On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix F, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3]

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider a new function $g(t)=e^{\sigma t_0}E_p(t+t_0)e^{-\sigma t}u(-t)+e^{\sigma t_0}E_p(t+t_0)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t)=e^{\sigma t_0}E_p(t+t_0)$ where $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t)=\frac{1}{2}[g(t)+g(-t)]$ given by $G_{even}(\omega)=G_R(\omega)$ must have **at least one zero** at $\omega=\omega_2(t_0)\neq 0$, for every value of t_0, to satisfy Statement 1, where $\omega_2(t_0)$ is real and finite.

1.3. *Step 3: On the zeros of the function* $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g_{even}(t)$ given by $G_R(\omega)$. We require $G_R(\omega)=0$ for $\omega=\omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**. In general, $\omega_2(t_0)\neq\omega_0$.

It is shown that $R(t_0)=G_R(\omega_2(t_0),t_0)=0$ for all t_0 as follows.

$$\begin{aligned}R(t_0)=e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau+\sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]\\+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0}E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau-\sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0}E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]=0\\R(t_0)=\int_{-\infty}^{0}[E_0(\tau+t_0)e^{-2\sigma\tau}+E_0(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau=0\end{aligned} \tag{3}$$

In Section 2.3, it is shown that $\omega_2(t_0)=\omega_2(-t_0)$.

1.4. *Step 4: First derivative of* $R(t_0)$

In Section 2.5, we derive the first derivative of $R(t_0)$ at $t_0 = 0$ as follows, where $\omega_{20} = [\omega_2(t_0)]_{t_0=0}$. $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$, $m_{0p} = \int_{-\infty}^{0} E_0(\tau)\cos(\omega_{20}\tau)d\tau$, $n_{0p} = \int_{-\infty}^{0} E_0(\tau)\sin(\omega_{20}\tau)d\tau$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} &= R_0 = m_0 + m_{0p} = 0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= R_1 = \omega_{20}[n_0 - n_{0p}] + 2\sigma m_0 = 0
\end{aligned} \tag{4}
$$

1.5. *Step 5: Next Step*

In Section 2.6, we replace $E_p(t)$ by $E_p'(t) = e^{\sigma t_2}E_p(t + t_2)$, for $|t_2| \leq \infty$ and derive as follows.

$$
\begin{aligned}
R_0'(t_2) &= m_0'(t_2) + m_{0p}'(t_2) = 0 \\
m_0'(t_2) &= e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau + \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau] \\
m_{0p}'(t_2) &= \cos(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2(t_2)\tau)d\tau
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
R_1'(t_2) &= \omega_2(t_2)[n_0'(t_2) - n_{0p}'(t_2)] + 2\sigma m_0'(t_2) = 0 \\
n_0'(t_2) &= e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
n_{0p}'(t_2) &= \cos(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2(t_2)\tau)d\tau + \sin(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2(t_2)\tau)d\tau
\end{aligned} \tag{6}
$$

1.6. *Step 6: Asymptotic Case and Final result*

In Section 2.7, we consider the asymptotic case $\lim_{t_2\to-\infty}$ and show that $\lim_{t_2\to-\infty}\omega_2(t_2) = \omega_z$ is a constant where ω_z is a zero on the critical line and derive as follows.

$$
\begin{aligned}
\lim_{t_2\to-\infty} m_{0p}'(t_2) &= 0 \\
\lim_{t_2\to-\infty} n_{0p}'(t_2) &= 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{i(\omega_z t)}dt &= 0
\end{aligned} \tag{7}
$$

Then we consider the asymptotic case $\lim_{t_2\to+\infty}$ and derive as follows.

$$
\begin{aligned}
\lim_{t_2\to\infty} m_0'(t_2) &= 0 \\
\lim_{t_2\to\infty} n_0'(t_2) &= 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{i(\omega_z t)}dt &= 0
\end{aligned}
$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration $n = 1, ...N$, we use $h(t) = e^{(2^n\sigma)t}u(-t)+e^{-(2^n\sigma)t}u(t)$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t}$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$ and we show a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$.

2. An Approach towards Riemann's Hypothesis: Method 3

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix C.1.

2.1. *New function* $g(t)$

Let us consider the function $f(t) = e^{\sigma t_0}E_p(t + t_0)$ where $|t_0| \leq \infty$ and we can see that the Fourier Transform of this function $F(\omega) = e^{\sigma t_0}E_{p\omega}(\omega)e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

We can see that $g(t)$ is a real L^1 integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma]. This is explained in detail in Appendix C.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}(\omega)e^{\sigma t_0}e^{i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_2(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ where $\omega_2(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_2(t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ for **every value** of t_0, where $\omega_2(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{\sigma t_0} E_p(t + t_0)$ and $h(t) = e^{\sigma t} u(-t) + e^{-\sigma t} u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}(\omega) e^{\sigma t_0} e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_2(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \tag{9}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 9 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{10}$$

We can split the integral in Eq. 10 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{11}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 11 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega') [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{12}$$

In Appendix C.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 12 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_2(t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

2.2. ***On the zeros of a related function*** $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for **every value** of t_0, to satisfy **Statement 1**. In general, $\omega_2(t_0) \neq \omega_0$.

First we compute the fourier transform of the function $g(t)$ given by $G(\omega) = G_R(\omega) + iG_I(\omega)$. We use $g(t) = e^{\sigma t_0} E_p(t+t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p(t+t_0) e^{\sigma t} u(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{0} g_-(t) e^{-i\omega t} dt + \int_{0}^{\infty} g_+(t) e^{-i\omega t} dt \\ G(\omega) &= \int_{-\infty}^{0} e^{\sigma t_0} E_p(t+t_0) e^{-\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{\sigma t_0} E_p(t+t_0) e^{\sigma t} e^{-i\omega t} dt \end{aligned} \tag{13}$$

We use $E_p(t) = E_0(t) e^{-\sigma t}$ and $E_p(t+t_0) = E_0(t+t_0) e^{-\sigma t} e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 13, we have

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} E_0(t+t_0) e^{-i\omega t} dt \\ G(\omega) &= \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_0(-t+t_0) e^{i\omega t} dt \end{aligned} \tag{14}$$

We define $E_{0m}(t) = E_0(-t)$ and get $E_0(-t+t_0) = E_{0m}(t-t_0)$ and write Eq. 14 as follows.

$$G(\omega) = \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_{0m}(t-t_0) e^{i\omega t} dt = G_R(\omega) + iG_I(\omega) \tag{15}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$G_R(\omega) = \int_{-\infty}^{0} E_0(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0m}(t-t_0)\cos(\omega t)dt \tag{16}$$

We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**. Hence we can see that $R(t_0) = G_R(\omega_2(t_0)) = 0$ and we can write as follows using $t = \tau$.

$$R(t_0) = \int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau = 0 \tag{17}$$

We can rewrite Eq. 17 as follows, using the substitution $\tau + t_0 = \tau'$ in the first integral and $\tau - t_0 = \tau''$ in the second integral and substituting back $\tau' = \tau$ and $\tau'' = \tau$.

$$\begin{aligned} R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\ +[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_{0m}(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_{0m}(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0 \end{aligned} \tag{18}$$

Now we replace t_0 by $-t_0$ in $f(t)$ and consider the function $f_2(t) = e^{-\sigma t_0}E_p(t - t_0)$ where $|t_0| \leq \infty$ and use the procedure in above section and we can write as follows.

$$\begin{aligned} R(-t_0) &= \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)]\cos(\omega_2(-t_0)\tau)d\tau = 0 \\ R(t_0) + R(-t_0) &= \int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau \\ &+ \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)]\cos(\omega_2(-t_0)\tau)d\tau = 0 \end{aligned} \tag{19}$$

2.3. $\omega_2(t_0)$ ***is an even function of variable*** t_0

Now we consider the function $f_T(t) = f(t) + f_2(t) = e^{\sigma t_0}E_p(t + t_0) + e^{-\sigma t_0}E_p(t - t_0)$ where $|t_0| \leq \infty$ and $g_T(t)h(t) = f_T(t)$ where $g_T(t) = f_T(t)e^{-\sigma t}u(-t) + f_T(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g_T(t)$ and compute its real part using the procedure in above section, similar to Eq. 16 and we can write as follows. We use $E_0(-\tau) = E_0(\tau)$.

$$\begin{aligned} G_{T_R}(\omega) &= G_1(\omega, t_0) + G_1(\omega, -t_0) \\ G_1(\omega, t_0) &= \int_{-\infty}^{0} E_0(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0m}(t-t_0)\cos(\omega t)dt \end{aligned} \tag{20}$$

We require $G_{T_R}(\omega) = 0$ for $\omega = \omega_0(t_0)$ for every value of t_0, to satisfy **Statement 1**. In general $\omega_0(t_0) \neq \omega_2(t_0)$. Hence we can see that $P(t_0) = G_{T_R}(\omega_0(t_0)) = 0$ and we can rewrite as follows using the substitution $t = \tau$.

$$
\begin{aligned}
P(t_0) = &\int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)] \cos{(\omega_0(t_0)\tau)} d\tau \\
&+ \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)] \cos{(\omega_0(t_0)\tau)} d\tau = 0
\end{aligned}
\tag{21}
$$

We see that $f_T(t) = e^{\sigma t_0} E_p(t+t_0) + e^{-\sigma t_0} E_p(t-t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_0(t_0)$ is an **even** function of variable t_0. Hence we can rewrite the second integral in Eq. 21 as follows using $\omega_0(t_0) = \omega_0(-t_0)$.

$$
\int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)] \cos{(\omega_0(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)] \cos{(\omega_0(-t_0)\tau)} d\tau = 0
\tag{22}
$$

We compare Eq. 22 and Eq. 19 as follows.

$$
\begin{aligned}
&\int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)] \cos{(\omega_0(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)] \cos{(\omega_0(-t_0)\tau)} d\tau = 0 \\
&\int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)] \cos{(\omega_2(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)] \cos{(\omega_2(-t_0)\tau)} d\tau = 0
\end{aligned}
\tag{23}
$$

We can see that there must be **at least one** common solution where $\omega_2(t_0) = \omega_0(t_0)$ to satisfy Eq. 23. Because $\omega_0(t_0)$ is an **even** function of variable t_0, we see that $\omega_2(t_0) = \omega_0(t_0)$ is also an **even** function of variable t_0.

2.4. $R(t_0)$ *at* $t_0 = 0$

Now we evaluate $R(t_0)$ in Eq. 17 at $t_0 = 0$. We define $\omega_{20} = [\omega_2(t_0)]_{t_0=0}$, $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau} \cos{(\omega_{20}\tau)} d\tau$, $m_{0p} = \int_{-\infty}^{0} E_{0m}(\tau) \cos{(\omega_{20}\tau)} d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau} \sin{(\omega_{20}\tau)} d\tau$ and $n_{0p} = \int_{-\infty}^{0} E_{0m}(\tau) \sin{(\omega_{20}\tau)} d\tau$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} = R_0 = m_0 + m_{0p} &= 0 \\
m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau} \cos{(\omega_{20}\tau)} d\tau \\
m_{0p} = \int_{-\infty}^{0} E_{0m}(\tau) \cos{(\omega_{20}\tau)} d\tau
\end{aligned}
\tag{24}
$$

2.5. ***First derivative of*** $R(t_0)$ ***at*** $t_0 = 0$

In Section 2.1, $\omega_2(t_0)$ is shown to be **finite** for all $|t_0| \leq \infty$. This means there are **no** Dirac delta functions present in $\omega_2(t_0)$. In Appendix D, we show that $\omega_2(t_0)$ is a continuous function around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$. In Appendix E, we show that $\omega_2(t_0)$ is differentiable **at least** once, in that interval. In Appendix F.3, it is shown that $\lim_{t_0 \to \infty} \omega_2(t_0)$ is a constant.

In Section 2.3, it is shown that $\omega_2(t_0) = \omega_2(-t_0)$ is an **even** function of variable t_0. Hence $\frac{d\omega_2(t_0)}{dt_0} = 0$ at $t_0 = 0$.

If $\omega_2(t_0)$ is a continuous function which is differentiable nowhere, given that $\omega_2(t_0) = \omega_2(-t_0)$ is shown to be an **even** function of variable t_0, $\frac{d\omega_2(t_0)}{dt_0} = 0$ at $t_0 = 0$ and it is sufficient for the computation below.

We take the first derivative of $R(t_0)$ in Eq. 17 and evaluate it at $t_0 = 0$. If $\frac{d\omega_2(t_0)}{dt_0}$ has Dirac delta functions at any $t_0 \neq 0$, it **does not** affect the computation below.

We consider $R(t_0)$ in Eq. 17 as follows.

$$R(t_0) = \int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau = 0 \tag{25}$$

We take the first derivative of $R(t_0)$ as follows. Given that the integrand in Eq. 25 is a continuous function which is well defined and bounded and the integral converges for all $|t_0| \leq \infty$, we can interchange the order of integration and differentiation using Leibnitz integral rule (link) and write as follows.(Details in Appendix E.1)

$$\begin{aligned}\frac{dR(t_0)}{dt_0} &= \int_{-\infty}^{0} [\frac{\partial}{\partial t_0} E_0(\tau + t_0)e^{-2\sigma\tau} + \frac{\partial}{\partial t_0} E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau \\ &- \frac{d\omega_2(t_0)}{dt_0} \int_{-\infty}^{0} \tau [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau = 0\end{aligned} \tag{26}$$

We can write as follows.

$$\begin{aligned}E_0(\tau) &= 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\ E_0(\tau + t_0) &= 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4t_0} - 3\pi n^2 e^{2\tau} e^{2t_0}] e^{-\pi n^2 e^{2\tau} e^{2t_0}} e^{\frac{\tau}{2}} e^{\frac{t_0}{2}}\end{aligned} \tag{27}$$

We can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_0)$ as follows.

$$\begin{aligned}\frac{\partial}{\partial t_0} E_0(\tau + t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2t_0}} e^{\frac{\tau}{2}} e^{\frac{t_0}{2}} [8\pi^2 n^4 e^{4\tau} e^{4t_0} - 6\pi n^2 e^{2\tau} e^{2t_0} \\ &+ (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2t_0})(2\pi^2 n^4 e^{4\tau} e^{4t_0} - 3\pi n^2 e^{2\tau} e^{2t_0})] \\ \frac{\partial}{\partial \tau} E_0(\tau + t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2t_0}} e^{\frac{\tau}{2}} e^{\frac{t_0}{2}} [8\pi^2 n^4 e^{4\tau} e^{4t_0} - 6\pi n^2 e^{2\tau} e^{2t_0} \\ &+ (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2t_0})(2\pi^2 n^4 e^{4\tau} e^{4t_0} - 3\pi n^2 e^{2\tau} e^{2t_0})]\end{aligned} \tag{28}$$

Similarly we can show that $\frac{\partial}{\partial t_0} E_{0m}(\tau - t_0) = -\frac{\partial}{\partial \tau} E_{0m}(\tau - t_0)$ and we can write Eq. 26 as follows.

$$\begin{aligned}\frac{dR(t_0)}{dt_0} &= \int_{-\infty}^{0} [\frac{\partial}{\partial \tau} E_0(\tau + t_0)e^{-2\sigma\tau} - \frac{\partial}{\partial \tau} E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau \\ &- \frac{d\omega_2(t_0)}{dt_0} \int_{-\infty}^{0} \tau [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau = 0\end{aligned} \tag{29}$$

We use the fact that $\int_{-\infty}^{0} \frac{\partial}{\partial \tau}(E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau = \int_{-\infty}^{0} \frac{\partial}{\partial \tau}E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ $+\int_{-\infty}^{0} E_0(\tau+t_0)\frac{\partial}{\partial \tau}(e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau$ for the first term in the first integral in Eq. 29. We use $E_0(\tau+t_0)e^{-2\sigma\tau}=0$ at $\tau=-\infty$ for finite t_0.

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{\partial}{\partial \tau}E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &= [E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)]_{-\infty}^{0} \\
&\quad - \int_{-\infty}^{0} E_0(\tau+t_0)\frac{\partial}{\partial \tau}(e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau \\
= E_0(t_0) + \omega_2(t_0)\int_{-\infty}^{0} E_0(\tau+t_0)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau &+ 2\sigma\int_{-\infty}^{0} E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{30}
$$

Similarly we can write the second term in the first integral in Eq. 29 as follows, using $\omega_2(-t_0)=\omega_2(t_0)$.

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{\partial}{\partial \tau}E_{0m}(\tau-t_0)\cos(\omega_2(t_0)\tau)d\tau &= [E_{0m}(\tau-t_0)\cos(\omega_2(t_0)\tau)]_{-\infty}^{0} - \int_{-\infty}^{0} E_{0m}(\tau-t_0)\frac{\partial}{\partial \tau}(\cos(\omega_2(t_0)\tau))d\tau \\
&= E_{0m}(-t_0) + \omega_2(t_0)\int_{-\infty}^{0} E_{0m}(\tau-t_0)\sin(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{31}
$$

Now we evaluate $\frac{dR(t_0)}{dt_0}$ in Eq. 29 at $t_0=0$, using Eq. 30 and Eq. 31 as follows. We see that the terms $E_0(t_0)$and $E_{0m}(-t_0)=E_0(t_0)$ cancel at $t_0=0$ and $[\frac{d\omega_2(t_0)}{dt_0}]_{t_0=0}=0$.

$$
\begin{aligned}
[\frac{dR(t_0)}{dt_0}]_{t_0=0} = \omega_{20}[\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau - \int_{-\infty}^{0} E_{0m}(\tau)\sin(\omega_{20}\tau)d\tau] &+ 2\sigma\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau) \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} = R_1 = \omega_{20}[n_0 - n_{0p}] &+ 2\sigma m_0 = 0
\end{aligned}
\tag{32}
$$

where $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau, n_{0p} = \int_{-\infty}^{0} E_{0m}(\tau)\sin(\omega_{20}\tau)d\tau$
and $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)$.

2.6. ***Next Step***

If we replace $E_p(t)$ in above section by $E_p^{'}(t) = e^{\sigma t_2}E_p(t+t_2) = E_0(t+t_2)e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, for $|t_2| \leq \infty$, where $E_0^{'}(t) = E_0(t+t_2)$, the location of the zeros in Fourier transform of $g(t,t_0,t_2)$ are represented by $\omega_2^{'}(t_2,t_0)$ and using method in the above section, we can get results similar to Eq. 24 and Eq. 32 with $E_0(t)$ replaced by $E_0^{'}(t)$ and ω_{20} replaced by $\omega_2^{'}(t_2)$ and other variables replaced with their **primed** versions as follows. We define $E_{0m}^{'}(t) = E_0^{'}(-t) = E_0(-t+t_2) = E_0(t-t_2)$, given that $E_0(t)=E_0(-t)$.

$$
\begin{aligned}
R_0^{'}(t_2) &= m_0^{'}(t_2) + m_{0p}^{'}(t_2) = 0 \\
m_0^{'}(t_2) &= \int_{-\infty}^{0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_2^{'}(t_2)\tau)d\tau, \quad m_{0p}^{'}(t_2) = \int_{-\infty}^{0} E_{0m}^{'}(\tau)\cos(\omega_2^{'}(t_2)\tau)d\tau
\end{aligned}
\tag{33}
$$

We use $E_0^{'}(\tau) = E_0(\tau+t_2)$ in Eq. 33 and then substitute $\tau+t_2=\tau'$ for the first term. We use $E_{0m}^{'}(\tau) = E_0(\tau-t_2)$ in Eq. 33 and then substitute $\tau-t_2=\tau'$ for the second term and write as follows.

$$m_0'(t_2) = e^{2\sigma t_2}[\cos(\omega_2'(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2'(t_2)\tau)d\tau + \sin(\omega_2'(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2'(t_2)\tau)d\tau]$$
$$m_{0p}'(t_2) = \cos(\omega_2'(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2'(t_2)\tau)d\tau - \sin(\omega_2'(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2'(t_2)\tau)d\tau \tag{34}$$

We compare Eq. 34 with Eq. 18 and see that $R(t_0)$ and $R_0'(t_2)$ are similar equations, given that $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$, with $t_0, \omega_2(t_0)$ in Eq. 18 replaced by $t_2, \omega_2'(t_2)$ in Eq. 34 and hence both equations **must have at least one** common solution. Hence we replace $\omega_2'(t_2)$ in Eq. 34 with $\omega_2(t_2)$ and write in a concise form as follows.

$$R_0'(t_2) = \int_{-\infty}^{0} [E_0(\tau + t_2)e^{-2\sigma\tau} + E_0(\tau - t_2)]\cos(\omega_2(t_2)\tau)d\tau = 0 \tag{35}$$

We can show that $\omega_2'(t_2, t_0) = \omega_0'(t_2, t_0)$ for **every value** of t_2, using the procedure and arguments outlined in Section 2.2 and Section 2.3 and hence $\omega_2'(t_2, t_0)$ is an **even** function of variable t_0 for **every value** of t_2. Hence $\frac{d\omega_2'(t_2,t_0)}{dt_0} = 0$ at $t_0 = 0$ for **every value** of t_2.

Using above method, we write Eq. 32 as follows. We replace $\omega_2'(t_2)$ with $\omega_2(t_2)$.

$$R_1{}'(t_2) = \omega_2(t_2)[n_0{}'(t_2) - n_{0p}{}'(t_2)] + 2\sigma m_0{}'(t_2) = 0$$
$$n_0'(t_2) = \int_{-\infty}^{0} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau$$
$$n_{0p}'(t_2) = \int_{-\infty}^{0} E_{0m}'(\tau)\sin(\omega_2(t_2)\tau)d\tau$$
$$n_0'(t_2) = e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau - \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau]$$
$$n_{0p}'(t_2) = \cos(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2(t_2)\tau)d\tau + \sin(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2(t_2)\tau)d\tau \tag{36}$$

2.7. *Asymptotic Fall off rate argument. Case 1:* $\lim_{t_0\to-\infty}$

In Appendix F.3, it is shown that $\lim_{t_0\to\infty}\omega_2(t_0) = \omega_z$ is a constant. In this section, we will show that ω_z is a zero on the critical line.

In this section, we can use dominated convergence theorem and interchange the order of limit and integration, given that $\lim_{t_0\to\pm\infty}\int_{-\infty}^{t_0}|E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)|d\tau < \infty$ and $\lim_{t_0\to\pm\infty}\int_{-\infty}^{t_0}|E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)|d\tau < \infty$ and the magnitude of the integrands are bounded and tend to an integrable function pointwise in the limit.

As $\lim_{t_0\to-\infty}$, we can compute $R(t_0)$ in Eq. 18 as follows. The first term goes to zero asymptotically as $\lim_{t_0\to-\infty}$. We use $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$.

$$R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]$$
$$+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0$$
$$\lim_{t_0\to-\infty} R(t_0) = \lim_{t_0\to-\infty}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0$$

(37)

As $\lim_{t_0 \to -\infty}$, we can compute $R_1{}'(t_0)$ in Eq. 36 by replacing t_2 with t_0. The terms $n_0{}'(t_2), m_0{}'(t_2)$ defined in Eq. 36 and Eq. 34 go to zero asymptotically as $\lim_{t_0 \to -\infty}$. We use the fact that $\lim_{t_0 \to -\infty} \omega_2(t_0) = \omega_z \neq 0$.

$$R_1{}'(t_0) = \omega_2(t_0)[n_0{}'(t_0) - n_{0p}{}'(t_0)] + 2\sigma m_0{}'(t_0) = 0$$

$$\lim_{t_0 \to -\infty} R_1{}'(t_0) = \lim_{t_0 \to -\infty} -\omega_2(t_0) n_{0p}{}'(t_0) = 0$$

$$\lim_{t_0 \to -\infty} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \sin(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \cos(\omega_2(t_0)\tau) d\tau] = 0 \tag{38}$$

We use $I_0(t_0) = \int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ and $Q_0(t_0) = \int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau$, we can write Eq. 37 and Eq. 38 as follows.

$$\lim_{t_0 \to -\infty} \cos(\omega_2(t_0)t_0) I_0(t_0) - \sin(\omega_2(t_0)t_0) Q_0(t_0) = 0$$
$$\lim_{t_0 \to -\infty} \cos(\omega_2(t_0)t_0) Q_0(t_0) + \sin(\omega_2(t_0)t_0) I_0(t_0) = 0$$
$$\lim_{t_0 \to -\infty} \frac{I_0(t_0)}{Q_0(t_0)} = \lim_{t_0 \to -\infty} \frac{\sin(\omega_2(t_0)t_0)}{\cos(\omega_2(t_0)t_0)} = \lim_{t_0 \to -\infty} -\frac{Q_0(t_0)}{I_0(t_0)} \tag{39}$$

For the general case of $\lim_{t_0 \to -\infty} \frac{\sin(\omega_2(t_0)t_0)}{\cos(\omega_2(t_0)t_0)} \neq 0, \pm\infty$, we get $\lim_{t_0 \to -\infty} I_0^2(t_0) + Q_0^2(t_0) = 0$. This implies that $\lim_{t_0 \to -\infty} I_0(t_0) = \lim_{t_0 \to -\infty} Q_0(t_0) = 0$ and $\lim_{t_0 \to -\infty} \int_{-\infty}^{\infty} E_0(\tau) e^{-i\omega_2(t_0)\tau} d\tau = 0$.

Given that the integrands in Eq. 37 and Eq. 38 are continuous functions which are well defined and bounded and the integral converges for all $|t_0| \leq \infty$, we can interchange the order of integration and limits, as follows, where $\lim_{t_0 \to -\infty} \omega_2(t_0) = \omega_z$ given that $\omega_2(t_0)$ is an even function of variable t_0.

$$\lim_{t_0 \to -\infty} \int_{-\infty}^{\infty} E_0(\tau) e^{-i\omega_2(t_0)\tau} d\tau = \int_{-\infty}^{\infty} \lim_{t_0 \to -\infty} E_0(\tau) e^{-i\omega_2(t_0)\tau} d\tau = \int_{-\infty}^{\infty} E_0(\tau) e^{-i\omega_z \tau} d\tau = 0 \tag{40}$$

We know that the Fourier transform of $E_0(\tau)$ given by $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$ on the **critical line**, has at least one real zero at $\omega = \omega_z \neq 0$ where ω_z is a **constant**.

*2.7.1. **Asymptotic Case 2:*** $\lim_{t_0 \to +\infty}$

Now we consider $R(t_0)$ in Eq. 37 and $R_1{}'(t_0)$ in Eq. 38, as $\lim_{t_0 \to \infty}$. The second term in $R(t_0)$ in Eq. 37 and $n_{0p}{}'(t_0)$ in $R_1{}'(t_0)$ in Eq. 38 go to zero asymptotically as $\lim_{t_0 \to \infty}$. Because $\omega_2(t_0) = \omega_2(-t_0)$, we use $\lim_{t_0 \to \infty} \omega_2(t_0) = \lim_{t_0 \to -\infty} \omega_2(t_0) = \omega_z \neq 0$ is the **same** constant. Given that $\lim_{t_0 \to \infty} e^{2\sigma t_0} \neq 0$, we can write as follows.

$$R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau]$$
$$+[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \cos(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \sin(\omega_2(t_0)\tau) d\tau] = m_0{}'(t_0) + m_{0p}{}'(t_0) = 0$$
$$\lim_{t_0 \to \infty} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] = 0$$

(41)

Similarly, the term $n_{0p}'(t_0)$ in $R_1'(t_0)$ in Eq. 38 goes to zero asymptotically as $\lim_{t_0\to\infty}$. We have shown that $\lim_{t_0\to\infty} m_0'(t_0) = 0$ in Eq. 41. Given that $\lim_{t_0\to\infty} e^{2\sigma t_0} \neq 0$, we can write as follows. We use the fact that $\lim_{t_0\to\infty} \omega_2(t_0) = \omega_z \neq 0$.

$$R_1'(t_0) = \omega_2(t_0)[n_0'(t_0) - n_{0p}'(t_0)] + 2\sigma m_0'(t_0) = 0$$

$$\lim_{t_0\to\infty}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau] = 0 \tag{42}$$

We use $I_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ and $Q_1(t_0) = \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau$, we can write Eq. 41 and Eq. 42 as follows.

$$\begin{aligned}
&\lim_{t_0\to\infty} \cos(\omega_2(t_0)t_0)I_1(t_0) + \sin(\omega_2(t_0)t_0)Q_1(t_0) = 0\\
&\lim_{t_0\to\infty} \cos(\omega_2(t_0)t_0)Q_1(t_0) - \sin(\omega_2(t_0)t_0)I_1(t_0) = 0\\
&\lim_{t_0\to\infty} \frac{Q_1(t_0)}{I_1(t_0)} = \lim_{t_0\to\infty} \frac{\sin(\omega_2(t_0)t_0)}{\cos(\omega_2(t_0)t_0)} = \lim_{t_0\to\infty} -\frac{I_1(t_0)}{Q_1(t_0)}
\end{aligned} \tag{43}$$

For the general case of $\lim_{t_0\to\infty} \frac{\sin(\omega_2(t_0)t_0)}{\cos(\omega_2(t_0)t_0)} \neq 0, \pm\infty$, we get $\lim_{t_0\to\infty} I_1^2(t_0) + Q_1^2(t_0) = 0$. This implies that $\lim_{t_0\to\infty} I_1(t_0) = \lim_{t_0\to\infty} Q_1(t_0) = 0$ and $\lim_{t_0\to\infty} \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_2(t_0)\tau}d\tau = 0$.

Given that the integrands in Eq. 41 and Eq. 42 are continuous functions which are well defined and bounded and the integral converges for all $|t_0| \leq \infty$, we can interchange the order of integration and limits, as follows, where $\lim_{t_0\to\infty} \omega_2(t_0) = \omega_z$.

$$\lim_{t_0\to\infty} \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_2(t_0)\tau}d\tau = \int_{-\infty}^{\infty} \lim_{t_0\to\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_2(t_0)\tau}d\tau = \int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0 \tag{44}$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

Now we can repeat the steps in Section 2, starting with the new result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$ and σ replaced by 2σ and derive the next result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-4\sigma\tau}e^{-i\omega_{(z1)}\tau}d\tau = 0$ where ω_{z1} is a real zero on the critical line.

We can repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration $n = 1,..N$, we use $h(t) = e^{(2^n\sigma)t}u(-t) + e^{-(2^n\sigma)t}u(t)$, $\omega_2(t_0)$ replaced by $\omega_{2n}(t_0)$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t} = \sum_{n=1}^{\infty}[4\pi^2n^4e^{4t} - 6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-(2^{N+1}\sigma)t}$ given by $E_{p\omega N}(\omega) = \xi(\frac{1}{2} + 2^{N+1}\sigma + i\omega)$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\displaystyle\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi\frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (Appendix F). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

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[6] Mathworld article on Hurwitz Zeta functions. (Link)

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). This is re-derived in Appendix F.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$$
$$E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix C.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\ F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\ F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega' \end{aligned} \tag{B.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.3}$$

Appendix B.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{B.4}$$

Appendix B.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \tag{B.5}$$

Appendix B.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. $E_p(t), h(t), g(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}}e^{\frac{5t}{2}}e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix C.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix C.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$, as $|\omega| \to \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \to \infty$, where $A, B > 0$ are real.

Appendix C.4. ***Payley-Weiner theorem and Exponential Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix C.3) and hence it should have **exponential fall-off** rates as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix D. $\omega_2(t_0)$ is a continuous function around $t_0 = 0$

This result is shown as follows.

• $G_R(\omega) = G_R(\omega, t_0)$ in Eq. 16 is copied below, which is a **continuous** function of ω which is differentiable **at least** once with respect to ω. (Eq. D.2 and Appendix D.3)

$$G_R(\omega) = G_R(\omega, t_0) = \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \tag{D.1}$$

Given that $E_0(t) \geq 0$ for $|t_0| \leq \infty$ (Appendix C.1), we see that $G_R(\omega) > 0$ at $\omega = 0$. **Set** $t_0 = 0$ and $G_R(\omega, t_0)$ passes through its **first zero** at $\omega = \omega_2(t_0) = \omega_2(0)$. In the rest of this section, we consider the **interval** $[-\delta t_0, \delta t_0]$ around $t_0 = 0$, in $\omega_2(t_0)$. There are 3 possibilities.

Case 1: $G_R(\omega) < 0$ for $\omega = \omega_2(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_2(0) - dw$ for infinitesimal dw (example plot)

In this case, we will show in Appendix D.1 that $\omega_2(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$.

Case 2: $G_R(\omega) > 0$ for $\omega = \omega_2(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_2(0) - dw$ (example plot)

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_2(0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_2(0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_2(0) + dw$.

$$\frac{dG_R(\omega)}{d\omega} = -\int_{-\infty}^{0} t[E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \sin(\omega t) dt \tag{D.2}$$

In this case, we will show Appendix D.2 that $\omega_2(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$.

Case 3: $G_R(\omega) = 0$ for $\omega = \omega_2(0)$ and $\omega = \omega_2(0) + dw$.

This is **not** possible because $G_R(\omega, t_0)$ in Eq. D.1 is an **analytic** function and infinitely differentiable with respect to ω(Appendix D.3). We know that analytic functions have **isolated** zeros. (link). Hence we cannot have $G_R(\omega) = 0$ for $\omega = \omega_2(0)$ and $\omega = \omega_2(0) + dw$ as $dw \to 0$.

Appendix D.1. ***Case 1:*** $G_R(\omega) < 0$ ***for*** $\omega = \omega_2(0) + dw$***,*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) - dw$

• Consider the **segment** S in $G_R(\omega, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega,t_0)}{d\omega} < 0$. (Segment S is the portion between the green lines in example plot)

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.1 is a **continuous** function of ω, for **each** value of t_0. Hence $G_R(\omega, t_0 - \delta t_0)$ and $G_R(\omega, t_0 + \delta t_0)$ are **continuous** functions of ω, which are differentiable **at least** once, and $G_R(\omega, t_0 \pm \delta t_0)$ tends to $G_R(\omega, t_0)$, as infinitesimal $\delta t_0 \to 0$.

$$G_R(\omega, t_0) = \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt$$
$$G_R(\omega, t_0 + \delta t_0) = \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos(\omega t) dt$$

(D.3)

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.3 is a **continuous** function of ω, for **each** value of t_0 and $\frac{dG_R(\omega,t_0)}{d\omega} < 0$ in the neighborhood around the **first zero crossing**. If we fix the X-coordinate ω, $G_R(\omega, t_0)$ is a **continuous** function of t_0, for **each** value of ω. Hence, for **each** value of ω, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0)$ moves towards $G_R(\omega, t_0 + \delta t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Every point in the segment S, moves continuously, as we change t_0 by an infinitesimal δt_0.

This also applies to the first **zero crossing** in $G_R(\omega, t_0)$ in the segment S, which corresponds to $\omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$ where $G_R(\omega, t_0) = 0$ in Eq. D.3. The zero crossing moves continuously, as we change t_0 by an infinitesimal δt_0. This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega, t_0)$ with zero crossing at point P_0, **Green** plot corresponds to $G_R(\omega, t_0 + \delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to $G_R(\omega, t_0 - \delta t_0)$ with zero crossing at point P_{21}.

We **define** the **point** P_{12} in $G_R(\omega, t_0 + \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_2(0)$. We **define** the **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_2(0)$.

We **define** the **point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 + \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{12} in $G_R(\omega, t_0 + \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_2(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_2(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

Similarly, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 - \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_2(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_2(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

$$
\begin{aligned}
G_R(\omega, t_0) &= \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 + \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 - \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 - \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 + \delta t_0)] \cos(\omega t) dt \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 + \delta t_0) &= G_R(\omega, t_0) \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 - \delta t_0) &= G_R(\omega, t_0)
\end{aligned}
\tag{D.4}
$$

• Hence in the **segment** S, $\omega_2(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$.

$$G_R(\omega_2(t_0), t_0) = \int_{-\infty}^{0} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \cos(\omega_2(t_0)t) dt = 0$$

$$G_R(\omega_2(t_0+\delta t_0), t_0+\delta t_0) = \int_{-\infty}^{0} [E_0(t+t_0+\delta t_0)e^{-2\sigma t} + E_{0m}(t-t_0-\delta t_0)] \cos((\omega_2(t_0+\delta t_0)t) dt = 0$$

(D.5)

Appendix D.2. ***Case 2:*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) + dw$***,*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) - dw$

• In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_2(t_0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_2(t_0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_2(t_0) + dw$.

• Consider the **segment** S' in $\frac{dG_R(\omega,t_0)}{d\omega}$ in the neighborhood around the first zero crossing where $\frac{d^2G_R(\omega,t_0)}{d\omega^2} > 0$. (Segment S' is the portion between the green lines in example plot) In this segment S', $\frac{dG_R(\omega,t_0)}{d\omega}$ is a **continuous** function of ω which is differentiable **at least** once.(Appendix D.3)

• In the **segment** S', $\frac{dG_R(\omega,t_0)}{d\omega} = 0$ at the **same** $\omega = \omega_2(t_0)$. The arguments in Appendix D.1 can be applied here, with $G_R(\omega, t_0)$ replaced by $\frac{dG_R(\omega,t_0)}{d\omega}$.

Hence $\omega_2(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$ in the **segment** S'.

Appendix D.3. ***Integral convergence in*** $\frac{dG_R(\omega)}{d\omega}$

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have exponential fall-off rates as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence the integrand $A_r(t) = \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \sin(\omega t)$ in Eq. D.2 copied below, is an absolutely **integrable function** and $\int_{-\infty}^{0} |A_r(t)|dt = \int_{-\infty}^{0} \frac{|t^r|}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]dt$ is **finite**, for $r = 0, 1, ..,$ given the **exponential** fall-off rate of $E_0(t)e^{-2\sigma t}$ and $E_0(t)$.

$$\frac{1}{!(r)} \frac{d^r G_R(\omega)}{d\omega^r} = (-1)^{\frac{r+1}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \sin(\omega t) dt, \quad r = odd$$
$$\frac{1}{!(r)} \frac{d^r G_R(\omega)}{d\omega^r} = (-1)^{\frac{r}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \cos(\omega t) dt, \quad r = even$$

(D.6)

Appendix E. $\omega_2(t_0)$ is differentiable at least once around $t_0 = 0$.

In Appendix D, we showed that $\omega_2(t_0)$ is a continuous function around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$. In this section, we show that $\omega_2(t_0)$ is differentiable **at least** once, in that interval. Thus we **rule out** the case of $\omega_2(t_0)$ as a **Weierstrass** type of function, which is continuous everywhere, but differentiable nowhere.

We take the first derivative of $R(t_0)$ in Eq. 17 as follows where $E_0'(\tau, t_0) = E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)$.

$$R(t_0) = \int_{-\infty}^{0} E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0}\int_{-\infty}^{0} E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = \lim_{\delta t_0 \to 0}\int_{-\infty}^{0} \frac{1}{\delta t_0}[E_0^{'}(\tau, t_0+\delta t_0)\cos(\omega_2(t_0+\delta t_0)\tau) - E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)]d\tau = 0 \tag{E.1}$$

The integrands in Eq. E.1 are continuous functions which are well defined and bounded and the integral converges. Hence we can use Leibnitz integral rule (link) and **interchange** the order of integration and differentiation and write as follows. (Details in Appendix E.1)

$$\frac{dR(t_0)}{dt_0} = \int_{-\infty}^{0} \frac{\partial}{\partial t_0}[E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)]d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0} \tau E_0^{'}(\tau, t_0)\sin(\omega_2(t_0)\tau)d\tau + \int_{-\infty}^{0} \frac{\partial}{\partial t_0}E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{d\omega_2(t_0)}{dt_0}P(t_0) = Q(t_0), \quad P(t_0) = \int_{-\infty}^{0} \tau E_0^{'}(\tau, t_0)\sin(\omega_2(t_0)\tau)d\tau, \quad Q(t_0) = \int_{-\infty}^{0} \frac{\partial}{\partial t_0}E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau \tag{E.2}$$

• We see that both integrals $P(t_0), Q(t_0)$ in Eq. E.2 are **continuous** functions, because integral of a well defined continuous function, is continuous. If we assume that $\omega_2(t_0)$ is a Weierstrass type of function which is **differentiable nowhere (Statement 5)**, $\frac{d\omega_2(t_0)}{dt_0}$ is not well defined in this case and we require $P(t_0) = Q(t_0) = 0$ for all $|t_0| \leq \infty$ to satisfy Eq. E.2. We will show that this leads to a contradiction and thus **rule out** this pathological case of $\omega_2(t_0)$ as follows.

$$Q(t_0) = \int_{-\infty}^{0} \frac{\partial}{\partial t_0}E_0^{'}(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0 \tag{E.3}$$

Using the procedure outlined in Section 2.5, we use the fact that $\frac{\partial}{\partial t_0}E_0(\tau+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_0)$ and $\frac{\partial}{\partial t_0}E_0(\tau-t_0) = -\frac{\partial}{\partial \tau}E_0(\tau-t_0)$ and write as follows. We use $E_{0m}(t) = E_0(-t) = E_0(t)$ and $E_0^{'}(\tau, t_0) = E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)$.

$$Q(t_0) = \int_{-\infty}^{0} [\frac{\partial}{\partial \tau}E_0(\tau+t_0)e^{-2\sigma\tau} - \frac{\partial}{\partial \tau}E_0(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau = 0 \tag{E.4}$$

We use the fact that $\int_{-\infty}^{0} \frac{\partial}{\partial \tau}(E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau = \int_{-\infty}^{0} \frac{\partial}{\partial \tau}E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$ $+ \int_{-\infty}^{0} E_0(\tau+t_0)\frac{\partial}{\partial \tau}(e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau$ and write the first term in Eq. E.4 as follows.

$$\int_{-\infty}^{0} \frac{\partial}{\partial \tau}E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau = [E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)]_{-\infty}^{0}$$
$$- \int_{-\infty}^{0} E_0(\tau+t_0)\frac{\partial}{\partial \tau}(e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau))d\tau$$
$$= E_0(t_0) + \omega_2(t_0)\int_{-\infty}^{0} E_0(\tau+t_0)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau + 2\sigma\int_{-\infty}^{0} E_0(\tau+t_0)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$$

$$(E.5)$$

Similarly we can write write the second term in Eq. E.4 as follows.

$$\begin{aligned}\int_{-\infty}^{0}\frac{\partial}{\partial\tau}E_0(\tau-t_0)\cos\left(\omega_2(t_0)\tau\right)d\tau &= [E_0(\tau-t_0)\cos\left(\omega_2(t_0)\tau\right)]_{-\infty}^{0}-\int_{-\infty}^{0}E_0(\tau-t_0)\frac{\partial}{\partial\tau}(\cos\left(\omega_2(t_0)\tau\right))d\tau\\ &= E_0(-t_0)+\omega_2(t_0)\int_{-\infty}^{0}E_0(\tau-t_0)\sin\left(\omega_2(t_0)\tau\right)d\tau\end{aligned} \tag{E.6}$$

Now we evaluate $Q(t_0)$ in Eq. E.4, using Eq. E.5 and Eq. E.6 as follows. We see that $E_0(t_0) = E_0(-t_0)$ and hence those terms cancel.

$$\begin{aligned}Q(t_0) = \omega_2(t_0)[\int_{-\infty}^{0}E_0(\tau+t_0)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau - \int_{-\infty}^{0}E_0(\tau-t_0)\sin\left(\omega_2(t_0)\tau\right)d\tau]\\ +2\sigma\int_{-\infty}^{0}E_0(\tau+t_0)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right) = 0\end{aligned} \tag{E.7}$$

We can substitute $\tau + t_0 = \tau'$ and $\tau - t_0 = \tau''$ in Eq. E.7 and expand it and we get the same equation as Eq. 36 as follows, with t_2 replaced by t_0.

$$\begin{aligned}R_1{}'(t_0) &= \omega_2(t_0)[n_0{}'(t_0) - n_{0p}{}'(t_0)] + 2\sigma m_0{}'(t_0) = 0\\ n_0^{'}(t_0) &= \int_{-\infty}^{0}E_0(\tau+t_0)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau, \quad n_{0p}^{'}(t_0) = \int_{-\infty}^{0}E_0(\tau-t_0)\sin\left(\omega_2(t_0)\tau\right)d\tau\\ m_0{}'(t_0) &= \int_{-\infty}^{0}E_0(\tau+t_0)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)\\ n_0^{'}(t_0) &= e^{2\sigma t_0}[\cos\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau - \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau]\\ n_{0p}^{'}(t_0) &= \cos\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\sin\left(\omega_2(t_0)\tau\right)d\tau + \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau\\ m_0{}'(t_0) &= e^{2\sigma t_0}[\cos\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau + \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin\left(\omega_2(t_0)\tau\right)d\tau]\end{aligned} \tag{E.8}$$

In Section 2.7, we have shown that the above equations which are the same as Eq. 36, lead to a **contradiction** for the asymptotic case $t_0 \to \pm\infty$, **if** Statement 1 is true. This suggests one of the following:

a) Statement 1 is true and above result **contradicts** Statement 5 and hence we can **rule out** pathological case for $\omega_2(t_0)$ **or**

b) Statement 5 is true and **Statement 1 is false** and we **complete the proof** of Theorem 1 at this point. We **do not** require to show that $\omega_2(t_0)$ is **not** pathological, for this case.

Hence the **pathological** case where $\omega_2(t_0)$ is a Weierstrass type of function, which is continuous everywhere but **differentiable nowhere**, leads to a **contradiction**, thus **ruling out** this pathological case.

In Section 2.3, it is shown that $\omega_2(t_0) = \omega_2(-t_0)$ is an **even** function of variable t_0. Hence $\frac{d\omega_2(t_0)}{dt_0} = 0$ at $t_0 = 0$.

We have shown that $\omega_2(t_0)$ is differentiable **at least** once around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$.

Appendix E.1. ***Interchanging order of differentiation and integration***

We consider $R(t_0)$ in Eq. 17 as follows.

$$R(t_0) = \int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_2(t_0)\tau)} d\tau = 0 \tag{E.9}$$

We see that the integrand in Eq. E.9 is a continuous function which is well defined and bounded. We take the first derivative of $R(t_0)$ as follows. We define $R(t_0, a) = \int_{-a}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_2(t_0)\tau)} d\tau$ and see that $\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0} \lim_{a\to\infty} R(t_0, a)$. We define $R_{abs}(t_0, a) = \int_{-a}^{0} |[E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_2(t_0)\tau)}| d\tau$ and see that $R_{abs}(t_0, a) \leq \int_{-a}^{0} |E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)| d\tau = \int_{-a}^{0} E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0) d\tau$ is **finite**, as $a \to \infty$, because $E_0(t) \geq 0$ for $|t_0| \leq \infty$ (Appendix C.1). We see that $|R(t_0, a)| \leq R_{abs}(t_0, a)$ is **finite**, as $a \to \infty$, because magnitude of the integral is less than or equal to the integral of the magnitude of the integrand.

Given that $R(t_0)$ converges for $|t_0| \leq \infty$, we see that $\lim_{a\to\infty} R(t_0, a) < \infty$. Given that $|R(t_0, a)| < \infty$, as $a \to \infty$, we see that $\frac{d}{dt_0} R(t_0, a) < \infty$ and $\lim_{a\to\infty} \frac{d}{dt_0} R(t_0, a) < \infty$ for $|t_0| \leq \infty$. Hence $\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0} \lim_{a\to\infty} R(t_0, a) = \lim_{a\to\infty} \frac{d}{dt_0} R(t_0, a)$ and we can interchange the order of integration and differentiation using Leibnitz integral rule (link) and write as follows.

$$\begin{aligned} \frac{dR(t_0)}{dt_0} = \int_{-\infty}^{0} [\frac{\partial}{\partial t_0} E_0(\tau + t_0)e^{-2\sigma\tau} + \frac{\partial}{\partial t_0} E_{0m}(\tau - t_0)] \cos{(\omega_2(t_0)\tau)} d\tau \\ - \frac{d\omega_2(t_0)}{dt_0} \int_{-\infty}^{0} \tau [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \sin{(\omega_2(t_0)\tau)} d\tau = 0 \end{aligned} \tag{E.10}$$

Appendix F. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned} \Gamma(\frac{s}{2}) &= \int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \end{aligned} \tag{F.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \tag{F.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and
$\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma(\frac{\sigma'}{2}) \pi^{-\frac{\sigma'}{2}} \zeta(\sigma')$.

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \tag{F.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. F.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1+2w(x))\sqrt{x}-1)}{2} dx \tag{F.4}$$

Hence we can simplify Eq. F.4 as follows.

$$\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}} w(x) dx \tag{F.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}] \tag{F.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix F.1. ***Derivation of*** $E_p(t)$ ***and*** $E_0(t)$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. F.6 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t}) dt] \tag{F.7}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \\ + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt]\end{aligned} \tag{F.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \quad \text{(F.9)}$$

We define $A(t)=[\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}
E_p(t)&=\frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2}\\
A(t)&=[\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}\\
\frac{dA(t)}{dt}&=\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2e^{-2t}]u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2e^{2t}]u(t)\\
\frac{d^2A(t)}{dt^2}&=\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2e^{-2t})^2]u(-t)\\
&+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty}e^{-\pi n^2}(1-4\pi n^2)]
\end{aligned} \quad \text{(F.10)}$$

We can simplify above equation as follows.

$$\begin{aligned}
\frac{d^2A(t)}{dt^2}&=\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2n^4e^{-4t}-6\pi n^2e^{-2t}-4\sigma\pi n^2e^{-2t}]u(-t)\\
&\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2n^4e^{4t}-6\pi n^2e^{2t}+4\sigma\pi n^2e^{2t}]u(t)+\delta(t)[\sum_{n=1}^{\infty}e^{-\pi n^2}(1-4\pi n^2)]
\end{aligned} \quad \text{(F.11)}$$

We use the fact that $F(x)=1+2w(x)=\frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x)=\sum_{n=1}^{\infty}e^{-\pi n^2x}$ and $x>0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x=1$. We see that $\sum_{n=1}^{\infty}e^{-\pi n^2}(1-4\pi n^2)=-\frac{1}{2}$ (Appendix F.2) and hence **dirac delta terms cancel each other** in equation below.

$$\begin{aligned}
E_p(t)&=\frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2}\\
E_p(t)&=\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-\frac{1}{4}+\sigma^2+2\sigma(\frac{1}{2}-\sigma-2\pi n^2e^{2t})+\frac{1}{4}+\sigma^2-\sigma+4\pi^2n^4e^{4t}-6\pi n^2e^{2t}+4\sigma\pi n^2e^{2t}]u(t)\\
&+\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{4}+\sigma^2+2\sigma(-\frac{1}{2}-\sigma+2\pi n^2e^{-2t})\\
&+\frac{1}{4}+\sigma^2+\sigma+4\pi^2n^4e^{-4t}-6\pi n^2e^{-2t}-4\sigma\pi n^2e^{-2t}]u(-t)
\end{aligned} \quad \text{(F.12)}$$

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{F.13}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{F.14}$$

Appendix F.2. **Derivation of** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}} \tag{F.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2})$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2} \tag{F.16}$$

Appendix F.3. ***Analytic Functions and Isolated Zeros.***

In this section, we show that $\lim_{t_0 \to \infty} g(t)$ is an analytic function, with the magnitude of the step discontinuity at $t = 0$ decreasing to zero, and its Fourier transform is an analytic function with isolated zeros, as $\lim_{t_0 \to \infty}$. Hence $\lim_{t_0 \to \infty} \omega_2(t_0) = \omega_z \neq 0$ which is a constant.

We see that $g(t) = e^{\sigma t_0} E_p(t + t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p(t + t_0) e^{\sigma t} u(t) = E_0(t + t_0) e^{-2\sigma t} u(-t) + E_0(t + t_0) u(t)$ and its first derivative has a **step** discontinuity at $t = 0$ with magnitude $\Delta_d = 2\sigma E_0(t_0)$. As $\lim_{t_0 \to \infty}$, $\Delta_d \to 0$ because $E_0(t_0)$ decreases to zero as $\lim_{t_0 \to \infty}$ and hence $\lim_{t_0 \to \infty} g(t) = \lim_{t_0 \to \infty} E_0(t + t_0) e^{-2\sigma t} + E_0(t + t_0)$ is an **analytic** function.

We use a **scale factor** and get $g_s(t) = g(t) e^{-2\sigma t_0}$, so that $\lim_{t_0 \to \infty} g_s(t)$ remains **finite** for all $|t| \leq \infty$. This scale factor **does not** affect the location of zeros in the Fourier transform of $g(t)$ and $g_s(t)$. Hence $\lim_{t_0 \to \infty} g_s(t) = \lim_{t_0 \to \infty} E_0(t + t_0) e^{-2\sigma t} e^{-2\sigma t_0} + E_0(t + t_0) e^{-2\sigma t_0} = E_0(t + t_0) e^{-2\sigma (t + t_0)}$ given that $\lim_{t_0 \to \infty} E_0(t + t_0) e^{-2\sigma t_0} = 0$.

The Fourier transform of $g_s(t)$ is given by $G_s(\omega)$ and $\lim_{t_0 \to \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma) e^{i\omega t_0}$.

Hence $\lim_{t_0 \to \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma) e^{i\omega t_0}$ is an **analytic function** for all $|\omega| \leq \infty$ because it is derived from the **entire function** $\xi(s)$ and we know that $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. The same statement holds for $\lim_{t_0 \to \infty} G(\omega)$ which differs only by a scale factor $e^{-2\sigma t_0}$.

We use the well known result that **analytic** functions have **isolated zeros**.(link) Hence $\lim_{t_0 \to \infty} G_s(\omega)$ and $\lim_{t_0 \to \infty} G(\omega)$ have **isolated zeros** at $\omega = \omega_2(t_0) = \omega_z$.

Appendix F.3.1. ***Isolated Zeros are single valued.***

We consider $g_s(t) = g(t) e^{-2\sigma t_0}$ and see that $\lim_{t_0 \to \infty} g_s(t) = E_0(t + t_0) e^{-2\sigma (t + t_0)}$ is an **analytic** function, whose Fourier transform is given by $\lim_{t_0 \to \infty} G_s(\omega) = E_{0\omega}(\omega - i2\sigma) e^{i\omega t_0}$ which is derived from the **entire** function $\xi(s)$ where $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$.

We know that **analytic** functions have **isolated zeros** (link) and each isolated zero has a **single value**. For example, the analytic function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$ corresponding to the **critical line**, is well known to have isolated zeros and each isolated zero has a **single value**. Hence we can expect the analytic function $\lim_{t_0 \to \infty} G_s(\omega)$ to have **isolated zeros** and each isolated zero to have a **single value**. Hence $\lim_{t_0 \to \infty} \omega_2(t_0) = \omega_z$ is a well defined constant.