\text{odd}}(t)$ is an odd function of variable t . If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ where ω_1 is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_1 \neq 0$, **then** the **real part** of $E_{p\omega}(\omega)$ given by $E_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, where $g(t)h(t) = E_p(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its real part given by $E_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_1 \neq 0$ and show that $E_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$E_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega')H(\omega - \omega')d\omega' \quad (\text{E.1})$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$. (Appendix D.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. E.1 and we get

$$E_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (\text{E.2})$$

We can split the integral in Eq. E.2 as follows.

$$E_R(\omega) = \frac{\sigma}{\pi} \left[\int_{-\infty}^0 G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right]$$

(E.3)

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix C.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. E.3 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_R(\omega) = \frac{\sigma}{\pi} \int_0^\infty G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (\text{E.4})$$

In Appendix D.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \rightarrow \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. ?? is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$.

• **Case 1:** $G_R(\omega') > 0$ for all finite $\omega' > 0$

We see that $E_R(\omega) > 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

• **Case 2:** $G_R(\omega') < 0$ for all finite $\omega' > 0$

We see that $E_R(\omega) < 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

Appendix F. On the zeros of a related function $G(\omega)$ Full version

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

Let us consider an even function $g(t) = f(t)e^{-\sigma t}u(-t) + f(-t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t , $f(t) = [e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)]$ and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)$ as shown in Section 2.1.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \rightarrow \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \rightarrow \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$, we get $\frac{1}{2\pi}[G(\omega)*H(\omega)] = F(\omega)$ as per convolution theorem (link), where $*$ denotes **convolution** operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = F_R(\omega) + iF_I(\omega)$ and $H(\omega) = H_R(\omega) + iH_I(\omega) = [\frac{\sigma}{(\sigma^2 + \omega^2)} + \frac{3\sigma}{(9\sigma^2 + \omega^2)}] + i\omega[\frac{1}{(\sigma^2 - \omega^2)} - \frac{1}{(9\sigma^2 + \omega^2)}]$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix C.1.

If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ for every value of t_0 , where $\omega_2(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H_R(\omega) = \frac{\sigma}{(\sigma^2 + \omega^2)} + \frac{3\sigma}{(9\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_2(t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega)*H_R(\omega)]$, obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$, for every value of t_0 , where $\omega_2(t_0)$ is real and finite, where $g(t)h(t) = f(t)$, $f(t) = [e^{-\sigma t_0}E_p(t - t_0) + e^{\sigma t_0}E_p(t + t_0)]$ and $h(t) = e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its real part given by $E_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_2(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H_R(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega')H_R(\omega - \omega')d\omega' \quad (\text{F.1})$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H_R(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$. (Appendix D.2)

We substitute $H_R(\omega) = \frac{\sigma}{(\sigma^2 + \omega^2)} + \frac{3\sigma}{(9\sigma^2 + \omega^2)}$ in Eq. F.1 and we get

$$F_R(\omega) = \frac{\sigma}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{3}{(9\sigma^2 + (\omega - \omega')^2)} \right] d\omega' \quad (\text{F.2})$$

We can split the integral in Eq. F.2 as follows.

$$F_R(\omega) = \frac{\sigma}{2\pi} \left[\int_{-\infty}^0 G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{3}{(9\sigma^2 + (\omega - \omega')^2)} \right] d\omega' \right. \\ \left. + \int_0^{\infty} G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{3}{(9\sigma^2 + (\omega - \omega')^2)} \right] d\omega' \right] \quad (\text{F.3})$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix C.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. F.3 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{2\pi} \int_0^{\infty} G_R(\omega') \left[\left(\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right) + \left(\frac{3}{(9\sigma^2 + (\omega - \omega')^2)} + \frac{3}{(9\sigma^2 + (\omega + \omega')^2)} \right) \right] d\omega' \quad (\text{F.4})$$

In Appendix D.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \rightarrow \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. F.4 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\left[\left(\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right) + \left(\frac{3}{(9\sigma^2 + (\omega - \omega')^2)} + \frac{3}{(9\sigma^2 + (\omega + \omega')^2)} \right) \right] > 0$.

• **Case 1:** $G_R(\omega') > 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix C.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where $\omega_2(t_0)$ is real and finite.

• **Case 2:** $G_R(\omega') < 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix C.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where $\omega_2(t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.