

On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s -plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function. [4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$. [2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < \text{Re}[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new

method discussed in this paper will lead to a correct solution by other researchers.

In Section 2 to Section 6, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 7, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$.

We present an **outline** of the new method below.

1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where ω is real. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$, where ω, t are real, as follows (link).[3] (Titchmarsh pp254-255) This is re-derived in Appendix B. We take the term $e^{\frac{t}{2}}$ out of the bracket and rearrange the terms as follows.

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t , given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ (link) and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$. (Details in Appendix C.8)

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t) = E_0(t) e^{-\sigma t} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that $E_p(t)$ is an analytic function for real t , given that the sum and product of exponential functions are analytic for real t and hence infinitely differentiable for real t .

1.2. Step 2: On the zeros of a related function $G(\omega, t_2, t_0)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t, t_2, t_0) = f(t, t_2, t_0) e^{-\sigma t} u(-t) + f(t, t_2, t_0) e^{\sigma t} u(t)$, where $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ and $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ and $E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$ and t_0, t_2 are real and $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can

see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, for every value of t_0 , for each nonzero value of t_2 , where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

1.3. Step 3: On the zeros of the function $G_R(\omega, t_2, t_0)$

In Section 2.3, we compute the Fourier transform of the function $g(t, t_2, t_0)$ and compute its real part given by $G_R(\omega, t_2, t_0)$ and we can write as follows.

$$G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \quad (3)$$

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **each non-zero value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = 0$.

1.4. Step 4: Zero Crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

In Section 2.4, we show the result in Eq. 4 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ and that $P_{odd}(t_2, t_0)$ is an **odd** function of t_0 , for each non-zero value of t_2 as follows.

$$P_{odd}(t_2, t_0) = [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau] \quad (4)$$

1.5. Step 5: Final Step

In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2, t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_{0n}(t, t_2) = E'_0(-t, t_2)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$$

(5)

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$, where $t_{0c} > 0$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We show this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we produce a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t , where $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$, $E_p(t) = E_0(t) e^{-\sigma t}$ and $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 \leq |\sigma| < \frac{1}{2}$. [3] (Titchmarsh pp30-31). This is shown in detail in first two paragraphs in Appendix C.1.

2.1. New function $g(t, t_2, t_0)$

Let us consider the function $E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2)) e^{-\sigma t} = E'_0(t, t_2) e^{-\sigma t}$, where t_2 is non-zero and real, and $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ (**Definition 1**). Its Fourier transform is given by $E'_{p\omega}(\omega, t_2) = E_{p\omega}(\omega) (e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$, using Statement 1 and linearity and time shift properties of the Fourier transform (link). (**Result 2.1.1**).

Let us consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega, t_2, t_0) = E'_{p\omega}(\omega, t_2) (e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$, using Result 2.1.1. (**Result 2.1.2**)

Let us consider a new function $g(t, t_2, t_0) = g_-(t, t_2, t_0) u(-t) + g_+(t, t_2, t_0) u(t)$ where $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t, t_2, t_0) = f(t, t_2, t_0) e^{-\sigma t}$ and $g_+(t, t_2, t_0) = f(t, t_2, t_0) e^{\sigma t}$. We can see that $g(t, t_2, t_0) h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$.

We can write the above equations as follows.

$$\begin{aligned}
E_p'(t, t_2) &= e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t, t_2)e^{-\sigma t} \\
f_1(t, t_2, t_0) &= e^{\sigma t_0} E_p'(t + t_0, t_2) \\
f_2(t, t_2, t_0) &= f_1(t, t_2, -t_0) = e^{-\sigma t_0} E_p'(t - t_0, t_2) \\
f(t, t_2, t_0) &= e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p'(t + t_0, t_2) + e^{\sigma t_0} E_p'(t - t_0, t_2) \\
g(t, t_2, t_0) &= [f(t, t_2, t_0)e^{-\sigma t}]u(-t) + [f(t, t_2, t_0)e^{\sigma t}]u(t) \\
g(t, t_2, t_0)h(t) &= f(t, t_2, t_0), \quad h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]
\end{aligned}$$

(6)

We can show that $E_p(t), E_p'(t, t_2), h(t)$ are absolutely integrable functions and go to zero as $t \rightarrow \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}'(\omega, t_2), H(\omega)$ are finite for real ω and go to zero as $|\omega| \rightarrow \infty$, as per Riemann Lebesgue Lemma (link). We can show that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions. These results are shown in Appendix C.1.

In Section 2.3 and Section 2.4, it is shown that $g(t, t_2, t_0)$ is a Fourier transformable function and its Fourier transform given by $G(\omega, t_2, t_0) = e^{-2\sigma t_0} G_1(\omega, t_2, t_0) + e^{2\sigma t_0} G_1(\omega, t_2, -t_0)$ converges. (Eq. 14 and Eq. 17)

If we take the Fourier transform of the equation $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, using Result 2.1.2, we get $\frac{1}{2\pi}[G(\omega, t_2, t_0) * H(\omega)] = F(\omega, t_2, t_0) = E_{p\omega}'(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega, t_2, t_0) + iF_I(\omega, t_2, t_0)$ as per **convolution theorem** (link), where $*$ denotes convolution operation given by $F(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega', t_2, t_0)H(\omega - \omega')d\omega'$.

We see that $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ (link). $G(\omega, t_2, t_0) = G_R(\omega, t_2, t_0) + iG_I(\omega, t_2, t_0)$ is the Fourier transform of the function $g(t, t_2, t_0)$. We can write $g(t, t_2, t_0) = g_{\text{even}}(t, t_2, t_0) + g_{\text{odd}}(t, t_2, t_0)$ where $g_{\text{even}}(t, t_2, t_0)$ is an even function and $g_{\text{odd}}(t, t_2, t_0)$ is an odd function of variable t .

If Statement 1 is true, then we require the Fourier transform of the function $f(t, t_2, t_0)$ given by $F(\omega, t_2, t_0)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0 , for each non-zero value of t_2 , using Result 2.1.2. This implies that the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$ (Appendix D.2) must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_2, t_0)$ is real and finite, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, explained below. We note that $\omega_z(t_2, t_0)$ can be different from ω_0 in general.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega, t_2, t_0)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, **then the real part** of $F(\omega, t_2, t_0)$ given by $F_R(\omega, t_2, t_0) = \frac{1}{2\pi}[G_R(\omega, t_2, t_0) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega, t_2, t_0)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against Result 2.1.2 and **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0 , for each non-zero value of t_2 , where $G_R(\omega, t_2, t_0)$ crosses

the zero line to the opposite sign and hence $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ and $\omega_z(t_2, t_0)$ is real and finite, where $g(t, t_2, t_0)h(t) = f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0}E'_p(t - t_0, t_2)$, $E'_p(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2)$, and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega, t_2, t_0) = E'_{p\omega}(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$, using Result 2.1.2 and its real part given by $F_R(\omega, t_2, t_0)$ also has a zero at the same location $\omega = \omega_0 \neq 0$ (**Result 2.1.3**).

We consider the case where $G_R(\omega, t_2, t_0)$ **does not** have at least one zero for finite $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign and will show that $F_R(\omega, t_2, t_0)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Result 2.1.3 and Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) H(\omega - \omega') d\omega' \quad (7)$$

We can show that the above integral converges for real ω , given that the integrand is absolutely integrable because $G(\omega, t_2, t_0)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t, t_2, t_0)$ and $h(t)$ are discontinuous at $t = 0$. (Appendix C.2 and Appendix C.6)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 7 and we get

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (8)$$

We can split the integral in Eq. 8 using $\int_{-\infty}^{\infty} = \int_{-\infty}^0 + \int_0^{\infty}$, as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \left[\int_{-\infty}^0 G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (9)$$

We see that $G_R(-\omega, t_2, t_0) = G_R(\omega, t_2, t_0)$ because $g(t, t_2, t_0)$ is a real function of variable t . (Appendix D.1) We can substitute $\omega' = -\omega''$ in the first integral in Eq. 9 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_0^{\infty} G_R(\omega', t_2, t_0) \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (10)$$

In Appendix C.2, it is shown that $G(\omega', t_2, t_0)$ is finite for real ω' and goes to zero as $|\omega'| \rightarrow \infty$. We can see that for $\omega' \rightarrow \infty$, the integrand in Eq. 10 is zero. For finite $\omega \geq 0$, and $0 \leq \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$, for $0 < \sigma < \frac{1}{2}$. We see that $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω' (Section 2.2). (**Result 2.1.4**)

• **Case 1:** $G_R(\omega', t_2, t_0) \geq 0$ for all finite $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix D.1) and link). Hence $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \leq 0$.

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This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$ where it crosses the zero line and becomes negative, where $\omega_z(t_2, t_0)$ is real and finite.

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• **Case 2:** $G_R(\omega', t_2, t_0) \leq 0$ for all finite $\omega' \geq 0$

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We see that $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix D.1) and link). Hence $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \leq 0$.

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This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$, where it crosses the zero line and becomes positive, where $\omega_z(t_2, t_0)$ is real and finite.

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We have shown that, $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**. It is shown in Section 4.1 that $G_R(\omega, t_2, t_0)$ is partially differentiable as a function of ω and hence a continuous function of ω , for a given value of t_0 and t_2 . Hence $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$. We call this **Result 2.1.5**.

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In the rest of the sections, we consider only the **first** zero crossing away from origin, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign. Hence $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2 , to satisfy **Statement 1**.

2.2. $G_R(\omega', t_2, t_0)$ **is not an all zero function of variable ω'**

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If $G_R(\omega', t_2, t_0)$ is an all zero function of variable ω' , for each given value of t_0, t_2 (**Statement 2**), then $F_R(\omega, t_2, t_0)$ in Eq. 7 is an all zero function of ω , for real ω . Hence $2f_{\text{even}}(t, t_2, t_0) = f(t, t_2, t_0) + f(-t, t_2, t_0)$ is an **all-zero** function of t , given that the Fourier transform of $f_{\text{even}}(t, t_2, t_0)$ is given by $F_R(\omega, t_2, t_0)$, using symmetry properties of Fourier transform(Appendix D.2) and link). Hence $f(t, t_2, t_0)$ is an **odd function** of variable t . (**Result 2.2**).

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From Eq. 6 we see that $E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = [E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t}$. Hence $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2) = [E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t}$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2) = [E_0(t - t_0 - t_2) - E_0(t - t_0 + t_2)]e^{-\sigma t}$. Hence we can write $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ in Eq. 6, as follows.

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$$f(t, t_2, t_0) = e^{-2\sigma t_0} [E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t} + e^{2\sigma t_0} [E_0(t - t_0 - t_2) - E_0(t - t_0 + t_2)]e^{-\sigma t} \quad (11)$$

Case 1: For $t_0 \neq 0$ and $t_2 \neq 0$, it is shown that Result 2.2 is false. We will compute $f(t, t_2, t_0)$ in Eq. 11 at $t = 0$ and show that it does not equal zero.

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We see that $f(0, t_2, t_0) = e^{-2\sigma t_0} [E_0(t_0 - t_2) - E_0(t_0 + t_2)] + e^{2\sigma t_0} [E_0(-t_0 - t_2) - E_0(-t_0 + t_2)] = -2 \sinh(2\sigma t_0) [E_0(t_0 - t_2) - E_0(t_0 + t_2)]$. We use the fact that $E_0(t_0) = E_0(-t_0)$ (Appendix C.8)

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and hence $E_0(t_0 - t_2) = E_0(-t_0 + t_2)$ and $E_0(t_0 + t_2) = E_0(-t_0 - t_2)$.

If Result 2.2 is true, then we require $f(0, t_2, t_0) = 0$ in Eq. 11. For our choice of $0 < \sigma < \frac{1}{2}$ and $t_0 \neq 0$, this implies that $E_0(t_0 - t_2) = E_0(t_0 + t_2)$. Given that $t_0 \neq 0$ and $t_2 \neq 0$, we set $t_2 = Kt_0$ for real $K \neq 0$ and we get $E_0((1 - K)t_0) = E_0((1 + K)t_0)$. This is **not** possible for $t_0 \neq 0$ because $E_0(t_0)$ is **strictly decreasing** for $t_0 > 0$ (Section 6) and $1 - K \neq 1 + K$ or $1 - K \neq -(1 + K)$ for $K \neq 0$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω' .

Case 2: For $t_0 = 0$ and $t_2 \neq 0$, we have $f(t, t_2, t_0) = 2[E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t} = 2D(t)e^{-\sigma t}$ in Eq. 11, where $D(t) = E_0(t - t_2) - E_0(t + t_2)$. We see that $D(t) + D(-t) = E_0(t - t_2) - E_0(t + t_2) + E_0(-t - t_2) - E_0(-t + t_2)$. Given that $E_0(t) = E_0(-t)$, we have $D(t) + D(-t) = E_0(t - t_2) - E_0(t + t_2) + E_0(t + t_2) - E_0(t - t_2) = 0$ and hence $D(t) = E_0(t - t_2) - E_0(t + t_2)$ is an **odd** function of variable t (**Result 2.2.1**).

If Result 2.2 is true, then we require $f(t, t_2, t_0) = 2D(t)e^{-\sigma t}$ to be an **odd** function of variable t . Using Result 2.2.1, we require $D(t)$ to be an **odd** function of variable t . This is possible only for $\sigma = 0$. This is **not** possible for our choice of $0 < \sigma < \frac{1}{2}$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω' .

Case 3: For $t_2 = 0$ and $|t_0| < \infty$, we have $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = 0$ and $f(t, t_2, t_0) = g(t, t_2, t_0) = 0$ for all t in Eq. 6 and Lemma 1 is not applicable for this case.

2.3. On the zeros of a related function $G(\omega, t_2, t_0)$

In this section, we compute the Fourier transform of the function $g_{\text{even}}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$ (Appendix D.2). We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0 , for each non-zero value of t_2 , to satisfy **Statement 1**, using Lemma 1 in Section 2.1.

We **define** $g_1(t, t_2, t_0) = f_1(t, t_2, t_0)e^{-\sigma t}u(-t) + f_1(t, t_2, t_0)e^{\sigma t}u(t) = e^{\sigma t_0}E_p'(t + t_0, t_2)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p'(t + t_0, t_2)e^{\sigma t}u(t)$, using Eq. 6 (**Definition 3**). First we compute the Fourier transform of the function $g_1(t, t_2, t_0)$ given by $G_1(\omega, t_2, t_0) = G_{1R}(\omega, t_2, t_0) + iG_{1I}(\omega, t_2, t_0)$.

$$\begin{aligned} G_1(\omega, t_2, t_0) &= \int_{-\infty}^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt = \int_{-\infty}^0 g_1(t, t_2, t_0)e^{-i\omega t}dt + \int_0^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt \\ G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 e^{\sigma t_0}E_p'(t + t_0, t_2)e^{-\sigma t}e^{-i\omega t}dt + \int_0^{\infty} e^{\sigma t_0}E_p'(t + t_0, t_2)e^{\sigma t}e^{-i\omega t}dt \end{aligned}$$

(12)

We use $E_p'(t, t_2) = E_0'(t, t_2)e^{-\sigma t}$ from Eq. 6, where $E_0'(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$, using Definition 1 in Section 2.1 and we get $E_p'(t + t_0, t_2) = E_0'(t + t_0, t_2)e^{-\sigma t}e^{-\sigma t_0}$ and write Eq. 12 as follows. Then we substitute $t = -t$ in the second integral in first line of Eq. 13.

$$\begin{aligned} G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 E_0'(t + t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_0^{\infty} E_0'(t + t_0, t_2)e^{-i\omega t}dt \\ G_1(\omega, t_2, t_0) &= \int_{-\infty}^0 E_0'(t + t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E_0'(-t + t_0, t_2)e^{i\omega t}dt \end{aligned}$$

$$(13)$$

We define $E'_{0n}(t, t_2) = E'_0(-t, t_2)$ (**Definition 2**) and get $E'_0(-t + t_0, t_2) = E'_{0n}(t - t_0, t_2)$ and write Eq. 13 as follows. The integral in Eq. 14 converges, given that $E_0(t)e^{-2\sigma t}$ is an absolutely **integrable** function (Appendix C.1) and its t_0, t_2 shifted versions are absolutely **integrable**, using $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ in Definition 1 in Section 2.1 and Definition 2.

$$G_1(\omega, t_2, t_0) = \int_{-\infty}^0 E'_0(t + t_0, t_2) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^0 E'_{0n}(t - t_0, t_2) e^{i\omega t} dt = G_{1R}(\omega, t_2, t_0) + iG_{1I}(\omega, t_2, t_0) \quad (14)$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G_1(\omega, t_2, t_0)$, we have

$$G_{1R}(\omega, t_2, t_0) = \int_{-\infty}^0 E'_0(t + t_0, t_2) e^{-2\sigma t} \cos(\omega t) dt + \int_{-\infty}^0 E'_{0n}(t - t_0, t_2) \cos(\omega t) dt \quad (15)$$

2.4. Zero crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0 , for a given t_2

Now we consider Eq. 6 and the function $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t + t_0, t_2) + e^{\sigma t_0} E'_p(t - t_0, t_2)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0} E'_p(t - t_0, t_2)$ and $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$. We can write the above equations and $g_1(t, t_2, t_0)$ from Definition 3 in Section 2.3, as follows. We define $g_2(t, t_2, t_0)$ below and write $g(t, t_2, t_0)$ as follows.

$$\begin{aligned} g_1(t, t_2, t_0) &= f_1(t, t_2, t_0)e^{-\sigma t}u(-t) + f_1(t, t_2, t_0)e^{\sigma t}u(t), & g_1(t, t_2, t_0)h(t) &= f_1(t, t_2, t_0) \\ g_2(t, t_2, t_0) &= f_2(t, t_2, t_0)e^{-\sigma t}u(-t) + f_2(t, t_2, t_0)e^{\sigma t}u(t), & g_2(t, t_2, t_0)h(t) &= f_2(t, t_2, t_0) \\ g(t, t_2, t_0) &= e^{-2\sigma t_0}g_1(t, t_2, t_0) + e^{2\sigma t_0}g_2(t, t_2, t_0) \end{aligned}$$

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$$(16)$$

We compute the Fourier transform of the function $g(t, t_2, t_0)$ in Eq. 16 and compute its real part $G_R(\omega, t_2, t_0)$ using the procedure in Section 2.3, similar to Eq. 15 and we can write as follows in Eq. 17. We use $G_{2R}(\omega, t_2, t_0) = G_{1R}(\omega, t_2, -t_0)$ given that $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0)$ and $g_2(t, t_2, t_0) = g_1(t, t_2, -t_0)$ and $G_2(\omega, t_2, t_0) = G_1(\omega, t_2, -t_0)$. We substitute $t = \tau$ in the equation for $G_{1R}(\omega, t_2, t_0)$ below, copied from Eq. 15.

$$\begin{aligned} G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0}G_{1R}(\omega, t_2, t_0) + e^{2\sigma t_0}G_{2R}(\omega, t_2, t_0) = e^{-2\sigma t_0}G_{1R}(\omega, t_2, t_0) + e^{2\sigma t_0}G_{1R}(\omega, t_2, -t_0) \\ G_{1R}(\omega, t_2, t_0) &= \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega \tau) d\tau \\ G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega \tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma \tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega \tau) d\tau \end{aligned}$$

(17)

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for each non-zero value of t_2 , to satisfy **Statement 1**, using Lemma 1 in Section 2.1. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = 0$ and we can rearrange the terms in Eq. 17 as follows. We take the first and fourth terms in $G_R(\omega, t_2, t_0)$ in Eq. 17 and include them in the first line in Eq. 18. We take the second and third terms in Eq. 17 and include them in the second line in Eq. 18.

$$P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_2, t_0) = \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\ + \int_{-\infty}^0 [e^{2\sigma t_0} E'_0(\tau - t_0, t_2) e^{-2\sigma\tau} + e^{-2\sigma t_0} E'_{0n}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0$$

(18)

We use the fact that $f(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t + t_0, t_2) + e^{\sigma t_0} E'_p(t - t_0, t_2) = f(t, t_2, -t_0)$ in Eq. 6, is **unchanged** by the substitution $t_0 = -t_0$. **If** $f(t, t_2, t_0) = f(t, t_2, -t_0)$ is unchanged by the substitution $t_0 = -t_0$, **then** $g(t, t_2, t_0) = g(t, t_2, -t_0)$ is unchanged by the substitution $t_0 = -t_0$, using the fact that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ and $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$.

Hence the Fourier transform of $g(t, t_2, t_0)$ given by $G(\omega, t_2, t_0) = G(\omega, t_2, -t_0)$ and its real part given by $G_R(\omega, t_2, t_0) = G_R(\omega, t_2, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and the zero crossing in $G_R(\omega, t_2, -t_0)$ given by $\omega_z(t_2, -t_0)$ is the **same** as the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ and we get $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0 , for each non-zero value of t_2 .

We can write Eq. 18 as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0 , for each non-zero value of t_2 . We use $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$.

$$P(t_2, t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\ P_{odd}(t_2, t_0) = \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0, t_2) e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau$$

(19)

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 19 as follows, using the substitution $\tau + t_0 = \tau'$. We get $\tau = \tau' - t_0$ and $d\tau = d\tau'$ and substitute back $\tau' = \tau$ in the second line below. We use $e^{-2\sigma t_0} e^{2\sigma t_0} = 1$ below.

$$\begin{aligned}
P_{odd}(t_2, t_0) &= \int_{-\infty}^{t_0} [e^{-2\sigma t_0} E'_0(\tau', t_2) e^{-2\sigma \tau'} e^{2\sigma t_0} + e^{2\sigma t_0} E'_{0n}(\tau', t_2)] \cos(\omega_z(t_2, t_0)(\tau' - t_0)) d\tau' \\
P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma \tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma \tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&\quad + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned} \tag{20}$$

In Section 2.1, it is shown that $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2 . In this section, we consider $t_0 > 0$ and $t_2 > 0$ only.

In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given that $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2 , we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2 and given that t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function and is positive for $t_0 > 0$ because $0 < \omega_z(t_2, t_0) < \infty$.

We see that $\omega_z(t_2, t_0) > 0$ and is a **continuous** function of variable t_0 and t_2 , as t_0 and t_2 increase to a larger and larger finite value without bounds and that the order of $\omega_z(t_2, t_0)t_0$ is greater than 1 (Section 5). As t_0 and t_2 increase from zero to a larger and larger finite value without bounds, the continuous function $\omega_z(t_2, t_0)t_0$ starts from zero and increases with order greater than $O[1]$ and will pass through $\frac{\pi}{2}$.

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 20 as follows. We use the fact that $\cos(\omega_z(t_{2c}, t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c})t_{0c}) = 1$ and $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.4.

$$\begin{aligned}
P_{odd}(t_{2c}, t_{0c}) &= \int_{-\infty}^{t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma \tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned} \tag{21}$$

We compute $P_{odd}(t_2, -t_0)$ in Eq. 20 as follows. We use $\omega_z(t_2, -t_0) = \omega_z(t_2, t_0)$ (Section 2.4).

$$\begin{aligned}
P_{odd}(t_2, -t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad - \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&+ e^{-2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau - \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned}
\tag{22}$$

369

370 We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 22 as follows. We use
371 $\cos(\omega_z(t_{2c}, t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c})t_{0c}) = 1$.

$$P_{odd}(t_{2c}, -t_{0c}) = - \int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\tag{23}$$

372

373 We compute $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ in Eq. 19, at $t_0 = t_{0c}$ and $t_2 = t_{2c}$ using Eq. 21 and
374 Eq. 23.

$$\begin{aligned}
&\int_{-\infty}^{t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&- \int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned}
\tag{24}$$

375

376 We split the first two integrals in the left hand side of Eq. 24 using $\int_{-\infty}^{t_{0c}} = \int_{-\infty}^{-t_{0c}} + \int_{-t_{0c}}^{t_{0c}}$ as follows.

$$\begin{aligned}
&[\int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&+ e^{2\sigma t_{0c}} [\int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&- \int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned}
\tag{25}$$

377

378 We cancel the common integral $\int_{-\infty}^{-t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 25 and rearrange
379 the terms as follows, using $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} E'_0(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&= -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned}$$

We can combine the integrals in the left hand side of Eq. 26 as follows.

$$\begin{aligned} & \int_{-t_{0c}}^{t_{0c}} [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &= -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

We denote the right hand side of Eq. 27 as RHS . We can split the integral in the left hand side of Eq. 27 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^0 + \int_0^{t_{0c}}$ as follows.

$$\begin{aligned} & \int_{-t_{0c}}^0 [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &+ \int_0^{t_{0c}} [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 28 as follows. We use $E'_0(-\tau, t_{2c}) = E'_{0n}(\tau, t_{2c})$ and $E'_{0n}(-\tau, t_{2c}) = E'_0(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$\begin{aligned} & \int_{t_{0c}}^0 [E'_{0n}(\tau, t_{2c})e^{2\sigma\tau} + E'_0(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &+ \int_0^{t_{0c}} [E'_0(\tau, t_{2c})e^{-2\sigma\tau} + E'_{0n}(\tau, t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned}$$

Given that $\int_{t_{0c}}^0 = -\int_0^{t_{0c}}$, we can simplify Eq. 29 as follows.

$$\int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 27 as follows. We use $E'_{0n}(-\tau, t_{2c}) = E'_0(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$RHS = 2 \sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$$

We split the integral on the right hand side in Eq. 31 using $\int_{t_{0c}}^{\infty} = \int_0^{\infty} - \int_0^{t_{0c}}$, as follows.

$$RHS = 2 \sinh(2\sigma t_{0c}) \left[\int_0^\infty E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \quad (32)$$

We consolidate the integrals of the form $\int_0^{t_{0c}} E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 30 and Eq. 32 as follows. We use $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (33)$$

We cancel the common term $e^{2\sigma t_{0c}}$ in the first integral in Eq. 33 as follows.

$$\begin{aligned} \int_0^{t_{0c}} [E'_0(\tau, t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty E'_0(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (34)$$

We substitute $E'_0(\tau, t_{2c}) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ (using Definition 1 in Section 2.1) and $E'_{0n}(\tau, t_{2c}) = E'_0(-\tau, t_{2c}) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$ (using Definition 2 in Section 2.3). We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$ (Appendix C.8). Hence we see that $E'_{0n}(\tau, t_{2c}) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E'_0(\tau, t_{2c})$ (**Result 3.1**) and write Eq. 34 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2 \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (35)$$

We substitute $2 \cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2 \cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 35 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (36)$$

Next Step:

We denote the right hand side of Eq. 36 as RHS' . We substitute $\tau - t_{2c} = \tau'$ and $\tau + t_{2c} = \tau''$ in the right hand side of Eq. 36 and then substitute $\tau' = \tau$ and $\tau'' = \tau$ in the second line below.

$$\begin{aligned}
RHS' &= \sinh(2\sigma t_{0c}) \left[\int_{-t_{2c}}^{\infty} E_0(\tau') \sin(\omega_z(t_{2c}, t_{0c})(\tau' + t_{2c})) d\tau' - \int_{t_{2c}}^{\infty} E_0(\tau'') \sin(\omega_z(t_{2c}, t_{0c})(\tau'' - t_{2c})) d\tau'' \right] \\
RHS' &= \sinh(2\sigma t_{0c}) \left[\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right. \\
&\quad \left. + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right. \\
&\quad \left. - \cos(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^{\infty} E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right]
\end{aligned} \tag{37}$$

In Eq. 37, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$. Hence we cancel common terms and write Eq. 37 and Eq. 36 as follows.

$$\begin{aligned}
&\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&= -\sinh(2\sigma t_{0c}) \left[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right]
\end{aligned} \tag{38}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 38 as follows. Given that $E_0(\tau)$ is an **even** function of variable τ (Appendix C.8) and $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ , we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

We see that $I = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^0 E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_0^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$. We substitute $\tau = -\tau$ in the first integral and get $I = \int_{t_{2c}}^0 E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_0^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = -\int_0^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_0^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$. We write Eq. 38 as follows.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{39}$$

We can multiply Eq. 39 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})] (\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \tag{40}$$

In Eq. 40, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $(0, t_{0c})$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $(0, \frac{\pi}{2})$ and the sinusoidal function is > 0 , in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 40, we see that the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0 , in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ (Section 6), we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 3.1) in the interval $0 < \tau < t_{0c}$. The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$.

The integrand is zero at $\tau = 0$ due to the term $\sin(\omega_z(t_{2c}, t_{0c})\tau)$ and the integrand is zero at $\tau = t_{0c}$ due to the term $\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 40. Hence this leads to a **contradiction**, for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 40 is zero, given the term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) = 0$ and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$. **If** the Fourier transform of $E_p(t) = E_0(t)e^{-\sigma t}$ given by $E_{p\omega}(\omega) = E_{pR\omega}(\omega) + iE_{pI\omega}(\omega)$ has a zero at $\omega = \omega_0$, **then** the real part $E_{pR\omega}(\omega)$ and imaginary part $E_{pI\omega}(\omega)$ **also** have a zero at $\omega = \omega_0$, to satisfy Statement 1.

Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is real, its Fourier transform $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ has symmetry properties and hence $E_{pR\omega}(-\omega) = E_{pR\omega}(\omega)$ and $E_{pI\omega}(-\omega) = -E_{pI\omega}(\omega)$ (Symmetry property) and hence $E_{p\omega}(-\omega) = \xi(\frac{1}{2} + \sigma - i\omega)$ **also** has a zero at $\omega = \omega_0$ to satisfy Statement 1.

Using the property $\xi(s) = \xi(1 - s)$, we get $\xi(\frac{1}{2} + \sigma - i\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ at $s = \frac{1}{2} + \sigma - i\omega$ and $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ **also** has a zero at $\omega = \omega_0$ to satisfy Statement 1. We see that $E_{q\omega}(\omega)$ is obtained by replacing σ in $E_{p\omega}(\omega)$ by $-\sigma$. Hence the results in above sections hold for $-\frac{1}{2} < \sigma < 0$ and for $0 < |\sigma| < \frac{1}{2}$.

Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Hence the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3.1. Result $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 6 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 40.

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t (Appendix C.8), and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c}) > E_0(t + t_{2c}) > E_0(3t_{0c})$ (**Result 6.3.1**) and $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c}) < E_0(t - t_{2c}) < E_0(-t_{0c})$ respectively. Given that $E_0(t) = E_0(-t)$, we see that $E_0(2t_{0c}) < E_0(t - t_{2c}) < E_0(t_{0c})$ in the interval $0 < t < t_{0c}$ (**Result 6.3.2**).

Using Result 6.3.1 and Result 6.3.2, we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 40, for $t_{0c} > 0$ and $t_{2c} = 2t_{0c}$.

479 **4. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2**

480

481 We see from Section 2.1 that $\omega_z(t_2, t_0)$ is shown to be **finite and non-zero** for all $|t_0| < \infty$ and
 482 for each non-zero value of t_2 and that $\omega_z(t_2, t_0)$ is an even function of variable t_0 , for a given value
 483 of t_2 (Section 2.4). For a given t_2 and t_0 , $\omega_z(t_2, t_0)$ can have more than one value, corresponding to
 484 multiple zero crossings in $G_R(\omega, t_2, t_0)$, but we consider only the first zero crossing away from origin in
 485 the section below, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, as detailed in **Lemma**
 486 **1** in Section 2.1 and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$. (example plot)

487

488 We consider the Fourier transform of the even part of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the
 489 section below and show that, under this Fourier transformation, as we change t_0 , the zero cross-
 490 ing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 , for all $0 < t_0 < \infty$, for **each**
 491 value of t_2 in the interval $0 < t_2 < \infty$. This is shown in the steps below. For a given **finite** value
 492 of t_2 , $G_R(\omega, t_2, t_0)$ is a function of two variables ω and t_0 , and we use Implicit Function Theorem in R^2 .

493

494 • It is shown in Section 4.1 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect
 495 to ω , as shown in Eq. 43.

496

497 • It is shown in Section 4.2 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect
 498 to t_0 , as shown in Eq. 44 and Eq. 49.

499

500 • It is shown in Section 4.3 that the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **contin-**
 501 **uous** function of t_0 , for a given t_2 , using **Implicit Function Theorem** in R^2 .

502

503 • It is shown in Section 4.4 that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 , for $0 < t_0 < \infty$
 504 and $0 < t_2 < \infty$, using **Implicit Function Theorem** in R^3 .

505 **4.1. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of ω**

506

507 $G_R(\omega, t_2, t_0)$ in Eq. 17 is copied below.

$$\begin{aligned} G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ &+ e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau = G'_{1R}(\omega, t_2, t_0) + G'_{1R}(\omega, t_2, -t_0) \\ G'_{1R}(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \end{aligned}$$

508

(41)

509 We can expand $G'_{1R}(\omega, t_2, t_0)$ in Eq. 41 by substituting $\tau + t_0 = \tau'$ in the first term in the integral
 510 and $\tau - t_0 = \tau''$ in the second term in the integral and expanding it, similar to Eq. 20 and substituting
 511 back $\tau' = \tau$ and $\tau'' = \tau$ in the second line below. We use $e^{-2\sigma t_0} e^{2\sigma t_0} = 1$ in the first integral below.

$$\begin{aligned}
G'_{1R}(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^{t_0} E'_0(\tau', t_2) e^{-2\sigma \tau'} e^{2\sigma t_0} \cos(\omega(\tau' - t_0)) d\tau' + e^{-2\sigma t_0} \int_{-\infty}^{-t_0} E'_{0n}(\tau'', t_2) \cos(\omega(\tau'' + t_0)) d\tau'' \\
G'_{1R}(\omega, t_2, t_0) &= [\cos(\omega t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma \tau} \cos(\omega \tau) d\tau + \sin(\omega t_0) \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma \tau} \sin(\omega \tau) d\tau] \\
&\quad + e^{-2\sigma t_0} [\cos(\omega t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \cos(\omega \tau) d\tau - \sin(\omega t_0) \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \sin(\omega \tau) d\tau]
\end{aligned}
\tag{42}$$

512

513 We could then use $E'_0(\tau, t_2) = (E_0(\tau - t_2) - E_0(\tau + t_2))$ (using Definition 1 in Section 2.1) and
514 $E'_{0n}(\tau, t_2) = E'_0(-\tau, t_2) = -E'_0(\tau, t_2)$ (using Definition 2 in Section 2.3 and Result 3.1 in Section 3) and
515 substitute $\tau + t_2 = t$ and $\tau - t_2 = t'$ and expanding it using the procedure used in Eq. 42. We see that
516 $E_0(\tau)$ in Eq. 1 and its t_0 and t_2 shifted versions are analytic functions of τ, t_0 and t_2 , given that the
517 sum and product of exponential functions are analytic and hence infinitely differentiable. (**Result 4.1**)

518

519 In Eq. 41, $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω and the integrals
520 converge in Eq. 41 and Eq. 43 for $0 < \sigma < \frac{1}{2}$, because the terms $\tau^r E'_0(\tau \pm t_0, t_2) e^{-2\sigma \tau}$ and $\tau^r E'_{0n}(\tau \pm$
521 $t_0, t_2) = -\tau^r E'_0(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \rightarrow \infty$, for $r = 0, 1, 2$
522 (Section 4.6). The integrands in Eq. 41 and Eq. 43 are absolutely integrable and are analytic functions
523 of variables ω and t_0 , for a given t_2 (using Result 4.1 and given that the terms $\cos(\omega \tau), \sin(\omega \tau)$ and
524 $e^{-2\sigma \tau}$ are analytic functions). The integrands have **exponential** asymptotic fall-off rate (Section 4.6
525) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is
526 absolutely integrable. (Section 4.7) Hence we can interchange the order of partial differentiation and
527 integration in Eq. 43 using theorem of differentiability of functions defined by Lebesgue integrals and
528 theorem of dominated convergence, recursively as follows. (theorem)

$$\begin{aligned}
\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \sin(\omega \tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau - t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau + t_0, t_2)] \sin(\omega \tau) d\tau] \\
\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial \omega^2} &= -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau + t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega \tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \tau^2 [E'_0(\tau - t_0, t_2) e^{-2\sigma \tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega \tau) d\tau]
\end{aligned}$$

529

(43)

530 **4.2. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_0**

531

532 In Eq. 41, $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals
533 converge in Eq. 44 and Eq. 49 shown as follows. The integrands in the equation for $G_R(\omega, t_2, t_0)$
534 in Eq. 44 are absolutely integrable because the terms $E'_0(\tau \pm t_0, t_2) e^{-2\sigma \tau}$ and $E'_{0n}(\tau \pm t_0, t_2) =$
535 $-E'_0(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6). We can ex-
536 pand $G_R(\omega, t_2, t_0)$ in Eq. 44 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 42. The
537 integrands in Eq. 41 and Eq. 43 are absolutely integrable and are analytic functions of variables

538 ω and t_0 , for a given t_2 (using Result 4.1). The integrands have **exponential** asymptotic fall-off
 539 rate(Section 4.6) and we can find a suitable dominating function with exponential asymptotic fall-off
 540 rate which is absolutely integrable.(Section 4.7) Hence we can interchange the order of partial differ-
 541 entiation and integration in Eq. 44 using theorem of differentiability of functions defined by Lebesgue
 542 integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
 G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
 &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
 \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
 &\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
 &\quad + 2\sigma e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
 &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau
 \end{aligned}
 \tag{44}$$

544 We show that the integrals in Eq. 44 converge, as follows. We see that $E'_0(\tau + t_0, t_2) = E_0(\tau +$
 545 $t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E'_{0n}(\tau - t_0, t_2) = -E'_0(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$
 546 (using Definition 1 in Section 2.1 and Result 3.1 in Section 3).We see that the first and third in-
 547 tegrals in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44 converge because the terms $E'_0(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and
 548 $E'_{0n}(\tau \pm t_0, t_2) = -E'_0(\tau \pm t_0, t_2)$ have exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6).

550 We consider the integrand in the second integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44 first and
 551 use the results in the above paragraph.

$$\begin{aligned}
 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_0} \\
 &\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_0}
 \end{aligned}
 \tag{45}$$

553 We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 45 and can show that the integrals converge in
 554 Eq. 44, as follows. We take the factor of 2 out of the summation in $E_0(\tau)$ in Eq. 1 copied below.

$$\begin{aligned}
 E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
 E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
 \end{aligned}$$

We can show that $\frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ as follows, given that the equation for $E_0(\tau + t_2 + t_0)$ in Eq. 46 has terms of the form $e^{\tau+t_0}$ and the equation is **invariant** if we interchange the variables τ and t_0 . (**Result A**)

$$\begin{aligned} \frac{\partial}{\partial t_0} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ &\quad + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ &\quad + (\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)}) (2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \end{aligned}$$

559

We can replace t_0 by $t'_0 = -t_0$ in Eq. 46 and see that $\frac{\partial}{\partial t'_0} E_0(\tau + t_2 + t'_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t'_0)$ (**Result E**) given that the equation is invariant if we interchange τ and t'_0 . Given that $\frac{\partial}{\partial t'_0} = \frac{\partial}{\partial t_0} \frac{dt_0}{dt'_0} = -\frac{\partial}{\partial t_0}$, we substitute it in Result E and get $\frac{\partial}{\partial t_0} E_0(\tau + t_2 - t_0) = -\frac{\partial}{\partial \tau} E_0(\tau + t_2 - t_0)$. (**Result B**)

563

We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 45, corresponding to the term in the second integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44, using Result A, as follows. We use the fact that $\int_{-\infty}^0 \frac{dA(\tau)}{d\tau} B(\tau) d\tau = \int_{-\infty}^0 \frac{d(A(\tau)B(\tau))}{d\tau} d\tau - \int_{-\infty}^0 A(\tau) \frac{dB(\tau)}{d\tau} d\tau$.

$$\begin{aligned} \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial t_0} e^{-2\sigma\tau} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\ &= \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0) \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\ &= [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\ &\quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0) e^{-2\sigma\tau} \cos(\omega\tau) d\tau \end{aligned}$$

567

We see that the integrals in Eq. 48 converge because the integrands are absolutely integrable because the terms $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \sin(\omega\tau)$ and $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6). The term $[E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0$ is finite, given that $E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \rightarrow -\infty$ (Appendix C.5). Hence the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau) d\tau$ in Eq. 48 and in Eq. 44 corresponding to the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 45, converges.

574

575 We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral
 576 $\int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_0} \cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau + t_2 - t_0)$ in Eq. 45 also con-
 577 verges, using Result B and the procedure used in Eq. 46 to Eq. 48.

578
 579 We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 46 to Eq. 48 and see that the integral
 580 $\int_{-\infty}^0 \frac{\partial(E_0(\tau-t_2+t_0)e^{-2\sigma\tau})}{\partial t_0} \cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 45
 581 also converges.

582
 583 We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral
 584 $\int_{-\infty}^0 \frac{\partial(E_0(\tau-t_2-t_0))}{\partial t_0} \cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 45 also con-
 585 verges, using Result B and the procedure used in Eq. 46 to Eq. 48. Hence the second integral in the
 586 equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44, also converges.

587
 588 We can see that the last integral in Eq. 44 converges, by setting $t_0 = -t_0$ in Eq. 45 and using
 589 Result B and using the procedure in Eq. 46 to Eq. 48. Hence all the integrals in Eq. 44 converge.

590 4.2.1. *Second Partial Derivative of $G_R(\omega, t_2, t_0)$ with respect to t_0*

591
 592 The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_0 is given by $\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} = \frac{\partial}{\partial t_0} \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$
 593 as follows. We use the result in Eq. 44 and the fact that the integrands are absolutely integrable
 594 using the results in Section 4.2 and are analytic functions of variables ω and t_0 for a given t_2 (using
 595 Result 4.1). The integrands have **exponential** asymptotic fall-off rate (Section 4.6) and we can
 596 find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely in-
 597 tegrable.(Section 4.7) Hence we can interchange the order of partial differentiation and integration in
 598 Eq. 49 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of
 599 dominated convergence as follows. (theorem)

$$\begin{aligned}
 \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau)d\tau \\
 &\quad - 4\sigma e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau)d\tau \\
 &\quad + e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0^2} \cos(\omega\tau)d\tau \\
 &\quad + 4\sigma^2 e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau)d\tau \\
 &\quad + 4\sigma e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau)d\tau \\
 &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_0^2} \cos(\omega\tau)d\tau
 \end{aligned}
 \tag{49}$$

601 The first two integrals and fourth and fifth integrals in Eq. 49 are the same as the integrals in the
 602 equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44 and have been shown to converge in Section 4.2. We will show that
 603 the third and sixth integrals in Eq. 49 converge, as follows.

604

605 We consider the integrand in the third integral in Eq. 49 first. We see that $E'_0(\tau + t_0, t_2) =$
606 $E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E'_{0n}(\tau - t_0, t_2) = -E'_0(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$
607 (using Definition 1 in Section 2.1 and Result 3.1 in Section 3). We write an equation similar to
608 Eq. 45.

$$\begin{aligned} \frac{\partial^2(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_0^2} &= \frac{\partial^2(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_0^2} \\ &+ \frac{\partial^2(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_0^2} \end{aligned} \quad (50)$$

610 We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 50 and copy Eq. 46 below.

$$\begin{aligned} E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\ E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \end{aligned} \quad (51)$$

612 We can see that $\frac{\partial^2}{\partial t_0^2} E_0(\tau + t_2 + t_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau + t_2 + t_0)$, given that the equation has terms of the
613 form $e^{\tau+t_0}$ and the equation is **invariant** if we interchange the variables τ and t_0 . (**Result A'**)

614 We can replace t_0 by $t'_0 = -t_0$ in Eq. 51 and see that $\frac{\partial^2}{\partial (t'_0)^2} E_0(\tau + t_2 + t'_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau + t_2 + t'_0)$
615 (**Result E'**) given that the equation has terms of the form $e^{\tau+t'_0}$ and the equation is **invariant** if we
616 interchange the variables τ and t'_0 .

618 Given that $\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t'_0} \frac{\partial t'_0}{\partial t_0} = -\frac{\partial}{\partial t'_0}$, we get $\frac{\partial^2}{\partial t_0^2} = \frac{\partial}{\partial t_0} (\frac{\partial}{\partial t_0}) = -\frac{\partial}{\partial t_0} (\frac{\partial}{\partial t'_0}) = \frac{\partial}{\partial t'_0} (\frac{\partial}{\partial t'_0}) = \frac{\partial^2}{\partial (t'_0)^2}$, we substi-
620 tute it in Result E' and get $\frac{\partial^2}{\partial t_0^2} E_0(\tau + t_2 - t_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau + t_2 - t_0)$. (**Result B'**)

622 We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 50, corresponding to the term in the third integral
623 in Eq. 49, using Result A', as follows. We use the fact that $\int_{-\infty}^0 \frac{dA(\tau)}{d\tau} B(\tau) d\tau = \int_{-\infty}^0 \frac{d(A(\tau)B(\tau))}{d\tau} d\tau -$
624 $\int_{-\infty}^0 A(\tau) \frac{dB(\tau)}{d\tau} d\tau$.

$$\begin{aligned} \int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial t_0^2} e^{-2\sigma\tau} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial \tau^2} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\ &= \int_{-\infty}^0 \frac{\partial(\frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\ &= [\frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\ &\quad + 2\sigma \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \end{aligned}$$

We see that the integral $\int_{-\infty}^0 \frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau$ in Eq. 52 converges, using Eq. 48 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0$ also converges, given that the Fourier transform of $\frac{dE_0(\tau)}{d\tau}$ given by $i\omega E_{0\omega}(\omega)$ (link) is finite for real ω and has exponential asymptotic fall-off rate as $|\omega| \rightarrow \infty$ (Appendix C.4) and hence absolutely integrable and hence $\frac{dE_0(\tau)}{d\tau}$ goes to zero as $|\tau| \rightarrow \infty$ as per Riemann-Lebesgue Lemma. (**Result 4.2.1.1**)

It is shown below that the remaining term $\int_{-\infty}^0 \frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau$ also converges.

$$\begin{aligned}
& \int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2+t_0))}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\
&= \int_{-\infty}^0 \frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau} \sin(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau+t_2+t_0) \frac{\partial(e^{-2\sigma\tau} \sin(\omega\tau))}{\partial \tau} d\tau \\
&= [E_0(\tau+t_2+t_0)e^{-2\sigma\tau} \sin(\omega\tau)]_{-\infty}^0 - \omega \int_{-\infty}^0 E_0(\tau+t_2+t_0) e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\
&\quad + 2\sigma \int_{-\infty}^0 E_0(\tau+t_2+t_0) e^{-2\sigma\tau} \sin(\omega\tau) d\tau
\end{aligned}$$

633

We see that the integrals in Eq. 53 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0)e^{-2\sigma\tau} \sin(\omega\tau)$ and $E_0(\tau+t_2+t_0)e^{-2\sigma\tau} \cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6). The term $[E_0(\tau+t_2+t_0)e^{-2\sigma\tau} \sin(\omega\tau)]_{-\infty}^0$ is finite, given that $E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \rightarrow -\infty$ (Appendix C.5). Hence the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_0^2} \cos(\omega\tau) d\tau$ in Eq. 52 and in Eq. 49 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 50, also converges.

640

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial t_0^2} \cos(\omega\tau) d\tau$ in Eq. 49 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. 50 also converges, using Result B' and the procedure used in Eq. 51 to Eq. 53.

644

We set $t_2 = -t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 51 to Eq. 53 and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau-t_2+t_0)e^{-2\sigma\tau})}{\partial t_0^2} \cos(\omega\tau) d\tau$ in Eq. 49 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in Eq. 50 also converges.

648

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau-t_2-t_0))}{\partial t_0^2} \cos(\omega\tau) d\tau$ in Eq. 49 corresponding to the term $E_0(\tau-t_2-t_0)$ in Eq. 50 also converges, using Result B' and the procedure used in Eq. 51 to Eq. 53. Hence the third integral in Eq. 49, also converges.

653

We can see that the sixth integral in Eq. 49 converges, by setting $t_0 = -t_0$ in Eq. 50 to Eq. 53 and using Result B' and the procedure used in Eq. 51 to Eq. 53. Hence all the integrals in Eq. 49 converge.

656

657 **4.3. Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 , for a given t_2 .**

658

659 We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that
 660 $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 , for a given value of t_2 , with continuous
 661 partial derivatives (Section 4.1 and Section 4.2) and given that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and
 662 $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1 in Section 2.1), we see that $\omega_z(t_2, t_0)$ is a differen-
 663 tiable function of t_0 , for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

664

665 Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for $0 < t_0 < \infty$, for each value of t_2 in the interval
 666 $0 < t_2 < \infty$.

667

668 • It is shown in Section 4.5 that $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect
 669 to t_2 . We can use the procedure in previous subsections and Implicit Function Theorem and show
 670 that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 , for $0 < t_2 < \infty$, for each value of t_0 in the interval
 671 $0 < t_0 < \infty$.

672 **4.4. Zero Crossings in $G_R(\omega, t_2, t_0)$ move continuously as a function of t_0 and t_2**

673

674 We can use the procedure in previous subsections and show that $\omega_z(t_2, t_0)$ is a **continuous** func-
 675 tion of t_2 and t_0 , for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using Implicit Function Theorem in R^3 .

676

677 We use **Implicit Function Theorem** for the three dimensional case (link and Theorem 3.2.1 in
 678 page 36). Given that $G_R(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2 , with con-
 679 tinuous partial derivatives (Section 4.1, Section 4.2 and Section 4.5) and given that $G_R(\omega, t_2, t_0) = 0$
 680 at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_R(\omega, t_2, t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1 in Section 2.1), we see that
 681 $\omega_z(t_2, t_0)$ is a differentiable function of t_0 and t_2 , for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

682

683 Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 , for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

684 **4.5. $G_R(\omega, t_2, t_0)$ is partially differentiable twice as a function of t_2**

685

686 In Eq. 41, $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals
 687 converge in Eq. 54 and Eq. 58 shown as follows. The integrands in the equation for $G_R(\omega, t_2, t_0)$ in
 688 Eq. 54 are absolutely integrable because the terms $E'_0(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $E'_{0n}(\tau \pm t_0, t_2) = -E'_0(\tau \pm$
 689 $t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6). The integrands are analytic
 690 functions of variables ω and t_2 , for a given t_0 (using Result 4.1) and we can expand $G_R(\omega, t_2, t_0)$ in
 691 Eq. 54 by substituting $\tau + t_0 = t$ and expanding it, similar to Eq. 42. The integrands have **exponential**
 692 asymptotic fall-off rate (Section 4.6) and we can find a suitable dominating function with exponential
 693 asymptotic fall-off rate which is absolutely integrable.(Section 4.7) Hence we can interchange the
 694 order of partial differentiation and integration in Eq. 54 using theorem of differentiability of functions
 695 defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial(E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau
\end{aligned} \tag{54}$$

696

697 We use the procedure outlined in Eq. 45 to Eq. 48, with t_0 replaced by t_2 and show that all the
698 integrals in Eq. 54 converge, as follows.

699

700 We see that $E'_0(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E'_{0n}(\tau - t_0, t_2) = -E'_0(\tau - t_0, t_2) =$
701 $E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 Result 3.1 in Section 3). We
702 consider the integrand in the first integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$ in Eq. 54 first.

$$\begin{aligned}
\frac{\partial(E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2} \\
&\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2}
\end{aligned} \tag{55}$$

703

704 We consider the term $E_0(\tau + t_0 + t_2)$ first and can show that the integrals converge in Eq. 54, as
705 follows. We copy Eq. 46 below.

$$\begin{aligned}
E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned} \tag{56}$$

706

707 We see that $\frac{\partial}{\partial t_2} E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_2 + t_0)$ given that the equation has terms of the form
708 $e^{\tau+t_2}$ and hence the equation is invariant if we interchange τ and t_2 . (**Result C**)

709

710 We can replace t_2 by $t'_2 = -t_2$ in Eq. 56 and see that $\frac{\partial}{\partial t'_2} E_0(\tau + t'_2 + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t'_2 + t_0)$ given
711 that the equation is invariant if we interchange τ and t'_2 (**Result F**). Given that $\frac{\partial}{\partial t'_2} = \frac{\partial}{\partial t_2} \frac{dt_2}{dt'_2} = -\frac{\partial}{\partial t_2}$,
712 we use it in Result F and we get $\frac{\partial}{\partial t_2} E_0(\tau - t_2 + t_0) = -\frac{\partial}{\partial \tau} E_0(\tau - t_2 + t_0)$. (**Result D**)

713

714 We consider the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ first in Eq. 55, corresponding to the term in the first
715 integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$ in Eq. 54 as follows, using Result C. We use the fact that
716 $\int_{-\infty}^0 \frac{dA(\tau)}{d\tau} B(\tau) d\tau = \int_{-\infty}^0 \frac{d(A(\tau)B(\tau))}{d\tau} d\tau - \int_{-\infty}^0 A(\tau) \frac{dB(\tau)}{d\tau} d\tau$.

$$\begin{aligned}
& \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial t_2} e^{-2\sigma\tau} \cos(\omega\tau) d\tau = \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\
& = \int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 E_0(\tau + t_2 + t_0) \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\
& = [E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\
& \quad + 2\sigma \int_{-\infty}^0 E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau) d\tau
\end{aligned}$$

(57)

We see that the integrals in Eq. 57 converge because the integrands are absolutely integrable because the terms $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \sin(\omega\tau)$ and $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau| \rightarrow \infty$ (Section 4.6). The term $[E_0(\tau + t_2 + t_0)e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0$ is finite, given that $E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \rightarrow -\infty$ (Appendix C.5). Hence the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 57 and Eq. 54 corresponding to the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 55 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau + t_2 - t_0))}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 54 corresponding to the term $E_0(\tau + t_2 - t_0)$ in Eq. 55 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau - t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 54 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 55 also converges, using Result D.

We $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^0 \frac{\partial(E_0(\tau - t_2 - t_0))}{\partial t_2} \cos(\omega\tau) d\tau$ in Eq. 54 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 55 also converges, using Result D. Hence the first integral in the equation for $\frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$ in Eq. 54 also converges.

We can see that the last integral in Eq. 54 converges, by setting $t_0 = -t_0$ in Eq. 57. Hence all the integrals in Eq. 54 converge.

4.5.1. *Second Partial Derivative of $G_R(\omega, t_2, t_0)$ with respect to t_2*

The second partial derivative of $G_R(\omega, t_2, t_0)$ with respect to t_2 is given by $\frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} = \frac{\partial}{\partial t_2} \frac{\partial G_R(\omega, t_2, t_0)}{\partial t_2}$ as follows. We use the result in Eq. 54 and the fact that the integrands are absolutely integrable using the results in Section 4.5 and are analytic functions of variables ω and t_2 for a given t_0 (using Result 4.1). The integrands have **exponential** asymptotic fall-off rate (Section 4.6) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable. (Section 4.7) Hence we can interchange the order of partial differentiation and integration in Eq. 58 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned} \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial t_2^2} &= e^{-2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2 (E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 \frac{\partial^2 (E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau \end{aligned} \quad (58)$$

We consider the first integral in Eq. 58 and using $E'_0(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E'_{0n}(\tau - t_0, t_2) = -E'_0(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3), we write an equation similar to Eq. 55.

$$\begin{aligned} \frac{\partial^2 (E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2))}{\partial t_2^2} &= \frac{\partial^2 (E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2^2} \\ &\quad + \frac{\partial^2 (E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2^2} \end{aligned} \quad (59)$$

We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 59 as follows. We copy Eq. 46 below.

$$\begin{aligned} E_0(\tau) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}] e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\ E_0(\tau + t_2 + t_0) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}] e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \end{aligned} \quad (60)$$

We can see that $\frac{\partial^2}{\partial t_2^2} E_0(\tau + t_2 + t_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau + t_2 + t_0)$, given that the equation has terms of the form $e^{\tau+t_2}$ and the equation is **invariant** if we interchange the variables τ and t_2 . (**Result C'**)

We can replace t_2 by $t'_2 = -t_2$ in Eq. 60 and see that $\frac{\partial^2}{\partial (t'_2)^2} E_0(\tau + t'_2 + t_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau + t'_2 + t_0)$ (**Result F'**) given that the equation has terms of the form $e^{\tau+t'_2}$ and the equation is **invariant** if we interchange the variables τ and t'_2 .

Given that $\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t'_2} \frac{\partial t'_2}{\partial t_2} = -\frac{\partial}{\partial t'_2}$, we get $\frac{\partial^2}{\partial t_2^2} = \frac{\partial}{\partial t_2} \left(\frac{\partial}{\partial t_2} \right) = -\frac{\partial}{\partial t_2} \left(\frac{\partial}{\partial t'_2} \right) = \frac{\partial}{\partial t'_2} \left(\frac{\partial}{\partial t'_2} \right) = \frac{\partial^2}{\partial (t'_2)^2}$, we substitute it in Result F' and get $\frac{\partial^2}{\partial t_2^2} E_0(\tau - t_2 + t_0) = \frac{\partial^2}{\partial \tau^2} E_0(\tau - t_2 + t_0)$. (**Result D'**)

We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 59, corresponding to the term in the first integral in Eq. 58, using Result C', as follows. We use the fact that $\int_{-\infty}^0 \frac{dA(\tau)}{d\tau} B(\tau) d\tau = \int_{-\infty}^0 \frac{d(A(\tau)B(\tau))}{d\tau} d\tau - \int_{-\infty}^0 A(\tau) \frac{dB(\tau)}{d\tau} d\tau$.

$$\begin{aligned}
\int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial t_2^2} e^{-2\sigma\tau} \cos(\omega\tau) d\tau &= \int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial \tau^2} e^{-2\sigma\tau} \cos(\omega\tau) d\tau \\
&= \int_{-\infty}^0 \frac{\partial(\frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau - \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} \frac{\partial(e^{-2\sigma\tau} \cos(\omega\tau))}{\partial \tau} d\tau \\
&= [\frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0 + \omega \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau \\
&\quad + 2\sigma \int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau
\end{aligned}$$

(61)

We see that the integral $\int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau) d\tau$ in Eq. 61 converges, using Eq. 57 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \cos(\omega\tau)]_{-\infty}^0$ also converges, using Result 4.2.1.1 in Section 4.2.1. It is shown in Eq. 53 that the remaining term $\int_{-\infty}^0 \frac{\partial E_0(\tau + t_2 + t_0)}{\partial \tau} e^{-2\sigma\tau} \sin(\omega\tau) d\tau$ also converges.

We see that the integrals in Eq. 61 converge and hence the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 58 corresponding to the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 59 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in Eq. 61 and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_2 - t_0))}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 58 corresponding to the term $E_0(\tau + t_2 - t_0)$ in Eq. 59 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ and use the procedure in Eq. 60 to Eq. 61 and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau})}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 58 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 59 converges, using Result D' .

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 60 to Eq. 61 and Result D' and see that the integral $\int_{-\infty}^0 \frac{\partial^2(E_0(\tau - t_0 - t_2))}{\partial t_2^2} \cos(\omega\tau) d\tau$ in Eq. 58 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 59 also converges. Hence the first integral in Eq. 58, also converges.

We can see that the second integral in Eq. 58 converge, by setting $t_0 = -t_0$ in Eq. 59 to Eq. 61. Hence all the integrals in Eq. 58 converge.

4.6. **Exponential Fall off rate of $B(t) = t^r E'_0(t \pm t_0, t_2)e^{-2\sigma t}$ for $r = 0, 1, 2$**

In this section, it is shown that the term $B(t) = t^r E'_0(t \pm t_0, t_2)e^{-2\sigma t}$ has exponential asymptotic fall-off rate as $|t| \rightarrow \infty$, for $r = 0, 1, 2$ where $E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Hence $B(t) = t^r e^{-2\sigma t} [E_0(t - t_2 \pm t_0) - E_0(t + t_2 \pm t_0)]$ (**Result B.6.1**).

We consider $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ for finite and real t_a . We see that $C(t + t_a) = (t + t_a)^r e^{-2\sigma(t + t_a)} E_0(t)$. We see that $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$ given that it has exponential fall-off rates as $|t| \rightarrow \infty$. (Appendix C.5 and Appendix C.6).

Hence $C(t + t_a) = (t + t_a)^r e^{-2\sigma t_a} E_0(t) e^{-2\sigma t}$ also has exponential fall-off rates as $|t| \rightarrow \infty$, for $r = 0, 1, 2$ and finite t_a and is an absolutely integrable function.

Hence $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ has exponential fall-off rates as $|t| \rightarrow \infty$, for finite t_a and is an absolutely integrable function. We set $t_a = t_2 \pm t_0$ and $t_a = -t_2 \pm t_0$ and see that $B(t)$ in Result B.6.1, has **exponential fall-off rates** as $|t| \rightarrow \infty$, for finite t_2, t_0 and is an absolutely integrable function.

4.7. Dominating function

We consider $x(t) = E_0(t) e^{-2\sigma t}$ which has asymptotic exponential fall-off rate of **at least** $O[e^{-0.5|t|}]$. (shown in Appendix C.5) We see that $x(t + t_a)$ also has the same asymptotic exponential fall-off rate, for finite shift of $t_a = t_2 \pm t_0$ and $y(t, t_a) = t^r x(t + t_a) e^{2\sigma t_a}$ also has the same asymptotic exponential fall-off rate, for $r = 0, 1, 2$. We consider the intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ where $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ are finite.

We consider $t_d \gg t_{a_{max}}$ where $y(t, t_a) = t^r x(t + t_a) e^{2\sigma t_a}$ falls off at the rate of at least $O[e^{0.5t}]$ for $t \ll -t_d$. We consider $f(t, t_a, \omega) = y(t, t_a) \cos(\omega t)$ and we get $\frac{\partial f(t, t_a, \omega)}{\partial \omega} = -ty(t, t_a) \sin(\omega t)$ which falls off at the rate of at least $O[e^{0.5t}]$ for $t \ll -t_d$. Let f_{max} be the maximum value of $\frac{\partial f(t, t_a, \omega)}{\partial \omega}$ in the interval $-\infty < t < \infty$.

We can find a suitable **dominating function** $D(t) = e^{-K|t|} f_{max} e^{Kt_d}$ with a fall off rate of $O[e^{-K|t|}]$ where $K < 0.5$ and hence $D(t)$ has a slower fall off rate than $\frac{\partial f(t, t_a, \omega)}{\partial \omega}$ and $D(t) = f_{max}$ at $t = -t_d$ and hence $D(t) > |\frac{\partial f(t, t_a, \omega)}{\partial \omega}|$ for $-\infty < t \leq 0$ and hence $|\frac{\partial f(t, t_a, \omega)}{\partial \omega}| \leq D(t)$ in the interval $(-\infty, 0]$ and $\int_{-\infty}^0 |D(t)| dt$ is finite. (**Result B.6.2**)

The first term in Eq. 43 given by $B(t) = t^r E'_0(t + t_0, t_2) e^{-2\sigma t} = t^r e^{-2\sigma t} [E_0(t - t_2 + t_0) - E_0(t + t_2 + t_0)]$ using Result B.6.1 in Section 4.6. We set $t_a = t_2 + t_0$ and $t_b = t_2 - t_0$ and get $B(t) = t^r e^{-2\sigma t} [E_0(t - t_b) - E_0(t + t_a)]$. Hence $y(t, t_a) = t^r x(t + t_a) e^{2\sigma t_a} = t^r E_0(t + t_a) e^{-2\sigma t}$ in the second para, corresponds to the second term in $B(t)$ and Result B.6.2 holds for this term. The first term in $B(t)$ is obtained by replacing t_a by $-t_b$ and Result B.6.2 holds for this term and hence for $B(t)$. We see that Result B.6.2 holds for the other 3 terms in Eq. 43 using arguments in above paragraphs and replacing t_0 by $-t_0$ and setting $\sigma = 0$ as needed.

As $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ increase to a larger and larger **finite value** without bounds, we consider larger intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ and f_{max} and t_d also increase to a larger and larger **finite value** without bounds and hence the results in above paragraphs are valid in these intervals.

Similarly, we consider $f(t, t_a, \omega) = y(t, t_a) \cos(\omega t) = t^r E_0(t + t_a) e^{-2\sigma t} \cos(\omega t) = t^r E_0(t + t_0 + t_2) e^{-2\sigma t} \cos(\omega t)$ and we see that $\frac{\partial f(t, t_a, \omega)}{\partial t_0}$ and $\frac{\partial f(t, t_a, \omega)}{\partial t_2}$ which fall off at the rate of **at least** $O[e^{0.5t}]$ for $t \ll -t_d$, using Eq. 47 and $E_0(t) = E_0(-t)$ and due to the term $e^{-\pi n^2 e^{-2t}}$ and we can use arguments in above paragraphs to get a result similar to Result B.6.2 for the terms in Eq. 44 and Eq. 54. We can use these arguments to get a result similar to Result B.6.2 for the second derivative terms $\frac{\partial^2 f(t, t_a, \omega)}{\partial t_0^2}$ and $\frac{\partial^2 f(t, t_a, \omega)}{\partial t_2^2}$ in Eq. 49 and Eq. 58.

846 5. Order of $\omega_z(t_2, t_0)t_0$ is greater than $O[1]$

847

848 It is noted that we **do not** use $\lim_{t_0 \rightarrow \infty}$ in this section. Instead we consider real $t_0 > 0$ which
 849 increases to a larger and larger finite value without bounds. We use $0 < \sigma < \frac{1}{2}$ below.

850

851 We write $P_{odd}(t_2, t_0)$ in Eq. 20 concisely as follows.

$$P_{odd}(t_2, t_0) = \int_{-\infty}^{t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau + e^{2\sigma t_0} \int_{-\infty}^{t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau$$

$$P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$$

852

(62)

853 We note that $E'_0(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$ and $E'_{0n}(\tau, t_2) = E'_0(-\tau, t_2) = -E'_0(\tau, t_2) =$
 854 $E_0(\tau + t_2) - E_0(\tau - t_2)$ (using Result 3.1 in Section 3). We choose $t_2 = 2t_0$ and we choose t_1 such
 855 that $E_0(t)$ approximates zero for $|t| > t_1$ and we choose $t_0 \gg t_1$ and hence $E_0(\tau - t_2) = E_0(\tau - 2t_0)$
 856 approximates zero in the interval $(-\infty, t_0]$. Hence in the interval $(-\infty, t_0]$, we see that $E'_0(\tau, t_2) \approx$
 857 $-E_0(\tau + t_2)$ and $E'_{0n}(\tau, t_2) \approx E_0(\tau + t_2)$, for sufficiently large t_0 . We can write Eq. 62 as follows. We
 858 use $\omega_z(t_2, -t_0) = \omega_z(t_2, t_0)$ (Section 2.4).

$$P_{odd}(t_2, t_0) \approx - \int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau$$

$$+ e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau$$

$$P_{odd}(t_2, -t_0) \approx \int_{-\infty}^{-t_0} E'_0(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau + t_0)) d\tau$$

$$+ e^{-2\sigma t_0} \int_{-\infty}^{-t_0} E'_{0n}(\tau, t_2) \cos(\omega_z(t_2, t_0)(\tau + t_0)) d\tau$$

859

(63)

860 We see that the term $P_{odd}(t_2, -t_0)$ in Eq. 63 approaches a value very close to zero, as real t_0
 861 increases to a larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals
 862 $\int_{-\infty}^{-t_0}$, given $0 < \sigma < \frac{1}{2}$ and $t_0 > 0$ and given that the integrands are absolutely integrable and finite
 863 because the terms $E'_0(\tau, t_2) e^{-2\sigma\tau}$ and $E'_{0n}(\tau, t_2) = -E'_0(\tau, t_2)$ have exponential asymptotic fall-off rate
 864 as $|\tau| \rightarrow \infty$ (Section 4.6) Hence we can ignore $P_{odd}(t_2, -t_0)$ for sufficiently large t_0 and write Eq. 62,
 865 using Eq. 63 and $t_2 = 2t_0$.

$$Q(t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) \approx - \int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau$$

$$+ e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau \approx 0$$

866

(64)

867 We substitute $\tau + 2t_0 = t$, $\tau = t - 2t_0$ and $d\tau = dt$ in Eq. 64 and write as follows.

$$\begin{aligned}
Q(t_0) &\approx -e^{4\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt \approx 0
\end{aligned}
\tag{65}$$

We multiply Eq. 65 by $e^{-3\sigma t_0}$ and ignore the last integral for sufficiently large t_0 , given that $e^{2\sigma t_0} e^{-3\sigma t_0} = e^{-\sigma t_0}$ and $|\int_{-\infty}^{3t_0} E_0(t) \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt| \leq \int_{-\infty}^{3t_0} |E_0(t)| dt$ (link) is finite. (Appendix C.1)

$$\begin{aligned}
S(t_0) &= Q(t_0) e^{-3\sigma t_0} \approx -e^{\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt = -e^{\sigma t_0} R(t_0) \approx 0 \\
R(t_0) &= \cos(\omega_z(t_2, t_0)3t_0) \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt + \sin(\omega_z(t_2, t_0)3t_0) \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \sin(\omega_z(t_2, t_0)t) dt
\end{aligned}
\tag{66}$$

Case 1: Order of $\omega_z(t_2, t_0)t_0$ less than 1

Let us assume that the order of $\omega_z(t_2, t_0)t_0$ is less than 1 and $\omega_z(t_2, t_0)t_0$ decreases to a very small finite value close to zero, as real t_0 increases to a larger and larger finite value without bounds. **(Statement B)** We see that t_0 is a real number and as it increases to a larger and larger finite value without bounds, we can use the approximations $\cos(\omega_z(t_2, t_0)3t_0) \approx 1$, $\sin(\omega_z(t_2, t_0)3t_0) \approx 3\omega_z(t_2, t_0)t_0 \approx 0$. We see that the integrals in the expression for $R(t_0)$ in Eq. 66 converge to a finite value, given that $|\int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt| \leq \int_{-\infty}^{3t_0} |E_0(t) e^{-2\sigma t}| dt$ (link) is finite. (Appendix C.1)

We choose t_3 such that $E_0(t) e^{-2\sigma t}$ approximates zero for $|t| > t_3$. As t_0 increases without bounds, we see that $t_3 \ll t_0$ and in the interval $[-t_3, t_3]$, we see that the term $\cos(\omega_z(t_2, t_0)t) = \cos(\omega_z(t_2, t_0)t_0 \frac{t}{t_0}) \approx 1$ given Statement B and $t_3 \ll t_0$. Hence we can write Eq. 66 as follows.

$$R(t_0) \approx \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} dt \approx \int_{-t_3}^{t_3} E_0(t) e^{-2\sigma t} dt
\tag{67}$$

For sufficiently large t_0 , the integral $R(t_0) \approx \int_{-t_3}^{t_3} E_0(t) e^{-2\sigma t} dt$ remains finite and non-zero and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds, given that $\int_{-\infty}^{\infty} E_0(t) e^{-2\sigma t} dt > 0$. (Appendix C.1) This is explained in detail in Section 5.1.

The term $e^{\sigma t_0}$ in $S(t_0) = -e^{\sigma t_0} R(t_0)$ in Eq. 66 increases to a larger and larger finite value **exponentially** and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ and $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ in Eq. 62 **cannot** equal zero, to satisfy Statement 1, in this case.

Hence **Statement B** is **false** and $\omega_z(t_2, t_0)t_0$ **does not** decrease towards zero, as finite t_0 increases without bounds. Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$ (Section 4), we see that the the order of $\omega_z(t_2, t_0)t_0$ is greater than or equal to 1, as finite t_0 increases without bounds. (**Result 5.1**)

Case 2: Order of $\omega_z(t_2, t_0)t_0$ is 1

Let us assume that the order of $\omega_z(t_2, t_0)t_0$ is 1, as real t_0 increases to a larger and larger finite value without bounds. (**Statement C**). In this case, the order of $\omega_z(t_2, t_0)$ is $O[\frac{1}{t_0}]$ and we consider $\omega_z(t_2, t_0) = \frac{K}{t_0}$ where $0 < K < \frac{\pi}{2}$. (We require $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ in Section 3. If $K \geq \frac{\pi}{2}$, we do not need the results in this section.)

We choose t_3 such that $Kt_3 \ll t_0$ and $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. As t_0 increase without bounds, in the interval $[-t_3, t_3]$, we see that the term $\cos(\omega_z(t_2, t_0)t) \approx 1$ and $\sin(\omega_z(t_2, t_0)t) \approx \omega_z(t_2, t_0)t \approx 0$, given that $\omega_z(t_2, t_0)t = \frac{Kt}{t_0} \leq \frac{Kt_3}{t_0} \ll 1$. Hence we can write Eq. 66 as follows.

$$R(t_0) \approx \cos(\omega_z(t_2, t_0)3t_0) \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} dt \approx \cos(3K) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t} dt \quad (68)$$

For sufficiently large t_0 , the integral $R(t_0) \approx \cos(3K) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t} dt$ remains finite, because the order of $\cos(\omega_z(t_2, t_0)3t_0)$ is 1 and $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t} dt > 0$ (Appendix C.1) and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds. This is explained in detail in Section 5.1.

The term $e^{\sigma t_0}$ in $S(t_0) = -e^{\sigma t_0} R(t_0)$ in Eq. 66 increases to a larger and larger finite value **exponentially** and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ and $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ in Eq. 62 **cannot** equal zero, to satisfy Statement 1, in this case.

Hence **Statement C** is **false** and the order of $\omega_z(t_2, t_0)t_0$ is **not** 1, as finite t_0 increases without bounds. Given that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2 , for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$ (Section 4) and given Result 5.1, we see that the the order of $\omega_z(t_2, t_0)t_0$ is **greater than** 1, as finite t_0 increases without bounds.

If we consider the case $\omega_z(t_2, t_0) = \frac{KD(t_2, t_0)}{t_0}$ where $0 < K < \frac{\pi}{2}$ and $D(t_2, t_0)$ is a function of order 1, whose maximum value is 1, the arguments in the above paragraphs still hold. If $K \geq \frac{\pi}{2}$, then $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached for suitable t_0 , which is required in Section 3.

5.1. $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt$ **does not have exponential fall off rate**

We compute the **minimum** value of the integral $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt$ in Eq. 66, for sufficiently large t_3 and $t_0 \gg t_3$ and $0 < \sigma < \frac{1}{2}$. We split $A(t_0)$ as follows.

$$\begin{aligned}
A(t_0) &= B(t_3, t_0) + C(t_3, t_0) + D(t_3, t_0) \\
B(t_3, t_0) &= \int_{-\infty}^{-t_3} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt, \quad C(t_3, t_0) = \int_{-t_3}^{t_3} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt \\
D(t_3, t_0) &= \int_{t_3}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt
\end{aligned}$$

(69)

We see that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ and $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function (Appendix C.1) and hence $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t} dt > 0$ (**Result 5.1.1**).

Given that $\omega_z(t_2, t_0) = \frac{K}{t_0}$ where $0 < K < \frac{\pi}{2}$ in Case 2 in previous subsection and $t_0 \gg t_3$, we see that $\omega_z(t_2, t_0)t \leq \frac{Kt_3}{t_0} \approx 0$ in the interval $|t| \leq t_3$ and hence $\cos(\omega_z(t_2, t_0)t) \approx 1$ and $\cos(\omega_z(t_2, t_0)t) > \frac{1}{2}$ in the interval $|t| \leq t_3$. The same result holds for Case 1 in previous subsection because $\omega_z(t_2, t_0)$ has a faster falloff rate. Hence we can write $C(t_3, t_0) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt > \frac{C_0(t_3)}{2} > 0$, using Result 5.1.1. (**Result 5.1.2**).

We see that $|B(t_3, t_0)| = |\int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt| \leq \int_{-\infty}^{-t_3} |E_0(t)e^{-2\sigma t}| dt \approx 0$ (link) and $|D(t_3, t_0)| = |\int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt| \leq \int_{t_3}^{3t_0} |E_0(t)e^{-2\sigma t}| dt \approx 0$, for sufficiently large t_3 and $t_0 \gg t_3$, given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-0.5|t|}]$ (Appendix C.5) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Appendix C.1).

As we increase t_3 to t'_3 and t_0 to $t'_0 \gg t'_3$, we see that $C(t'_3, t'_0) > C(t_3, t_0) > 0$, using Result 5.1.1 and Result 5.1.2, given that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (**Result 5.1.3**).

As we increase t_3 to t'_3 and t_0 to $t'_0 \gg t'_3$, we see that $|B(t'_3, t'_0)| < |B(t_3, t_0)|$ and $|D(t'_3, t'_0)| < |D(t_3, t_0)|$ approach zero (**Result 5.1.4**), given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-0.5|t|}]$ (Appendix C.5) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Appendix C.1).

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt > \frac{C_0(t_3)}{2} - |B(t_3, t_0)| - |D(t_3, t_0)| \approx \frac{C_0(t_3)}{2}$ using Result 5.1.2, Result 5.1.3 and Result 5.1.4.

For example, we choose $t_3 = 10$ such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. Given that $E_0(t) > 0$ for $|t| < \infty$ (Appendix C.7) and the term $e^{-2\sigma t}$ has a minimum value of $e^{-|t|}$ for $0 < \sigma < \frac{1}{2}$, we see that the integral $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t} dt > 2 \int_0^{t_3} E_0(t)e^{-|t|} dt > C_{00} = 0.42$ where C_{00} is computed by considering the first 5 terms $n = 1, 2, 3, 4, 5$ in $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. Hence $C_0(t_3) > 0.42$.

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t) dt > \frac{C_0(t_3)}{2} - |B(t_3, t_0)| - |D(t_3, t_0)| \approx 0.21$. As t_0 increases without bounds, we see that $A(t_0)$ **does not** have exponential fall off rate.

968 6. Strictly decreasing $E_0(t)$ for $t > 0$

969

970 Let us consider $E_0(t) = \Phi(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ in Eq. 1, whose Fourier
 971 Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. It is known that $\Phi(t)$ is positive for
 972 $|t| < \infty$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function
 973 for $t > 0$. (link). This is shown below. We take the term $2\pi n^2$ out of the brackets.

$$E_0(t) = \Phi(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$$

974

(70)

975 We show that $X(t) = \frac{E_0(t)}{2}$ is a **strictly decreasing** function for $t > 0$ as follows.

976

977 • In Section 6.1, it is shown that the first derivative of $X(t)$, given by $\frac{dX(t)}{dt} < 0$ for $t > t_z$ where
 978 $t_z = \frac{1}{2} \log \frac{y_z}{\pi}$ and $y_z = 3.16$.

979

980 • In Section 6.2, it is shown that, $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$.

981

982 Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is strictly decreasing for all $t > 0$ and $E_0(t) = 2X(t)$
 983 is **strictly decreasing** for all $t > 0$.

984 6.1. $\frac{dX(t)}{dt} < 0$ **for** $t > t_z$

985

986 We consider $X(t) = \frac{E_0(t)}{2} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$ in Eq. 70 and take the first
 987 derivative of $X(t)$. We note that $E_0(t)$ and $X(t)$ are analytic functions for real t and infinitely
 988 differentiable in that interval. We compute $\frac{dX(t)}{dt}$ below and take the term e^{2t} out, in the last line
 989 below.

$$\begin{aligned} \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t}) (\frac{1}{2} - 2\pi n^2 e^{2t})] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}] \\ \frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}] \end{aligned}$$

990

(71)

991 We substitute $y = \pi e^{2t}$ in Eq. 71 and define $A(y)$ such that $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$. [8]

$$A(y) = \sum_{n=1}^{\infty} n^2 e^{-n^2 y} [-4n^4 y^2 + 15n^2 y - \frac{15}{2}] \quad (72)$$

We see that $A(y) = 0$ at $y = \pi$ which corresponds to $t = 0$ given $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, given that $\frac{dX(t)}{dt} = 0$ at $t = 0$. Because $X(t) = \frac{E_0(t)}{2}$ is an even function of variable t (Appendix C.8) and hence $\frac{dX(t)}{dt}$ is an **odd** function of variable t .

The quadratic expression $B(y, n) = (-4n^4 y^2 + 15n^2 y - \frac{15}{2})$ in Eq. 72 has roots at $y = \frac{-15n^2 \pm \sqrt{225n^4 - 120n^4}}{-8n^4} = \frac{(15 \pm \sqrt{105})}{8n^2}$. We see that the first derivative of $B(y, n)$ is given by $\frac{dB(y, n)}{dy} = -8n^4 y + 15n^2$ is zero at $y = \frac{15}{8n^2}$. The second derivative of $B(y, n)$ given by $\frac{d^2 B(y, n)}{dy^2} = -8n^4$, is negative for all y and $n \geq 1$ and hence $B(y, n)$ is a **concave down** function for each n , which reaches a maximum at $y = \frac{15}{8n^2}$ and given the dominant term $-4n^4 y^2$ in Eq. 72, we see that $B(y, n) < 0$, for $y > \frac{(15 + \sqrt{105})}{8} > 3.16 = y_z$, for $n \geq 1$ and hence $A(y) < 0$ for $y > y_z$. Using $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, we see that $\frac{dX(t)}{dt} < 0$ for $t > \frac{1}{2} \log \frac{y_z}{\pi} = t_z$ (**Result 1**). (concave down function)

We show in the next section that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. It suffices to show that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq y_z = 3.16$ and hence $A(y) < 0$ for $\pi < y \leq y_z = 3.16$, given that $A(y) = 0$ at $y = \pi$. [We use $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$ and $\frac{dX(t)}{dt} = 0$ at $t = 0$.]

6.2. $\frac{dX(t)}{dt} < 0$ **for** $0 < t \leq t_z$

It is shown in this section that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq 3.16$ and hence $A(y) < 0$ for $\pi < y \leq 3.16$ [8], given that $A(y) = 0$ at $y = \pi$. We take the derivative of $A(y)$ in Eq. 72 and take the factor n^2 out of the brackets in the last line below.

$$\begin{aligned} \frac{dA(y)}{dy} &= \sum_{n=1}^{\infty} n^2 e^{-n^2 y} [-8n^4 y + 15n^2 + (-4n^4 y^2 + 15n^2 y - \frac{15}{2})(-n^2)] \\ \frac{dA(y)}{dy} &= \sum_{n=1}^{\infty} n^4 e^{-n^2 y} [-8n^2 y + 15 + 4n^4 y^2 - 15n^2 y + \frac{15}{2}] = \sum_{n=1}^{\infty} n^4 e^{-n^2 y} [4n^4 y^2 - 23n^2 y + \frac{45}{2}] \end{aligned} \quad (73)$$

We examine the term $C(y, n) = n^4 e^{-n^2 y} (4n^4 y^2 - 23n^2 y + \frac{45}{2})$ in Eq. 73 in the interval $\pi \leq y \leq 3.16$ and show that $\frac{dA(y)}{dy} = C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < 0$, as follows. We want the maximum value of $C(y, n)$ and we consider the maximum value of positive terms and minimum value of absolute value of negative terms in the paragraphs below.

For $n = 1$, we see that $C(y, 1) = e^{-y} (4y^2 - 23y + \frac{45}{2}) = 4y^2 e^{-y} - 23y e^{-y} + \frac{45}{2} e^{-y} < 0$ in the interval $\pi \leq y \leq 3.16$ as follows. Given that $3.16^2 < 10$ and $\pi > 3.14$, in the interval $\pi \leq y \leq 3.16$, we see that $C(y, 1) < 4 * 10e^{-3.14} - 23 * 3.14e^{-3.16} + \frac{45}{2} e^{-3.14} = -0.3588 < -6e^{-3} = C_{max}(1)$ where $C_{max}(1)$ is the maximum value of $C(y, 1)$ in the interval $\pi \leq y \leq 3.16$.

$$C(y, 1) = e^{-y} (4y^2 - 23y + \frac{45}{2}) < -6e^{-3}, \quad \pi \leq y \leq 3.16 \quad (74)$$

For $n > 1$, in the interval $\pi \leq y \leq 3.16$, we can write $C(y, n)$ as follows, given that $\pi > 3.14$ and $3.16^2 < 10$ and the term $-23n^2 y < 0$ is omitted below, given that we want the maximum value of $C(y, n)$. We write the term $\frac{45}{2} < 4n^4 * 0.5$ and $e^{-0.14n^2} * 10.5 < 10$ for $n \geq 2$.

$$C(y, n) = n^4 e^{-n^2 y} (4n^4 y^2 - 23n^2 y + \frac{45}{2}) < n^4 e^{-\pi n^2} (4n^4 ((3.16)^2 + 0.5)) < 4n^8 e^{-3n^2} e^{-0.14n^2} * 10.5 < 40n^8 e^{-3n^2}$$

1025

(75)

1026 We want to show that $\frac{dA(y)}{dy} = C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < 0$ in the interval $\pi \leq y \leq 3.16$. Using
 1027 Eq. 74 and Eq. 75, we write as follows. We multiply both sides by e^3 in the second line below.

$$\begin{aligned} \frac{dA(y)}{dy} &= C(y, 1) + \sum_{n=2}^{\infty} C(y, n) < -6e^{-3} + \sum_{n=2}^{\infty} 40n^8 e^{-3n^2} \\ e^3 \frac{dA(y)}{dy} &< -6 + \sum_{n=2}^{\infty} 40n^8 e^{3-3n^2} \end{aligned}$$

1028

(76)

1029 We want to show that $e^3 \frac{dA(y)}{dy} < 0$ in the interval $\pi \leq y \leq 3.16$. We compute $\log(n^8 e^{3-3n^2})$ as
 1030 follows. We note that $f(x) = \log x$ is a **concave down** function whose second derivative given by
 1031 $-\frac{1}{x^2} < 0$ for $|x| < \infty$ and we can write $f(x) = \log x \leq f(x_0) + f'(x_0)(x - x_0)$ using its **tangent line**
 1032 equation. We see that $f'(x) = \frac{1}{x}$. We set $x = n$ and $x_0 = 2$ and get $\log n \leq \log 2 + \frac{1}{2}(n - 2)$ below.

$$\begin{aligned} \log(n^8 e^{3-3n^2}) &= 8 \log n + (3 - 3n^2) \leq 8(\log 2 + \frac{1}{2}(n - 2)) + (3 - 3n^2) \\ \log(n^8 e^{3-3n^2}) &\leq 8 \log 2 + 4n - 5 - 3n^2 \end{aligned}$$

1033

(77)

1034 We note that $g(x) = 4x - 5 - 3x^2$ in Eq. 77 is a **concave down** function (concave down function),
 1035 whose second derivative given by $-6 < 0$ for all x and we can write $g(x) \leq g(x_0) + g'(x_0)(x - x_0)$
 1036 using its **tangent line** equation. We see that $g'(x) = 4 - 6x$. We set $x = n$ and $x_0 = 2$ and get
 1037 $g(n) \leq g(2) + [4 - 6x]_{x=2}(n - 2) = -9 - 8(n - 2)$ and write Eq. 77 as follows. We take the exponent
 1038 e on both sides in the second line below.

$$\begin{aligned} \log(n^8 e^{3-3n^2}) &\leq 8 \log 2 - 9 - 8(n - 2) \leq 8 \log 2 - 1 + 8(1 - n) \\ n^8 e^{3-3n^2} &\leq e^{8 \log 2 - 1 + 8(1 - n)} = 2^8 e^{-1} e^{8(1 - n)} \end{aligned}$$

1039

(78)

1040 We substitute the result in Eq. 78 in Eq. 76 and simplify as follows.

$$\begin{aligned}
e^3 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 e^{-1} \sum_{n=2}^{\infty} e^{8(1-n)} \\
e^3 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 e^{-1} * e^8 \sum_{n=2}^{\infty} e^{-8n} \\
e^3 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 e^{-1} * e^8 \frac{e^{-8*2}}{1 - e^{-8}} \\
e^3 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 e^{-1} * \frac{e^{-8}}{1 - e^{-8}} \\
e^3 \frac{dA(y)}{dy} &< -6 + 40 * 2^8 e^{-1} * \frac{1}{e^8 - 1}
\end{aligned}$$

(79)

We multiply Eq. 79 by $\frac{(e^8-1)}{6}$ and write as follows.

$$e^3 \frac{dA(y)}{dy} \frac{(e^8 - 1)}{6} < -e^8 + 1 + 40e^{-1} * \frac{256}{6} \approx -2352 \quad (80)$$

We see that $e^3 \frac{dA(y)}{dy} \frac{(e^8-1)}{6} < 0$ in Eq. 80, given that $e > 2$ and hence $\frac{dA(y)}{dy} < 0$, in the interval $\pi \leq y \leq 3.16$, given that $e^3 \frac{(e^8-1)}{6} > 0$. Given that $A(y) = 0$ at $y = \pi$, we see that $A(y) < 0$ in Eq. 72, for $\pi < y \leq 3.16$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y) < 0$ in the interval $0 < t \leq t_z$. (**Result 2**)

In Section 6.1, it is shown that $\frac{dX(t)}{dt} < 0$ for $t > t_z$ (from Result 1). In this section, we have shown that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$.

Hence $E_0(t) = 2X(t)$ is a **strictly decreasing function** for $t > 0$.

7. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ (Appendix C.8) where $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied [6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi) \xi(1-s, \bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to

1068 them. This proof does not need or use Euler product.

1069

1070 We know that $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and

1071 entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$,

1072 where $x > 0$ is real [4](link) and then derive $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. In the case
1073 of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical
1074 line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of
1075 a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and
1076 analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta
1077 function and related zeta functions.

1078

1079 The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Rie-
1080 mann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$
1081 and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses
1082 the fact that $E_0(t)$ is an **even** function of variable t and $E_0(t) > 0$ for $|t| < \infty$ (Appendix C.7) and
1083 $E_0(t)$ is **strictly decreasing** function for $t > 0$ (Section 6). These conditions may **not** be satisfied
1084 for many other functions including those which have non-trivial zeros away from the critical line and
1085 hence the new method may **not** be applicable to such functions.

1086

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1099 Appendix A. Derivation of $E_p(t)$

1100

1101 Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) =$
 1102 $E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} -$
 1103 $6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ using Eq. 1.

1104

1105 We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) =$
 1106 $E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t) e^{-\sigma t}$ where $0 < |\sigma| < \frac{1}{2}$ is real. We use $E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$ below.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega \end{aligned}$$

1107

(A.1)

1108 We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows. We get $\omega = \omega' + i\sigma$ and $d\omega = d\omega'$.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \quad (A.2)$$

1109 We can evaluate the above integral in the complex plane using contour integration, substituting
 1110 $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $z = [-\infty, \infty]$, C_2
 1111 along the line $z = [\infty, \infty - i\sigma]$, C_3 along the line $z = [-\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line
 1112 $z = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded
 1113 by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

1114

1115 We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$,
 1116 **goes to zero** as $x \rightarrow \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because
 1117 $E_0(t) e^{yt}$ is a absolutely integrable function for real t (Appendix A.1). Hence the integral in Eq. A.2
 1118 **vanishes** along the contours C_2 and C_4 . Using Cauchy's Integral theroem, we can write Eq. A.2 as
 1119 follows.

$$\begin{aligned} E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \end{aligned}$$

1120

(A.3)

1121 Thus we have arrived at the desired result $E_p(t) = E_0(t) e^{-\sigma t}$. **Alternate** derivation of $E_0(t)$ and
 1122 $E_p(t)$ are in Appendix B.1.

1123 *Appendix A.1. $E_y(t) = E_0(t) e^{yt}$ is an absolutely integrable function*

1124

1125 We see that $E_0(t) > 0$ and finite for $-\infty < t < \infty$ (Appendix C.7). Hence $E_y(t) = E_0(t) e^{yt} > 0$
 1126 and finite for all $-\infty < t < \infty$, for $-\sigma \leq y \leq 0$ and $0 \leq |\sigma| < \frac{1}{2}$ (**Result 11**).

1127

1128 $E_0(t)$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-1.5|t|}]$ (Appendix C.5) and hence
 1129 $E_y(t) = E_0(t)e^{yt}$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5+y)|t|}] > O[e^{-|t|}]$, for
 1130 $-\sigma \leq y \leq 0$ and $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_y(t) = E_0(t)e^{yt}$ decays exponentially, at $t \rightarrow \pm\infty$. (**Result 12**)

1131
 1132 Using Result 11 and 12, we can write $\int_{-\infty}^{\infty} |E_y(t)|dt$ is finite and $E_y(t)$ is an absolutely **integrable**
 1133 **function** (Appendix C.6) and its Fourier transform $E_{y\omega}(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per
 1134 Riemann Lebesgue Lemma (link).

1135 Appendix B. Derivation of entire function $\xi(s)$

1136

1137 In this section, we will start with Riemann's Xi function $\xi(s)$ and take the inverse Fourier Trans-
 1138 form of $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

1139

1140 We will use the equation for $\xi(s)$ derived in Ellison's book "Prime Numbers" pages 151-152 which
 1141 uses **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is
 1142 real.[4] (link).

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}]$$

1143

(B.1)

1144 We see that $\xi(s)$ is an entire function, for all values of s in the complex plane and hence we get
 1145 an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

1146 Appendix B.1. Derivation of $E_p(t)$ and $E_0(t)$

1147

1148 Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. B.1 and evaluate at $s =$
 1149 $\frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2} + \sigma + i\omega)(-\frac{1}{2} + \sigma + i\omega) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t}) dt] \quad (\text{B.2})$$

1150 We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) & \left[\int_{-\infty}^0 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right. \\ & \left. + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right] \end{aligned}$$

1151

(B.3)

1152 We can write this as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) \int_{-\infty}^{\infty} [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} e^{-i\omega t} dt \quad (\text{B.4})$$

1153 We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t}$ and get the **inverse Fourier**
 1154 **transform** of $\xi(\frac{1}{2} + \sigma + i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function
 1155 $\delta(t)$.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2 A(t)}{dt^2}$$

$$A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t}$$

(B.5)

1157 We compute the derivatives of $A(t)$ as follows.

$$\frac{dA(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}] u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}] u(t)$$

$$\frac{d^2 A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-4\pi n^2 e^{-2t} + (-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})^2] u(-t)$$

$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [-4\pi n^2 e^{2t} + (\frac{1}{2} - \sigma - 2\pi n^2 e^{2t})^2] u(t) + A_0 \delta(t)$$

(B.6)

1158 We use $A_0 = [\frac{dA(t)}{dt}]_{t=0+} - [\frac{dA(t)}{dt}]_{t=0-} = \sum_{n=1}^{\infty} e^{-\pi n^2} (\frac{1}{2} - \sigma - 2\pi n^2 - (-\frac{1}{2} - \sigma + 2\pi n^2)) = \sum_{n=1}^{\infty} e^{-\pi n^2} (1 -$
 1159 $4\pi n^2)$. We can simplify above equation as follows.

$$\frac{d^2 A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma \pi n^2 e^{-2t}] u(-t)$$

$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma \pi n^2 e^{2t}] u(t) + \delta(t) [\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2)]$$

(B.7)

1162 We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real
 1163 $[4]$, and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) =$
 1164 $-\frac{1}{2}$ (Appendix B.2) and hence **dirac delta terms cancel each other** in Eq. B.5 written as follows.

$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + \left(-\frac{1}{4} + \sigma^2\right)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}\right) \right. \\
&\quad \left. + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}\right] u(-t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}\right) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} \right. \\
&\quad \left. - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}\right] u(t) \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} D(t, n) u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} C(t, n) u(t)
\end{aligned}
\tag{B.8}$$

We cancel the common terms in Eq. B.8 and simplify above equation as follows.

$$\begin{aligned}
C(t, n) &= -\frac{1}{4} + \sigma^2 + \sigma - 2\sigma^2 - 4\sigma\pi n^2 e^{2t} + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t} \\
D(t, n) &= -\frac{1}{4} + \sigma^2 - \sigma - 2\sigma^2 + 4\sigma\pi n^2 e^{-2t} + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t} \\
C(t, n) &= 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} \\
D(t, n) &= 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t}
\end{aligned}
\tag{B.9}$$

We see that $D(t, n) = C(-t, n)$. Hence we can write as follows.

$$\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= \sum_{n=1}^{\infty} C(t, n) e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned}
\tag{B.10}$$

We use the fact that $E_0(t) = E_0(-t)$ (Appendix C.8) we arrive at the desired result for $E_p(t)$ as follows.

$$\begin{aligned}
E_0(t) &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) &= E_0(t) e^{-\sigma t} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{B.11}$$

1173 *Appendix B.2. Derivation of* $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$
 1174

1175 In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) =$
 1176 $\frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real [4], and we take the first derivative of $F(x)$
 1177 and evaluate it at $x = 1$.

$$\begin{aligned}
 F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
 F(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
 \frac{dF(x)}{dx} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2 \frac{1}{x}} (\frac{1}{x^2}) + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) (\frac{-1}{2}) \frac{1}{x^{\frac{3}{2}}}
 \end{aligned}$$

(B.12)

1179 We evaluate the above equation at $x = 1$ and we simplify as follows.

$$\begin{aligned}
 [\frac{dF(x)}{dx}]_{x=1} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2} = \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2} + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}) (\frac{-1}{2}) \\
 &\quad \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}
 \end{aligned}$$

(B.13)

1181 Appendix C. Properties of Fourier Transforms

1182

1183 Appendix C.1. $E_p(t), h(t)$ are absolutely integrable functions and their Fourier Trans- 1184 forms are finite.

1185

1186 The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) =$
1187 $E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. In Eq. 1, we see that $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} >$
1188 0 and finite for all $-\infty < t < \infty$ (Appendix C.7). Hence $E_p(t) = E_0(t)e^{-\sigma t} > 0$ and finite for all
1189 $-\infty < t < \infty$.

1190

1191 It is shown in Appendix C.5 that $E_0(t)$ has an asymptotic **exponential** fall-off rate of **at least**
1192 $O[e^{-1.5|t|}]$ and hence $E_p(t)$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5-\sigma)|t|}] >$
1193 $O[e^{-|t|}]$, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t}$ goes to zero, at $t \rightarrow \pm\infty$ and we showed that
1194 $E_p(t) > 0$ and finite for all $-\infty < t < \infty$ in the last paragraph. (**Result 21**) Hence $E_{p\omega}(\omega) =$
1195 $\int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at
1196 $\omega = 0$ and hence $\omega_0 \neq 0$.

1197

1198 Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω
1199 and also for $\omega = 0$. Hence $E_{p\omega}(0) = \int_{-\infty}^{\infty} E_p(t)dt$ is finite. Using Result 21, we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$
1200 is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to
1201 zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue Lemma (link).

1202

1203 Using the arguments in above paragraph, we replace σ in $E_p(t)$ by 0 and 2σ respectively and see
1204 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$
1205 and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$.

1206

1207 Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, its shifted versions are abso-
1208 lutely integrable and we see that $E_p'(t, t_2) = e^{-\sigma t_2} E_p(t-t_2) - e^{\sigma t_2} E_p(t+t_2) = (E_0(t-t_2) - E_0(t+t_2))e^{-\sigma t}$
1209 in Eq. 6 is an absolutely integrable function, for a finite shift of t_2 . (We substitute $t - t_2 = \tau$ and
1210 $dt = d\tau$ and get $\int_{-\infty}^{\infty} |E_p(t - t_2)|dt = \int_{-\infty}^{\infty} |E_p(\tau)|d\tau$ and hence $E_p(t - t_2)$ is an absolutely integrable
1211 function, given that $E_p(t)$ is absolutely integrable. Same argument holds for $E_p(t + t_2)$.)

1212

1213 We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $h(t) > 0$
1214 for real t and $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for
1215 $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \rightarrow \pm\infty$, as per Riemann Lebesgue
1216 Lemma (link).

1217

1218 Appendix C.2. Convolution integral convergence

1219

1220 Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose first derivative given by $\frac{dh(t)}{dt} = \sigma e^{\sigma t}u(-t) -$
1221 $\sigma e^{-\sigma t}u(t)$ and $A_0 = [\frac{dh(t)}{dt}]_{t=0+} - [\frac{dh(t)}{dt}]_{t=0-} = -2\sigma$ and hence $\frac{dh(t)}{dt}$ is **discontinuous** at $t = 0$, for
1222 $0 < \sigma < \frac{1}{2}$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where
1223 $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0 , corresponding to the Dirac delta
1224 function.

1225

1226 This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term
 1227 $\frac{A_0}{(i\omega)^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges. (**Result B.2**)

1228
 1229 Let us consider the function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ in Eq. 6 and
 1230 its first derivative given by $\frac{dg(t, t_2, t_0)}{dt} = [-\sigma e^{-\sigma t}f(t, t_2, t_0) + e^{-\sigma t}\frac{df(t, t_2, t_0)}{dt}]u(-t) + [\sigma e^{\sigma t}f(t, t_2, t_0) +$
 1231 $e^{\sigma t}\frac{df(t, t_2, t_0)}{dt}]u(t)$. We get $[\frac{dg(t, t_2, t_0)}{dt}]_{t=0-} = -\sigma f(0, t_2, t_0) + [\frac{df(t, t_2, t_0)}{dt}]_{t=0-}$ and $[\frac{dg(t, t_2, t_0)}{dt}]_{t=0+} = \sigma f(0, t_2, t_0) +$
 1232 $[\frac{df(t, t_2, t_0)}{dt}]_{t=0+}$ (**Result B.2.1**).

1233
 1234 We note that $f(t, t_2, t_0)$ is a continuous function in Eq. 6 and get $[\frac{df(t, t_2, t_0)}{dt}]_{t=0+} = [\frac{df(t, t_2, t_0)}{dt}]_{t=0-}$
 1235 and get $[\frac{dg(t, t_2, t_0)}{dt}]_{t=0+} - [\frac{dg(t, t_2, t_0)}{dt}]_{t=0-} = 2\sigma f(0, t_2, t_0)$ using Result B.2.1. Hence $\frac{dg(t, t_2, t_0)}{dt}$ is **discon-**
 1236 **tinuous** at $t = 0$, for $0 < \sigma < \frac{1}{2}$, if $f(0, t_2, t_0) \neq 0$.

1237
 1238 We can see that the **first derivatives** of $g(t, t_2, t_0), h(t)$ are **discontinuous** at $t = 0$ and hence
 1239 $G(\omega, t_2, t_0), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$, using Result B.2. Hence the convolution
 1240 integral below converges to a finite value for real ω , for the case $f(0, t_2, t_0) \neq 0$.

$$F(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega', t_2, t_0) H(\omega - \omega') d\omega' = \frac{1}{2\pi} [G(\omega, t_2, t_0) * H(\omega)] \quad (C.1)$$

1241 If $f(0, t_2, t_0) = 0$, and if the N^{th} **derivative** of $g(t, t_2, t_0)$ is **discontinuous** at $t = 0$ where $N > 1$,
 1242 we see that $G(\omega, t_2, t_0)$ has **fall-off rate** of $\frac{1}{\omega^{(N+1)}}$ as $|\omega| \rightarrow \infty$ (Appendix C.3). $G(\omega, t_2, t_0)$ has a
 1243 minimum **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ for this case. Hence the convolution integral in Eq. C.1
 1244 converges to a finite value for real ω .

1245 Appendix C.3. *Fall off rate of Fourier Transform of functions*

1246
 1247 Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose
 1248 $(N - 1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$
 1249 has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform
 1250 $P_{N\omega}(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

1251
 1252 This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P_\omega(\omega)$ has a
 1253 term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

1254
 1255 We have shown that if the $(N - 1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$
 1256 then its Fourier transform $P_\omega(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

1257 Appendix C.4. *Exponential Fall off rate of analytic functions.*

1258
 1259 We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by
 1260 $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant [3] (Titchmarsh pp256-257 and Titchmarsh pp28-31).

1261
 1262 We consider $x(t) = E_0(t)e^{-2\sigma t}$ and its Fourier transform is given by $X(\omega) = \int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{-i\omega t}dt =$
 1263 $\int_{-\infty}^{\infty} E_0(t)e^{-i(\omega - i2\sigma)t}dt = E_{0\omega}(\omega - i2\sigma) = \xi(\frac{1}{2} + i(\omega - i2\sigma)) = \xi(\frac{1}{2} + 2\sigma + i\omega) = E_{0\omega}(\omega - i2\sigma)$. Hence
 1264 both $E_{0\omega}(\omega)$ and $X(\omega) = E_{0\omega}(\omega - i2\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \rightarrow \infty$
 1265 and they are absolutely integrable (Appendix C.6) and Fourier transformable, given that they are

1266 derived from an entire function $\xi(s)$.

1267

1268 Given that $\xi(s)$ is an entire function in the s -plane, we see that $X(\omega)$ is an **analytic** function
 1269 which is infinitely differentiable which produces no discontinuities for real ω and $0 < \sigma < \frac{1}{2}$. Hence
 1270 its **inverse Fourier transform** $x(t)$ has fall-off rate faster than $\lim_{M \rightarrow \infty} \frac{1}{t^M}$, as $|t| \rightarrow \infty$ (Appendix
 1271 C.3) and hence $x(t) = E_0(t)e^{-2\sigma t}$ should have **exponential fall-off** rate of $e^{-B|t|}$, as $|t| \rightarrow \infty$, where
 1272 $B > 0$ is real.

1273 *Appendix C.5. Exponential Fall off rate of $x(t) = E_0(t)e^{-2\sigma t}$*

1274

1275 We can write $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ in Eq. 1 as follows. We take the term
 1276 $2\pi n^2 e^{2t}$ out of the brackets below. In the term $e^{-\pi n^2 e^{2t}}$, we use Taylor series expansion around $t = 0$
 1277 for $e^{2t} = \sum_{r=0}^{\infty} \frac{(2t)^r}{r!}$, given that e^{2t} is an analytic function for real t .

$$\begin{aligned} E_0(t) &= \sum_{n=1}^{\infty} 2\pi n^2 e^{2t} [2\pi n^2 e^{2t} - 3] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\ &= \sum_{n=1}^{\infty} 2\pi n^2 e^{2t} [2\pi n^2 e^{2t} - 3] e^{-\pi n^2 (1+2t)} e^{-\pi n^2 (\frac{(2t)^2}{12} + \frac{(2t)^3}{13} \dots)} e^{\frac{t}{2}} \end{aligned}$$

1278

(C.2)

1279 We take the term $e^{-2\pi t}$ out of the summation, corresponding to $n = 1$ and then take the term
 1280 $2\pi e^{4t} e^{\frac{t}{2}} = 2\pi e^{\frac{9t}{2}}$ out and write Eq. C.2 as follows.

$$E_0(t) = 2\pi e^{-2\pi t} e^{\frac{9t}{2}} \sum_{n=1}^{\infty} n^2 [2\pi n^2 - 3e^{-2t}] e^{-\pi n^2} e^{-2\pi(n^2-1)t} e^{-\pi n^2 (\frac{(2t)^2}{12} + \frac{(2t)^3}{13} \dots)} \quad (C.3)$$

1281 For $t > 0$, we see that the term corresponding to $n = 1$ in Eq. C.3 has an asymptotic fall-off rate
 1282 of **at least** $O[e^{-(2\pi - \frac{9}{2})t}] > O[e^{-1.5t}]$. The terms corresponding to $n > 1$ have fall-off rates **higher**
 1283 than $O[e^{-1.5t}]$, due to the term $e^{-2\pi(n^2-1)t}$.

1284

1285 Hence we see that $E_0(t)$ has an asymptotic fall-off rate of **at least** $O[e^{-1.5t}]$, for $t > 0$. Given that
 1286 $E_0(t) = E_0(-t)$ (Appendix C.8), we see that $E_0(t)$ has an **exponential** asymptotic fall-off rate of
 1287 at least $O[e^{-1.5|t|}]$.

1288

1289 Similarly, $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of **at least** $O[e^{-(1.5-2\sigma)|t|}] >$
 1290 $O[e^{-0.5|t|}]$, for $0 \leq |\sigma| < \frac{1}{2}$.

1291

1292 Using a second method, it is shown that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate
 1293 in Appendix C.4.

1294 *Appendix C.6. Absolutely integrable functions*

1295

1296 We see that a real function $y(t)$ which is finite for all t and has an asymptotic falloff rate of **at**
 1297 **least** $O[\frac{1}{t^2}]$ is an absolutely integrable function, given that $\int_{-\infty}^{\infty} |y(t)| dt = \int_{-\infty}^{-T} |y(t)| dt + \int_{-T}^T |y(t)| dt +$
 1298 $\int_T^{\infty} |y(t)| dt$ is finite, for non-zero and finite T , because when we integrate the integrand $|y(t)|$ with
 1299 order $O[\frac{1}{t^2}]$, we get the result $O[\frac{1}{t}]$, which is finite at the limit $t = \pm T$ and the result $O[\frac{1}{t}]$ is zero at

the limit $t \rightarrow \pm\infty$. If $y(t)$ has an exponential asymptotic falloff rate, when we integrate the integrand $|y(t)|$ with order $O[e^{-A|t|}]$ for real $A > 0$, we get the result $O[\frac{1}{A}e^{-A|t|}]$, which is finite at the limit $t = \pm T$ and the result is zero at the limit $t \rightarrow \pm\infty$ and hence $y(t)$ is an absolutely integrable function.

*Appendix C.7. $E_0(t) > 0$ **for** $-\infty < t < \infty$*

For $0 \leq t < \infty$, we can show that $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ where $f(t, n) = [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = 2\pi n^2 e^{2t} [2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ as follows.

The sum is positive because each summand $f(t, n)$ is positive for finite n , and each summand is positive because the term $2\pi n^2 e^{2t} - 3 > 0$ for all $t \geq 0$ and $n \geq 1$, given that $\pi > 3$ and $2\pi n^2 e^{2t} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for $0 \leq t < \infty$ and finite $n \geq 1$. (**Result B.7.1**)

For $t = 0$ and $n = 1$, we see that $f(0, 1) = 2\pi[2\pi - 3]e^{-\pi} > 0$.

For $t = 0$ and for **each finite** $n \geq 1$, we see that $f(0, n) = 2\pi n^2 [2\pi n^2 - 3]e^{-\pi n^2} > 0$.

For $0 < t < \infty$ and for **each finite** $n \geq 1$, we see that $f(t, n) = 2\pi n^2 e^{2t} [2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$, using Result B.7.1.

As $n \rightarrow \infty$, $f(t, n)$ tends to zero, for $0 \leq t < \infty$ due to the term $e^{-\pi n^2 e^{2t}}$. We do summation over n and see that the sum of the terms $\sum_{n=1}^{\infty} f(t, n) > 0$.

Hence $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ for $0 \leq t < \infty$.

Given that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $E_{0\omega}(0) = \int_{-\infty}^{\infty} E_0(t) dt$ is finite. We see that $E_0(t)$ is an analytic function for real t . Hence $E_0(t) = \sum_{n=1}^{\infty} f(t, n) > 0$ is finite for $0 \leq t < \infty$.

Given that $E_0(t) = E_0(-t)$ (Appendix C.8), we see that $E_0(t) > 0$ and finite for all $-\infty < t < \infty$.

*Appendix C.8. $E_0(t)$ **is real and even***

We see that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ (**Result 13**) because $\xi(s) = \xi(1-s)$ (link) and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

We take the Inverse Fourier transform of $E_{0\omega}(\omega)$ and use $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ from Result 13 and then substitute $\omega = -\omega'$ in the integrand, as follows.

$$\begin{aligned} E_0(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(-\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{-i\omega' t} d\omega' = E_0(-t) \end{aligned}$$

(C.4)

We see that $E_0(t)$ in Eq. 1 is real and $E_0(t)$ in Eq. C.4 is even and hence we have derived the result that $E_0(t)$ is a **real and even** function of variable t .

1340 Appendix D. Properties of Fourier Transforms Part 1

1341

1342 In this section, some well-known properties of Fourier transforms are re-derived.

1343 Appendix D.1. *Fourier transform of Real $g(t)$*

1344

1345 In this section, we show that the Fourier transform of a **real** function $g(t)$, given by $G(\omega) =$
 1346 $G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$. We use the
 1347 fact that $g(t)$ is real and $\cos(\omega t)$ is an **even** function of ω and $\sin(\omega t)$ is an **odd** function of ω below.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t) \cos(\omega t) dt = G_R(-\omega) \\ G_I(\omega) &= - \int_{-\infty}^{\infty} g(t) \sin(\omega t) dt = -G_I(-\omega) \end{aligned}$$

1348

(D.1)

1349 Appendix D.2. *Even part of $g(t)$ corresponds to real part of Fourier transform $G(\omega)$*

1350

1351 In this section, we take the **even part** of real function $g(t)$, given by $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$
 1352 and show that its Fourier transform is given by the **real part** of $G(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t} dt = \frac{G(\omega)}{2} + \frac{1}{2} \int_{-\infty}^{\infty} g(-t)e^{-i\omega t} dt \end{aligned}$$

1353

(D.2)

1354 We substitute $t = -t$ in the second integral in Eq. D.2. We use the fact that $G_R(-\omega) = G_R(\omega)$
 1355 and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix D.1)

$$\begin{aligned} \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t} dt &= \frac{G(\omega)}{2} + \frac{1}{2} \int_{-\infty}^{\infty} g(t)e^{i\omega t} dt = \frac{G(\omega)}{2} + \frac{G(-\omega)}{2} \\ &= \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(-\omega) + iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(\omega) - iG_I(\omega)] = G_R(\omega) \end{aligned}$$

1356

(D.3)

1357 *Appendix D.3. Odd part of $g(t)$ corresponds to imaginary part of Fourier transform*
 1358 $G(\omega)$
 1359

1360 In this section, we take the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ and
 1361 show that its Fourier transform is given by the **imaginary part** of $G(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t} dt = \frac{G(\omega)}{2} - \frac{1}{2} \int_{-\infty}^{\infty} g(-t)e^{-i\omega t} dt \end{aligned}$$

1362 (D.4)

1363 We substitute $t = -t$ in the second integral in Eq. D.4. We use the fact that $G_R(-\omega) = G_R(\omega)$
 1364 and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix D.1)

$$\begin{aligned} \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t} dt &= \frac{G(\omega)}{2} - \frac{1}{2} \int_{-\infty}^{\infty} g(t)e^{i\omega t} dt = \frac{G(\omega)}{2} - \frac{G(-\omega)}{2} \\ &= \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(-\omega) - iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(\omega) + iG_I(\omega)] = iG_I(\omega) \end{aligned}$$

1365 (D.5)

1366 *Appendix D.4. Fourier transform of a real and even function $g(t)$*
 1367

1368 In this section, we show that the Fourier transform of a **real and even** function $g(t)$, given by
 1369 $G(\omega)$ is also **real and even**. We use the fact that $\int_{-\infty}^{\infty} g(t) \sin \omega t dt = 0$ because $g(t)$ is even and the
 1370 integrand is an **odd function** of variable t .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = \int_{-\infty}^{\infty} g(t) \cos \omega t dt - i \int_{-\infty}^{\infty} g(t) \sin \omega t dt \\ &= \int_{-\infty}^{\infty} g(t) \cos \omega t dt \end{aligned}$$

1371 (D.6)

1372 We see that $G(\omega) = \int_{-\infty}^{\infty} g(t) \cos \omega t dt$ is **real** function of ω , given that $g(t)$ and the integrand are
 1373 real functions. We see that $G(\omega)$ is an **even** function of ω because $\cos \omega t$ is a **even** function of ω .