On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix D, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3]

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider a new function $g(t)=f(t)e^{-\sigma t}u(-t)+f(t)e^{\sigma t}u(t)$, where $f(t)=e^{-2\sigma t_0}f_1(t)+e^{2\sigma t_0}f_2(t)$ and $f_1(t)=e^{\sigma t_0}E_p^{'}(t+t_0)$ and $f_2(t)=e^{-\sigma t_0}E_p^{'}(t-t_0)$ and $E_p^{'}(t)=e^{-\sigma t_2}E_p(t-t_2)-e^{\sigma t_2}E_p(t+t_2)$ and t_0,t_2 are real and $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t)=f(t)$ where $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t)=\frac{1}{2}[g(t)+g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega=\omega_z(t_2,t_0)\neq0$, for every value of t_0, to satisfy Statement 1, where $\omega_z(t_2,t_0)$ is real and finite.

1.3. *Step 3: On the zeros of the function* $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g(t)$ and compute its real part given by $G_R(\omega)=G_R(\omega,t_2,t_0)$ and we can write as follows.

$$\begin{aligned}G_R(\omega,t_2,t_0)=e^{-2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0)]\cos(\omega\tau)d\tau\\+e^{2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0)]\cos(\omega\tau)d\tau\end{aligned} \tag{3}$$

We require $G_R(\omega,t_2,t_0)=0$ for $\omega=\omega_z(t_2,t_0)$ for every value of t_0, for **each fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2,t_0)\neq\omega_0$. Hence we can see that $P(t_2,t_0)=G_R(\omega_z(t_2,t_0),t_0)=0$.

1.4. *Step 4: Zero Crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0*

In Section 2.3, we show the result in Eq. 4 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ is an **odd** function of t_0, for all t_0, for a given value of t_2 as follows.

$$\begin{aligned} P_{odd}(t_2, t_0) = [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0'(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau)d\tau \\ + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0'(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau)d\tau] \\ + e^{2\sigma t_0}[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}'(\tau) \cos(\omega_z(t_2, t_0)\tau)d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}'(\tau) \sin(\omega_z(t_2, t_0)\tau)d\tau] \end{aligned} \tag{4}$$

1.5. *Step 5: Final Step*

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2. In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2, t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use $E_0'(t) = E_0(t - t_2) - E_0(t + t_2)$ and $E_{0n}'(t) = E_0'(-t)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0 \tag{5}$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $\tau = [0, t_{0c}]$ where $t_{0c} > 0$. For $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, we see that $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi}{2t_{0c}}\tau$ lies in the range $[0, \frac{\pi}{2}]$ and hence $\sin(\omega_{c1}\tau) > 0$ in that interval $\tau = [0, t_{0c}]$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix C.1.

2.1. ***New function*** $g(t)$

Let us consider the function $E_p'(t) = E_p'(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t)e^{-\sigma t}$, where t_2 is finite and real, and $E_0'(t) = E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E_{p\omega}'(\omega) = E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0)$ and $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega) = F(\omega, t_2, t_0) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$.

Let us consider a new function $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), F(\omega)$ and $G(\omega)$ which are also functions of variables t_2, t_0. Similarly we use the shorthand notation for the functions $E_p'(t), E_0'(t)$ and $E_{p\omega}'(\omega)$ which are also functions of variable t_2.

We can show that $E_p(t), E_p'(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}'(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_z(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by

$G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0, where $\omega_z(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = e^{\sigma t_0} E_p^{'}(t + t_0)$ and $f_2(t) = e^{-\sigma t_0} E_p^{'}(t - t_0)$, $E_p^{'}(t) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$, and $h(t) = e^{\sigma t} u(-t) + e^{-\sigma t} u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}^{\prime}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \tag{6}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 6 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{7}$$

We can split the integral in Eq. 7 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{8}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 8 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega') [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{9}$$

In Appendix C.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 9 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

Corollary A: Given that $G_R(\omega)$ is not an all zero function, we see that there must exist at least one zero at finite $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega)$ **crosses the zero line** to the opposite sign. If $G_R(\omega) > 0$ at $\omega = \omega_z(t_2, t_0) - d\omega$, then $G_R(\omega) < 0$ at $\omega = \omega_z(t_2, t_0) + d\omega$. If $G_R(\omega) < 0$ at $\omega = \omega_z(t_2, t_0) - d\omega$, then $G_R(\omega) > 0$ at $\omega = \omega_z(t_2, t_0) + d\omega$. Because if $G_R(\omega)$ does not cross the zero line at $\omega = \omega_z(t_2, t_0) \neq 0$, then $F_R(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega')H(\omega - \omega')d\omega'$ and $E_R(\omega)$ cannot have a zero at $\omega = \omega_0$ which contradicts Statement 1.

2.2. *On the zeros of a related function* $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, for a given value of t_2, to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the Fourier transform of the function $g_1(t)$ given by $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$. We use $g_1(t) = f_1(t)e^{-\sigma t}u(-t) + f_1(t)e^{\sigma t}u(t) = e^{\sigma t_0}E_p'(t + t_0)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p'(t + t_0)e^{\sigma t}u(t)$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$ and $G_1(\omega)$ which are also functions of variables t_2, t_0. Similarly we use the shorthand notation for the functions $E_p'(t), E_0'(t)$ and $E_{0n}'(t)$ which are also functions of variable t_2.

$$G_1(\omega) = \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^{0} g_1(t)e^{-i\omega t}dt + \int_{0}^{\infty} g_1(t)e^{-i\omega t}dt$$
$$G_1(\omega) = \int_{-\infty}^{0} e^{\sigma t_0}E_p'(t + t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0}E_p'(t + t_0)e^{\sigma t}e^{-i\omega t}dt \tag{10}$$

We use $E_p'(t) = E_0'(t)e^{-\sigma t}$ where $E_0'(t) = E_0(t-t_2) - E_0(t+t_2)$ and $E_p'(t+t_0) = E_0'(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 10, we have

$$G_1(\omega) = \int_{-\infty}^{0} E_0'(t + t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_0'(t + t_0)e^{-i\omega t}dt$$
$$G_1(\omega) = \int_{-\infty}^{0} E_0'(t + t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_0'(-t + t_0)e^{i\omega t}dt \tag{11}$$

We define $E_{0n}'(t) = E_0'(-t)$ and get $E_0'(-t + t_0) = E_{0n}'(t - t_0)$ and write Eq. 11 as follows.

$$G_1(\omega) = \int_{-\infty}^{0} E_0'(t + t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_{0n}'(t - t_0)e^{i\omega t}dt = G_R(\omega) + iG_I(\omega) \tag{12}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^{0} E_0'(t + t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0n}'(t - t_0)\cos(\omega t)dt \tag{13}$$

2.3. *Zero crossing function* $\omega_z(t_2, t_0)$ *is an even function of variable* t_0

Now we consider the function $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t) = e^{-\sigma t_0} E_p^{'}(t+t_0) + e^{\sigma t_0} E_p^{'}(t-t_0)$ where $f_1(t) = e^{\sigma t_0} E_p^{'}(t+t_0)$ and $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0} E_p^{'}(t-t_0)$ and $g(t)h(t) = f(t)$ where $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g(t)$ and compute its real part using the procedure in above section, similar to Eq. 13 and we can write as follows. We substitute $t = \tau$.

$$
\begin{aligned}
G_R(\omega, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_0) + e^{2\sigma t_0} G_{1R}(\omega, -t_0) \\
G_{1R}(\omega, t_0) &= \int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)] \cos(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0)] \cos(\omega\tau) d\tau
\end{aligned}
\tag{14}
$$

We require $G_R(\omega, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ and we can rearrange the terms as follows.

$$
\begin{aligned}
P(t_2, t_0) = &\int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau+t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\
+ &\int_{-\infty}^{0} [e^{2\sigma t_0} E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + e^{-2\sigma t_0} E_{0n}^{'}(\tau-t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0
\end{aligned}
\tag{15}
$$

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0.

$$
\begin{aligned}
P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\
P_{odd}(t_2, t_0) &= \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau+t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau
\end{aligned}
\tag{16}
$$

We see that $f(t, t_0) = e^{-\sigma t_0} E_p^{'}(t+t_0) + e^{\sigma t_0} E_p^{'}(t-t_0) = f(t, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2.

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 16 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E_{0n}^{'}(\tau) = E_0^{'}(-\tau)$ and $E_0^{'}(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

We **note** that we use the **shorthand** notation for the functions $E_0^{'}(t)$ and $E_{0n}^{'}(t)$ which are also functions of variable t_2.

$$
\begin{aligned}
P_{odd}(t_2, t_0) = [&\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&+ \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
+ e^{2\sigma t_0} [&\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}
$$

(17)

In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for a given value of t_2. This means there are **no** Dirac delta functions present in $\omega_z(t_2, t_0)$.

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2.

In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0, for every value of t_2, and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, we see that $\omega_z(t_2, t_0)t_0$ will **certainly pass through** π, as t_0 is increased from zero to ∞.

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 17 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\begin{aligned}
&\int_{-\infty}^{t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}}\int_{-\infty}^{t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
&-\int_{-\infty}^{-t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0
\end{aligned} \tag{18}$$

We split the integral in the left hand side of Eq. 18 and write as follows.

$$\begin{aligned}
&[\int_{-\infty}^{-t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \\
&+e^{2\sigma t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \\
&-\int_{-\infty}^{-t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0
\end{aligned} \tag{19}$$

We combine the terms with common integrals and cancel common terms in Eq. 19 as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] + e^{2\sigma t_{0c}}\int_{-t_{0c}}^{t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
&= -2\sinh(2\sigma t_{0c})\int_{-\infty}^{-t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau
\end{aligned} \tag{20}$$

We can rearrange the terms in Eq. 20 as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} [E_0'(\tau)e^{-2\sigma\tau} + E_{0n}'(\tau)e^{2\sigma t_{0c}}]\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
&= -2\sinh(2\sigma t_{0c})\int_{-\infty}^{-t_{0c}} E_{0n}'(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau
\end{aligned} \tag{21}$$

We denote the right hand side of Eq. 21 as RHS. We can split the integral in Eq. 21 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\int_{-t_{0c}}^{0} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ + \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \tag{22}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 22 as follows. We use $E_0^{'}(-\tau) = E_{0n}^{'}(\tau)$ and $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$\int_{t_{0c}}^{0} [E_{0n}^{'}(\tau)e^{2\sigma\tau} + E_0^{'}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ + \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \tag{23}$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \tag{24}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 21 as follows. We use $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$RHS = 2\sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E_0^{'}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{25}$$

We split the integral on the right hand side in Eq. 25 as follows.

$$RHS = 2\sinh(2\sigma t_{0c})[\int_{0}^{\infty} E_0^{'}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{0}^{t_{0c}} E_0^{'}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{26}$$

We consolidate the integrals with the term $\int_{0}^{t_{0c}} E_0^{'}(\tau)$ in Eq. 24 and Eq. 26 as follows. We use $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_{0}^{\infty} E_0^{'}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{27}$$

We cancel common terms in Eq. 27 as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_{0}^{\infty} E_0^{'}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$

$$(28)$$

We substitute $E_0^{'}(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E_{0n}^{'}(\tau) = E_0^{'}(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$. Hence we see that $E_{0n}^{'}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0^{'}(\tau)$. We can write Eq. 28 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= 2\sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{29}$$

We substitute $2\cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2\cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 29 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= \sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{30}$$

Next Step:

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 30 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$RHS = \sinh(2\sigma t_{0c})[\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$+ \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$- \cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \tag{31}$$

In Eq. 31, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$. Hence we cancel common terms and write Eq. 31 and Eq. 30 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= -\sinh(2\sigma t_{0c})[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \tag{32}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 32 as follows. Given that $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ, we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{33}$$

We can multiply Eq. 33 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{34}$$

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and hence the sinusoidal function varies over a **half cycle** and is > 0, in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 34, we see that in the interval $0 < \tau < t_{0c}$, the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t \geq \frac{1}{8}$, we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 4.7) . The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 34. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 34 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

4. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. (link)

$$
\begin{aligned}
E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_0(t) = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]
\end{aligned}
\tag{35}
$$

We show that $X(t) = \frac{E_0(t)}{2\pi}$ is a **strictly decreasing** function for $t \geq 0$ as follows.

• In Section 4.1, it is shown that the second derivative of $X(t)$, given by $X_2(t) = \frac{d^2X(t)}{dt^2} < 0$ for $t = 0$.

• In Section 4.2, it is shown that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. (example plot) Hence $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_{min}$.

• In Section 4.3, it is shown that $\frac{dX(t)}{dt} < 0$ for $t \geq t'_{min}$, where t'_{min} corresponds to the minima of $X_{11}(t)$ which is the partial term in $\frac{dX(t)}{dt}$ corresponding to $n = 1$.

• In Section 4.4, it is shown that $\frac{dX(t)}{dt} < 0$ for $t_{min} < t < t'_{min}$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is **strictly decreasing** for all $t > 0$ and $E_0(t) = 2\pi X(t)$ is **strictly decreasing** for all $t > 0$.

4.1. $\frac{d^2X(t)}{dt^2} < 0$ *for* $t = 0$

We consider $X(t) = \frac{E_0(t)}{2\pi} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$ and take the first derivative of $X(t)$ as follows. We note that $E_0(t)$ is an analytic function for $|t| \leq \infty$ and is infinitely differentiable in that interval.

$$
\begin{aligned}
\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t})(\frac{1}{2} - 2\pi n^2 e^{2t})] \\
\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})] \\
\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}] \\
\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]
\end{aligned}
\tag{36}
$$

We take the second derivative of $X(t)$ as follows.

$$\frac{d^2X(t)}{dt^2} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}}[-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t} + (-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2})(\frac{5}{2} - 2\pi n^2 e^{2t})]$$

$$\frac{d^2X(t)}{dt^2} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}}[-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t}$$

$$+(-10\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t} - \frac{75}{4} + 8\pi^3 n^6 e^{6t} - 30\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t})]$$

$$\frac{d^2X(t)}{dt^2} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t), \quad Z(t) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4} \tag{37}$$

- **Case 1:** $n = 1, t = 0$: The partial term for $n = 1$ and $t = 0$ in $\frac{d^2X(t)}{dt^2}$ in Eq. 37, given by $A_1 < 0$ shown as follows.

$$A_1 = e^{-\pi}(8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4}) < 0 \tag{38}$$

At $n = 1, t = 0$, we see that $Z(t) = 8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4} = \pi(8\pi^2 - 56\pi + \frac{165}{2}) - \frac{75}{4} \leq 3*(8*10-56*3+83)-18 = -33 < 0$ (**Result K**), because $3 < \pi < 3.1429$ and $\pi^2 < 10$. We note that the term in the brackets is negative and hence we multiply it by the minimum value of π. Hence we see that the partial term for $n = 1$ and $t = 0$ in $\frac{d^2X(t)}{dt^2}$ given by $A_1 < 0$.

- **Case 2:** $n > 1, t = 0$: the partial terms for $n > 1$ and $t = 0$ in $\frac{d^2X(t)}{dt^2}$ in Eq. 37, given by $A_2 > 0$ and $A_2 < |A_1|$ shown as follows.

$$A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2}(8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4}) > 0 \tag{39}$$

At $n = 2, t = 0$, we see that $Z(t) = 8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4} = \pi n^2(8\pi^2 n^4 - 56\pi n^2 + \frac{165}{2}) - \frac{75}{4} > 3*2^2(8*3^2*2^4 - 56*4*2^2 + 82) - 19 = 4037 > 0$, because $3 < \pi < 4$. At $n > 2, t = 0$, we see that $Z(t) > 0$ due to the dominant term $8\pi^2 n^4$. Hence the partial terms for $n > 1$ and $t = 0$ in $\frac{d^2X(t)}{dt^2}$ in Eq. 37, given by $A_2 > 0$.

- **We can show** that at $t = 0$, $A_1 \leq -0.4283$ and $A_2 < |A_1|$ and hence $\frac{d^2X(t)}{dt^2} < 0$ in Eq. 37.

At $n = 1, t = 0$, we see that, given $3 < \pi < \frac{22}{7} = 3.1429$ (link) and $e^{-\pi} \geq e^{-0.1429}(e^{-\frac{1}{2}})^6$ and $e^{-0.1429} \geq 1 - 0.1429 = 0.8571$ and $e^{-\frac{1}{2}} \geq 1 - \frac{1}{2} = \frac{1}{2}$, we can write $e^{-\pi} \geq \frac{0.8571}{2^6}$ and $|A_1| = e^{-\pi}|Z(t)| \geq 33 * \frac{0.8571}{2^6} \geq 32 * \frac{0.8571}{64} = 0.4283$(using Result K). Hence we see that the partial term for $n = 1, t = 0$ in $\frac{d^2X(t)}{dt^2}$ in Eq. 37, given by $A_1 \leq -0.4283$ (**Result A**).

At $n > 1, t = 0$, we see that $A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} Z(0) \leq \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} * (8\pi^3 n^6 + \frac{165}{2}\pi n^2)$ and we **want to show** $A_2 < |A_1|$ where $A_1 \leq -0.4283$. We use the fact that $(n^2)^4 e^{-\pi n^2} < e^{-(\pi-1)n^2}$ and $(n^2)^2 e^{-\pi n^2} < e^{-(\pi-1)n^2}$.

$$A_2 \leq \sum_{n=2}^{\infty} e^{-(\pi-1)n^2}(8\pi^3 + \frac{165}{2}\pi) \tag{40}$$

We use $3 < \pi < 3.1429$ and $\sum_{n=2}^{\infty} f(n) \leq \int_{u=2}^{\infty} f(u)du$.

$$A_2 \le (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-(\pi-1)u^2} du \le (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-2u^2} du \tag{41}$$

We substitute $\sqrt{2}u = t$ as follows. We use $8\pi^3 + \frac{165}{2}\pi = \pi(8\pi^2 + \frac{165}{2}) \le 3.1429(8 * 10 + 83) \le 514$. We use the complementary error function $erfc(z)$ and $\frac{\sqrt{\pi}}{2\sqrt{2}} \le 1$ as follows. (link)

$$\begin{aligned} erfc(z) &= \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt \\ A_2 \le \frac{513}{\sqrt{2}} \int_{u=2\sqrt{2}}^{\infty} e^{-t^2} dt \le \frac{514}{\sqrt{2}}(\frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2})) &\le \frac{514}{\sqrt{2}}\frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2}) \\ A_2 \le 514 * erfc(2\sqrt{2}) &= 0.0326 \end{aligned} \tag{42}$$

We can see that $A_1 + A_2 < 0$ given that $A_1 \le -0.4283$ (using Result A).

We use the fact that $erfc(z) \le \frac{e^{-z^2}}{z\sqrt{\pi}}$ (link) . Hence $A_2 \le 514\frac{e^{-8}}{2\sqrt{2\pi}} \le 514\frac{e^{-8}}{4} \le 129e^{-8}$. We can show that $A_1 + A_2 < 0$ as follows, using Eq. 38.

$$\begin{aligned} A_2 \le 129e^{-8}, \quad A_1 \le -33e^{-\pi} \le -33e^{-3}, \quad A_2 &< |A_1| \\ A_1 + A_2 \le -33e^{-3} + 129e^{-8} = e^{-3}(-33 + 129e^{-5}) \\ A_1 + A_2 \le e^{-3}(-33 + 129 * 2^{-5}) = e^{-3}(-33 + \frac{129}{32}) &< 0 \\ A_1 + A_2 &< 0 \\ (\frac{d^2X(t)}{dt^2})_{t=0} = A_1 + A_2 &< 0 \end{aligned} \tag{43}$$

We have shown that the second derivative of $X(t)$, given by $X_2(t) = \frac{d^2X(t)}{dt^2} < 0$ for $t = 0$. (**Result H**)

4.2. $\frac{dX(t)}{dt} < 0$ *for* $0 < t \le t_{min}$

• In the sections below, **we will show** that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. Thus we will show that $X(t)$ is a **strictly decreasing** function of t. We will also show that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. (example plot)

• We see that $\frac{d^2X(t)}{dt^2} < 0$ at $t = 0$ (using Result H) . Hence, as t increases from zero (point A), $\frac{dX(t)}{dt}$ reaches a **negative minimum** value at $t = t_{min}$(point B) and then starts increasing towards zero, in the neighborhood of $t > t_{min}$. We see that $\frac{d^2X(t)}{dt^2}$ remains negative in $0 \le t < t_{min}$ and then reaches a **zero** at $t = t_{min}$ and then becomes **positive** in the neighborhood of $t > t_{min}$. (example plot) Hence $\frac{dX(t)}{dt} < 0$ for $0 < t \le t_{min}$. (**Result I**)

4.3. $\frac{dX(t)}{dt} < 0$ *for* $t \ge t'_{min}$

We note that $\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t)$ where $X_{11}(t)$ has partial term corresponding to $n = 1$ and $X_{12}(t)$ has partial terms corresponding to $n > 1$.

$$\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t}[-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]$$
$$X_{11}(t) = e^{-\pi e^{2t}} e^{\frac{t}{2}} e^{2t}[-4\pi^2 e^{4t} + 15\pi e^{2t} - \frac{15}{2}]$$
$$X_{12}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t}[-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}] \tag{44}$$

In the interval $t > t_{min}$, $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ reaches a **positive maximum** at $t = t_{max2}$ (point C) and then starts decreasing to zero, in the neighborhood $t > t_{max2}$. $X_{21}(t)$ has partial term corresponding to $n = 1$ and $X_{22}(t)$ has partial terms corresponding to $n > 1$. This means that $\frac{dX(t)}{dt}$ **increases** towards zero, in the neighborhood $t > t_{min}$. (example plot)

$$\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t), \quad Z(t) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4}$$
$$X_{21}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} (8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2}\pi e^{2t} - \frac{75}{4})$$
$$X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4}) \tag{45}$$

As t increases from zero (point A), $X_{11}(t)$ in Eq. 44 starts from a positive value and **decreases** to a negative value, passing through **zero** at $t = t_z$. Given that $\lim_{t\to\infty} X_{11}(t) = 0$, it reaches a **negative minimum** value at $t = t'_{min}$(point B') and then starts **increasing** towards zero, for $t > t'_{min}$ (**Result L**). (example plot) We note that $X_{11}(t)$ **remains negative** for $t > t_z$ due to the dominant term $-4\pi^2 e^{4t}$ in Eq. 44.

This means that the derivative of $X_{11}(t)$ given by $X_{21}(t)$ in Eq. 45 is **negative** at $0 \le t < t'_{min}$ and $X_{21}(t)$ reaches a **zero** at $t = t'_{min}$ (**Result N**) and then becomes **positive** for $t > t'_{min}$ (using Result L) and falls to zero as $t \to \infty$, given that $\lim_{t\to\infty} X_{21}(t) = 0$. We note that, when $X_{21}(t)$ in Eq. 45 becomes positive after crossing $t = t'_{min}$, it **remains positive** as $t \to \infty$ due to the dominant term $8\pi^3 e^{6t}$ in Eq. 45(**Result B**).

Hence $X_{11}(t)$ is a **strictly decreasing** function for $t < t'_{min}$ and then it starts **increasing** towards zero as $t \to \infty$. We note that $X_{11}(t)$ **cannot** become positive at some $t > t'_{min}$, because if it did become positive again, then it would have to decrease to zero as $t \to \infty$, which would **require** $X_{21}(t)$ to become negative again, which is **not** the case as shown in Result B.

We will show that $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) = 0$ **only for** $t = t_{min}$, and then it becomes positive and then starts decreasing towards zero, as $t \to \infty$.

In Section 4.5, it is shown that $X_{22}(t)$ is **strictly decreasing** for $t > 0$ (using Result E) and hence $X_{22}(t) > 0$ given that $\lim_{t\to\infty} X_{22}(t) = 0$. (**Result C**).

We note that $t_{min} < t'_{min}$ given that $X_{21}(t) > 0$ for $t > t'_{min}$ (using Result B) and $X_{22}(t) > 0$ for $t > t'_{min}$ (using Result C) and hence $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ for $t > t'_{min}$ **(Result D)**.

We see that $X_{12}(t) < 0$ for all $t > 0$ in Eq. 44 due to the dominant term $-4\pi^2 n^4 e^{4t}$. Because $\frac{dX(t)}{dt} = X_{11}(t)+X_{12}(t)$ is negative at $t = t'_{min}$, we see that $\frac{dX(t)}{dt}$ starts **increasing** from a negative value towards zero. (using Result D) Hence $\frac{dX(t)}{dt} < 0$ for $t \ge t'_{min}$. **(Result G)**

We note that $\frac{dX(t)}{dt}$ **cannot** become positive at some $t > t'_{min}$, because if it did become positive again, then it would have to decrease to zero as $t \to \infty$, which would **require** $\frac{d^2X(t)}{dt^2}$ to become negative again, which is **not** the case as shown in Result D.

4.4. $\frac{dX(t)}{dt} < 0$ *for* $t_{min} < t < t'_{min}$

We can show that the second derivative $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ becomes zero **only once** at $t = t_{min}$, in the interval $0 < t < t'_{min}$. At $t = t_{min}$, we see that $X_{22}(t_{min}) = X_0 > 0$ and hence $X_{21}(t_{min}) = -X_0 < 0$. In the interval $t_{min} < t < t'_{min}$, we see that $X_{22}(t)$ is **strictly decreasing** and remains positive (using Result E in Section 4.5) and decreases from X_0 further towards zero at a **slower** rate, while $X_{21}(t)$ increases from the negative value $-X_0$ towards zero **faster**, to make $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$.

We see that $X_{21}(t)$ is **strictly increasing** for $t_{min} < t < t'_{min}$(using Result F in Section 4.6) and **reaches zero** at $t = t'_{min}$ at a **faster** rate than $X_{22}(t)$. Hence $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ in the interval $t_{min} < t < t'_{min}$(**Result M**). Hence $\frac{dX(t)}{dt}$ is **increasing** from a negative value towards zero, in the interval $t_{min} < t < t'_{min}$.

We can **rule out** the Case A that $\frac{dX(t)}{dt} > 0$ somewhere in the interval $t > t_{min}$ and reaches a maximum at $t = t_{max}$, as follows. We see that $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$ becomes zero **only once** at $t = t_{min}$, in the interval $0 < t < t'_{min}$ and is positive for $t_{min} < t < t'_{min}$ (using Result M) and **does not** become zero again at $t = t_{max}$, which is required for Case A. Hence $\frac{dX(t)}{dt} < 0$ in the interval $t_{min} < t < t'_{min}$. **(Result J)**

We have shown in earlier sections that $\frac{dX(t)}{dt} < 0$ in the interval $0 < t \leq t_{min}$ (using Result I) and $t \geq t'_{min}$ (using Result G). We see that $\frac{dX(t)}{dt} < 0$ in the interval $t_{min} < t < t'_{min}$ (using Result J) and hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is **strictly decreasing** for all $t > 0$. Hence $E_0(t) = 2\pi X(t)$ is **strictly decreasing** for all $t > 0$.

4.5. *Second derivative given by* X_{22} *is a strictly decreasing function for* $t > 0$

We consider $X_{22}(t)$ as follows.

$$X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4}) \tag{46}$$

We see that $X_{22}(t)$ in Eq. 46, is **positive** for each $n = 2, 3, ..$ and hence for **all** $t \geq 0$, as follows. We see that $X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (\pi^2 n^4 e^{4t}(8\pi n^2 e^{2t} - 56) + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})$. For $n = 2, t = 0$, we see that $X_{22}(t) = 2^2 e^{-\pi 2^2}(8\pi^3 2^6 - 56\pi^2 2^4 + \frac{165}{2}\pi 2^2 - \frac{75}{4}) = 4e^{-4\pi}(\pi^2 2^4(32\pi - 56) + 165 * 2\pi - \frac{75}{4}) > 0$. For $n \geq 2$ and $t \geq 0$, $X_{22}(t) > 0$ due to the **dominant term** $8\pi^3 n^6 e^{6t}$.

We compute the **derivative** of $X_{22}(t)$ as follows.

$$X_{32}(t) = \frac{dX_{22}(t)}{dt} = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t}$$
$$+ (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})(\frac{5}{2} - 2\pi n^2 e^{2t})]$$
$$X_{32}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t}$$
$$+ (20\pi^3 n^6 e^{6t} - 28 * 5\pi^2 n^4 e^{4t} + \frac{165 * 5}{4}\pi n^2 e^{2t} - \frac{75 * 5}{8} - 16\pi^4 n^8 e^{8t} + 56 * 2\pi^3 n^6 e^{6t} - 165\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t})]$$
$$X_{32}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Y'(t), \quad Y'(t) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75 * 5}{8}$$

(47)

We examine the term $Y'(t) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75*5}{8}$ in Eq. 47. We see that, for $n = 2$ and $t = 0$, $Y'(t) = -\pi^3 2^6(16\pi * 4 - 180) - \pi 2^2(529\pi * 2^2 - 408.75) - \frac{75*5}{8} < 0$. We see that, **for all** $n > 1$ and $t \geq 0$, $Y'(t) < 0$. Hence $X_{32}(t) < 0$ for all $t \geq 0$.

Hence $X_{22}(t)$ is a **strictly decreasing function** for $t > 0$.(**Result E**) (example plot)

4.6. *Second derivative given by* X_{21} *is a strictly increasing function for* $t_{min} < t < t'_{min}$

We can show that $X_{21}(t)$ is a **strictly increasing function** for $t_{min} < t < t'_{min}$. We take the derivative of $X_{21}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}}(8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2}\pi e^{2t} - \frac{75}{4})$, given by $X_{31}(t)$ and set $n = 1$ in the summand in Eq. 47 as follows.

$$X_{31}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} Z'(t), \quad Z'(t) = -16\pi^4 e^{8t} + 180\pi^3 e^{6t} - 529\pi^2 e^{4t} + 408.75\pi e^{2t} - \frac{75*5}{8} \tag{48}$$

We see that $X_{21}(t)$ in Eq. 45 is **negative** at $0 \leq t < t'_{min}$ (using Result N). In the interval $t_{min} < t < t'_{min}$, we consider 3 cases for $X_{21}(t)$ as follows.

Case 1: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and increases to zero at $t = t'_{min}$. This is **possible** because we know that $X_{21}(t)$ in Eq. 45 is **negative** at $0 \leq t < t'_{min}$ and $X_{21}(t) = 0$ at $t = t'_{min}$ (using Result N).

Case 2: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and decreases to a more negative value and reaches a negative minimum at $t = t_c < t'_{min}$ and then starts increasing towards zero. This requires $X_{31}(t) < 0$ in the interval $t_{min} < t < t_c$ and $X_{31}(t) > 0$ in the interval $t_c < t < t'_{min}$, which is **NOT** possible because of the dominant term $-16\pi^4 e^{8t}$ in Eq. 48.

Case 3: $X_{21}(t)$ starts from the negative value at $t = t_{min}$ and increases to a less negative value and has a point of inflection at $t = t_d < t'_{min}$ and then starts decreasing towards a more negative value and has a second point of inflection at $t = t_e < t'_{min}$ and then starts increasing towards zero. This requires $X_{31}(t) < 0$ in the interval $t_d < t < t_e$ and $X_{31}(t) > 0$ in the interval $t_e < t < t'_{min}$, which is **NOT** possible because of the dominant term $-16\pi^4 e^{8t}$ in Eq. 48.

Similarly, we can extend the argument to the case where $X_{21}(t)$ has many points of inflection in the interval $t_{min} < t < t'_{min}$.

Hence we see that only **Case 1** is possible and hence $X_{21}(t)$ is a **strictly increasing function** for $t_{min} < t < t'_{min}$.(**Result F**) (example plot)

4.7. *Result* $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 4 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 34 .

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t, and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c})$ to $E_0(3t_{0c})$, which is **less than** $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c})$ to $E_0(-t_{0c})$. Hence we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 34 , for $t_{0c} > 0$.

5. $\omega_z(t_2, t_0)$ is a continuous function of t_0

We see from Section 2.1 that $\omega_z(t_2, t_0)$ is shown to be **finite and non-zero** for all $|t_0| < \infty$ and that $\omega_z(t_2, t_0)$ is an even function of variable t_0, for a given value of t_2. For a given t_2 and t_0, $\omega_z(t_2, t_0)$ can have more than one value, but we consider only the first zero crossing away from origin in the section below, where $G_R(\omega)$ crosses the zero line to the opposite sign, as detailed in **Corollary A** in Section 2.1 and $\frac{\partial G_R(\omega,t_2,t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$. We consider the case $G_R(\omega) > 0$ at $\omega = \omega_z(t_2, t_0) - d\omega$ in the section below. example plot) The arguments are similar for the case $G_R(\omega) < 0$ for $\omega = \omega_z(t_2, t_0) - d\omega$.

We consider the Fourier transform of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0, the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$, for **each** fixed value of t_2. This is shown in 3 steps below.

- It is shown in Section 5.1 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω, for **each** value of t_0 and t_2 as shown in Eq. 49.

- For each **fixed** value of ω, $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_0, as shown in Section 5.2 and Eq. 50. This is also true for the zero crossing point given by $\omega_z(t_2, t_0)$.

- It is shown in Section 5.3 that the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **continuous** function of t_0, for a given t_2, using **Implicit Function Theorem**.

5.1. $G_R(\omega, t_2, t_0)$ ***is partially differentiable twice as a function of*** ω***, for each fixed value of*** t_0 ***and*** t_2

$G_R(\omega) = G_R(\omega, t_2, t_0)$ in Eq. 14 is copied below, which is a **continuous** function of ω which is partially differentiable **at least** twice with respect to ω and the integrals converge in Eq. 49 for $0 < \sigma < \frac{1}{2}$, because the term $\tau^2 E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau}$ has asymptotic fall-off rate of $\tau^2 e^{(\frac{5}{2} - 2\sigma)}$ and the integrands are absolutely integrable.

$$
\begin{aligned}
G_R(\omega) = G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} &= -[e^{-2\sigma t_0} \int_{-\infty}^{0} \tau [E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} \tau [E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \\
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial \omega^2} &= -[e^{-2\sigma t_0} \int_{-\infty}^{0} \tau^2 [E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} \tau^2 [E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau]
\end{aligned}
\tag{49}
$$

5.2. $G_R(\omega, t_2, t_0)$ ***is partially differentiable twice as a function of*** t_0***, for each fixed value of*** ω ***and*** t_2

Consider the **segment** S in $G_R(\omega, t_2, t_0)$ in the neighborhood around the first zero crossing where $\frac{\partial G_R(\omega,t_2,t_0)}{\partial \omega} < 0$. (Segment S is the portion between the majenta lines in example plot)

In the **segment** S, $G_R(\omega, t_2, t_0)$ in Eq. 49 is a **continuous** function of ω, for **each** value of t_0 and t_2 as shown in Eq. 49 and $\frac{\partial G_R(\omega,t_2,t_0)}{\partial \omega} < 0$ in the neighborhood around the **first zero crossing**.

• If we **fix** the X-coordinate ω and t_2, $G_R(\omega, t_2, t_0)$ is a **continuous** function of t_0, for **each** fixed value of ω, given that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice and the integrals converge in Eq. 50 and Eq. 51. Hence, for **each** fixed value of ω, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_2, t_0 - \delta t_0)$ and $G_R(\omega, t_2, t_0 + \delta t_0)$, move towards $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0} \int_{-\infty}^{0} \frac{d(E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{dt_0} \cos(\omega\tau) d\tau \\
&\quad + 2\sigma e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{0} \frac{d(E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{dt_0} \cos(\omega\tau) d\tau
\end{aligned} \tag{50}$$

The second derivative of $G_R(\omega, t_2, t_0)$ is given by

$$\begin{aligned}
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad - 4\sigma e^{-2\sigma t_0} \int_{-\infty}^{0} \frac{d(E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{dt_0} \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0} \int_{-\infty}^{0} \frac{d^2(E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{dt_0^2} \cos(\omega\tau) d\tau \\
&\quad + 4\sigma^2 e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + 4\sigma e^{2\sigma t_0} \int_{-\infty}^{0} \frac{d(E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{dt_0} \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{0} \frac{d^2(E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{dt_0^2} \cos(\omega\tau) d\tau
\end{aligned} \tag{51}$$

We consider the term $E_0^{'}(\tau + t_0, t_2)$ first. We see that $E_0^{'}(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$. We consider the term $E_0(\tau + t_2)$ first and can show that the integrals converge in Eq. 50 and Eq. 51, as follows.

$$\begin{aligned}
E_0(\tau) &= 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau + t_2 + t_0) &= 2\sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned} \tag{52}$$

We can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows.

$$\frac{\partial}{\partial t_0}E_0(\tau+t_2+t_0) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau}e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})]$$

$$\frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau}e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \tag{53}$$

Similarly we can show that $\frac{\partial}{\partial t_0}E_0(\tau+t_2-t_0) = -\frac{\partial}{\partial \tau}E_0(\tau+t_2-t_0)$ and we can write one of the terms corresponding to the term $E_0(\tau+t_2)$ in the second integral in Eq. 51 as follows.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{d(E_0(\tau+t_2+t_0,t_2)e^{-2\sigma\tau})}{dt_0}\cos(\omega\tau)d\tau &= \int_{-\infty}^{0} \frac{d(E_0(\tau+t_2+t_0,t_2))}{d\tau}e^{-2\sigma\tau}\cos(\omega\tau)d\tau \\
= \int_{-\infty}^{0} \frac{d(E_0(\tau+t_2+t_0,t_2)e^{-2\sigma\tau}\cos(\omega\tau))}{d\tau}d\tau &- \int_{-\infty}^{0} E_0(\tau+t_2+t_0,t_2))\frac{d(e^{-2\sigma\tau}\cos(\omega\tau)}{d\tau}d\tau \\
= [E_0(\tau+t_2+t_0,t_2)e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0} E_0(\tau+t_2+t_0,t_2))e^{-2\sigma\tau}\sin(\omega\tau)d\tau \\
&+ 2\sigma\int_{-\infty}^{0} E_0(\tau+t_2+t_0,t_2))e^{-2\sigma\tau}\cos(\omega\tau)d\tau
\end{aligned} \tag{54}$$

We see that the integrals in Eq. 54 converge and hence the integral $\int_{-\infty}^{0} \frac{d(E_0(\tau+t_2+t_0,t_2)e^{-2\sigma\tau})}{dt_0}\cos(\omega\tau)d\tau$ also converges. Similarly, we can use above procedure for the term $E_0(\tau-t_2)$ in the second integral in Eq. 51 and show that $\int_{-\infty}^{0} \frac{d(E_0(\tau-t_2+t_0,t_2)e^{-2\sigma\tau})}{dt_0}\cos(\omega\tau)d\tau$ also converges. Similarly, we can use above procedure for the term $E_{0n}^{'}(\tau-t_0,t_2)$ in Eq. 51 and show that the corresponding integral converges.

We can use the above procedure in Eq. 52 to Eq. 54 for the term $\frac{d^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{dt_0^2}$ and show that all the terms and integrals in Eq. 50 and Eq. 51 converge.

• Every point in the segment S (plot), moves continuously, as we change t_0 by an infinitesimal δt_0, for each fixed value of ω. This is also true for the zero crossing point given by $\omega_z(t_2,t_0)$ as shown below.

• It is shown in Section 5.3 that the zero crossing in $G_R(\omega,t_2,t_0)$ given by $\omega_z(t_2,t_0)$, is a **continuous** function of t_0 for a given t_2, using Implicit Function Theorem.

5.3. *Zero Crossings in $G_R(\omega,t_2,t_0)$ move continuously as a function of t_0, for a given t_2.*

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_R(\omega,t_2,t_0)$ is partially differentiable with respect to ω and t_0, for a given fixed value of t_2, with continuous partial derivatives (Section 5.1 and Section 5.2) and given that $G_R(\omega,t_2,t_0)=0$ at $\omega=\omega_z(t_2,t_0)$ and $\frac{\partial G_R(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega=\omega_z(t_2,t_0)$ (using Corollary A), we see that $\omega_z(t_2,t_0)$ is differentiable in the interval $[t_0-\delta t_0, t_0+\delta t_0]$ for all $|t_0|<\infty$.

Hence in the **segment** S, $\omega_z(t_2,t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega=\omega_z(t_2,t_0)$.

5.4. ***Further Points***

• Using arguments in previous subsections, we see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 in the neighbourhood $[t_2 - \delta t_2, t_2 + \delta t_2]$ for all $|t_2| < \infty$, for **each** fixed value of t_0.

• We **set** $t_2 = 2t_0$. Using arguments in previous subsections, we see that $\omega_z(2t_0, t_0)$ is a **continuous** function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$.

6. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi)\xi(1-s, \bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \dfrac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (Appendix D). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function which is **strictly decreasing** function for $t \geq \frac{1}{8}$. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

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[6] Mathworld article on Hurwitz Zeta functions. (Link)

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ (link). This is re-derived in Appendix D.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$$
$$E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix C.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'
\end{aligned} \tag{B.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega - \omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.3}$$

Appendix B.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned} \tag{B.4}$$

Appendix B.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \tag{B.5}$$

Appendix B.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. $E_p(t), h(t), g(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \le |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t \to \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \ne 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \le \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \ge 0$ for all $|t| \le \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix C.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $[\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t)+e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

Appendix C.4. ***Payley-Weiner theorem and Exponential Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix C.3) and hence it should have **exponential fall-off** rates as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix D. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{D.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{D.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^{\infty}|x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{D.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. D.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty}\frac{x^{-(\frac{s}{2}-1)}}{x^2}\frac{(1+2w(x))\sqrt{x}-1)}{2}dx \tag{D.4}$$

Hence we can simplify Eq. D.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}}w(x)dx \tag{D.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1+s(s-1)\int_1^{\infty}(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}]$$

(D.6)

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix D.1. ***Derivation of*** $E_p(t)$ ***and*** $E_0(t)$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. D.6 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{D.7}$$

We can substitute $t=-t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt \\ +\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{D.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{D.9}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}E_p(t) &= \frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\ &+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned} \tag{D.10}$$

We can simplify above equation as follows.

$$\begin{aligned}\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t}-6\pi n^2 e^{-2t}-4\sigma\pi n^2 e^{-2t}]u(-t) \\ &\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t}-6\pi n^2 e^{2t}+4\sigma\pi n^2 e^{2t}]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned}$$

(D.11)

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix D.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$

$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+ \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$

(D.12)

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

(D.13)

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$

(D.14)

Appendix D.2. ***Derivation of*** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}}$$

$$\tag{D.15}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$
\begin{aligned}
[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} &= \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2}) \\
\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) &= -\frac{1}{2}
\end{aligned}
\tag{D.16}
$$

Appendix D.3. ***Derivation of Result***

In this section, we derive the result $\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}$ for $|t| < \infty$. We use the the well known theorem $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4]. We substitute $x = e^{2t}$ as follows.

$$
\begin{aligned}
F(x) = 1 + 2w(x) &= \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(e^{2t}) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} &= \frac{1}{e^t}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}})
\end{aligned}
\tag{D.17}
$$

We multiply above equation by $\frac{1}{2}e^{\frac{t}{2}}$ and derive the result as follows.

$$
\begin{aligned}
\frac{1}{2}e^{\frac{t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \frac{1}{2}e^{\frac{-t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} \\
\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}
\end{aligned}
\tag{D.18}
$$