1. Compton Effect Derivation with Non-Zero electron velocity before collision with photon

Compton Scattering equations have been derived assuming electron is at rest before collision with photon. (link).] This is a theoretical case, in practice, electrons are always moving inside the matter, even before collision with photon. In this section, we will derive the equations for the general case of **Non-Zero electron velocity before** collision with photon.

Let $f, f^{'}$ be the frequency of the light before and after collision with an electron with rest mass m_e. Let $v, v^{'}$ be the velocity of electron before and after collision. Let $p_e = \frac{m_e v}{\sqrt{1-\frac{v^2}{c^2}}}, p_e^{'} = \frac{m_e v^{'}}{\sqrt{1-\frac{(v^{'})^2}{c^2}}}$ be the momentum of electron before and after collision. Let $K = \sqrt{1 - \frac{v^2}{c^2}}$ and $K^{'} = \sqrt{1 - \frac{(v^{'})^2}{c^2}}$.

Energy of electron before and after collision is given by $E_e = m_T c^2$ and $E_e^{'} = m_T^{'} c^2$ and Energy of photon before and after collision is given by $E_\gamma = hf$ and $E_\gamma^{'} = hf^{'}$. Using **Conservation of Energy** we have as follows.

$$\begin{aligned}
E_\gamma + E_e &= E_\gamma^{'} + E_e^{'} \\
hf + \sqrt{(m_e c^2)^2 + (p_e c)^2} &= hf^{'} + \sqrt{(m_e c^2)^2 + (p_e^{'} c)^2} \\
(m_e c^2)^2 + (p_e^{'} c)^2 &= (hf - hf^{'} + \sqrt{(m_e c^2)^2 + (p_e c)^2})^2 \\
(p_e^{'} c)^2 = -(m_e c^2)^2 + (hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} &+ (m_e c^2)^2 + (p_e c)^2 + 2h(f - f^{'}) \sqrt{(m_e c^2)^2 + (p_e c)^2} \\
(p_e^{'} c)^2 &= (hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} + (p_e c)^2 + 2h(f - f^{'}) \sqrt{(m_e c^2)^2 + (p_e c)^2}
\end{aligned} \tag{1}$$

Using **Conservation of Momentum** we have as follows.

$$\begin{aligned}
\vec{p_\gamma} + \vec{p_e} &= \vec{p_\gamma^{'}} + \vec{p_e^{'}} \\
\vec{p_e^{'}} &= \vec{p_\gamma} + \vec{p_e} - \vec{p_\gamma^{'}} \\
(p_e^{'})^2 &= \vec{p_e^{'}} \cdot \vec{p_e^{'}} = (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \cdot (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \\
(p_e^{'})^2 &= [(p_\gamma)^2 + (p_\gamma^{'})^2 - 2 p_\gamma p_\gamma^{'} \cos\theta] + [p_e^2 + 2 p_e p_\gamma \cos\theta_1 - 2 p_e p_\gamma^{'} \cos\theta_2]
\end{aligned} \tag{2}$$

We multiply both sides of above equation by c^2 and use $p_\gamma = \frac{hf}{c}, p_\gamma^{'} = \frac{hf^{'}}{c}$ and write as follows.

$$(p_e^{'})^2 c^2 = [(hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} \cos\theta] + [p_e^2 c^2 + 2 p_e h f c \cos\theta_1 - 2 p_e h f^{'} c \cos\theta_2] \tag{3}$$

Equating Eq. 1 and Eq. 3 and cancelling common terms, we have

$$2h^2ff^{'}[1-\cos\theta] = 2h(f-f^{'})\sqrt{(m_ec^2)^2+(p_ec)^2} - 2hcp_e[f\cos\theta_1 - f^{'}\cos\theta_2] \tag{4}$$

Dividing both sides of above equation by the term $2hff^{'}m_ec$, we use $p_e = \frac{m_ev}{\sqrt{(1-\frac{v^2}{c^2})}}$, $\lambda = \frac{c}{f}, \lambda^{'} = \frac{c}{f'}$ we have

$$\frac{h}{m_ec}[1-\cos\theta] = (\lambda^{'}-\lambda)\sqrt{1+(\frac{p_e}{m_ec})^2} - \frac{1}{ff'm_e}p_e[f\cos\theta_1 - f^{'}\cos\theta_2] \tag{5}$$

If **electron is at rest** before collision, $p_e = 0$ and we get the familiar **Compton effect equation** as follows.

$$\frac{h}{m_ec}[1-\cos\theta] = (\lambda^{'}-\lambda) \tag{6}$$

Thus we can see that Eq. 5 has **extra terms** when electron has **non-zero velocity** before collision with photon.

Now we substitute $\theta = \pi, \theta_1 = 0, \theta_2 = \pi$ in Eq. 3 , assuming the case where electron direction is the same before and after collision and is aligned with photon direction before collision and photon is reflected back at angle π after collision.

$$\begin{aligned}
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 + 2h^2ff^{'}] + [p_e^2c^2 + 2p_ehfc + 2p_ehf^{'}c] \\
(p_e^{'})^2c^2 &= h^2(f+f^{'})^2 + p_ec[p_ec + 2h(f+f^{'})] \\
(p_e^{'})^2 &= \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_ec + 2h(f+f^{'})]
\end{aligned} \tag{7}$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle, which uses Compton scattering of an electron by photon, **more complicated**, compared to the case where $p_e = 0$.

2. Compton Effect and Heisenberg's uncertainty principle

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle, which uses Compton scattering of an electron by photon, **more complicated**, compared to the case where $p_e = 0$.

$$\begin{aligned}
(p_e^{'})^2 = \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_e c + 2h(f+f^{'})] = \frac{h^2}{c^2}(f+f^{'})^2 X(f,f^{'},p_e) \\
X(f,f^{'},p_e) = 1 + \frac{p_e}{c}(p_e c + 2h(f+f^{'})) = [1 + Z(f,f^{'},p_e)] \\
A(f,f^{'},p_e) = [\sqrt{X(f,f^{'},p_e)} - 1] = ([1 + \frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] - 1) \\
= [\frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] \\
Y(f,f^{'},p_e) = \frac{h}{c}(f+f^{'})A(f,f^{'},p_e)
\end{aligned} \tag{8}$$

We can see that $Y(f,f^{'},p_e) > 0$ for all f.

$$\begin{aligned}
p_e' = m_e^{'}v = \frac{h}{c}(f+f^{'}) + Y(f,f^{'},p_e) \\
\frac{d(p_e')}{df} = \frac{h}{c} + \frac{d(Y)}{df}
\end{aligned} \tag{9}$$

Replacing $d(p_e')$ by $\triangle p$, we have

$$\begin{aligned}
\triangle x \triangle p = c \triangle t \triangle p = c \triangle t \triangle f[\frac{h}{c} + \frac{d(Y)}{df}] \\
\triangle t \triangle f \geq \frac{1}{2\pi} \\
\triangle x \triangle p \geq \frac{h}{2\pi} + \frac{c}{2\pi}\frac{d(Y)}{df}
\end{aligned} \tag{10}$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle **more complicated**, which uses Compton scattering of an electron by photon, compared to the case where $p_e = 0$.