On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function. [4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2 to Section 6, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 7, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s=\frac{1}{2}$.

We present an **outline** of the new method below.

1.1. ***Step 1: Inverse Fourier Transform of*** $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where ω is real. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3] (Titchmarsh pp254-255) This is re-derived in Appendix D. We take the term $e^{\frac{t}{2}}$ out of the bracket and rearrange the terms as follows.

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=\sum_{n=1}^{\infty}[4\pi^2n^4e^{4t}-6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.(Details in Appendix B.9)

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t)=E_0(t)e^{-\sigma t}=\sum_{n=1}^{\infty}[4\pi^2n^4e^{4t}-6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function for real t, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. ***Step 2: On the zeros of a related function*** $G(\omega,t_2,t_0)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider a new function $g(t,t_2,t_0)=f(t,t_2,t_0)e^{-\sigma t}u(-t)+f(t,t_2,t_0)e^{\sigma t}u(t)$, where $f(t,t_2,t_0)=e^{-2\sigma t_0}f_1(t,t_2,t_0)+e^{2\sigma t_0}f_2(t,t_2,t_0)$ and $f_1(t,t_2,t_0)=e^{\sigma t_0}E_p'(t+t_0,t_2)$ and $f_2(t,t_2,t_0)=e^{-\sigma t_0}E_p'(t-t_0,t_2)$ and $E_p'(t,t_2)=e^{-\sigma t_2}E_p(t-t_2)-e^{\sigma t_2}E_p(t+t_2)$ and t_0,t_2 are real and $g(t,t_2,t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_0,t_2)$ must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$, for every value of t_0, for a given value of t_2, where $G_R(\omega,t_0,t_2)$ crosses the zero line to the opposite sign, to satisfy Statement 1, where $\omega_z(t_2,t_0)$ is real and finite.

1.3. ***Step 3: On the zeros of the function*** $G_R(\omega,t_0,t_2)$

In Section 2.3, we compute the Fourier transform of the function $g(t,t_2,t_0)$ and compute its real part given by $G_R(\omega,t_2,t_0)$ and we can write as follows.

$$\begin{aligned} G_R(\omega,t_2,t_0) = e^{-2\sigma t_0} &\int_{-\infty}^{0} [E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\ +e^{2\sigma t_0} &\int_{-\infty}^{0} [E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \end{aligned} \tag{3}$$

We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for every value of t_0, for **each non-zero value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2,t_0) \neq \omega_0$. Hence we can see that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = 0$.

1.4. ***Step 4: Zero Crossing function*** $\omega_z(t_2,t_0)$ ***is an even function of variable*** t_0

In Section 2.4, we show the result in Eq. 4 and that $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$. It is shown that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0$ and that $P_{odd}(t_2,t_0)$ is an **odd** function of t_0, for each non-zero value of t_2 as follows.

$$\begin{aligned} P_{odd}(t_2,t_0) = [\cos(\omega_z(t_2,t_0)t_0) &\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau \\ + \sin(\omega_z(t_2,t_0)t_0) &\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau] \\ +e^{2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)\tau)d\tau &+ \sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\sin(\omega_z(t_2,t_0)\tau)d\tau] \end{aligned} \tag{4}$$

1.5. ***Step 5: Final Step***

In Section 4, it is shown that $\omega_z(t_2,t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2,t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use $E_0'(t,t_2) = E_0(t-t_2) - E_0(t+t_2)$ and $E_{0n}'(t,t_2) = E_0'(-t,t_2)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma t_{0c})} - \cosh{(2\sigma\tau)}) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = 0 \tag{5}$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$, where $t_{0c} > 0$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t} d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] (Titchmarsh pp30-31). This is shown in detail in first two paragraphs in Appendix B.1.

2.1. ***New function*** $g(t, t_2, t_0)$

Let us consider the function $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0^{'}(t, t_2)e^{-\sigma t}$, where t_2 is non-zero and real, and $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ (**Definition 1**). Its Fourier transform is given by $E_{p\omega}^{'}(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$, using Statement 1. (**Result 2.1.1**).

Let us consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p^{'}(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0}E_p^{'}(t - t_0, t_2)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega, t_2, t_0) = E_{p\omega}^{'}(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$, using Result 2.1.1. (**Result 2.1.2**)

Let us consider a new function $g(t,t_2,t_0) = g_-(t,t_2,t_0)u(-t) + g_+(t,t_2,t_0)u(t)$ where $g(t,t_2,t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t,t_2,t_0) = f(t,t_2,t_0)e^{-\sigma t}$ and $g_+(t,t_2,t_0) = f(t,t_2,t_0)e^{\sigma t}$. We can see that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can write the above equations as follows.

$$
\begin{aligned}
E_p^{'}(t,t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = (E_0(t-t_2) - E_0(t+t_2))e^{-\sigma t} = E_0^{'}(t,t_2)e^{-\sigma t} \\
f_1(t,t_2,t_0) = e^{\sigma t_0}E_p^{'}(t+t_0,t_2) \\
f_2(t,t_2,t_0) = f_1(t,t_2,-t_0) = e^{-\sigma t_0}E_p^{'}(t-t_0,t_2) \\
f(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0) + e^{2\sigma t_0}f_2(t,t_2,t_0) = e^{-\sigma t_0}E_p^{'}(t+t_0,t_2) + e^{\sigma t_0}E_p^{'}(t-t_0,t_2) \\
g(t,t_2,t_0) = [f(t,t_2,t_0)e^{-\sigma t}]u(-t) + [f(t,t_2,t_0)e^{\sigma t}]u(t) \\
g(t,t_2,t_0)h(t) = f(t,t_2,t_0), \quad h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]
\end{aligned}
\tag{6}
$$

We can show that $E_p(t), E_p^{'}(t,t_2), h(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}^{'}(\omega,t_2), H(\omega)$ are finite for real ω and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). We can show that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions. These results are shown in Appendix B.1.

In Section 2.3 and Section 2.4, it is shown that $g(t,t_2,t_0)$ is a Fourier transformable function and its Fourier transform given by $G(\omega,t_2,t_0) = e^{-2\sigma t_0}G_1(\omega,t_2,t_0) + e^{2\sigma t_0}G_1(\omega,t_2,-t_0)$ converges. (Eq. 13 and Eq. 16)

If we take the Fourier transform of the equation $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, using Result 2.1.2, we get $\frac{1}{2\pi}[G(\omega,t_2,t_0) * H(\omega)] = F(\omega,t_2,t_0) = E_{p\omega}^{'}(\omega,t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega,t_2,t_0) + iF_I(\omega,t_2,t_0)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega,t_2,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega',t_2,t_0)H(\omega-\omega')d\omega'$.

We see that $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ (link). $G(\omega,t_2,t_0) = G_R(\omega,t_0,t_2) + iG_I(\omega,t_2,t_0)$ is the Fourier transform of the function $g(t,t_2,t_0)$. We can write $g(t,t_2,t_0) = g_{even}(t,t_2,t_0) + g_{odd}(t,t_2,t_0)$ where $g_{even}(t,t_2,t_0)$ is an even function and $g_{odd}(t,t_2,t_0)$ is an odd function of variable t.

If Statement 1 is true, then we require the Fourier transform of the function $f(t,t_2,t_0)$ given by $F(\omega,t_2,t_0)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0, for each non-zero value of t_2. This implies that the **real** part of the Fourier transform of the **even function** $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_0,t_2)$(Appendix C.2) must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$ where $\omega_z(t_2,t_0)$ is real and finite, where $G_R(\omega,t_0,t_2)$ crosses the zero line to the opposite sign. We note that $\omega_z(t_2,t_0)$ can be different from ω_0 in general.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega,t_0,t_2)$ does not have at least one zero for some $\omega = \omega_z(t_2,t_0) \neq 0$, where $G_R(\omega,t_0,t_2)$ crosses the zero line to the opposite sign, **then** the **real part** of $F(\omega,t_2,t_0)$ given by $F_R(\omega,t_2,t_0) = \frac{1}{2\pi}[G_R(\omega,t_0,t_2) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega, t_0, t_2)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against Result 2.1.2 and **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_0, t_2)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0, for each non-zero value of t_2, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign and $\omega_z(t_2, t_0)$ is real and finite, where $g(t, t_2, t_0)h(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0, t_2)$, $E_p'(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$, and $h(t) = e^{\sigma t} u(-t) + e^{-\sigma t} u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega, t_2, t_0) = E'_{p\omega}(\omega, t_2)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$, using Result 2.1.2 and its real part given by $F_R(\omega, t_2, t_0)$ also has a zero at the same location $\omega = \omega_0 \neq 0$(**Result 2.1.3**).

Let us consider the case where $G_R(\omega, t_2, t_0)$ **does not** have at least one zero for finite $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign and show that $F_R(\omega, t_2, t_0)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Result 2.1.3 and Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) H(\omega - \omega') d\omega' \tag{7}$$

We can show that the above integral converges for real ω, given that the integrand is absolutely integrable because $G(\omega, t_2, t_0)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t, t_2, t_0)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix B.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 7 and we get

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{8}$$

We can split the integral in Eq. 8 as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{9}$$

We see that $G_R(-\omega, t_2, t_0) = G_R(\omega, t_2, t_0)$ because $g(t, t_2, t_0)$ is a real function of variable t. (Appendix C.1) We can substitute $\omega' = -\omega''$ in the first integral in Eq. 9 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega', t_2, t_0) [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{10}$$

In Appendix B.2, it is shown that $G(\omega', t_2, t_0)$ is finite for real ω' and goes to zero as $|\omega'| \to \infty$. We can see that for $\omega' \to \infty$, the integrand in Eq. 10 is zero. For finite $\omega \geq 0$, and $0 \leq \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$, for $0 < \sigma < \frac{1}{2}$. We see that $G_R(\omega', t_0, t_2)$ is **not** an all zero function of variable ω' (Section 2.2).

- **Case 1:** $G_R(\omega', t_2, t_0) \geq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix C.1) and link). Hence $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line and becomes negative, where $\omega_z(t_2, t_0)$ is real and finite.

- **Case 2:** $G_R(\omega', t_2, t_0) \leq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \geq 0$. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix C.1) and link). Hence $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where it crosses the zero line and becomes positive, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**. We call this **Result 2.1.4**. In the rest of the sections, we consider only the **first** zero crossing away from origin, where $G_R(\omega, t_0, t_2)$ crosses the zero line to the opposite sign. Hence $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2.

2.2. $G_R(\omega', t_0, t_2)$ ***is not an all zero function of variable*** ω'

If $G_R(\omega', t_0, t_2)$ is an all zero function of variable ω', for each given value of t_0, t_2 (**Statement 2**), then $F_R(\omega, t_2, t_0)$ in Eq. 7 is an all zero function of ω for real ω. Hence $2f_{even}(t, t_2, t_0) = f(t, t_2, t_0) + f(-t, t_2, t_0)$ is an **all-zero** function of t, given that the Fourier transform of $f_{even}(t, t_2, t_0)$ is given by $F_R(\omega, t_2, t_0)$, using symmetry properties of Fourier transform(Appendix C.2) and link). Hence $f(t, t_2, t_0)$ is an **odd function** of variable t.(**Result 2.2**).

From Eq. 6 we see that $E_p'(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = [E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t}$.
Hence $f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0, t_2) = [E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t}$ and
$f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0, t_2) = [E_0(t - t_0 - t_2) - E_0(t - t_0 + t_2)]e^{-\sigma t}$.

Hence $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$
$= e^{-2\sigma t_0}[E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t} + e^{2\sigma t_0}[E_0(t - t_0 - t_2) - E_0(t - t_0 + t_2)]e^{-\sigma t}$.

Case 1: For $t_0 \neq 0$ and $t_2 \neq 0$, it is shown that Result 2.2 is false. We will compute $f(t, t_2, t_0)$ at $t = 0$ and show that it does not equal zero.

We see that $f(0,t_2,t_0) = e^{-2\sigma t_0}[E_0(t_0-t_2) - E_0(t_0+t_2)] + e^{2\sigma t_0}[E_0(-t_0-t_2) - E_0(-t_0+t_2)]$ $= -2\sinh(2\sigma t_0)[E_0(t_0-t_2) - E_0(t_0+t_2)]$. We use the fact that $E_0(t) = E_0(-t)$ and hence $E_0(t_0-t_2) = E_0(-t_0+t_2)$ and $E_0(t_0+t_2) = E_0(-t_0-t_2)$(Appendix B.9).

If Result 2.2 is true, then we require $f(0,t_2,t_0) = 0$. For our choice of $0 < \sigma < \frac{1}{2}$ and $t_0 \neq 0$, this implies that $E_0(t_0-t_2) = E_0(t_0+t_2)$. Given that $t_0 \neq 0$ and $t_2 \neq 0$, we set $t_2 = Kt_0$ for real $K \neq 0$ and we get $E_0((1-K)t_0) = E_0((1+K)t_0)$. This is not possible for $t_0 \neq 0$ because $E_0(t_0)$ is **strictly decreasing** for $t_0 > 0$ (Section 6) and $1-K \neq 1+K$ or $1-K \neq -(1+K)$ for $K \neq 0$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_0, t_2)$ is **not** an all zero function of variable ω'.

Case 2: For $t_0 = 0$ and $t_2 \neq 0$, we have $f(t,t_2,t_0) = 2[E_0(t-t_2) - E_0(t+t_2)]e^{-\sigma t}$. We define $D(t) = E_0(t-t_2) - E_0(t+t_2)$ and see that $D(t) + D(-t) = E_0(t-t_2) - E_0(t+t_2) + E_0(-t-t_2) - E_0(-t+t_2)$. Given that $E_0(t) = E_0(-t)$, we have $D(t)+D(-t) = E_0(t-t_2) - E_0(t+t_2) + E_0(t+t_2) - E_0(t-t_2) = 0$ and hence $D(t) = E_0(t-t_2) - E_0(t+t_2)$ is an **odd** function of variable t (**Result 2.2.1**).

If Result 2.2 is true, then we require $f(t,t_2,t_0) = 2D(t)e^{-\sigma t}$ to be an **odd** function of variable t. Using Result 2.2.1, this is possible only for $\sigma = 0$. This is **not** possible for our choice of $0 < \sigma < \frac{1}{2}$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_0, t_2)$ is **not** an all zero function of variable ω'.

Case 3: For $t_2 = 0$ and $|t_0| < \infty$, we have $E'_p(t,t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = 0$ and $f(t,t_2,t_0) = g(t,t_2,t_0) = 0$ for all t and Lemma 1 is not applicable for this case.

2.3. ***On the zeros of a related function*** $G(\omega, t_2, t_0)$

We can compute the fourier transform of the function $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_2,t_0)$(Appendix C.2). We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for **every value** of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Result 2.1.4 in Section 2.1. In general, $\omega_z(t_2,t_0) \neq \omega_0$.

We **define** $g_1(t,t_2,t_0) = f_1(t,t_2,t_0)e^{-\sigma t}u(-t) + f_1(t,t_2,t_0)e^{\sigma t}u(t) = e^{\sigma t_0}E'_p(t+t_0,t_2)e^{-\sigma t}u(-t) + e^{\sigma t_0}E'_p(t+t_0,t_2)e^{\sigma t}u(t)$, using Eq. 6 (**Definition 3**). First we compute the Fourier transform of the function $g_1(t,t_2,t_0)$ given by $G_1(\omega,t_2,t_0) = G_{1R}(\omega,t_2,t_0) + iG_{1I}(\omega,t_2,t_0)$.

$$\begin{aligned} G_1(\omega,t_2,t_0) = \int_{-\infty}^{\infty} g_1(t,t_2,t_0)e^{-i\omega t}dt = \int_{-\infty}^{0} g_1(t,t_2,t_0)e^{-i\omega t}dt + \int_{0}^{\infty} g_1(t,t_2,t_0)e^{-i\omega t}dt \\ G_1(\omega,t_2,t_0) = \int_{-\infty}^{0} e^{\sigma t_0}E'_p(t+t_0,t_2)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0}E'_p(t+t_0,t_2)e^{\sigma t}e^{-i\omega t}dt \end{aligned} \tag{11}$$

We use $E'_p(t,t_2) = E'_0(t,t_2)e^{-\sigma t}$ from Eq. 6, where $E'_0(t,t_2) = E_0(t-t_2) - E_0(t+t_2)$, using Definition 1 in Section 2.1 and we get $E'_p(t+t_0,t_2) = E'_0(t+t_0,t_2)e^{-\sigma t}e^{-\sigma t_0}$ and write Eq. 11 as follows. Substituting $t = -t$ in the second integral in first line of Eq. 12, we get

$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{0}^{\infty}E_0^{'}(t+t_0,t_2)e^{-i\omega t}dt$$
$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{-\infty}^{0}E_0^{'}(-t+t_0,t_2)e^{i\omega t}dt \tag{12}$$

We define $E_{0n}^{'}(t,t_2)=E_0^{'}(-t,t_2)$ (**Definition 2**) and get $E_0^{'}(-t+t_0,t_2)=E_{0n}^{'}(t-t_0,t_2)$ and write Eq. 12 as follows. The integral in Eq. 13 converges, given that $E_0(t)e^{-2\sigma t}$ is an absolutely **integrable** function (Appendix B.1) and its t_0,t_2 shifted versions are absolutely **integrable**.

$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{-\infty}^{0}E_{0n}^{'}(t-t_0,t_2)e^{i\omega t}dt=G_{1R}(\omega,t_2,t_0)+iG_{1I}(\omega,t_2,t_0) \tag{13}$$

The above equations can be expanded as follows using the identity $e^{i\omega t}=\cos(\omega t)+i\sin(\omega t)$. Comparing the **real parts** of $G_1(\omega,t_2,t_0)$, we have

$$G_{1R}(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}\cos(\omega t)dt+\int_{-\infty}^{0}E_{0n}^{'}(t-t_0,t_2)\cos(\omega t)dt \tag{14}$$

2.4. ***Zero crossing function*** $\omega_z(t_2,t_0)$ ***is an even function of variable*** t_0***, for a given*** t_2

Now we consider Eq. 6 and the function $f(t,t_2,t_0)=e^{-2\sigma t_0}f_1(t,t_2,t_0)+e^{2\sigma t_0}f_2(t,t_2,t_0)=e^{-\sigma t_0}E_p^{'}(t+t_0,t_2)+e^{\sigma t_0}E_p^{'}(t-t_0,t_2)$ where $f_1(t,t_2,t_0)=e^{\sigma t_0}E_p^{'}(t+t_0,t_2)$ and $f_2(t,t_2,t_0)=f_1(t,t_2,-t_0)=e^{-\sigma t_0}E_p^{'}(t-t_0,t_2)$ and $g(t,t_2,t_0)h(t)=f(t,t_2,t_0)$ where $g(t,t_2,t_0)=f(t,t_2,t_0)e^{-\sigma t}u(-t)+f(t,t_2,t_0)e^{\sigma t}u(t)$ and $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$. We can write the above equations and $g_1(t,t_2,t_0)$ from Definition 3 in Section 2.3, as follows. We define $g_2(t,t_2,t_0)$ below and write $g(t,t_2,t_0)$ as follows.

$$\begin{aligned}g_1(t,t_2,t_0)&=f_1(t,t_2,t_0)e^{-\sigma t}u(-t)+f_1(t,t_2,t_0)e^{\sigma t}u(t),\quad g_1(t,t_2,t_0)h(t)=f_1(t,t_2,t_0)\\ g_2(t,t_2,t_0)&=f_2(t,t_2,t_0)e^{-\sigma t}u(-t)+f_2(t,t_2,t_0)e^{\sigma t}u(t),\quad g_2(t,t_2,t_0)h(t)=f_2(t,t_2,t_0)\\ g(t,t_2,t_0)&=e^{-2\sigma t_0}g_1(t,t_2,t_0)+e^{2\sigma t_0}g_2(t,t_2,t_0)\end{aligned} \tag{15}$$

We compute the Fourier transform of the function $g(t,t_2,t_0)$ and compute its real part $G_R(\omega,t_2,t_0)$ using the procedure in Section 2.3, similar to Eq. 14 and we can write as follows. We use $G_{2R}(\omega,t_2,t_0)=G_{1R}(\omega,t_2,-t_0)$ given that $f_2(t,t_2,t_0)=f_1(t,t_2,-t_0)$ and $g_2(t,t_2,t_0)=g_1(t,t_2,-t_0)$. We substitute $t=\tau$ in the equation for $G_{1R}(\omega,t_2,t_0)$ below, copied from Eq. 14.

$$G_R(\omega,t_2,t_0)=e^{-2\sigma t_0}G_{1R}(\omega,t_2,t_0)+e^{2\sigma t_0}G_{2R}(\omega,t_2,t_0)=e^{-2\sigma t_0}G_{1R}(\omega,t_2,t_0)+e^{2\sigma t_0}G_{1R}(\omega,t_2,-t_0)$$
$$G_{1R}(\omega,t_2,t_0)=\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau$$
$$\begin{aligned}G_R(\omega,t_2,t_0)=e^{-2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau\\ +e^{2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau\end{aligned}$$

(16)

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Lemma 1 in Section 2.1. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = 0$ and we can rearrange the terms in Eq. 16 as follows.

$$P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau$$
$$+ \int_{-\infty}^{0} [e^{2\sigma t_0} E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + e^{-2\sigma t_0} E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0$$

(17)

We use the fact that $f(t, t_2, t_0) = e^{-\sigma t_0} E_p^{'}(t + t_0, t_2) + e^{\sigma t_0} E_p^{'}(t - t_0, t_2) = f(t, t_2, -t_0)$ in Eq. 6, is **unchanged** by the substitution $t_0 = -t_0$. **If** $f(t, t_2, t_0) = f(t, t_2, -t_0)$ is unchanged by the substitution $t_0 = -t_0$, **then** $g(t, t_2, t_0) = g(t, t_2, -t_0)$ is unchanged by the substitution $t_0 = -t_0$, using the fact that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ and $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$.

Hence $G_R(\omega, t_2, t_0) = G_R(\omega, t_2, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and the zero crossing in $G_R(\omega, t_2, -t_0)$ given by $\omega_z(t_2, -t_0)$ is the **same** as the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ and we get $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0, for each non-zero value of t_2.

We can write Eq. 17 as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0, for each non-zero value of t_2. We use $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$.

$$P(t_2, t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$$
$$P_{odd}(t_2, t_0) = \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau$$

(18)

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 18 as follows, using the substitution $\tau + t_0 = \tau'$. We get $\tau = \tau' - t_0$ and $d\tau = d\tau'$ and substitute back $\tau' = \tau$.

$$P_{odd}(t_2, t_0) = \int_{-\infty}^{t_0} [e^{-2\sigma t_0} E_0^{'}(\tau^{'}, t_2) e^{-2\sigma\tau^{'}} e^{2\sigma t_0} + e^{2\sigma t_0} E_{0n}^{'}(\tau^{'}, t_2)] \cos(\omega_z(t_2, t_0)(\tau^{'} - t_0) d\tau$$
$$P_{odd}(t_2, t_0) = [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau$$
$$+ \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau]$$
$$+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau]$$

$$(19)$$

In Section 2.1, it is shown that $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2. In this section, we consider $t_0 > 0$ and $t_2 > 0$ only.

In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given that $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2 and given that t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function and is positive for $t_0 > 0$ because $0 < \omega_z(t_2, t_0) < \infty$.

We see that $\omega_z(t_2, t_0) > 0$ and is a **continuous** function of variable t_0 and t_2, as t_0 and t_2 increase to a larger and larger finite value without bounds and that the order of $\omega_z(t_2, t_0)t_0$ is greater than 1 (Section 5). As t_0 and t_2 increase from zero to a larger and larger finite value without bounds, the continuous function $\omega_z(t_2, t_0)t_0$ starts from zero and increases with order greater than $O[1]$ and will pass through $\frac{\pi}{2}$.

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 19 as follows. We use the fact that $\cos(\omega_z(t_{2c}, t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c})t_{0c}) = 1$ and $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.4.

$$P_{odd}(t_{2c}, t_{0c}) = \int_{-\infty}^{t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$P_{odd}(t_{2c}, -t_{0c}) = -\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \tag{20}$$

We compute $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ in Eq. 18, at $t_0 = t_{0c}$ and $t_2 = t_{2c}$ using Eq. 20.

$$\int_{-\infty}^{t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0 \tag{21}$$

We split the first two integrals in the left hand side of Eq. 21 using $\int_{-\infty}^{t_{0c}} = \int_{-\infty}^{-t_{0c}} + \int_{-t_{0c}}^{t_{0c}}$ as follows.

$$[\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau]$$
$$+e^{2\sigma t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau]$$
$$-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0$$

$$(22)$$

We cancel the common integral $\int_{-\infty}^{-t_{0c}} E_0'(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$ in Eq. 22 and rearrange the terms as follows, using $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned}\int_{-t_{0c}}^{t_{0c}} E_0'(\tau, t_{2c})e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E_{0n}'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\ = -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau\end{aligned} \tag{23}$$

We can combine the integrals in the left hand side of in Eq. 23 as follows.

$$\begin{aligned}\int_{-t_{0c}}^{t_{0c}} [E_0'(\tau, t_{2c})e^{-2\sigma\tau} + E_{0n}'(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\ = -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau\end{aligned} \tag{24}$$

We denote the right hand side of Eq. 24 as RHS. We can split the integral in Eq. 24 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\begin{aligned}\int_{-t_{0c}}^{0} [E_0'(\tau, t_{2c})e^{-2\sigma\tau} + E_{0n}'(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\ + \int_{0}^{t_{0c}} [E_0'(\tau, t_{2c})e^{-2\sigma\tau} + E_{0n}'(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = RHS\end{aligned} \tag{25}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 25 as follows. We use $E_0'(-\tau, t_{2c}) = E_{0n}'(\tau, t_{2c})$ and $E_{0n}'(-\tau, t_{2c}) = E_0'(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$\begin{aligned}\int_{t_{0c}}^{0} [E_{0n}'(\tau, t_{2c})e^{2\sigma\tau} + E_0'(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\ + \int_{0}^{t_{0c}} [E_0'(\tau, t_{2c})e^{-2\sigma\tau} + E_{0n}'(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = RHS\end{aligned} \tag{26}$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify Eq. 26 as follows.

$$\int_{0}^{t_{0c}} [E_0'(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E_{0n}'(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = RHS \tag{27}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 24 as follows. We use $E_{0n}^{'}(-\tau, t_{2c}) = E_0^{'}(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$RHS = 2\sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{28}$$

We split the integral on the right hand side in Eq. 28 using $\int_{t_{0c}}^{\infty} = \int_0^{\infty} - \int_0^{t_{0c}}$, as follows.

$$RHS = 2\sinh(2\sigma t_{0c})[\int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{29}$$

We consolidate the integrals of the form $\int_0^{t_{0c}} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 27 and Eq. 29 as follows. We use $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned}\int_0^{t_{0c}} [E_0^{'}(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau\end{aligned} \tag{30}$$

We cancel the common term $e^{2\sigma t_{0c}}$ in Eq. 30 as follows.

$$\begin{aligned}\int_0^{t_{0c}} [E_0^{'}(\tau, t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau\end{aligned} \tag{31}$$

We substitute $E_0^{'}(\tau, t_{2c}) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ (using Definition 1 in Section 2.1) and $E_{0n}^{'}(\tau, t_{2c}) = E_0^{'}(-\tau, t_{2c}) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$ (using Definition 2 in Section 2.3). We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$(Appendix B.9). Hence we see that $E_{0n}^{'}(\tau, t_{2c}) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0^{'}(\tau, t_{2c})$ (**Result 3.1**) and write Eq. 31 as follows.

$$\begin{aligned}\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau\end{aligned} \tag{32}$$

We substitute $2\cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2\cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 32 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= \sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{33}$$

Next Step:

We denote the right hand side of Eq. 33 as RHS. We substitute $\tau - t_{2c} = \tau'$ and $\tau + t_{2c} = \tau''$ in the right hand side of Eq. 33 and then substitute $\tau' = \tau$ and $\tau'' = \tau$.

$$RHS = \sinh(2\sigma t_{0c})[\int_{-t_{2c}}^{\infty} E_0(\tau') \sin(\omega_z(t_{2c}, t_{0c})(\tau' + t_{2c})) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau'') \sin(\omega_z(t_{2c}, t_{0c})(\tau'' - t_{2c})) d\tau]$$
$$RHS = \sinh(2\sigma t_{0c})[\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$+ \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$- \cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \tag{34}$$

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$. Hence we cancel common terms and write Eq. 34 and Eq. 33 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= -\sinh(2\sigma t_{0c})[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \tag{35}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 35 as follows. Given that $E_0(\tau)$ is an **even** function of variable τ (Appendix B.9) and $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ, we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{36}$$

We can multiply Eq. 36 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{37}$$

In Eq. 37, given that $\omega_z(t_{2c}, t_{0c}) t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and the sinusoidal function is > 0, in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 37, we see that the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$(Section 6), we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 6.3) in the interval $0 < \tau < t_{0c}$. The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$.

The integrand is zero at $\tau = 0$ due to the term $\sin(\omega_z(t_{2c}, t_{0c})\tau)$ and the integrand is zero at $\tau = t_{0c}$ due to the term $\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 37. Hence this leads to a **contradiction**, for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 37 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(s) = \xi(1 - s)$ and hence $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. We consider $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_q(t) = E_p(-t) = E_0(-t)e^{\sigma t} = E_0(t)e^{\sigma t}$. Their Fourier transforms are given by $E_{p\omega}(\omega) = E_{q\omega}(-\omega)$.(link) We see that $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ and $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ by definition (Section 1.1) and hence $E_{q\omega}(-\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$. Given that $E_{p\omega}(\omega) = E_{q\omega}(-\omega)$, we get $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$.

This means that, **if** the Fourier transform of $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$, **then** the Fourier transform of $E_q(t) = E_0(t)e^{\sigma t}$ **also** has a zero at $\omega = \omega_0$. Hence the results in above sections hold for $-\frac{1}{2} < \sigma < 0$ and for $0 < |\sigma| < \frac{1}{2}$.

Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

4. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2

We see from Section 2.1 that $\omega_z(t_2, t_0)$ is shown to be **finite and non-zero** for all $|t_0| < \infty$ and for each non-zero value of t_2 and that $\omega_z(t_2, t_0)$ is an even function of variable t_0, for a given value of t_2(Section 2.4). For a given t_2 and t_0, $\omega_z(t_2, t_0)$ can have more than one value, but we consider only the first zero crossing away from origin in the section below, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, as detailed in **Lemma 1** in Section 2.1 and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$.

(example plot)

We consider the Fourier transform of the even part of $g(t,t_2,t_0)$ given by $G_R(\omega,t_2,t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0, the zero crossing in $G_R(\omega,t_2,t_0)$ given by $\omega_z(t_2,t_0)$ is a continuous function of t_0, for all $0 < t_0 < \infty$, for **each** value of t_2 in the interval $0 < t_2 < \infty$. This is shown in the steps below. For a given **finite** value of t_2, $G_R(\omega,t_2,t_0)$ is a function of two variables ω and t_0, and we use Implicit Function Theorem in R^2.

- It is shown in Section 4.1 that $G_R(\omega,t_2,t_0)$ is partially differentiable at least twice with respect to ω, as shown in Eq. 38.

- It is shown in Section 4.2 that $G_R(\omega,t_2,t_0)$ is partially differentiable at least twice with respect to t_0, as shown in Eq. 41 and Eq. 46.

- It is shown in Section 4.3 that the zero crossing in $G_R(\omega,t_2,t_0)$ given by $\omega_z(t_2,t_0)$, is a **continuous** function of t_0, for a given t_2, using **Implicit Function Theorem** in R^2.

- It is shown in Section 4.4 that $\omega_z(t_2,t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using **Implicit Function Theorem** in R^3.

4.1. $G_R(\omega,t_2,t_0)$ is partially differentiable twice as a function of ω

$G_R(\omega,t_2,t_0)$ in Eq. 16 is copied below.

$$G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau$$
$$+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau = G_{1R}^{'}(\omega,t_2,t_0) + G_{1R}^{'}(\omega,t_2,-t_0)$$
$$G_{1R}^{'}(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \tag{38}$$

We can expand $G_{1R}^{'}(\omega,t_2,t_0)$ in Eq. 38 by substituting $\tau+t_0=\tau'$ in the first term in the integral and $\tau-t_0=\tau''$ in the second term in the integral and expanding it, similar to Eq. 19 and substituting back $\tau'=\tau$ and $\tau''=\tau$. We use $e^{-2\sigma t_0}e^{2\sigma t_0}=1$ in the first term below.

$$G_{1R}^{'}(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{t_0}E_0^{'}(\tau',t_2)e^{-2\sigma\tau}e^{2\sigma t_0}\cos(\omega(\tau'-t_0))d\tau + e^{-2\sigma t_0}\int_{-\infty}^{-t_0}E_{0n}^{'}(\tau'',t_2)\cos(\omega(\tau''+t_0))d\tau$$
$$G_{1R}^{'}(\omega,t_2,t_0) = [\cos(\omega t_0)\int_{-\infty}^{t_0}E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega\tau)d\tau + \sin(\omega t_0)\int_{-\infty}^{t_0}E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega\tau)d\tau]$$
$$+e^{-2\sigma t_0}[\cos(\omega t_0)\int_{-\infty}^{-t_0}E_{0n}^{'}(\tau,t_2)\cos(\omega\tau)d\tau - \sin(\omega t_0)\int_{-\infty}^{-t_0}E_{0n}^{'}(\tau,t_2)\sin(\omega\tau)d\tau] \tag{39}$$

We could then use $E_0^{'}(t,t_2) = (E_0(t-t_2) - E_0(t+t_2)$ (using Definition 1 in Section 2.1) and $E_{0n}^{'}(t,t_2) = E_0^{'}(-t,t_2) = -E_0^{'}(t,t_2)$ (using Definition 2 in Section 2.3 and Result 3.1 in Section 3) and substitute $t+t_2=t$ and $t-t_2=t'$ and expanding it using the procedure used in Eq. 39. The integrands are absolutely integrable and we could then use theorem of dominated convergence as follows.

$G_R(\omega,t_2,t_0)$ is partially differentiable at least twice with respect to ω and the integrals converge in Eq. 40 for $0<\sigma<\frac{1}{2}$, because the terms $\tau^r E_0'(\tau\pm t_0,t_2)e^{-2\sigma\tau}$ and $\tau^r E_{0n}'(\tau\pm t_0,t_2)=-\tau^r E_0'(\tau\pm t_0,t_2)$ have exponential asymptotic fall-off rate as $|\tau|\to\infty$, for $r=0,1,2$ (Appendix B.6). The integrands are absolutely integrable and the integrands are analytic functions of variables ω and t_0, for a given t_2. We can interchange the order of partial differentiation and integration in Eq. 40 using theorem of dominated convergence, recursively as follows.(link) (We could also use theorem 3 in link and link.)

$$
\begin{aligned}
\frac{\partial G_R(\omega,t_2,t_0)}{\partial\omega} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau[E_0'(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau-t_0,t_2)]\sin(\omega\tau)d\tau \\
&\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau[E_0'(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau+t_0,t_2)]\sin(\omega\tau)d\tau] \\
\frac{\partial^2 G_R(\omega,t_2,t_0)}{\partial\omega^2} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^2[E_0'(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
&\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau^2[E_0'(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau+t_0,t_2)]\cos(\omega\tau)d\tau]
\end{aligned}
\tag{40}
$$

4.2. $G_R(\omega,t_2,t_0)$ *is partially differentiable twice as a function of* t_0

$G_R(\omega,t_2,t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals converge in Eq. 41 and Eq. 46 shown as follows. The integrands in the equation for $G_R(\omega,t_2,t_0)$ in Eq. 41 are absolutely integrable because the terms $E_0'(\tau\pm t_0,t_2)e^{-2\sigma\tau}$ and $E_{0n}'(\tau\pm t_0,t_2)=-E_0'(\tau\pm t_0,t_2)$ have exponential asymptotic fall-off rate as $|\tau|\to\infty$(Appendix B.6). The integrands are analytic functions of variables ω and t_0, for a given t_2 and we can expand $G_R(\omega,t_2,t_0)$ in Eq. 41 by substituting $\tau+t_0=t$ and expanding it, similar to Eq. 39. We can interchange the order of partial differentiation and integration in Eq. 41 using theorem of dominated convergence as follows. (link) (We could also use theorem 3 in link and link)

$$
\begin{aligned}
G_R(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0'(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
&\quad +e^{2\sigma t_0}\int_{-\infty}^{0}[E_0'(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \\
\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0'(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
&\quad +e^{-2\sigma t_0}\int_{-\infty}^{0}\frac{\partial(E_0'(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau-t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&\quad +2\sigma e^{2\sigma t_0}\int_{-\infty}^{0}[E_0'(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \\
&\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\frac{\partial(E_0'(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}'(\tau+t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau
\end{aligned}
$$

$$ \tag{41} $$

We show that the integrals in Eq. 41 converge, as follows. We see that $E_0^{'}(\tau+t_0,t_2) = E_0(\tau+t_0-t_2)-E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2) = -E_0^{'}(\tau-t_0,t_2) = E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2)$ (using Definition 1 in Section 2.1 and Definition 2 in Section 2.3).We see that the first integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 41 converges because the terms $E_0^{'}(\tau\pm t_0,t_2)e^{-2\sigma\tau}$ and $E_{0n}^{'}(\tau\pm t_0,t_2) = -E_0^{'}(\tau\pm t_0,t_2)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Appendix B.6).

We consider the integrand in the second integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 41 first and use the results in the above paragraph.

$$
\begin{aligned}
\frac{\partial(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0} &= \frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_0} \\
&\quad + \frac{\partial(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_0}
\end{aligned}
\tag{42}
$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. 42 and can show that the integrals converge in Eq. 41, as follows. We take the factor of 2 out of the summation in $E_0(t)$ below.

$$
\begin{aligned}
E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned}
\tag{43}
$$

We can show that $\frac{\partial}{\partial t_0}E_0(\tau+t_2+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0)$ as follows, given that the equation has terms of the form $e^{\tau+t_0}$ and the equation is **invariant** if we interchange the variables τ and t_0. (**Result A**)

$$
\begin{aligned}
\frac{\partial}{\partial t_0}E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\
&\quad + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\
\frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\
&\quad + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})]
\end{aligned}
\tag{44}
$$

We can replace t_0 by $t_0^{'} = -t_0$ in Eq. 44 and see that $\frac{\partial}{\partial t_0^{'}}E_0(\tau+t_2+t_0^{'}) = \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0^{'})$ (**Result E**) given that the equation is invariant if we interchange τ and $t_0^{'}$. Given that $\frac{\partial}{\partial t_0^{'}} = \frac{\partial}{\partial t_0}\frac{dt_0}{dt_0^{'}} = -\frac{\partial}{\partial t_0}$, we substitute it in Result E and get $\frac{\partial}{\partial t_0}E_0(\tau+t_2-t_0) = -\frac{\partial}{\partial \tau}E_0(\tau+t_2-t_0)$.(**Result B**)

We can write the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. 42, corresponding to the term in the second integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 41, using Result A, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}
\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial t_0}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau &= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial \tau}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau \\
= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau &- \int_{-\infty}^{0}E_0(\tau+t_2+t_0))\frac{\partial(e^{-2\sigma\tau}\cos{(\omega\tau)}}{\partial\tau}d\tau \\
= [E_0(\tau+t_2+t_0)e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0}E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau \\
&+2\sigma\int_{-\infty}^{0}E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau
\end{aligned} \tag{45}$$

We see that the integrals in Eq. 45 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\sin{(\omega\tau)}$ and $E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\cos{(\omega\tau)}$ have exponential asymptotic fall-off rate as $|\tau|\to\infty$(Appendix B.6). Hence the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_0}\cos{(\omega\tau)}d\tau$ in Eq. 45 also converges.

We set $\sigma=0$ and $t_0=-t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_0}\cos{(\omega\tau)}d\tau$ in Eq. 42 also converges, using Result B.

We set $t_2=-t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 43 to Eq. 45 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_0}\cos{(\omega\tau)}d\tau$ in Eq. 42 also converges.

We set $\sigma=0$ and $t_0=-t_0$ in the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau-t_0-t_2))}{\partial t_0}\cos{(\omega\tau)}d\tau$ in Eq. 42 also converges, using Result B. Hence the second integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 41 corresponding to the terms in Eq. 42, also converges.

We can see that the last two integrals in Eq. 41 converge, by setting $t_0=-t_0$ in Eq. 42 and using Result B and using the procedure in Eq. 43 to Eq. 45. Hence all the integrals in Eq. 41 converge.

4.2.1. ***Second Partial Derivative of*** $G_R(\omega,t_2,t_0)$ ***with respect to*** t_0

The second partial derivative of $G_R(\omega,t_2,t_0)$ with respect to t_0 is given by $\frac{\partial^2 G_R(\omega,t_2,t_0)}{\partial t_0^2} = \frac{\partial}{\partial t_0}\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ as follows. We use the result in Eq. 41 and the fact that the integrands are absolutely integrable using the results in Section 4.2 and we can interchange the order of partial differentiation and integration in Eq. 46 using theorem of dominated convergence as follows.

$$
\begin{aligned}
\frac{\partial^2 G_R(\omega,t_2,t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
&-4\sigma e^{-2\sigma t_0}\int_{-\infty}^{0}\frac{\partial(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&+e^{-2\sigma t_0}\int_{-\infty}^{0}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau \\
&+4\sigma^2 e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \\
&+4\sigma e^{2\sigma t_0}\int_{-\infty}^{0}\frac{\partial(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&+e^{2\sigma t_0}\int_{-\infty}^{0}\frac{\partial^2(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau
\end{aligned}
\tag{46}
$$

The first two integrals and fourth and fifth integrals in Eq. 46 are the same as the integrals in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 41 and have been shown to converge in Section 4.2. We will show that the third and sixth integrals in Eq. 46 converge, as follows.

We consider the integrand in the third integral in Eq. 46 first. We see that $E_0^{'}(\tau+t_0,t_2) = E_0(\tau+t_0-t_2)-E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2) = -E_0^{'}(\tau-t_0,t_2) = E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2)$ (using Definition 1 in Section 2.1 and Definition 2 in Section 2.3). We write an equation similar to Eq. 42.

$$
\begin{aligned}
\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0^2} &= \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_0^2} \\
&+\frac{\partial^2(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_0^2}
\end{aligned}
\tag{47}
$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. 47 as follows.

$$
\begin{aligned}
E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau}-3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}}e^{\frac{\tau}{2}} \\
E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau}e^{4(t_2+t_0)}-3\pi n^2 e^{2\tau}e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau}e^{2(t_2+t_0)}}e^{\frac{\tau}{2}}e^{\frac{(t_2+t_0)}{2}}
\end{aligned}
\tag{48}
$$

We can see that $\frac{\partial^2}{\partial t_0^2}E_0(\tau+t_2+t_0) = \frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_0}$ and the equation **is invariant** if we interchange the variables τ and t_0.(**Result A'**)

We can replace t_0 by $t_0^{'} = -t_0$ in Eq. 48 and see that $\frac{\partial^2}{\partial(t_0^{'})^2}E_0(\tau + t_2 + t_0^{'}) = \frac{\partial^2}{\partial\tau^2}E_0(\tau + t_2 + t_0^{'})$ (**Result E'**) given that the equation has terms of the form $e^{\tau+t_0^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_0^{'}$.

Given that $\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t_0^{'}}\frac{\partial t_0^{'}}{\partial t_0} = -\frac{\partial}{\partial t_0^{'}}$, we get $\frac{\partial^2}{\partial t_0^2} = \frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0}) = -\frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0^{'}}) = \frac{\partial}{\partial t_0^{'}}(\frac{\partial}{\partial t_0^{'}}) = \frac{\partial^2}{\partial(t_0^{'})^2}$, we substitute it in Result E' and get $\frac{\partial^2}{\partial t_0^2}E_0(\tau + t_2 - t_0) = \frac{\partial^2}{\partial\tau^2}E_0(\tau + t_2 - t_0)$.(**Result B'**)

We can write the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. 47, corresponding to the term in the third integral in Eq. 46, using Result $A^{'}$, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$
\begin{aligned}
\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial t_0^2}e^{-2\sigma\tau}\cos(\omega\tau)d\tau &= \int_{-\infty}^{0}\frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial\tau^2}e^{-2\sigma\tau}\cos(\omega\tau)d\tau \\
= \int_{-\infty}^{0}\frac{\partial(\frac{dE_0(\tau+t_2+t_0)}{d\tau}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial\tau}d\tau &- \int_{-\infty}^{0}\frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\frac{\partial(e^{-2\sigma\tau}\cos(\omega\tau)}{\partial\tau}d\tau \\
= [\frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0}\frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}e^{-2\sigma\tau}\sin(\omega\tau)d\tau \\
&+2\sigma\int_{-\infty}^{0}\frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)d\tau
\end{aligned}
\tag{49}
$$

We see that the integral $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$ in Eq. 49 converges, using Eq. 45 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}$ also converges, given that the Fourier transform of $\frac{dE_0(\tau)}{d\tau}$ given by $i\omega E_{0\omega}(\omega)$ is finite for real ω and has exponential asymptotic fall-off rate as $|\omega| \to \infty$(Appendix B.4) and hence absolutely integrable and hence $\frac{dE_0(\tau)}{d\tau}$ goes to zero as $|\tau| \to \infty$ as per Riemann-Lebesgue Lemma. (**Result 4.2.1.1**)

It is shown below that the remaining term $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$ also converges.

$$
\begin{aligned}
&\int_{-\infty}^{0}\frac{\partial(E_0(\tau + t_2 + t_0))}{\partial\tau}e^{-2\sigma\tau}\sin(\omega\tau)d\tau \\
= \int_{-\infty}^{0}\frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\sin(\omega\tau))}{\partial\tau}d\tau &- \int_{-\infty}^{0}E_0(\tau + t_2 + t_0))\frac{\partial(e^{-2\sigma\tau}\sin(\omega\tau)}{\partial\tau}d\tau \\
= [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\sin(\omega\tau)]_{-\infty}^{0} &- \omega\int_{-\infty}^{0}E_0(\tau + t_2 + t_0))e^{-2\sigma\tau}\cos(\omega\tau)d\tau \\
&+2\sigma\int_{-\infty}^{0}E_0(\tau + t_2 + t_0))e^{-2\sigma\tau}\sin(\omega\tau)d\tau
\end{aligned}
\tag{50}
$$

We see that the integrals in Eq. 50 converge and hence the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_0^2}\cos(\omega\tau)d\tau$ in Eq. 49 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_2-t_0))}{\partial t_0^2} \cos(\omega\tau)d\tau$ in Eq. 47 also converges, using Result B'.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 48 to Eq. 50 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_0^2} \cos(\omega\tau)d\tau$ in Eq. 47 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau-t_0-t_2))}{\partial t_0^2} \cos(\omega\tau)d\tau$ in Eq. 47 also converges, using Result B'. Hence the third integral in Eq. 46, also converges.

We can see that the sixth integral in Eq. 46 converge, by setting $t_0 = -t_0$ in Eq. 47 to Eq. 50 and using Result B'. Hence all the integrals in Eq. 46 converge.

4.3. ***Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0, for a given t_2.***

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0, for a given value of t_2, with continuous partial derivatives (Section 4.1 and Section 4.2) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0, for $0 < t_0 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

• It is shown in Section 4.5 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2. We can use the procedure in previous subsections and Implicit Function Theorem and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2, for $0 < t_2 < \infty$, for each value of t_0 in the interval $0 < t_0 < \infty$.

4.4. ***Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 and t_2***

We can use the procedure in previous subsections and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 and t_0, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using Implicit Function Theorem in R^3.

We use **Implicit Function Theorem** for the three dimensional case (link). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2, with continuous partial derivatives (Section 4.1, Section 4.2 and Section 4.5) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1), we see that $\omega_z(t_2, t_0)$ is differentiable function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

4.5. $G_R(\omega, t_2, t_0)$ ***is partially differentiable twice as a function of t_2***

$G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals converge in Eq. 51 and Eq. 55 shown as follows. The integrands in the equation for $G_R(\omega, t_2, t_0)$ in Eq. 51 are absolutely integrable because the terms $E_0'(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $E_{0n}'(\tau \pm t_0, t_2) = -E_0'(\tau \pm t_0, t_2)$

have exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$(Appendix B.6). The integrands are analytic functions of variables ω and t_2, for a given t_0 and we can expand $G_R(\omega, t_2, t_0)$ in Eq. 51 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 39. We can interchange the order of partial differentiation and integration in Eq. 51 using theorem of dominated convergence as follows. (link) (We could also use theorem 3 in link and link)

$$
\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos{(\omega\tau)} d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos{(\omega\tau)} d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0} \int_{-\infty}^{0} \frac{\partial (E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2} \cos{(\omega\tau)} d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} \frac{\partial (E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_2} \cos{(\omega\tau)} d\tau
\end{aligned}
\tag{51}
$$

We use the procedure outlined in Eq. 42 to Eq. 45, with t_0 replaced by t_2 and show that all the integrals in Eq. 51 converge, as follows.

We see that $E_0^{'}(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E_{0n}^{'}(\tau - t_0, t_2) = -E_0^{'}(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 and Definition 2 in Section 2.3). We consider the integrand in the first integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$ in Eq. 51 first.

$$
\begin{aligned}
\frac{\partial (E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2} &= \frac{\partial (E_0(\tau + t_0 - t_2) e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2) e^{-2\sigma\tau})}{\partial t_2} \\
&+ \frac{\partial (E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2}
\end{aligned}
\tag{52}
$$

We consider the term $E_0(\tau + t_0 + t_2)$ first and can show that the integrals converge in Eq. 51, as follows. We consider Eq. 43 below.

$$
\begin{aligned}
E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned}
\tag{53}
$$

We see that $\frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ given that the equation is invariant if we interchange τ and t_2.(**Result C**)

We can replace t_2 by $t_2^{'} = -t_2$ in Eq. 43 and see that $\frac{\partial}{\partial t_2^{'}} E_0(\tau + t_2^{'} + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2^{'} + t_0)$ given that the equation is invariant if we interchange τ and $t_2^{'}$. Given that $\frac{\partial}{\partial t_2^{'}} = \frac{\partial}{\partial t_2} \frac{dt_2}{dt_2^{'}} = -\frac{\partial}{\partial t_2}$, we get

$\frac{\partial}{\partial t_2}E_0(\tau - t_2 + t_0) = -\frac{\partial}{\partial \tau}E_0(\tau - t_2 + t_0)$.(**Result D**)

We consider the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ first in Eq. 52, corresponding to the term in the first integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_2}$ in Eq. 51 as follows, using Result C. We use the fact that $\int_{-\infty}^{0} \frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0} \frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0} A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial t_2} e^{-2\sigma\tau}\cos(\omega\tau)d\tau &= \int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial \tau} e^{-2\sigma\tau}\cos(\omega\tau)d\tau \\
= \int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\cos(\omega\tau))}{\partial \tau}d\tau &- \int_{-\infty}^{0} E_0(\tau + t_2 + t_0)\frac{\partial(e^{-2\sigma\tau}\cos(\omega\tau)}{\partial \tau}d\tau \\
= [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0} E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\sin(\omega\tau)d\tau \\
&+ 2\sigma\int_{-\infty}^{0} E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}\cos(\omega\tau)d\tau
\end{aligned} \tag{54}$$

We see that the integrals in Eq. 54 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\sin(\omega\tau)$ and $E_0(\tau+t_2+t_0))e^{-2\sigma\tau}\cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Appendix B.6). Hence the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_2}\cos(\omega\tau)d\tau$ in Eq. 54 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and in Eq. 54 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_2}\cos(\omega\tau)d\tau$ in Eq. 52 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 52 to Eq. 54 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_2}\cos(\omega\tau)d\tau$ in Eq. 52 also converges, using Result D.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ and in Eq. 54 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau-t_0-t_2))}{\partial t_2}\cos(\omega\tau)d\tau$ in Eq. 52 also converges. Hence the first integral in the equation for $\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_2}$ in Eq. 51 corresponding to the terms in Eq. 52, also converges.

We can see that the last integral in Eq. 51 converge, by setting $t_0 = -t_0$ in Eq. 54. Hence all the integrals in Eq. 51 converge.

4.5.1. ***Second Partial Derivative of*** $G_R(\omega, t_2, t_0)$ ***with respect to*** t_2

The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_2 is given by $\frac{\partial^2 G_R(\omega,t_2,t_0)}{\partial t_2^2} = \frac{\partial}{\partial t_2}\frac{\partial G_R(\omega,t_2,t_0)}{\partial t_2}$ as follows. We use the result in Eq. 51 and the fact that the integrands are absolutely integrable using the results in Section 4.5 and we can interchange the order of partial differentiation and integration in Eq. 55 using theorem of dominated convergence as follows.

$$\begin{aligned}
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} &= e^{-2\sigma t_0}\int_{-\infty}^{0} \frac{\partial^2(E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau \\
&+ e^{2\sigma t_0}\int_{-\infty}^{0} \frac{\partial^2(E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau
\end{aligned}$$

(55)

We consider the first integral in Eq. 55 and using $E_0^{'}(\tau+t_0,t_2)=E_0(\tau+t_0-t_2)-E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2)=-E_0^{'}(\tau-t_0,t_2)=E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2)$(using Definition 1 in Section 2.1 and Definition 2 in Section 2.3), we write an equation similar to Eq. 52.

$$\begin{aligned}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_2^2}&=\frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_2^2}\\&+\frac{\partial^2(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_2^2}\end{aligned} \tag{56}$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. 56 as follows.

$$\begin{aligned}E_0(\tau)&=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}-3\pi n^2e^{2\tau}]e^{-\pi n^2e^{2\tau}}e^{\frac{\tau}{2}}\\E_0(\tau+t_2+t_0)&=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}e^{4(t_2+t_0)}-3\pi n^2e^{2\tau}e^{2(t_2+t_0)}]e^{-\pi n^2e^{2\tau}e^{2(t_2+t_0)}}e^{\frac{\tau}{2}}e^{\frac{(t_2+t_0)}{2}}\end{aligned} \tag{57}$$

We can see that $\frac{\partial^2}{\partial t_2^2}E_0(\tau+t_2+t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_2}$ and the equation **is invariant** if we interchange the variables τ and t_2.(**Result C'**)

We can replace t_2 by $t_2^{'}=-t_2$ in Eq. 57 and see that $\frac{\partial^2}{\partial(t_2^{'})^2}E_0(\tau+t_0+t_2^{'})=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_0+t_2^{'})$ (**Result F'**) given that the equation has terms of the form $e^{\tau+t_2^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_2^{'}$.

Given that $\frac{\partial}{\partial t_2}=\frac{\partial}{\partial t_2^{'}}\frac{\partial t_2^{'}}{\partial t_2}=-\frac{\partial}{\partial t_2^{'}}$, we get $\frac{\partial^2}{\partial t_2^2}=\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2})=-\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial}{\partial t_2^{'}}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial^2}{\partial(t_2^{'})^2}$, we substitute it in Result F' and get $\frac{\partial^2}{\partial t_2^2}E_0(\tau+t_0-t_2)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_0-t_2)$.(**Result D'**)

We can write the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. 56, corresponding to the term in the first integral in Eq. 55, using Result $C^{'}$, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau=\int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau-\int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial t_2^2}e^{-2\sigma\tau}\cos(\omega\tau)d\tau&=\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial\tau^2}e^{-2\sigma\tau}\cos(\omega\tau)d\tau\\=\int_{-\infty}^{0}\frac{\partial(\frac{dE_0(\tau+t_2+t_0)}{d\tau}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial\tau}d\tau&-\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\frac{\partial(e^{-2\sigma\tau}\cos(\omega\tau)}{\partial\tau}d\tau\\=[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}&+\omega\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\sin(\omega\tau)d\tau\\&+2\sigma\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}e^{-2\sigma\tau}\cos(\omega\tau)d\tau\end{aligned}$$

$$(58)$$

We see that the integral $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau$ in Eq. 58 converges, using Eq. 54 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^{0}$ also converges, using Result 4.2.1.1 in Section 4.2.1. It is shown in Eq. 50 that the remaining term $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau$ also converges.

We see that the integrals in Eq. 58 converge and hence the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau+t_2+t_0) e^{-2\sigma\tau})}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 55 and Eq. 56 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in Eq. 58 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau+t_2-t_0))}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 55 and Eq. 56 also converges.

We set $t_2 = -t_2$ in Eq. 57 to Eq. 58 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau+t_0-t_2) e^{-2\sigma\tau})}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 55 and Eq. 56 also converges, using Result D'.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_0 - t_2) e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau-t_0-t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 55 and Eq. 56 also converges. Hence the first integral in Eq. 55, also converges.

We can see that the second integral in Eq. 55 converge, by setting $t_0 = -t_0$. Hence all the integrals in Eq. 55 converge.

5. Order of $\omega_z(t_2, t_0) t_0$ is greater than $O[1]$

It is noted that we **do not** use $\lim_{t_0 \to \infty}$ in this section. Instead we consider real $t_0 > 0$ which increases to a larger and larger finite value without bounds.

We write $P_{odd}(t_2, t_0)$ in Eq. 19 concisely as follows.

$$P_{odd}(t_2, t_0) = \int_{-\infty}^{t_0} E_0'(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau + e^{2\sigma t_0} \int_{-\infty}^{t_0} E_{0n}'(\tau, t_2) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau$$
$$P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \qquad (59)$$

We note that $E_0'(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$ and $E_{0n}'(\tau, t_2) = E_0'(-\tau, t_2) = -E_0'(\tau, t_2) = E_0(\tau + t_2) - E_0(\tau - t_2)$ (using Result 3.1 in Section 3). We choose $t_2 = 2t_0$ and we choose t_1 such that $E_0(t)$ approximates zero for $|t| > t_1$ and we choose $t_0 >> t_1$ and hence $E_0(\tau - t_2) = E_0(\tau - 2t_0)$ approximates zero in the interval $(-\infty, t_0]$. Hence in the interval $(-\infty, t_0]$, we see that $E_0'(\tau, t_2) \approx -E_0(\tau + t_2)$ and $E_{0n}'(\tau, t_2) \approx E_0(\tau + t_2)$, for sufficiently large t_0.

We see that the term $P_{odd}(t_2, -t_0)$ in Eq. 59 approaches a value very close to zero, as real t_0 increases to a larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals $\int_{-\infty}^{-t_0}$. Hence we can write Eq. 59 as follows using $t_2 = 2t_0$ and results in previous paragraph.

$$Q(t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) \approx -\int_{-\infty}^{t_0} E_0(\tau + 2t_0)e^{-2\sigma\tau} \cos\left(\omega_z(t_2, t_0)(\tau - t_0)\right)d\tau$$
$$+e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos\left(\omega_z(t_2, t_0)(\tau - t_0)\right)d\tau \approx 0 \tag{60}$$

We substitute $\tau + 2t_0 = t$ in Eq. 60 and write as follows.

$$Q(t_0) \approx -e^{4\sigma t_0} \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos\left(\omega_z(t_2, t_0)(t - 3t_0)\right)dt$$
$$+e^{2\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) \cos\left(\omega_z(t_2, t_0)(t - 3t_0)\right)dt \approx 0 \tag{61}$$

We multiply Eq. 61 by $e^{-3\sigma t_0}$ and ignore the last integral for sufficiently large t_0, given that $e^{2\sigma t_0}e^{-3\sigma t_0} = e^{-\sigma t_0}$ and $|\int_{-\infty}^{3t_0} E_0(t) \cos\left(\omega_z(t_2, t_0)(t - 3t_0)\right)dt| \leq \int_{-\infty}^{3t_0} |E_0(t)|dt$ is finite.(Appendix B.1)

$$S(t_0) = Q(t_0)e^{-3\sigma t_0} \approx -e^{\sigma t_0} \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos\left(\omega_z(t_2, t_0)(t - 3t_0)\right)dt = -e^{\sigma t_0}R(t_0) \approx 0$$
$$R(t_0) = \cos\left(\omega_z(t_2, t_0)3t_0\right) \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos\left(\omega_z(t_2, t_0)t\right)dt + \sin\left(\omega_z(t_2, t_0)3t_0\right) \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \sin\left(\omega_z(t_2, t_0)t\right)dt \tag{62}$$

Case 1: Order of $\omega_z(t_2, t_0)t_0$ less than 1

Let us assume that the order of $\omega_z(t_2, t_0)t_0$ is less than 1 and $\omega_z(t_2, t_0)t_0$ decreases to a very small finite value close to zero, as real t_0 increases to a larger and larger finite value without bounds. (**Statement B**) We see that t_0 is a real number and as it increases to a larger and larger finite value without bounds, we can use the approximations $\cos\left(\omega_z(t_2, t_0)3t_0\right) \approx 1$, $\sin\left(\omega_z(t_2, t_0)3t_0\right) \approx 3\omega_z(t_2, t_0)t_0 \approx 0$. We see that $\cos\left(\omega_z(t_2, t_0)t\right)$ and $\sin\left(\omega_z(t_2, t_0)t\right)$ are finite and the integrals in the expression for $R(t_0)$ in Eq. 62 converge to a finite value, given that $|\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos\left(\omega_z(t_2, t_0)(t - 3t_0)\right)dt| \leq \int_{-\infty}^{3t_0} |E_0(t)e^{-2\sigma t}|dt$ is finite.(Appendix B.1)

We choose t_3 such that $E_0(t)e^{-2\sigma t}$ approximates zero for $|t| > t_3$. As t_0 increase without bounds, we see that $t_3 << t_0$ and in the interval $[-t_3, t_3]$, we see that the term $\cos\left(\omega_z(t_2, t_0)t\right) = \cos\left(\omega_z(t_2, t_0)t_0\frac{t}{t_0}\right) \approx 1$ given Statement B and $t_3 << t_0$. Hence we can write Eq. 62 as follows.

$$R(t_0) \approx \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}dt \approx \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt \tag{63}$$

For sufficiently large t_0, the integral $R(t_0) \approx \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt$ remains finite and non-zero and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds, given that $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}dt > 0$. (Appendix B.1) This is explained in detail in Section 5.1.

The term $e^{\sigma t_0}$ in $S(t_0)$ in Eq. 62 increases to a larger and larger finite value **exponentially** and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ and $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ **cannot** equal zero in this case.

Hence **Statement B** is **false** and $\omega_z(t_2, t_0)t_0$ **does not** decrease towards zero, as finite t_0 increases without bounds. Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$ (Section 4), we see that the the order of $\omega_z(t_2, t_0)t_0$ is greater than or equal to 1, as finite t_0 increases without bounds.(**Result 5.1**)

Case 2: Order of $\omega_z(t_2, t_0)t_0$ is 1

Let us assume that the order of $\omega_z(t_2, t_0)t_0$ is 1, as real t_0 increases to a larger and larger finite value without bounds. (**Statement C**). In this case, the order of $\omega_z(t_2, t_0)$ is $O[\frac{1}{t_0}]$ and we consider $\omega_z(t_2, t_0) = \frac{K}{t_0}$ where $K < \frac{\pi}{2}$.(We require $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ in Section 3)

We choose t_3 such that $Kt_3 << t_0$ and $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. As t_0 increase without bounds, in the interval $[-t_3, t_3]$, we see that the term $\cos(\omega_z(t_2, t_0)t) \approx 1$ and $\sin(\omega_z(t_2, t_0)t) \approx \omega_z(t_2, t_0)t \approx 0$, given that $\omega_z(t_2, t_0)t = \frac{Kt_3}{t_0} << 1$. Hence we can write Eq. 62 as follows.

$$R(t_0) \approx \cos(\omega_z(t_2, t_0)3t_0) \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}dt \approx \cos(3K) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt \tag{64}$$

For sufficiently large t_0, the integral $R(t_0) \approx \cos(3K) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt$ remains finite, because the order of $\cos(\omega_z(t_2, t_0)3t_0)$ is 1 and $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}dt > 0$ (Appendix B.1) and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds. This is explained in detail in Section 5.1.

The term $e^{\sigma t_0}$ in $S(t_0)$ in Eq. 62 increases to a larger and larger finite value **exponentially** and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ and $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ **cannot** equal zero in this case.

Hence **Statement C** is **false** and the order of $\omega_z(t_2, t_0)t_0$ is **not** 1, as finite t_0 increases without bounds. Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$ (Section 4) and given Result 5.1, we see that the the order of $\omega_z(t_2, t_0)t_0$ is **greater than** 1, as finite t_0 increases without bounds.

If we consider the case $\omega_z(t_2, t_0) = \frac{KD(t_2,t_0)}{t_0}$ where $K < \frac{\pi}{2}$ and $D(t_2, t_0)$ is a function of order 1, whose maximum value is 1, the arguments in the above paragraphs still hold. If $K \geq \frac{\pi}{2}$, then $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached for suitable t_0, which is required in Section 3.

5.1. $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$ ***does not have exponential fall off rate***

In this section, we compute the **minimum** value of the integral $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$ for sufficiently large t_3 and $t_0 >> t_3$ and $0 < \sigma < \frac{1}{2}$. We split $A(t_0)$ as follows.

$$
\begin{aligned}
A(t_0) &= A_1(t_0) + A_2(t_0) + A_3(t_0) \\
A_1(t_0) = \int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt, \quad A_2(t_0) &= \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt \\
A_3(t_0) &= \int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt
\end{aligned}
\tag{65}
$$

We will show that $A(t_0) \geq K_0 - K_1 - K_2$ where K_0 is the minimum value of $A_2(t_0)$ and K_1 is the maximum value of $A_3(t_0)$ and K_2 is the maximum value of $A_1(t_0)$.

We choose $t_3 = 10$ such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. Given that $E_0(t) > 0$ for $|t| < \infty$ (Appendix B.1), for $0 < \sigma < \frac{1}{2}$, we see that the integral $\int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt > 2\int_0^{t_3} E_0(t)e^{-|t|}dt > K_{00} = 0.42$ where K_{00} is computed by considering the first 5 terms $n = 1,2,3,4,5$ in $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

Given that $\omega_z(t_2,t_0) = \frac{K}{t_0}$ where $K < \frac{\pi}{2}$ in Case 2 in previous subsection and $t_0 >> t_3$, we see that $\omega_z(t_2,t_0)t \leq \frac{Kt_3}{t_0} \approx 0$ in the interval $|t| \leq t_3$ and hence $\cos(\omega_z(t_2,t_0)t) \approx 1$ and $\cos(\omega_z(t_2,t_0)t) > \frac{1}{2}$ in the interval $|t| \leq t_3$. The same result holds for Case 1 in previous subsection because $\omega_z(t_2,t_0)$ has a faster falloff rate. Hence we can write $A_2(t_0) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{K_{00}}{2} = K_0 = 0.21$.

Next we consider the integral $A_3(t_0) = \int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$ for $0 < \sigma < \frac{1}{2}$. Given that $E_0(t) > 0$ for $|t| < \infty$, we have $A_3(t_0) \leq \int_{t_3}^{3t_0} |E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)|dt = \int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}dt < \int_{t_3}^{\infty} E_0(t)e^{-2\sigma t}dt < \int_{t_3}^{\infty} E_0(t)dt = K_{10}$.

We see that $E_0(t)$ has a fall-off rate of $O[e^{-1.5t}]$(Appendix B.5) which is higher than a **minimum** fall-off rate of e^{-t}. Hence we can write $K_{10} < E_0(t_3)e^{t_3}\int_{t_3}^{\infty} e^{-t}dt = -E_0(t_3)e^{t_3}[e^{-t}]_{t_3}^{\infty} = E_0(t_3)e^{t_3}e^{-t_3} = E_0(t_3) = K_1$. For $t_3 = 10$, we see that $K_1 = E_0(t_3) << 1 \approx 0$, given that $E_0(0) < 1$ and $E_0(t)$ is a strictly decreasing function for $t > 0$. (Section 6)

Similarly, we see that $A_1(t_0) = \int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt = \int_{t_3}^{\infty} E_0(t)e^{2\sigma t}\cos(\omega_z(t_2,t_0)t)dt \leq \int_{t_3}^{\infty} E_0(t)e^{t}dt = K_{20}$. We see that $E_0(t)$ has a **minimum** fall-off rate of $e^{-1.5t}$(Appendix B.5). Hence we can write $K_{20} < E_0(t_3)e^{t_3}e^{0.5t_3}\int_{t_3}^{\infty} e^{-0.5t}dt = -2E_0(t_3)e^{t_3}e^{0.5t_3}[e^{-0.5t}]_{t_3}^{\infty} = 2E_0(t_3)e^{t_3} = K_2$. For $t_3 = 10$, we see that $K_2 = 2E_0(t_3)e^{t_3} << 1 \approx 0$, given that $E_0(0) < 1$ and $E_0(t)$ is a strictly decreasing function for $t > 0$ (Section 6).

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > K_0 - K_1 - K_2 = 0.21 - K_1 - K_2 \approx 0.21$. As t_0 increases without bounds, we see that $A(t_0) > 0.21$ and **does not** have exponential fall off rate.

6. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_0(t) = \Phi(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. It is known that $\Phi(t)$ is positive for $|t| < \infty$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t > 0$. (link). This is shown below.

$$E_0(t) = \Phi(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi n^2 e^{4t} - 3e^{2t}] \tag{66}$$

We show that $X(t) = \frac{E_0(t)}{2}$ is a **strictly decreasing** function for $t > 0$ as follows.

• In Section 6.1, it is shown that the first derivative of $X(t)$, given by $\frac{dX(t)}{dt} < 0$ for $t > t_z$ where $t_z = \frac{1}{2}\log\frac{y_z}{\pi}$ and $y_z = 3.16$.

• In Section 6.2, it is shown that, $\frac{dX(t)}{dt} < 0$ for $0 < t \le t_z$.

Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is strictly decreasing for all $t > 0$ and $E_0(t) = 2X(t)$ is **strictly decreasing** for all $t > 0$.

6.1. $\frac{dX(t)}{dt} < 0$ *for* $t > t_z$

We consider $X(t) = \frac{E_0(t)}{2} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi n^2 e^{4t} - 3e^{2t}]$ and take the first derivative of $X(t)$ as follows. We note that $E_0(t)$ and $X(t)$ are analytic functions for real t and infinitely differentiable in that interval. We compute $\frac{dX(t)}{dt}$ below and take the term e^{2t} out.

$$\begin{aligned}
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t})(\frac{1}{2} - 2\pi n^2 e^{2t})] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t}[-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]
\end{aligned} \tag{67}$$

We substitute $y = \pi e^{2t}$ in Eq. 67 and define $A(y)$ such that $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$. [8]

$$A(y) = \sum_{n=1}^{\infty} n^2 e^{-n^2 y}[-4n^4 y^2 + 15n^2 y - \frac{15}{2}] \tag{68}$$

We see that $A(y) = 0$ at $y = \pi$ which corresponds to $t = 0$ given $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, given that $\frac{dX(t)}{dt} = 0$ at $t = 0$. Because $X(t) = \frac{E_0(t)}{2}$ is an even function of variable t(Appendix B.9) and hence $\frac{dX(t)}{dt}$ is an **odd** function of variable t.

The quadratic expression $B(y,n) = (-4n^4y^2 + 15n^2y - \frac{15}{2})$ in Eq. 68 has roots at $y = \frac{-15n^2 \pm \sqrt{225n^4 - 120n^4}}{-8n^4} = \frac{(15 \pm \sqrt{105})}{8n^2}$. We see that the first derivative of $B(y,n)$ is given by $\frac{dB(y,n)}{dy} = -8n^4y + 15n^2$ is zero at $y = \frac{15}{8n^2}$. The second derivative of $B(y,n)$ given by $\frac{d^2B(y,n)}{dy^2} = -8n^4$, is negative for all y and $n \geq 1$ and hence $B(y,n)$ is a **concave down** function for each n, which reaches a maximum at $y = \frac{15}{8n^2}$ and given the dominant term $-4n^4y^2$ in Eq. 68 , we see that $B(y,n) < 0$, for $y > \frac{(15+\sqrt{105})}{8} > 3.16 = y_z$, for $n \geq 1$ and hence $A(y) < 0$ for $y > y_z$. Using $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, we see that $\frac{dX(t)}{dt} < 0$ for $t > \frac{1}{2} \log \frac{y_z}{\pi} = t_z$(**Result 1**). (concave down function)

We show in the next section that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. It suffices to show that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq y_z = 3.16$ and hence $A(y) < 0$ for $\pi < y \leq y_z = 3.16$, given that $A(y) = 0$ at $y = \pi$. [We use $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$ and $\frac{dX(t)}{dt} = 0$ at $t = 0$.]

6.2. $\frac{dX(t)}{dt} < 0$ ***for*** $0 < t \leq t_z$

It is shown in this section that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq 3.16$ and hence $A(y) < 0$ for $\pi < y \leq 3.16$ [8] , given that $A(y) = 0$ at $y = \pi$. We take the derivative of $A(y)$ in Eq. 68 and take the factor n^2 out of the brackets, as follows.

$$\frac{dA(y)}{dy} = \sum_{n=1}^{\infty} n^2 e^{-n^2y}[-8n^4y + 15n^2 + (-4n^4y^2 + 15n^2y - \frac{15}{2})(-n^2)]$$

$$\frac{dA(y)}{dy} = \sum_{n=1}^{\infty} n^4 e^{-n^2y}[-8n^2y + 15 + 4n^4y^2 - 15n^2y + \frac{15}{2}] = \sum_{n=1}^{\infty} n^4 e^{-n^2y}[4n^4y^2 - 23n^2y + \frac{45}{2}] \tag{69}$$

We examine the term $C(y,n) = n^4 e^{-n^2y}(4n^4y^2 - 23n^2y + \frac{45}{2})$ in Eq. 69 in the interval $\pi \leq y \leq 3.16$ and show that $\frac{dA(y)}{dy} = C(y,1) + \sum_{n=2}^{\infty} C(y,n) < 0$, as follows.

For $n = 1$, we see that $C(y,1) = e^{-y}(4y^2 - 23y + \frac{45}{2}) < 0$ in the interval $\pi \leq y \leq 3.16$ as follows. Given that $3.16 < 4$ and $3.16^2 < 10$ and $\pi > 3$, in the interval $\pi \leq y \leq 3.16$, we see that $C(y,1) < e^{-3}(4*10 - 23*3 + \frac{45}{2}) < e^{-3}(40 - 69 + 23) = -6e^{-3} = C_{max}(1)$ where $C_{max}(1)$ is the maximum value of $C(y,1)$ in the interval $\pi \leq y \leq 3.16$.

$$C(y,1) = e^{-y}(4y^2 - 23y + \frac{45}{2}) < -6e^{-3}, \quad \pi \leq y \leq 3.16 \tag{70}$$

For $n > 1$, in the interval $\pi \leq y \leq 3.16$, we can write $C(y,n)$ as follows, given that $\pi > 3$ and $3.16^2 < 10$ and the term $-23n^2y + \frac{45}{2} < -23*3 + 23 < 0$ is ignored below.

$$C(y,n) = n^4 e^{-n^2y}(4n^4y^2 - 23n^2y + \frac{45}{2}) < n^4 e^{-\pi n^2}(4n^4(3.16)^2) < 40n^8 e^{-\pi n^2} < 40n^8 e^{-3n^2} \tag{71}$$

We want to show that $\frac{dA(y)}{dy} = C(y,1) + \sum_{n=2}^{\infty} C(y,n) < 0$ in the interval $\pi \leq y \leq 3.16$. Using Eq. 70 and Eq. 71, we write

$$\frac{dA(y)}{dy} = C(y,1) + \sum_{n=2}^{\infty} C(y,n) < -6e^{-3} + \sum_{n=2}^{\infty} 40n^8 e^{-3n^2}$$
$$e^3 \frac{dA(y)}{dy} < -6 + \sum_{n=2}^{\infty} 40n^8 e^{3-3n^2} \tag{72}$$

We want to show that $e^3 \frac{dA(y)}{dy} < 0$ in the interval $\pi \leq y \leq 3.16$. We compute $\log\left(n^8 e^{3-3n^2}\right)$ as follows. We note that $f(x) = \log x$ is a **concave down** function whose second derivative given by $-\frac{1}{x^2} < 0$ for $|x| < \infty$ and we can write $f(x) = \log x \leq f(x_0) + f^{'}(x_0)(x - x_0)$ using its **tangent line** equation. We see that $f^{'}(x) = \frac{1}{x}$. We set $x = n$ and $x_0 = 2$ and get $\log n \leq \log 2 + \frac{1}{2}(n-2)$ below.

$$\log\left(n^8 e^{3-3n^2}\right) = 8\log n + (3 - 3n^2) \leq 8(\log 2 + \frac{1}{2}(n-2)) + (3 - 3n^2)$$
$$\log\left(n^8 e^{3-3n^2}\right) \leq 8\log 2 + 4n - 5 - 3n^2 \tag{73}$$

We note that $g(x) = 4x - 5 - 3x^2$ in Eq. 73 is a **concave down** function whose second derivative given by $-6 < 0$ for all x and we can write $g(x) \leq g(x_0) + g^{'}(x_0)(x - x_0)$ using its **tangent line** equation. We see that $g^{'}(x) = 4 - 6x$. We set $x = n$ and $x_0 = 2$ and get $g(n) \leq g(2) + [4 - 6n]_{n=2}(n - 2) = -9 - 8(n-2)$ and write Eq. 73 as follows. We take the exponent e on both sides.

$$\log\left(n^8 e^{3-3n^2}\right) \leq 8\log 2 - 9 - 8(n-2) \leq 8\log 2 - 1 + 8(1-n)$$
$$n^8 e^{3-3n^2} \leq e^{8\log 2 - 1 + 8(1-n)} = 2^8 e^{-1} e^{8(1-n)} \tag{74}$$

We substitute the result in Eq. 74 in Eq. 72 and simplify as follows.

$$e^3 \frac{dA(y)}{dy} < -6 + 40 * 2^8 e^{-1} \sum_{n=2}^{\infty} e^{8(1-n)}$$
$$e^3 \frac{dA(y)}{dy} < -6 + 40 * 2^8 e^{-1} * e^8 \sum_{n=2}^{\infty} e^{-8n}$$
$$e^3 \frac{dA(y)}{dy} < -6 + 40 * 2^8 e^{-1} * e^8 \frac{e^{-8*2}}{1 - e^{-8}}$$
$$e^3 \frac{dA(y)}{dy} < -6 + 40 * 2^8 e^{-1} * \frac{e^{-8}}{1 - e^{-8}}$$
$$e^3 \frac{dA(y)}{dy} < -6 + 40 * 2^8 e^{-1} * \frac{1}{e^8 - 1} \tag{75}$$

We multiply Eq. 75 by $\frac{(e^8-1)}{6}$ and write as follows.

$$e^3 \frac{dA(y)}{dy} \frac{(e^8 - 1)}{6} < -e^8 + 1 + 40e^{-1} * \frac{256}{6} \approx -2352 \tag{76}$$

We see that $e^3 \frac{dA(y)}{dy} \frac{(e^8-1)}{6} < 0$ in Eq. 76 and hence $\frac{dA(y)}{dy} < 0$, in the interval $\pi \leq y \leq 3.16$, given that $e^3 \frac{(e^8-1)}{6} > 0$. Given that $A(y) = 0$ at $y = \pi$, we see that $A(y) < 0$ in Eq. 68 , for $\pi < y \leq 3.16$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y) < 0$ in the interval $0 < t \leq t_z$.(**Result 2**)

In Section 6.1, it is shown that $\frac{dX(t)}{dt} < 0$ for $t > t_z$ (from Result 1). In this section, we have shown that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$.

Hence $E_0(t) = 2X(t)$ is a **strictly decreasing function** for $t > 0$.

6.3. ***Result*** $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 6 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 37 .

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t (Appendix B.9), and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t+t_{2c}) = E_0(t+2t_{0c})$ ranges from $E_0(2t_{0c}) > E_0(t + t_{2c}) > E_0(3t_{0c})$(**Result 6.3.1**) and $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c}) < E_0(t-t_{2c}) < E_0(-t_{0c})$ respectively. Given that $E_0(t) = E_0(-t)$, we see that $E_0(2t_{0c}) < E_0(t - t_{2c}) < E_0(t_{0c})$ in the interval $0 < t < t_{0c}$ (**Result 6.3.2**).

Using Result 6.3.1 and Result 6.3.2, we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 37 , for $t_{0c} > 0$ and $t_{2c} = 2t_{0c}$.

7. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1 - s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ (Appendix B.9) where $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1 - s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied [6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi)\xi(1 - s, \bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them. This proof does not need or use Euler product.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \dfrac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real [4] and then derive $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ Appendix D. In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function of variable t and $\int_{-\infty}^{\infty} E_0(t)dt > 0$ for $|t| < \infty$ (Appendix B.1) and $E_0(t)$ is **strictly decreasing** function for $t > 0$ (Section 6). These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

References

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[2] Hardy, G.H., Littlewood, J.E. The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift volume 10, pp.283 to 317 (1921).

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[4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152

[5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)

[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

[8] Thomas Browning. draft article

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ using Eq. 1. This is re-derived in Appendix D.1 .

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 < |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) &= \xi(\frac{1}{2}+i(\omega-i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega-i\sigma)\\ E_p(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega-i\sigma)e^{i\omega t}d\omega\end{aligned} \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x+iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [0, -i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-i\sigma, 0]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2}+iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2}+iz)$ is an entire function in the Z-plane.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2}+iz) = \xi(\frac{1}{2}-y+ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \le y \le 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function for real t(Appendix B.8). Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned}E_p(t) &= e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'\\ E_p(t) = E_0(t)e^{-\sigma t} &= \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}\end{aligned} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$. **Alternate** derivation is in Appendix D.1.

Appendix B. Properties of Fourier Transforms

Appendix B.1. $E_p(t), h(t)$ *are absolutely integrable functions and their Fourier Transforms are finite.*

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$(Appendix B.7). Given that $E_0(t) = E_0(-t)$(Appendix B.9), we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

It is shown in Appendix B.5 that $E_0(t)$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-1.5|t|}]$ and hence $E_p(t)$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5-\sigma)|t|}] > O[e^{-|t|}]$, for $0 \le |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t}$ goes to zero, at $t \to \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$ in the last paragraph.(**Result 21**) Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \ge 0$ for all t, using Result 21. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Using the arguments in above paragraph, we replace σ by 0 and 2σ respectively and see that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $\frac{A_0}{(i\omega)^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

Let us consider the function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$. We can see that $G(\omega, t_2, t_0), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t, t_2, t_0), h(t)$ are **discontinuous** at $t = 0$. Hence the convolution integral below converges to a finite value for real ω.

$$F(\omega, t_2, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega', t_2, t_0)H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega, t_2, t_0) * H(\omega)] \tag{B.1}$$

Appendix B.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_{N\omega}(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P_\omega(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

Appendix B.4. ***Exponential Fall off rate of analytic functions.***

We know that the order of Riemann's Xi function $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant [3] (Titchmarsh pp256-257).

We consider $x(t) = E_0(t)e^{-2\sigma t}$ and its Fourier transform is given by $X(\omega) = \int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{-i\omega t}dt = \int_{-\infty}^{\infty} E_0(t)e^{-i(\omega - i2\sigma)t}dt = E_{0\omega}(\omega - i2\sigma) = \xi(\frac{1}{2}+i(\omega - i2\sigma)) = \xi(\frac{1}{2}+2\sigma+i\omega) = E_{0\omega}(\omega - i2\sigma)$. Hence both $E_{0\omega}(\omega)$ and $X(\omega) = E_{0\omega}(\omega - i2\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $X(\omega)$ is an **analytic** function which is infinitely differentiable which produce no discontinuities for real ω and $0 < \sigma < \frac{1}{2}$. Hence its **inverse Fourier transform** $x(t)$ has fall-off rate faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix B.3) and hence $x(t) = E_0(t)e^{-2\sigma t}$ should have **exponential fall-off** rate as $|t| \to \infty$.

Appendix B.5. ***Exponential Fall off rate of*** $x(t) = E_0(t)e^{-2\sigma t}$

We can write $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ in Eq. 1 as follows. In the term $e^{-\pi n^2 e^{2t}}$, we use Taylor series expansion around $t = 0$ for $e^{2t} = \displaystyle\sum_{r=0}^{\infty}\frac{(2t)^r}{!r}$, given that e^{2t} is an analytic function for real t.

$$
\begin{aligned}
E_0(t) &= \sum_{n=1}^{\infty} 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \\
&= \sum_{n=1}^{\infty} 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2(1+2t)}e^{-\pi n^2(\frac{(2t)^2}{!2}+\frac{(2t)^3}{!3}....)}e^{\frac{t}{2}}
\end{aligned}
\tag{B.2}
$$

We take the term $e^{-2\pi t}$ out of the summation, corresponding to $n = 1$ and then take the term $2\pi e^{4t}e^{\frac{t}{2}} = 2\pi e^{\frac{9t}{2}}$ out and write as follows.

$$
E_0(t) = 2\pi e^{-2\pi t}e^{\frac{9t}{2}}\sum_{n=1}^{\infty} n^2[2\pi n^2 - 3e^{-2t}]e^{-\pi n^2}e^{-2\pi(n^2-1)t}e^{-\pi n^2(\frac{(2t)^2}{!2}+\frac{(2t)^3}{!3}....)}
\tag{B.3}
$$

For $t > 0$, we see that the term corresponding to $n = 1$ in Eq. B.3 has an asymptotic fall-off rate of **at least** $O[e^{-(2\pi - \frac{9}{2})t}] > O[e^{-1.5t}]$. The terms corresponding to $n > 1$ have fall-off rates **higher** than $O[e^{-1.5t}]$, due to the term $e^{-2\pi(n^2-1)t}$.

Hence we see that $E_0(t)$ has an asymptotic fall-off rate of **at least** $O[e^{-1.5t}]$, for $t > 0$. Given that $E_0(t) = E_0(-t)$(Appendix B.9), we see that $E_0(t)$ has an **exponential** asymptotic fall-off rate of at least $O[e^{-1.5|t|}]$.

Similarly, $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5-2\sigma)|t|}] > O[e^{-0.5|t|}]$, for $0 \le |\sigma| < \frac{1}{2}$.

Using a second method, it is shown that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate in Appendix B.4.

Appendix B.6. ***Exponential Fall off rate of*** $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ ***for*** $r = 0, 1, 2$

In this section, it is shown that the term $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ has exponential asymptotic fall-off rate as $|t| \to \infty$, for $r = 0, 1, 2$ where $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Hence $B(t) = t^r e^{-2\sigma t}[E_0(t - t_2 \pm t_0) - E_0(t + t_2 \pm t_0)]$.

We consider $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ for finite and real t_a. We see that $C(t + t_a) = (t + t_a)^r e^{-2\sigma t} e^{-2\sigma t_a} E_0(t)$. We see that $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$ with exponential fall-off rates as $|t| \to \infty$. (Appendix B.5).

Hence $C(t + t_a) = (t + t_a)^r e^{-2\sigma t_a} E_0(t)e^{-2\sigma t}$ also has exponential fall-off rates as $|t| \to \infty$, for $r = 0, 1, 2$ and finite t_a and is an absolutely integrable function.

Hence $C(t)$ has exponential fall-off rates as $|t| \to \infty$, for finite t_a and is an absolutely integrable function. We set $t_a = t_2 \pm t_0$ and $t_a = -t_2 \pm t_0$ and see that $B(t)$ has **exponential fall-off rates** as $|t| \to \infty$, for finite t_2, t_0 and is an absolutely integrable function.

Appendix B.7. $E_0(t) > 0$ ***for*** $0 \leq t < \infty$

For $0 \leq t < \infty$, we can show that $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ where $f(t, n) = [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ as follows.

The sum is positive because each summand $f(t, n)$ is positive for finite n, and each summand is positive because the term $2\pi n^2 e^{2t} - 3 > 0$ for all $t \geq 0$ and $n \geq 1$, given that $\pi > 3$ and $2\pi n^2 e^{2t} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for $0 \leq t < \infty$ and finite $n \geq 1$.(**Statement 8**)

For $t = 0$ and $n = 1$, we see that $f(0, 1) = 2\pi[2\pi - 3]e^{-\pi} > 0$.

For $t = 0$ and for **each finite** $n \geq 1$, we see that $f(0, n) = 2\pi n^2[2\pi n^2 - 3]e^{-\pi n^2} > 0$.

For $0 < t < \infty$ and for **each finite** $n \geq 1$, we see that $f(t, n) = 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$, using Statement 8.

As $n \to \infty$, $f(t, n)$ tends to zero, for $0 \leq t < \infty$ due to the term $e^{-\pi n^2 e^{2t}}$. We do summation over n and see that the sum of the terms $\sum_{n=1}^{\infty} f(t, n) > 0$.

Hence $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ for $0 \leq t < \infty$.

Given that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_0(t)dt$ is finite. We see that $E_0(t)$ is an analytic function for real t. Hence $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ is finite for $0 \leq t < \infty$.

Appendix B.8. $E_y(t) = E_0(t)e^{yt}$ ***is an absolutely integrable function***

The Fourier transform of $E_y(t) = E_0(t)e^{yt}$ is given by $E_{y\omega}(\omega) = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-i\omega t}dt = \int_{-\infty}^{\infty} E_0(t)e^{-i(\omega+iy)t}dt =$ $E_{0\omega}(\omega + iy) = \xi(\frac{1}{2} + i(\omega + iy)) = \xi(\frac{1}{2} - y + i\omega)$.

Given that $\xi(\frac{1}{2} - y + i\omega) = E_{y\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_y(t)dt$ is finite, where $E_y(t) = E_0(t)e^{yt}$ and $-\sigma \leq y \leq 0$ and $0 \leq |\sigma| < \frac{1}{2}$(**Result 11**).

We see that $E_0(t) > 0$ for $0 \leq t < \infty$ (Appendix B.7). Given that $E_0(t) = E_0(-t)$(Appendix B.9), we see that $E_0(t) > 0$ for all $-\infty < t < \infty$. Hence $E_y(t) = E_0(t)e^{yt} > 0$ for all $-\infty < t < \infty$.

$E_y(t)$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5+y)|t|}] > O[e^{-|t|}]$, for $-\sigma \leq y \leq 0$ and $0 \leq |\sigma| < \frac{1}{2}$.(Appendix B.5). Hence $E_y(t)$ goes to zero, at $t \to \pm\infty$ and we showed that $E_y(t) > 0$ for all $-\infty < t < \infty$.(**Result 12**)

Using Result 11 and 12, we can write $\int_{-\infty}^{\infty} |E_y(t)|dt$ is finite and $E_y(t)$ is an absolutely **integrable function** and its Fourier transform $E_{y\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B.9. $E_0(t)$ ***is real and even***

We see that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ (**Result 13**) because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

We take the Inverse Fourier transform of $E_{0\omega}(\omega)$ and use $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ from Result 13 and then substitute $\omega = -\omega'$ in the integrand, as follows.

$$
\begin{aligned}
E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(-\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{-i\omega' t}d\omega' = E_0(-t)
\end{aligned}
\tag{B.4}
$$

Hence we have derived the result that $E_0(t)$ is a **real and even** function of variable t.

Appendix C. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix C.1. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a **real** function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$. We use the fact that $\cos(\omega t)$ is an **even** function of ω and $\sin(\omega t)$ is an **odd** function of ω .

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned}
\tag{C.1}
$$

Appendix C.2. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t)+g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t)+g(-t)]e^{-i\omega t}dt &= \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt
\end{aligned}
\tag{C.2}
$$

We substitute $t = -t$ in the second integral in Eq. C.2. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix C.1)

$$
\begin{aligned}
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} + \frac{G(-\omega)}{2} \\
= \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(-\omega) + iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(\omega) - iG_I(\omega)] = G_R(\omega)
\end{aligned}
\tag{C.3}
$$

Appendix C.3. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t)-g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t)-g(-t)]e^{-i\omega t}dt &= \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt
\end{aligned}
\tag{C.4}
$$

We substitute $t = -t$ in the second integral in Eq. C.4. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix C.1)

$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} - \frac{G(-\omega)}{2}$$
$$= \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(-\omega) - iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(\omega) + iG_I(\omega)] = iG_I(\omega) \tag{C.5}$$

Appendix C.4. ***Fourier transform of a real and even function*** $g(t)$

In this section, we show that the Fourier transform of a **real and even** function $g(t)$, given by $G(\omega)$ is also **real and even**. We use the fact that $\int_{-\infty}^{\infty} g(t)\sin\omega t dt = 0$ because $g(t)$ is even and the integrand is an **odd function** of variable t.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)\cos\omega t dt - i\int_{-\infty}^{\infty} g(t)\sin\omega t dt$$
$$G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt \tag{C.6}$$

We see that $G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt$ is **real** function of ω, given that $g(t)$ and the integrand are real functions. We see that $G(\omega)$ is an **even** function of ω because $\cos\omega t$ is a **even** function of ω.

Appendix D. Derivation of entire function $\boldsymbol{\xi(s)}$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below [4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{D.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{D.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma(\frac{\sigma'}{2}) \pi^{-\frac{\sigma'}{2}} \zeta(\sigma')$.

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \tag{D.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. D.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $x > 0$ is real. [4] (link)

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{((1+2w(x))\sqrt{x} - 1)}{2} dx \tag{D.4}$$

Hence we can simplify Eq. D.4 as follows. We use $\int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(\sqrt{x}-1)}{2} dx = \frac{1}{s(s-1)}$ for $Re[s] > 1$.

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}} w(x) dx \tag{D.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}] \tag{D.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix D.1. ***Derivation of $E_p(t)$ and $E_0(t)$***

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. D.6 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2} + \sigma + i\omega)(-\frac{1}{2} + \sigma + i\omega) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t}) dt] \tag{D.7}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt] \tag{D.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} e^{-i\omega t} dt \tag{D.9}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4}+\sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} \\
\frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-4\pi n^2 e^{-2t} + (-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\
&+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t) + A_0\delta(t)
\end{aligned} \tag{D.10}$$

We use $A_0 = [\frac{dA(t)}{dt}]_{t=0+} - [\frac{dA(t)}{dt}]_{t=0-} = \sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)$. We can simplify above equation as follows.

$$\begin{aligned}
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \\
&+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]
\end{aligned} \tag{D.11}$$

We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real [4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$ (Appendix D.2) and hence **dirac delta terms cancel each other** in equation below.

$$
\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}) \\
&\quad + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} C(t)u(t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} D(t)u(-t)
\end{aligned}
\tag{D.12}
$$

We can simplify above equation as follows. We see that $C(t) = -\frac{1}{4} + \sigma^2 + \sigma - 2\sigma^2 - 4\sigma\pi n^2 e^{2t} + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t} = 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}$ and $D(t) = -\frac{1}{4} + \sigma^2 - \sigma - 2\sigma^2 + 4\sigma\pi n^2 e^{-2t} + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t} = 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t}$. We see that $D(t) = C(-t)$. Hence we can write as follows.

$$
\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned}
\tag{D.13}
$$

We use the fact that $E_0(t) = E_0(-t)$ (Appendix B.9) we arrive at the desired result for $E_p(t)$ as follows.

$$
\begin{aligned}
E_0(t) &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) = E_0(t)e^{-\sigma t} &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{D.14}
$$

Appendix D.2. ***Derivation of*** $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real [4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}} \tag{D.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2})$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2} \tag{D.16}$$