On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix F, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows (link).[3] This is re-derived in Appendix B.

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: Taylor's series representation of* $E_p(t)$

We can substitute $z=e^{2t}$ in Eq. 2 as follows.

$$\begin{aligned}E_p(t)=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}=E_0(t)e^{-\sigma t}=f(e^{2t})e^{\frac{t}{2}}e^{-\sigma t}\\ f(z)=2\sum_{n=1}^{\infty}[2\pi^2n^4z^2-3\pi n^2z]e^{-\pi n^2z}\end{aligned} \tag{3}$$

We can expand the real analytic function $f(z)$ using Taylor series expansion **around** $z=1$ as follows.

$$\begin{aligned}f(z)=\sum_{n=1}^{\infty}a_nz^2[\sum_{k=0}^{\infty}d_{nk}(z-1)^k]-b_nz[\sum_{k=0}^{\infty}d_{nk}(z-1)^k]\\ a_n=4\pi^2n^4e^{-\pi n^2},\quad b_n=6\pi n^2e^{-\pi n^2},\quad d_{nk}=\frac{(-\pi n^2)^k}{!(k)}\end{aligned} \tag{4}$$

Now we substitute $z=e^{2t}$ in Eq. 7 and we can write the Taylor series expansion of $E_p(t)$ as follows and we use binomial series expansion $(e^{2t}-1)^v=\sum_{p=0}^{v}\binom{v}{p}(-1)^pe^{2t(v-p)}$ for v is a positive integer.

$$E_p(t)=[\sum_{n=1}^{\infty}a_ne^{4t}[\sum_{k=0}^{\infty}d_{nk}(e^{2t}-1)^k]-b_ne^{2t}[\sum_{k=0}^{\infty}d_{nk}(e^{2t}-1)^k]]e^{\frac{t}{2}}e^{-\sigma t}$$

$$E_p(t)=[\sum_{n=1}^{\infty}a_n\sum_{k=0}^{\infty}d_{nk}[\sum_{p=0}^{k}\binom{k}{p}(-1)^pe^{2t(k+2-p)}]-b_n\sum_{k=0}^{\infty}d_{nk}[\sum_{p=0}^{k}\binom{k}{p}(-1)^pe^{2t(k+1-p)}]]e^{\frac{t}{2}}e^{-\sigma t}$$

$$(5)$$

This equation can be simplified as follows, using shorthand notation.

$$E_p(t) = \sum_{n,k,r,p} c_{nkrp} e^{b_{krp}t} e^{-\sigma t}$$

$$b_{krp} = (2k + \frac{5}{2} + 2r - 2p), \quad \sum_{n,k,r,p} c_{nkrp} = \sum_{r=0}^{1} \sum_{n=1}^{\infty} e_{nr} \sum_{k=0}^{\infty} d_{nk} \sum_{p=0}^{k} \binom{k}{p} (-1)^p, \quad e_{n1} = a_n, \quad e_{n0} = -b_n, \tag{6}$$

In Section 1.1, we showed that $E_0(t) = E_0(-t)$ and we can write $E_p(t) = E_0(t)e^{-\sigma t}$ as an **infinite summation** of two-sided decaying exponential functions as follows.

$$E_p(t) = [\sum_{n,k,r,p} c_{nkrp} e^{b_{krp}t} u(-t) + \sum_{n,k,r,p} c_{nkrp} e^{-b_{krp}t} u(t)] e^{-\sigma t} \tag{7}$$

*1.3. **Step 3: Two-sided decaying exponentials and zeros in their Fourier transform***

We know that a real two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, where $u(t)$ is Heaviside unit step function and $a, b > 0$ and t are real, has Fourier Transform $G_0(\omega)$, where ω is real. (link)

$$\begin{aligned} G_0(\omega) = \int_{-\infty}^{\infty} g_0(t) e^{-i\omega t} dt = \frac{1}{b - i\omega} + \frac{1}{a + i\omega} = \frac{b + i\omega}{b^2 + \omega^2} + \frac{a - i\omega}{a^2 + \omega^2} \\ = [\frac{b}{b^2 + \omega^2} + \frac{a}{a^2 + \omega^2}] + i\omega[\frac{1}{b^2 + \omega^2} - \frac{1}{a^2 + \omega^2}] \end{aligned} \tag{8}$$

We can see that the real part of $G_0(\omega)$ given by $\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}$ **does not have zeros** for any finite real value of ω and hence $G_0(\omega)$ does not have zeros for any finite value of ω.

Given that the inverse Fourier Transform of Riemann Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ given by $E_p(t)$ is expressed as an **infinite summation of two-sided decaying exponential functions** in previous subsection, we could investigate if $E_{p\omega}(\omega)$ also does not have zeros for any finite real value of ω.

*1.4. **Step 4: On the zeros of a related function** $G(\omega)$*

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider **a toy example** with a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In **Appendix F**, we will show that the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_{even}(\omega) = G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite.

As an **example**, consider $E_p(t) = e^{bt}u(t) + e^{-at}u(-t)$ where $a, b > \sigma > 0$ are real and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$. We see that $g(t) = e^{(b-\sigma)t}u(-t) + e^{-(a-\sigma)t}u(t)$. The real part of Fourier transform of $g(t)$ is given by $G_R(\omega) = \frac{(b-\sigma)}{(b-\sigma)^2+\omega^2} + \frac{(a-\sigma)}{(a-\sigma)^2+\omega^2}$ **does not** have any zeros for real and finite ω. The Fourier transform of $h(t)$ is given by

$H(\omega) = \frac{2\sigma}{(\sigma)^2+\omega^2}$ also **does not** have any zeros for real and finite ω.

Because $g(t)h(t) = E_p(t)$ corresponds to **convolution** of the respective Fourier transforms (plot), therefore real part of Fourier transform of $E_p(t)$ given by $Re[E_{p\omega}(\omega)]$ **cannot** have zeros for real and finite ω, which **contradicts** Statement 1. Therefore $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite.

Similarly, in Section 2.1, we consider a **modified even symmetric** function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{3\sigma t}u(t)$ for $|t_0| \leq \infty$ where $f(t) = e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$ where $g(t)h(t) = f(t)$ and show that Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$, for **every value** of t_0, to satisfy Statement 1, where $\omega_2(t_0)$ is real and finite. (Appendix G).

If there is more than one solution for $\omega_2(t_0)$, these different solutions can remain distinct. This is shown by an example video simulation in link. In Section 3, it is shown that $\omega_2(t_0)$ is a well defined continuous function, which is **at least** differentiable twice.

1.5. ***Step 5: On the zeros of the function*** $G_R(\omega)$

In Section 2.1, we compute the Fourier transform of the even function $g(t)$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**. In general, $\omega_2(t_0) \neq \omega_0$.

It is shown that $R(t_0) = G_R(\omega_2(t_0), t_0)$ is an **odd** function of variable t_0 as follows.

$$R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \tag{9}$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r,p} c_{nkrp}e^{b_{krp}t}e^{-\sigma t}$, we use the fact that $E_0(t) = E_0(-t)$, we can write as follows.

$$R(t_0) = \sum_{n,k,r,p} c_{nkrp}\frac{(b_{krp}-2\sigma)e^{(b_{krp})t_0}}{(\omega_2^2(t_0) + (b_{krp}-2\sigma)^2)} \tag{10}$$

We see that there is a **one to one correspondence** between the integral representation in Eq. 9 and Taylor series representation in Eq. 10. Given that it is easier to show integral convergence, we use only the integral representation in subsequent steps.

1.6. ***Step 6: First 2 derivatives of*** $R(t_0)$

In Section 3.1, we derive the first 2 derivatives of $R(t_0)$ at $t_0 = 0$ as follows, where
$e_0 = E_0(0), \omega_{20} = [\omega_2(t_0)]_{t_0=0}$. $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$,
$m_2 = -\omega_{22}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$.

$$\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= e_0 + n_0\omega_{20} + 2\sigma m_0 \\
(\frac{d^2R(t_0)}{dt_0^2})_{t_0=0} &= m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - m_0\frac{\omega_{20}^2}{2}
\end{aligned} \tag{11}$$

Given that $R(t_0) = G_R(\omega_2(t_0), t_0)$ is an **odd** function of variable t_0, we get $m_0 = 0$ and $m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} = 0$.

1.7. ***Step 7: Next Step***

In Section 3.2, we replace $E_p(t)$ by $E_{pp}(t) = e^{\sigma t_2}E_p(t+t_2) + e^{-\sigma t_2}E_p(t-t_2)$, for $|t_2| \leq \infty$ and derive as follows.

$$
\begin{aligned}
m_0^{'}(t_2) &= R'(t_2) + R'(-t_2) = 0 \\
R'(t_2) = e^{2\sigma t_2}[\cos(\omega_{20}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}(t_2)\tau)d\tau &+ \sin(\omega_{20}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}(t_2)\tau)d\tau] \\
A(t_2) &= m_2^{'}(t_2) + \sigma e_0^{'}(t_2) + 2\sigma n_0^{'}(t_2)\omega_2(t_2) = 0 \\
e_0^{'}(t_2) &= E_0(t_2) + E_0(-t_2) \\
n_0^{'}(t_2) &= n_{0p}(t_2) + n_{0p}(-t_2) \\
n_{0p}(t_2) = e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau &- \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
m_2^{'}(t_2) &= m_{2p}(t_2) + m_{2p}(-t_2) \\
m_{2p}(t_2) = -\frac{1}{2}\frac{d^2\omega_2(t_2)}{dt_2^2}e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2) &\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau \\
&- \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau]
\end{aligned} \tag{12}
$$

1.8. ***Step 8: Asymptotic Case and Final result***

In Section 3.3, we consider the asymptotic case and show that $\lim_{t_2\to\infty}\omega_2(t_2) = \omega_z$ and derive as follows.

$$
\begin{aligned}
\lim_{t_2\to\infty} A(t_2) = \lim_{t_2\to\infty} 2\sigma\omega_z n_0^{'}(t_2) &= 0 \\
\lim_{t_2\to\infty} n_0^{'}(t_2) &= 0 \\
\lim_{t_2\to\infty} m_0^{'}(t_2) &= 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{i(\omega_z t)}dt &= 0
\end{aligned} \tag{13}
$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration n, we use $h(t) = e^{(2^{N+1}\sigma)t}u(-t) + e^{-3*(2^{N+1}\sigma)t}u(t)$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t}$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$ and we show a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$.

2. An Approach towards Riemann's Hypothesis: Method 1

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2}+\sigma+i\omega) = \xi(\frac{1}{2}-\sigma-i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2}+\sigma+i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix D.1.

2.1. *On a related function* $G(\omega)$

Let us form a new function $f(t) = e^{-\sigma t_0}E_p(t-t_0)+e^{\sigma t_0}E_p(t+t_0) = [E_0(t+t_0)+E_0(t-t_0)]e^{-\sigma t} = E_{0n}(t)e^{-\sigma t}$, where $|t_0| \leq \infty$, $E_{0n}(t) = E_{0n}(-t) = E_0(t+t_0) + E_0(t-t_0)$. Its Fourier Transform given by $F(\omega) = E_{p\omega}(\omega)[e^{-\sigma t_0}e^{-i\omega t_0} + e^{\sigma t_0}e^{i\omega t_0}]$ also has a zero at $\omega = \omega_0$.

Let us consider a real and **even symmetric** function $g(t) = g(-t) = g_-(t)u(-t) + g_+(t)u(t)$ where $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = g_-(-t) = f(-t)e^{\sigma t} = f(t)e^{3\sigma t}$, because $f(t) = E_{0n}(t)e^{-\sigma t}$, $f(-t)e^{\sigma t} = E_{0n}(t)e^{2\sigma t}$, $f(t)e^{3\sigma t} = E_{0n}(t)e^{2\sigma t}$ and $E_{0n}(t) = E_{0n}(-t)$. We see that $g(t) = E_{0n}(t)e^{-2\sigma t}u(-t) + E_{0n}(t)e^{2\sigma t}u(t)$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$.

We can see that $g(t)$ is a real L^1 integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma]. This is explained in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega)$ where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{3\sigma+i\omega}] = [\frac{\sigma}{(\sigma^2+\omega^2)} + \frac{3\sigma}{(9\sigma^2+\omega^2)}] + i\omega[\frac{1}{(\sigma^2-\omega^2)} - \frac{1}{(9\sigma^2+\omega^2)}]$ is the Fourier transform of the function $h(t)$.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_2(t_0)$ for **every value** of t_0. Because the real part of $H(\omega)$ given by $H_R(\omega) = \frac{\sigma}{(\sigma^2+\omega^2)} + \frac{3\sigma}{(9\sigma^2+\omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

This is explained in detail in Appendix G.

Next Step

Let us compute the Fourier transform of the function $g(t)$ given by $G(\omega)$.

$$
\begin{aligned}
g(t) &= g_-(t)u(-t) + g_+(t)u(t) = g_-(t)u(-t) + g_-(-t)u(t) \\
g(t) &= [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]e^{-\sigma t}u(-t) + [e^{-\sigma t_0}E_p(-t-t_0) + e^{\sigma t_0}E_p(-t+t_0)]e^{\sigma t}u(t) \\
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]e^{-\sigma t}e^{-i\omega t}dt \\
&+ \int_{0}^{\infty} [e^{-\sigma t_0}E_p(-t-t_0) + e^{\sigma t_0}E_p(-t+t_0)]e^{\sigma t}e^{-i\omega t}dt
\end{aligned} \tag{14}
$$

In the second integral in above equation ,we can substitute $t = -t$ and we get

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]e^{-\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]e^{-\sigma t}e^{i\omega t}dt \\
G(\omega) &= 2\int_{-\infty}^{0} [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]e^{-\sigma t}\cos\omega t dt = G_R(\omega) + iG_I(\omega) = G_R(\omega)
\end{aligned} \tag{15}
$$

Using the substitutions $t - t_0 = \tau, dt = d\tau$ and $t + t_0 = \tau, dt = d\tau$, we can write the above equation as follows. We use $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$.

$$
\begin{aligned}
G_R(\omega) &= G_R(\omega, t_0) = G_2(\omega, t_0) + G_2(\omega, -t_0) \\
G_2(\omega, t_0) &= 2e^{\sigma t_0}e^{\sigma t_0}[\cos(\omega t_0)\int_{-\infty}^{t_0} E_p(\tau)e^{-\sigma\tau}\cos(\omega\tau)d\tau + \sin(\omega t_0)\int_{-\infty}^{t_0} E_p(\tau)e^{-\sigma\tau}\sin(\omega\tau)d\tau] \\
G_2(\omega, t_0) &= 2e^{2\sigma t_0}[\cos(\omega t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega\tau)d\tau + \sin(\omega t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega\tau)d\tau]
\end{aligned} \tag{16}
$$

We require $G(\omega) = G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for **every value** of t_0, to satisfy **Statement 1**. Hence we can see that $R(t_0) = \frac{1}{2}G_2(\omega_2(t_0), t_0)$ is an **odd function** of variable t_0.

$$
\begin{aligned}
G(\omega_2(t_0), t_0) &= G_2(\omega_2(t_0), t_0) + G_2(\omega_2(t_0), -t_0) = 0 \\
R(t_0) &= \frac{1}{2}G_2(\omega_2(t_0), t_0) \\
R(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
S(t_0) &= R(t_0) + R(-t_0) = 0
\end{aligned} \tag{17}
$$

We see that $f(t) = e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_2(t_0)$ is an **even** function of variable t_0.

2.2. *Method 1: Asymptotic Fall off rate argument.*

This method **does not** require differentability of $\omega_2(t_0)$ and is **independent** of Method 2 in Section 3.

In Section 3.4.1, we show that $\lim_{t_0\to\infty} g(t)$ is an **analytic** function, with the **magnitude** of the step discontinuity at $t=0$ **decreasing to zero**, and its Fourier transform is an analytic function with **isolated zeros** and each isolated zero has a single value, as $\lim_{t_0\to\infty}$.

In Section 2.3, we show that $\lim_{t_0\to\infty}\omega_2(t_0)=\omega_z$ is a constant and we **rule out** the pathological case of $\omega_2(t_0)$ which is discontinuous everywhere and/or ill-defined. It is shown that the integrals $I_1(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ and $I_2(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau$ in Eq. 18 **converge** as $\lim_{t_0\to\infty}$.

As $\lim_{t_0\to\infty}$, we can compute $S(t_0)$ in Eq. 17 as follows. The expression for $R(-t_0)$ goes to zero as $\lim_{t_0\to\infty}$, due to the term $e^{-2\sigma t_0}$. In the equation for $R(t_0)$, the term $\lim_{t_0\to\infty}e^{2\sigma t_0}=\infty$. Hence we require $\lim_{t_0\to\infty}\cos(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_z\tau)d\tau+\sin(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_z\tau)d\tau=0$.
We use $\lim_{t_0\to\infty}\omega_2(t_0)=\omega_z$ and write as follows.

$$\lim_{t_0\to\infty}S(t_0)=\lim_{t_0\to\infty}e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau+\sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]=0$$
$$\lim_{t_0\to\infty}\cos(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_z\tau)d\tau+\sin(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_z\tau)d\tau=0 \tag{18}$$

We define $I_1(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ and $I_2(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau$ in Eq. 18 and note that $\lim_{t_0\to\infty}I_1(t_0)$ and $\lim_{t_0\to\infty}I_2(t_0)$ tend to a constant, which is finite and determinate, given that $\lim_{t_0\to\infty}\omega_2(t_0)=\omega_z$. We see that the terms $I_1(t_0)$ and $I_2(t_0)$ have an **asymptotic fall-off** rate of e^{-Kt_0}, as $\lim_{t_0\to\infty}$, where $K>2\sigma$, to satisfy the equation $S(t_0)=R(t_0)+R(-t_0)=0$. Hence we can write a **new equation** by interchanging $I_1(t_0)$ and $I_2(t_0)$ in Eq. 18 as follows.

$$\lim_{t_0\to\infty}\cos(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_z\tau)d\tau-\sin(\omega_z t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_z\tau)d\tau=0 \tag{19}$$

We use $I_1(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_z\tau)d\tau$ and $I_2(t_0)=\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_z\tau)d\tau$, we can write Eq. 18 and Eq. 19 as follows.

$$\begin{aligned}\lim_{t_0\to\infty}\cos(\omega_z t_0)I_1(t_0)+\lim_{t_0\to\infty}\sin(\omega_z t_0)I_2(t_0)=0\\ \lim_{t_0\to\infty}\cos(\omega_z t_0)I_2(t_0)-\lim_{t_0\to\infty}\sin(\omega_z t_0)I_1(t_0)=0\\ \lim_{t_0\to\infty}\frac{I_2(t_0)}{I_1(t_0)}=\lim_{t_0\to\infty}\frac{\sin(\omega_z t_0)}{\cos(\omega_z t_0)}=\lim_{t_0\to\infty}-\frac{I_1(t_0)}{I_2(t_0)}\end{aligned} \tag{20}$$

For the general case of $\lim_{t_0\to\infty}\frac{\sin(\omega_z t_0)}{\cos(\omega_z t_0)}\neq 0,\pm\infty$, we get $\lim_{t_0\to\infty}I_1(t_0)^2+I_2(t_0)^2=0$. This implies that $\lim_{t_0\to\infty}I_1(t_0)=\lim_{t_0\to\infty}I_2(t_0)=0$ and $\int_{-\infty}^{\infty}E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau=0$.

We started with **Statement 1** that the Fourier Transform of the function $E_p(t)=E_0(t)e^{-\sigma t}$ has a zero at $\omega=\omega_0$ which means that $\int_{-\infty}^{\infty}E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau=0$ and we derived the result that $\int_{-\infty}^{\infty}E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau=0$.

Now we can repeat the steps in Section 2, starting with the new result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$ and σ replaced by 2σ and derive the next result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-4\sigma\tau}e^{-i\omega_{(z1)}\tau}d\tau = 0$.

We can repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration n, we use $h(t) = e^{(2^{N+1}\sigma)t}u(-t) + e^{-3*(2^{N+1}\sigma)t}u(t)$, $\omega_2(t_0)$ replaced by $\omega_{2n}(t_0)$ and ω_z replaced by $\omega_{(zn)}$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t} = \sum_{n=1}^{\infty}[4\pi^2n^4e^{4t} - 6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-(2^{N+1}\sigma)t}$ given by $E_{p\omega N}(\omega) = \xi(\frac{1}{2} + 2^N\sigma + i\omega)$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2.3. ***Integral convergence***

In this section, we show that $\lim_{t_0\to\infty}\omega_2(t_0)$ equals a well defined constant and that the integrals $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ and $I_2(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau$ in Eq. 18 **converge** as $\lim_{t_0\to\infty}$.

We take Eq. 17 and write it as follows using $E_0^{'}(t) = E_0(t+t_0) + E_0(t-t_0)$. If we substitute $t + t_0 = \tau$ and $t - t_0 = \tau$, we get Eq. 17 copied below.

$$S(t_0) = R(t_0) + R(-t_0) = \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \tag{21}$$

We consider the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined (**Statement 2**). Then $S(t_0) = \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ is ill-defined everywhere as a result (**Statement 3**) and we can show that this pathological case **does not** apply to $\omega_2(t_0)$.

We see that the integral $S(t_0) = 0$ in Eq. 21 **converges** for $|t_0| \le \infty$ and this result is derived from **Statement 1** (Riemann's Xi function has a zero in the critical strip excluding the critical line). This contradicts Statement 2 and 3.

Given that $\lim_{t_0\to\infty} R(-t_0) = 0$, we have $\lim_{t_0\to\infty} R(t_0) = \lim_{t_0\to\infty} S(t_0) = 0$.

- We can **rule out** the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined, as follows. **If Statements 1, 2 and 3** were true, **then** the result that the integral in Eq. 21 converges, suggests one of the following:

a) Statement 1 is true and above result **contradicts** Statement 2 and 3 and hence we can **rule out** pathological case for $\omega_2(t_0)$ **or**

b) Statements 2 and 3 are true and **Statement 1 is false** and we complete the proof of theorem 1 at this point. We **do not** require to show that $\omega_2(t_0)$ is **not** pathological, for this case.

- Let us consider the **pathological** case where $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined. In Eq. 18 copied below, we see that $\cos(\omega_2(t_0)t_0)$ and $\sin(\omega_2(t_0)t_0)$ are ill-defined and the integrals are also ill-defined functions.

$$\lim_{t_0\to\infty}\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau = 0 \tag{22}$$

Hence we see that (ill-defined function) * (ill-defined function) + (ill-defined function) * (ill-defined function) = 0, as $\lim_{t_0\to\infty}$, and this **does not** make sense. Therefore the assumption that $\omega_2(t_0)$ is **discontinuous everywhere** and/or ill-defined is **false**, if **Statement 1** is true.

• Let us consider the case where $\omega_2(t_0)$ is well defined but **first derivative** $\frac{d\omega_2(t_0)}{dt_0}$ is **discontinuous everywhere** and/or ill-defined. We take the first derivative of Eq. 21.

$$\frac{dS(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0}\tau E_0^{'}(\tau)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau + \int_{-\infty}^{0}\frac{dE_0^{'}(\tau)}{dt_0}e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau = 0 \tag{23}$$

Hence we see that (ill-defined function) * (ill-defined function) + (ill-defined function) = 0, for all $|t_0| \leq \infty$, and this **does not** make sense. Therefore the assumption that $\frac{d\omega_2(t_0)}{dt_0}$ is **discontinuous everywhere** and/or ill-defined is **false**, if **Statement 1** is true.

• **If** Statement 1 is true, **then** we have shown that **Statement 2 is false** and hence $\omega_2(t_0)$ is a well-defined function and $\lim_{t_0\to\infty}\omega_2(t_0) = \omega_z$ is a well defined constant (also shown in Section 3.4.1). Hence the results derived in Section 2.2 are **valid**, with constant ω_z. Hence the integrals $I_1(t_0), I_2(t_0)$ in Eq. 18 converge.

3. Method 2: $\omega_2(t_0)$, $R(t_0)$ are at least differentiable twice.

In this section, which is applicable for the **non-pathological**, well defined case of $\omega_2(t_0)$ and $\frac{d\omega_2(t_0)}{dt_0}$, it is shown that $\omega_2(t_0)$, $R(t_0)$ and $M(t_0), N(t_0)$ are well defined continuous functions, which are **at least** differentiable twice. This method is **independent** of Method 1 in Section 2.2.

In Appendix G, $\omega_2(t_0)$ is shown to be **finite** for all $|t_0| \leq \infty$. This means there are **no** Dirac delta functions present in $\omega_2(t_0)$.

There is a well known equation describing derivatives of Dirac delta function $t^{2r}\delta^{2r}(t) = (-1)^{2r}(!(2r))\delta(t) = (!(2r))\delta(t)$ (Eq. 17 in link).

We take the first 2 derivatives of $S(t_0)$, **even if** we **assume** that the first 2 derivatives contains Dirac delta functions and we show that the **assumption** that $\frac{d\omega_2(t_0)}{dt_0}$ or $\frac{d^2\omega_2(t_0)}{dt_0^2}$ has a Dirac delta function is **false**.

We take the first derivative of $S(t_0)$ in Eq. 21 as follows where $E_0^{'}(t) = E_0(t+t_0) + E_0(t-t_0)$.

$$\begin{gathered} S(t_0) = \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau = 0 \\ \frac{dS(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0}\tau E_0^{'}(\tau)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau + \int_{-\infty}^{0}\frac{dE_0^{'}(\tau)}{dt_0}e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau = 0 \\ \frac{d\omega_2(t_0)}{dt_0}P(t_0) = Q(t_0), \quad P(t_0) = \int_{-\infty}^{0}\tau E_0^{'}(\tau)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau, \quad Q(t_0) = \int_{-\infty}^{0}\frac{dE_0^{'}(\tau)}{dt_0}e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau \end{gathered} \tag{24}$$

• Let us consider the case $\omega_2(t_0)$ has a **step discontinuity** at $t_0 = \pm t_A$ of magnitude A_0 and continuous everywhere else. In this case, $\frac{d\omega_2(t_0)}{dt_0} = A_0(\delta(t-t_A) - \delta(t+t_A)) + B(t_0)$ has a **Dirac delta** function at $t_0 = \pm t_A$ given that $\omega_2(t_0)$ has even symmetry and $B(t_0)$ does not have Dirac delta function components. We see that both integrals $P(t_0), Q(t_0)$ in Eq. 24 are **continuous** functions, because integral of a rectangular function with step discontinuity is a triangular function which is continuous.

It is possible that the Dirac delta function at $t = t_A$ in $\frac{d\omega_2(t_0)}{dt_0}$ is cancelled if $P(t_A) = 0$. We see that the term $\frac{d\omega_2(t_0)}{dt_0}$ **does not** have any other step discontinuity other than the Dirac delta function at $t = t_A$, given that we **require** $\frac{d\omega_2(t_0)}{dt_0}P(t_0) = Q(t_0)$ (**Result A**).

We take $M(t_0)$ and its first derivative in Eq. E.10 as follows.

$$M(t_0) = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$$
$$\frac{dM(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau = -\frac{d\omega_2(t_0)}{dt_0}C(t_0) \tag{25}$$

• Let us consider the case $\omega_2(t_0)$ has a **step discontinuity** at $t_0 = \pm t_A$ of magnitude A_0. In this case, $\frac{d\omega_2(t_0)}{dt_0} = A_0(\delta(t - t_A) - \delta(t + t_A)) + B(t_0)$ has a **Dirac delta** function at $t_0 = \pm t_A$ given that $\omega_2(t_0)$ has even symmetry and $B(t_0)$ does not have Dirac delta function components. We see that $M(t_0)$ is a **continuous** function whose first derivative $\frac{dM(t_0)}{dt_0}$ has a **step discontinuity** at $t_0 = \pm t_A$, because $M(t_0)$ is obtained by **integrating** terms containing $\omega_2(t_0)$. We see that $C(t_0)$ is also a **continuous** function for the same reason.

In Eq. 25, we see that, at $t_0 = t_A$, the left hand side of the equation $\frac{dM(t_0)}{dt_0}$ has a **step discontinuity** at $t_0 = t_A$, while the terms on the right hand side $C(t_0)$ is continuous, and $\frac{d\omega_2(t_0)}{dt_0}$ has a Dirac delta function at $t_0 = \pm t_A$. Hence the right hand side is **either** a continuous function if $C(t_A) = 0$ **or** has a Dirac delta at $t_0 = t_A$. This is **not** possible. Hence we infer that $\omega_2(t_0)$ **does not** have a step discontinuity at $t_0 = \pm t_A$.

For example, $\frac{dM(t_0)}{dt_0}$ has a left limit value of M_0 and a **different** right limit value of $M_1 \neq M_0$ at $t_0 = t_A$. In the right hand side of Eq. 25, if $C(t_A) = 0$, then the Dirac delta function term contribution vanishes at $t_0 = t_A$ and left limit value and right limit value are the **same** at $t_0 = t_A$ because $C(t_0)$ is a a continuous function and $\frac{d\omega_2(t_0)}{dt_0}$ is continuous in the vicinity of $t_0 = t_A$ (**Result A**), besides the Dirac delta function at $t_0 = t_A$.

• Let us consider the case $\omega_2(t_0)$ is a continuous function but $\frac{d\omega_2(t_0)}{dt_0}$ has a **step discontinuity** at $t_0 = \pm t_A$ of magnitude A_1. In Eq. 25, we see that $\frac{dM(t_0)}{dt_0}$ on the left hand side, is a continuous function, because it is obtained by integrating a continuous $\omega_2(t_0)$. On the right hand side, $C(t_0)$ is a continuous function, while $\frac{d\omega_2(t_0)}{dt_0}$ has a **step discontinuity** at $t_0 = \pm t_A$. This is clearly **not** possible. Hence we infer that $\frac{d\omega_2(t_0)}{dt_0}$ **does not** have a step discontinuity at $t_0 = \pm t_A$.

The above arguments apply to the case of one or more **isolated step discontinuitities** in $\omega_2(t_0)$ and $\frac{d\omega_2(t_0)}{dt_0}$.

Hence we have shown that $\omega_2(t_0)$, $R(t_0)$ and $M(t_0), N(t_0)$ are well defined continuous functions, which are **at least** differentiable twice.

3.1. ***First 2 derivatives of*** $R(t_0)$

In Appendix E, we derive the first 2 derivatives of $R(t_0)$ at $t_0 = 0$ as follows, where $m_0 = M(0), m_2 = [\frac{d^2M(t_0)}{dt_0^2}]_{t_0=0}$ and $n_0 = N(0), n_2 = [\frac{d^2N(t_0)}{dt_0^2}]_{t_0=0}$ and $M(t_0) = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ and $N(t_0) = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau$ $e_0 = [E_0(t)]_{t_0=0}$ and $[\omega_2(t_0)]_{t_0=0} = \omega_{20}$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= e_0 + n_0\omega_{20} + 2\sigma m_0 \\
(\frac{d^2R(t_0)}{dt_0^2})_{t_0=0} &= m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - m_0\frac{\omega_{20}^2}{2}
\end{aligned}
\tag{26}
$$

The equations for m_0, m_2, n_0 are described in Appendix E.2. Given that $R(t_0)$ is an **odd function** of variable t_0, we get

$$
\begin{aligned}
m_0 &= 0 \\
m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - m_0\frac{\omega_{20}^2}{2} = 0, \quad m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} &= 0 \\
m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau, \quad n_0 &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau \\
m_2 = -\omega_{22}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau))d\tau, \quad e_0 &= E_0(0)
\end{aligned}
\tag{27}
$$

3.2. ***Next Step***

If we replace $E_p(t)$ in above section by $E_{pp}(t) = e^{\sigma t_2}E_p(t+t_2) + e^{-\sigma t_2}E_p(t-t_2) = [E_0(t+t_2) + E_0(t-t_2)]e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, for $|t_2| \leq \infty$, where $E_0^{'}(t) = E_0(t+t_2) + E_0(t-t_2)$, the location of the zeros in Fourier transform of $g(t, t_0, t_2)$ are represented by $\omega_2^{'}(t_2, t_0)$ and using method in the above section, we can get results similar to Eq. 27 with $E_0(t)$ replaced by $E_0^{'}(t)$ and ω_{20} replaced by $\omega_{20}^{'}(t_2)$ and other variables replaced with their **primed** versions as follows. We use $\omega_2^{'}(t_2, t_0) = w_{20}^{'}(t_2) + w_{22}^{'}(t_2)t_0^2 + ...$

$$
\begin{aligned}
m_0^{'}(t_2) &= \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_{20}^{'}(t_2)\tau)d\tau = 0 \\
m_2^{'}(t_2) + \sigma e_0^{'}(t_2) + 2\sigma n_0^{'}(t_2)\omega_{20}^{'}(t_2) &= 0 \\
n_0^{'}(t_2) &= \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_{20}^{'}(t_2)\tau)d\tau \\
m_2^{'}(t_2) = -\omega_{22}^{'}(t_2)\int_{-\infty}^{0} \tau E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_{20}^{'}(t_2)\tau))d\tau, \quad e_0^{'}(t_2) &= E_0^{'}(0) = E_0(t_2) + E_0(-t_2)
\end{aligned}
\tag{28}
$$

We use $E_0^{'}(t) = E_0(t+t_2) + E_0(t-t_2)$ in Eq. 28 and then substitute $t + t_2 = \tau$ for the first term and $t - t_2 = \tau$ for the second term and get $m_0^{'}(t_2)$ as follows.

$$
\begin{aligned}
m_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}^{'}(t_2)\tau)d\tau + \sin(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}^{'}(t_2)\tau)d\tau] \\
+e^{-2\sigma t_2}[\cos(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}^{'}(t_2)\tau)d\tau - \sin(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}^{'}(t_2)\tau)d\tau] = 0 \\
m_0^{'}(t_2) = R'(t_2) + R'(-t_2) = 0 \\
R'(t_2) = e^{2\sigma t_2}[\cos(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}^{'}(t_2)\tau)d\tau + \sin(\omega_{20}^{'}(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}^{'}(t_2)\tau)d\tau]
\end{aligned}
$$

(29)

We compare Eq. 29 with Eq. 17 and see that $R(t_0)$ and $R'(t_2)$ are similar equations, with $t_0, \omega_2(t_0)$ replaced by $t_2, \omega_{20}^{'}(t_2)$ and hence both equations **must have at least one** common solution with $\omega_2(t_0) = \omega_{20}^{'}(t_2)$. Hence we replace $\omega_{20}^{'}(t_2)$ in Eq. 28 with $\omega_2(t_2)$ and use $E_0^{'}(t) = E_0(t+t_2) + E_0(t-t_2)$ and write as follows.

$$
\begin{aligned}
n_0^{'}(t_2) &= n_{0p}(t_2) + n_{0p}(-t_2) \\
n_{0p}(t_2) &= e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
m_2^{'}(t_2) &= m_{2p}(t_2) + m_{2p}(-t_2) \\
m_{2p}(t_2) &= -\frac{1}{2}\frac{d^2\omega_2(t_2)}{dt_2^2}e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau \\
&\quad - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2}(\tau - t_2)E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
e_0^{'}(t_2) &= E_0(t_2) + E_0(-t_2) \\
A(t_2) &= m_2^{'}(t_2) + \sigma e_0^{'}(t_2) + 2\sigma n_0^{'}(t_2)\omega_2(t_2) = 0
\end{aligned}
$$

(30)

The term $\frac{d^2\omega_2(t_0)}{dt_0^2}$ in Eq. 30 is obtained as follows. We see that $f^{'}(t) = e^{\sigma t_0}E_{pp}(t+t_0) + e^{-\sigma t_0}E_{pp}(t-t_0)$ remains the **same**, when we **interchange** t_0 with t_2, where $E_{pp}(t) = e^{\sigma t_2}E_p(t+t_2) + e^{-\sigma t_2}E_p(t-t_2)$. Because the Fourier transform of $f^{'}(t)$ given by $F^{'}(\omega) = E_{pp\omega}(\omega)(e^{\sigma t_0}e^{i\omega t_0} + e^{-\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{\sigma t_2}e^{i\omega t_2} + e^{-\sigma t_2}e^{-i\omega t_2})(e^{\sigma t_0}e^{i\omega t_0} + e^{-\sigma t_0}e^{-i\omega t_0})$ remains the **same**, when we **interchange** t_0 with t_2.

Hence $\omega_2(t_2, t_0) = \omega_2(t_0, t_2)$. The second derivative is given by $\frac{d^2\omega_2(t_2,t_0)}{dt_0^2} = \frac{d^2\omega_2(t_0,t_2)}{dt_2^2}$. In Eq. E.10, we computed ω_{22} by evaluating $\frac{1}{2}\frac{d^2\omega_2(t_0)}{dt_0^2}$ at $t_0 = 0$ to obtain m_2. Similarly, we compute $\omega_{22}^{'}(t_2)$ in Eq. 28 , by evaluating the term $\frac{1}{2}\frac{d^2\omega_2(t_2,t_0)}{dt_0^2}$ at $t_0 = 0$, hence this is the **same** as evaluating $\frac{1}{2}\frac{d^2\omega_2(t_0,t_2)}{dt_2^2}$ at $t_0 = 0$ which equals $\frac{1}{2}\frac{d^2\omega_2(t_2)}{dt_2^2}$. in Eq. 30.

*3.3. **Asymptotic Case and Final result***

In Section 3.4.1, we show that $\lim_{t_2\to\infty} g(t)$ is an **analytic** function, with the **magnitude** of the step discontinuity at $t = 0$ **decreasing to zero**, and its Fourier transform is an analytic function with **isolated zeros**, as $\lim_{t_2\to\infty}$. Hence $\lim_{t_2\to\infty}\omega_2(t_2) = \omega_z \neq 0$ which is a constant and $\lim_{t_2\to\infty}\frac{d^2\omega_2(t_2)}{dt_2^2} = 0$. Hence $\lim_{t_2\to\infty} m_2^{'}(t_2) = 0$. We see that $\lim_{t_2\to\infty} e_0^{'}(t_2) = 0$ and $\lim_{t_2\to\infty} n_{0p}(-t_2), m_{2p}(-t_2) = 0$ and we write Eq. 30 as follows given $\sigma, \omega_z \neq 0$.

$$
\begin{aligned}
\lim_{t_2\to\infty} A(t_2) &= \lim_{t_2\to\infty} 2\sigma\omega_z n_0^{'}(t_2) = 0 \\
\lim_{t_2\to\infty} n_0^{'}(t_2) &= \lim_{t_2\to\infty} e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] = 0 \\
&\lim_{t_2\to\infty}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] = 0
\end{aligned}
$$

(31)

Similarly, we can write Eq. 29 in the asymptotic case $\lim_{t_2\to\infty}$ as follows.

$$\lim_{t_2\to\infty} m_0^{'}(t_2) = \lim_{t_2\to\infty} e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau$$
$$+\sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau] = 0 \tag{32}$$

If we write $I_1(t_2) = \int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_z\tau)d\tau$ and $I_2(t_2) = \int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_z\tau)d\tau$, and $\lim_{t_2\to\infty}(\omega_2(t_2) = \omega_z$ we can write Eq. 31 and Eq. 32 as follows.

$$\lim_{t_2\to\infty}\cos(\omega_z t_2)I_2(t_2) - \lim_{t_2\to\infty}\sin(\omega_z t_2)I_1(t_2) = 0$$
$$\lim_{t_2\to\infty}\cos(\omega_z t_2)I_1(t_2) + \lim_{t_2\to\infty}\sin(\omega_z t_2)I_2(t_2) = 0$$
$$\lim_{t_2\to\infty}\frac{I_2(t_2)}{I_1(t_2)} = \lim_{t_2\to\infty}\frac{\sin(\omega_z t_2)}{\cos(\omega_z t_2)} = \lim_{t_2\to\infty} -\frac{I_1(t_2)}{I_2(t_2)} \tag{33}$$

For the general case of $\lim_{t_2\to\infty}\frac{\sin(\omega_z t_2)}{\cos(\omega_z t_2)} \neq 0, \pm\infty$, we get $\lim_{t_2\to\infty}(I_1(t_2)^2 + I_2(t_2)^2) = 0$. This implies that $\lim_{t_2\to\infty} I_1(t_2) = \lim_{t_2\to\infty} I_2(t_2) = 0$ and $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

Now we can repeat the steps in Section 2, starting with the new result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$ and σ replaced by 2σ and derive the next result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-4\sigma\tau}e^{-i\omega_{(z1)}\tau}d\tau = 0$.

We can repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration n, we use $h(t) = e^{(2^{N+1}\sigma)t}u(-t) + e^{-3*(2^{N+1}\sigma)t}u(t)$, $\omega_2(t_2)$ replaced by $\omega_{2n}(t_2)$ and ω_z replaced by $\omega_{(zn)}$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t} = \sum_{n=1}^{\infty}[4\pi^2n^4e^{4t} - 6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-(2^{N+1}\sigma)t}$ given by $E_{p\omega N}(\omega) = \xi(\frac{1}{2} + 2^N\sigma + i\omega)$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

3.4. *Analytic Functions and Isolated Zeros.*

In this section, we show that $\lim_{t_0\to\infty} g(t)$ is an analytic function, with the magnitude of the step discontinuity at $t = 0$ decreasing to zero, and its Fourier transform is an analytic function with isolated zeros, as $\lim_{t_0\to\infty}$. Hence $\lim_{t_0\to\infty} \omega_2(t_0) = \omega_z \neq 0$ which is a constant and $\lim_{t_0\to\infty} \frac{d^2\omega_2(t_0)}{dt_0^2} = 0$.

We see that $g(t) = E_0^{'}(t)e^{-2\sigma t}u(-t) + E_0^{'}(t)e^{2\sigma t}u(t)$ where $E_0^{'}(t) = E_0^{'}(-t) = E_0(t+t_0) + E_0(t-t_0)$ and its first derivative has a **step** discontinuity at $t = 0$ with magnitude $\Delta_d = 4\sigma E_0^{'}(0) = 4\sigma(E_0(t_0)+E_0(-t_0))$. As $\lim_{t_0\to\infty}$, $\Delta_d \to 0$ because $E_0(t_0)$ and $E_0(-t_0)$ decrease to zero as $\lim_{t_0\to\infty}$ and hence $\lim_{t_0\to\infty} g(t) = \lim_{t_0\to\infty}[E_0^{'}(t)e^{-2\sigma t} + E_0^{'}(t)e^{2\sigma t}]$ is an **analytic** function.

We use a **scale factor** and get $g_s(t) = g(t)e^{-2\sigma t_0}$, so that $\lim_{t_0\to\infty} g_s(t)$ remains **finite** for all $|t| \leq \infty$. This scale factor **does not** affect the location of zeros in the Fourier transform of $g(t)$ and $g_s(t)$. Hence $\lim_{t_0\to\infty} g_s(t) = \lim_{t_0\to\infty} E_0^{'}(t)[e^{-2\sigma t} + e^{2\sigma t}]e^{-2\sigma t_0} = E_0(t+t_0)e^{-2\sigma(t+t_0)} + E_0(t-t_0)e^{2\sigma(t-t_0)} + E_0(t-t_0)e^{-2\sigma(t+t_0)} + E_0(t+t_0)e^{2\sigma(t-t_0)}$.

The Fourier transform of $g_s(t)$ is given by $G_s(\omega)$ and $\lim_{t_0\to\infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0} + E_{0\omega}(\omega - i2\sigma)e^{-i\omega t_0}e^{-4\sigma t_0} + E_{0\omega}(\omega + i2\sigma)e^{i\omega t_0}e^{-4\sigma t_0}$. As $\lim_{t_0\to\infty}$, the last two terms in $\lim_{t_0\to\infty} G_s(\omega)$ go to zero.

Hence $\lim_{t_0\to\infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0}$ is an **analytic function** for all $|\omega| \leq \infty$ because it is derived from the **entire function** $\xi(s)$ and we know that $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. The same statement holds for $\lim_{t_0\to\infty} G(\omega)$ which differs only by a scale factor $e^{-2\sigma t_0}$.

We use the well known result that **analytic** functions have **isolated zeros**.(link) Hence $\lim_{t_0\to\infty} G_s(\omega)$ and $\lim_{t_0\to\infty} G(\omega)$ have **isolated zeros** at $\omega = \omega_2(t_0) = \omega_z$ and the **second derivative** given by $\lim_{t_0\to\infty} \frac{d^2\omega_2(t_0)}{dt_0^2} = 0$.

3.4.1. *Isolated Zeros are single valued.*

We consider $g_s(t) = g(t)e^{-2\sigma t_0}$ and see that $\lim_{t_0\to\infty} g_s(t) = E_0(t+t_0)e^{-2\sigma(t+t_0)} + E_0(t-t_0)e^{2\sigma(t-t_0)} + E_0(t-t_0)e^{-2\sigma(t+t_0)} + E_0(t+t_0)e^{2\sigma(t-t_0)}$ is an **analytic** function, whose Fourier transform is given by $\lim_{t_0\to\infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma)e^{i\omega t_0} + E_{0\omega}(\omega + i2\sigma)e^{-i\omega t_0}$ which is derived from the **entire** function $\xi(s)$ where $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$.

We know that **analytic** functions have **isolated zeros** (link) and each isolated zero has a **single value**. For example, the analytic function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$ corresponding to the **critical line**, is well known to have isolated zeros and each isolated zero has a **single value**. Hence we can expect the analytic function $\lim_{t_0\to\infty} G_s(\omega)$ to have **isolated zeros** and each isolated zero to have a **single value**. Hence $\lim_{t_0\to\infty} \omega_2(t_0)$ is **not** a pathological function with multiple values or ill-defined, but $\lim_{t_0\to\infty} \omega_2(t_0) = \omega_z$ is a well defined constant.

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[4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152

[5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)

[6] Mathworld article on Hurwitz Zeta functions. (Link)

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ (link). This is re-derived in Appendix B.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \le |\sigma| < \frac{1}{2}$ is real.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$$
$$E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix D.1**, we show that $\int_{-\infty}^{\infty}|E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \le |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \le y \le 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \le t \le \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$. **Alternate** derivation is in Appendix B.1.

Appendix B. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^\infty y^{\frac{s}{2}-1}e^{-y}dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{B.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{B.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^\infty |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}w(x)dx \tag{B.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. B.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty \frac{x^{-(\frac{s}{2}-1)}}{x^2}\frac{(1+2w(x))\sqrt{x}-1)}{2}dx \tag{B.4}$$

Hence we can simplify Eq. B.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{s}{2}-1}w(x)dx + \int_1^\infty x^{\frac{-(s+1)}{2}}w(x)dx \tag{B.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{B.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix B.1. ***Derivation of*** $E_p(t)$ ***and*** $E_0(t)$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. B.6 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{B.7}$$

We can substitute $t=-t$ in the first term in above integral and simplify above equation as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt$$
$$+\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt] \tag{B.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{B.9}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$E_p(t) = \frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2}$$
$$A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$$
$$\frac{dA(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t)$$
$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t)$$
$$+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)] \tag{B.10}$$

We can simplify above equation as follows.

$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t}-6\pi n^2 e^{-2t}-4\sigma\pi n^2 e^{-2t}]u(-t)$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t}-6\pi n^2 e^{2t}+4\sigma\pi n^2 e^{2t}]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)] \tag{B.11}$$

We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x>0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x=1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$ (Appendix B.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4}+\sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$

$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \tag{B.12}$$

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{B.13}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{B.14}$$

Appendix B.2. ***Derivation of*** $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}} \tag{B.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2})$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2} \tag{B.16}$$

Appendix C. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix C.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{C.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'
\end{aligned} \tag{C.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega - \omega')$ in Eq. C.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{C.3}$$

Appendix C.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned} \tag{C.4}$$

Appendix C.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega)
\end{aligned}
\tag{C.5}
$$

Appendix C.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega)
\end{aligned}
\tag{C.6}
$$

Appendix D. Properties of Fourier Transforms Part 2

Appendix D.1. $E_p(t), h(t), g(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}}e^{\frac{5t}{2}}e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix D.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix D.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix D.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{D.1}$$

Appendix D.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$, as $|\omega| \to \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \to \infty$, where $A, B > 0$ are real.

Appendix D.4. ***Payley-Weiner theorem and Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix D.3) and hence it should have a fall-off rate **at least** $\frac{1}{t^2}$ as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix E. First 2 derivatives of $R(t_0)$

In this section, we derive the first 2 derivatives of $R(t_0)$. We use the result in Section 3 that $\omega_2(t_0)$ is **at least** differentiable twice.

We expand a few terms in $R(t_0)$ which are analytic functions, using Taylor series as follows. We use $E_0(t) = E_0(-t)$, $E_0(t)e^{-2\sigma t} = [e_0 + e_2\frac{t^2}{!2} + e_4\frac{t^4}{!4} + ...][1 - 2\sigma t + 2\sigma^2 t^2 + ...] = e_0 - 2\sigma e_0 t + t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...$

We use $(f_c(t))_{-\infty} = [\int E_0(t)e^{-2\sigma t}\cos(\omega_2(t_0)t)dt]_{t=-\infty} - K_{1c} = -M(t_0) - K_{1c}$ and $(f_s(t))_{-\infty} = [\int E_0(t)e^{-2\sigma t}\sin(\omega_2(t_0)t)dt]_{t=-\infty} - K_{1s} = -N(t_0) - K_{1s}$ as derived in Appendix E.3 and split each integral in Eq. 17 copied below, into two integrals evaluated at upper and lower limits. $M(t_0), N(t_0)$ are defined in Appendix E.2. Integration constants K_{1c}, K_{1s} get **cancelled** at upper and lower limits of the integrals.

$$\begin{aligned}
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\cos(\omega_2(t_0)t)dt + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\sin(\omega_2(t_0)t)dt]\\
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}[e_0 - 2\sigma e_0 t + t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\cos(\omega_2(t_0)t)dt\\
+\sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}[e_0 - 2\sigma e_0 t + t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\sin(\omega_2(t_0)t)dt]\\
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)[\int(e_0 - 2\sigma e_0 t + t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...)\cos(\omega_2(t_0)t)dt]_{t=t_0}\\
+\sin(\omega_2(t_0)t_0)[\int(e_0 - 2\sigma e_0 t + t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...)\sin(\omega_2(t_0)t)dt]_{t=t_0}]\\
+e^{2\sigma t_0}((M(t_0) + K_{1c})\cos(\omega_2(t_0)t_0) + (N(t_0) + K_{1s})\sin(\omega_2(t_0)t_0))
\end{aligned} \tag{E.1}$$

Using **repeated** integration by parts, for the first two terms t^0, t^1 in the two integrals in above equation, this can be simplified as follows. For the **first** integral $I_1(t_0) = \int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\cos(\omega_2(t_0)t)dt$, we use $u = \cos(\omega_2(t_0)t), dv = t^r dt, v = \frac{t^{(r+1)}}{(r+1)}, du = -\omega_2(t_0)\sin(\omega_2(t_0)t)dt$ for $r = 0, 1$. For the **second** integral $Q_1(t_0) = \int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\sin(\omega_2(t_0)t)dt$, we use $u = \sin(\omega_2(t_0)t), dv = t^r dt, v = \frac{t^{(r+1)}}{(r+1)}, du = \omega_2(t_0)\cos(\omega_2(t_0)t)dt$ for $r = 0, 1$.

$$\begin{aligned}
I_1(t_0) = \int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\cos(\omega_2(t_0)t)dt = e_0[(t_0\cos(\omega_2(t_0)t_0) + \frac{t_0^2}{!2}\sin(\omega_2(t_0)t_0)\omega_2(t_0) + ...)]\\
-2\sigma e_0[(\frac{t_0^2}{2}\cos(\omega_2(t_0)t_0) + \frac{t_0^3}{!3}\sin(\omega_2(t_0)t_0)\omega_2(t_0) + ...)\\
+\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\cos(\omega_2(t_0)t)dt]_{t=t_0}] + (M(t_0) + K_{1c})\\
Q_1(t_0) = \int_{-\infty}^{t_0} E_0(t)e^{-2\sigma t}\sin(\omega_2(t_0)t)dt = e_0[(t_0\sin(\omega_2(t_0)t_0) - \frac{t_0^2}{!2}\cos(\omega_2(t_0)t_0)\omega_2(t_0) + ...)]\\
-2\sigma e_0[(\frac{t_0^2}{2}\sin(\omega_2(t_0)t_0) - \frac{t_0^3}{!3}\cos(\omega_2(t_0)t_0)\omega_2(t_0) + ...)]\\
+\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\sin(\omega_2(t_0)t)dt]_{t=t_0}] + (N(t_0) + K_{1s})
\end{aligned} \tag{E.2}$$

We can simplify $R(t_0)$ in eq. E.1 as follows.

$$
\begin{aligned}
R(t_0) = e^{2\sigma t_0}[e_0[&\cos(\omega_2(t_0)t_0)(t_0\cos(\omega_2(t_0)t_0) + \frac{t_0^2}{!2}\sin(\omega_2(t_0)t_0)\omega_2(t_0) + ...) \\
&+ \sin(\omega_2(t_0)t_0)(t_0\sin(\omega_2(t_0)t_0) - \frac{t_0^2}{!2}\cos(\omega_2(t_0)t_0)\omega_2(t_0) + ...)] \\
&-2\sigma e_0[\cos(\omega_2(t_0)t_0)(\frac{t_0^2}{2}\cos(\omega_2(t_0)t_0) + \frac{t_0^3}{!3}\sin(\omega_2(t_0)t_0)\omega_2(t_0) + ...) \\
&+ \sin(\omega_2(t_0)t_0)(\frac{t_0^2}{2}\sin(\omega_2(t_0)t_0) - \frac{t_0^3}{!3}\cos(\omega_2(t_0)t_0)\omega_2(t_0) + ...)] \\
&+ \cos(\omega_2(t_0)t_0)[\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\cos(\omega_2(t_0)t)dt]_{t=t_0} \\
&+ \sin(\omega_2(t_0)t_0)[\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\sin(\omega_2(t_0)t)dt]_{t=t_0}] \\
+e^{2\sigma t_0}[(K_{1c}\cos(\omega_2(t_0)t_0) + K_{1s}\sin(\omega_2(t_0)t_0)] &+ e^{2\sigma t_0}[(M(t_0)\cos(\omega_2(t_0)t_0) + N(t_0)\sin(\omega_2(t_0)t_0)]
\end{aligned}
\tag{E.3}
$$

This can be further simplified as follows by cancelling common terms in the term involving e_0 and $2\sigma e_0$. Using $e^{2\sigma t_0} = 1 + 2\sigma t_0 + 2\sigma^2 t_0^2 + ... = \sum_{k=0}^{\infty}(2\sigma)^k \frac{t_0^k}{!k}$, we get

$$
\begin{aligned}
R(t_0) = (1 + 2\sigma t_0 + 2\sigma^2 t_0^2 + ...)[e_0[t_0 + \frac{t_0^3}{!3}\omega_2^2(t_0) + ...] &- 2\sigma e_0[\frac{t_0^2}{!2} + \frac{t_0^4}{!4}\omega_2^2(t_0) + ...] \\
&+ \cos(\omega_2(t_0)t_0)[\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\cos(\omega_2(t_0)t)dt]_{t=t_0} \\
&+ \sin(\omega_2(t_0)t_0)[\int[t^2(\frac{e_2}{!2} + 2e_0\sigma^2) + ...]\sin(\omega_2(t_0)t)dt]_{t=t_0}] \\
+e^{2\sigma t_0}[(K_{1c}\cos(\omega_2(t_0)t_0) + K_{1s}\sin(\omega_2(t_0)t_0)] &+ e^{2\sigma t_0}[(M(t_0)\cos(\omega_2(t_0)t_0) + N(t_0)\sin(\omega_2(t_0)t_0)]
\end{aligned}
\tag{E.4}
$$

Integration constants K_{1c}, K_{1s} get **cancelled** at upper and lower limits of the integrals. The terms inside the integrals in above equation can be shown to have terms of the order of t_0^3 and above. Hence we can write as follows, where a_k are the coefficients of the terms $\frac{t_0^k}{!k}$.

$$
\begin{aligned}
R(t_0) &= (1 + 2\sigma t_0 + 2\sigma^2 t_0^2 + ...)[(e_0 t_0 - 2\sigma e_0\frac{t_0^2}{2} + a_3\frac{t_0^3}{3} + a_4\frac{t_0^4}{4} + ...)] \\
&\quad e^{2\sigma t_0}[(M(t_0)\cos(\omega_2(t_0)t_0) + N(t_0)\sin(\omega_2(t_0)t_0)] \\
R(t_0) &= (1 + 2\sigma t_0 + 2\sigma^2 t_0^2 + ...)[(e_0 t_0 - \sigma e_0 t_0^2 + a_3\frac{t_0^3}{3} + a_4\frac{t_0^4}{4} + ...)] \\
&\quad +e^{2\sigma t_0}[(M(t_0)\cos(\omega_2(t_0)t_0) + N(t_0)\sin(\omega_2(t_0)t_0)] \\
R(t_0) &= (e_0 t_0 + t_0^2(-\sigma e_0 + 2\sigma e_0) + t_0^3() + ...) + e^{2\sigma t_0}[(M(t_0)\cos(\omega_2(t_0)t_0) + N(t_0)\sin(\omega_2(t_0)t_0)]
\end{aligned}
\tag{E.5}
$$

We want to evaluate the first and second derivative of $R(t_0)$ in section below.

Appendix E.1. ***Computation of first two derivatives of*** $M(t_0), N(t_0)$*:*

Define $\theta(t_0) = \omega_2(t_0)t_0$, we have $\frac{d\theta(t_0)}{dt_0} = t_0\frac{d\omega_2(t_0)}{dt_0} + \omega_2(t_0)$ which equals ω_{20} at $t_0 = 0$. $\frac{d^2\theta(t_0)}{dt_0^2} = t_0\frac{d^2\omega_2(t_0)}{dt_0^2} + 2\frac{d\omega_2(t_0)}{dt_0}$ which equals zero at $t_0 = 0$, given that $\omega_2(t_0)$ is an even function of t_0. We substitute $(\frac{dM(t_0)}{dt_0})_{t_0=0} = 0$ and $(\frac{dN(t_0)}{dt_0})_{t_0=0} = 0$ from Eq. E.10 and Eq. E.11 in Eq. E.6. We can write Eq. E.5 as follows.

$$\begin{aligned}
R(t_0) &= (e_0 t_0 + t_0^2(\sigma e_0) + t_0^3() + ...) + MN(t_0) \\
MN(t_0) &= e^{2\sigma t_0}(M(t_0)\cos(\theta(t_0)) + N(t_0)\sin(\theta(t_0))) \\
MN(0) &= m_0 \\
\frac{dMN(t_0)}{dt_0} &= e^{2\sigma t_0}[\cos(\theta(t_0))[2\sigma M(t_0) + \frac{dM(t_0)}{dt_0} + N(t_0)\frac{d\theta(t_0)}{dt_0}] + \sin(\theta(t_0))[2\sigma N(t_0) + \frac{dN(t_0)}{dt_0} - M(t_0)\frac{d\theta(t_0)}{dt_0}]] \\
(\frac{dMN(t_0)}{dt_0})_{t_0=0} &= 2\sigma M(0) + (\frac{dM(t_0)}{dt_0})_{t_0=0} + N(0)\omega_{20} = 2\sigma m_0 + n_0\omega_{20}
\end{aligned} \tag{E.6}$$

Now we compute the second derivative as follows. We use $m_2 = \frac{1}{2}(\frac{d^2M(t_0)}{dt_0^2})_{t_0=0}$.

$$\begin{aligned}
\frac{d^2MN(t_0)}{dt_0^2} &= e^{2\sigma t_0}[\cos(\theta(t_0))[2\sigma(2\sigma M(t_0) + \frac{dM(t_0)}{dt_0} + N(t_0)\frac{d\theta(t_0)}{dt_0}) + 2\sigma\frac{dM(t_0)}{dt_0} + \frac{d^2M(t_0)}{dt_0^2} + N(t_0)\frac{d^2\theta(t_0)}{dt_0^2} \\
&\quad + \frac{d\theta(t_0)}{dt_0}\frac{dN(t_0)}{dt_0} + \frac{d\theta(t_0)}{dt_0}(2\sigma N(t_0) + \frac{dN(t_0)}{dt_0} - M(t_0)\frac{d\theta(t_0)}{dt_0})] \\
&\quad + \sin(\theta(t_0))[2\sigma(2\sigma N(t_0) + \frac{dN(t_0)}{dt_0} - M(t_0)\frac{d\theta(t_0)}{dt_0}) - \frac{d\theta(t_0)}{dt_0}(2\sigma M(t_0) + \frac{dM(t_0)}{dt_0} + N(t_0)\frac{d\theta(t_0)}{dt_0}) \\
&\quad + 2\sigma\frac{dN(t_0)}{dt_0} + \frac{d^2N(t_0)}{dt_0^2} - M(t_0)\frac{d^2\theta(t_0)}{dt_0^2} - \frac{d\theta(t_0)}{dt_0}\frac{dM(t_0)}{dt_0}]] \\
\frac{1}{2}(\frac{d^2MN(t_0)}{dt_0^2})_{t_0=0} &= \sigma(2\sigma m_0 + n_0\omega_{20}) + m_2 + \frac{1}{2}\omega_{20}(2\sigma n_0 - m_0\omega_{20}) \\
\frac{1}{2}(\frac{d^2MN(t_0)}{dt_0^2})_{t_0=0} &= m_2 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - \frac{m_0}{2}\omega_{20}^2
\end{aligned} \tag{E.7}$$

We substitute above result in Eq. E.5 and derive as follows.

$$\begin{aligned}
R(t_0) &= (e_0 t_0 + t_0^2(\sigma e_0) + t_0^3() + ...) + MN(t_0) \\
R(0) &= MN(0) = m_0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= e_0 + (\frac{dMN(t_0)}{dt_0})_{t_0=0} = e_0 + 2\sigma m_0 + n_0\omega_{20} \\
\frac{1}{2}(\frac{d^2R(t_0)}{dt_0^2})_{t_0=0} = \sigma e_0 + \frac{1}{2}(\frac{d^2MN(t_0)}{dt_0^2})_{t_0=0} &= \sigma e_0 + m_2 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - \frac{m_0}{2}\omega_{20}^2
\end{aligned} \tag{E.8}$$

We can simplify as follows and get the result in Eq. 26.

$$\begin{aligned}
[R(t_0)]_{t_0=0} &= m_0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= e_0 + n_0\omega_{20} + 2\sigma m_0 \\
(\frac{d^2R(t_0)}{dt_0^2})_{t_0=0} &= m_2 + \sigma e_0 + 2\sigma n_0\omega_{20} + 2\sigma^2 m_0 - m_0\frac{\omega_{20}^2}{2}
\end{aligned} \tag{E.9}$$

Appendix E.2. ***Computation of*** $m_0, m_1, m_2, n_0, n_1, n_2$

In Section 3.1, we see that $f(t) = e^{-\sigma t_0} E_p(t - t_0) + e^{\sigma t_0} E_p(t + t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_2(t_0)$ is an **even** function of variable t_0. Hence $\frac{d\omega_2(t_0)}{dt_0}$ is an **odd** function of variable t_0. We define the first 2 derivatives of $\omega_2(t_0)$ as $\omega_2(0) = \omega_{20}$ and $[\frac{d\omega_2(t_0)}{dt_0}]_{t_0=0} = \omega_{21} = 0$ and $[\frac{d^2\omega_2(t_0)}{dt_0^2}]_{t_0=0} = 2\omega_{22}$.

We can compute $m_0, m_1, m_2, n_0, n_1, n_2$ as follows. We define $[M(t_0)]_{t_0=0} = m_0$, $[\frac{dM(t_0)}{dt_0}]_{t_0=0} = m_1$, $[\frac{d^2M(t_0)}{dt_0^2}]_{t_0=0} = 2m_2$ and $[N(t_0)]_{t_0=0} = n_0$, $[\frac{dN(t_0)}{dt_0}]_{t_0=0} = n_1$, $[\frac{d^2N(t_0)}{dt_0^2}]_{t_0=0} = 2n_2$. Define $\theta(t_0) = \omega_2(t_0)\tau$, we have $\frac{d\theta(t_0)}{dt_0} = \tau\frac{d\omega_2(t_0)}{dt_0}$ and equals $\omega_{21}\tau = 0$ at $t_0 = 0$. $\frac{d^2\theta(t_0)}{dt_0^2} = \tau\frac{d^2\omega(t_0)}{dt_0^2}$ and equals $2\omega_{22}\tau$ at $t_0 = 0$.

$$\begin{aligned}
M(t_0) &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau \\
m_0 &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau \\
\frac{dM(t_0)}{dt_0} &= -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)\frac{d\theta(t_0)}{dt_0}d\tau = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau \\
m_1 &= (\frac{dM(t_0)}{dt_0})_{t_0=0} = -\omega_{21}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau = 0 \\
\frac{d^2M(t_0)}{dt_0^2} &= -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau))\frac{d^2\theta(t_0)}{dt_0^2}d\tau - \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))(\frac{d\theta(t_0)}{dt_0})^2 d\tau \\
m_2 &= \frac{1}{2}(\frac{d^2M(t_0)}{dt_0^2})_{t_0=0} = -\omega_{22}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau))d\tau
\end{aligned} \tag{E.10}$$

Similarly, we can compute n_0, n_1, n_2 as follows.

$$\begin{aligned}
N(t_0) &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau \\
n_0 &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau \\
\frac{dN(t_0)}{dt_0} &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)\frac{d\theta(t_0)}{dt_0}d\tau = \frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau \\
n_1 &= (\frac{dN(t_0)}{dt_0})_{t_0=0} = \omega_{21}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau = 0 \\
\frac{d^2N(t_0)}{dt_0^2} &= \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))\frac{d^2\theta(t_0)}{dt_0^2}d\tau - \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau))(\frac{d\theta(t_0)}{dt_0})^2 d\tau \\
n_2 &= \frac{1}{2}(\frac{d^2N(t_0)}{dt_0^2})_{t_0=0} = \omega_{22}\int_{-\infty}^{0} \tau E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau))d\tau
\end{aligned} \tag{E.11}$$

Appendix E.3. ***Derivation of*** $f_c(t), f_s(t)$ ***at*** $t = -\infty$

In this section, we compare $(f_c(t))_{-\infty} = [\int E_0(t)e^{-2\sigma t}\cos(\omega_2(t_0)t)dt]_{t=-\infty} - K_{1c}$ and
$f_s(t) = [\int E_0(t)e^{-2\sigma t}\sin(\omega_2(t_0)t)dt]_{t=-\infty} - K_{1s}$ in para 3 of Appendix E with corresponding version $f_{c0}(t), f_{s0}(t)$ using Taylor series representation of $E_0(t)$ in Eq. 1.2 as follows and obtain the values of $f_c(t), f_s(t)$ at $t = -\infty$. We use the fact that $[f_{c0}(t)]_{-\infty} = [f_{s0}(t)]_{-\infty} = 0$. We copy $f_c(t), f_s(t)$ from Eq. E.2.

$$
\begin{aligned}
f_{c0}(t) &= \sum_{n,k,r,p} c_{nkrp}\frac{e^{(b_{krp}-2\sigma)t}}{(b_{krp}^2+\omega_2^2(t_0))}[(b_{krp}-2\sigma)\cos(\omega_2(t_0)t)+\omega_2(t_0)\sin(\omega_2(t)t)] \\
f_c(t) &= e_0(t\cos(\omega_2(t_0)t)+\frac{t^2}{!2}\sin(\omega_2(t_0)t)\omega_2(t_0))-2\sigma e_0((\frac{t^2}{2}\cos(\omega_2(t_0)t))+\frac{t^3}{3}()+... \\
K_{1c}(t_0)+f_c(t) &= K_{0c}(t_0)+f_{c0}(t) \\
(f_c(t))_{-\infty} &= [f_{c0}(t)]_{-\infty}+K_{0c}(t_0)-K_{1c}(t_0) = K_{0c}(t_0)-K_{1c}(t_0)
\end{aligned}
\tag{E.12}
$$

Similarly, we get

$$
\begin{aligned}
f_{s0}(t) &= \sum_{n,k,r,p} c_{nkrp}\frac{e^{(b_{krp}-2\sigma)t}}{(b_{krp}^2+\omega_2^2(t_0))}[(b_{krp}-2\sigma)\sin(\omega_2(t_0)t)-\omega_2(t_0)\cos(\omega_2(t_0)t)] \\
f_s(t) &= e_0(t\sin(\omega_2(t_0)t)-\frac{t^2}{!2}\cos(\omega_2(t_0)t)\omega_2(t_0))-2\sigma e_0((\frac{t^2}{2}\sin(\omega_2(t_0)t))+\frac{t^3}{3}()+... \\
K_{1s}(t_0)+f_s(t) &= K_{0s}(t_0)+f_{s0}(t) \\
(f_s(t))_{-\infty} &= [f_{s0}(t)]_{-\infty}+K_{0s}(t_0)-K_{1s}(t_0) = K_{0s}(t_0)-K_{1s}(t_0)
\end{aligned}
\tag{E.13}
$$

We can evaluate integration constants $K_{0c}(t_0), K_{0s}(t_0)$, $K_{1c}(t_0), K_{1s}(t_0)$ by comparing above equations for $f_{c0}(t)$ and $f_c(t)$, at $t=0$ and similarly for $f_{s0}(t)$ and $f_s(t)$, at $t=0$. We see that $(f_c(t))_{t=0} = (f_s(t))_{t=0} = 0$.

$$
\begin{aligned}
(f_c(t))_{-\infty} = K_{0c}(t_0)-K_{1c}(t_0) = (f_c(t))_{t=0}-(f_{c0}(t))_{t=0} = -(f_{c0}(t))_{t=0} &= -\sum_{n,k,r,p} c_{nkrp}\frac{(b_{krp}-2\sigma)}{((b_{krp}-2\sigma^2)+\omega_2^2(t_0))} \\
&= -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau \\
(f_s(t))_{-\infty} = K_{0s}(t_0)-K_{1s}(t_0) = (f_s(t))_{t=0}-(f_{s0}(t))_{t=0} = -(f_{s0}(t))_{t=0} &= \sum_{n,k,r,p} c_{nkrp}\frac{\omega_2(t_0)}{((b_{krp}-2\sigma)^2+\omega_2^2(t_0))} \\
&= -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau \\
(f_c(t))_{-\infty} = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &= -M(t_0) \\
(f_s(t))_{-\infty} = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau &= -N(t_0)
\end{aligned}
\tag{E.14}
$$

Appendix F. On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t)+e^{-\sigma t}u(t)$.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = E_p(t)$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = E_{p\omega}(\omega)$ as per convolution theorem (link), where $*$ denotes **convolution** operation given by $E_{p\omega}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix C.1.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ where ω_1 is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_1 \neq 0$, **then** the **real part** of $E_{p\omega}(\omega)$ given by $E_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t)+g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, where $g(t)h(t) = E_p(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its real part given by $E_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_1 \neq 0$ and show that $E_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$E_R(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega')H(\omega-\omega')d\omega' \tag{F.1}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix D.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. F.1 and we get

$$E_R(\omega) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} G_R(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' \tag{F.2}$$

We can split the integral in Eq. F.2 as follows.

$$E_R(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_R(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' + \int_{0}^{\infty} G_R(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega']$$

(F.3)

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix C.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. F.3 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_R(\omega) = \frac{\sigma}{\pi}\int_0^\infty G_R(\omega')[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}]d\omega' \tag{F.4}$$

In Appendix D.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. **??** is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $E_R(\omega) > 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $E_R(\omega) < 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

Appendix G. On the zeros of a related function $G(\omega)$ Full version

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

Let us consider an even function $g(t) = f(t)e^{-\sigma t}u(-t) + f(-t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t, $f(t) = [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]$ and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)$ as shown in Section 2.1.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix D.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega)$ as per convolution theorem (link), where $*$ denotes **convolution** operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = H_R(\omega) + iH_I(\omega) = [\frac{\sigma}{(\sigma^2+\omega^2)} + \frac{3\sigma}{(9\sigma^2+\omega^2)}] + i\omega[\frac{1}{(\sigma^2-\omega^2)} - \frac{1}{(9\sigma^2+\omega^2)}]$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix C.1.

If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ for every value of t_0, where $\omega_2(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H_R(\omega) = \frac{\sigma}{(\sigma^2+\omega^2)} + \frac{3\sigma}{(9\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_2(t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H_R(\omega)]$, obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g(t)$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$, for every value of t_0, where $\omega_2(t_0)$ is real and finite, where $g(t)h(t) = f(t)$, $f(t) = [e^{-\sigma t_0}E_p(t-t_0) + e^{\sigma t_0}E_p(t+t_0)]$ and $h(t) = e^{\sigma t}u(-t) + e^{-3\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its real part given by $E_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_2(t_0) \neq 0$ and show that $E_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H_R(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$E_R(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega')H_R(\omega-\omega')d\omega' \tag{G.1}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H_R(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix D.2)

We substitute $H_R(\omega) = \frac{\sigma}{(\sigma^2+\omega^2)} + \frac{3\sigma}{(9\sigma^2+\omega^2)}$ in Eq. G.1 and we get

$$E_R(\omega) = \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} G_R(\omega')[\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{3}{(9\sigma^2+(\omega-\omega')^2)}]d\omega' \tag{G.2}$$

We can split the integral in Eq. G.2 as follows.

$$E_R(\omega) = \frac{\sigma}{2\pi}[\int_{-\infty}^{0} G_R(\omega')[\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{3}{(9\sigma^2+(\omega-\omega')^2)}]d\omega'$$
$$+ \int_{0}^{\infty} G_R(\omega')[\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{3}{(9\sigma^2+(\omega-\omega')^2)}]d\omega'] \tag{G.3}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix C.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. G.3 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_R(\omega) = \frac{\sigma}{2\pi}\int_{0}^{\infty} G_R(\omega')[(\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)}) + (\frac{3}{(9\sigma^2+(\omega-\omega')^2)} + \frac{3}{(9\sigma^2+(\omega+\omega')^2)})]d\omega' \tag{G.4}$$

In Appendix D.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. G.4 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $[(\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)}) + (\frac{3}{(9\sigma^2+(\omega-\omega')^2)} + \frac{3}{(9\sigma^2+(\omega+\omega')^2)})] > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $E_R(\omega) > 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where $\omega_2(t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $E_R(\omega) < 0$ for all finite $\omega > 0$. We see that $E_R(-\omega) = E_R(\omega)$ because $E_p(t)$ is a real function (Appendix C.2). Hence $E_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where $\omega_2(t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.