On a new method towards proof of Riemann's Hypothesis

Akhila Raman

University of California at Berkeley. Email: akhila.raman@berkeley.edu.

Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

---

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix D, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0s\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_{0s}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3]

$$E_{0s}(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_{0s}(t)=E_{0s}(-t)$ is a real and **even** function of t, given that $E_{0s\omega}(\omega)=E_{0s\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{ps\omega}(\omega)$ is given by the real function $E_{ps}(t)$. We can write $E_{ps}(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_{ps}(t)=E_{0s}(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_{ps}(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: Modified Function* $E_0(t)$

The proof presented in following sections **requires** $\Phi(t)=E_{0s}(t)$ to be a **strictly decreasing** function for $t>0$. It is stated in Conrey's article (link) that $\Phi(t)$ is positive for $t>0$ and its first derivative is negative for $t>0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t>0$. The proof to show that $\Phi(t)$ is **strictly decreasing** function for $t>0$ is not available in textbooks.

Hence we consider $\xi'(s)=-\frac{\xi(s)}{s(s-1)}$ evaluated at $s=\frac{1}{2}+i\omega$ given by $E_{0\omega}(\omega)=\frac{E_{0s\omega}(\omega)}{(\frac{1}{4}+\omega^2)}$, whose Inverse Fourier transform is given by the **modified** function $E_0(t)$. We will show that this modified function $E_0(t)$ is a **strictly decreasing** function for $t>0$ in Section 4. We form $E_p(t)=E_0(t)e^{-\sigma t}$ whose Fourier transform is given by $E_{p\omega}(\omega)=\frac{E_{ps\omega}(\omega)}{(\frac{1}{4}-\sigma^2+\omega^2-i2\sigma\omega)}$. **If** Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{ps\omega}(\omega)$ has a zero at $\omega=\omega_0$, **then** we see that $E_{p\omega}(\omega)$ also has a zero at $\omega=\omega_0$.

We start with the well known equation for Riemann's Xi function derived in Eq. D.6 as follows.

$$\xi(s)=\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)=\frac{1}{2}[1+s(s-1)\int_1^{\infty}(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{3}$$

We substitute $x=e^{2t}$, $\frac{dx}{x}=2dt$ and $w(x)=\sum_{n=1}^{\infty}e^{-\pi n^2x}$ in Eq. 3 and evaluate at $s=\frac{1}{2}+i\omega$ as follows. We define $A(t)=[\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}u(t)]$ whose Fourier transform is given by $A(\omega)$. This is re-derived in Eq. D.9 in Appendix D.

$$\begin{aligned}\xi(\frac{1}{2}+i\omega)=E_{0s\omega}(\omega)=\frac{1}{2}+(-\frac{1}{4}-\omega^2)\int_{-\infty}^{\infty}A(t)e^{-i\omega t}dt=\frac{1}{2}+(-\frac{1}{4}-\omega^2)A(\omega)\\ E_{0\omega}(\omega)=\frac{E_{0s\omega}(\omega)}{(\frac{1}{4}+\omega^2)}=\frac{1}{2(\frac{1}{4}+\omega^2)}-A(\omega)\end{aligned} \tag{4}$$

The Inverse Fourier Transform of $E_{0\omega}(\omega)$ is given by $E_0(t)$ as follows.

$$
\begin{aligned}
E_0(t) &= \frac{1}{2}(e^{\frac{t}{2}}u(-t) + e^{-\frac{t}{2}}u(t)) - A(t) \\
E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} &- \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t) \\
E_p(t) &= E_0(t)e^{-\sigma t}
\end{aligned}
\tag{5}
$$

We use these modified functions $E_0(t)$ and $E_p(t)$ in the subsequent sections.

1.3. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{4}$ at first. Let us consider a new function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$, where $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ and $f_1(t) = e^{\sigma t_0} E_p^{'}(t+t_0)$ and $f_2(t) = e^{-\sigma t_0} E_p^{'}(t-t_0)$ and $E_p^{'}(t) = e^{-\sigma t_2} E_p(t-t_2) - e^{\sigma t_2} E_p(t+t_2)$ and t_0, t_2 are real and $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, for every value of t_0, to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

1.4. *Step 3: On the zeros of the function* $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g(t)$ and compute its real part given by $G_R(\omega) = G_R(\omega, t_2, t_0)$ and we can write as follows.

$$
\begin{aligned}
G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} &\int_{-\infty}^{0} [E_0^{'}(\tau + t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0)] \cos(\omega\tau) d\tau \\
+ e^{2\sigma t_0} &\int_{-\infty}^{0} [E_0^{'}(\tau - t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0)] \cos(\omega\tau) d\tau
\end{aligned}
\tag{6}
$$

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for **each fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$.

1.5. *Step 4: Zero Crossing function* $\omega_z(t_2, t_0)$ *is an even function of variable* t_0

In Section 2.3, we show the result in Eq. 7 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ is an **odd** function of t_0, for all t_0, for a given value of t_2 as follows.

$$
\begin{aligned}
P_{odd}(t_2, t_0) = [\cos(\omega_z(t_2, t_0)t_0) &\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
+ \sin(\omega_z(t_2, t_0)t_0) &\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
+ e^{2\sigma t_0}[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau &+ \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}
\tag{7}
$$

1.6. *Step 5: Final Step*

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = Kt_{0c}$, for positive even integer K, such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2, t_0)$ in Eq. 7 and show that this leads to the result in Eq. 8. We use $E_0'(t) = E_0(t - t_2) - E_0(t + t_2)$ and $E_{0n}'(t) = E_0'(-t)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma t_{0c})} - \cosh{(2\sigma\tau)})\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau = 0 \tag{8}$$

We show that the **each** of the terms in the integrand in Eq. 8 are **greater than zero**, in the interval $\tau = [0, t_{0c}]$ where $t_{0c} > 0$. For $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, we see that $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi}{2t_{0c}}\tau$ lies in the range $[0, \frac{\pi}{2}]$ and hence $\sin{(\omega_{c1}\tau)} > 0$ in that interval $\tau = [0, t_{0c}]$.

Hence the result in Eq. 8 leads to a **contradiction** for $0 < \sigma < \frac{1}{4}$.

We have shown this result for $0 < \sigma < \frac{1}{4}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{4} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{4}$.

In Section 3.2, the outline of the proof for the case of $0 < |\sigma| < \frac{1}{2}$ is presented.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{ps\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ and $E_{p\omega}(\omega) = \frac{E_{ps\omega}(\omega)}{(\frac{1}{4} - \sigma^2 + \omega^2 - i2\sigma\omega)}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3]

2.1. *New function* $g(t)$

In this section, we consider $0 < \sigma < \frac{1}{4}$ first and then in Section 3.2, we outline a method to prove Theorem 1 for the region $\frac{1}{4} \leq \sigma < \frac{1}{2}$.

Let us consider the function $E_p'(t) = E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t)e^{-\sigma t}$, where t_2 is finite and real, and $E_0'(t) = E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E_{p\omega}'(\omega) = E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0} E_p'(t + t_0)$ and $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t - t_0)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega) = F(\omega, t_2, t_0) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$.

Let us consider a new function $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), F(\omega)$ and $G(\omega)$ which are also functions of variables t_2, t_0. Similarly we use the shorthand notation for the functions $E_p'(t), E_0'(t)$ and $E_{p\omega}'(\omega)$ which are also functions of variable t_2.

We see that $E_0(t) = [\frac{1}{2} e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2} e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ goes to zero, as $|t| \to \infty$. In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$. This implies that $E_0(t) > 0$ for all $|t| < \infty$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $|t| < \infty$.

We can see that $g(t)$ is a real L^1 integrable function as follows. $E_p(t)$ has a fall off rate of $e^{-t(\frac{1}{2} + \sigma)}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2} - \sigma)}$ as $t \to -\infty$. Hence $g(t)$ has a fall off rate of $e^{-\frac{t}{2}}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2} - 2\sigma)}$ as $t \to -\infty$. Hence **for** $0 < \sigma < \frac{1}{4}$, we see that $g(t)$ is a real L^1 integrable function. Its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma].

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_z(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{ps\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0, where $\omega_z(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = e^{\sigma t_0} E_p'(t + t_0)$ and $f_2(t) = e^{-\sigma t_0} E_p'(t - t_0)$, $E_p'(t) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$, and $h(t) = e^{\sigma t} u(-t) + e^{-\sigma t} u(t)$ and $0 < \sigma < \frac{1}{4}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}'(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \tag{9}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.1)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 9 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{10}$$

We can split the integral in Eq. 10 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{11}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 11 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega') [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{12}$$

We showed earlier in this section that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 12 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

2.2. *On the zeros of a related function* $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, for a given value of t_2, to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the Fourier transform of the function $g_1(t)$ given by $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$. We use $g_1(t) = f_1(t)e^{-\sigma t}u(-t) + f_1(t)e^{\sigma t}u(t) = e^{\sigma t_0}E_p^{'}(t+t_0)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p^{'}(t+t_0)e^{\sigma t}u(t)$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$ and $G_1(\omega)$ which are also functions of variables t_2, t_0. Similarly we use the shorthand notation for the functions $E_p^{'}(t), E_0^{'}(t)$ and $E_{0n}^{'}(t)$ which are also functions of variable t_2.

$$
\begin{aligned}
G_1(\omega) &= \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^{0} g_1(t)e^{-i\omega t}dt + \int_{0}^{\infty} g_1(t)e^{-i\omega t}dt \\
G_1(\omega) &= \int_{-\infty}^{0} e^{\sigma t_0}E_p^{'}(t+t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0}E_p^{'}(t+t_0)e^{\sigma t}e^{-i\omega t}dt
\end{aligned}
\tag{13}
$$

We use $E_p^{'}(t) = E_0^{'}(t)e^{-\sigma t}$ where $E_0^{'}(t) = E_0(t-t_2) - E_0(t+t_2)$ and $E_p^{'}(t+t_0) = E_0^{'}(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 13, we have

$$
\begin{aligned}
G_1(\omega) &= \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_0^{'}(t+t_0)e^{-i\omega t}dt \\
G_1(\omega) &= \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_0^{'}(-t+t_0)e^{i\omega t}dt
\end{aligned}
\tag{14}
$$

We define $E_{0n}^{'}(t) = E_0^{'}(-t)$ and get $E_0^{'}(-t+t_0) = E_{0n}^{'}(t-t_0)$ and write Eq. 14 as follows.

$$
G_1(\omega) = \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_{0n}^{'}(t-t_0)e^{i\omega t}dt = G_R(\omega) + iG_I(\omega)
\tag{15}
$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$
G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0n}^{'}(t-t_0)\cos(\omega t)dt
\tag{16}
$$

2.3. *Zero crossing function* $\omega_z(t_2, t_0)$ *is an even function of variable* t_0

Now we consider the function $f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t) = e^{-\sigma t_0}E_p^{'}(t+t_0) + e^{\sigma t_0}E_p^{'}(t-t_0)$ where $f_1(t) = e^{\sigma t_0}E_p^{'}(t+t_0)$ and $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0}E_p^{'}(t-t_0)$ and $g(t)h(t) = f(t)$ where $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g(t)$ and compute its real part using the procedure in above section, similar to Eq. 16 and we can write as follows. We substitute $t = \tau$.

$$
\begin{aligned}
G_R(\omega, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_0) + e^{2\sigma t_0} G_{1R}(\omega, -t_0) \\
G_{1R}(\omega, t_0) &= \int_{-\infty}^{0} [E_0^{'}(\tau + t_0) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0)] \cos(\omega\tau) d\tau \\
&+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0)] \cos(\omega\tau) d\tau
\end{aligned}
\tag{17}
$$

We require $G_R(\omega, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ and we can rearrange the terms as follows.

$$
\begin{aligned}
P(t_2, t_0) &= \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau + t_0) e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&+ \int_{-\infty}^{0} [e^{2\sigma t_0} E_0^{'}(\tau - t_0) e^{-2\sigma\tau} + e^{-2\sigma t_0} E_{0n}^{'}(\tau - t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0
\end{aligned}
\tag{18}
$$

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0.

$$
\begin{aligned}
P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\
P_{odd}(t_2, t_0) &= \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0^{'}(\tau + t_0) e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}^{'}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau
\end{aligned}
\tag{19}
$$

We see that $f(t, t_0) = e^{-\sigma t_0} E_p^{'}(t + t_0) + e^{\sigma t_0} E_p^{'}(t - t_0) = f(t, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2.

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 19 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E_{0n}^{'}(\tau) = E_0^{'}(-\tau)$ and $E_0^{'}(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

We **note** that we use the **shorthand** notation for the functions $E_0^{'}(t)$ and $E_{0n}^{'}(t)$ which are also functions of variable t_2.

$$
\begin{aligned}
P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&+ \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}
\tag{20}
$$

In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for a given value of t_2. This means there are **no** Dirac delta functions present in $\omega_z(t_2, t_0)$.

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2.

In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0, for every value of t_2, and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, we see that $\omega_z(t_2, t_0)t_0$ will **certainly pass through** π, as t_0 is increased from zero to ∞.

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 20 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\begin{aligned}
&\int_{-\infty}^{t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}}\int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
-&\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0
\end{aligned} \tag{21}$$

We split the integral in the left hand side of Eq. 21 and write as follows.

$$\begin{aligned}
&[\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \\
&+e^{2\sigma t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \\
-&\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0
\end{aligned} \tag{22}$$

We combine the terms with common integrals and cancel common terms in Eq. 64 as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] + e^{2\sigma t_{0c}}\int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
&\qquad = -2\sinh(2\sigma t_{0c})\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau
\end{aligned} \tag{23}$$

We can rearrange the terms in Eq. 23 as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}]\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\
&= -2\sinh(2\sigma t_{0c})\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau
\end{aligned} \tag{24}$$

We denote the right hand side of Eq. 24 as RHS. We can split the integral in Eq. 24 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\int_{-t_{0c}}^{0} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$+ \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{25}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 25 as follows. We use $E_0^{'}(-\tau) = E_{0n}^{'}(\tau)$ and $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$\int_{t_{0c}}^{0} [E_{0n}^{'}(\tau)e^{2\sigma\tau} + E_0^{'}(\tau)e^{2\sigma t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$+ \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-2\sigma\tau} + E_{0n}^{'}(\tau)e^{2\sigma t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{26}$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{27}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 24 as follows. We use $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$RHS = 2\sinh{(2\sigma t_{0c})} \int_{t_{0c}}^{\infty} E_0^{'}(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{28}$$

We split the integral on the right hand side in Eq. 28 as follows.

$$RHS = 2\sinh{(2\sigma t_{0c})}[\int_{0}^{\infty} E_0^{'}(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau - \int_{0}^{t_{0c}} E_0^{'}(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{29}$$

We consolidate the integrals with the term $\int_{0}^{t_{0c}} E_0^{'}(\tau)$ in Eq. 27 and Eq. 29 as follows. We use $2\sinh{(2\sigma t_{0c})} = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$= 2\sinh{(2\sigma t_{0c})} \int_{0}^{\infty} E_0^{'}(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{30}$$

We cancel common terms in Eq. 30 as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$= 2\sinh{(2\sigma t_{0c})} \int_{0}^{\infty} E_0^{'}(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$

(31)

We substitute $E_0^{'}(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E_{0n}^{'}(\tau) = E_0^{'}(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$. Hence we see that $E_{0n}^{'}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0^{'}(\tau)$. We can write Eq. 31 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= 2\sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$

(32)

We substitute $2\cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2\cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 32 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= \sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$

(33)

Next Step:

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 33 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$RHS = \sinh(2\sigma t_{0c})[\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$+ \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$- \cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau]$$

(34)

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) = -1$. Hence we cancel common terms and write Eq. 34 and Eq. 33 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$
$$= -\sinh(2\sigma t_{0c})[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau]$$

(35)

We use $\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 35 as follows. Given that $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ, we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$$

(36)

We can multiply Eq. 36 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{37}$$

In Eq. 37, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and hence the sinusoidal function varies over a **half cycle** and is > 0, in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 37, we see that in the interval $0 < \tau < t_{0c}$, the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t \geq \frac{1}{8}$, we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 3.1) . The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 37. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{4}$.

For $\sigma = 0$, both sides of Eq. 37 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{4}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{4} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{4}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{4}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3.1. ***Result*** $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 4 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 37.

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t, and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c})$ to $E_0(3t_{0c})$, which is **less than** $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c})$ to $E_0(-t_{0c})$. Hence we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{2c} = 2t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 37, for $t_{0c} > 0$.

3.2. ***Case:*** $\frac{1}{4} \leq |\sigma| < \frac{1}{2}$

In the previous section, we proved Theorem 1 for $0 < |\sigma| < \frac{1}{4}$. To prove Theorem 1 for the region $0 < |\sigma| < \frac{1}{2}$, we need to **repeat** the procedure in the previous subsections by considering $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$ where $0 < \Delta < \frac{1}{2} - \sigma$.

Given that $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$, this **ensures** that $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ is an **absolutely integrable** function with a fall off rate of $e^{-t(\frac{1}{2}+\sigma-\Delta)}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2}-\sigma-\Delta)}$ as $t \to -\infty$.

This will be shown in detail in Section 5.4.

4. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_{0s}(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0s\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. (link)

We consider $\xi'(s) = -\frac{\xi(s)}{s(s-1)}$ evaluated at $s = \frac{1}{2} + i\omega$ given by $E_{0\omega}(\omega) = \frac{E_{0s\omega}(\omega)}{(\frac{1}{4}+\omega^2)}$, which is a **holomorphic** function in the entire s plane, except at $s = 0$ and $s = 1$, whose Inverse Fourier Transform is given by $E_0(t)$ as follows. (Section 1.2)

$$E_0(t) = [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t) + [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) \tag{38}$$

We define $X(t) = E_0(t)$ and consider **only** $t \geq 0$ in the rest of this section. For $t > 0$, we see that the first two derivatives of $X(t)$ are continuous functions. At $t = 0$, if the second derivative of $X(t)$ has a dirac delta function, it does not affect the arguments below.

$$X(t) = E_0(t) = \frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = A_0(t) - B_0(t), \quad t \geq 0$$
$$A_0(t) = \frac{1}{2}e^{\frac{-t}{2}}, \quad B_0(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{39}$$

We show that $X(t) = E_0(t)$ is a **strictly decreasing** function for $t \geq 0$ as follows.

- In Section 4.1 it is shown that the second derivative of $B_0(t)$ in Eq. 39, given by $B_2(t)$ is a **strictly decreasing** function for $t \geq t_{min}$.

- In Section 4.2, it is shown that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. (example plot)

- In Section 4.3 it is shown that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$.

4.1. *Second derivative of $B_0(t)$ given by $B_2(t)$ is a strictly decreasing function for $t > t'_{max}$ and $t \geq t_{min}$*

We take the first derivative of $B_0(t)$ for $t \geq 0$ as follows.

$$B_1(t) = \frac{dB_0(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}] \tag{40}$$

We take the second derivative of $B_0(t)$ for $t \geq 0$ as follows.

$$B_2(t) = \frac{dB_1(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi n^2 e^{2t} + (\frac{1}{2} - 2\pi n^2 e^{2t})^2] = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi n^2 e^{2t} + \frac{1}{4} - 2\pi n^2 e^{2t} + 4\pi^2 n^4 e^{4t}]$$
$$B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}]$$

(41)

We see that $B_0(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ is a **strictly decreasing** function of t, given that its first derivative given by $B_1(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}]$, is **negative** for $t \geq 0$ for each $n = 1, 2, 3, ..$ and hence for **all** $t \geq 0$. (example plot)

We see that $B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}]$ in Eq. 41, is **positive** for for each $n = 1, 2, 3, ..$ and hence for **all** $t \geq 0$. This implies that, as t increases from zero, $B_1(t)$ starts from a **negative** value and increases towards zero.

We define $B_2'(t) = B_2(t)e^{\frac{t}{2}}$ and compute its **first derivative** as follows.

$$B_3'(t) = \frac{dB_2'(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t[16\pi^2 n^4 e^{4t} - 12\pi n^2 e^{2t} + (4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4})(1 - 2\pi n^2 e^{2t})]$$
$$B_3'(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t[16\pi^2 n^4 e^{4t} - 12\pi n^2 e^{2t} + (4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4} - 8\pi^3 n^6 e^{6t} + 12\pi^2 n^4 e^{4t} - \frac{1}{2}\pi n^2 e^{2t})]$$
$$B_3'(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t Y'(t), \quad Y'(t) = -8\pi^3 n^6 e^{6t} + 32\pi^2 n^4 e^{4t} - \frac{37}{2}\pi n^2 e^{2t} + \frac{1}{4}$$

(42)

We examine the term $Y'(t) = -8\pi^3 n^6 e^{6t} + 32\pi^2 n^4 e^{4t} - \frac{37}{2}\pi n^2 e^{2t} + \frac{1}{4}$ in Eq. 42. We see that, for $n = 2$ and $t = 0$, $Y'(t) < -8\pi^3 * 64 + 32\pi^2 * 16 + \frac{1}{4} < 8 * 64 * \pi^2(1 - \pi) + \frac{1}{4} < 0$. We see that, **for all** $n > 1$ and $t \geq 0$, $Y'(t) < 0$.

For $n = 1$, we want to find the **minimum value** of t for which $Y'(t) < 0$. We choose a **loose bound** $t_a = 1$ and for $t \geq t_a$ and $n = 1$, we see that $Y'(t) \leq 8\pi^2 e^{4t_a}(4 - \pi e^{2t_a}) + \frac{1}{4}$. Given that $\pi > 3$ and $e^{2t_a} = e^2 > 2^2 = 4$, we see that $Y'(t) \leq 8\pi^2 e^4(4 - 12) + \frac{1}{4}$ which is **negative**. Hence $B_3'(t) < 0$ for $t > t_a = 1$ and $B_2'(t)$ is **strictly decreasing** for all $t > t_a$.

At $t = t'_{max}$(**point B'**), which lies **somewhere** in the interval $0 \leq t \leq t_a$, $B_2'(t)$ reaches a **maximum** and then starts decreasing for $t > t'_{max}$. We require $B_3'(t) = 0$ at $t = t'_{max}$ and the partial term in $B_3'(t)$ corresponding to $n = 1$ **equals** a positive value B'_{30} which **cancels** the negative value $-B'_{30}$ due to the partial terms corresponding to $n > 1$, to make $B_3'(t) = 0$.

For $t > t'_{max}$, the partial term in $B_3'(t)$ corresponding to $n = 1$ continues to **decrease below** B'_{30} due to the dominant term $-8\pi^3 n^6 e^{6t}$, while the partial terms corresponding to $n > 1$ become **more negative** than $-B'_{30}$. Hence $B_3'(t) < 0$ for $t > t'_{max}$. Hence $B_2'(t)$ is **strictly decreasing** for all $t > t'_{max}$.(**Result 1**)

Given that $B_2(t) = B_2'(t)e^{-\frac{t}{2}}$, we see that $B_2(t)$ is **strictly decreasing** for all $t > t'_{max}$.(**Result 2**) In Section 4.3, it is shown that $t'_{max} < t_{min}$ and hence $B_2(t)$ is a **strictly decreasing** function for $t \geq t_{min}$.

4.2. First and Second derivatives of $X(t)$

We take the first derivative of $X(t)$ for $t \geq 0$ in Eq. 39 and substitute $B_1(t)$ in Eq. 40 as follows.

$$X(t) = \frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = A_0(t) - B_0(t)$$
$$\frac{dX(t)}{dt} = -\frac{1}{4}e^{\frac{-t}{2}} - B_1(t) = -\frac{1}{4}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}]$$

(43)

We take the second derivative of $X(t)$ for $t \geq 0$ and substitute $B_2(t)$ in Eq. 41 as follows.

$$\frac{d^2X(t)}{dt^2} = \frac{1}{8}e^{\frac{-t}{2}} - B_2(t) = \frac{1}{8}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}] \tag{44}$$

• In the section below, **we will show** that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. Thus we will show that $X(t)$ is a **strictly decreasing** function of t. We will also show that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. (example plot)

• We see that in $\frac{d^2X(t))}{dt^2}$ in Eq. 44, partial term corresponding to $t = 0$ and $n = 1$, equals $a_1 = \frac{1}{8} - a_2 < 0, a_2 = e^{-\pi}(4\pi^2 - 6\pi + \frac{1}{4})$. Because $3 < \pi < \frac{22}{7} = 3.1429$ (link) and $e^{-\pi} \geq e^{-0.1429}(e^{-\frac{1}{2}})^6$ and $e^{-0.1429} \geq 1 - 0.1429 = 0.8571$ and $e^{-\frac{1}{2}} \geq 1 - \frac{1}{2} = \frac{1}{2}$, we can write $a_1 = 0.125 - a_2, a_2 \geq \frac{0.8571}{2^6}(\pi(4\pi - 6) + \frac{1}{4})$. We see that the **minimum** value of a_2 is given by $a_2 \geq \frac{0.8571}{64} * (3(4*3-6)) = \frac{0.8571*18}{64} \geq \frac{0.8571*16}{64} = \frac{0.8571}{4} \geq 0.21$ and the **maximum** value of a_1 is given by $a_1 = 0.125 - 0.21 = -0.085$. For $n \geq 1$ and $t = 0$, partial terms in $B_2(t)$ are positive and the **minimum** value of $B_2(t)$ is given by 0.21.

• Hence the **maximum** value of $\frac{d^2X(t))}{dt^2}$ at $t = 0$ is given by $0.125 - 0.21 = -0.085$, which is **negative**. Hence $\frac{d^2X(t))}{dt^2}$ is **negative** at $t = 0$ and remains **negative** for small values of positive t close to zero.

• We know that $\frac{dX(t)}{dt} = 0$ at $t = 0$, given that $X(t) = E_0(t)$ is an **even** function. This implies that $\frac{dX(t)}{dt}$ is **negative** for small values of positive t close to zero.

• This implies that $X(t)$ is a **strictly decreasing** function of t, for small values of positive t close to zero.

• As t increases from zero (point A), $\frac{dX(t)}{dt}$ reaches a **negative minimum** value at $t = t_{min}$(point C) and then starts increasing towards zero, for $t > t_{min}$. This means that $\frac{d^2X(t)}{dt^2}$ reaches a **zero** at $t = t_{min}$, at which point $B_2(t) = \frac{1}{8}e^{\frac{-t_{min}}{2}}$ and then becomes **positive** for $t > t_{min}$, given that $B_2(t)$ decreases towards zero for $t > t_{min}$ (**from Result 1**).(example plot)

• Then $\frac{d^2X(t)}{dt^2}$ reaches a **positive maximum** at $t = t_{max2}$ (point D) and then starts decreasing to zero, as $t \to \infty$. This means that $\frac{dX(t)}{dt}$ **increases** towards zero, as $t \to \infty$. (example plot)

4.3. $\frac{dX(t)}{dt}$ ***does not become positive for any*** $t > 0$

• $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. We can **rule out** the case of $\frac{dX(t)}{dt}$ becoming positive, as $t \to \infty$, as follows. Let us assume that $\frac{dX(t)}{dt}$ becomes positive at $t = t_p$ as t increases. (**Statement A**). In this case, we **require** $\frac{dX(t)}{dt}$ to reach a maximum at $t = t_m$ and then fall back towards zero, as $t \to \infty$, given that $X(t)$ and $\frac{dX(t)}{dt}$ approach zero, as $t \to \infty$.(example plot)

This means $\frac{d^2X(t)}{dt^2}$ becomes **negative** for $t > t_m$. This is **not** possible, because $B_2(t)$ is decreasing towards zero for $t > t_{min}$ (shown in Eq. 46) and hence $\frac{d^2X(t)}{dt^2} = \frac{1}{8}e^{\frac{-t}{2}} - B_2(t) > 0$ for $t > t_{min}$. Hence $\frac{dX(t)}{dt}$ remains **negative** for all $t > 0$ and hence $X(t)$ is a **strictly decreasing** function of t, as $t \to \infty$.

$$B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad \frac{d^2X(t)}{dt^2} = \frac{1}{8}e^{\frac{-t}{2}} - B_2(t)$$

$$B_2'(t) = B_2(t)e^{\frac{t}{2}}$$

(45)

In Eq. 43 and Eq. 45, we see that $B_2(t)$ at $t = 0$ (point A) equals $B_2(0) > e^{-\pi}(4\pi^2 - 6\pi) > \pi(4*3-6)*\frac{0.8571}{2^6} = \frac{0.8571*18}{64} \geq \frac{0.8571*16}{64} \geq 0.21$, which is **greater** than $\frac{1}{8} = 0.125$ and hence $|B_2(0)| > \frac{1}{8}$ and $|B_2'(0)| > \frac{1}{8}$ (**Result 3**) and $\frac{d^2X(t)}{dt^2} < 0$ at $t = 0$.

As t increases from zero, $B_2(t)$ increases and reaches a **maximum** inflection point at $t = t_{max}$ (point B) **if** $t_{max} > 0$ and then $B_2(t)$ starts **decreasing** and reaches $B_2(t) = \frac{1}{8}e^{\frac{-t}{2}}$ at $t = t_{min}$ (point C) and hence $\frac{d^2X(t)}{dt^2} = 0$ at $t = t_{min}$(point C). We see that $t_{max} < t_{min}$ and t_{max} can be zero as well. We see that $\frac{d^2X(t)}{dt^2} < 0$ in $0 \leq t \leq t_{min}$ and **cannot** equal zero in the segment $0 \leq t \leq t_{max}$ or in the segment $t_{max} \leq t \leq t_{min}$. We see that $\frac{d^2X(t)}{dt^2} = 0$ at $t = t_{min}$. (example plot)

Similarly, as t increases from zero, $B_2'(t)$ increases and reaches a **maximum** inflection point at $t = t'_{max}$ **if** $t'_{max} > 0$ and then $B_2'(t)$ starts **decreasing** and reaches $B_2'(t) = \frac{1}{8}$ at $t = t_{min}$ (point C). We see that $t'_{max} < t_{min}$. (**Result 4**) We **note** that t'_{max} can be zero as well.

We see that $B_2(t) = \frac{1}{8}e^{\frac{-t}{2}}$ and $\frac{d^2X(t)}{dt^2} = 0$ at $t = t_{min}$ (point C) and $\frac{d^2X(t)}{dt^2}$ reaches a **positive maximum** at $t = t_{max2}$ (point D). For $t_{min} \leq t \leq t_{max2}$, in the segment between points C and D, we see that $\frac{d^2X(t)}{dt^2} = \frac{1}{8}e^{\frac{-t}{2}} - B_2(t) > 0$, given that $\frac{dX(t)}{dt}$ is negative and increasing towards zero. (example plot) This means $B_2(t) < \frac{1}{8}e^{\frac{-t}{2}}$ in Eq. 45 in the segment $t_{min} \leq t \leq t_{max2}$ (**Result 5**), which continues in the segment $t > t_{max2}$. Given the fact that $B_2(t)$ is **strictly decreasing** for all $t > t'_{max}$(given **Result 2**), and $t'_{max} < t_{min}$ (given **Result 4**), $B_2(t)$ **cannot** increase suddenly to greater than $\frac{1}{8}e^{\frac{-t}{2}}$, shown as follows.

At $t = t_{min}$, we can write as follows.

$$
\begin{aligned}
\frac{1}{8}e^{\frac{-t}{2}} = B_2(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad t = t_{min} \\
\frac{1}{8} = B_2(t)e^{\frac{t}{2}} = B_2'(t) &= \sum_{n=2}^{\infty} e^{-\pi n^2 e^{2t}} e^{t}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad t = t_{min} \\
, \quad \frac{d^2X(t)}{dt^2} &= \frac{1}{8}e^{\frac{-t}{2}} - B_2(t)
\end{aligned}
\tag{46}
$$

Given **Result 5** in the segment $t_{min} \leq t \leq t_{max2}$, we see that $B_2(t) < \frac{1}{8}e^{\frac{-t}{2}}$. For $t > t_{max2}$, $B_2'(t) = B_2(t)e^{\frac{t}{2}}$ is positive and decreasing further towards zero (from **Result 1 and 4**) and the right hand side of Eq. 46 is **decreasing lower** than $\frac{1}{8}$ and $B_2'(t) < \frac{1}{8}$.

Given the fact that $B_2'(t)$ is **strictly decreasing** for all $t > t'_{max}$(given **Result 1**), and $t'_{max} < t_{min}$ (given **Result 4**), $B_2'(t)$ is **strictly decreasing** for all $t > t_{min}$ as well and $B_2'(t) < \frac{1}{8}$ in the segment $t > t_{min}$ and $B_2'(t)$ **cannot** suddenly increase to a value larger than $\frac{1}{8}$, in order to make $B_2(t)e^{\frac{t}{2}} > \frac{1}{8}$, which is required to make $\frac{d^2X(t)}{dt^2} < 0$ for some value of $t > t_{min}$. Hence $\frac{d^2X(t)}{dt^2} > 0$ for $t > t_{min}$ and **does not** become negative.

We **note** that $B_2'(t) = B_2(t)e^{\frac{t}{2}}$ reaches a maximum at $t = t'_{max}$ where $|B_2'(t)| > \frac{1}{8}$ (**given Result 3**) and hence $B_2'(t) - \frac{1}{8} > 0$ at $t = t'_{max}$. Then $B_2'(t)$ starts decreasing and equals $\frac{1}{8}$ at $t = t_{min}$. Hence $t_{min} > t'_{max}$ and $B_2'(t)$ is **strictly decreasing** for all $t > t_{min}$ as well and $B_2'(t) < \frac{1}{8}$ and $B_2'(t)$ **cannot** suddenly increase to a value larger than $\frac{1}{8}$. Hence $\frac{d^2X(t)}{dt^2} = \frac{1}{8}e^{\frac{-t}{2}} - B_2(t) = (\frac{1}{8} - B_2'(t))e^{\frac{-t}{2}} > 0$ for $t > t_{min}$ and **does not** become negative.

Thus we have **ruled out** the case of $\frac{dX(t)}{dt}$ becoming positive, as $t \to \infty$. This implies that $E_0(t) = X(t)$ is a **strictly decreasing** function of t for all $t > 0$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t)$ is a **strictly decreasing** function of t for all $t < 0$, going from $t = 0$ towards $t = -\infty$.

5. $\omega_z(t_2, t_0)$ is a continuous function of t_0

It is shown in this section that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$, for **each** fixed value of t_2.

• $G_R(\omega) = G_R(\omega, t_2, t_0)$ in Eq. 17 is copied below, which is a **continuous** function of ω which is differentiable **at least** once with respect to ω. (Eq. 48).

$$\begin{aligned} G_R(\omega) = G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} & \int_{-\infty}^{0} [E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ +e^{2\sigma t_0} & \int_{-\infty}^{0} [E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \end{aligned} \tag{47}$$

Given that $E_0(\tau) > 0$ for $|\tau| < \infty$ and $\lim_{\tau \to \pm\infty} E_0(\tau) = 0$ (**??**), we see that $G_R(\omega) > 0$ at $\omega = 0$. **Set** $t_0 = 0$ and $G_R(\omega)$ passes through its **first zero** at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$. In the rest of this section, we consider the **interval** $[-\delta t_0, \delta t_0]$ around $t_0 = 0$, in $\omega_z(t_2, t_0)$. There are 3 possibilities.

Case 1: $G_R(\omega) < 0$ for $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) - dw$ for infinitesimal dw (example plot)

In this case, we will show in Section 5.1 that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

Case 2: $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) - dw$ (example plot)

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, 0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, 0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(t_2, 0) + dw$.

$$\begin{aligned} \frac{dG_R(\omega, t_2, t_0)}{d\omega} = -[e^{-2\sigma t_0} & \int_{-\infty}^{0} \tau[E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\ +e^{2\sigma t_0} & \int_{-\infty}^{0} \tau[E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \end{aligned} \tag{48}$$

In this case, we will show in Section 5.2 that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

Case 3: $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, 0)$ and $\omega = \omega_z(t_2, 0) + dw$.

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, 0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, 0) - dw$ and $\frac{dG_R(\omega)}{d\omega} = 0$ at $\omega = \omega_z(t_2, 0)$. The arguments are similar to that of Case 2 presented in Section 5.2 where it is shown that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

*5.1. **Case 1:** $G_R(\omega) < 0$ **for** $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ **for** $\omega = \omega_z(t_2, 0) - dw$*

• Consider the **segment** S in $G_R(\omega, t_2, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega, t_2, t_0)}{d\omega} < 0$. (Segment S is the portion between the green lines in example plot)

• In the **segment** S, $G_R(\omega, t_2, t_0)$ in Eq. 47 is a **continuous** function of ω, for **each** value of t_0 and t_2. Hence $G_R(\omega, t_2, t_0 - \delta t_0)$ and $G_R(\omega, t_2, t_0 + \delta t_0)$ are **continuous** functions of ω, which are differentiable **at least** once, and $G_R(\omega, t_2, t_0 \pm \delta t_0)$ tends to $G_R(\omega, t_2, t_0)$, as infinitesimal $\delta t_0 \to 0$.

$$
\begin{aligned}
G_R(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau\\
G_R(\omega,t_2,t_0+\delta t_0) &= e^{-2\sigma(t_0+\delta t_0)}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0+\delta t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0-\delta t_0,t_2)]\cos(\omega\tau)d\tau\\
&+e^{2\sigma(t_0+\delta t_0)}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0-\delta t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0+\delta t_0,t_2)]\cos(\omega\tau)d\tau\\
G_R(\omega,t_2,t_0-\delta t_0) &= e^{-2\sigma(t_0-\delta t_0)}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0-\delta t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0+\delta t_0,t_2)]\cos(\omega\tau)d\tau\\
&+e^{2\sigma((t_0-\delta t_0))}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0+\delta t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0-\delta t_0,t_2)]\cos(\omega\tau)d\tau\\
&\lim_{\delta t_0\to 0}G_R(\omega,t_2,t_0+\delta t_0)=G_R(\omega,t_2,t_0)\\
&\lim_{\delta t_0\to 0}G_R(\omega,t_2,t_0-\delta t_0)=G_R(\omega,t_2,t_0)
\end{aligned}
\tag{49}
$$

• In the **segment** S, $G_R(\omega,t_2,t_0)$ in Eq. 49 is a **continuous** function of ω, for **each** value of t_0 and t_2 and $\frac{dG_R(\omega,t_2,t_0)}{d\omega}<0$ in the neighborhood around the **first zero crossing**. If we **fix** the X-coordinate ω and t_2, $G_R(\omega,t_2,t_0)$ is a **continuous** function of t_0, for **each** fixed value of ω. Hence, for **each** fixed value of ω, as we change t_0 by an infinitesimal δt_0, $G_R(\omega,t_2,t_0-\delta t_0)$ and $G_R(\omega,t_2,t_0+\delta t_0)$ in Eq. 49, move towards $G_R(\omega,t_2,t_0)$ in a **continuous** manner, as $\delta t_0\to 0$. Every point in the segment S, moves continuously, as we change t_0 by an infinitesimal δt_0.

This also applies to the first **zero crossing** in $G_R(\omega,t_2,t_0)$ in the segment S, which corresponds to $\omega_z(t_2,t_0)=\omega_z(t_2,0)$ at $t_0=0$ where $G_R(\omega,t_2,t_0)=0$ in Eq. 49. The **zero crossing** moves **continuously**, as we change t_0 by an infinitesimal δt_0. This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega,t_2,t_0)$ with zero crossing at point P_0, **Green** plot corresponds to $G_R(\omega,t_2,t_0+\delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to $G_R(\omega,t_2,t_0-\delta t_0)$ with zero crossing at point P_{21}.

We **define** the **point** P_{12} in $G_R(\omega,t_2,t_0+\delta t_0)$ as the point which has the **fixed X-coordinate** $\omega=\omega_z(t_2,0)$. We **define** the **point** P_{22} in $G_R(\omega,t_2,t_0-\delta t_0)$ as the point which has the **fixed X-coordinate** $\omega=\omega_z(t_2,0)$.

We **define** the **point** P_{11} in $G_R(\omega,t_2,t_0+\delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega,t_2,t_0-\delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change t_0 by an infinitesimal δt_0, $G_R(\omega,t_2,t_0+\delta t_0)$ in Eq. 49 moves towards $G_R(\omega,t_2,t_0)$ in a **continuous** manner, for **each fixed** value of ω and t_2, including the zero crossing point, as follows. The **point** P_{12} in $G_R(\omega,t_2,t_0+\delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega=\omega_z(t_2,0)$, moves towards corresponding point P_0 in $G_R(\omega,t_2,t_0)$, for the **same** $\omega=\omega_z(t_2,0)$ in a **continuous** manner, as $\delta t_0\to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega,t_2,t_0)$, this is equivalent to the **Zero crossing point** P_{11} in $G_R(\omega,t_2,t_0+\delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega,t_2,t_0)$ in a **continuous** manner, as $\delta t_0\to 0$.

Similarly, as we change t_0 by an infinitesimal δt_0, $G_R(\omega,t_2,t_0-\delta t_0)$ in Eq. 49 moves towards $G_R(\omega,t_2,t_0)$ in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega,t_2,t_0-\delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega=\omega_z(t_2,0)$, moves towards corresponding point P_0 in $G_R(\omega,t_2,t_0)$, for the **same** $\omega=\omega_z(t_2,0)$ in a **continuous** manner, as $\delta t_0\to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega,t_2,t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega,t_2,t_0-\delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega,t_2,t_0)$ in a **contin-**

uous manner, as $\delta t_0 \to 0$.

• Hence in the **segment** S, $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ at $t_0 = 0$.

$$
\begin{aligned}
G_R(\omega_z(t_2,t_0),t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau \\
+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau = 0 \\
G_R(\omega_z(t_2,t_0+\delta t_0),t_2,t_0+\delta t_0) = \\
e^{-2\sigma(t_0+\delta t_0)}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0+\delta t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0-\delta t_0,t_2)]\cos(\omega_z(t_2,t_0+\delta t_0)\tau)d\tau \\
+e^{2\sigma(t_0+\delta t_0)}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0-\delta t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0+\delta t_0,t_2)]\cos(\omega_z(t_2,t_0+\delta t_0)\tau)d\tau = 0
\end{aligned}
\tag{50}
$$

5.2. *Case 2:* $G_R(\omega) > 0$ *for* $\omega = \omega_z(t_2, 0) + dw$*,* $G_R(\omega) > 0$ *for* $\omega = \omega_z(t_2, 0) - dw$

• In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, t_0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, t_0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(t_2, t_0) + dw$.

• Consider the **segment** S' in $\frac{dG_R(\omega,t_2,t_0)}{d\omega}$ in the neighborhood around the first zero crossing where $\frac{d^2G_R(\omega,t_2,t_0)}{d\omega^2} > 0$. (Segment S' is the portion between the green lines in example plot) In this segment S', $\frac{dG_R(\omega,t_2,t_0)}{d\omega}$ is a **continuous** function of ω which is differentiable **at least** once.(Section **??**)

• In the **segment** S', $\frac{dG_R(\omega,t_2,t_0)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, t_0)$. The arguments in Section 5.1 can be applied here, with $G_R(\omega, t_2, t_0)$ replaced by $\frac{dG_R(\omega,t_2,t_0)}{d\omega}$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ at $t_0 = 0$ in the **segment** S'.

We can use similar arguments and see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$, for **each** fixed value of t_2.

5.3. *Further Points*

• Using arguments in previous subsections, we see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 in the neighbourhood $[t_2 - \delta t_2, t_2 + \delta t_2]$ for all $|t_2| < \infty$, for **each** fixed value of t_0.

• We **set** $t_2 = Kt_0$ for even positive integer K. Using arguments in previous subsections, we see that $\omega_z(Kt_0, t_0)$ is a **continuous** function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$.

5.4. **New function** $g(t)$

In this section, we prove Theorem 1 for the region $0 < |\sigma| < \frac{1}{2}$. We need to **repeat** the procedure in Section 2.1 by considering $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$ where $0 < \Delta < \frac{1}{2} - \sigma$.

Given that $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$, this **ensures** that $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ is an **absolutely integrable** function with a fall off rate of $e^{-t(\frac{1}{2}+\sigma-\Delta)}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2}-\sigma-\Delta)}$ as $t \to -\infty$.

Let us consider the function $E_p'(t) = E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t)e^{-\sigma t}$, where t_2 is finite and real, and $E_0'(t) = E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E_{p\omega}'(\omega) = E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t)$ where $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0}E_p'(t + t_0)$ and $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0}E_p'(t - t_0)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega) = F(\omega, t_2, t_0) = E_{p\omega}'(\omega)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$.

Let us consider a new function $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$ and $0 < \Delta < \frac{1}{2} - \sigma$. We note that $g(t), h(t)$ are **different** from the $g(t), h(t)$ in Section 2.1.

We see that $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ goes to zero, as $|t| \to \infty$. In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$. This implies that $E_0(t) > 0$ for all $|t| < \infty$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $|t| < \infty$.

We can see that $g(t)$ is a real L^1 integrable function as follows. $E_p(t)$ has a fall off rate of $e^{-t(\frac{1}{2}+\sigma)}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2}-\sigma)}$ as $t \to -\infty$. Hence $g(t)$ has a fall off rate of $e^{-t(\frac{1}{2}+\sigma-\Delta)}$ as $t \to \infty$ and a fall off rate of $e^{t(\frac{1}{2}-\sigma-\Delta)}$ as $t \to -\infty$. Hence **for** $0 < \sigma < \frac{1}{2}$ and $0 < \Delta < \frac{1}{2} - \sigma$, we see that $g(t)$ is a real L^1 integrable function. Its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma].

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}'(\omega)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\Delta - i\omega} + \frac{1}{\Delta + i\omega}] = \frac{2\Delta}{(\Delta^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_z(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\Delta}{(\Delta^2+\omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\Delta}{(\Delta^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t)+g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$ for **every value** of t_0, where $\omega_z(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t)$ where $f_1(t) = e^{\sigma t_0}E_p^{'}(t+t_0)$ and $f_2(t) = e^{-\sigma t_0}E_p^{'}(t-t_0)$, $E_p^{'}(t) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2)$, and $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}^{\prime}(\omega)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega')H(\omega-\omega')d\omega' \tag{51}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.1)

We substitute $H(\omega) = \frac{2\Delta}{(\Delta^2+\omega^2)}$ in Eq. 51 and we get

$$F_R(\omega) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} G_R(\omega')\frac{1}{(\Delta^2+(\omega-\omega')^2)}d\omega' \tag{52}$$

We can split the integral in Eq. 52 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_R(\omega')\frac{1}{(\Delta^2+(\omega-\omega')^2)}d\omega' + \int_{0}^{\infty} G_R(\omega')\frac{1}{(\Delta^2+(\omega-\omega')^2)}d\omega'] \tag{53}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 53 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi}\int_{0}^{\infty} G_R(\omega')[\frac{1}{(\Delta^2+(\omega-\omega')^2)} + \frac{1}{(\Delta^2+(\omega+\omega')^2)}]d\omega' \tag{54}$$

In **??** last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 54 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\Delta^2+(\omega-\omega')^2)} + \frac{1}{(\Delta^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2,t_0) \neq 0$, where $\omega_z(t_2,t_0)$ is real and finite.

- **Case 2: $G_R(\omega') < 0$ for all finite $\omega' > 0$**

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

5.5. *On the zeros of a related function* $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, for a given value of t_2, to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the Fourier transform of the function $g_1(t)$ given by $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$. We use $g_1(t) = f_1(t)e^{-\Delta t}u(-t) + f_1(t)e^{\Delta t}u(t) = e^{\sigma t_0} E'_p(t + t_0)e^{-\Delta t}u(-t) + e^{\sigma t_0} E'_p(t + t_0)e^{\Delta t}u(t)$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$ and $G_1(\omega)$ which are also functions of variables t_2, t_0. Similarly we use the shorthand notation for the functions $E'_p(t), E'_0(t)$ and $E'_{0n}(t)$ which are also functions of variable t_2.

$$
\begin{aligned}
G_1(\omega) &= \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^{0} g_1(t)e^{-i\omega t}dt + \int_{0}^{\infty} g_1(t)e^{-i\omega t}dt \\
G_1(\omega) &= \int_{-\infty}^{0} e^{\sigma t_0} E'_p(t + t_0)e^{-\Delta t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0} E'_p(t + t_0)e^{\Delta t}e^{-i\omega t}dt
\end{aligned}
\tag{55}
$$

We use $E'_p(t) = E'_0(t)e^{-\sigma t}$ where $E'_0(t) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_p(t + t_0) = E'_0(t + t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 55, we have

$$
\begin{aligned}
G_1(\omega) &= \int_{-\infty}^{0} E'_0(t + t_0)e^{-(\sigma+\Delta)t}e^{-i\omega t}dt + \int_{0}^{\infty} E'_0(t + t_0)e^{-(\sigma-\Delta)t}e^{-i\omega t}dt \\
G_1(\omega) &= \int_{-\infty}^{0} E'_0(t + t_0)e^{-(\sigma+\Delta)t}e^{-i\omega t}dt + \int_{-\infty}^{0} E'_0(-t + t_0)e^{(\sigma-\Delta)t}e^{i\omega t}dt
\end{aligned}
\tag{56}
$$

We define $E'_{0n}(t) = E'_0(-t)$ and get $E'_0(-t + t_0) = E'_{0n}(t - t_0)$ and write Eq. 56 as follows.

$$
G_1(\omega) = \int_{-\infty}^{0} E'_0(t + t_0)e^{-(\sigma+\Delta)t}e^{-i\omega t}dt + \int_{-\infty}^{0} E'_{0n}(t - t_0)e^{(\sigma-\Delta)t}e^{i\omega t}dt = G_R(\omega) + iG_I(\omega)
\tag{57}
$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$
G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^{0} E'_0(t + t_0)e^{-(\sigma+\Delta)t}\cos(\omega t)dt + \int_{-\infty}^{0} E'_{0n}(t - t_0)e^{(\sigma-\Delta)t}\cos(\omega t)dt
\tag{58}
$$

5.6. *Zero crossing function* $\omega_z(t_2, t_0)$ *is an even function of variable* t_0

Now we consider the function $f(t) = e^{-(\sigma+\Delta)t_0} f_1(t) + e^{(\sigma+\Delta)t_0} f_2(t) = e^{-\Delta t_0} E_p'(t+t_0) + e^{\Delta t_0} E_p'(t-t_0)$ where $f_1(t) = e^{\sigma t_0} E_p'(t+t_0)$ and $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0} E_p'(t-t_0)$ and $g(t)h(t) = f(t)$ where $g(t) = f(t)e^{-\Delta t}u(-t) + f(t)e^{\Delta t}u(t)$ and $h(t) = [e^{\Delta t}u(-t) + e^{-\Delta t}u(t)]$ and compute the Fourier transform of the function $g(t)$ and compute its real part using the procedure in above section, similar to Eq. 58 and we can write as follows. We substitute $t = \tau$.

$$\begin{aligned}
G_R(\omega, t_0) &= e^{-(\sigma+\Delta)t_0} G_{1R}(\omega, t_0) + e^{(\sigma+\Delta)t_0} G_{1R}(\omega, -t_0) \\
G_{1R}(\omega, t_0) &= \int_{-\infty}^{0} [E_0'(\tau + t_0)e^{-(\sigma+\Delta)\tau} + E_{0n}'(\tau - t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega\tau) d\tau \\
G_R(\omega, t_0) &= e^{-(\sigma+\Delta)t_0} \int_{-\infty}^{0} [E_0'(\tau + t_0)e^{-(\sigma+\Delta)\tau} + E_{0n}'(\tau - t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega\tau) d\tau \\
&+ e^{(\sigma+\Delta)t_0} \int_{-\infty}^{0} [E_0'(\tau - t_0)e^{-(\sigma+\Delta)\tau} + E_{0n}'(\tau + t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega\tau) d\tau
\end{aligned} \tag{59}$$

We require $G_R(\omega, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ and we can rearrange the terms as follows.

$$\begin{aligned}
P(t_2, t_0) = &\int_{-\infty}^{0} [e^{-(\sigma+\Delta)t_0} E_0'(\tau + t_0)e^{-(\sigma+\Delta)\tau} + e^{(\sigma+\Delta)t_0} E_{0n}'(\tau + t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega_z(t_2, t_0)\tau) d\tau \\
+ &\int_{-\infty}^{0} [e^{(\sigma+\Delta)t_0} E_0'(\tau - t_0)e^{-(\sigma+\Delta)\tau} + e^{-(\sigma+\Delta)t_0} E_{0n}'(\tau - t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0
\end{aligned} \tag{60}$$

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0.

$$\begin{aligned}
P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\
P_{odd}(t_2, t_0) &= \int_{-\infty}^{0} [e^{-(\sigma+\Delta)t_0} E_0'(\tau + t_0)e^{-(\sigma+\Delta)\tau} + e^{(\sigma+\Delta)t_0} E_{0n}'(\tau + t_0)e^{(\sigma-\Delta)\tau}] \cos(\omega_z(t_2, t_0)\tau) d\tau
\end{aligned} \tag{61}$$

We see that $f(t, t_0) = e^{-\sigma t_0} E_p'(t+t_0) + e^{\sigma t_0} E_p'(t-t_0) = f(t, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2.

6. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 61 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E_{0n}'(\tau) = E_0'(-\tau)$ and $E_0'(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

We **note** that we use the **shorthand** notation for the functions $E_0'(t)$ and $E_{0n}'(t)$ which are also functions of variable t_2.

$$\begin{aligned}
P_{odd}(t_2, t_0) = &[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0'(\tau)e^{-(\sigma+\Delta)\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&+ \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0'(\tau)e^{-(\sigma+\Delta)\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
+ e^{2\Delta t_0}[\cos(\omega_z(t_2, t_0)t_0) &\int_{-\infty}^{t_0} E_{0n}'(\tau)e^{(\sigma-\Delta)\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_{0n}'(\tau)e^{(\sigma-\Delta)\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}$$

(62)

In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for a given value of t_2. This means there are **no** Dirac delta functions present in $\omega_z(t_2, t_0)$.

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2.

In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0, for every value of t_2, and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, we see that $\omega_z(t_2, t_0)t_0$ will **certainly pass through** π, as t_0 is increased from zero to ∞.

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 62 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\int_{-\infty}^{t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau + e^{2\Delta t_{0c}}\int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau$$
$$-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau - e^{-2\Delta t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau = 0 \tag{63}$$

We split the integral in the left hand side of Eq. 63 and write as follows.

$$[\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau + \int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau]$$
$$+e^{2\Delta t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau]$$
$$-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau - e^{-2\Delta t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau = 0 \tag{64}$$

We combine the terms with common integrals and cancel common terms in Eq. 64 as follows.

$$\int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau] + e^{2\Delta t_{0c}}\int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau$$
$$= -2\sinh{(2\Delta t_{0c})}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau \tag{65}$$

We can rearrange the terms in Eq. 65 as follows.

$$\int_{-t_{0c}}^{t_{0c}} [E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau} + E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}]\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau$$
$$= -2\sinh{(2\Delta t_{0c})}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}\sin{(\omega_z(t_{2c}, t_{0c})\tau)}d\tau \tag{66}$$

We denote the right hand side of Eq. 66 as RHS. We can split the integral in Eq. 66 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\int_{-t_{0c}}^{0} [E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau} + E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$+ \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau} + E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{67}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 67 as follows. We use $E_0^{'}(-\tau) = E_{0n}^{'}(\tau)$ and $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$\int_{t_{0c}}^{0} [E_{0n}^{'}(\tau)e^{(\sigma+\Delta)\tau} + E_{0}^{'}(\tau)e^{-(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$+ \int_{0}^{t_{0c}} [E_0^{'}(\tau)e^{-(\sigma+\Delta)\tau} + E_{0n}^{'}(\tau)e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{68}$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-(\sigma+\Delta)\tau} - e^{-(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}) + E_{0n}^{'}(\tau)(-e^{(\sigma+\Delta)\tau} + e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = RHS \tag{69}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 66 as follows. We use $E_{0n}^{'}(-\tau) = E_0^{'}(\tau)$.

$$RHS = 2\sinh{(2\Delta t_{0c})} \int_{t_{0c}}^{\infty} E_0^{'}(\tau)e^{-(\sigma-\Delta)\tau} \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{70}$$

We split the integral on the right hand side in Eq. 70 as follows.

$$RHS = 2\sinh{(2\Delta t_{0c})}[\int_{0}^{\infty} E_0^{'}(\tau)e^{-(\sigma-\Delta)\tau} \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau - \int_{0}^{t_{0c}} E_0^{'}(\tau)e^{-(\sigma-\Delta)\tau} \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{71}$$

We consolidate the integrals with the term $\int_0^{t_{0c}} E_0^{'}(\tau)$ in Eq. 69 and Eq. 71 as follows. We use $2\sinh{(2\sigma t_{0c})} = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-(\sigma+\Delta)\tau} - e^{-(\sigma-\Delta)\tau}e^{2\Delta t_{0c}} + e^{2\Delta t_{0c}}e^{-(\sigma-\Delta)\tau} - e^{-2\Delta t_{0c}}e^{-(\sigma-\Delta)\tau})$$
$$+E_{0n}^{'}(\tau)(-e^{(\sigma+\Delta)\tau} + e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau = 2\sinh{(2\Delta t_{0c})} \int_{0}^{\infty} E_0^{'}(\tau)e^{-(\sigma-\Delta)\tau} \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \tag{72}$$

We cancel common terms in Eq. 72 as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau)(e^{-(\sigma+\Delta)\tau} - e^{-(\sigma-\Delta)\tau}e^{-2\Delta t_{0c}}) + E_{0n}^{'}(\tau)(-e^{(\sigma+\Delta)\tau} + e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}})] \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$
$$= 2\sinh{(2\Delta t_{0c})} \int_{0}^{\infty} E_0^{'}(\tau)e^{-(\sigma-\Delta)\tau} \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau$$

$$(73)$$

We substitute $E_0^{'}(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E_{0n}^{'}(\tau) = E_0^{'}(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$. Hence we see that $E_{0n}^{'}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0^{'}(\tau)$. We can write Eq. 73 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-(\sigma+\Delta)\tau} - e^{-(\sigma-\Delta)\tau}e^{-2\Delta t_{0c}} + e^{(\sigma+\Delta)\tau} - e^{(\sigma-\Delta)\tau}e^{2\Delta t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$= 2\sinh(2\Delta t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \tag{74}$$

We substitute $2\cosh((\sigma + \Delta)\tau) = e^{(\sigma+\Delta)\tau} + e^{-(\sigma+\Delta)\tau}$ and $2\cosh(((\sigma - \Delta)\tau + 2\Delta t_{0c})t_{0c}) = e^{((\sigma-\Delta)\tau+2\Delta t_{0c})t_{0c}} + e^{-((\sigma-\Delta)\tau+2\Delta t_{0c})t_{0c}}$ and cancel the common factor of 2 in Eq. 74 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh((\sigma + \Delta)\tau) - \cosh((\sigma - \Delta)\tau + 2\Delta t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$= \sinh(2\Delta t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \tag{75}$$

Next Step:

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 75 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$RHS = \sinh(2\Delta t_{0c})[\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$+ \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \cos(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$- \cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \cos(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \tag{76}$$

In Eq. 76, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c}))t_{2c}) = -1$. Hence we cancel common terms and write Eq. 76 and Eq. 75 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh((\sigma + \Delta)\tau) - \cosh((\sigma - \Delta)\tau + 2\Delta t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$= -\sinh(2\Delta t_{0c})[\int_{-t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - \int_{t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau] \tag{77}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$ $+\int_{t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$ in Eq. 77 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh((\sigma + \Delta)\tau) - \cosh((\sigma - \Delta)\tau + 2\Delta t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$
$$= -\sinh(2\Delta t_{0c}) \int_{-t_{2c}}^{t_{2c}} E_0(\tau)e^{-(\sigma-\Delta)\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau$$

$$(78)$$

We can multiply Eq. 78 by a factor of -1 as follows.

$$\begin{aligned}&\int_0^{t_{0c}}[E_0(\tau-t_{2c})-E_0(\tau+t_{2c})](\cosh((\sigma-\Delta)\tau+2\Delta t_{0c})-\cosh((\sigma+\Delta)\tau))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\\&=\sinh(2\Delta t_{0c})\int_{-t_{2c}}^{t_{2c}}E_0(\tau)e^{-(\sigma-\Delta)\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\\&=\sinh(2\Delta t_{0c})[\int_{-t_{2c}}^{0}E_0(\tau)e^{-(\sigma-\Delta)\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau+\int_0^{t_{2c}}E_0(\tau)e^{-(\sigma-\Delta)\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau]\end{aligned} \tag{79}$$

We can substitute $\tau=-\tau$ in the first integral on the right hand side of Eq. 79 and simplify as follows, using $E_0(\tau)=E_0(-\tau)$.

$$\begin{aligned}&\int_0^{t_{0c}}[E_0(\tau-t_{2c})-E_0(\tau+t_{2c})](\cosh((\sigma-\Delta)\tau+2\Delta t_{0c})-\cosh((\sigma+\Delta)\tau))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\\&=\sinh(2\Delta t_{0c})[\int_{t_{2c}}^{0}E_0(\tau)e^{(\sigma-\Delta)\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau+\int_0^{t_{2c}}E_0(\tau)e^{-(\sigma-\Delta)\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau]\\&=\sinh(2\Delta t_{0c})\int_0^{t_{2c}}E_0(\tau)(e^{-(\sigma-\Delta)\tau}-e^{(\sigma-\Delta)\tau})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\\&=-2\sinh(2\Delta t_{0c})\int_0^{t_{2c}}E_0(\tau)\sinh(\sigma-\Delta)\tau\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\end{aligned} \tag{80}$$

We can write as follows.

$$\begin{aligned}&\int_0^{t_{0c}}[E_0(\tau-t_{2c})-E_0(\tau+t_{2c})](\cosh((\sigma-\Delta)\tau+2\Delta t_{0c})-\cosh((\sigma+\Delta)\tau))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\\&=-2\sinh(2\Delta t_{0c})\int_0^{t_{2c}}E_0(\tau)\sinh(\sigma-\Delta)\tau\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\end{aligned} \tag{81}$$

In Eq. 81, given that $\omega_z(t_{2c},t_{0c})t_{0c}=\frac{\pi}{2}$, as τ varies over the interval $[0,t_{0c}]$, $\omega_z(t_{2c},t_{0c})\tau=\frac{\pi\tau}{2t_{0c}}$ varies from $[0,\frac{\pi}{2}]$ and hence the sinusoidal function varies over a **half cycle** and is >0, in the interval $0<\tau<t_{0c}$, for $t_{0c}>0$.

In Eq. 81, we see that in the interval $0<\tau<t_{0c}$, the integral on the **left hand side** is >0 for $t_{0c}>0$, because each of the terms in the integrand are >0, in the interval $0<\tau<t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t\geq\frac{1}{8}$, we see that $E_0(\tau-t_{2c})-E_0(\tau+t_{2c})$ is >0 (Section 3.1) . The term $(\cosh(2\sigma t_{0c})-\cosh(2\sigma\tau))$ is >0 in the interval $0<\tau<t_{0c}$ and the integrand is zero at $\tau=0$ and $\tau=t_{0c}$.

On the other hand, the integral on the **right hand side** of Eq. 81 is <0 for $t_{0c}>0$, because the integral $\int_0^{t_{2c}}E_0(\tau)\sinh(\sigma-\Delta)\tau\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau>0$ given that $\sigma>\Delta$. Hence this leads to a **contradiction** for $0<\sigma<\frac{1}{4}$.

For $\sigma=0$, the integral on the **right hand side** of Eq. 81 is >0 for $t_{0c}>0$, because the integral $\int_0^{t_{2c}}E_0(\tau)\sinh(-\Delta)\tau\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau<0$. Both sides of Eq. 81 are positive and **does not** lead to a contradiction.

We have shown this result for $0<\sigma<\frac{1}{2}$ and then use the property $\xi(\frac{1}{2}+\sigma+i\omega)=\xi(\frac{1}{2}-\sigma-i\omega)$ to show the result for $-\frac{1}{2}<\sigma<0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t)=E_0(t)e^{-\sigma t}$ has a zero at $\omega=\omega_0$ for $0<|\sigma|<\frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

7. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ (Appendix D). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function which is **strictly decreasing** function for $t \geq \frac{1}{8}$. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

References

[1] Bernhard Riemann, On the Number of Prime Numbers less than a Given Quantity.(Ueber die Anzahl der Primzahlen untereiner gegebenen Grosse.) Monatsberichte der Berliner Akademie,November 1859. (Link to Riemann's 1859 paper)

[2] Hardy, G.H., Littlewood, J.E. The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift volume 10, pp.283 to 317 (1921).

[3] E. C. Titchmarsh, The Theory of the Riemann Zeta Function. (1986) pp.254 to 255

[4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152

[5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)

[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). This is re-derived in Appendix D.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$
\begin{aligned}
\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\
E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega
\end{aligned}
\tag{A.1}
$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$
E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega'
\tag{A.2}
$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **??**, we show that $\int_{-\infty}^{\infty}|E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$
\begin{aligned}
E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \\
E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{A.3}
$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'
\end{aligned} \tag{B.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.3}$$

Appendix B.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned} \tag{B.4}$$

Appendix B.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t)+g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega)
\end{aligned} \tag{B.5}$$

Appendix B.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{4}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.2. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

Appendix D. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned}\Gamma(\frac{s}{2}) &= \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx\end{aligned} \tag{D.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{D.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^{\infty} |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{D.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. D.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0,1]$ and $[1,\infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0,1]$. We use **the well known theorem** $1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty}\frac{x^{-(\frac{s}{2}-1)}}{x^2}\frac{(1+2w(x))\sqrt{x}-1)}{2}dx \tag{D.4}$$

Hence we can simplify Eq. D.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}}w(x)dx \tag{D.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1+s(s-1)\int_1^{\infty}(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}]$$

$$(D.6)$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix D.1. ***Derivation of $E_p(t)$ and $E_0(t)$***

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. D.6 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{D.7}$$

We can substitute $t=-t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt \\ +\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{D.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{D.9}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}E_p(t) &= \frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\ &+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned} \tag{D.10}$$

We can simplify above equation as follows.

$$\begin{aligned}\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t}-6\pi n^2 e^{-2t}-4\sigma\pi n^2 e^{-2t}]u(-t) \\ &\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t}-6\pi n^2 e^{2t}+4\sigma\pi n^2 e^{2t}]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned}$$

(D.11)

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix D.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$

$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+ \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$

(D.12)

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

(D.13)

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$

(D.14)

Appendix D.2. ***Derivation of*** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}}$$

(D.15)

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2})$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$$

(D.16)