

# On a new method towards proof of Riemann's Hypothesis

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## Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function**  $\xi(s)$  which is evaluated at  $s = \frac{1}{2} + \sigma + i\omega$ , given by  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ , where  $\sigma, \omega$  are real and  $-\infty \leq \omega \leq \infty$  and compute its inverse Fourier transform given by  $E_p(t)$ .

We use a new method and show that the Fourier Transform of  $E_p(t)$  given by  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$  **does not have zeros** for finite and real  $\omega$  when  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of  $s = \frac{1}{2}$  and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

*Keywords:* Riemann, Hypothesis, Zeta, Xi, exponential functions

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## 1. Introduction

It is well known that Riemann's Zeta function given by  $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$  converges in the half-plane where the real part of  $s$  is greater than 1. Riemann proved that  $\zeta(s)$  has an analytic continuation to the whole s-plane apart from a simple pole at  $s = 1$  and that  $\zeta(s)$  satisfies a symmetric functional equation given by  $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$  where  $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$  is the Gamma function.<sup>[4] [5]</sup> We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of  $\zeta(s)$  lie on the critical line with real part of  $s = \frac{1}{2}$ , which is called the Riemann Hypothesis.<sup>[1]</sup>

Hardy and Littlewood later proved that infinitely many of the zeros of  $\zeta(s)$  are on the critical line with real part of  $s = \frac{1}{2}$ .<sup>[2]</sup> It is well known that  $\zeta(s)$  does not have non-trivial zeros when real part of  $s = \frac{1}{2} + \sigma + i\omega$ , given by  $\frac{1}{2} + \sigma \geq 1$  and  $\frac{1}{2} + \sigma \leq 0$ . In this paper, **critical strip**  $0 < \text{Re}[s] < 1$  corresponds to  $0 \leq |\sigma| < \frac{1}{2}$ .

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  and compute inverse Fourier transform of  $E_{p\omega}(\omega)$  given by  $E_p(t)$  and show that its Fourier transform  $E_{p\omega}(\omega)$  does not have zeros for finite and real  $\omega$  when  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix D, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

### 1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function  $\xi(s)$  evaluated at  $s = \frac{1}{2} + i\omega$  given by  $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$ , where  $-\infty \leq \omega \leq \infty$ . Its inverse Fourier Transform is given by  $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$ , where  $\omega, t$  are real, as follows (link).<sup>[3]</sup>

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that  $E_0(t) = E_0(-t)$  is a real and **even** function of  $t$ , given that  $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  because  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when evaluated at  $s = \frac{1}{2} + i\omega$ .

The inverse Fourier Transform of  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  is given by the real function  $E_p(t)$ . We can write  $E_p(t)$  as follows for  $0 < |\sigma| < \frac{1}{2}$  and this is shown in detail in Appendix A using contour integration.

$$E_p(t) = E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that  $E_p(t)$  is an analytic function in the interval  $|t| \leq \infty$ , given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

### 1.2. Step 2: On the zeros of a related function $G(\omega)$

**Statement 1:** Let us assume that Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$  where  $\omega_0$  is real and finite and  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider  $0 < \sigma < \frac{1}{2}$  at first. Let us consider a new function  $g(t) = f(t) e^{-\sigma t} u(-t) + f(t) e^{\sigma t} u(t)$ , where  $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$  and  $f_1(t) = e^{\sigma t_0} E'_p(t+t_0)$  and  $f_2(t) = e^{-\sigma t_0} E'_p(t-t_0)$  and  $E'_p(t) = e^{-\sigma t_2} E_p(t-t_2) - e^{\sigma t_2} E_p(t+t_2)$  and  $t_0, t_2$  are real and  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function. We can see that  $g(t)h(t) = f(t)$  where  $h(t) = [e^{\sigma t} u(-t) + e^{-\sigma t} u(t)]$ .

In Section 2.1, we will show that the Fourier transform of the **even function**  $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$  given by  $G_R(\omega)$  must have **at least one zero** at  $\omega = \omega_z(t_2, t_0) \neq 0$ , for every value of  $t_0$ , to satisfy Statement 1, where  $\omega_z(t_2, t_0)$  is real and finite.

### 1.3. Step 3: On the zeros of the function $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function  $g(t)$  and compute its real part given by  $G_R(\omega) = G_R(\omega, t_2, t_0)$  and we can write as follows.

$$G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0) e^{-2\sigma \tau} + E'_{0n}(\tau - t_0)] \cos(\omega \tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0) e^{-2\sigma \tau} + E'_{0n}(\tau + t_0)] \cos(\omega \tau) d\tau \quad (3)$$

We require  $G_R(\omega, t_2, t_0) = 0$  for  $\omega = \omega_z(t_2, t_0)$  for every value of  $t_0$ , for **each fixed value** of  $t_2$ , to satisfy **Statement 1**. In general  $\omega_z(t_2, t_0) \neq \omega_0$ . Hence we can see that  $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ .

1.4. **Step 4: Zero Crossing function**  $\omega_z(t_2, t_0)$  **is an even function of variable**  $t_0$

In Section 2.3, we show the result in Eq. 4 and that  $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$ . It is shown that  $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = P_{\text{odd}}(t_2, t_0) + P_{\text{odd}}(t_2, -t_0)$  is an **odd** function of  $t_0$ , for all  $t_0$ , for a given value of  $t_2$  as follows.

$$\begin{aligned} P_{\text{odd}}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &\quad + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau] \end{aligned} \quad (4)$$

1.5. **Step 5: Final Step**

In Section 5, it is shown that  $\omega_z(t_2, t_0)$  is a **continuous** function of variable  $t_0$  for all  $|t_0| < \infty$ , for **every given** fixed value of  $t_2$ . In Section 4, it is shown that  $E_0(t)$  is **strictly decreasing** for  $t > 0$ .

In Section 3, we set  $t_0 = t_{0c}$  and  $t_2 = t_{2c} = 2t_{0c}$ , such that  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$  and substitute in the equation for  $P_{\text{odd}}(t_2, t_0)$  in Eq. 4 and show that this leads to the result in Eq. 5. We use  $E'_0(t) = E_0(t - t_2) - E_0(t + t_2)$  and  $E'_{0n}(t) = E'_0(-t)$ .

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) (\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \quad (5)$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval  $\tau = [0, t_{0c}]$  where  $t_{0c} > 0$ . For  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ , we see that  $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi}{2t_{0c}}\tau$  lies in the range  $[0, \frac{\pi}{2}]$  and hence  $\sin(\omega_{c1}\tau) > 0$  in that interval  $\tau = [0, t_{0c}]$ .

Hence the result in Eq. 5 leads to a **contradiction** for  $0 < \sigma < \frac{1}{2}$ .

We have shown this result for  $0 < \sigma < \frac{1}{2}$  and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$ . Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function  $E_p(t) = E_0(t)e^{-\sigma t}$  has a zero at  $\omega = \omega_0$  for  $0 < |\sigma| < \frac{1}{2}$ .

## 2. An Approach towards Riemann's Hypothesis

**Theorem 1:** Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  does not have zeros for any real value of  $-\infty < \omega < \infty$ , for  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line, given that  $E_0(t) = E_0(-t)$  is an even function of variable  $t$ , where  $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$ ,  $E_p(t) = E_0(t)e^{-\sigma t}$  and  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ .

**Proof:** We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

**Statement 1:** Let us assume that Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$  where  $\omega_0$  is real and finite and  $0 < |\sigma| < \frac{1}{2}$ , corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for  $0 < \sigma < \frac{1}{2}$  first and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$  and hence show the result for  $0 < |\sigma| < \frac{1}{2}$ .

We know that  $\omega_0 \neq 0$ , because  $\zeta(s)$  has no zeros on the real axis between 0 and 1, when  $s = \frac{1}{2} + \sigma + i\omega$  is real,  $\omega = 0$  and  $0 < |\sigma| < \frac{1}{2}$ . [3] This is shown in detail in first two paragraphs in Appendix C.1.

### 2.1. New function $g(t)$

Let us consider the function  $E'_p(t) = E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E'_0(t)e^{-\sigma t}$ , where  $t_2$  is finite and real, and  $E'_0(t) = E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ . Its Fourier transform is given by  $E'_{p\omega}(\omega) = E'_{p\omega}(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$  which has a zero at the **same**  $\omega = \omega_0$ .

Let us consider the function  $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$  where  $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0)$  and  $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0)$  where  $t_0$  is finite and real and we can see that the Fourier Transform of this function  $F(\omega) = F(\omega, t_2, t_0) = E'_{p\omega}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$  also has a zero at the **same**  $\omega = \omega_0$ .

Let us consider a new function  $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $g_-(t) = f(t)e^{-\sigma t}$  and  $g_+(t) = f(t)e^{\sigma t}$ . We can see that  $g(t)h(t) = f(t)$  where  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ .

We **note** that we use the **shorthand** notation for the functions  $f(t), g(t), f_1(t), f_2(t), F(\omega)$  and  $G(\omega)$  which are also functions of variables  $t_2, t_0$ . Similarly we use the shorthand notation for the functions  $E'_p(t), E'_0(t)$  and  $E'_{p\omega}(\omega)$  which are also functions of variable  $t_2$ .

We can show that  $E_p(t), E'_p(t), h(t), g(t)$  are real absolutely integrable functions and go to zero as  $t \rightarrow \pm\infty$ . Hence their respective Fourier transforms given by  $E_{p\omega}(\omega), E'_{p\omega}(\omega), H(\omega), G(\omega)$  are finite for  $|\omega| \leq \infty$  and go to zero as  $|\omega| \rightarrow \infty$ , as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

If we take the Fourier transform of the equation  $g(t)h(t) = f(t)$  where  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ , we get  $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E'_{p\omega}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$  as per convolution theorem (link), where  $*$  denotes convolution operation given by  $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$  and  $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  is real and is the Fourier transform of the function  $h(t)$  and  $G(\omega) = G_R(\omega) + iG_I(\omega)$  is the Fourier transform of the function  $g(t)$ . This is shown in detail in Appendix B.1.

For **every value** of  $t_0$ , we require the Fourier transform of the function  $f(t)$  given by  $F(\omega)$  to have a zero at  $\omega = \omega_0$ . This implies that the Fourier transform of the **even** function  $g(t)$  given by  $G(\omega) = G_R(\omega)$  must have **at least one real zero** at  $\omega = \omega_z(t_0)$  for **every value** of  $t_0$ . Because  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  does not have real zeros, if  $G_R(\omega)$  does not have real zeros, then  $F_R(\omega) = G_R(\omega) * H_R(\omega)$  obtained by the convolution of  $H_R(\omega)$  and  $G_R(\omega)$ , cannot possibly have real zeros, which goes against **Statement 1**.

We can write  $g(t) = g_{\text{even}}(t) + g_{\text{odd}}(t)$  where  $g_{\text{even}}(t)$  is an even function and  $g_{\text{odd}}(t)$  is an odd function of variable  $t$ . If Statement 1 is true, then the **real** part of the Fourier transform of the **even function**  $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$  given by  $G_R(\omega)$  must have **at least one zero** at  $\omega = \omega_z(t_2, t_0) \neq 0$  where  $\omega_z(t_0)$  is real and finite and can be different from  $\omega_0$  in general. We call this **Statement 2**.

Because  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  is real and does not have zeros for any finite value of  $\omega$ , **if**  $G_R(\omega)$  does not have at least one zero for some  $\omega = \omega_z(t_2, t_0) \neq 0$ , **then** the **real part** of  $F(\omega)$  given by  $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$ , obtained by the convolution of  $H(\omega)$  and  $G_R(\omega)$ , **cannot** possibly have zeros for any non-zero finite value of  $\omega$ , which goes against **Statement 1**. This is shown in detail in Lemma 1.

**Lemma 1:** If Riemann's Xi function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0 \neq 0$  where  $\omega_0$  is real and finite, then the **real** part of the Fourier transform of the **even function**  $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$  given by

166  $G_R(\omega)$  must have **at least one zero** at  $\omega = \omega_z(t_2, t_0) \neq 0$  for **every value** of  $t_0$ , where  $\omega_z(t_0)$  is real and fi-  
 167 nite, where  $g(t)h(t) = f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t)$  where  $f_1(t) = e^{\sigma t_0}E'_p(t + t_0)$  and  $f_2(t) = e^{-\sigma t_0}E'_p(t - t_0)$ ,  
 168  $E'_p(t) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2)$ , and  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  and  $0 < \sigma < \frac{1}{2}$ .

169  
 170 **Proof:** If  $E_{p\omega}(\omega)$  has a zero at finite  $\omega = \omega_0 \neq 0$  to satisfy Statement 1, then  $F(\omega) = E'_{p\omega}(\omega)(e^{-\sigma t_0}e^{i\omega t_0} +$   
 171  $e^{\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$  also has a zero at  $\omega = \omega_0$  and its real part given  
 172 by  $F_R(\omega)$  also has a zero at the same location  $\omega = \omega_0 \neq 0$ .

173  
 174 Let us consider the case where  $G_R(\omega)$  **does not** have at least one zero for finite  $\omega = \omega_z(t_0) \neq 0$  and show that  
 175  $F_R(\omega)$  does not have at least one zero at finite  $\omega \neq 0$  for this case, which **contradicts** Statement 1. Given that  $H(\omega)$   
 176 is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \quad (6)$$

177 We can show that the above integral converges for all  $|\omega| \leq \infty$ , given that  $G(\omega)$  and  $H(\omega)$  have fall-off rate of  $\frac{1}{\omega^2}$   
 178 as  $|\omega| \rightarrow \infty$  because the first derivatives of  $g(t)$  and  $h(t)$  are discontinuous at  $t = 0$ . (Appendix C.2)

179  
 180 We substitute  $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$  in Eq. 6 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (7)$$

181 We can split the integral in Eq. 7 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \left[ \int_{-\infty}^0 G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (8)$$

182  
 183 We see that  $G_R(-\omega) = G_R(\omega)$  because  $g(t)$  is a real function (Appendix B.2). We can substitute  $\omega' = -\omega''$  in  
 184 the first integral in Eq. 8 and substituting  $\omega'' = \omega'$  in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_0^{\infty} G_R(\omega') \left[ \frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (9)$$

185  
 186 In Appendix C.1 last paragraph, it is shown that  $G(\omega)$  is finite for  $|\omega| \leq \infty$  and goes to zero as  $|\omega| \rightarrow \infty$ . We  
 187 can see that for  $\omega' = 0$  and  $\omega' = \infty$ , the integrand in Eq. 9 is zero. For finite  $\omega > 0$ , and  $0 < \omega' < \infty$ , we can see that  
 188 the term  $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$ .

189  
 190 **• Case 1:**  $G_R(\omega') > 0$  for all finite  $\omega' > 0$

191  
 192 We see that  $F_R(\omega) > 0$  for all finite  $\omega > 0$ . We see that  $F_R(-\omega) = F_R(\omega)$  because  $f(t)$  is a real function (Ap-  
 193 pendix B.2). Hence  $F_R(\omega) > 0$  for all finite  $\omega < 0$ .

194  
 195 This **contradicts** Statement 1 which requires  $F_R(\omega)$  to have at least one zero at finite  $\omega \neq 0$  because we showed  
 196 that  $\omega_0 \neq 0$  in **Section 2** paragraph 5. Therefore  $G_R(\omega')$  must have **at least one zero** at  $\omega' = \omega_z(t_2, t_0) \neq 0$ , where  
 197  $\omega_z(t_2, t_0)$  is real and finite.

198  
 199 **• Case 2:**  $G_R(\omega') < 0$  for all finite  $\omega' > 0$

200  
 201 We see that  $F_R(\omega) < 0$  for all finite  $\omega > 0$ . We see that  $F_R(-\omega) = F_R(\omega)$  because  $f(t)$  is a real function (Ap-  
 202 pendix B.2). Hence  $F_R(\omega) < 0$  for all finite  $\omega < 0$ .

203

This **contradicts** Statement 1 which requires  $F_R(\omega)$  to have at least one zero at finite  $\omega \neq 0$ . Therefore  $G_R(\omega')$  must have **at least one zero** at  $\omega' = \omega_z(t_2, t_0) \neq 0$ , where  $\omega_z(t_2, t_0)$  is real and finite.

We have shown that,  $G_R(\omega)$  must have **at least one zero** at finite  $\omega = \omega_z(t_2, t_0) \neq 0$  to satisfy **Statement 1**. We call this **Statement 2**.

## 2.2. On the zeros of a related function $G(\omega)$

We can compute the fourier transform of the function  $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$  given by  $G_R(\omega)$ . We require  $G_R(\omega) = 0$  for  $\omega = \omega_z(t_2, t_0)$  for **every value** of  $t_0$ , for a given value of  $t_2$ , to satisfy **Statement 1**. In general,  $\omega_z(t_2, t_0) \neq \omega_0$ .

First we compute the Fourier transform of the function  $g_1(t)$  given by  $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$ . We use  $g_1(t) = f_1(t)e^{-\sigma t}u(-t) + f_1(t)e^{\sigma t}u(t) = e^{\sigma t_0}E'_p(t+t_0)e^{-\sigma t}u(-t) + e^{\sigma t_0}E'_p(t+t_0)e^{\sigma t}u(t)$ .

We **note** that we use the **shorthand** notation for the functions  $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$  and  $G_1(\omega)$  which are also functions of variables  $t_2, t_0$ . Similarly we use the shorthand notation for the functions  $E'_p(t), E'_0(t)$  and  $E'_{0n}(t)$  which are also functions of variable  $t_2$ .

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^0 g_1(t)e^{-i\omega t}dt + \int_0^{\infty} g_1(t)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 e^{\sigma t_0}E'_p(t+t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_0^{\infty} e^{\sigma t_0}E'_p(t+t_0)e^{\sigma t}e^{-i\omega t}dt \end{aligned} \quad (10)$$

We use  $E'_p(t) = E'_0(t)e^{-\sigma t}$  where  $E'_0(t) = E_0(t-t_2) - E_0(t+t_2)$  and  $E'_p(t+t_0) = E'_0(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$ . Substituting  $t = -t$  in the second integral in Eq. 10, we have

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_0^{\infty} E'_0(t+t_0)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_0(-t+t_0)e^{i\omega t}dt \end{aligned} \quad (11)$$

We define  $E'_{0n}(t) = E'_0(-t)$  and get  $E'_0(-t+t_0) = E'_{0n}(t-t_0)$  and write Eq. 11 as follows.

$$G_1(\omega) = \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_{0n}(t-t_0)e^{i\omega t}dt = G_R(\omega) + iG_I(\omega) \quad (12)$$

The above equations can be expanded as follows using the identity  $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ . Comparing the **real parts** of  $G(\omega)$ , we have

$$G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^0 E'_{0n}(t-t_0)\cos(\omega t)dt \quad (13)$$

2.3. **Zero crossing function**  $\omega_z(t_2, t_0)$  **is an even function of variable**  $t_0$

Now we consider the function  $f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t) = e^{-\sigma t_0} E'_p(t + t_0) + e^{\sigma t_0} E'_p(t - t_0)$  where  $f_1(t) = e^{\sigma t_0} E'_p(t + t_0)$  and  $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0} E'_p(t - t_0)$  and  $g(t)h(t) = f(t)$  where  $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$  and  $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$  and compute the Fourier transform of the function  $g(t)$  and compute its real part using the procedure in above section, similar to Eq. 13 and we can write as follows. We substitute  $t = \tau$ .

$$\begin{aligned} G_R(\omega, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_0) + e^{2\sigma t_0} G_{1R}(\omega, -t_0) \\ G_{1R}(\omega, t_0) &= \int_{-\infty}^0 [E'_0(\tau + t_0)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0)] \cos(\omega\tau) d\tau \\ G_R(\omega, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0)] \cos(\omega\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0)] \cos(\omega\tau) d\tau \end{aligned} \tag{14}$$

We require  $G_R(\omega, t_0) = 0$  for  $\omega = \omega_z(t_2, t_0)$  for every value of  $t_0$ , for **every given fixed value** of  $t_2$ , to satisfy **Statement 1**. In general  $\omega_z(t_2, t_0) \neq \omega_0$ . Hence we can see that  $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$  and we can rearrange the terms as follows.

$$\begin{aligned} P(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \int_{-\infty}^0 [e^{2\sigma t_0} E'_0(\tau - t_0)e^{-2\sigma\tau} + e^{-2\sigma t_0} E'_{0n}(\tau - t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \end{aligned} \tag{15}$$

We can write as follows, where  $P_{odd}(t_2, t_0)$  is an **odd** function of variable  $t_0$ .

$$\begin{aligned} P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\ P_{odd}(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \end{aligned} \tag{16}$$

We see that  $f(t, t_0) = e^{-\sigma t_0} E'_p(t + t_0) + e^{\sigma t_0} E'_p(t - t_0) = f(t, -t_0)$  is **unchanged** by the substitution  $t_0 = -t_0$  and hence  $\omega_z(t_2, t_0)$  is an **even** function of variable  $t_0$ , for **every fixed value** of  $t_2$ .

### 3. Final Step

We expand  $P_{odd}(t_2, t_0)$  in Eq. 16 as follows, using the substitution  $\tau + t_0 = \tau'$  and substituting back  $\tau' = \tau$ . We use  $E'_{0n}(\tau) = E'_0(-\tau)$  and  $E'_0(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$ .

We **note** that we use the **shorthand** notation for the functions  $E'_0(t)$  and  $E'_{0n}(t)$  which are also functions of variable  $t_2$ .

$$\begin{aligned} P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\ &\quad + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau] \end{aligned}$$

In Section 2.1,  $\omega_z(t_2, t_0)$  is shown to be **finite** for all  $|t_0| < \infty$ , for a given value of  $t_2$ . This means there are **no** Dirac delta functions present in  $\omega_z(t_2, t_0)$ .

In Section 5, it is shown that  $\omega_z(t_2, t_0)$  is a **continuous** function of variable  $t_0$  for all  $|t_0| < \infty$ , for **every given** fixed value of  $t_2$ .

In Section 4, it is shown that  $E_0(t)$  is **strictly decreasing** for  $t > 0$ .

Given  $\omega_z(t_2, t_0)$  is a continuous function of both  $t_0$  and  $t_2$ , we can find a suitable value of  $t_0 = t_{0c}$  and  $t_2 = t_{2c} = 2t_{0c}$  such that  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ . Given that  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$ , for every value of  $t_2$ , and  $t_0$  is a continuous function, we see that the **product** of two continuous functions  $\omega_z(t_2, t_0)t_0$  is a **continuous** function as well. Given that  $0 < \omega_z(t_2, t_0) < \infty$ , we see that  $\omega_z(t_2, t_0)t_0$  will **certainly pass through**  $\pi$ , as  $t_0$  is increased from zero to  $\infty$ .

We use  $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$  as follows. We set  $t_0 = t_{0c}$  and  $t_2 = t_{2c} = 2t_{0c}$  such that  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$  in Eq. 17 as follows. We use the fact that  $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$  shown in Section 2.3.

$$\begin{aligned} & \int_{-\infty}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & - \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \end{aligned} \quad (18)$$

We split the integral in the left hand side of Eq. 18 and write as follows.

$$\begin{aligned} & \left[ \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \\ & + e^{2\sigma t_{0c}} \left[ \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \\ & - \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \end{aligned} \quad (19)$$

We combine the terms with common integrals and cancel common terms in Eq. 19 as follows.

$$\begin{aligned} & \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (20)$$

We can rearrange the terms in Eq. 20 as follows.

$$\begin{aligned} & \int_{-t_{0c}}^{t_{0c}} [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$



We denote the right hand side of Eq. 21 as  $RHS$ . We can split the integral in Eq. 21 using  $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^0 + \int_0^{t_{0c}}$  as follows.

$$\begin{aligned} & \int_{-t_{0c}}^0 [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \int_0^{t_{0c}} [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned} \quad (22)$$

We substitute  $\tau = -\tau$  in the first integral in Eq. 22 as follows. We use  $E'_0(-\tau) = E'_{0n}(\tau)$  and  $E'_{0n}(-\tau) = E'_0(\tau)$ .

$$\begin{aligned} & \int_{t_{0c}}^0 [E'_{0n}(\tau)e^{2\sigma\tau} + E'_0(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \int_0^{t_{0c}} [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned} \quad (23)$$

Given that  $\int_{t_{0c}}^0 = -\int_0^{t_{0c}}$ , we can simplify as follows.

$$\int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \quad (24)$$

We substitute  $\tau = -\tau$  in the right hand side of Eq. 21 as follows. We use  $E'_{0n}(-\tau) = E'_0(\tau)$ .

$$RHS = 2 \sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \quad (25)$$

We split the integral on the right hand side in Eq. 25 as follows.

$$RHS = 2 \sinh(2\sigma t_{0c}) \left[ \int_0^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \quad (26)$$

We consolidate the integrals with the term  $\int_0^{t_{0c}} E'_0(\tau)$  in Eq. 24 and Eq. 26 as follows. We use  $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$ .

$$\begin{aligned} & \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = 2 \sinh(2\sigma t_{0c}) \int_0^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (27)$$

We cancel common terms in Eq. 27 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = 2 \sinh(2\sigma t_{0c}) \int_0^\infty E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

(28)

We substitute  $E'_0(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$  and  $E'_{0n}(\tau) = E'_0(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$ . We see that  $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$  and  $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$  given that  $E_0(\tau) = E_0(-\tau)$ . Hence we see that  $E'_{0n}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E'_0(\tau)$ . We can write Eq. 28 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = 2 \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

(29)

We substitute  $2 \cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$  and  $2 \cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$  and cancel the common factor of 2 in Eq. 29 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

(30)

### Next Step:

We substitute  $\tau + t_{2c} = \tau'$  in the right hand side of Eq. 30 and then substitute  $\tau' = \tau$ . Similarly we substitute  $\tau - t_{2c} = \tau'$  as follows.

$$\begin{aligned} RHS = & \sinh(2\sigma t_{0c}) [\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{-t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & - \cos(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c})t_{2c}) \int_{t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \end{aligned}$$

(31)

In Eq. 31, given that  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$  and  $t_{2c} = 2t_{0c}$  and hence  $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$  and  $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$  and  $\cos(\omega_z(t_{2c}, t_{0c})t_{2c}) = -1$ . Hence we cancel common terms and write Eq. 31 and Eq. 30 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = -\sinh(2\sigma t_{0c}) [\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \end{aligned}$$

(32)

We use  $\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$  and cancel the common term  $\int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$  in Eq. 32 as follows. Given that  $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$  is an **odd** function of variable  $\tau$ , we get  $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$ .

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$$

(33)

We can multiply Eq. 33 by a factor of  $-1$  as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$$

(34)

In Eq. 34, given that  $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ , as  $\tau$  varies over the interval  $[0, t_{0c}]$ ,  $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$  varies from  $[0, \frac{\pi}{2}]$  and hence the sinusoidal function varies over a **half cycle** and is  $> 0$ , in the interval  $0 < \tau < t_{0c}$ , for  $t_{0c} > 0$ .

In Eq. 34, we see that in the interval  $0 < \tau < t_{0c}$ , the integral on the left hand side is  $> 0$  for  $t_{0c} > 0$ , because each of the terms in the integrand are  $> 0$ , in the interval  $0 < \tau < t_{0c}$  as follows. Given that  $E_0(t)$  is a **strictly decreasing** function for  $t \geq \frac{1}{8}$ , we see that  $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$  is  $> 0$  (Section 4.7). The term  $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$  is  $> 0$  in the interval  $0 < \tau < t_{0c}$  and the integrand is zero at  $\tau = 0$  and  $\tau = t_{0c}$  and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 34. Hence this leads to a **contradiction** for  $0 < \sigma < \frac{1}{2}$ .

For  $\sigma = 0$ , both sides of Eq. 34 is zero and **does not** lead to a contradiction.

We have shown this result for  $0 < \sigma < \frac{1}{2}$  and then use the property  $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$  to show the result for  $-\frac{1}{2} < \sigma < 0$ . Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function  $E_p(t) = E_0(t)e^{-\sigma t}$  has a zero at  $\omega = \omega_0$  for  $0 < |\sigma| < \frac{1}{2}$ .

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  has a zero at  $\omega = \omega_0$ , where  $\omega_0$  is real and finite, leads to a **contradiction** for the region  $0 < |\sigma| < \frac{1}{2}$  which corresponds to the critical strip excluding the critical line. This means  $\zeta(s)$  does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

#### 4. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  whose Fourier Transform is given by the entire function  $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$ . (link)

$$E_0(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

$$E_0(t) = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$$

(35)

We show that  $X(t) = \frac{E_0(t)}{2\pi}$  is a **strictly decreasing** function for  $t \geq 0$  as follows.

- In Section 4.1, it is shown that the second derivative of  $X(t)$ , given by  $X_2(t) = \frac{d^2 X(t)}{dt^2} < 0$  for  $t = 0$ .

- In Section 4.2, it is shown that, as  $t$  increases from zero,  $\frac{dX(t)}{dt}$  starts from zero and reaches a **negative minimum** value at  $t = t_{min}$  and then starts increasing towards zero, for  $t > t_{min}$ . (example plot) Hence  $\frac{dX(t)}{dt} < 0$  for  $0 < t \leq t_{min}$ .

- In Section 4.3, it is shown that  $\frac{dX(t)}{dt} < 0$  for  $t \geq t'_{min}$ , where  $t'_{min}$  corresponds to the minima of  $X_{11}(t)$  which is the partial term in  $\frac{dX(t)}{dt}$  corresponding to  $n = 1$ .

- In Section 4.4, it is shown that  $\frac{dX(t)}{dt} < 0$  for  $t_{min} < t < t'_{min}$ . Hence  $\frac{dX(t)}{dt} < 0$  for all  $t > 0$  and hence  $X(t)$  is **strictly decreasing** for all  $t > 0$  and  $E_0(t) = 2\pi X(t)$  is **strictly decreasing** for all  $t > 0$ .

4.1.  $\frac{d^2 X(t)}{dt^2} < 0$  for  $t = 0$

We consider  $X(t) = \frac{E_0(t)}{2\pi} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [2\pi n^2 e^{4t} - 3e^{2t}]$  and take the first derivative of  $X(t)$  as follows. We note that  $E_0(t)$  is an analytic function for  $|t| \leq \infty$  and is infinitely differentiable in that interval.

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t}) (\frac{1}{2} - 2\pi n^2 e^{2t})]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}]$$

$$\frac{dX(t)}{dt} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]$$

(36)

We take the second derivative of  $X(t)$  as follows.

$$\frac{d^2 X(t)}{dt^2} = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t} + (-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2})(\frac{5}{2} - 2\pi n^2 e^{2t})]$$

$$\begin{aligned} \frac{d^2 X(t)}{dt^2} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [-16\pi^2 n^4 e^{4t} + 30\pi n^2 e^{2t} \\ &\quad + (-10\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t} - \frac{75}{4} + 8\pi^3 n^6 e^{6t} - 30\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t})] \\ \frac{d^2 X(t)}{dt^2} &= \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t), \quad Z(t) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4} \end{aligned}$$

(37)

• **Case 1:**  $n = 1, t = 0$ : The partial term for  $n = 1$  and  $t = 0$  in  $\frac{d^2 X(t)}{dt^2}$  in Eq. 37, given by  $A_1 < 0$  shown as follows.

$$A_1 = e^{-\pi}(8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4}) < 0$$

(38)

At  $n = 1, t = 0$ , we see that  $Z(t) = 8\pi^3 - 56\pi^2 + \frac{165}{2}\pi - \frac{75}{4} = \pi(8\pi^2 - 56\pi + \frac{165}{2}) - \frac{75}{4} \leq 3*(8*10 - 56*3 + 83) - 18 = -33 < 0$  (**Result K**), because  $3 < \pi < 3.1429$  and  $\pi^2 < 10$ . We note that the term in the brackets is negative and hence we multiply it by the minimum value of  $\pi$ . Hence we see that the partial term for  $n = 1$  and  $t = 0$  in  $\frac{d^2 X(t)}{dt^2}$  given by  $A_1 < 0$ .

• **Case 2:**  $n > 1, t = 0$ : the partial terms for  $n > 1$  and  $t = 0$  in  $\frac{d^2 X(t)}{dt^2}$  in Eq. 37, given by  $A_2 > 0$  and  $A_2 < |A_1|$  shown as follows.

$$A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} (8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4}) > 0$$

(39)

At  $n = 2, t = 0$ , we see that  $Z(t) = 8\pi^3 n^6 - 56\pi^2 n^4 + \frac{165}{2}\pi n^2 - \frac{75}{4} = \pi n^2(8\pi^2 n^4 - 56\pi n^2 + \frac{165}{2}) - \frac{75}{4} > 3*2^2(8*3^2*2^4 - 56*4*2^2 + 82) - 19 = 4037 > 0$ , because  $3 < \pi < 4$ . At  $n > 2, t = 0$ , we see that  $Z(t) > 0$  due to the dominant term  $8\pi^2 n^4$ . Hence the partial terms for  $n > 1$  and  $t = 0$  in  $\frac{d^2 X(t)}{dt^2}$  in Eq. 37, given by  $A_2 > 0$ .

• **We can show** that at  $t = 0$ ,  $A_1 \leq -0.4283$  and  $A_2 < |A_1|$  and hence  $\frac{d^2 X(t)}{dt^2} < 0$  in Eq. 37.

At  $n = 1, t = 0$ , we see that, given  $3 < \pi < \frac{22}{7} = 3.1429$  (link) and  $e^{-\pi} \geq e^{-0.1429}(e^{-\frac{1}{2}})^6$  and  $e^{-0.1429} \geq 1 - 0.1429 = 0.8571$  and  $e^{-\frac{1}{2}} \geq 1 - \frac{1}{2} = \frac{1}{2}$ , we can write  $e^{-\pi} \geq \frac{0.8571}{2^6}$  and  $|A_1| = e^{-\pi}|Z(t)| \geq 33 * \frac{0.8571}{2^6} \geq 32 * \frac{0.8571}{64} = 0.4283$  (using Result K). Hence we see that the partial term for  $n = 1, t = 0$  in  $\frac{d^2 X(t)}{dt^2}$  in Eq. 37, given by  $A_1 \leq -0.4283$  (**Result A**).

At  $n > 1, t = 0$ , we see that  $A_2 = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} Z(0) \leq \sum_{n=2}^{\infty} n^2 e^{-\pi n^2} * (8\pi^3 n^6 + \frac{165}{2}\pi n^2)$  and we **want to show**  $A_2 < |A_1|$  where  $A_1 \leq -0.4283$ . We use the fact that  $(n^2)^4 e^{-\pi n^2} < e^{-(\pi-1)n^2}$  and  $(n^2)^2 e^{-\pi n^2} < e^{-(\pi-1)n^2}$ .

$$A_2 \leq \sum_{n=2}^{\infty} e^{-(\pi-1)n^2} (8\pi^3 + \frac{165}{2}\pi)$$

(40)

We use  $3 < \pi < 3.1429$  and  $\sum_{n=2}^{\infty} f(n) \leq \int_{u=2}^{\infty} f(u) du$ .

$$A_2 \leq (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-(\pi-1)u^2} du \leq (8\pi^3 + \frac{165}{2}\pi) \int_{u=2}^{\infty} e^{-2u^2} du$$

(41)

We substitute  $\sqrt{2}u = t$  as follows. We use  $8\pi^3 + \frac{165}{2}\pi = \pi(8\pi^2 + \frac{165}{2}) \leq 3.1429(8 * 10 + 83) \leq 514$ . We use the complementary error function  $erfc(z)$  and  $\frac{\sqrt{\pi}}{2\sqrt{2}} \leq 1$  as follows. (link)

$$\begin{aligned} erfc(z) &= \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt \\ A_2 &\leq \frac{513}{\sqrt{2}} \int_{u=2\sqrt{2}}^{\infty} e^{-t^2} dt \leq \frac{514}{\sqrt{2}} (\frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2})) \leq \frac{514}{\sqrt{2}} \frac{\sqrt{\pi}}{2} * erfc(2\sqrt{2}) \\ A_2 &\leq 514 * erfc(2\sqrt{2}) = 0.0326 \end{aligned}$$

(42)

We can see that  $A_1 + A_2 < 0$  given that  $A_1 \leq -0.4283$  (using Result A).

We use the fact that  $erfc(z) \leq \frac{e^{-z^2}}{z\sqrt{\pi}}$  (link) . Hence  $A_2 \leq 514 \frac{e^{-8}}{2\sqrt{2}\pi} \leq 514 \frac{e^{-8}}{4} \leq 129e^{-8}$ . We can show that  $A_1 + A_2 < 0$  as follows, using Eq. 38.

$$\begin{aligned} A_2 &\leq 129e^{-8}, \quad A_1 \leq -33e^{-\pi} \leq -33e^{-3}, \quad A_2 < |A_1| \\ A_1 + A_2 &\leq -33e^{-3} + 129e^{-8} = e^{-3}(-33 + 129e^{-5}) \\ A_1 + A_2 &\leq e^{-3}(-33 + 129 * 2^{-5}) = e^{-3}(-33 + \frac{129}{32}) < 0 \\ A_1 + A_2 &< 0 \\ (\frac{d^2X(t)}{dt^2})_{t=0} &= A_1 + A_2 < 0 \end{aligned}$$

(43)

We have shown that the second derivative of  $X(t)$ , given by  $X_2(t) = \frac{d^2X(t)}{dt^2} < 0$  for  $t = 0$ . (**Result H**)

4.2.  $\frac{dX(t)}{dt} < 0$  **for**  $0 < t \leq t_{min}$

• In the sections below, **we will show** that, as  $t$  increases from zero,  $\frac{dX(t)}{dt}$  starts from zero and reaches a **negative minimum** value at  $t = t_{min}$  and then starts increasing towards zero, for  $t > t_{min}$ . Thus we will show that  $X(t)$  is a **strictly decreasing** function of  $t$ . We will also show that  $\frac{dX(t)}{dt}$  **does not** become positive for any  $t > 0$ . (example plot)

• We see that  $\frac{d^2X(t)}{dt^2} < 0$  at  $t = 0$  (using Result H) . Hence, as  $t$  increases from zero (point A),  $\frac{dX(t)}{dt}$  reaches a **negative minimum** value at  $t = t_{min}$  (point B) and then starts increasing towards zero, in the neighborhood of  $t > t_{min}$ . We see that  $\frac{d^2X(t)}{dt^2}$  remains negative in  $0 \leq t < t_{min}$  and then reaches a **zero** at  $t = t_{min}$  and then becomes **positive** in the neighborhood of  $t > t_{min}$ . (example plot) Hence  $\frac{dX(t)}{dt} < 0$  for  $0 < t \leq t_{min}$ . (**Result I**)

4.3.  $\frac{dX(t)}{dt} < 0$  **for**  $t \geq t'_{min}$

We note that  $\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t)$  where  $X_{11}(t)$  has partial term corresponding to  $n = 1$  and  $X_{12}(t)$  has partial terms corresponding to  $n > 1$ .

$$\begin{aligned}\frac{dX(t)}{dt} &= X_{11}(t) + X_{12}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}] \\ X_{11}(t) &= e^{-\pi e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 e^{4t} + 15\pi e^{2t} - \frac{15}{2}] \\ X_{12}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t} [-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]\end{aligned}$$

(44)

In the interval  $t > t_{min}$ ,  $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$  reaches a **positive maximum** at  $t = t_{max2}$  (point C) and then starts decreasing to zero, in the neighborhood  $t > t_{max2}$ .  $X_{21}(t)$  has partial term corresponding to  $n = 1$  and  $X_{22}(t)$  has partial terms corresponding to  $n > 1$ . This means that  $\frac{dX(t)}{dt}$  **increases** towards zero, in the neighborhood  $t > t_{min}$ . (example plot)

$$\begin{aligned}\frac{d^2 X(t)}{dt^2} &= X_{21}(t) + X_{22}(t) = \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Z(t), \quad Z(t) = 8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2} \pi n^2 e^{2t} - \frac{75}{4} \\ X_{21}(t) &= e^{-\pi e^{2t}} e^{\frac{5t}{2}} (8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2} \pi e^{2t} - \frac{75}{4}) \\ X_{22}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2} \pi n^2 e^{2t} - \frac{75}{4})\end{aligned}$$

(45)

As  $t$  increases from zero (point A),  $X_{11}(t)$  in Eq. 44 starts from a positive value and **decreases** to a negative value, passing through **zero** at  $t = t_z$ . Given that  $\lim_{t \rightarrow \infty} X_{11}(t) = 0$ , it reaches a **negative minimum** value at  $t = t'_{min}$  (point B') and then starts **increasing** towards zero, for  $t > t'_{min}$  (**Result L**). (example plot) We note that  $X_{11}(t)$  **remains negative** for  $t > t_z$  due to the dominant term  $-4\pi^2 e^{4t}$  in Eq. 44.

This means that the derivative of  $X_{11}(t)$  given by  $X_{21}(t)$  in Eq. 45 is **negative** at  $0 \leq t < t'_{min}$  and  $X_{21}(t)$  reaches a **zero** at  $t = t'_{min}$  (**Result N**) and then becomes **positive** for  $t > t'_{min}$  (using Result L) and falls to zero as  $t \rightarrow \infty$ , given that  $\lim_{t \rightarrow \infty} X_{21}(t) = 0$ . We note that, when  $X_{21}(t)$  in Eq. 45 becomes positive after crossing  $t = t'_{min}$ , it **remains positive** as  $t \rightarrow \infty$  due to the dominant term  $8\pi^3 e^{6t}$  in Eq. 45 (**Result B**).

Hence  $X_{11}(t)$  is a **strictly decreasing** function for  $t < t'_{min}$  and then it starts **increasing** towards zero as  $t \rightarrow \infty$ . We note that  $X_{11}(t)$  **cannot** become positive at some  $t > t'_{min}$ , because if it did become positive again, then it would have to decrease to zero as  $t \rightarrow \infty$ , which would **require**  $X_{21}(t)$  to become negative again, which is **not** the case as shown in Result B.

We will show that  $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) = 0$  **only for**  $t = t_{min}$ , and then it becomes positive and then starts decreasing towards zero, as  $t \rightarrow \infty$ .

In Section 4.5, it is shown that  $X_{22}(t)$  is **strictly decreasing** for  $t > 0$  (using Result E) and hence  $X_{22}(t) > 0$  given that  $\lim_{t \rightarrow \infty} X_{22}(t) = 0$ . (**Result C**).

We note that  $t_{min} < t'_{min}$  given that  $X_{21}(t) > 0$  for  $t > t'_{min}$  (using Result B) and  $X_{22}(t) > 0$  for  $t > t'_{min}$  (using Result C) and hence  $\frac{d^2 X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$  for  $t > t'_{min}$  (**Result D**).

We see that  $X_{12}(t) < 0$  for all  $t > 0$  in Eq. 44 due to the dominant term  $-4\pi^2 n^4 e^{4t}$ . Because  $\frac{dX(t)}{dt} = X_{11}(t) + X_{12}(t)$  is negative at  $t = t'_{min}$ , we see that  $\frac{dX(t)}{dt}$  starts **increasing** from a negative value towards zero. ( using Result D) Hence  $\frac{dX(t)}{dt} < 0$  for  $t \geq t'_{min}$ . (**Result G**)

We note that  $\frac{dX(t)}{dt}$  **cannot** become positive at some  $t > t'_{min}$ , because if it did become positive again, then it would have to decrease to zero as  $t \rightarrow \infty$ , which would **require**  $\frac{d^2X(t)}{dt^2}$  to become negative again, which is **not** the case as shown in Result D.

4.4.  $\frac{dX(t)}{dt} < 0$  **for**  $t_{min} < t < t'_{min}$

We can show that the second derivative  $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$  becomes zero **only once** at  $t = t_{min}$ , in the interval  $0 < t < t'_{min}$ . At  $t = t_{min}$ , we see that  $X_{22}(t_{min}) = X_0 > 0$  and hence  $X_{21}(t_{min}) = -X_0 < 0$ . In the interval  $t_{min} < t < t'_{min}$ , we see that  $X_{22}(t)$  is **strictly decreasing** and remains positive (using Result E in Section 4.5) and decreases from  $X_0$  further towards zero at a **slower** rate, while  $X_{21}(t)$  increases from the negative value  $-X_0$  towards zero **faster**, to make  $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$ .

We see that  $X_{21}(t)$  is **strictly increasing** for  $t_{min} < t < t'_{min}$  (using Result F in Section 4.6) and **reaches zero** at  $t = t'_{min}$  at a **faster** rate than  $X_{22}(t)$ . Hence  $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t) > 0$  in the interval  $t_{min} < t < t'_{min}$  (**Result M**). Hence  $\frac{dX(t)}{dt}$  is **increasing** from a negative value towards zero, in the interval  $t_{min} < t < t'_{min}$ .

We can **rule out** the Case A that  $\frac{dX(t)}{dt} > 0$  somewhere in the interval  $t > t_{min}$  and reaches a maximum at  $t = t_{max}$ , as follows. We see that  $\frac{d^2X(t)}{dt^2} = X_{21}(t) + X_{22}(t)$  becomes zero **only once** at  $t = t_{min}$ , in the interval  $0 < t < t'_{min}$  and is positive for  $t_{min} < t < t'_{min}$  ( using Result M) and **does not** become zero again at  $t = t_{max}$ , which is required for Case A. Hence  $\frac{dX(t)}{dt} < 0$  in the interval  $t_{min} < t < t'_{min}$ . (**Result J**)

We have shown in earlier sections that  $\frac{dX(t)}{dt} < 0$  in the interval  $0 < t \leq t_{min}$  (using Result I) and  $t \geq t'_{min}$  (using Result G). We see that  $\frac{dX(t)}{dt} < 0$  in the interval  $t_{min} < t < t'_{min}$  (using Result J) and hence  $\frac{dX(t)}{dt} < 0$  for all  $t > 0$  and hence  $X(t)$  is **strictly decreasing** for all  $t > 0$ . Hence  $E_0(t) = 2\pi X(t)$  is **strictly decreasing** for all  $t > 0$ .

4.5. *Second derivative given by  $X_{22}$  is a strictly decreasing function for  $t > 0$*

We consider  $X_{22}(t)$  as follows.

$$X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})$$

(46)

We see that  $X_{22}(t)$  in Eq. 46, is **positive** for each  $n = 2, 3, \dots$  and hence for **all**  $t \geq 0$ , as follows. We see that  $X_{22}(t) = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} (\pi^2 n^4 e^{4t} (8\pi n^2 e^{2t} - 56) + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})$ . For  $n = 2, t = 0$ , we see that  $X_{22}(t) = 2^2 e^{-\pi 2^2} (8\pi^3 2^6 - 56\pi^2 2^4 + \frac{165}{2}\pi 2^2 - \frac{75}{4}) = 4e^{-4\pi} (\pi^2 2^4 (32\pi - 56) + 165 * 2\pi - \frac{75}{4}) > 0$ . For  $n \geq 2$  and  $t \geq 0$ ,  $X_{22}(t) > 0$  due to the **dominant term**  $8\pi^3 n^6 e^{6t}$ .

We compute the **derivative** of  $X_{22}(t)$  as follows.

$$\begin{aligned} X_{32}(t) &= \frac{dX_{22}(t)}{dt} = \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t} \\ &\quad + (8\pi^3 n^6 e^{6t} - 56\pi^2 n^4 e^{4t} + \frac{165}{2}\pi n^2 e^{2t} - \frac{75}{4})(\frac{5}{2} - 2\pi n^2 e^{2t})] \\ X_{32}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} [48\pi^3 n^6 e^{6t} - 56 * 4\pi^2 n^4 e^{4t} + 165\pi n^2 e^{2t} \\ &\quad + (20\pi^3 n^6 e^{6t} - 28 * 5\pi^2 n^4 e^{4t} + \frac{165 * 5}{4}\pi n^2 e^{2t} - \frac{75 * 5}{8} - 16\pi^4 n^8 e^{8t} + 56 * 2\pi^3 n^6 e^{6t} - 165\pi^2 n^4 e^{4t} + \frac{75}{2}\pi n^2 e^{2t})] \\ X_{32}(t) &= \sum_{n=2}^{\infty} n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} Y'(t), \quad Y'(t) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75 * 5}{8} \end{aligned}$$



We examine the term  $Y'(t) = -16\pi^4 n^8 e^{8t} + 180\pi^3 n^6 e^{6t} - 529\pi^2 n^4 e^{4t} + 408.75\pi n^2 e^{2t} - \frac{75*5}{8}$  in Eq. 47. We see that, for  $n = 2$  and  $t = 0$ ,  $Y'(t) = -\pi^3 2^6 (16\pi * 4 - 180) - \pi^2 (529\pi * 2^2 - 408.75) - \frac{75*5}{8} < 0$ . We see that, **for all**  $n > 1$  and  $t \geq 0$ ,  $Y'(t) < 0$ . Hence  $X_{32}(t) < 0$  for all  $t \geq 0$ .

Hence  $X_{22}(t)$  is a **strictly decreasing function** for  $t > 0$ . **(Result E)** (example plot)

#### 4.6. Second derivative given by $X_{21}$ is a strictly increasing function for $t_{min} < t < t'_{min}$

We can show that  $X_{21}(t)$  is a **strictly increasing function** for  $t_{min} < t < t'_{min}$ . We take the derivative of  $X_{21}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} (8\pi^3 e^{6t} - 56\pi^2 e^{4t} + \frac{165}{2}\pi e^{2t} - \frac{75}{4})$ , given by  $X_{31}(t)$  and set  $n = 1$  in the summand in Eq. 47 as follows.

$$X_{31}(t) = e^{-\pi e^{2t}} e^{\frac{5t}{2}} Z'(t), \quad Z'(t) = -16\pi^4 e^{8t} + 180\pi^3 e^{6t} - 529\pi^2 e^{4t} + 408.75\pi e^{2t} - \frac{75 * 5}{8}$$

(48)

We see that  $X_{21}(t)$  in Eq. 45 is **negative** at  $0 \leq t < t'_{min}$  (using Result N). In the interval  $t_{min} < t < t'_{min}$ , we consider 3 cases for  $X_{21}(t)$  as follows.

**Case 1:**  $X_{21}(t)$  starts from the negative value at  $t = t_{min}$  and increases to zero at  $t = t'_{min}$ . This is **possible** because we know that  $X_{21}(t)$  in Eq. 45 is **negative** at  $0 \leq t < t'_{min}$  and  $X_{21}(t) = 0$  at  $t = t'_{min}$  (using Result N).

**Case 2:**  $X_{21}(t)$  starts from the negative value at  $t = t_{min}$  and decreases to a more negative value and reaches a negative minimum at  $t = t_c < t'_{min}$  and then starts increasing towards zero. This requires  $X_{31}(t) < 0$  in the interval  $t_{min} < t < t_c$  and  $X_{31}(t) > 0$  in the interval  $t_c < t < t'_{min}$ , which is **NOT** possible because of the dominant term  $-16\pi^4 e^{8t}$  in Eq. 48.

**Case 3:**  $X_{21}(t)$  starts from the negative value at  $t = t_{min}$  and increases to a less negative value and has a point of inflection at  $t = t_d < t'_{min}$  and then starts decreasing towards a more negative value and has a second point of inflection at  $t = t_e < t'_{min}$  and then starts increasing towards zero. This requires  $X_{31}(t) < 0$  in the interval  $t_d < t < t_e$  and  $X_{31}(t) > 0$  in the interval  $t_e < t < t'_{min}$ , which is **NOT** possible because of the dominant term  $-16\pi^4 e^{8t}$  in Eq. 48.

Similarly, we can extend the argument to the case where  $X_{21}(t)$  has many points of inflection in the interval  $t_{min} < t < t'_{min}$ .

Hence we see that only **Case 1** is possible and hence  $X_{21}(t)$  is a **strictly increasing function** for  $t_{min} < t < t'_{min}$ . **(Result F)** (example plot)

#### 4.7. Result $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 4 that  $E_0(t)$  is **strictly decreasing** for  $t > 0$ . In this section, it is shown that  $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ , for  $0 < t < t_{0c}$  and  $t_{2c} = 2t_{0c}$  in Eq. 34 .

Given that  $E_0(t)$  is a **strictly decreasing** function for  $t > 0$  and  $E_0(t)$  is an **even** function of variable  $t$ , and  $t_{2c} = 2t_{0c}$ , we see that, in the interval  $0 < t < t_{0c}$ ,  $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$  ranges from  $E_0(2t_{0c})$  to  $E_0(3t_{0c})$ , which is **less than**  $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$  which ranges from  $E_0(-2t_{0c})$  to  $E_0(-t_{0c})$ . Hence we see that  $E_0(t - t_{2c}) > E_0(t + t_{2c})$ , in the interval  $0 < t < t_{0c}$ . At  $t = 0$ ,  $E_0(t - t_{2c}) = E_0(t + t_{2c})$ . At  $t = t_{0c}$ ,  $E_0(t - t_{2c}) > E_0(t + t_{2c})$  because  $E_0(-t_{0c}) > E_0(3t_{0c})$ .

Hence  $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$  for  $0 < t < t_{0c}$  in Eq. 34 , for  $t_{0c} > 0$ .

5.  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$

It is shown in this section that  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$  in the neighbourhood  $[t_0 - \delta t_0, t_0 + \delta t_0]$  for all  $|t_0| < \infty$ , for **each** fixed value of  $t_2$ .

•  $G_R(\omega) = G_R(\omega, t_2, t_0)$  in Eq. 14 is copied below, which is a **continuous** function of  $\omega$  which is differentiable **at least** once with respect to  $\omega$ . (Eq. 50).

$$G_R(\omega) = G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \quad (49)$$

Given that  $E_0(\tau) > 0$  for  $|\tau| < \infty$  and  $\lim_{\tau \rightarrow \pm\infty} E_0(\tau) = 0$  ( Appendix C.1), we see that  $G_R(\omega) > 0$  at  $\omega = 0$ . Set  $t_0 = 0$  and  $G_R(\omega)$  passes through its **first zero** at  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ . In the rest of this section, we consider the **interval**  $[-\delta t_0, \delta t_0]$  around  $t_0 = 0$ , in  $\omega_z(t_2, t_0)$ . There are 3 possibilities.

**Case 1:**  $G_R(\omega) < 0$  for  $\omega = \omega_z(t_2, 0) + dw$ ,  $G_R(\omega) > 0$  for  $\omega = \omega_z(t_2, 0) - dw$  for infinitesimal  $dw$  (example plot)

In this case, we will show in Section 5.1 that  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$  in the interval  $[-\delta t_0, \delta t_0]$ , in the neighborhood around the first zero crossing at  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ .

**Case 2:**  $G_R(\omega) > 0$  for  $\omega = \omega_z(t_2, 0) + dw$ ,  $G_R(\omega) > 0$  for  $\omega = \omega_z(t_2, 0) - dw$  (example plot)

In this case,  $\frac{dG_R(\omega)}{d\omega} = 0$  at the **same**  $\omega = \omega_z(t_2, 0)$  because  $\frac{dG_R(\omega)}{d\omega} < 0$  at  $\omega = \omega_z(t_2, 0) - dw$  and  $\frac{dG_R(\omega)}{d\omega} > 0$  at  $\omega = \omega_z(t_2, 0) + dw$ .

$$\frac{dG_R(\omega, t_2, t_0)}{d\omega} = -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \quad (50)$$

In this case, we will show in Section 5.2 that  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$  in the interval  $[-\delta t_0, \delta t_0]$ , in the neighborhood around the first zero crossing at  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ .

**Case 3:**  $G_R(\omega) = 0$  for  $\omega = \omega_z(t_2, 0)$  and  $\omega = \omega_z(t_2, 0) + dw$ .

In this case,  $\frac{dG_R(\omega)}{d\omega} = 0$  at the **same**  $\omega = \omega_z(t_2, 0)$  because  $\frac{dG_R(\omega)}{d\omega} < 0$  at  $\omega = \omega_z(t_2, 0) - dw$  and  $\frac{dG_R(\omega)}{d\omega} = 0$  at  $\omega = \omega_z(t_2, 0)$ . The arguments are similar to that of Case 2 presented in Section 5.2 where it is shown that  $\omega_z(t_2, t_0)$  is a continuous function of  $t_0$  in the interval  $[-\delta t_0, \delta t_0]$ , in the neighborhood around the first zero crossing at  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ .

5.1. **Case 1:**  $G_R(\omega) < 0$  **for**  $\omega = \omega_z(t_2, 0) + dw$ ,  $G_R(\omega) > 0$  **for**  $\omega = \omega_z(t_2, 0) - dw$

• Consider the **segment S** in  $G_R(\omega, t_2, t_0)$  in the neighborhood around the first zero crossing where  $\frac{dG_R(\omega, t_2, t_0)}{d\omega} < 0$ . (Segment S is the portion between the green lines in example plot)

• In the **segment S**,  $G_R(\omega, t_2, t_0)$  in Eq. 49 is a **continuous** function of  $\omega$ , for **each** value of  $t_0$  and  $t_2$ . Hence  $G_R(\omega, t_2, t_0 - \delta t_0)$  and  $G_R(\omega, t_2, t_0 + \delta t_0)$  are **continuous** functions of  $\omega$ , which are differentiable **at least** once, and  $G_R(\omega, t_2, t_0 \pm \delta t_0)$  tends to  $G_R(\omega, t_2, t_0)$ , as infinitesimal  $\delta t_0 \rightarrow 0$ .

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_2, t_0 + \delta t_0) &= e^{-2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 - \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 + \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_2, t_0 - \delta t_0) &= e^{-2\sigma(t_0 - \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 + \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma(t_0 - \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 - \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_2, t_0 + \delta t_0) &= G_R(\omega, t_2, t_0) \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_2, t_0 - \delta t_0) &= G_R(\omega, t_2, t_0)
\end{aligned}$$

(51)

• In the **segment S**,  $G_R(\omega, t_2, t_0)$  in Eq. 51 is a **continuous** function of  $\omega$ , for **each** value of  $t_0$  and  $t_2$  and  $\frac{dG_R(\omega, t_2, t_0)}{d\omega} < 0$  in the neighborhood around the **first zero crossing**. If we **fix** the X-coordinate  $\omega$  and  $t_2$ ,  $G_R(\omega, t_2, t_0)$  is a **continuous** function of  $t_0$ , for **each** fixed value of  $\omega$ . Hence, for **each** fixed value of  $\omega$ , as we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_2, t_0 - \delta t_0)$  and  $G_R(\omega, t_2, t_0 + \delta t_0)$  in Eq. 51, move towards  $G_R(\omega, t_2, t_0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Every point in the segment S, moves continuously, as we change  $t_0$  by an infinitesimal  $\delta t_0$ .

This also applies to the first **zero crossing** in  $G_R(\omega, t_2, t_0)$  in the segment S, which corresponds to  $\omega_z(t_2, t_0) = \omega_z(t_2, 0)$  at  $t_0 = 0$  where  $G_R(\omega, t_2, t_0) = 0$  in Eq. 51. The **zero crossing** moves **continuously**, as we change  $t_0$  by an infinitesimal  $\delta t_0$ . This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to  $G_R(\omega, t_2, t_0)$  with zero crossing at point  $P_0$ , **Green** plot corresponds to  $G_R(\omega, t_2, t_0 + \delta t_0)$  with zero crossing at point  $P_{11}$  and **Blue** plot corresponds to  $G_R(\omega, t_2, t_0 - \delta t_0)$  with zero crossing at point  $P_{21}$ .

We **define** the **point**  $P_{12}$  in  $G_R(\omega, t_2, t_0 + \delta t_0)$  as the point which has the **fixed X-coordinate**  $\omega = \omega_z(t_2, 0)$ . We **define** the **point**  $P_{22}$  in  $G_R(\omega, t_2, t_0 - \delta t_0)$  as the point which has the **fixed X-coordinate**  $\omega = \omega_z(t_2, 0)$ .

We **define** the **point**  $P_{11}$  in  $G_R(\omega, t_2, t_0 + \delta t_0)$  as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point**  $P_{21}$  in  $G_R(\omega, t_2, t_0 - \delta t_0)$  as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_2, t_0 + \delta t_0)$  in Eq. 51 moves towards  $G_R(\omega, t_2, t_0)$  in a **continuous** manner, for **each fixed** value of  $\omega$  and  $t_2$ , including the zero crossing point, as follows. The **point**  $P_{12}$  in  $G_R(\omega, t_2, t_0 + \delta t_0)$  which corresponds to the **fixed X-coordinate**  $\omega = \omega_z(t_2, 0)$ , moves towards corresponding point  $P_0$  in  $G_R(\omega, t_2, t_0)$ , for the **same**  $\omega = \omega_z(t_2, 0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Given that  $P_0$  is a **zero crossing point** in  $G_R(\omega, t_2, t_0)$ , this is equivalent to the **Zero crossing point**  $P_{11}$  in  $G_R(\omega, t_2, t_0 + \delta t_0)$  moving towards corresponding **zero crossing point**  $P_0$  in  $G_R(\omega, t_2, t_0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ .

Similarly, as we change  $t_0$  by an infinitesimal  $\delta t_0$ ,  $G_R(\omega, t_2, t_0 - \delta t_0)$  in Eq. 51 moves towards  $G_R(\omega, t_2, t_0)$  in a **continuous** manner as follows. The **point**  $P_{22}$  in  $G_R(\omega, t_2, t_0 - \delta t_0)$  which corresponds to the **fixed X-coordinate**  $\omega = \omega_z(t_2, 0)$ , moves towards corresponding point  $P_0$  in  $G_R(\omega, t_2, t_0)$ , for the **same**  $\omega = \omega_z(t_2, 0)$  in a **continuous** manner, as  $\delta t_0 \rightarrow 0$ . Given that  $P_0$  is a **zero crossing point** in  $G_R(\omega, t_2, t_0)$ , this is equivalent to the **Zero crossing point**  $P_{21}$  in  $G_R(\omega, t_2, t_0 - \delta t_0)$  moving towards corresponding **zero crossing point**  $P_0$  in  $G_R(\omega, t_2, t_0)$  in a **contin-**

599 **uous** manner, as  $\delta t_0 \rightarrow 0$ .

600

601 • Hence in the **segment** S,  $\omega_z(t_2, t_0)$  is a **continuous** function of  $t_0$  in the neighborhood  $[-\delta t_0, \delta t_0]$  around the  
602 first zero crossing at  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$  at  $t_0 = 0$ .

$$\begin{aligned} G_R(\omega_z(t_2, t_0), t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \\ G_R(\omega_z(t_2, t_0 + \delta t_0), t_2, t_0 + \delta t_0) &= \\ e^{-2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 - \delta t_0, t_2)] \cos(\omega_z(t_2, t_0 + \delta t_0)\tau) d\tau \\ &\quad + e^{2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 + \delta t_0, t_2)] \cos(\omega_z(t_2, t_0 + \delta t_0)\tau) d\tau = 0 \end{aligned}$$

603

(52)

604 5.2. **Case 2:**  $G_R(\omega) > 0$  **for**  $\omega = \omega_z(t_2, 0) + dw$ ,  $G_R(\omega) > 0$  **for**  $\omega = \omega_z(t_2, 0) - dw$

605

606 • In this case,  $\frac{dG_R(\omega)}{d\omega} = 0$  at the **same**  $\omega = \omega_z(t_2, t_0)$  because  $\frac{dG_R(\omega)}{d\omega} < 0$  at  $\omega = \omega_z(t_2, t_0) - dw$  and  $\frac{dG_R(\omega)}{d\omega} > 0$   
607 at  $\omega = \omega_z(t_2, t_0) + dw$ .

608

609 • Consider the **segment** S' in  $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$  in the neighborhood around the first zero crossing where  $\frac{d^2G_R(\omega, t_2, t_0)}{d\omega^2} > 0$ .  
610 (Segment S' is the portion between the green lines in example plot) In this segment S',  $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$  is a **continuous**  
611 function of  $\omega$  which is differentiable **at least** once. (Section ??)

612

613 • In the **segment** S',  $\frac{dG_R(\omega, t_2, t_0)}{d\omega} = 0$  at the **same**  $\omega = \omega_z(t_2, t_0)$ . The arguments in Section 5.1 can be applied  
614 here, with  $G_R(\omega, t_2, t_0)$  replaced by  $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$ .

615

616 Hence  $\omega_z(t_2, t_0)$  is a **continuous** function of  $t_0$  in the neighborhood  $[-\delta t_0, \delta t_0]$  around the first zero crossing at  
617  $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$  at  $t_0 = 0$  in the **segment** S'.

618

619 We can use similar arguments and see that  $\omega_z(t_2, t_0)$  is a **continuous** function of  $t_0$  in the neighbourhood  $[t_0 -$   
620  $\delta t_0, t_0 + \delta t_0]$  for all  $|t_0| < \infty$ , for **each** fixed value of  $t_2$ .

### 621 5.3. Further Points

622

623 • Using arguments in previous subsections, we see that  $\omega_z(t_2, t_0)$  is a **continuous** function of  $t_2$  in the neighbour-  
624 hood  $[t_2 - \delta t_2, t_2 + \delta t_2]$  for all  $|t_2| < \infty$ , for **each** fixed value of  $t_0$ .

625

626 • We **set**  $t_2 = Kt_0$  for even positive integer  $K$ . Using arguments in previous subsections, we see that  $\omega_z(Kt_0, t_0)$   
627 is a **continuous** function of  $t_0$  in the neighbourhood  $[t_0 - \delta t_0, t_0 + \delta t_0]$  for all  $|t_0| < \infty$ .

## 628 6. Hurwitz Zeta Function and related functions

629

630 We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and  
631 **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of  $s = \frac{1}{2}$ .  
632 The new method requires the **symmetry** relation  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when eval-  
633 uated at the critical line  $s = \frac{1}{2} + i\omega$ . This means  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  and  $E_0(t) = E_0(-t)$  where

$E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by  $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$  satisfies a symmetry relation similar to  $\xi(s) = \xi(1-s)$  where  $\xi(s)$  is an entire function, for  $a \neq 1$  and hence the condition  $E_0(t) = E_0(-t)$  is **not** known to be satisfied<sup>[6]</sup>. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation  $\xi(s, \chi) = \epsilon(\chi) \xi(1-s, \bar{\chi})$  <sup>[7]</sup> which does **not** translate to  $E_0(t) = E_0(-t)$  required by the new method and hence this proof is **not** applicable to them.

We know that  $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$  diverges for real part of  $s \leq 1$ . Hence we derive a convergent and entire function  $\xi(s)$  using the well known theorem  $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$ , where  $x > 0$  is real and then derive  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  ( Appendix D). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to  $F(x)$  exists, which enables derivation of a convergent and entire function  $\xi(s)$  and results in  $E_0(t)$  as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip**  $0 \leq |\sigma| < \frac{1}{2}$ . This proof requires both  $E_p(t)$  and  $E_{p\omega}(\omega)$  to be Fourier transformable where  $E_p(t) = E_0(t) e^{-\sigma t}$  is a real analytic function and uses the fact that  $E_0(t)$  is an **even** function which is **strictly decreasing** function for  $t \geq \frac{1}{8}$ . These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

## References

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- [4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152
- [5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)
- [6] Mathworld article on Hurwitz Zeta functions. (Link)
- [7] Wikipedia article on Dirichlet L-functions. (Link)

## Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function  $\xi(s)$  evaluated at  $s = \frac{1}{2} + i\omega$  given by  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ . Its inverse Fourier Transform is given by  $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$  (link). This is

re-derived in Appendix D.

We will show in this section that the inverse Fourier Transform of the function  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ , is given by  $E_p(t) = E_0(t)e^{-\sigma t}$  where  $0 \leq |\sigma| < \frac{1}{2}$  is real.

$$\begin{aligned}\xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega\end{aligned}$$

(A.1)

We substitute  $\omega' = \omega - i\sigma$  in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega'$$

(A.2)

We can evaluate the above integral in the complex plane using contour integration, substituting  $\omega' = z = x + iy$  and we use a rectangular contour comprised of  $C_1$  along the line  $x = [-\infty, \infty]$ ,  $C_2$  along the line  $y = [\infty, \infty - i\sigma]$ ,  $C_3$  along the line  $x = [\infty - i\sigma, -\infty - i\sigma]$  and then  $C_4$  along the line  $y = [-\infty - i\sigma, -\infty]$ . We can see that  $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$  has no singularities in the region bounded by the contour because  $\xi(\frac{1}{2} + iz)$  is an entire function in the  $Z$ -plane.

In **Appendix C.1**, we show that  $\int_{-\infty}^{\infty} |E_p(t)| dt$  is finite and  $E_p(t) = E_0(t)e^{-\sigma t}$  is an absolutely integrable function, for  $0 \leq |\sigma| < \frac{1}{2}$ .

We use the fact that  $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$ , **goes to zero** as  $x \rightarrow \pm\infty$  when  $-\sigma \leq y \leq 0$ , as per Riemann-Lebesgue Lemma (link), because  $E_0(t)e^{yt}$  is a absolutely integrable function in the interval  $-\infty \leq t \leq \infty$ . Hence the integral in Eq. A.2 **vanishes** along the contours  $C_2$  and  $C_4$ . Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned}E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}\end{aligned}$$

(A.3)

Thus we have arrived at the desired result  $E_p(t) = E_0(t)e^{-\sigma t}$ .

## Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

### Appendix B.1. Convolution Theorem: Multiplication of $g(t)$ and $h(t)$ corresponds to convolution in Fourier transform domain

We start with the Fourier transform equation  $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$  where  $f(t) = g(t)h(t)$  and show that  $F(\omega) = \frac{1}{2\pi} [G(\omega) * H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega'$  obtained by the **convolution** of the functions  $G(\omega)$  and  $H(\omega)$  which correspond to the Fourier transforms of  $g(t)$  and  $h(t)$  respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t} dt \quad (\text{B.1})$$

We use the inverse Fourier transform equation  $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')e^{i\omega' t} d\omega'$  and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} G(\omega')e^{i\omega' t} d\omega' \right] h(t)e^{-i\omega t} dt \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[ \int_{-\infty}^{\infty} e^{i\omega' t} h(t)e^{-i\omega t} dt \right] d\omega' \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[ \int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t} dt \right] d\omega' \end{aligned} \quad (\text{B.2})$$

We substitute  $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t} dt = H(\omega - \omega')$  in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t} dt \quad (\text{B.3})$$

### Appendix B.2. *Fourier transform of Real $g(t)$*

In this section, we show that the Fourier transform of a real function  $g(t)$ , given by  $G(\omega) = G_R(\omega) + iG_I(\omega)$  has the properties given by  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t) \cos(\omega t) dt = G_R(-\omega) \\ G_I(\omega) &= - \int_{-\infty}^{\infty} g(t) \sin(\omega t) dt = -G_I(-\omega) \end{aligned} \quad (\text{B.4})$$

### Appendix B.3. *Even part of $g(t)$ corresponds to real part of Fourier transform $G(\omega)$*

In this section, we show that the **even part** of real function  $g(t)$ , given by  $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ , corresponds to **real part** of its Fourier transform  $G(\omega)$ . We use the fact that  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$  for a real function  $g(t)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t} dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \quad (\text{B.5})$$

#### Appendix B.4. Odd part of $g(t)$ corresponds to imaginary part of Fourier transform $G(\omega)$

In this section, we show that the **odd part** of real function  $g(t)$ , given by  $g_{\text{odd}}(t) = \frac{1}{2}[g(t) - g(-t)]$ , corresponds to **imaginary part** of its Fourier transform  $G(\omega)$ . We use the fact that  $G_R(-\omega) = G_R(\omega)$  and  $G_I(-\omega) = -G_I(\omega)$  for a real function  $g(t)$ .

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{odd}}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{B.6}$$

### Appendix C. Properties of Fourier Transforms Part 2

#### Appendix C.1. $E_p(t), h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite.

The inverse Fourier Transform of the function  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$  is given by  $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$ . We see that  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$  for all  $0 \leq t < \infty$ . Given that  $E_0(t) = E_0(-t)$ , we see that  $E_0(t) > 0$  and  $E_p(t) = E_0(t)e^{-\sigma t} > 0$  for all  $-\infty < t < \infty$ .

As  $t \rightarrow \infty$ ,  $E_p(t)$  goes to zero, due to the term  $e^{-\pi n^2 e^{2t}}$ . As  $t \rightarrow -\infty$ ,  $E_p(t)$  goes to zero, because for every value of  $n$ , the term  $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$  goes to zero, for  $0 \leq |\sigma| < \frac{1}{2}$ . Hence  $E_p(t) = E_0(t)e^{-\sigma t} = 0$  at  $t \rightarrow \pm\infty$  and we showed that  $E_p(t) > 0$  for all  $-\infty < t < \infty$ . Hence  $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$ , evaluated at  $\omega = 0$  **cannot** be zero. Hence  $E_{p\omega}(\omega)$  **does not have a zero** at  $\omega = 0$  and hence  $\omega_0 \neq 0$ .

Given that  $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$  is an entire function in the whole of s-plane, it is finite for  $|\omega| \leq \infty$  and also for  $\omega = 0$ . Hence  $\int_{-\infty}^{\infty} E_p(t)dt$  is finite. We see that  $E_p(t) \geq 0$  for all  $|t| \leq \infty$ . Hence we can write  $\int_{-\infty}^{\infty} |E_p(t)|dt$  is finite and  $E_p(t)$  is an absolutely **integrable function** and its Fourier transform  $E_{p\omega}(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

Let us consider a new function  $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $0 < \sigma < \frac{1}{2}$ . We can see that  $g(t)h(t) = f(t)$  where  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ .

We can see that  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  is an absolutely **integrable function** because  $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}]_{\omega=0} = \frac{2}{\sigma}$ , is finite for  $0 < \sigma < \frac{1}{2}$  and its Fourier transform  $H(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that  $E_0(t)$  and  $E_0(t)e^{-2\sigma t}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$  and hence are absolutely **integrable** functions and the integrals  $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$  and  $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$ . Hence  $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$  is an absolutely **integrable function** and  $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$  is finite and its Fourier transform  $G(\omega)$  goes to zero as  $\omega \rightarrow \pm\infty$ , as per Riemann Lebesgue Lemma (link).

#### Appendix C.2. Convolution integral convergence

Let us consider  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$  whose **first derivative is discontinuous** at  $t = 0$ . The second derivative of  $h(t)$  given by  $h_2(t)$  has a Dirac delta function  $A_0\delta(t)$  where  $A_0 = -2\sigma$  and its Fourier transform  $H_2(\omega)$  has a constant term  $A_0$ , corresponding to the Dirac delta function.



This means  $h(t)$  is obtained by integrating  $h_2(t)$  twice and its Fourier transform  $H(\omega)$  has a term  $-\frac{A_0}{\omega^2}$  (link) and has a **fall off rate** of  $\frac{1}{\omega^2}$  as  $|\omega| \rightarrow \infty$  and  $\int_{-\infty}^{\infty} H(\omega) d\omega$  converges.

We see that  $E_p(t) = E_0(t)e^{-\sigma t}$  where  $[\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ .

Let us consider a new function  $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$  where  $g(t)$  is a real function of variable  $t$  and  $u(t)$  is Heaviside unit step function and  $0 < \sigma < \frac{1}{2}$ . We can see that  $g(t)h(t) = E_p(t)$  where  $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ .

We can see that  $G(\omega), H(\omega)$  have **fall-off rate** of  $\frac{1}{\omega^2}$  as  $|\omega| \rightarrow \infty$  because the **first derivatives** of  $g(t), h(t)$  are **discontinuous** at  $t = 0$ . Also,  $h(t), g(t)$  are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for  $|\omega| \leq \infty$ .

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') H(\omega - \omega') d\omega' = \frac{1}{2\pi} [G(\omega) * H(\omega)] \quad (C.1)$$

### Appendix C.3. **Fall off rate of Fourier Transform of functions**

Let us consider a real Fourier transformable function  $P(t) = P_+(t)u(t) + P_-(t)u(-t)$  whose  $(N-1)^{th}$  **derivative** is **discontinuous** at  $t = 0$ . The  $(N)^{th}$  derivative of  $P(t)$  given by  $P_N(t)$  has a Dirac delta function  $A_0\delta(t)$  where  $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$  and its Fourier transform  $P_N(\omega)$  has a constant term  $A_0$ , corresponding to the Dirac delta function.

This means  $P(t)$  is obtained by integrating  $P_N(t)$ ,  $N$  times and its Fourier transform  $P(\omega)$  has a term  $\frac{A_0}{(i\omega)^N}$  (link) and has a **fall off rate** of  $\frac{1}{\omega^N}$  as  $|\omega| \rightarrow \infty$ .

We have shown that if the  $(N-1)^{th}$  **derivative** of the function  $P(t)$  is **discontinuous** at  $t = 0$  then its Fourier transform  $P(\omega)$  has a **fall-off rate** of  $\frac{1}{\omega^N}$  as  $|\omega| \rightarrow \infty$ .

### Appendix C.4. **Payley-Weiner theorem and Exponential Fall off rate of analytic functions.**

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions  $E_0(t), E_p(t)$  and  $x(t) = E_0(t)e^{-2\sigma t}$  and  $\frac{d^{2r}x(t)}{dt^{2r}}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$  for  $0 < \sigma < \frac{1}{2}$ .

We know that the order of Riemann's Xi function  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$  is given by  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  where  $A$  is a constant<sup>[3]</sup> (link). Hence both  $E_{0\omega}(\omega)$  and  $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$  have **exponential fall-off** rate  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  as  $|\omega| \rightarrow \infty$  and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function  $\xi(s)$ .

Given that  $\xi(s)$  is an entire function in the  $s$ -plane, we see that  $E_{0\omega}(\omega)$  and  $E_{p\omega}(\omega)$  are **analytic** functions which are infinitely differentiable which produce no discontinuities for all  $|\omega| \leq \infty$  and  $0 < \sigma < \frac{1}{2}$ . Hence their respective **inverse Fourier transforms**  $E_0(t), E_p(t)$  have fall-off rates faster than  $\frac{1}{t^M}$  as  $M \rightarrow \infty$ , as  $|t| \rightarrow \infty$  ( Appendix C.3) and hence it should have **exponential fall-off** rates as  $|t| \rightarrow \infty$ .

We can use similar arguments to show that  $x(t) = E_0(t)e^{-2\sigma t}$  and  $\frac{d^{2r}x(t)}{dt^{2r}}$  have fall-off rates **at least**  $\frac{1}{t^2}$  as  $|t| \rightarrow \infty$ , because their Fourier transforms are **analytic** functions for all  $|\omega| \leq \infty$  with **exponential fall-off** rate  $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$  as  $|\omega| \rightarrow \infty$ .

## Appendix D. **Derivation of entire function $\xi(s)$**

In this section, we will re-derive Riemann's Xi function  $\xi(s)$  and the inverse Fourier Transform of  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$  and show the result  $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ .

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below<sup>[4]</sup> (link).  
 We start with the gamma function  $\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy$  and substitute  $y = \pi n^2 x$  and derive as follows.

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma\left(\frac{s}{2}\right)(\pi n^2)^{-\frac{s}{2}} &= \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\end{aligned}\tag{D.1}$$

For real part of  $s$  greater than 1, we can do a summation of both sides of above equation for all positive integers  $n$  and obtain as follows. We note that  $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ .

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\tag{D.2}$$

For real part of  $s$  ( $\sigma'$ ) greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that  $w(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$  and

$$\sum_{n=1}^\infty \int_0^\infty |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma\left(\frac{\sigma'}{2}\right)\pi^{-\frac{\sigma'}{2}}\zeta(\sigma').$$

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} w(x) dx\tag{D.3}$$

For real part of  $s$  less than or equal to 1,  $\zeta(s)$  **diverges**. Hence we do the following. In Eq. D.3, first we consider real part of  $s$  greater than 1 and we divide the range of integration into two parts:  $(0, 1]$  and  $[1, \infty)$  and make the substitution  $x \rightarrow \frac{1}{x}$  in the first interval  $(0, 1]$ . We use **the well known theorem**  $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $x > 0$  is real.<sup>[4]</sup>

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_1^\infty \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1}{2} dx\tag{D.4}$$

Hence we can simplify Eq. D.4 as follows.

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_1^\infty x^{\frac{-(s+1)}{2}} w(x) dx\tag{D.5}$$

We multiply above equation by  $\frac{1}{2}s(s-1)$  and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}]\tag{D.6}$$

We see that  $\xi(s)$  is an entire function, for all values of  $Re[s]$  in the complex plane and hence we get an analytic continuation of  $\xi(s)$  over the entire complex plane. We see that  $\xi(s) = \xi(1-s)$  <sup>[4]</sup>.

### Appendix D.1. Derivation of $E_p(t)$ and $E_0(t)$

Given that  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ , we substitute  $x = e^{2t}$ ,  $\frac{dx}{x} = 2dt$  in Eq. D.6 and evaluate at  $s = \frac{1}{2} + \sigma + i\omega$  as follows.

$$\xi\left(\frac{1}{2} + \sigma + i\omega\right) = \frac{1}{2} \left[ 1 + 2\left(\frac{1}{2} + \sigma + i\omega\right) \left(-\frac{1}{2} + \sigma + i\omega\right) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} \left( e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} \right) dt \right] \quad (D.7)$$

We can substitute  $t = -t$  in the first term in above integral and simplify above equation as follows.

$$\begin{aligned} \xi\left(\frac{1}{2} + \sigma + i\omega\right) &= \frac{1}{2} + \left(-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)\right) \left[ \int_{-\infty}^0 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right. \\ &\quad \left. + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right] \end{aligned} \quad (D.8)$$

We can write this as follows.

$$\xi\left(\frac{1}{2} + \sigma + i\omega\right) = \frac{1}{2} + \left(-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)\right) \int_{-\infty}^{\infty} \left[ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t} e^{-i\omega t} dt \quad (D.9)$$

We define  $A(t) = \left[ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t}$  and get the **inverse Fourier transform** of  $\xi\left(\frac{1}{2} + \sigma + i\omega\right)$  in above equation given by  $E_p(t)$  as follows. We use dirac delta function  $\delta(t)$ .

$$\begin{aligned} E_p(t) &= \frac{1}{2} \delta(t) + \left(-\frac{1}{4} + \sigma^2\right) A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2 A(t)}{dt^2} \\ A(t) &= \left[ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[ -\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t} \right] u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[ \frac{1}{2} - \sigma - 2\pi n^2 e^{2t} \right] u(t) \\ \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[ -4\pi n^2 e^{-2t} + \left(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}\right)^2 \right] u(-t) \\ &\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[ -4\pi n^2 e^{2t} + \left(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}\right)^2 \right] u(t) + \delta(t) \left[ \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) \right] \end{aligned} \quad (D.10)$$

We can simplify above equation as follows.

$$\begin{aligned} \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[ \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma \pi n^2 e^{-2t} \right] u(-t) \\ &\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[ \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma \pi n^2 e^{2t} \right] u(t) + \delta(t) \left[ \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) \right] \end{aligned} \quad (D.11)$$

845 We use the fact that  $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and  $x > 0$  is real<sup>[4]</sup>, and we  
846 take the first derivative of  $F(x)$  and evaluate it at  $x = 1$ . We see that  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$  ( Appendix D.2)  
847 and hence **dirac delta terms cancel each other** in equation below.

$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}) \\
&\quad + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)
\end{aligned}
\tag{D.12}$$

849 We can simplify above equation as follows.

$$\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned}
\tag{D.13}$$

851 We use the fact that  $E_0(t) = E_0(-t)$  because  $\xi(s) = \xi(1-s)$  and hence  $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$  when evaluated at  
852 the critical line  $s = \frac{1}{2} + i\omega$ . This means  $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$  and  $E_0(t) = E_0(-t)$  and we arrive at the  
853 desired result for  $E_p(t)$  as follows.

$$\begin{aligned}
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) &= E_0(t)e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{D.14}$$

855 *Appendix D.2. Derivation of*  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$   
856

857 In this section, we derive  $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$ . We use the fact that  $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ ,  
858 where  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and  $x > 0$  is real<sup>[4]</sup>, and we take the first derivative of  $F(x)$  and evaluate it at  $x = 1$ .

$$\begin{aligned}
F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2 \frac{1}{x}} (\frac{1}{x^2}) + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) (\frac{-1}{2}) \frac{1}{x^{\frac{3}{2}}}
\end{aligned}
\tag{D.15}$$

We evaluate the above equation at  $x = 1$  and we simplify as follows.

$$\begin{aligned} \left[\frac{dF(x)}{dx}\right]_{x=1} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2} = \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2} + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}) \left(\frac{-1}{2}\right) \\ &\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2} \end{aligned}$$

(D.16)

### Appendix D.3. Derivation of Result 1

In this section, we derive the result  $\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}$  for  $|t| < \infty$ . We use the well known theorem  $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$ , where  $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$  and  $x > 0$  is real<sup>[4]</sup>. We substitute  $x = e^{2t}$  as follows.

$$\begin{aligned} F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\ F(e^{2t}) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} = \frac{1}{e^t}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}) \end{aligned}$$

(D.17)

We multiply above equation by  $\frac{1}{2}e^{\frac{t}{2}}$  and derive the result as follows.

$$\begin{aligned} \frac{1}{2}e^{\frac{t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \frac{1}{2}e^{\frac{-t}{2}} + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} \\ \frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} &= \frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} \end{aligned}$$

(D.18)