

# On a new method towards proof of Riemann's Hypothesis

Akhila Raman

University of California at Berkeley. Email: akhila.raman@berkeley.edu.

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## Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function**  $\xi(s)$  which is evaluated at  $s = \frac{1}{2} + \sigma + i\omega$ , given by  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ , where  $\sigma, \omega$  are real and  $-\infty \leq \omega \leq \infty$  and compute its inverse Fourier transform given by  $E_p(t)$ .

We use a new method and show that the Fourier Transform of  $E_p(t)$  given by  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$  **does not have zeros** for finite and real  $\omega$  when  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of  $s = \frac{1}{2}$  and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

*Keywords:* Riemann, Hypothesis, Zeta, Xi, exponential functions

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## 1. Introduction

It is well known that Riemann's Zeta function given by  $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$  converges in the half-plane where the real part of  $s$  is greater than 1. Riemann proved that  $\zeta(s)$  has an analytic continuation to the whole s-plane apart from a simple pole at  $s = 1$  and that  $\zeta(s)$  satisfies a symmetric functional equation given by  $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$  where  $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$  is the Gamma function.<sup>[4] [5]</sup> We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of  $\zeta(s)$  lie on the critical line with real part of  $s = \frac{1}{2}$ , which is called the Riemann Hypothesis.<sup>[1]</sup>

Hardy and Littlewood later proved that infinitely many of the zeros of  $\zeta(s)$  are on the critical line with real part of  $s = \frac{1}{2}$ .<sup>[2]</sup> It is well known that  $\zeta(s)$  does not have non-trivial zeros when real part of  $s = \frac{1}{2} + \sigma + i\omega$ , given by  $\frac{1}{2} + \sigma \geq 1$  and  $\frac{1}{2} + \sigma \leq 0$ . In this paper, **critical strip**  $0 < \text{Re}[s] < 1$  corresponds to  $0 \leq |\sigma| < \frac{1}{2}$ .

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  and compute inverse Fourier transform of  $E_{p\omega}(\omega)$  given by  $E_p(t)$  and show that its Fourier transform  $E_{p\omega}(\omega)$  does not have zeros for finite and real  $\omega$  when  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix E, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

### 1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function  $\xi(s)$  evaluated at  $s = \frac{1}{2} + i\omega$  given by  $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$ , where  $-\infty \leq \omega \leq \infty$ . Its inverse Fourier Transform is given by  $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$ , where  $\omega, t$  are real, as follows (link).<sup>[3]</sup>

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that  $E_0(t) = E_0(-t)$  is a real and **even** function of  $t$ , given that  $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  because  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when evaluated at  $s = \frac{1}{2} + i\omega$ .

The inverse Fourier Transform of  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  is given by the real function  $E_p(t)$ . We can write  $E_p(t)$  as follows for  $0 < |\sigma| < \frac{1}{2}$  and this is shown in detail in Appendix A using contour integration.

$$E_p(t) = E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that  $E_p(t)$  is an analytic function in the interval  $|t| \leq \infty$ , given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

### 1.2. Step 2: On the zeros of a related function $G(\omega)$

**Statement 1:** Let us assume that Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$  where  $\omega_0$  is real and finite and  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider  $0 < \sigma < \frac{1}{2}$  at first. Let us consider a new function  $g(t) = e^{\sigma t_0} E_p(t + t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p(t + t_0) e^{\sigma t} u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function. We can see that  $g(t)h(t) = e^{\sigma t_0} E_p(t + t_0)$  where  $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$ .

In Section 2.1, we will show that the Fourier transform of the **odd function**  $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$  given by  $G_{odd}(\omega) = G_I(\omega)$  must have **at least one zero** at  $\omega = \omega_2(t_0) \neq 0$ , for every value of  $t_0$ , to satisfy Statement 1, where  $0 < \omega_2(t_0) < \infty$  is real.

### 1.3. Step 3: On the zeros of the function $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function  $g_{odd}(t)$  given by  $G_I(\omega)$ . We require  $G_I(\omega) = 0$  for  $\omega = \omega_2(t_0)$  for every value of  $t_0$ , to satisfy **Statement 1**. In general,  $\omega_2(t_0) \neq \omega_0$ .

It is shown that  $R(t_0) = -G_I(\omega_2(t_0), t_0) = 0$  for all  $t_0$  as follows.

$$\begin{aligned} R(t_0) &= e^{2\sigma t_0} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau] \\ &\quad - [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \sin(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \cos(\omega_2(t_0)\tau) d\tau] = 0 \\ R(t_0) &= \int_{-\infty}^0 [E_0(\tau + t_0) e^{-2\sigma\tau} - E_0(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau = 0 \end{aligned}$$

(3)

1.4. **Step 4:  $\omega_2(t_0)$  is an even function of variable  $t_0$**

In Section 2.3, we show the result in Eq. 4 and that  $\omega_0(t_0) = \omega_2(t_0) = \omega_2(-t_0)$ .

$$P(t_0) = \int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_0(\tau - t_0)] \sin(\omega_0(t_0)\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_0(\tau + t_0)] \sin(\omega_0(t_0)\tau) d\tau = 0 \quad (4)$$

1.5. **Step 5: Final Step**

In Section 3.2, it is shown that we can set  $t_0 = t_1$  where  $0 < t_1 < \infty$  and make sure that  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  where  $0 < M < \infty$  is real. In Section 3, we **choose**  $t_0 = t_1$  and  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  in the equation for  $R(Mt_0)$  in Eq. 3 and show that this leads to the result in Eq. 5.

$$\int_0^{Mt_1} E_0(\tau) [\cosh(2M\sigma t_1) - \cosh(2\sigma\tau)] \cos(\omega_2(t_1)\tau) d\tau = 0 \quad (5)$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than or equal to zero**, in the interval  $\tau = [0, Mt_1]$  where  $t_1 > 0$ . For  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$ , we see that  $\omega_2(t_1)\tau = \frac{\pi}{2Mt_1}\tau$  lies in the range  $[0, \frac{\pi}{2}]$  and hence  $\cos(\omega_2(t_1)\tau) \geq 0$  in that interval  $\tau = [0, Mt_1]$ .

Hence the result  $\int_0^{Mt_1} E_0(\tau) [\cosh(2M\sigma t_1) - \cosh(2\sigma\tau)] \cos(\omega_2(t_1)\tau) d\tau = 0$  in Eq. 5 leads to a **contradiction** for  $0 < \sigma < \frac{1}{2}$ .

We have shown this result for  $0 < \sigma < \frac{1}{2}$  and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$ . Hence we have produced a **contradiction of Statement 1** that the Fourier Transform of the function  $E_p(t) = E_0(t)e^{-\sigma t}$  has a zero at  $\omega = \omega_0$  for  $0 < |\sigma| < \frac{1}{2}$ .

## 2. An Approach towards Riemann's Hypothesis

**Theorem 1:** Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  does not have zeros for any real value of  $-\infty < \omega < \infty$ , for  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line, given that  $E_0(t) = E_0(-t)$  is an even function of variable  $t$ , where  $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$ ,  $E_p(t) = E_0(t)e^{-\sigma t}$  and  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ .

**Proof:** We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

**Statement 1:** Let us assume that Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$  where  $\omega_0$  is real and finite and  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for  $0 < \sigma < \frac{1}{2}$  first and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$  and hence show the result for  $0 < |\sigma| < \frac{1}{2}$ .

We know that  $\omega_0 \neq 0$ , because  $\zeta(s)$  has no zeros on the real axis between 0 and 1, when  $s = \frac{1}{2} + \sigma + i\omega$  is real,  $\omega = 0$  and  $0 < |\sigma| < \frac{1}{2}$ . [3] This is shown in detail in first two paragraphs in Appendix C.1.

### 2.1. New function $g(t)$

Let us consider the function  $f(t) = e^{\sigma t_0} E_p(t + t_0)$  where  $|t_0| \leq \infty$  and we can see that the Fourier Transform of this function  $F(\omega) = e^{\sigma t_0} E_{p\omega}(\omega) e^{i\omega t_0}$  also has a zero at  $\omega = \omega_0$ .

Let us consider a new function  $g(t) = g_-(t)u(-t) + g_+(t)u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $g_-(t) = f(t)e^{-\sigma t}$  and  $g_+(t) = f(t)e^{\sigma t}$ . We can see that  $g(t)h(t) = f(t)$  where  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ .

We can show that  $E_p(t), h(t), g(t)$  are real absolutely integrable functions and go to zero as  $t \rightarrow \pm\infty$ . Hence their respective Fourier transforms given by  $E_{p\omega}(\omega), H(\omega), G(\omega)$  are finite for  $|\omega| \leq \infty$  and go to zero as  $|\omega| \rightarrow \infty$ , as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

We can see that  $g(t)$  is a real  $L^1$  integrable function, its Fourier transform  $G(\omega)$  is finite for  $|\omega| < \infty$  and goes to zero as  $\omega \rightarrow \pm\infty$ , as per **Riemann-Lebesgue Lemma** [Riemann Lebesgue Lemma]. This is explained in detail in Appendix C.1.

If we take the Fourier transform of the equation  $g(t)h(t) = f(t)$  where  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ , we get  $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}(\omega) e^{\sigma t_0} e^{i\omega t_0} = F_R(\omega) + iF_I(\omega)$  as per convolution theorem (link), where  $*$  denotes convolution operation given by  $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega'$  and  $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  is real and is the Fourier transform of the function  $h(t)$  and  $G(\omega) = G_R(\omega) + iG_I(\omega)$  is the Fourier transform of the function  $g(t)$ . This is shown in detail in Appendix B.1.

For **every value** of  $t_0$ , we require the Fourier transform of the function  $f(t)$  given by  $F(\omega)$  to have a zero at  $\omega = \omega_0$ . This implies that the Fourier transform of the **odd** function  $g_{odd}(t)$  given by  $G_I(\omega)$  must have **at least one real zero** at  $\omega = \omega_2(t_0)$  for **every value** of  $t_0$ . Because  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  does not have real zeros, if  $G_I(\omega)$  does not have real zeros, then  $F_I(\omega) = G_I(\omega) * H_R(\omega)$  obtained by the convolution of  $H_R(\omega)$  and  $G_I(\omega)$ , cannot possibly have real zeros, which goes against **Statement 1**.

We can write  $g(t) = g_{even}(t) + g_{odd}(t)$  where  $g_{even}(t)$  is an even function and  $g_{odd}(t)$  is an odd function of variable  $t$ . If Statement 1 is true, then the **imaginary** part of the Fourier transform of the **odd function**  $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$  given by  $G_I(\omega)$  must have **at least one zero** at  $\omega = \omega_2(t_0) \neq 0$  where  $\omega_2(t_0)$  is real and finite and can be different from  $\omega_0$  in general. We call this **Statement 2**.

Because  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  is real and does not have zeros for any finite value of  $\omega$ , **if**  $G_I(\omega)$  does not have at least one zero for some  $\omega = \omega_2(t_0) \neq 0$ , **then** the **imaginary part** of  $F(\omega)$  given by  $F_I(\omega) = \frac{1}{2\pi}[G_I(\omega) * H(\omega)]$ , obtained by the convolution of  $H(\omega)$  and  $G_I(\omega)$ , **cannot** possibly have zeros for any non-zero finite value of  $\omega$ , which goes against **Statement 1**. This is shown in detail in Lemma 1.

**Lemma 1:** If Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0 \neq 0$  where  $\omega_0$  is real and finite, then the **imaginary** part of the Fourier transform of the **odd function**  $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$  given by  $G_I(\omega)$  must have **at least one zero** at  $\omega = \omega_2(t_0) \neq 0$  for **every value** of  $t_0$ , where  $0 < \omega_2(t_0) < \infty$  is real, where  $g(t)h(t) = f(t) = e^{\sigma t_0} E_p(t + t_0)$  and  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  and  $0 < \sigma < \frac{1}{2}$ .

**Proof:** If  $E_{p\omega}(\omega)$  has a zero at finite  $\omega = \omega_0 \neq 0$  to satisfy Statement 1, then  $F(\omega) = E_{p\omega}(\omega) e^{\sigma t_0} e^{i\omega t_0}$  also has a zero at  $\omega = \omega_0$  and its imaginary part given by  $F_I(\omega)$  also has a zero at the same location  $\omega = \omega_0 \neq 0$ .

Let us consider the case where  $G_I(\omega)$  **does not** have at least one zero for finite  $\omega = \omega_2(t_0) \neq 0$  and show that  $F_I(\omega)$  does not have at least one zero at finite  $\omega \neq 0$  for this case, which **contradicts** Statement 1. Given that  $H(\omega)$  is real, we can write the convolution theorem only for the imaginary parts as follows.

$$F_I(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_I(\omega') H(\omega - \omega') d\omega' \quad (6)$$

We can show that the above integral converges for all  $|\omega| \leq \infty$ , given that  $G(\omega)$  and  $H(\omega)$  have fall-off rate of  $\frac{1}{\omega^2}$  as  $|\omega| \rightarrow \infty$  because the first derivatives of  $g(t)$  and  $h(t)$  are discontinuous at  $t = 0$ . (Appendix C.2)

We substitute  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  in Eq. 6 and we get

$$F_I(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (7)$$

We can split the integral in Eq. 7 as follows.

$$F_I(\omega) = \frac{\sigma}{\pi} \left[ \int_{-\infty}^0 G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (8)$$

We see that  $G_I(-\omega) = -G_I(\omega)$  because  $g(t)$  is a real function (Appendix B.2). We can substitute  $\omega' = -\omega''$  in the first integral in Eq. 8 and substituting  $\omega'' = \omega'$  in the result, we can write as follows.

$$F_I(\omega) = \frac{\sigma}{\pi} \int_0^{\infty} G_I(\omega') \left[ \frac{1}{(\sigma^2 + (\omega - \omega')^2)} - \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (9)$$

In Appendix C.1 last paragraph, it is shown that  $G(\omega)$  is finite for  $|\omega| \leq \infty$  and goes to zero as  $|\omega| \rightarrow \infty$ . We can see that for  $\omega' = 0$  and  $\omega' = \infty$ , the integrand in Eq. 9 is zero. For finite  $\omega > 0$ , and  $0 < \omega' < \infty$ , we can see that the term  $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} - \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$ .

• **Case 1:**  $G_I(\omega') > 0$  for all finite  $\omega' > 0$

We see that  $F_I(\omega) > 0$  for all finite  $\omega > 0$ . We see that  $F_I(-\omega) = -F_I(\omega)$  because  $E_p(t)$  is a real function (Appendix B.2). Hence  $F_I(\omega) < 0$  for all finite  $\omega < 0$ .

This **contradicts** Statement 1 which requires  $F_I(\omega)$  to have at least one zero at finite  $\omega \neq 0$  because we showed that  $\omega_0 \neq 0$  in **Section 2** paragraph 5. Therefore  $G_I(\omega')$  must have **at least one zero** at  $\omega' = \omega_2(t_0) \neq 0$ , where  $\omega_2(t_0)$  is real and finite.

• **Case 2:**  $G_I(\omega') < 0$  for all finite  $\omega' > 0$

We see that  $F_I(\omega) < 0$  for all finite  $\omega > 0$ . We see that  $F_I(-\omega) = -F_I(\omega)$  because  $E_p(t)$  is a real function (Appendix B.2). Hence  $F_I(\omega) > 0$  for all finite  $\omega < 0$ .

This **contradicts** Statement 1 which requires  $F_I(\omega)$  to have at least one zero at finite  $\omega \neq 0$ . Therefore  $G_I(\omega')$  must have **at least one zero** at  $\omega' = \omega_2(t_0) \neq 0$ , where  $\omega_2(t_0)$  is real and finite.

We have shown that,  $G_I(\omega)$  must have **at least one zero** at real  $\omega = \omega_2(t_0)$  where  $0 < \omega_2(t_0) < \infty$  is real, to satisfy **Statement 1**. We call this **Statement 2**. We will investigate if Statement 2 leads to a contradiction for  $0 < \sigma < \frac{1}{2}$ .

## 2.2. On the zeros of a related function $G(\omega)$

We can compute the fourier transform of the function  $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$  given by  $G_I(\omega)$ . We require  $G_I(\omega) = 0$  for  $\omega = \omega_2(t_0)$  for **every value** of  $t_0$ , to satisfy **Statement 1**. In general,  $\omega_2(t_0) \neq \omega_0$ .

First we compute the fourier transform of the function  $g(t)$  given by  $G(\omega) = G_R(\omega) + iG_I(\omega)$ . We use  $g(t) = e^{\sigma t_0} E_p(t + t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p(t + t_0) e^{\sigma t} u(t)$ .

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = \int_{-\infty}^0 g_-(t)e^{-i\omega t} dt + \int_0^{\infty} g_+(t)e^{-i\omega t} dt \\
G(\omega) &= \int_{-\infty}^0 e^{\sigma t_0} E_p(t+t_0)e^{-\sigma t} e^{-i\omega t} dt + \int_0^{\infty} e^{\sigma t_0} E_p(t+t_0)e^{\sigma t} e^{-i\omega t} dt
\end{aligned} \tag{10}$$

We use  $E_p(t) = E_0(t)e^{-\sigma t}$  and  $E_p(t+t_0) = E_0(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$ . Substituting  $t = -t$  in the second integral in Eq. 10, we have

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^0 E_0(t+t_0)e^{-2\sigma t} e^{-i\omega t} dt + \int_0^{\infty} E_0(t+t_0)e^{-i\omega t} dt \\
G(\omega) &= \int_{-\infty}^0 E_0(t+t_0)e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^0 E_0(-t+t_0)e^{i\omega t} dt
\end{aligned} \tag{11}$$

We define  $E_{0m}(t) = E_0(-t)$  and get  $E_0(-t+t_0) = E_{0m}(t-t_0)$  and write Eq. 11 as follows.

$$G(\omega) = \int_{-\infty}^0 E_0(t+t_0)e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^0 E_{0m}(t-t_0)e^{i\omega t} dt = G_R(\omega) + iG_I(\omega) \tag{12}$$

The above equations can be expanded as follows using the identity  $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ . Comparing the **imaginary parts** of  $G(\omega)$ , we have

$$G_I(\omega) = - \int_{-\infty}^0 E_0(t+t_0)e^{-2\sigma t} \sin(\omega t) dt + \int_{-\infty}^0 E_{0m}(t-t_0) \sin(\omega t) dt \tag{13}$$

We require  $G_I(\omega) = 0$  for  $\omega = \omega_2(t_0)$  for every value of  $t_0$ , to satisfy **Statement 1**. Hence we can see that  $R(t_0) = -G_I(\omega_2(t_0)) = 0$  and we can write as follows using  $t = \tau$ .

$$R(t_0) = \int_{-\infty}^0 [E_0(\tau+t_0)e^{-2\sigma\tau} - E_{0m}(\tau-t_0)] \sin(\omega_2(t_0)\tau) d\tau = 0 \tag{14}$$

We can rewrite Eq. 14 as follows, using the substitution  $\tau+t_0 = \tau'$  in the first integral and  $\tau-t_0 = \tau''$  in the second integral and substituting back  $\tau' = \tau$  and  $\tau'' = \tau$ .

$$\begin{aligned}
R(t_0) &= e^{2\sigma t_0} [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau] \\
&\quad - [\cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_{0m}(\tau) \sin(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_{0m}(\tau) \cos(\omega_2(t_0)\tau) d\tau] = 0
\end{aligned} \tag{15}$$

Now we replace  $t_0$  by  $-t_0$  in  $f(t)$  and consider the function  $f_2(t) = e^{-\sigma t_0} E_p(t-t_0)$  where  $|t_0| \leq \infty$  and use the procedure in above section and we can write as follows.

$$\begin{aligned}
R(-t_0) &= \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_2(-t_0)\tau) d\tau = 0 \\
R(t_0) + R(-t_0) &= \int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau \\
&\quad + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_2(-t_0)\tau) d\tau = 0
\end{aligned} \tag{16}$$

2.3.  $\omega_2(t_0)$  *is an even function of variable  $t_0$*

Now we consider the function  $f_T(t) = f(t) + f_2(t) = e^{\sigma t_0} E_p(t + t_0) + e^{-\sigma t_0} E_p(t - t_0)$  where  $|t_0| \leq \infty$  and  $g_T(t)h(t) = f_T(t)$  where  $g_T(t) = f_T(t)e^{-\sigma t}u(-t) + f_T(t)e^{\sigma t}u(t)$  and  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$  and compute the Fourier transform of the function  $g_T(t)$  and compute its imaginary part using the procedure in above section, similar to Eq. 13 and we can write as follows. We use  $E_0(-\tau) = E_0(\tau)$ .

$$\begin{aligned}
G_{T_I}(\omega, t_0) &= G_1(\omega, t_0) + G_1(\omega, -t_0) \\
G_1(\omega, t_0) &= - \int_{-\infty}^0 E_0(t + t_0)e^{-2\sigma t} \sin(\omega t) dt + \int_{-\infty}^0 E_{0m}(t - t_0) \sin(\omega t) dt \\
G_{T_I}(\omega, t_0) &= - \left[ \int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega\tau) d\tau \right]
\end{aligned} \tag{17}$$

We require  $G_{T_I}(\omega, t_0) = 0$  for  $\omega = \omega_0(t_0)$  for every value of  $t_0$ , to satisfy **Statement 1**. In general  $\omega_0(t_0) \neq \omega_2(t_0)$ . Hence we can see that  $P(t_0) = -G_{T_I}(\omega_0(t_0)) = 0$  and we can rewrite as follows using the substitution  $t = \tau$ .

$$P(t_0) = \int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega_0(t_0)\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_0(t_0)\tau) d\tau = 0 \tag{18}$$

We see that  $f_T(t) = e^{\sigma t_0} E_p(t + t_0) + e^{-\sigma t_0} E_p(t - t_0)$  is **unchanged** by the substitution  $t_0 = -t_0$  and hence  $\omega_0(t_0)$  is an **even** function of variable  $t_0$ . Hence we can rewrite the second integral in Eq. 18 as follows using  $\omega_0(t_0) = \omega_0(-t_0)$ .

$$\int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega_0(t_0)\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_0(-t_0)\tau) d\tau = 0 \tag{19}$$

We compare Eq. 19 and Eq. 16 as follows.

$$\begin{aligned}
&\int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega_0(t_0)\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_0(-t_0)\tau) d\tau = 0 \\
&\int_{-\infty}^0 [E_0(\tau + t_0)e^{-2\sigma\tau} - E_{0m}(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau + \int_{-\infty}^0 [E_0(\tau - t_0)e^{-2\sigma\tau} - E_{0m}(\tau + t_0)] \sin(\omega_2(-t_0)\tau) d\tau = 0
\end{aligned} \tag{20}$$

We can see that there must be **at least one** common solution where  $\omega_2(t_0) = \omega_0(t_0)$  to satisfy Eq. 20. Because  $\omega_0(t_0)$  is an **even** function of variable  $t_0$ , we see that  $\omega_2(t_0) = \omega_0(t_0)$  is also an **even** function of variable  $t_0$ .

The results in this section apply **only** for the case  $0 < \sigma < \frac{1}{2}$ . For  $\sigma = 0$ ,  $g_T(t) = E_0(t + t_0) + E_0(t - t_0)$  is an even function of variable  $t_0$  and  $2g_{T_{odd}}(t) = g_T(t) - g_T(-t) = 0$  and hence  $G_{T_I}(\omega) = 0$  for all  $|\omega| \leq \infty$ .

In Section 2.1,  $\omega_2(t_0)$  is shown to be **finite** for all  $|t_0| \leq \infty$ . This means there are **no** Dirac delta functions present in  $\omega_2(t_0)$ . In Appendix D, we show that  $\omega_2(t_0)$  is a continuous function around  $t_0 = 0$  in the interval  $[-\delta t_0, \delta t_0]$ , though we do not require this result for the proof in the next section.

### 3. Final Proof

We define  $A(t_0) = e^{2\sigma t_0} [\cos(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau]$  and  $B(t_0) = \cos(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_{0m}(\tau) \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_{0m}(\tau) \cos(\omega_2(t_0)\tau) d\tau$  and we write  $R(t_0) = A(t_0) - B(-t_0) = 0$  in Eq. 15 as follows. We use  $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$ .

$$\begin{aligned} R(t_0) &= A(t_0) - B(-t_0) = 0 \\ R(t_0) &= e^{2\sigma t_0} [\cos(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau] \\ &\quad - [\cos(\omega_2(t_0))t_0 \int_{-\infty}^{-t_0} E_0(\tau) \sin(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0))t_0 \int_{-\infty}^{-t_0} E_0(\tau) \cos(\omega_2(t_0)\tau) d\tau] \end{aligned} \quad (21)$$

In Section 2.1, it is shown that  $0 < \omega_2(t_0) < \infty$  for all  $|t_0| < \infty$ . We choose  $t_0 = t_1$  where  $0 < t_1 < \infty$  is a finite constant and we use the **scaled** function  $E'_0(t) = E_0(Mt)$  where  $0 < M < \infty$  is real and is chosen such that  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  ((link)). In Section 3.2, it is shown that we can make sure that  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$ , by using a scaled function. We write as follows.

$$\begin{aligned} R(Mt_0) &= e^{2M\sigma t_0} [\cos(M\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(M\tau) e^{-2M\sigma\tau} \sin(M\omega_2(t_0)\tau) d\tau \\ &\quad - \sin(M\omega_2(t_0))t_0 \int_{-\infty}^{t_0} E_0(M\tau) e^{-2M\sigma\tau} \cos(M\omega_2(t_0)\tau) d\tau] \\ &\quad - [\cos(M\omega_2(t_0))t_0 \int_{-\infty}^{-t_0} E_0(M\tau) \sin(M\omega_2(t_0)\tau) d\tau + \sin(M\omega_2(t_0))t_0 \int_{-\infty}^{-t_0} E_0(M\tau) \cos(M\omega_2(t_0)\tau) d\tau] = 0 \end{aligned} \quad (22)$$

We **set**  $t_0 = t_1$  and  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  in the scaled function  $R(Mt_0)$  in Eq. 22 and we get  $\cos(M\omega_2(t_1))t_1 = 0$  and  $\sin(M\omega_2(t_1))t_1 = 1$ .

$$e^{2M\sigma t_1} \int_{-\infty}^{t_1} E_0(M\tau) e^{-2M\sigma\tau} \cos(M\omega_2(t_1)\tau) d\tau = - \int_{-\infty}^{-t_1} E_0(M\tau) \cos(M\omega_2(t_1)\tau) d\tau \quad (23)$$

We substitute  $M\tau = \tau'$  and  $d\tau = \frac{d\tau'}{M}$  in Eq. 23 and then substitute  $\tau' = \tau$  as follows.

$$\frac{1}{M} e^{2M\sigma t_1} \int_{-\infty}^{Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau = - \frac{1}{M} \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \quad (24)$$

We cancel the common term  $\frac{1}{M}$ , which is finite, on both sides of Eq. 24 and then split the integral in the left hand side and write as follows.



$$\int_{-\infty}^{-Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau + \int_{-Mt_1}^{Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau = -e^{-2M\sigma t_1} \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \quad (25)$$

In Eq. 22, we substitute  $t_0 = -t_1$  for which  $M\omega_2(t_1)(-t_1) = -\frac{\pi}{2}$ . We use  $\omega_2(-t_1) = \omega_2(t_1)$  and derive a result similar to Eq. 24 and simplify as follows.

$$\begin{aligned} e^{-2M\sigma t_1} \int_{-\infty}^{-Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau &= - \int_{-\infty}^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\ \int_{-\infty}^{-Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau &= -e^{2M\sigma t_1} \int_{-\infty}^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\ &= -e^{2M\sigma t_1} \left[ \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau + \int_{-Mt_1}^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \right] \end{aligned} \quad (26)$$

We substitute Eq. 26 in Eq. 25 as follows.

$$\begin{aligned} \int_{-Mt_1}^{Mt_1} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_1)\tau) d\tau &= -e^{-2M\sigma t_1} \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\ &\quad + e^{2M\sigma t_1} \left[ \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau + \int_{-Mt_1}^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \right] \end{aligned} \quad (27)$$

We can rearrange the terms in Eq. 27 as follows, using  $e^{-2\sigma\tau} = \cosh(2\sigma\tau) - \sinh(2\sigma\tau)$  and we note that  $\int_{-Mt_1}^{Mt_1} E_0(\tau) \sinh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau = 0$  because the integrand is an **odd** function of variable  $\tau$ . We see that  $\int_{-Mt_1}^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau = 2 \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau$  and  $\int_{-Mt_1}^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau = 2 \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau$  because the integrands are **even** functions of variable  $\tau$ .

$$\begin{aligned} 2 \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau &= 2 \sinh(2M\sigma t_1) \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\ &\quad + 2e^{2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \end{aligned} \quad (28)$$

We cancel the common factor 2 on both sides of Eq. 28 as follows.

$$\begin{aligned} \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau &= \sinh(2M\sigma t_1) \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\ &\quad + e^{2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \end{aligned} \quad (29)$$

We substitute  $\tau = -\tau$  in the first integral in the right hand side of Eq. 29 and split it into two integrals, as follows.

$$\begin{aligned}
& \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau \\
&= \sinh(2M\sigma t_1) \int_{Mt_1}^{\infty} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau + e^{2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\
& \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau \\
&= \sinh(2M\sigma t_1) \left[ \int_0^{\infty} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau - \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \right] + e^{2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau
\end{aligned} \tag{30}$$

We consolidate the integrals  $\int_0^{Mt_1}$  in Eq. 30 as follows.

$$\begin{aligned}
& \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau + (\sinh(2M\sigma t_1) - e^{2M\sigma t_1}) \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\
&= \sinh(2M\sigma t_1) \int_0^{\infty} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau
\end{aligned} \tag{31}$$

We use  $\sinh(2M\sigma t_1) - e^{2M\sigma t_1} = -\cosh(2M\sigma t_1)$ .

$$\int_0^{Mt_1} E_0(\tau) [\cosh(2\sigma\tau) - \cosh(2M\sigma t_1)] \cos(\omega_2(t_1)\tau) d\tau = \sinh(2M\sigma t_1) \int_0^{\infty} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \tag{32}$$

### 3.1. Next Step

We multiply both sides of Eq. 29 by the term  $e^{-2M\sigma t_1}$  as follows.

$$\begin{aligned}
e^{-2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau &= e^{-2M\sigma t_1} \sinh(2M\sigma t_1) \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\
&+ \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\
e^{-2M\sigma t_1} \int_0^{Mt_1} E_0(\tau) \cosh(2\sigma\tau) \cos(\omega_2(t_1)\tau) d\tau &= \frac{1}{2}(1 - e^{-4M\sigma t_1}) \int_{-\infty}^{-Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau \\
&+ \int_0^{Mt_1} E_0(\tau) \cos(\omega_2(t_1)\tau) d\tau
\end{aligned} \tag{33}$$

Now we **fix** the value of  $t_1$  for which  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  and then use the **scaled** function  $E_{p_N}(\tau) = 3^N E_p(3^N \tau) = E_{0_N}(\tau) e^{-\sigma_N \tau}$  where  $E_{0_N}(\tau) = 3^N E_0(3^N \tau)$  where  $N$  is a positive integer and  $\sigma_N = 3^N \sigma$  and we get  $\omega_{2_N}(t_1) = 3^N \omega_2(t_1)$  is the location of zeros in the imaginary part of Fourier transform of the corresponding scaled function  $g_N(t)$  and substitute in Eq. 22 as follows.

We note that  $t_1$  remains the **same** in the scaled function, for which  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$ . We get  $M\omega_{2_N}(t_1)t_1 = 3^N M\omega_2(t_1)t_1 = \frac{3^N \pi}{2}$  and  $\cos(3^N M\omega_2(t_1)t_1) = 0$  and  $\sin(3^N M\omega_2(t_1)t_1) = -1$ . Using the procedure outlined in the above paras, we get a result similar to Eq. 33 as follows.

$$\begin{aligned}
& 3^N e^{-3^N * 2M\sigma t_1} \int_0^{Mt_1} E_0(3^N \tau) \cosh(3^N * 2\sigma \tau) \cos(3^N \omega_2(t_1) \tau) d\tau \\
&= 3^N \frac{1}{2} (1 - e^{-3^N * 4M\sigma t_1}) \int_{-\infty}^{-Mt_1} E_0(3^N \tau) \cos(3^N \omega_2(t_1) \tau) d\tau + 3^N \int_0^{Mt_1} E_0(3^N \tau) \cos(3^N \omega_2(t_1) \tau) d\tau
\end{aligned} \tag{34}$$

We substitute  $3^N \tau = \tau'$  and then  $\tau' = \tau$  in Eq. 34 as follows. We get  $d\tau = \frac{d\tau'}{3^N}$ .

$$\begin{aligned}
e^{-3^N * 2M\sigma t_1} \int_0^{3^N Mt_1} E_0(\tau) \cosh(2\sigma \tau) \cos(\omega_2(t_1) \tau) d\tau &= \frac{1}{2} (1 - e^{-3^N * 4M\sigma t_1}) \int_{-\infty}^{-3^N Mt_1} E_0(\tau) \cos(\omega_2(t_1) \tau) d\tau \\
&\quad + \int_0^{3^N Mt_1} E_0(\tau) \cos(\omega_2(t_1) \tau) d\tau
\end{aligned} \tag{35}$$

We can rearrange the terms in Eq. 35 as follows.

$$\begin{aligned}
\int_0^{3^N Mt_1} E_0(\tau) \cos(\omega_2(t_1) \tau) d\tau &= e^{-3^N * 2M\sigma t_1} \int_0^{3^N Mt_1} E_0(\tau) \cosh(2\sigma \tau) \cos(\omega_2(t_1) \tau) d\tau \\
&\quad - \frac{1}{2} (1 - e^{-3^N * 4M\sigma t_1}) \int_{-\infty}^{-3^N Mt_1} E_0(\tau) \cos(\omega_2(t_1) \tau) d\tau
\end{aligned} \tag{36}$$

As  $N \rightarrow \infty$ , the **scaled** function  $E_{0_N}(\tau) = 3^N E_0(3^N \tau)$  tends towards a Dirac Delta function, similar to the case of a gaussian function (link). This is shown in detail in Section 3.3.

The integrals in Eq. 36 **converge** to a finite value and we see that  $E_0(\tau)$  and  $E_0(\tau) \cosh(2\sigma \tau)$  approach zero exponentially, as  $N$  increases towards  $\infty$  and the difference term  $\int_0^\infty - \int_0^{3^N Mt_1}$  approaches zero exponentially, as the upper limit of the integral  $\int_0^{3^N Mt_1}$  in Eq. 36 approaches towards  $\infty$ .

The right hand side of Eq. 36 approaches towards **zero**, given that the upper limit of the first integral in the right hand side of Eq. 36 approaches towards  $\infty$  and the upper limit of the second integral in the right hand side of Eq. 36 approaches towards **zero**.

As  $N \rightarrow \infty$ , the integral  $\lim_{N \rightarrow \infty} \int_0^{3^N Mt_1} = \int_0^\infty$ , and the integral  $\lim_{N \rightarrow \infty} \int_{-\infty}^{-3^N Mt_1}$  tends to zero. We see that  $\lim_{N \rightarrow \infty} e^{-3^N * 4M\sigma t_1} = 0$  and  $\lim_{N \rightarrow \infty} e^{-3^N * 2M\sigma t_1} = 0$  and the right hand side of Eq. 36 approaches towards zero and we can write as follows.

$$\int_0^\infty E_0(\tau) \cos(\omega_2(t_1) \tau) d\tau = 0 \tag{37}$$

We can substitute the result in Eq. 37 into Eq. 32 and write as follows.

$$\int_0^{Mt_1} E_0(\tau) [\cosh(2M\sigma t_1) - \cosh(2\sigma \tau)] \cos(\omega_2(t_1) \tau) d\tau = 0 \tag{38}$$

In Eq. 38, given that  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$ , as  $\tau$  varies over the interval  $[0, Mt_1]$ ,  $\omega_2(t_1)\tau = \frac{\pi\tau}{2Mt_1}$  varies from  $[0, \frac{\pi}{2}]$  where  $M \geq 1$  and hence  $\cos(\omega_2(t_1)\tau) \geq 0$ . We see that  $E_0(t) \geq 0$  (First para in Appendix C) and  $\cosh(2M\sigma t_1) - \cosh(2\sigma\tau) > 0$  in the interval  $[0, Mt_1]$ . Each of the terms in the integrand are  $\geq 0$  in the interval  $[0, Mt_1]$  and the integrand is positive for  $\tau = 0$  and hence the integral in Eq. 38 is positive and **cannot** equal zero. This leads to a **contradiction** for  $0 < \sigma < \frac{1}{2}$ .

For  $\sigma = 0$ , both sides of Eq. 38 is zero and **does not** lead to a contradiction. It should be noted that the results from Section 2.3 to Section 3 are valid only for  $\sigma \neq 0$ .

We have shown this result for  $0 < \sigma < \frac{1}{2}$  and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$ . Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function  $E_p(t) = E_0(t)e^{-\sigma t}$  has a zero at  $\omega = \omega_0$  for  $0 < |\sigma| < \frac{1}{2}$ .

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$ , where  $\omega_0$  is real and finite, leads to a **contradiction** for the region  $0 < |\sigma| < \frac{1}{2}$  which corresponds to the critical strip excluding the critical line. This means  $\zeta(s)$  does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

### 3.2. Details of $M\omega_2(t_1)t_1 = \frac{\pi}{2}$

We consider the case  $\omega_2(t_1)t_1 \neq \frac{\pi}{2}$ . In this case, we consider the "t" scaled function  $E_{pm}(t) = E_p(Mt)$  whose Fourier transform  $E_{pm\omega}(\omega) = \frac{1}{M}E_{p\omega}(\frac{\omega}{M})$  has a real zero at  $\omega = M\omega_0$  and  $\sigma_m = M\sigma$  and  $0 < M < \infty$  is real ((link)). We consider  $f_m(t) = e^{\sigma t_1}E_{pm}(t+t_1)$  and  $g_m(t) = f_m(t)e^{-\sigma t}u(-t) + f_m(t)e^{\sigma t}u(t)$ . and see that  $G_{Rm}(\omega)$  has a real zero at  $\omega = \omega_{2m}(t_1) = M\omega_2(t_1)$  and we can derive similar results in earlier sections as follows. We use  $E_{pm}(t) = E_{0M}(t)e^{-\sigma_m t}$  where  $E_{0M}(t) = E_0(Mt) = E_0(-Mt)$ .

$$\begin{aligned} R_m(t_1) &= e^{2\sigma_m t_1} [\cos(\omega_{2m}(t_1)t_1) \int_{-\infty}^{t_1} E_{0M}(\tau) e^{-2\sigma_m \tau} \sin(\omega_{2m}(t_1)\tau) d\tau \\ &\quad - \sin(\omega_{2m}(t_1)t_1) \int_{-\infty}^{t_1} E_{0M}(\tau) e^{-2\sigma_m \tau} \cos(\omega_{2m}(t_1)\tau) d\tau] \\ &\quad - [\cos(\omega_{2m}(t_1)t_1) \int_{-\infty}^{-t_1} E_{0M}(\tau) \sin(\omega_{2m}(t_1)\tau) d\tau + \sin(\omega_{2m}(t_1)t_1) \int_{-\infty}^{-t_1} E_{0M}(\tau) \cos(\omega_{2m}(t_1)\tau) d\tau] = 0 \end{aligned} \quad (39)$$

We can derive the result in Eq. 22 as follows.

$$\begin{aligned} R_m(t_1) &= e^{2M\sigma t_1} [\cos(M\omega_2(t_1)t_1) \int_{-\infty}^{t_1} E_0(M\tau) e^{-2M\sigma \tau} \sin(M\omega_2(t_1)\tau) d\tau \\ &\quad - \sin(M\omega_2(t_1)t_1) \int_{-\infty}^{t_1} E_0(M\tau) e^{-2M\sigma \tau} \cos(M\omega_2(t_1)\tau) d\tau] \\ &\quad - [\cos(M\omega_2(t_1)t_1) \int_{-\infty}^{-t_1} E_0(M\tau) \sin(M\omega_2(t_1)\tau) d\tau + \sin(M\omega_2(t_1)t_1) \int_{-\infty}^{-t_1} E_0(M\tau) \cos(M\omega_2(t_1)\tau) d\tau] = 0 \end{aligned} \quad (40)$$

In this case, we can choose  $M$  such that  $M\omega_2(t_1)t_1 = \frac{\pi}{2}$  in Eq. 40 and get  $\cos(M\omega_2(t_1)t_1) = 0$  and  $\sin(M\omega_2(t_1)t_1) = 1$  and derive the results in in Eq. 23.

$$e^{2M\sigma t_1} \int_{-\infty}^{t_1} E_0(M\tau) e^{-2M\sigma \tau} \cos(M\omega_2(t_1)\tau) d\tau = - \int_{-\infty}^{-t_1} E_0(M\tau) \cos(M\omega_2(t_1)\tau) d\tau \quad (41)$$

3.3.  $\lim_{N \rightarrow \infty} E_{0_N}(\tau) = \lim_{N \rightarrow \infty} 3^N E_0(3^N \tau)$  **approaches Dirac Delta function**

Let us consider  $E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ . (link)

It is shown in this section that, as  $N \rightarrow \infty$ , the **scaled** function  $E_{0_N}(\tau) = 3^N E_0(3^N \tau)$  tends towards a Dirac Delta function. Let  $M = 3^N$  and  $L = \frac{1}{M}$  and we get  $E_{0_N}(\tau) = M E_0(M\tau) = \frac{E_0(\frac{\tau}{L})}{L}$ .

$$\begin{aligned} E_0(t) = \Phi(t) &= 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] \\ E_{0_N}(\tau) &= 3^N E_0(3^N \tau) = M E_0(M\tau) = \frac{E_0(\frac{\tau}{L})}{L} = \frac{1}{L} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{\frac{2\tau}{L}}} e^{\frac{\tau}{2L}} [4\pi^2 n^4 e^{\frac{4\tau}{L}} - 6\pi n^2 e^{\frac{2\tau}{L}}] \end{aligned} \quad (42)$$

• For  $\tau > 0$ , we want to compute  $\lim_{N \rightarrow \infty} E_{0_N}(\tau) = \lim_{L \rightarrow 0} \frac{E_0(\frac{\tau}{L})}{L}$ , given that  $L = \frac{1}{M} = \frac{1}{3^N}$ . For  $\tau > 0$ , the numerator and denominator go to zero, as  $\lim_{L \rightarrow 0}$  and we can use **L'Hospital's Rule** as follows. The first derivative of the denominator with respect to  $L$  equals 1.

$$\begin{aligned} \lim_{L \rightarrow 0} \frac{E_0(\frac{\tau}{L})}{L} &= \lim_{L \rightarrow 0} \frac{d}{dL} E_0(\frac{\tau}{L}) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{\frac{2\tau}{L}}} e^{\frac{\tau}{2L}} [4\pi^2 n^4 e^{\frac{4\tau}{L}} (-\frac{4\tau}{L^2}) - 6\pi n^2 e^{\frac{2\tau}{L}} (-\frac{2\tau}{L^2}) \\ &\quad + (4\pi^2 n^4 e^{\frac{4\tau}{L}} - 6\pi n^2 e^{\frac{2\tau}{L}}) ((-\frac{\tau}{2L^2}) - \pi n^2 (-\frac{2\tau}{L^2}) e^{\frac{2\tau}{L}})] \end{aligned} \quad (43)$$

For  $\tau > 0$ , we can see that  $\lim_{L \rightarrow 0} \frac{E_0(\frac{\tau}{L})}{L} = \lim_{L \rightarrow 0} \frac{d}{dL} E_0(\frac{\tau}{L}) = 0$  due to the **dominant** term  $e^{-\pi n^2 e^{\frac{2\tau}{L}}}$ . Given that  $E_0(\tau)$  is an **even function** of variable  $\tau$ , we see that  $\lim_{L \rightarrow 0} \frac{E_0(\frac{\tau}{L})}{L} = 0$  for  $\tau < 0$  also.

• For  $\tau = 0$ ,  $E_0(\tau) = \sum_{n=1}^{\infty} e^{-\pi n^2} [4\pi^2 n^4 - 6\pi n^2] = e_0$  is a finite positive constant and we see that  $\lim_{M \rightarrow \infty} E_0(M\tau) = e_0$  for  $\tau = 0$ . Hence for  $\tau = 0$ ,  $\lim_{M \rightarrow \infty} M E_0(M\tau) = \infty$  and  $\lim_{N \rightarrow \infty} E_{0_N}(\tau)$  goes to  $\infty$ .

$$E_0(M\tau) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2M\tau}} e^{\frac{M\tau}{2}} [4\pi^2 n^4 e^{4M\tau} - 6\pi n^2 e^{2M\tau}] \quad (44)$$

• We can show that  $d_0(t) = \lim_{N \rightarrow \infty} E_{0_N}(t)$  has the **sifting property** of Dirac delta function, inside an integral, given by  $y(t) = \int_{-\infty}^{\infty} d_0(\tau) f(t - \tau) d\tau = f(t)$ , where  $f(t)$  is an integrable and Fourier transformable function. We compute the Fourier transform of  $y(t)$  given by  $Y(\omega) = D_0(\omega) F(\omega)$  as follows, where  $D_0(\omega), F(\omega)$  are the Fourier transforms of  $d_0(t)$  and  $f(t)$  respectively. We use the result that the Fourier transform of  $E_0(3^N t)$  is given by  $\frac{1}{3^N} E_{0\omega}(\frac{\omega}{3^N})$  in (link).

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} d_0(\tau) f(t - \tau) d\tau \\ Y(\omega) &= \int_{-\infty}^{\infty} y(t) e^{-i\omega t} dt = D_0(\omega) F(\omega) \\ D_0(\omega) &= \int_{-\infty}^{\infty} d_0(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} [\lim_{N \rightarrow \infty} E_{0_N}(t)] e^{-i\omega t} dt \\ D_0(\omega) &= \int_{-\infty}^{\infty} [\lim_{N \rightarrow \infty} 3^N E_0(3^N t)] e^{-i\omega t} dt \end{aligned}$$

(45)

We see that  $E_0(t)$  and  $E_0(3^N t)$  are well defined analytic functions and we can interchange the order of integration and limits in Eq. 45.

$$\begin{aligned}
 D_0(\omega) &= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} [3^N E_0(3^N t)] e^{-i\omega t} dt = \lim_{N \rightarrow \infty} E_{0\omega}(\frac{\omega}{3^N}) = E_{0\omega}(0) \\
 Y(\omega) &= D_0(\omega) F(\omega) = E_{0\omega}(0) F(\omega) \\
 y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega = E_{0\omega}(0) f(t) = \int_{-\infty}^{\infty} d_0(\tau) f(t - \tau) d\tau
 \end{aligned}$$

(46)

We see that  $d_0(t)$  has a **sifting property** inside the integral in Eq. 46, similar to Dirac delta function (link). and we infer that  $d_0(t) = \lim_{N \rightarrow \infty} E_{0_N}(t) = E_{0\omega}(0) \delta(t)$  .

Hence as  $N \rightarrow \infty$ , the **scaled** function  $E_{0_N}(\tau) = 3^N E_0(3^N \tau)$  tends towards a **Dirac Delta function** and hence  $d_0(t) = \lim_{N \rightarrow \infty} E_{0_N}(t) = E_{0\omega}(0) \delta(t)$ .

#### 4. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of  $s = \frac{1}{2}$ . The new method requires the **symmetry** relation  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when evaluated at the critical line  $s = \frac{1}{2} + i\omega$ . This means  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  and  $E_0(t) = E_0(-t)$  where  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by  $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$  satisfies a symmetry relation similar to  $\xi(s) = \xi(1-s)$  where  $\xi(s)$  is an entire function, for  $a \neq 1$  and hence the condition  $E_0(t) = E_0(-t)$  is **not** known to be satisfied<sup>[6]</sup>. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation  $\xi(s, \chi) = \epsilon(\chi) \xi(1-s, \bar{\chi})$  <sup>[7]</sup> which does **not** translate to  $E_0(t) = E_0(-t)$  required by the new method and hence this proof is **not** applicable to them.

We know that  $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$  diverges for real part of  $s \leq 1$ . Hence we derive a convergent and entire function  $\xi(s)$  using the well known theorem  $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$ , where  $x > 0$  is real and then derive  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  ( Appendix E). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to  $F(x)$  exists, which enables derivation of a convergent and entire function  $\xi(s)$  and results in  $E_0(t)$  as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip**  $0 \leq |\sigma| < \frac{1}{2}$ . This proof requires both  $E_p(t)$  and  $E_{p\omega}(\omega)$  to be Fourier transformable where  $E_p(t) = E_0(t)e^{-\sigma t}$  is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

#### References

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- [4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152
- [5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)
- [6] Mathworld article on Hurwitz Zeta functions. (Link)
- [7] Wikipedia article on Dirichlet L-functions. (Link)

## Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function  $\xi(s)$  evaluated at  $s = \frac{1}{2} + i\omega$  given by  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ . Its inverse Fourier Transform is given by  $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  (link). This is re-derived in Appendix E.

We will show in this section that the inverse Fourier Transform of the function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ , is given by  $E_p(t) = E_0(t) e^{-\sigma t}$  where  $0 \leq |\sigma| < \frac{1}{2}$  is real.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega \end{aligned} \tag{A.1}$$

We substitute  $\omega' = \omega - i\sigma$  in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting  $\omega' = z = x + iy$  and we use a rectangular contour comprised of  $C_1$  along the line  $x = [-\infty, \infty]$ ,  $C_2$  along the line  $y = [\infty, \infty - i\sigma]$ ,  $C_3$  along the line  $x = [\infty - i\sigma, -\infty - i\sigma]$  and then  $C_4$  along the line  $y = [-\infty - i\sigma, -\infty]$ . We can see that  $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$  has no singularities in the region bounded by the contour because  $\xi(\frac{1}{2} + iz)$  is an entire function in the Z-plane.

In **Appendix C.1**, we show that  $\int_{-\infty}^{\infty} |E_p(t)| dt$  is finite and  $E_p(t) = E_0(t) e^{-\sigma t}$  is an absolutely integrable function, for  $0 \leq |\sigma| < \frac{1}{2}$ .

We use the fact that  $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$ , **goes to zero** as  $x \rightarrow \pm\infty$  when  $-\sigma \leq y \leq 0$ , as per Riemann-Lebesgue Lemma (link), because  $E_0(t) e^{yt}$  is a absolutely integrable function in the interval  $-\infty \leq t \leq \infty$ . Hence the integral in Eq. A.2 **vanishes** along the contours  $C_2$  and  $C_4$ . Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned} E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \end{aligned} \tag{A.3}$$

Thus we have arrived at the desired result  $E_p(t) = E_0(t) e^{-\sigma t}$ .

## Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.



*Appendix B.1. Convolution Theorem: Multiplication of  $g(t)$  and  $h(t)$  corresponds to convolution in Fourier transform domain*

We start with the Fourier transform equation  $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$  where  $f(t) = g(t)h(t)$  and show that  $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$  obtained by the **convolution** of the functions  $G(\omega)$  and  $H(\omega)$  which correspond to the Fourier transforms of  $g(t)$  and  $h(t)$  respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \quad (\text{B.1})$$

We use the inverse Fourier transform equation  $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$  and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega' \right] h(t)e^{-i\omega t}dt \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[ \int_{-\infty}^{\infty} e^{i\omega' t} h(t)e^{-i\omega t}dt \right] d\omega' \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[ \int_{-\infty}^{\infty} h(t)e^{-i(\omega - \omega')t}dt \right] d\omega' \end{aligned} \quad (\text{B.2})$$

We substitute  $\int_{-\infty}^{\infty} h(t)e^{-i(\omega - \omega')t}dt = H(\omega - \omega')$  in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \quad (\text{B.3})$$

*Appendix B.2. Fourier transform of Real  $g(t)$*

In this section, we show that the Fourier transform of a real function  $g(t)$ , given by  $G(\omega) = G_R(\omega) + iG_I(\omega)$  has the properties given by  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t) \cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= - \int_{-\infty}^{\infty} g(t) \sin(\omega t)dt = -G_I(-\omega) \end{aligned} \quad (\text{B.4})$$

*Appendix B.3. Even part of  $g(t)$  corresponds to real part of Fourier transform  $G(\omega)$*

In this section, we show that the **even part** of real function  $g(t)$ , given by  $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ , corresponds to **real part** of its Fourier transform  $G(\omega)$ . We use the fact that  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$  for a real function  $g(t)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \quad (\text{B.5})$$

#### Appendix B.4. Odd part of $g(t)$ corresponds to imaginary part of Fourier transform $G(\omega)$

In this section, we show that the **odd part** of real function  $g(t)$ , given by  $g_{\text{odd}}(t) = \frac{1}{2}[g(t) - g(-t)]$ , corresponds to **imaginary part** of its Fourier transform  $G(\omega)$ . We use the fact that  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$  for a real function  $g(t)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{odd}}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t} dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{B.6}$$

### Appendix C. Properties of Fourier Transforms Part 2

#### Appendix C.1. $E_p(t), h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite.

The inverse Fourier Transform of the function  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$  is given by  $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t} d\omega$ . We see that  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$  for all  $0 \leq t < \infty$ . Given that  $E_0(t) = E_0(-t)$ , we see that  $E_0(t) > 0$  and  $E_p(t) = E_0(t)e^{-\sigma t} > 0$  for all  $-\infty < t < \infty$ .

As  $t \rightarrow \infty$ ,  $E_p(t)$  goes to zero, due to the term  $e^{-\pi n^2 e^{2t}}$ . As  $t \rightarrow -\infty$ ,  $E_p(t)$  goes to zero, because for every value of  $n$ , the term  $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$  goes to zero, for  $0 \leq |\sigma| < \frac{1}{2}$ . Hence  $E_p(t) = E_0(t)e^{-\sigma t} = 0$  at  $t = \pm\infty$  and we showed that  $E_p(t) > 0$  for all  $-\infty < t < \infty$ . Hence  $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t} dt$ , evaluated at  $\omega = 0$  **cannot** be zero. Hence  $E_{p\omega}(\omega)$  **does not have a zero** at  $\omega = 0$  and hence  $\omega_0 \neq 0$ .

Given that  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  is an entire function in the whole of  $s$ -plane, it is finite for  $|\omega| \leq \infty$  and also for  $\omega = 0$ . Hence  $\int_{-\infty}^{\infty} E_p(t) dt$  is finite. We see that  $E_p(t) \geq 0$  for all  $|t| \leq \infty$ . Hence we can write  $\int_{-\infty}^{\infty} |E_p(t)| dt$  is finite and  $E_p(t)$  is an absolutely **integrable function** and its Fourier transform  $E_{p\omega}(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

Let us consider a new function  $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $0 < \sigma < \frac{1}{2}$ . We can see that  $g(t)h(t) = E_p(t)$  where  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ .

We can see that  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  is an absolutely **integrable function** because  $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$ , is finite for  $0 < \sigma < \frac{1}{2}$  and its Fourier transform  $H(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that  $E_0(t)$  and  $E_0(t)e^{-2\sigma t}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$  and hence are absolutely **integrable** functions and the integrals  $\int_{-\infty}^{\infty} |E_0(t)| dt < \infty$  and  $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}| dt < \infty$ . Hence  $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$  is an absolutely **integrable function** and  $\int_{-\infty}^{\infty} |g(t)| dt = \int_{-\infty}^{\infty} g(t) dt$  is finite and its Fourier transform  $G(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

#### Appendix C.2. Convolution integral convergence

Let us consider  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  whose **first derivative is discontinuous** at  $t = 0$ . The second derivative of  $h(t)$  given by  $h_2(t)$  has a Dirac delta function  $A_0\delta(t)$  where  $A_0 = -2\sigma$  and its Fourier transform  $H_2(\omega)$  has a constant term  $A_0$ , corresponding to the Dirac delta function.

This means  $h(t)$  is obtained by integrating  $h_2(t)$  twice and its Fourier transform  $H(\omega)$  has a term  $-\frac{A_0}{\omega^2}$  (link) and has a **fall off rate** of  $\frac{1}{\omega^2}$  as  $|\omega| \rightarrow \infty$  and  $\int_{-\infty}^{\infty} H(\omega) d\omega$  converges.

We see that  $E_p(t) = E_0(t)e^{-\sigma t}$  where  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ .

Let us consider a new function  $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $0 < \sigma < \frac{1}{2}$ . We can see that  $g(t)h(t) = E_p(t)$  where  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ .

We can see that  $G(\omega), H(\omega)$  have **fall-off rate** of  $\frac{1}{\omega^2}$  as  $|\omega| \rightarrow \infty$  because the **first derivatives** of  $g(t), h(t)$  are **discontinuous** at  $t = 0$ . Also,  $h(t), g(t)$  are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix C.1. Hence the convolution integral below converges to a finite value for  $|\omega| \leq \infty$ .

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega' = \frac{1}{2\pi} [G(\omega) * H(\omega)] \quad (C.1)$$

### Appendix C.3. *Fall off rate of Fourier Transform of functions*

Let us consider a real Fourier transformable function  $P(t) = P_+(t)u(t) + P_-(t)u(-t)$  whose  $(N-1)^{th}$  **derivative** is **discontinuous** at  $t = 0$ . The  $(N)^{th}$  derivative of  $P(t)$  given by  $P_N(t)$  has a Dirac delta function  $A_0\delta(t)$  where  $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$  and its Fourier transform  $P_N(\omega)$  has a constant term  $A_0$ , corresponding to the Dirac delta function.

This means  $P(t)$  is obtained by integrating  $P_N(t)$ ,  $N$  times and its Fourier transform  $P(\omega)$  has a term  $\frac{A_0}{(i\omega)^N}$  (link) and has a **fall off rate** of  $\frac{1}{\omega^N}$  as  $|\omega| \rightarrow \infty$ .

We have shown that if the  $(N-1)^{th}$  **derivative** of the function  $P(t)$  is **discontinuous** at  $t = 0$  then its Fourier transform  $P(\omega)$  has a **fall-off rate** of  $\frac{1}{\omega^N}$  as  $|\omega| \rightarrow \infty$ .

In Section 1.1, we showed that  $E_0(t)$  is an analytic function which is infinitely differentiable which produces no discontinuities in  $|t| \leq \infty$ . Hence its Fourier transform  $E_{0\omega}(\omega)$  has a fall-off rate faster than  $\frac{1}{\omega^M}$  as  $M \rightarrow \infty$ , as  $|\omega| \rightarrow \infty$  and it should have a fall-off rate **at least** of the order of  $\omega^A e^{-B|\omega|}$  as  $|\omega| \rightarrow \infty$ , where  $A, B > 0$  are real.

### Appendix C.4. *Payley-Weiner theorem and Exponential Fall off rate of analytic functions.*

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions  $E_0(t), E_p(t)$  and  $x(t) = E_0(t)e^{-2\sigma t}$  and  $\frac{d^{2r}x(t)}{dt^{2r}}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$  for  $0 < \sigma < \frac{1}{2}$ .

We know that the order of Riemann's Xi function  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$  is given by  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  where  $A$  is a constant<sup>[3]</sup> (link). Hence both  $E_{0\omega}(\omega)$  and  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$  have **exponential fall-off** rate  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  as  $|\omega| \rightarrow \infty$  and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function  $\xi(s)$ .

Given that  $\xi(s)$  is an entire function in the  $s$ -plane, we see that  $E_{0\omega}(\omega)$  and  $E_{p\omega}(\omega)$  are **analytic** functions which are infinitely differentiable which produce no discontinuities for all  $|\omega| \leq \infty$  and  $0 < \sigma < \frac{1}{2}$ . Hence their respective **inverse Fourier transforms**  $E_0(t), E_p(t)$  have fall-off rates faster than  $\frac{1}{t^M}$  as  $M \rightarrow \infty$ , as  $|t| \rightarrow \infty$  (Appendix C.3) and hence it should have **exponential fall-off** rates as  $|t| \rightarrow \infty$ .

We can use similar arguments to show that  $x(t) = E_0(t)e^{-2\sigma t}$  and  $\frac{d^{2r}x(t)}{dt^{2r}}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$ , because their Fourier transforms are **analytic** functions for all  $|\omega| \leq \infty$  with **exponential fall-off** rate  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  as  $|\omega| \rightarrow \infty$ .

## Appendix D. $\omega_2(t_0)$ is a continuous function around $t_0 = 0$

This result is shown as follows.

- $G_R(\omega) = G_R(\omega, t_0)$  in Eq. 13 is copied below, which is a **continuous** function of  $\omega$  which is differentiable **at least** once with respect to  $\omega$ . (Eq. D.2 and Appendix D.3)

$$G_R(\omega) = G_R(\omega, t_0) = \int_{-\infty}^0 [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \quad (\text{D.1})$$

Given that  $E_0(t) \geq 0$  for  $|t_0| \leq \infty$  (Appendix C.1), we see that  $G_R(\omega) > 0$  at  $\omega = 0$ . **Set**  $t_0 = 0$  and  $G_R(\omega, t_0)$  passes through its **first zero** at  $\omega = \omega_2(t_0) = \omega_2(0)$ . In the rest of this section, we consider the **interval**  $[-\delta t_0, \delta t_0]$  around  $t_0 = 0$ , in  $\omega_2(t_0)$ . There are 3 possibilities.

**Case 1:**  $G_R(\omega) < 0$  for  $\omega = \omega_2(0) + dw$ ,  $G_R(\omega) > 0$  for  $\omega = \omega_2(0) - dw$  for infinitesimal  $dw$  (example plot)

In this case, we will show in Appendix D.1 that  $\omega_2(t_0)$  is a continuous function of  $t_0$  in the interval  $[-\delta t_0, \delta t_0]$ , in the neighborhood around the first zero crossing at  $\omega = \omega_2(t_0) = \omega_2(0)$ .

**Case 2:**  $G_R(\omega) > 0$  for  $\omega = \omega_2(0) + dw$ ,  $G_R(\omega) > 0$  for  $\omega = \omega_2(0) - dw$  (example plot)

In this case,  $\frac{dG_R(\omega)}{d\omega} = 0$  at the **same**  $\omega = \omega_2(0)$  because  $\frac{dG_R(\omega)}{d\omega} < 0$  at  $\omega = \omega_2(0) - dw$  and  $\frac{dG_R(\omega)}{d\omega} > 0$  at  $\omega = \omega_2(0) + dw$ .

$$\frac{dG_R(\omega)}{d\omega} = - \int_{-\infty}^0 t [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \sin(\omega t) dt \quad (\text{D.2})$$

In this case, we will show Appendix D.2 that  $\omega_2(t_0)$  is a continuous function of  $t_0$  in the interval  $[-\delta t_0, \delta t_0]$ , in the neighborhood around the first zero crossing at  $\omega = \omega_2(t_0) = \omega_2(0)$ .

**Case 3:**  $G_R(\omega) = 0$  for  $\omega = \omega_2(0)$  and  $\omega = \omega_2(0) + dw$ .

This is **not** possible because  $G_R(\omega, t_0)$  in Eq. D.1 is an **analytic** function and infinitely differentiable with respect to  $\omega$  (Appendix D.3). We know that analytic functions have **isolated** zeros. (link). Hence we cannot have  $G_R(\omega) = 0$  for  $\omega = \omega_2(0)$  and  $\omega = \omega_2(0) + dw$  as  $dw \rightarrow 0$ .

*Appendix D.1. Case 1:*  $G_R(\omega) < 0$  **for**  $\omega = \omega_2(0) + dw$ ,  $G_R(\omega) > 0$  **for**  $\omega = \omega_2(0) - dw$

- Consider the **segment** S in  $G_R(\omega, t_0)$  in the neighborhood around the first zero crossing where  $\frac{dG_R(\omega, t_0)}{d\omega} < 0$ . (Segment S is the portion between the green lines in example plot)

- In the **segment** S,  $G_R(\omega, t_0)$  in Eq. D.1 is a **continuous** function of  $\omega$ , for **each** value of  $t_0$ . Hence  $G_R(\omega, t_0 - \delta t_0)$  and  $G_R(\omega, t_0 + \delta t_0)$  are **continuous** functions of  $\omega$ , which are differentiable **at least** once, and  $G_R(\omega, t_0 \pm \delta t_0)$  tends to  $G_R(\omega, t_0)$ , as infinitesimal  $\delta t_0 \rightarrow 0$ .

$$G_R(\omega, t_0) = \int_{-\infty}^0 [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt$$

$$G_R(\omega, t_0 + \delta t_0) = \int_{-\infty}^0 [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos(\omega t) dt$$

• In the **segment S**,  $G_R(\omega, t_0)$  in Eq. D.3 is a **continuous** function of  $\omega$ , for **each** value of  $t_0$  and  $\frac{dG_R(\omega, t_0)}{d\omega} < 0$  in the neighborhood around the **first zero crossing**. If we fix the X-coordinate  $\omega$ ,  $G_R(\omega, t_0)$  is a **continuous** function of  $t_0$ , for **each** value of  $\omega$ . Hence, for **each** value of  $\omega$ , as we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_0)$  moves towards  $G_R(\omega, t_0 + \delta t_0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Every point in the segment S, moves continuously, as we change  $t_0$  by an infinitesimal  $\delta t_0$ .

This also applies to the first **zero crossing** in  $G_R(\omega, t_0)$  in the segment S, which corresponds to  $\omega_2(t_0) = \omega_2(0)$  at  $t_0 = 0$  where  $G_R(\omega, t_0) = 0$  in Eq. D.3. The zero crossing moves continuously, as we change  $t_0$  by an infinitesimal  $\delta t_0$ . This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to  $G_R(\omega, t_0)$  with zero crossing at point  $P_0$ , **Green** plot corresponds to  $G_R(\omega, t_0 + \delta t_0)$  with zero crossing at point  $P_{11}$  and **Blue** plot corresponds to  $G_R(\omega, t_0 - \delta t_0)$  with zero crossing at point  $P_{21}$ .

We **define** the **point**  $P_{12}$  in  $G_R(\omega, t_0 + \delta t_0)$  as the point which has the **fixed X-coordinate**  $\omega = \omega_2(0)$ . We **define** the **point**  $P_{22}$  in  $G_R(\omega, t_0 - \delta t_0)$  as the point which has the **fixed X-coordinate**  $\omega = \omega_2(0)$ .

We **define** the **point**  $P_{11}$  in  $G_R(\omega, t_0 + \delta t_0)$  as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point**  $P_{21}$  in  $G_R(\omega, t_0 - \delta t_0)$  as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_0 + \delta t_0)$  in Eq. D.4 moves towards  $G_R(\omega, t_0)$  in a **continuous** manner as follows. The **point**  $P_{12}$  in  $G_R(\omega, t_0 + \delta t_0)$  which corresponds to the **fixed X-coordinate**  $\omega = \omega_2(0)$ , moves towards corresponding point  $P_0$  in  $G_R(\omega, t_0)$ , for the **same**  $\omega = \omega_2(0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Given that  $P_0$  is a **zero crossing point** in  $G_R(\omega, t_0)$ , this is equivalent to the **Zero crossing point**  $P_{11}$  in  $G_R(\omega, t_0 + \delta t_0)$  moving towards corresponding **zero crossing point**  $P_0$  in  $G_R(\omega, t_0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ .

Similarly, as we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_0 - \delta t_0)$  in Eq. D.4 moves towards  $G_R(\omega, t_0)$  in a **continuous** manner as follows. The **point**  $P_{22}$  in  $G_R(\omega, t_0 - \delta t_0)$  which corresponds to the **fixed X-coordinate**  $\omega = \omega_2(0)$ , moves towards corresponding point  $P_0$  in  $G_R(\omega, t_0)$ , for the **same**  $\omega = \omega_2(0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Given that  $P_0$  is a **zero crossing point** in  $G_R(\omega, t_0)$ , this is equivalent to the **Zero crossing point**  $P_{21}$  in  $G_R(\omega, t_0 - \delta t_0)$  moving towards corresponding **zero crossing point**  $P_0$  in  $G_R(\omega, t_0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ .

$$\begin{aligned}
G_R(\omega, t_0) &= \int_{-\infty}^0 [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 + \delta t_0) &= \int_{-\infty}^0 [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 - \delta t_0) &= \int_{-\infty}^0 [E_0(t + t_0 - \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 + \delta t_0)] \cos(\omega t) dt \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_0 + \delta t_0) &= G_R(\omega, t_0) \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_0 - \delta t_0) &= G_R(\omega, t_0)
\end{aligned}$$

• Hence in the **segment S**,  $\omega_2(t_0)$  is a **continuous** function of  $t_0$  in the neighborhood  $[-\delta t_0, \delta t_0]$  around the first zero crossing at  $\omega = \omega_2(t_0) = \omega_2(0)$  at  $t_0 = 0$ .

$$G_R(\omega_2(t_0), t_0) = \int_{-\infty}^0 [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \cos(\omega_2(t_0)t) dt = 0$$

$$G_R(\omega_2(t_0 + \delta t_0), t_0 + \delta t_0) = \int_{-\infty}^0 [E_0(t+t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t-t_0 - \delta t_0)] \cos((\omega_2(t_0 + \delta t_0)t) dt = 0$$

(D.5)

*Appendix D.2. Case 2:*  $G_R(\omega) > 0$  **for**  $\omega = \omega_2(0) + dw$ ,  $G_R(\omega) > 0$  **for**  $\omega = \omega_2(0) - dw$

- In this case,  $\frac{dG_R(\omega)}{d\omega} = 0$  at the **same**  $\omega = \omega_2(t_0)$  because  $\frac{dG_R(\omega)}{d\omega} < 0$  at  $\omega = \omega_2(t_0) - dw$  and  $\frac{dG_R(\omega)}{d\omega} > 0$  at  $\omega = \omega_2(t_0) + dw$ .

- Consider the **segment** S' in  $\frac{dG_R(\omega, t_0)}{d\omega}$  in the neighborhood around the first zero crossing where  $\frac{d^2G_R(\omega, t_0)}{d\omega^2} > 0$ . (Segment S' is the portion between the green lines in example plot) In this segment S',  $\frac{dG_R(\omega, t_0)}{d\omega}$  is a **continuous** function of  $\omega$  which is differentiable **at least** once. (Appendix D.3)

- In the **segment** S',  $\frac{dG_R(\omega, t_0)}{d\omega} = 0$  at the **same**  $\omega = \omega_2(t_0)$ . The arguments in Appendix D.1 can be applied here, with  $G_R(\omega, t_0)$  replaced by  $\frac{dG_R(\omega, t_0)}{d\omega}$ .

Hence  $\omega_2(t_0)$  is a **continuous** function of  $t_0$  in the neighborhood  $[-\delta t_0, \delta t_0]$  around the first zero crossing at  $\omega = \omega_2(t_0) = \omega_2(0)$  at  $t_0 = 0$  in the **segment** S'.

*Appendix D.3. Integral convergence in  $\frac{dG_R(\omega)}{d\omega}$*

It is shown in Appendix C.4 that  $E_0(t)$  and  $E_0(t)e^{-2\sigma t}$  have exponential fall-off rates as  $|t| \rightarrow \infty$  and hence are absolutely **integrable** functions and the integrals  $\int_{-\infty}^{\infty} |E_0(t)| dt < \infty$  and  $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}| dt < \infty$ . Hence the integrand  $A_r(t) = \frac{t^r}{r!} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \sin(\omega t)$  in Eq. D.2 copied below, is an absolutely **integrable function** and  $\int_{-\infty}^0 |A_r(t)| dt = \int_{-\infty}^0 \frac{|t^r|}{r!} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] dt$  is **finite**, for  $r = 0, 1, \dots$ , given the **exponential** fall-off rate of  $E_0(t)e^{-2\sigma t}$  and  $E_0(t)$ .

$$\begin{aligned} \frac{1}{r!} \frac{d^r G_R(\omega)}{d\omega^r} &= (-1)^{\frac{r+1}{2}} \int_{-\infty}^0 \frac{t^r}{r!} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \sin(\omega t) dt, \quad r = \text{odd} \\ \frac{1}{r!} \frac{d^r G_R(\omega)}{d\omega^r} &= (-1)^{\frac{r}{2}} \int_{-\infty}^0 \frac{t^r}{r!} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)] \cos(\omega t) dt, \quad r = \text{even} \end{aligned}$$

(D.6)

## Appendix E. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function  $\xi(s)$  and the inverse Fourier Transform of  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$  and show the result  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ .

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below<sup>[4]</sup> (link). We start with the gamma function  $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$  and substitute  $y = \pi n^2 x$  and derive as follows.

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right) &= \int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma\left(\frac{s}{2}\right) (\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \end{aligned}$$

(E.1)

For real part of  $s$  greater than 1, we can do a summation of both sides of above equation for all positive integers  $n$  and obtain as follows. We note that  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ .

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \quad (\text{E.2})$$

For real part of  $s$  ( $\sigma'$ ) greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and

$$\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma\left(\frac{\sigma'}{2}\right)\pi^{-\frac{\sigma'}{2}} \zeta(\sigma').$$

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \quad (\text{E.3})$$

For real part of  $s$  less than or equal to 1,  $\zeta(s)$  **diverges**. Hence we do the following. In Eq. E.3, first we consider real part of  $s$  greater than 1 and we divide the range of integration into two parts:  $(0, 1]$  and  $[1, \infty)$  and make the substitution  $x \rightarrow \frac{1}{x}$  in the first interval  $(0, 1]$ . We use **the well known theorem**  $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $x > 0$  is real.<sup>[4]</sup>

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1}{2} dx \quad (\text{E.4})$$

Hence we can simplify Eq. E.4 as follows.

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}} w(x) dx \quad (\text{E.5})$$

We multiply above equation by  $\frac{1}{2}s(s-1)$  and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}] \quad (\text{E.6})$$

We see that  $\xi(s)$  is an entire function, for all values of  $Re[s]$  in the complex plane and hence we get an analytic continuation of  $\xi(s)$  over the entire complex plane. We see that  $\xi(s) = \xi(1-s)$  <sup>[4]</sup>.

Appendix E.1. **Derivation of  $E_p(t)$  and  $E_0(t)$**

Given that  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ , we substitute  $x = e^{2t}$ ,  $\frac{dx}{x} = 2dt$  in Eq. E.6 and evaluate at  $s = \frac{1}{2} + \sigma + i\omega$  as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} [1 + 2(\frac{1}{2} + \sigma + i\omega)(-\frac{1}{2} + \sigma + i\omega) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} (e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t}) dt] \quad (\text{E.7})$$

We can substitute  $t = -t$  in the first term in above integral and simplify above equation as follows.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) & [\int_{-\infty}^0 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \\ & + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt] \end{aligned} \quad (\text{E.8})$$

We can write this as follows.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)) \int_{-\infty}^{\infty} [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} e^{-i\omega t} dt \quad (\text{E.9})$$

We define  $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t}$  and get the **inverse Fourier transform** of  $\xi(\frac{1}{2} + \sigma + i\omega)$  in above equation given by  $E_p(t)$  as follows. We use dirac delta function  $\delta(t)$ .

$$\begin{aligned} E_p(t) &= \frac{1}{2} \delta(t) + (-\frac{1}{4} + \sigma^2) A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2 A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}] u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}] u(t) \\ \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-4\pi n^2 e^{-2t} + (-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})^2] u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [-4\pi n^2 e^{2t} + (\frac{1}{2} - \sigma - 2\pi n^2 e^{2t})^2] u(t) + \delta(t) [\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2)] \end{aligned} \quad (\text{E.10})$$

We can simplify above equation as follows.

$$\begin{aligned} \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma \pi n^2 e^{-2t}] u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma \pi n^2 e^{2t}] u(t) + \delta(t) [\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2)] \end{aligned} \quad (\text{E.11})$$

We use the fact that  $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and  $x > 0$  is real<sup>[4]</sup>, and we take the first derivative of  $F(x)$  and evaluate it at  $x = 1$ . We see that  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$  ( Appendix E.2) and hence **dirac delta terms cancel each other** in equation below.



$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + \left(-\frac{1}{4} + \sigma^2\right)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}\right) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}\right] u(t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[-\frac{1}{4} + \sigma^2 + 2\sigma\left(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}\right) + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}\right] u(-t)
\end{aligned} \tag{E.12}$$

We can simplify above equation as follows.

$$\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned} \tag{E.13}$$

We use the fact that  $E_0(t) = E_0(-t)$  because  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when evaluated at the critical line  $s = \frac{1}{2} + i\omega$ . This means  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  and  $E_0(t) = E_0(-t)$  and we arrive at the desired result for  $E_p(t)$  as follows.

$$\begin{aligned}
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) &= E_0(t)e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned} \tag{E.14}$$

*Appendix E.2. Derivation of  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$*

In this section, we derive  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$ . We use the fact that  $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and  $x > 0$  is real<sup>[4]</sup>, and we take the first derivative of  $F(x)$  and evaluate it at  $x = 1$ .

$$\begin{aligned}
F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2 \frac{1}{x}} \left(\frac{1}{x^2}\right) + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \left(\frac{-1}{2}\right) \frac{1}{x^{\frac{3}{2}}}
\end{aligned} \tag{E.15}$$

We evaluate the above equation at  $x = 1$  and we simplify as follows.

$$\begin{aligned}
\left[\frac{dF(x)}{dx}\right]_{x=1} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2} = \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2} + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}) \left(\frac{-1}{2}\right) \\
&\quad \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}
\end{aligned}$$

(E.16)