On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function. [4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2 to Section 6, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 7, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega) = \Xi(\omega) = E_{0\omega}(\omega)$, where ω is real. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows (link).[3] (Titchmarsh pp254-255) We take the term $e^{\frac{t}{2}}$ out of the bracket and rearrange the terms as follows.

$$E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2n^4\pi^2 e^{\frac{9t}{2}} - 3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2 e^{2t}} = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t) = E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ (link) and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at $s = \frac{1}{2}+i\omega$.(Details in Appendix C.8)

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t) = E_0(t)e^{-\sigma t} = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function for real t, given that the sum and product of exponential functions are analytic for real t and hence infinitely differentiable for real t.

1.2. *Step 2: On the zeros of a related function* $G(\omega, t_2, t_0)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$, where $f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ and $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p'(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0}E_p'(t - t_0, t_2)$ and $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2)$ and t_0, t_2 are real and $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_2,t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$, for every value of t_0, for each nonzero value of t_2, where $G_R(\omega,t_2,t_0)$ crosses the zero line to the opposite sign, to satisfy Statement 1, where $\omega_z(t_2,t_0)$ is real and finite.

1.3. *Step 3: On the zeros of the function* $G_R(\omega,t_2,t_0)$

In Section 2.3, we compute the Fourier transform of the function $g(t,t_2,t_0)$ and compute its real part given by $G_R(\omega,t_2,t_0)$ and we can write as follows.

$$\begin{aligned} G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0} [E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\ +e^{2\sigma t_0}\int_{-\infty}^{0} [E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \end{aligned} \tag{3}$$

We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for every value of t_0, for **each non-zero value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2,t_0) \neq \omega_0$. Hence we can see that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = 0$.

1.4. *Step 4: Zero Crossing function* $\omega_z(t_2,t_0)$ *is an even function of variable* t_0

In Section 2.4, we show the result in Eq. 4 and that $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$. It is shown that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0$ and that $P_{odd}(t_2,t_0)$ is an **odd** function of t_0, for each non-zero value of t_2 as follows.

$$\begin{aligned} P_{odd}(t_2,t_0) = [\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau \\ +\sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau] \\ +e^{2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)\tau)d\tau + \sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\sin(\omega_z(t_2,t_0)\tau)d\tau] \end{aligned} \tag{4}$$

1.5. *Step 5: Final Step*

In Section 4, it is shown that $\omega_z(t_2,t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2,t_0)$ in Eq. 4 and show that this leads to the result in Eq. 5. We use $E_0'(t,t_2) = E_0(t-t_2) - E_0(t+t_2)$ and $E_{0n}'(t,t_2) = E_0'(-t,t_2)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0 \tag{5}$$

We show that **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$, where $t_{0c} > 0$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We show this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we produce a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 \leq |\sigma| < \frac{1}{2}$. [3] (Titchmarsh pp30-31). This is shown in detail in first two paragraphs in Appendix C.1.

2.1. *New function* $g(t, t_2, t_0)$

Let us consider the function $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0'(t, t_2)e^{-\sigma t}$, where t_2 is non-zero and real, and $E_0'(t, t_2) = E_0(t-t_2) - E_0(t+t_2)$ (**Definition 1**). Its Fourier transform is given by $E_{p\omega}'(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$, using Statement 1 and linearity and time shift properties of the Fourier transform (link). (**Result 2.1.1**).

Let us consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p'(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0}E_p'(t - t_0, t_2)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega, t_2, t_0) = E_{p\omega}'(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$, using Result 2.1.1. (**Result 2.1.2**)

Let us consider a new function $g(t, t_2, t_0) = g_-(t, t_2, t_0)u(-t) + g_+(t, t_2, t_0)u(t)$ where $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}$ and $g_+(t, t_2, t_0) = f(t, t_2, t_0)e^{\sigma t}$. We can see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can write the above equations as follows.

$$
\begin{aligned}
E_p^{'}(t,t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = (E_0(t-t_2)-E_0(t+t_2))e^{-\sigma t} = E_0^{'}(t,t_2)e^{-\sigma t} \\
f_1(t,t_2,t_0) = e^{\sigma t_0}E_p^{'}(t+t_0,t_2) \\
f_2(t,t_2,t_0) = f_1(t,t_2,-t_0) = e^{-\sigma t_0}E_p^{'}(t-t_0,t_2) \\
f(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0) + e^{2\sigma t_0}f_2(t,t_2,t_0) = e^{-\sigma t_0}E_p^{'}(t+t_0,t_2) + e^{\sigma t_0}E_p^{'}(t-t_0,t_2) \\
g(t,t_2,t_0) = [f(t,t_2,t_0)e^{-\sigma t}]u(-t) + [f(t,t_2,t_0)e^{\sigma t}]u(t) \\
g(t,t_2,t_0)h(t) = f(t,t_2,t_0), \quad h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]
\end{aligned}
\tag{6}
$$

We can show that $E_p(t), E_p^{'}(t,t_2), h(t)$ are absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}^{'}(\omega,t_2), H(\omega)$ are finite for real ω and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). We can show that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions. These results are shown in Appendix C.1.

In Section 2.3 and Section 2.4, it is shown that $g(t,t_2,t_0)$ is a Fourier transformable function and its Fourier transform given by $G(\omega,t_2,t_0) = e^{-2\sigma t_0}G_1(\omega,t_2,t_0) + e^{2\sigma t_0}G_1(\omega,t_2,-t_0)$ converges. (Eq. 14 and Eq. 17)

If we take the Fourier transform of the equation $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, using Result 2.1.2, we get $\frac{1}{2\pi}[G(\omega,t_2,t_0) * H(\omega)] = F(\omega,t_2,t_0) = E_{p\omega}^{'}(\omega,t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega,t_2,t_0) + iF_I(\omega,t_2,t_0)$ as per **convolution theorem** (link), where $*$ denotes convolution operation given by $F(\omega,t_2,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega',t_2,t_0)H(\omega-\omega')d\omega'$.

We see that $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ (link). $G(\omega,t_2,t_0) = G_R(\omega,t_2,t_0) + iG_I(\omega,t_2,t_0)$ is the Fourier transform of the function $g(t,t_2,t_0)$. We can write $g(t,t_2,t_0) = g_{even}(t,t_2,t_0) + g_{odd}(t,t_2,t_0)$ where $g_{even}(t,t_2,t_0)$ is an even function and $g_{odd}(t,t_2,t_0)$ is an odd function of variable t.

If Statement 1 is true, then we require the Fourier transform of the function $f(t,t_2,t_0)$ given by $F(\omega,t_2,t_0)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0, for each non-zero value of t_2, using Result 2.1.2. This implies that the **real** part of the Fourier transform of the **even function** $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_2,t_0)$(Appendix D.2) must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$ where $\omega_z(t_2,t_0)$ is real and finite, where $G_R(\omega,t_2,t_0)$ crosses the zero line to the opposite sign, explained below. We note that $\omega_z(t_2,t_0)$ can be different from ω_0 in general.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega,t_2,t_0)$ does not have at least one zero for some $\omega = \omega_z(t_2,t_0) \neq 0$, where $G_R(\omega,t_2,t_0)$ crosses the zero line to the opposite sign, **then** the **real part** of $F(\omega,t_2,t_0)$ given by $F_R(\omega,t_2,t_0) = \frac{1}{2\pi}[G_R(\omega,t_2,t_0) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega,t_2,t_0)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against Result 2.1.2 and **Statement 1**. This is shown in detail in Lemma 1.

The proof for Lemma 1 below is shown for **a fixed value** of $t_0 = t_{0f}$ and $t_2 = t_{2f}$, in the interval $|t_0| < \infty$ and $0 < |t_2| < \infty$ (**Interval A**), where $G_R(\omega, t_2, t_0)$ is a function of ω **only**. The proof continues to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

Lemma 1: Let $t_0, t_2 \in \Re$ be fixed values and $t_2 \neq 0$ and $\xi(\frac{1}{2} + \sigma + i\omega_0) = E_{p\omega}(\omega_0) = 0$ using Statement 1. Then the Fourier transform of the **even function** $g_{even}(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign and $\omega_z(t_2, t_0)$ is real.

Proof: If $E_{p\omega}(\omega_0) = 0$ to satisfy Statement 1, then $F(\omega_0, t_2, t_0) = 0$, using Result 2.1.2 and its real part given by $F_R(\omega_0, t_2, t_0) = 0$, where $\omega_0 \neq 0$(**Result 2.1.3**).

We do not have a closed form solution for $G_R(\omega, t_2, t_0)$ and do not know the exact location of its zeros at $\omega = \omega_z(t_2, t_0)$, for each fixed choice of t_2, t_0. For a specific choice of t_2, t_0, **only one** of the 2 cases is possible: **Case B:** $G_R(\omega, t_2, t_0)$ has at least one zero crossing for a specific $\omega \neq 0$ or
Case A: $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any choice of $\omega \neq 0$. **If** Statement 1 is true, **then** Case B is the **only** possibility and Case A is **ruled out**, as shown below.

We want to show the **Result 2.1.5** that $G_R(\omega, t_2, t_0)$ **must have at least one** zero crossing at **some value** of $\omega = \omega_z(t_2, t_0) \neq 0$ (**Case B**), to satisfy **Statement 1**, for this choice of fixed t_0, t_2.

To show Result 2.1.5, we **assume the opposite Case A**, that $G_R(\omega, t_2, t_0)$ **does not** have at least one zero for **any** value of $\omega \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign (zero crossing) and will show that $F_R(\omega, t_2, t_0)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Result 2.1.3 and Statement 1 and hence prove Result 2.1.5 and Case B.

This **does not** mean that, proof of Lemma 1 will work **only if** $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any value of $\omega \neq 0$, for any choice of t_2, t_0. The device **Proof by Contradiction** is used here to **rule out** Case A and arrive at Case B. (Details of other cases in Section 2.1.1)

The arguments above and following proof continue to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) H(\omega - \omega') d\omega' \tag{7}$$

We can show that the above integral converges for real ω, given that the integrand is absolutely integrable because $G(\omega, t_2, t_0)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t, t_2, t_0)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2 and Appendix C.6)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 7 and we get

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{8}$$

We can split the integral in Eq. 8 using $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$, as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_R(\omega', t_2, t_0)\frac{1}{(\sigma^2 + (\omega - \omega')^2)}d\omega' + \int_{0}^{\infty} G_R(\omega', t_2, t_0)\frac{1}{(\sigma^2 + (\omega - \omega')^2)}d\omega'] \tag{9}$$

We see that $G_R(-\omega, t_2, t_0) = G_R(\omega, t_2, t_0)$ because $g(t, t_2, t_0)$ is a real function of variable t. (Appendix D.1) We can substitute $\omega' = -\omega''$ in the first integral in Eq. 9 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi}\int_{0}^{\infty} G_R(\omega', t_2, t_0)[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}]d\omega' \tag{10}$$

We note that t_0 and t_2 are **fixed** in Eq. 10 and $G_R(\omega, t_2, t_0)$ is a function of ω **only** and the integrand in Eq. 10 is integrated over the variable ω **only**.

In Appendix C.2, it is shown that $G(\omega', t_2, t_0)$ is finite for real ω' and goes to zero as $|\omega'| \to \infty$. We can see that for $\omega' \to \infty$, the integrand in Eq. 10 goes to zero. For finite $\omega \geq 0$, and $0 \leq \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$, for $0 < \sigma < \frac{1}{2}$. We see that $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω' (Section 2.2). (**Result 2.1.4**)

- **Case 1:** $G_R(\omega', t_2, t_0) \geq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix D.1) and link). Hence $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$ where it crosses the zero line and becomes negative, where $\omega_z(t_2, t_0)$ is real and finite.

- **Case 2:** $G_R(\omega', t_2, t_0) \leq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix D.1) and link). Hence $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$, where it crosses the zero line and becomes positive, where $\omega_z(t_2, t_0)$ is real.

We have shown that, $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**, for specific choices of fixed t_0, t_2. We call this **Result 2.1.5**.

The arguments above and the proof continue to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

In the rest of the sections, we consider only the **first** zero crossing away from origin, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign. Hence $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2, to satisfy **Statement 1**.

2.1.1. ***Discussion of Lemma 1***

Result 2.1.5: $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**.

For each fixed value of t_0, t_2, only 2 cases are possible for $G_R(\omega, t_2, t_0)$. **Case A:** $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any choice of $\omega \neq 0$. **Case B:** $G_R(\omega, t_2, t_0)$ has at least one zero crossing for a specific $\omega \neq 0$. Proof of Lemma 1 assumes Case A and uses **Proof by Contradiction** to rule out Case A and arrive at Case B, for each choice of fixed t_0, t_2. This does not mean that Proof of Lemma 1 does not work for Case B. For Case B, we **do not** use Proof of Lemma 1 and jump to the end of the proof because we already have the desired Result 2.1.5 which is the same as Case B.

The logic used is this proof is as follows: **If** Statement 1 is true(RH is false), **then** Result 2.1.5 is true (Case B), for **each and every** combination of **fixed** values of t_0, t_2 in interval A ($|t_0| < \infty$ and $0 < |t_2| < \infty$)and hence Case A is **ruled out** and only Case B is possible for $G_R(\omega, t_2, t_0)$. Then we proceed with Result 2.1.5 to Section 2.3, 2.4 and Section 3, to produce a **contradiction** of Statement 1 in Eq. 40 and thus prove the truth of RH.

We present an **alternate method** of analyzing all possible cases of $G_R(\omega, t_2, t_0)$ below. We can arrive at Result 2.1.5, for **each and every** combination of **fixed** values of t_0, t_2 in interval A, using Proof of Lemma 1 for Case C and Case D or using Case E, as explained below.

It is noted that $F_R(\omega, t_2, t_0)$ and $G_R(\omega, t_2, t_0)$ may have more zeros than $F(\omega, t_2, t_0)$ and $G(\omega, t_2, t_0)$ respectively. That **does not** affect the proof of Lemma 1, as explained below.

We do not have a closed form solution for $G_R(\omega, t_2, t_0)$ and do not know the exact location of its zeros at $\omega = \omega_z(t_2, t_0)$, for each fixed choice of t_2, t_0. We consider 3 possible cases of $G_R(\omega, t_2, t_0)$ below.

• **Case C:** We consider the case that $G_R(\omega, t_2, t_0)$ **does not** have at least one zero crossing, for any value of $\omega \neq 0$, for **each and every** choice of t_2, t_0 and we use Proof of Lemma 1 for each and every choice of t_2, t_0, to show that it leads to a **contradiction** of Statement 1, and hence prove Result 2.1.5.

Hence Case C is **ruled out**, **if** Statement 1 is true.

• **Case D:** We consider the case $G_R(\omega, t_2^{'}, t_0^{'})$ has a zero crossing, for a specific value of $\omega = \omega_z(t_2', t_0')$, corresponding to **specific** choices of $t_2^{'}, t_0^{'}$.(**Not** for all possible choices of $t_2^{'}, t_0^{'}$)

For Case D, this means that $G_R(\omega, t_2^{'}, t_0^{'})$ has **at least one zero crossing** at $\omega = \omega_z(t_2', t_0')$ which is the desired **Result 2.1.5** and hence we **do not** go through the arguments in this proof and we can jump to end of Proof of Lemma 1. In this case, we **have not** assumed Statement 1 and yet arrived at Result 2.1.5, for **specific** choices of $t_2^{'}, t_0^{'}$.

For Case D, there may be **at least one** choice of t_{2f}, t_{0f} for which $G_R(\omega, t_{2f}, t_{0f})$ **does not** have at least one zero crossing, for any value of $\omega \neq 0$. For this choice of t_{2f}, t_{0f}, we use Proof of Lemma 1 to show that it leads to a **contradiction** of Statement 1, and hence prove Result 2.1.5.

Hence Case D is **ruled out**, **if** Statement 1 is true.

• **Case E:** We consider the case $G_R(\omega, t_2, t_0)$ has at least one zero crossing, for a specific value of $\omega = \omega_z(t_2, t_0)$, corresponding to **each and every** choices of t_2, t_0. We call this **Statement 3**.

For Case E, this means that $G_R(\omega, t_2, t_0)$ has **at least one zero crossing** at $\omega = \omega_z(t_2, t_0)$, for **each and every** choices of t_2, t_0 which is the desired **Result 2.1.5** and hence we **do not** go through the arguments in this proof and we can jump to end of Proof of Lemma 1. In this case, we **have not** assumed Statement 1 and yet arrived at Result 2.1.5, for **each and every** choices of t_2, t_0.

For Case E, we see that we arrive at Result 2.1.5 by **assuming** Statement 3 only. Then we proceed with Result 2.1.5 to Section 2.3, 2.4 and Section 3, to produce a **contradiction** of Statement 3 in Eq. 40. Hence Statement 3 is false and Case E is **ruled out**.

There are **only 3** possible cases for $G_R(\omega, t_2, t_0)$ given by Case C,D and E. We have ruled out Case E in above para. **If** Statement 1 is true, Case C and Case D have been **ruled out**. This means **Statement 1 is false**.

Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Hence the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

2.2. $G_R(\omega', t_2, t_0)$ ***is not an all zero function of variable*** ω'

If $G_R(\omega', t_2, t_0)$ is an all zero function of variable ω', for each given value of t_0, t_2 (**Statement 2**), then $F_R(\omega, t_2, t_0)$ in Eq. 7 is an all zero function of ω, for real ω. Hence $2f_{even}(t, t_2, t_0) = f(t, t_2, t_0) + f(-t, t_2, t_0)$ is an **all-zero** function of t, given that the Fourier transform of $f_{even}(t, t_2, t_0)$ is given by $F_R(\omega, t_2, t_0)$, using symmetry properties of Fourier transform(Appendix D.2) and link). Hence $f(t, t_2, t_0)$ is an **odd function** of variable t.(**Result 2.2**).

From Eq. 6 we see that $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = [E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t}$. Hence $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p^{'}(t + t_0, t_2) = [E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t}$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0}E_p^{'}(t - t_0, t_2) = [E_0(t - t_0 - t_2) - E_0(t - t_0 + t_2)]e^{-\sigma t}$. Hence we can write $f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ in Eq. 6, as follows.

$$f(t, t_2, t_0) = e^{-2\sigma t_0}[E_0(t+t_0-t_2) - E_0(t+t_0+t_2)]e^{-\sigma t} + e^{2\sigma t_0}[E_0(t-t_0-t_2) - E_0(t-t_0+t_2)]e^{-\sigma t} \quad (11)$$

Case 1: For $t_0 \neq 0$ and $t_2 \neq 0$, it is shown that Result 2.2 is false. We will compute $f(t, t_2, t_0)$ in

Eq. 11 at $t = 0$ and show that it does not equal zero.

We see that $f(0, t_2, t_0) = e^{-2\sigma t_0}[E_0(t_0 - t_2) - E_0(t_0 + t_2)] + e^{2\sigma t_0}[E_0(-t_0 - t_2) - E_0(-t_0 + t_2)]$ $= -2\sinh(2\sigma t_0)[E_0(t_0 - t_2) - E_0(t_0 + t_2)]$. We use the fact that $E_0(t_0) = E_0(-t_0)$ (Appendix C.8) and hence $E_0(t_0 - t_2) = E_0(-t_0 + t_2)$ and $E_0(t_0 + t_2) = E_0(-t_0 - t_2)$.

If Result 2.2 is true, then we require $f(0, t_2, t_0) = 0$ in Eq. 11. For our choice of $0 < \sigma < \frac{1}{2}$ and $t_0 \neq 0$, this implies that $E_0(t_0 - t_2) = E_0(t_0 + t_2)$. Given that $t_0 \neq 0$ and $t_2 \neq 0$, we set $t_2 = Kt_0$ for real $K \neq 0$ and we get $E_0((1 - K)t_0) = E_0((1 + K)t_0)$. This is **not** possible for $t_0 \neq 0$ because $E_0(t_0)$ is **strictly decreasing** for $t_0 > 0$ (Section 6) and $1 - K \neq 1 + K$ or $1 - K \neq -(1 + K)$ for $K \neq 0$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω'.

Case 2: For $t_0 = 0$ and $t_2 \neq 0$, we have $f(t, t_2, t_0) = 2[E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t} = 2D(t)e^{-\sigma t}$ in Eq. 11, where $D(t) = E_0(t - t_2) - E_0(t + t_2)$. We see that $D(t) + D(-t) = E_0(t - t_2) - E_0(t + t_2) + E_0(-t - t_2) - E_0(-t + t_2)$. Given that $E_0(t) = E_0(-t)$, we have $D(t) + D(-t) = E_0(t - t_2) - E_0(t + t_2) + E_0(t + t_2) - E_0(t - t_2) = 0$ and hence $D(t) = E_0(t - t_2) - E_0(t + t_2)$ is an **odd** function of variable t (**Result 2.2.1**).

If Result 2.2 is true, then we require $f(t, t_2, t_0) = 2D(t)e^{-\sigma t}$ to be an **odd** function of variable t. Using Result 2.2.1, we require $D(t)$ to be an **odd** function of variable t. This is possible only for $\sigma = 0$. This is **not** possible for our choice of $0 < \sigma < \frac{1}{2}$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω'.

Case 3: For $t_2 = 0$ and $|t_0| < \infty$, we have $E_p'(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = 0$ and $f(t, t_2, t_0) = g(t, t_2, t_0) = 0$ for all t in Eq. 6 and Lemma 1 is not applicable for this case.

2.3. ***On the zeros of a related function*** $G(\omega, t_2, t_0)$

In this section, we compute the Fourier transform of the function $g_{even}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$(Appendix D.2). We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Lemma 1 in Section 2.1.

We **define** $g_1(t, t_2, t_0) = f_1(t, t_2, t_0)e^{-\sigma t}u(-t) + f_1(t, t_2, t_0)e^{\sigma t}u(t) = e^{\sigma t_0}E_p^{'}(t + t_0, t_2)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p^{'}(t + t_0, t_2)e^{\sigma t}u(t)$, using Eq. 6 (**Definition 3**). First we compute the Fourier transform of the function $g_1(t, t_2, t_0)$ given by $G_1(\omega, t_2, t_0) = G_{1R}(\omega, t_2, t_0) + iG_{1I}(\omega, t_2, t_0)$.

$$G_1(\omega, t_2, t_0) = \int_{-\infty}^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt = \int_{-\infty}^{0} g_1(t, t_2, t_0)e^{-i\omega t}dt + \int_{0}^{\infty} g_1(t, t_2, t_0)e^{-i\omega t}dt$$
$$G_1(\omega, t_2, t_0) = \int_{-\infty}^{0} e^{\sigma t_0}E_p^{'}(t + t_0, t_2)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} e^{\sigma t_0}E_p^{'}(t + t_0, t_2)e^{\sigma t}e^{-i\omega t}dt \tag{12}$$

We use $E_p^{'}(t, t_2) = E_0^{'}(t, t_2)e^{-\sigma t}$ from Eq. 6, where $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$, using Definition 1 in Section 2.1 and we get $E_p^{'}(t + t_0, t_2) = E_0^{'}(t + t_0, t_2)e^{-\sigma t}e^{-\sigma t_0}$ and write Eq. 12 as follows. Then we substitute $t = -t$ in the second integral in first line of Eq. 13.

$$G_1(\omega, t_2, t_0) = \int_{-\infty}^{0} E_0^{'}(t + t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_0^{'}(t + t_0, t_2)e^{-i\omega t}dt$$
$$G_1(\omega, t_2, t_0) = \int_{-\infty}^{0} E_0^{'}(t + t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_0^{'}(-t + t_0, t_2)e^{i\omega t}dt \tag{13}$$

We define $E_{0n}^{'}(t, t_2) = E_0^{'}(-t, t_2)$ (**Definition 2**) and get $E_0^{'}(-t + t_0, t_2) = E_{0n}^{'}(t - t_0, t_2)$ and write Eq. 13 as follows. The integral in Eq. 14 converges, given that $E_0(t)e^{-2\sigma t}$ is an absolutely **integrable** function (Appendix C.1) and its t_0, t_2 shifted versions are absolutely **integrable**, using $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ in Definition 1 in Section 2.1 and Definition 2.

$$G_1(\omega, t_2, t_0) = \int_{-\infty}^{0} E_0^{'}(t + t_0, t_2)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^{0} E_{0n}^{'}(t - t_0, t_2)e^{i\omega t}dt = G_{1R}(\omega, t_2, t_0) + iG_{1I}(\omega, t_2, t_0) \tag{14}$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G_1(\omega, t_2, t_0)$, we have

$$G_{1R}(\omega, t_2, t_0) = \int_{-\infty}^{0} E_0^{'}(t + t_0, t_2)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0n}^{'}(t - t_0, t_2)\cos(\omega t)dt \tag{15}$$

2.4. *Zero crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0, for a given t_2*

Now we consider Eq. 6 and the function $f(t,t_2,t_0) = e^{-2\sigma t_0} f_1(t,t_2,t_0) + e^{2\sigma t_0} f_2(t,t_2,t_0) = e^{-\sigma t_0} E_p'(t+t_0,t_2) + e^{\sigma t_0} E_p'(t-t_0,t_2)$ where $f_1(t,t_2,t_0) = e^{\sigma t_0} E_p'(t+t_0,t_2)$ and $f_2(t,t_2,t_0) = f_1(t,t_2,-t_0) = e^{-\sigma t_0} E_p'(t-t_0,t_2)$ and $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $g(t,t_2,t_0) = f(t,t_2,t_0)e^{-\sigma t}u(-t) + f(t,t_2,t_0)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$. We can write the above equations and $g_1(t,t_2,t_0)$ from Definition 3 in Section 2.3, as follows. We define $g_2(t,t_2,t_0)$ below and write $g(t,t_2,t_0)$ as follows.

$$\begin{aligned}
g_1(t,t_2,t_0) = f_1(t,t_2,t_0)e^{-\sigma t}u(-t) + f_1(t,t_2,t_0)e^{\sigma t}u(t), \quad g_1(t,t_2,t_0)h(t) = f_1(t,t_2,t_0) \\
g_2(t,t_2,t_0) = f_2(t,t_2,t_0)e^{-\sigma t}u(-t) + f_2(t,t_2,t_0)e^{\sigma t}u(t), \quad g_2(t,t_2,t_0)h(t) = f_2(t,t_2,t_0) \\
g(t,t_2,t_0) = e^{-2\sigma t_0} g_1(t,t_2,t_0) + e^{2\sigma t_0} g_2(t,t_2,t_0)
\end{aligned} \tag{16}$$

We compute the Fourier transform of the function $g(t,t_2,t_0)$ in Eq. 16 and compute its real part $G_R(\omega,t_2,t_0)$ using the procedure in Section 2.3, similar to Eq. 15 and we can write as follows in Eq. 17. We use $G_{2R}(\omega,t_2,t_0) = G_{1R}(\omega,t_2,-t_0)$ given that $f_2(t,t_2,t_0) = f_1(t,t_2,-t_0)$ and $g_2(t,t_2,t_0) = g_1(t,t_2,-t_0)$ and $G_2(\omega,t_2,t_0) = G_1(\omega,t_2,-t_0)$. We substitute $t = \tau$ in the equation for $G_{1R}(\omega,t_2,t_0)$ below, copied from Eq. 15.

$$\begin{aligned}
G_R(\omega,t_2,t_0) = e^{-2\sigma t_0} G_{1R}(\omega,t_2,t_0) + e^{2\sigma t_0} G_{2R}(\omega,t_2,t_0) = e^{-2\sigma t_0} G_{1R}(\omega,t_2,t_0) + e^{2\sigma t_0} G_{1R}(\omega,t_2,-t_0) \\
G_{1R}(\omega,t_2,t_0) = \int_{-\infty}^{0} [E_0'(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}'(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
G_R(\omega,t_2,t_0) = e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0'(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}'(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0'(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}'(\tau+t_0,t_2)]\cos(\omega\tau)d\tau
\end{aligned} \tag{17}$$

We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for every value of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Lemma 1 in Section 2.1. In general $\omega_z(t_2,t_0) \neq \omega_0$. Hence we can see that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = 0$ and we can rearrange the terms in Eq. 17 as follows. We take the first and fourth terms in $G_R(\omega,t_2,t_0)$ in Eq. 17 and include them in the first line in Eq. 18. We take the second and third terms in Eq. 17 and include them in the second line in Eq. 18.

$$\begin{aligned}
P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0'(\tau+t_0,t_2)e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}'(\tau+t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau \\
+ \int_{-\infty}^{0} [e^{2\sigma t_0} E_0'(\tau-t_0,t_2)e^{-2\sigma\tau} + e^{-2\sigma t_0} E_{0n}'(\tau-t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau = 0
\end{aligned} \tag{18}$$

We use the fact that $f(t,t_2,t_0) = e^{-\sigma t_0} E_p'(t+t_0,t_2) + e^{\sigma t_0} E_p'(t-t_0,t_2) = f(t,t_2,-t_0)$ in Eq. 6, is **unchanged** by the substitution $t_0 = -t_0$. **If** $f(t,t_2,t_0) = f(t,t_2,-t_0)$ is unchanged by the substitution $t_0 = -t_0$, **then** $g(t,t_2,t_0) = g(t,t_2,-t_0)$ is unchanged by the substitution $t_0 = -t_0$, using the fact that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

Hence the Fourier transform of $g(t,t_2,t_0)$ given by $G(\omega,t_2,t_0) = G(\omega,t_2,-t_0)$ and its real part given by $G_R(\omega,t_2,t_0) = G_R(\omega,t_2,-t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and the zero crossing in $G_R(\omega,t_2,-t_0)$ given by $\omega_z(t_2,-t_0)$ is the **same** as the zero crossing in $G_R(\omega,t_2,t_0)$ given by $\omega_z(t_2,t_0)$ and we get $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$ and hence $\omega_z(t_2,t_0)$ is an **even** function of variable t_0, for each non-zero value of t_2.

We can write Eq. 18 as follows, where $P_{odd}(t_2,t_0)$ is an **odd** function of variable t_0, for each non-zero value of t_2. We use $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$.

$$
\begin{aligned}
P(t_2,t_0) &= P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0 \\
P_{odd}(t_2,t_0) &= \int_{-\infty}^{0} [e^{-2\sigma t_0} E_0'(\tau+t_0,t_2)e^{-2\sigma\tau} + e^{2\sigma t_0} E_{0n}'(\tau+t_0,t_2)] \cos{(\omega_z(t_2,t_0)\tau)} d\tau
\end{aligned}
\tag{19}
$$

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 19 as follows, using the substitution $\tau + t_0 = \tau'$. We get $\tau = \tau' - t_0$ and $d\tau = d\tau'$ and substitute back $\tau' = \tau$ in the second line below. We use $e^{-2\sigma t_0} e^{2\sigma t_0} = 1$ below.

$$P_{odd}(t_2, t_0) = \int_{-\infty}^{t_0} [e^{-2\sigma t_0} E_0^{'}(\tau^{'}, t_2) e^{-2\sigma\tau^{'}} e^{2\sigma t_0} + e^{2\sigma t_0} E_{0n}^{'}(\tau^{'}, t_2)] \cos(\omega_z(t_2, t_0)(\tau^{'} - t_0) d\tau'$$

$$P_{odd}(t_2, t_0) = [\cos(\omega_z(t_2, t_0) t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau$$

$$+ \sin(\omega_z(t_2, t_0) t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau]$$

$$+ e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0) t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau, t_2) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0) t_0) \int_{-\infty}^{t_0} E_{0n}^{'}(\tau, t_2) \sin(\omega_z(t_2, t_0)\tau) d\tau] \quad (20)$$

In Section 2.1, it is shown that $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2. In this section, we consider $t_0 > 0$ and $t_2 > 0$ only.

In Section 4, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

In Section 6, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given that $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c}) t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2 and given that t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0) t_0$ is a **continuous** function and is positive for $t_0 > 0$ because $0 < \omega_z(t_2, t_0) < \infty$.

We see that $\omega_z(t_2, t_0) > 0$ and is a **continuous** function of variable t_0 and t_2, and that $\omega_z(t_2, t_0) t_0 = \frac{\pi}{2}$ can be reached for specific values of t_0 and $t_2 = 2t_0$, as finite t_0 increases without bounds.(Section 5). As t_0 and t_2 increase from zero to a larger and larger finite value without bounds, the continuous function $\omega_z(t_2, t_0) t_0$ starts from zero and will pass through $\frac{\pi}{2}$, for specific values of t_0 and $t_2 = 2t_0$.

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c}) t_{0c} = \frac{\pi}{2}$ in Eq. 20 as follows. We use the fact that $\cos(\omega_z(t_{2c}, t_{0c}) t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c}) t_{0c}) = 1$ and $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.4.

$$P_{odd}(t_{2c}, t_{0c}) = \int_{-\infty}^{t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \quad (21)$$

We compute $P_{odd}(t_2, -t_0)$ in Eq. 20 as follows. We use $\omega_z(t_2, -t_0) = \omega_z(t_2, t_0)$ (Section 2.4).

$$
\begin{aligned}
P_{odd}(t_2,-t_0) = [\cos(\omega_z(t_2,t_0)t_0) \int_{-\infty}^{-t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau \\
- \sin(\omega_z(t_2,t_0)t_0) \int_{-\infty}^{-t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau] \\
+e^{-2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0) \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)\tau)d\tau - \sin(\omega_z(t_2,t_0)t_0) \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau,t_2)\sin(\omega_z(t_2,t_0)\tau)d\tau]
\end{aligned}
\tag{22}
$$

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 22 as follows. We use $\cos(\omega_z(t_{2c},t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c},t_{0c})t_{0c}) = 1$.

$$
P_{odd}(t_{2c},-t_{0c}) = -\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau
\tag{23}
$$

We compute $P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0$ in Eq. 19, at $t_0 = t_{0c}$ and $t_2 = t_{2c}$ using Eq. 21 and Eq. 23.

$$
\begin{aligned}
\int_{-\infty}^{t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \\
-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0
\end{aligned}
\tag{24}
$$

We split the first two integrals in the left hand side of Eq. 24 using $\int_{-\infty}^{t_{0c}} = \int_{-\infty}^{-t_{0c}} + \int_{-t_{0c}}^{t_{0c}}$ as follows.

$$
\begin{aligned}
[\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau] \\
+e^{2\sigma t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau] \\
-\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0
\end{aligned}
\tag{25}
$$

We cancel the common integral $\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$ in Eq. 25 and rearrange the terms as follows, using $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$
\begin{aligned}
\int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \\
= -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c})\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau
\end{aligned}
$$

$$\tag{26}$$

We can combine the integrals in the left hand side of Eq. 26 as follows.

$$\begin{aligned}\int_{-t_{0c}}^{t_{0c}} [E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau} + E_{0n}^{'}(\tau,t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \\ = -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau,t_{2c}) \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau\end{aligned} \tag{27}$$

We denote the right hand side of Eq. 27 as RHS. We can split the integral in the left hand side of Eq. 27 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\begin{aligned}\int_{-t_{0c}}^{0} [E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau} + E_{0n}^{'}(\tau,t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \\ + \int_{0}^{t_{0c}} [E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau} + E_{0n}^{'}(\tau,t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = RHS\end{aligned} \tag{28}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 28 as follows. We use $E_0^{'}(-\tau,t_{2c}) = E_{0n}^{'}(\tau,t_{2c})$ and $E_{0n}^{'}(-\tau,t_{2c}) = E_0^{'}(\tau,t_{2c})$ using Definition 2 in Section 2.3.

$$\begin{aligned}\int_{t_{0c}}^{0} [E_{0n}^{'}(\tau,t_{2c})e^{2\sigma\tau} + E_{0}^{'}(\tau,t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \\ + \int_{0}^{t_{0c}} [E_0^{'}(\tau,t_{2c})e^{-2\sigma\tau} + E_{0n}^{'}(\tau,t_{2c})e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = RHS\end{aligned} \tag{29}$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify Eq. 29 as follows.

$$\int_{0}^{t_{0c}} [E_0^{'}(\tau,t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E_{0n}^{'}(\tau,t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = RHS \tag{30}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 27 as follows. We use $E_{0n}^{'}(-\tau,t_{2c}) = E_0^{'}(\tau,t_{2c})$ using Definition 2 in Section 2.3.

$$RHS = 2\sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E_0^{'}(\tau,t_{2c}) \sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \tag{31}$$

We split the integral on the right hand side in Eq. 31 using $\int_{t_{0c}}^{\infty} = \int_{0}^{\infty} - \int_{0}^{t_{0c}}$, as follows.

$$RHS = 2\sinh(2\sigma t_{0c})[\int_0^{\infty} E_0'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E_0'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{32}$$

We consolidate the integrals of the form $\int_0^{t_{0c}} E_0'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 30 and Eq. 32 as follows. We use $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned} \int_0^{t_{0c}} [E_0'(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E_{0n}'(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \tag{33}$$

We cancel the common term $e^{2\sigma t_{0c}}$ in the first integral in Eq. 33 as follows.

$$\begin{aligned} \int_0^{t_{0c}} [E_0'(\tau, t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E_{0n}'(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0'(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \tag{34}$$

We substitute $E_0'(\tau, t_{2c}) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ (using Definition 1 in Section 2.1) and $E_{0n}'(\tau, t_{2c}) = E_0'(-\tau, t_{2c}) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$ (using Definition 2 in Section 2.3). We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$(Appendix C.8). Hence we see that $E_{0n}'(\tau, t_{2c}) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0'(\tau, t_{2c})$ (**Result 3.1**) and write Eq. 34 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \tag{35}$$

We substitute $2\cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2\cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 35 as follows.

$$\begin{aligned} \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = \sinh(2\sigma t_{0c}) \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \tag{36}$$

Next Step:

We denote the right hand side of Eq. 36 as $RHS^{'}$. We substitute $\tau - t_{2c} = \tau'$ and $\tau + t_{2c} = \tau''$ in the right hand side of Eq. 36 and then substitute $\tau' = \tau$ and $\tau'' = \tau$ in the second line below.

$$\begin{aligned}
RHS^{'} &= \sinh{(2\sigma t_{0c})}[\int_{-t_{2c}}^{\infty} E_0(\tau')\sin{(\omega_z(t_{2c},t_{0c})(\tau'+t_{2c}))}d\tau' - \int_{t_{2c}}^{\infty} E_0(\tau'')\sin{(\omega_z(t_{2c},t_{0c})(\tau''-t_{2c}))}d\tau'']\\
RHS^{'} &= \sinh{(2\sigma t_{0c})}[\cos{(\omega_z(t_{2c},t_{0c}))t_{2c})}\int_{-t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau\\
&+\sin{(\omega_z(t_{2c},t_{0c})t_{2c})}\int_{-t_{2c}}^{\infty} E_0(\tau)\cos{(\omega_z(t_{2c},t_{0c})\tau)}d\tau\\
&-\cos{(\omega_z(t_{2c},t_{0c}))t_{2c})}\int_{t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau + \sin{(\omega_z(t_{2c},t_{0c})t_{2c})}\int_{t_{2c}}^{\infty} E_0(\tau)\cos{(\omega_z(t_{2c},t_{0c})\tau)}d\tau]
\end{aligned} \tag{37}$$

In Eq. 37, given that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c},t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin{(\omega_z(t_{2c},t_{0c})t_{2c})} = 0$ and $\cos{(\omega_z(t_{2c},t_{0c})t_{2c})} = -1$. Hence we cancel common terms and write Eq. 37 and Eq. 36 as follows.

$$\begin{aligned}
&\int_{0}^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma\tau)} - \cosh{(2\sigma t_{0c})})\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau\\
&= -\sinh{(2\sigma t_{0c})}[\int_{-t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau - \int_{t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau]
\end{aligned} \tag{38}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau + \int_{t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau$ in Eq. 38 as follows. Given that $E_0(\tau)$ is an **even** function of variable τ (Appendix C.8) and $E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}$ is an **odd** function of variable τ, we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = 0$.

We see that $I = \int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = \int_{-t_{2c}}^{0} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau$ $+ \int_{0}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau$. We substitute $\tau = -\tau$ in the first integral and get $I = \int_{t_{2c}}^{0} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau + \int_{0}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau$ $= -\int_{0}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau + \int_{0}^{t_{2c}} E_0(\tau)\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = 0$. We write Eq. 38 as follows.

$$\int_{0}^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma\tau)} - \cosh{(2\sigma t_{0c})})\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = 0 \tag{39}$$

We can multiply Eq. 39 by a factor of -1 as follows.

$$\int_{0}^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh{(2\sigma t_{0c})} - \cosh{(2\sigma\tau)})\sin{(\omega_z(t_{2c},t_{0c})\tau)}d\tau = 0 \tag{40}$$

In Eq. 40, given that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $(0, t_{0c})$, $\omega_z(t_{2c},t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $(0, \frac{\pi}{2})$ and the sinusoidal function is > 0, in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 40, we see that the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$(Section 6), we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 3.1) in the interval $0 < \tau < t_{0c}$. The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$.

The integrand is zero at $\tau = 0$ due to the term $\sin(\omega_z(t_{2c}, t_{0c})\tau)$ and the integrand is zero at $\tau = t_{0c}$ due to the term $\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 40. Hence this leads to a **contradiction**, for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 40 is zero, given the term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) = 0$ and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$. **If** the Fourier transform of $E_p(t) = E_0(t)e^{-\sigma t}$ given by $E_{p\omega}(\omega) = E_{pR\omega}(\omega) + iE_{pI\omega}(\omega)$ has a zero at $\omega = \omega_0$, **then** the real part $E_{pR\omega}(\omega)$ and imaginary part $E_{pI\omega}(\omega)$ **also** have a zero at $\omega = \omega_0$, to satisfy Statement 1.

Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is real, its Fourier transform $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ has symmetry properties and hence $E_{pR\omega}(-\omega) = E_{pR\omega}(\omega)$ and $E_{pI\omega}(-\omega) = -E_{pI\omega}(\omega)$ (Symmetry property) and hence $E_{p\omega}(-\omega) = \xi(\frac{1}{2} + \sigma - i\omega)$ **also** has a zero at $\omega = \omega_0$ to satisfy Statement 1.

Using the property $\xi(s) = \xi(1 - s)$, we get $\xi(\frac{1}{2} + \sigma - i\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ at $s = \frac{1}{2} + \sigma - i\omega$ and $E_{q\omega}(\omega) = \xi(\frac{1}{2} - \sigma + i\omega)$ **also** has a zero at $\omega = \omega_0$ to satisfy Statement 1. We see that $E_{q\omega}(\omega)$ is obtained by replacing σ in $E_{p\omega}(\omega)$ by $-\sigma$. Hence the results in above sections hold for $-\frac{1}{2} < \sigma < 0$ and for $0 < |\sigma| < \frac{1}{2}$.

Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Hence the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3.1. ***Result*** $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 6 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 40 .

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t (Appendix C.8), and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t+t_{2c}) = E_0(t+2t_{0c})$ ranges from $E_0(2t_{0c}) > E_0(t + t_{2c}) > E_0(3t_{0c})$(**Result 6.3.1**) and $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c}) < E_0(t - t_{2c}) < E_0(-t_{0c})$ respectively. Given that $E_0(t) = E_0(-t)$, we see that $E_0(2t_{0c}) < E_0(t - t_{2c}) < E_0(t_{0c})$ in the interval $0 < t < t_{0c}$ (**Result 6.3.2**).

Using Result 6.3.1 and Result 6.3.2, we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$, in the interval $0 < t < t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 40 , for $t_{0c} > 0$ and $t_{2c} = 2t_{0c}$.

4. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2

It is shown in **Lemma 1** in Section 2.1 that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ where it crosses the zero line to the opposite sign, if Statement 1 is true, and that $\omega_z(t_2, t_0)$ is **finite and non-zero** for all $|t_0| < \infty$ and for each non-zero value of t_2 and that $\omega_z(t_2, t_0)$ is an even function of variable t_0, for a given value of t_2(Section 2.4). For a given t_2 and t_0, $\omega_z(t_2, t_0)$ can have more than one value, corresponding to multiple zero crossings in $G_R(\omega, t_2, t_0)$, but we consider only the first zero crossing away from origin in the section below, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, as detailed in **Lemma 1** in Section 2.1.

We consider the Fourier transform of the even part of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0 and t_2, the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. This is shown in the steps below using **Implicit Function Theorem**.

• It is shown in Section 4.1 that $G_R(\omega, t_2, t_0)$ and $G_{R,2r}(\omega, t_2, t_0)$ are partially differentiable at least twice with respect to ω, for some value of $r \in W$ (element of set of whole numbers including zero.)

• It is shown in Section 4.4 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_0. It is shown in Section 4.5 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2.

• In Section 4.8, it is shown in proof of Lemma 2 that, **if** $G_R(\omega, t_2, t_0) = 0$ at $\omega = \pm\omega_z(t_2, t_0)$, for each fixed choice of $t_0, t_2 \in \Re$ and $(2r + 1)$ is the highest order of the zero at $\omega = \pm\omega_z(t_2, t_0)$ for some value of $r \in W$ (element of set of whole numbers including zero), **then** $G_{R,2r}(\omega, t_2, t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} = 0$ at $\omega = \pm\omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} = \frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z(t_2, t_0)$.

• It is shown in Section 4.6 that the zero crossing in $G_{R,2r}(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **continuous** function of t_0, for a given t_2, for $0 < t_0 < \infty$, using **Implicit Function Theorem** in $\Re^2$.

• It is shown in Section 4.7 that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using **Implicit Function Theorem** in $\Re^3$.

4.1. $G_R(\omega, t_2, t_0)$ *and* $G_{R,2r}(\omega, t_2, t_0)$ *are partially differentiable twice as a function of* ω

$G_R(\omega, t_2, t_0)$ in Eq. 17 is copied below.

$$\begin{aligned} G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} &\int_{-\infty}^{0} [E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ +e^{2\sigma t_0} &\int_{-\infty}^{0} [E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \end{aligned} \tag{41}$$

We could then use $E_0'(\tau, t_2) = (E_0(\tau - t_2) - E_0(\tau + t_2)$ (using Definition 1 in Section 2.1) and $E_{0n}'(\tau, t_2) = E_0'(-\tau, t_2) = -E_0'(\tau, t_2)$ (using Definition 2 in Section 2.3 and Result 3.1 in Section 3).

We see that $E_0(\tau)$ in Eq. 1 and its t_0 and t_2 shifted versions are analytic functions of τ, t_0 and t_2, given that the sum and product of exponential functions are analytic and hence infinitely differentiable.(**Result 4.1**)

In Eq. 41, $G_R(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to ω and the integrals converge in Eq. 41 and Eq. 42 for $0 < \sigma < \frac{1}{2}$, because the terms $\tau^r E_0^{'}(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $\tau^r E_{0n}^{'}(\tau \pm t_0, t_2) = -\tau^r E_0^{'}(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$, for $r \in W$ (Section 4.2). The integrands in Eq. 41 and Eq. 42 are analytic functions of variables ω and t_0, for a given t_2(using Result 4.1 in Section 4.1 and given that the terms $\cos(\omega\tau), \sin(\omega\tau)$ and $e^{-2\sigma\tau}$ are analytic functions). The integrands have **exponential** asymptotic fall-off rate (Section 4.2) and absolutely integrable and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Section 4.3) Hence we can interchange the order of partial differentiation and integration in Eq. 42 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence, recursively as follows.(theorem)

$$\begin{aligned}\frac{\partial G_R(\omega, t_2, t_0)}{\partial\omega} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0} \tau[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)]\sin(\omega\tau)d\tau \\ &\quad + e^{2\sigma t_0}\int_{-\infty}^{0} \tau[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)]\sin(\omega\tau)d\tau] \\ \frac{\partial^2 G_R(\omega, t_2, t_0)}{\partial\omega^2} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0} \tau^2[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)]\cos(\omega\tau)d\tau \\ &\quad + e^{2\sigma t_0}\int_{-\infty}^{0} \tau^2[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)]\cos(\omega\tau)d\tau]\end{aligned} \tag{42}$$

We can use the arguments in the above paras and derive the $(2r)^{th}$ derivative of $G_R(\omega, t_2, t_0)$, for $r \in W$, which is differentiable at least twice, as follows.

$$\begin{aligned}G_{R,2r}(\omega, t_2, t_0) = \frac{\partial^{2r} G_R(\omega, t_2, t_0)}{\partial\omega^{2r}} &= (-1)^r[e^{-2\sigma t_0}\int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)]\cos(\omega\tau)d\tau \\ &\quad + e^{2\sigma t_0}\int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)]\cos(\omega\tau)d\tau]\end{aligned} \tag{43}$$

4.2. ***Exponential Fall off rate of*** $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ ***for*** $r \in W$

In this section, it is shown that the term $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ has exponential asymptotic fall-off rate as $|t| \to \infty$, for $r \in W$ where $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Hence $B(t) = t^r e^{-2\sigma t}[E_0(t - t_2 \pm t_0) - E_0(t + t_2 \pm t_0)]$ (**Result B.6.1**).

We consider $C(t) = t^r e^{-2\sigma t}E_0(t - t_a)$ for finite and real t_a. We see that $C(t + t_a) = (t + t_a)^r e^{-2\sigma t}e^{-2\sigma t_a}E_0(t)$. We see that $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function, for $0 \le |\sigma| < \frac{1}{2}$ given that it has exponential fall-off rates as $|t| \to \infty$. (Appendix C.5 and Appendix C.6).

Hence $C(t+t_a) = (t+t_a)^r e^{-2\sigma t_a} E_0(t) e^{-2\sigma t}$ also has exponential fall-off rates as $|t| \to \infty$, for $r \in W$ and finite t_a and is an absolutely integrable function.

Hence $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ has exponential fall-off rates as $|t| \to \infty$, for finite t_a and is an absolutely integrable function. We set $t_a = t_2 \pm t_0$ and $t_a = -t_2 \pm t_0$ and see that $B(t)$ in Result B.6.1, has **exponential fall-off rates** as $|t| \to \infty$, for finite t_2, t_0 and is an absolutely integrable function.

4.3. ***Dominating function***

We consider $x(t) = E_0(t)e^{-2\sigma t}$ which has asymptotic exponential fall-off rate of $o[e^{-0.5|t|}]$.(Appendix C.5) We see that $x(t + t_a)$ also has the same asymptotic exponential fall-off rate, for finite shift of $t_a = t_2 \pm t_0$ and $y(t, t_a) = t^r x(t + t_a)e^{2\sigma t_a}$ also has the same asymptotic exponential fall-off rate, for $r \in W$. We consider the intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ where $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ are finite.

We consider $t_d >> t_{a_{max}}$ where $y(t, t_a) = t^r x(t+t_a)e^{2\sigma t_a}$ falls off at the rate of $o[e^{0.5t}]$ for $t << -t_d$. We consider $f(t, t_a, \omega) = y(t, t_a)\cos(\omega t)$ and we get $\frac{\partial f(t,t_a,\omega)}{\partial \omega} = -ty(t, t_a)\sin(\omega t)$ which falls off at the rate of $o[e^{0.5t}]$ for $t << -t_d$. Let $f_{max} > 0$ be the maximum value of $|\frac{\partial f(t,t_a,\omega)}{\partial \omega}|$ in the interval $-\infty < t < \infty$.

We can find a suitable **dominating function** $D(t) = e^{-K|t|} f_{max} e^{Kt_d} > 0$ with a fall off rate of $O[e^{-K|t|}]$ where $0 < K < 0.5$ and hence $D(t)$ has a slower fall off rate than $\frac{\partial f(t,t_a,\omega)}{\partial \omega}$ and $D(t) = f_{max}$ at $t = -t_d$ and hence $D(t) > |\frac{\partial f(t,t_a,\omega)}{\partial \omega}|$ for $-\infty < t \leq 0$ and hence $|\frac{\partial f(t,t_a,\omega)}{\partial \omega}| \leq D(t)$ in the interval $(-\infty, 0]$ and $\int_{-\infty}^{0} |D(t)|dt = \int_{-\infty}^{0} e^{Kt} f_{max} e^{Kt_d} dt = \frac{1}{K} f_{max} e^{Kt_d} [e^{Kt}]_{-\infty}^{0} = \frac{1}{K} f_{max} e^{Kt_d}$ is finite.(**Result B.6.2**)

The first term in Eq. 42 given by $B(t) = t^r E_0'(t+t_0, t_2)e^{-2\sigma t} = t^r e^{-2\sigma t}[E_0(t-t_2+t_0) - E_0(t+t_2+t_0)]$ using Result B.6.1 in Section 4.2. We set $t_a = t_2 + t_0$ and $t_b = t_2 - t_0$ and get $B(t) = t^r e^{-2\sigma t}[E_0(t - t_b) - E_0(t + t_a)]$. Hence $y(t, t_a) = t^r x(t + t_a)e^{2\sigma t_a} = t^r E_0(t + t_a)e^{-2\sigma t}$ in the second para, corresponds to the second term in $B(t)$ and Result B.6.2 holds for this term. The first term in $B(t)$ is obtained by replacing t_a by $-t_b$ and Result B.6.2 holds for this term and hence for $B(t)$. We see that Result B.6.2 holds for the other 3 terms in Eq. 42 using arguments in above paragraphs and replacing t_0 by $-t_0$ and setting $\sigma = 0$ as needed.

As $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ increase to a larger and larger **finite value** without bounds, we consider larger intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ and f_{max} and t_d also increase correspondingly and the results in above paragraphs are valid in these intervals.

Similarly, we consider $f(t, t_a, \omega) = y(t, t_a)\cos(\omega t) = t^r E_0(t + t_a)e^{-2\sigma t}\cos(\omega t) = t^r E_0(t + t_0 + t_2)e^{-2\sigma t}\cos(\omega t)$ and we see that $\frac{\partial f(t,t_a,\omega)}{\partial t_0}$ and $\frac{\partial f(t,t_a,\omega)}{\partial t_2}$ which fall off at the rate of $o[e^{0.5t}]$ for $t << -t_d$, using Eq. 47 and $E_0(t) = E_0(-t)$ and due to the term $e^{-\pi n^2 e^{-2t}}$ and we can use arguments in above paragraphs to get a result similar to Result B.6.2 for the terms in Eq. 44 and Eq. 54. We can use these arguments to get a result similar to Result B.6.2 for the second derivative terms $\frac{\partial^2 f(t,t_a,\omega)}{\partial t_0^2}$ and $\frac{\partial^2 f(t,t_a,\omega)}{\partial t_2^2}$ in Eq. 49 and Eq. 58.

4.4. $G_{R,2r}(\omega, t_2, t_0)$ ***are partially differentiable twice as a function of*** t_0, $r \in W$

In Eq. 43, $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals converge in Eq. 44 and Eq. 49 shown as follows. The integrands in the equation for $G_{R,2r}(\omega, t_2, t_0)$ in Eq. 44 are absolutely integrable because the terms $\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $\tau^{2r}E_{0n}^{'}(\tau \pm t_0, t_2) = -\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$, for $r \in W$ (Section 4.2). The integrands in Eq. 44 are absolutely integrable and are analytic functions of variables ω and t_0, for a given t_2 (using Result 4.1 in Section 4.1). The integrands have **exponential** asymptotic fall-off rate(Section 4.2) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Section 4.3) Hence we can interchange the order of partial differentiation and integration in Eq. 44 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
G_{R,2r}(\omega, t_2, t_0) &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau \\
&\quad + 2\sigma e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_0} \cos(\omega\tau) d\tau
\end{aligned} \tag{44}$$

We show that the integrals in Eq. 44 converge, as follows. We see that $E_0^{'}(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E_{0n}^{'}(\tau - t_0, t_2) = -E_0^{'}(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3).We see that the first and third integrals in the equation for $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44 converge because the terms $\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $\tau^{2r}E_{0n}^{'}(\tau \pm t_0, t_2) = -\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2).

We consider the integrand in the second integral in the equation for $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44 first and use the results in the above paragraph.

$$\begin{aligned}
\frac{\partial(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_0} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_0} \\
&\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_0}
\end{aligned} \tag{45}$$

We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 45 and can show that the integrals converge in Eq. 44, as follows. We take the factor of 2 out of the summation in $E_0(\tau)$ in Eq. 1 copied below.

$$E_0(\tau) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}}$$
$$E_0(\tau + t_2 + t_0) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \tag{46}$$

We can show that $\frac{\partial}{\partial t_0}E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau}E_0(\tau + t_2 + t_0)$ as follows, given that the equation for $E_0(\tau + t_2 + t_0)$ in Eq. 46 has terms of the form $e^{\tau + t_0}$ and the equation is **invariant** if we interchange the variables τ and t_0. (**Result A**)

$$\begin{aligned}\frac{\partial}{\partial t_0}E_0(\tau + t_2 + t_0) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ +(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau}E_0(\tau + t_2 + t_0) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ +(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})]\end{aligned} \tag{47}$$

We can replace t_0 by $t_0^{'} = -t_0$ in Eq. 46 and see that $\frac{\partial}{\partial t_0^{'}}E_0(\tau + t_2 + t_0^{'}) = \frac{\partial}{\partial \tau}E_0(\tau + t_2 + t_0^{'})$ (**Result E**) given that the equation is invariant if we interchange τ and $t_0^{'}$. Given that $\frac{\partial}{\partial t_0^{'}} = \frac{\partial}{\partial t_0}\frac{dt_0}{dt_0^{'}} = -\frac{\partial}{\partial t_0}$, we substitute it in Result E and get $\frac{\partial}{\partial t_0}E_0(\tau + t_2 - t_0) = -\frac{\partial}{\partial \tau}E_0(\tau + t_2 - t_0)$.(**Result B**)

We can write the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. 45, corresponding to the term in the second integral in the equation for $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_0}$ in Eq. 44, using Result A, as follows. We use the fact that $\int_{-\infty}^{0} \frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0} \frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0} A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}\int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0))}{\partial t_0}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau = \int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0)}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau \\ = \int_{-\infty}^{0} \frac{\partial(E_0(\tau + t_2 + t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial \tau}d\tau - \int_{-\infty}^{0} E_0(\tau + t_2 + t_0)\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}}{\partial \tau}d\tau \\ = [E_0(\tau + t_2 + t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0} + \omega\int_{-\infty}^{0} E_0(\tau + t_2 + t_0))\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau \\ +2\sigma\int_{-\infty}^{0} E_0(\tau + t_2 + t_0))\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau - 2r\int_{-\infty}^{0} E_0(\tau + t_2 + t_0))\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau\end{aligned} \tag{48}$$

We see that the integrals in Eq. 48 converge because the integrands are absolutely integrable because the terms $E_0(\tau + t_2 + t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}$ and $E_0(\tau + t_2 + t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2). The term $[E_0(\tau + t_2 + t_0)\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \to -\infty$(Appendix C.5). Hence the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0}\cos(\omega\tau)d\tau$ in Eq. 48 and in Eq. 44 corresponding to the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 45, converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau + t_2 - t_0)$ in Eq. 45 also converges, using Result B and the procedure used in Eq. 46 to Eq. 48.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 46 to Eq. 48 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau-t_2+t_0)e^{-2\sigma\tau})}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 45 also converges.

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau-t_2-t_0))}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 44 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 45 also converges, using Result B and the procedure used in Eq. 46 to Eq. 48. Hence the second integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 44, also converges.

We can see that the last integral in Eq. 44 converges, by setting $t_0 = -t_0$ in Eq. 45 and using Result B and using the procedure in Eq. 46 to Eq. 48. Hence all the integrals in Eq. 44 converge.

4.4.1. ***Second Partial Derivative of*** $G_{R,2r}(\omega, t_2, t_0)$ ***with respect to*** t_0

The second partial derivative of $G_{R,2r}(\omega, t_2, t_0)$ with respect to t_0 is given by $\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_0^2} = \frac{\partial}{\partial t_0}\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ as follows. We use the result in Eq. 44 and the fact that the integrands are absolutely integrable using the results in Section 4.4 and are analytic functions of variables ω and t_0 for a given t_2 (using Result 4.1 in Section 4.1). The integrands have **exponential** asymptotic fall-off rate (Section 4.2) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Section 4.3) Hence we can interchange the order of partial differentiation and integration in Eq. 49 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
\frac{\partial^2 G_{R,2r}(\omega, t_2, t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)]\cos(\omega\tau)d\tau \\
&\quad -4\sigma e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&\quad +e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau \\
&\quad +4\sigma^2 e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)]\cos(\omega\tau)d\tau \\
&\quad +4\sigma e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&\quad +e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau
\end{aligned} \tag{49}$$

The first two integrals and fourth and fifth integrals in Eq. 49 are the same as the integrals in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. 44 and have been shown to converge in Section 4.4. We will show that the third and sixth integrals in Eq. 49 converge, as follows.

We consider the integrand in the third integral in Eq. 49 first. We see that $E_0^{'}(\tau+t_0,t_2) = E_0(\tau+t_0-t_2)-E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2) = -E_0^{'}(\tau-t_0,t_2) = E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3). We write an equation similar to Eq. 45.

$$\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0^2} = \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_0^2} + \frac{\partial^2(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_0^2} \tag{50}$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. 50 and copy Eq. 46 below.

$$\begin{aligned} E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}-3\pi n^2e^{2\tau}]e^{-\pi n^2e^{2\tau}}e^{\frac{\tau}{2}} \\ E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}e^{4(t_2+t_0)}-3\pi n^2e^{2\tau}e^{2(t_2+t_0)}]e^{-\pi n^2e^{2\tau}e^{2(t_2+t_0)}}e^{\frac{\tau}{2}}e^{\frac{(t_2+t_0)}{2}} \end{aligned} \tag{51}$$

We can see that $\frac{\partial^2}{\partial t_0^2}E_0(\tau+t_2+t_0) = \frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_0}$ and the equation **is invariant** if we interchange the variables τ and t_0.(**Result A'**)

We can replace t_0 by $t_0^{'}=-t_0$ in Eq. 51 and see that $\frac{\partial^2}{\partial (t_0^{'})^2}E_0(\tau+t_2+t_0^{'}) = \frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2+t_0^{'})$ (**Result E'**) given that the equation has terms of the form $e^{\tau+t_0^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_0^{'}$.

Given that $\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t_0^{'}}\frac{\partial t_0^{'}}{\partial t_0} = -\frac{\partial}{\partial t_0^{'}}$, we get $\frac{\partial^2}{\partial t_0^2} = \frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0}) = -\frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0^{'}}) = \frac{\partial}{\partial t_0^{'}}(\frac{\partial}{\partial t_0^{'}}) = \frac{\partial^2}{\partial (t_0^{'})^2}$, we substitute it in Result E' and get $\frac{\partial^2}{\partial t_0^2}E_0(\tau+t_2-t_0) = \frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2-t_0)$.(**Result B'**)

We can write the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. 50, corresponding to the term in the third integral in Eq. 49, using Result $A^{'}$, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial t_0^2}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau = \int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial \tau^2}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$$
$$= \int_{-\infty}^{0} \frac{\partial(\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial\tau}d\tau - \int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial\tau}d\tau$$
$$= [\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} + \omega\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$$
$$+2\sigma\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau - 2r\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos(\omega\tau)d\tau \quad (52)$$

We see that the integral $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$ in Eq. 52 converges, using Eq. 48 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}$ also converges, given that $E_0(\tau) = E_0(-\tau)$ and $E_0(\tau+t_2+t_0) = E_0(-\tau-t_2-t_0)$ and we consider $\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau} = \frac{\partial E_0(-\tau-t_2-t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}$ using Eq. 47 and see that the term $e^{-\pi n^2 e^{-2\tau}}$ goes to zero faster than the rising term $\tau^{2r}e^{-2\sigma\tau}e^{-6\tau}e^{-\frac{\tau}{2}}$, as $\tau \to -\infty$. (**Result 4.2.1.1**)

It is shown below that the remaining term $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$ also converges.

$$\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$$
$$= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau))}{\partial\tau}d\tau - \int_{-\infty}^{0}E_0(\tau+t_2+t_0))\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)}{\partial\tau}d\tau$$
$$= [E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)]_{-\infty}^{0} - \omega\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$$
$$+2\sigma\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau - 2r\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r-1}e^{-2\sigma\tau}\sin(\omega\tau)d\tau \quad (53)$$

We see that the integrals in Eq. 53 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)$ and $E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2). The term $[E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \to -\infty$(Appendix C.5). Hence the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0^2}\cos(\omega\tau)d\tau$ in Eq. 52 and in Eq. 49 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 50, also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2-t_0))}{\partial t_0^2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 49 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. 50 also converges, using Result B' and the procedure used in Eq. 51 to Eq. 53.

We set $t_2 = -t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 51 to Eq. 53 and see that the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau-t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0^2}\cos(\omega\tau)d\tau$ in Eq. 49 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in Eq. 50

also converges.

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau-t_2-t_0))}{\partial t_0^2} \tau^{2r} \cos(\omega\tau) d\tau$ in Eq. 49 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 50 also converges, using Result B' and the procedure used in Eq. 51 to Eq. 53. Hence the third integral in Eq. 49, also converges.

We can see that the sixth integral in Eq. 49 converges, by setting $t_0 = -t_0$ in Eq. 50 to Eq. 53 and using Result B' and the procedure used in Eq. 51 to Eq. 53. Hence all the integrals in Eq. 49 converge.

4.5. $G_{R,2r}(\omega, t_2, t_0)$ *is partially differentiable twice as a function of* t_2 *for* $r \in W$

In Eq. 43, $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals converge in Eq. 54 and Eq. 58 shown as follows. The integrands in the equation for $G_{R,2r}(\omega, t_2, t_0)$ in Eq. 54 are absolutely integrable because the terms $\tau^{2r} E_0'(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $\tau^{2r} E_{0n}'(\tau \pm t_0, t_2) = -\tau^{2r} E_0'(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2). The integrands are analytic functions of variables ω and t_2, for a given t_0 (using Result 4.1 in Section 4.1). The integrands have **exponential** asymptotic fall-off rate (Section 4.2) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Section 4.3) Hence we can interchange the order of partial differentiation and integration in Eq. 54 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
G_{R,2r}(\omega, t_2, t_0) &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2))}{\partial t_2} \cos(\omega\tau) d\tau
\end{aligned} \tag{54}$$

We use the procedure outlined in Eq. 45 to Eq. 48, with t_0 replaced by t_2 and show that all the integrals in Eq. 54 converge, as follows.

We see that $E_0'(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E_{0n}'(\tau - t_0, t_2) = -E_0'(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3). We consider the integrand in the first integral in the equation for $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_2}$ in Eq. 54 first.

$$\begin{aligned}
\frac{\partial(E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2))}{\partial t_2} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2} \\
&\quad + \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2}
\end{aligned}$$

$$(55)$$

We consider the term $E_0(\tau+t_0+t_2)$ first and can show that the integrals converge in Eq. 54, as follows. We copy Eq. 46 below.

$$E_0(\tau) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}}$$

$$E_0(\tau+t_2+t_0) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \tag{56}$$

We see that $\frac{\partial}{\partial t_2}E_0(\tau+t_2+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0)$ given that the equation has terms of the form $e^{\tau+t_2}$ and hence the equation is invariant if we interchange τ and t_2.(**Result C**)

We can replace t_2 by $t_2^{'} = -t_2$ in Eq. 56 and see that $\frac{\partial}{\partial t_2^{'}}E_0(\tau+t_2^{'}+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_2^{'}+t_0)$ given that the equation is invariant if we interchange τ and $t_2^{'}$(**Result F**). Given that $\frac{\partial}{\partial t_2^{'}} = \frac{\partial}{\partial t_2}\frac{dt_2}{dt_2^{'}} = -\frac{\partial}{\partial t_2}$, we use it in Result F and we get $\frac{\partial}{\partial t_2}E_0(\tau-t_2+t_0) = -\frac{\partial}{\partial \tau}E_0(\tau-t_2+t_0)$.(**Result D**)

We consider the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ first in Eq. 55, corresponding to the term in the first integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ in Eq. 54 as follows, using Result C. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{gathered}
\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial t_2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau = \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau \\
= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau - \int_{-\infty}^{0}E_0(\tau+t_2+t_0)\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}}{\partial\tau}d\tau \\
= [E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0} + \omega\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau \\
+2\sigma\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau - 2r\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau
\end{gathered} \tag{57}$$

We see that the integrals in Eq. 57 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}$ and $E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2). The term $[E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \to -\infty$(Appendix C.5). Hence the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. 57 and Eq. 54 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 55 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. 54 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. 55 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau - t_2 + t_0)e^{-2\sigma\tau})}{\partial t_2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 54 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. 55 also converges, using Result D.

We $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 56 to Eq. 57 and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau - t_2 - t_0))}{\partial t_2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. 54 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 55 also converges, using Result D. Hence the first integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ in Eq. 54 also converges.

We can see that the last integral in Eq. 54 converges, by setting $t_0 = -t_0$ in Eq. 57. Hence all the integrals in Eq. 54 converge.

4.5.1. ***Second Partial Derivative of*** $G_{R,2r}(\omega, t_2, t_0)$ ***with respect to*** t_2 ***for*** $r \in W$

The second partial derivative of $G_{R,2r}(\omega, t_2, t_0)$ with respect to t_2 is given by $\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_2^2} = \frac{\partial}{\partial t_2}\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ as follows. We use the result in Eq. 54 and the fact that the integrands are absolutely integrable using the results in Section 4.5 and the integrands are analytic functions of variables ω and t_2 for a given t_0 (using Result 4.1 in Section 4.1). The integrands have **exponential** asymptotic fall-off rate(Section 4.2) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Section 4.3) Hence we can interchange the order of partial differentiation and integration in Eq. 58 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}\frac{\partial^2 G_{R,2r}(\omega, t_2, t_0)}{\partial t_2^2} &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau \\ &+ e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau\end{aligned} \tag{58}$$

We consider the first integral in Eq. 58 and using $E_0^{'}(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E_{0n}^{'}(\tau - t_0, t_2) = -E_0^{'}(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$(using Definition 1 in Section 2.1 and Result 3.1 in Section 3), we write an equation similar to Eq. 55.

$$\begin{aligned}\frac{\partial^2(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2^2} &= \frac{\partial^2(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2^2} \\ &+ \frac{\partial^2(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2^2}\end{aligned} \tag{59}$$

We consider the term $E_0(\tau + t_0 + t_2)$ first in Eq. 59 as follows. We copy Eq. 46 below.

$$E_0(\tau) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}}$$

$$E_0(\tau + t_2 + t_0) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}$$

(60)

We can see that $\frac{\partial^2}{\partial t_2^2}E_0(\tau+t_2+t_0)=\frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_2}$ and the equation **is invariant** if we interchange the variables τ and t_2.(**Result C'**)

We can replace t_2 by $t_2^{'}=-t_2$ in Eq. 60 and see that $\frac{\partial^2}{\partial (t_2^{'})^2}E_0(\tau+t_2^{'}+t_0)=\frac{\partial^2}{\partial \tau^2}E_0(\tau+t_2^{'}+t_0)$ (**Result F'**) given that the equation has terms of the form $e^{\tau+t_2^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_2^{'}$.

Given that $\frac{\partial}{\partial t_2}=\frac{\partial}{\partial t_2^{'}}\frac{\partial t_2^{'}}{\partial t_2}=-\frac{\partial}{\partial t_2^{'}}$, we get $\frac{\partial^2}{\partial t_2^2}=\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2})=-\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial}{\partial t_2^{'}}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial^2}{\partial (t_2^{'})^2}$, we substitute it in Result F' and get $\frac{\partial^2}{\partial t_2^2}E_0(\tau-t_2+t_0)=\frac{\partial^2}{\partial \tau^2}E_0(\tau-t_2+t_0)$.(**Result D'**)

We can write the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. 59, corresponding to the term in the first integral in Eq. 58, using Result $C^{'}$, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau=\int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau-\int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}
&\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial t_2^2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau=\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial \tau^2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau\\
&=\int_{-\infty}^{0}\frac{\partial(\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau-\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}}{\partial\tau}d\tau\\
&=[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0}+\omega\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau\\
&+2\sigma\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau-2r\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau
\end{aligned}\tag{61}$$

We see that the integral $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau$ in Eq. 61 converges, using Eq. 57 in the previous subsection. We see that the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0}$ also converges, using Result 4.2.1.1 in Section 4.4.1. It is shown in Eq. 53 that the remaining term $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau$ also converges.

We see that the integrals in Eq. 61 converge and hence the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_2^2}\cos{(\omega\tau)}d\tau$ in Eq. 58 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. 59 also converges.

We set $\sigma=0$ and $t_0=-t_0$ in Eq. 61 and see that the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2-t_0))}{\partial t_2^2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. 58 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. 59 also converges.

We set $t_2=-t_2$ in the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ and use the procedure in Eq. 60 to Eq. 61 and see that the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_2^2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. 58 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in Eq. 59 converges, using Result $D^{'}$.

We set $t_2=-t_2$, $\sigma=0$ and $t_0=-t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. 60 to Eq. 61 and Result $D^{'}$ and see that the integral $\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau-t_0-t_2))}{\partial t_2^2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. 58

corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. 59 also converges. Hence the first integral in Eq. 58, also converges.

We can see that the second integral in Eq. 58 converge, by setting $t_0 = -t_0$ in Eq. 59 to Eq. 61 . Hence all the integrals in Eq. 58 converge.

4.6. ***Zero Crossings in*** $G_{R,2r}(\omega, t_2, t_0)$ ***move continuously as a function of*** t_0***, for a given*** t_2***, for*** $r \in W$***.***

Result 4.7.1: It is shown in **Lemma 1** in Section 2.1 that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ where it crosses the zero line to the opposite sign, if Statement 1 is true. It is shown in Section 4.8 that $G_{R,2r}(\omega, t_2, t_0) = 0$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where $(2r+1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$. (example plot)

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0, for a given value of t_2, with continuous partial derivatives (Section 4.1 and Section 4.4) and given that $G_{R,2r}(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where $(2r + 1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1 in Section 2.1 , Lemma 2 in Section 4.8 and Result 4.7.1), we see that $\omega_z(t_2, t_0)$ is a differentiable function of t_0, for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

• It is shown in Section 4.5 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2. We can use the procedure in previous subsections and Implicit Function Theorem and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2, for $0 < t_2 < \infty$, for each value of t_0 in the interval $0 < t_0 < \infty$.

4.7. ***Zero Crossings in*** $G_{R,2r}(\omega, t_2, t_0)$ ***move continuously as a function of*** t_0 ***and*** t_2***, for*** $r \in W$

We can use the procedure in previous subsections and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 and t_0, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using Implicit Function Theorem in $\Re^3$.

We use **Implicit Function Theorem** for the three dimensional case (link and Theorem 3.2.1 in page 36). Given that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2, with continuous partial derivatives, for $r \in W$ (Section 4.1, Section 4.4 and Section 4.5) and given that $G_{R,2r}(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where $(2r + 1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$ (using Lemma 1 in Section 2.1,Lemma 2 in Section 4.8 and Result 4.7.1), we see that $\omega_z(t_2, t_0)$ is a differentiable function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

4.8. ***Proof of Lemma 2***

In this section, it is shown that, **if** $G_R(\omega, t_2, t_0) = 0$ at $\omega = \pm\omega_z(t_2, t_0)$, for each fixed choice of positive $t_0, t_2 \in \Re$ and $(2r+1)$ is the highest order of the zero at $\omega = \pm\omega_z(t_2, t_0)$ for some value of $r \in W$ (element of set of whole numbers including zero), **then** $G_{R,2r}(\omega, t_2, t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial \omega^{2r}} = 0$ at $\omega = \pm\omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial \omega} = \frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial \omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z(t_2, t_0)$.

In Section 4.1, it is shown that $G_R(\omega, t_2, t_0)$ is partially differentiable $(2r+2)$ times, as a function of ω, where $r \in W$.

We see that $G_R(\omega, t_2, t_0)$ is a real and even function of ω because $g(t, t_2, t_0)$ is a real function of variable t (Appendix D.1) and hence $G_R(\omega, t_2, t_0)$ has its first **zero crossing** at $\omega = \pm\omega_z(t_2, t_0) \neq 0$ where it changes sign, for each fixed $t_0, t_2 \in \Re$ and $t_2 \neq 0$.(Result 2.1.5 in Section 2.1) Hence we can write $G_R(\omega, t_2, t_0) = (\omega_z(t_2, t_0)^2 - \omega^2)^{2r+1} N'(\omega, t_2, t_0)$, for $r \in W$, where $N'(\omega_z(t_2, t_0), t_2, t_0) \neq 0$, for each fixed positive $t_0, t_2 \in \Re$ and $(2r+1)$ is the highest order of the zero at $\omega = \omega_z(t_2, t_0)$.

The case of $(\omega_z(t_2, t_0)^2 - \omega^2)^{2r}$ is **ruled out** because $G_R(\omega, t_2, t_0)$ changes sign at $\omega = \pm\omega_z(t_2, t_0)$ and $N'(\omega, t_2, t_0)$ does not change sign at $\omega = \pm\omega_z(t_2, t_0)$ and $(\omega_z(t_2, t_0)^2 - \omega^2)^{2r} \geq 0$ for real ω and does not change sign at $\omega = \pm\omega_z(t_2, t_0)$.

It is noted that the order of the zero given by $(2r+1)$ is finite because $G_R(\omega, t_2, t_0)$ is finite.

For a fixed positive $t_0, t_2 \in \Re$, let $G_R(\omega, t_2, t_0) = M(\omega)$,$N'(\omega, t_2, t_0) = N(\omega)$ and $\omega_z(t_2, t_0) = \omega_z$.

We consider the case of $M(\omega) = M_r(\omega) = (\omega_z^2 - \omega^2)^{2r+1} N_r(\omega)$ for each $r \in W$, where $N_r(\omega_z) \neq 0$.

Lemma 2: If $M_r(\omega) = (\omega_z^2 - \omega^2)^{2r+1} N_r(\omega)$ where $N_r(\omega_z) \neq 0$ and $r \in W$ and $(2r+1)$ is the highest order of the zero at $\omega = \omega_z$ and $M_r(\omega)$ is differentiable $(2r+1)$ times as a function of ω, **then** $\frac{d^{2r} M_r(\omega)}{d\omega^{2r}} = 0$ and $\frac{d^{2r+1} M_r(\omega)}{d\omega^{2r+1}} \neq 0$ at $\omega = \omega_z$ using principle of mathematical induction.

Proof: For **r=0**, we see that $M_0(\omega) = (\omega_z^2 - \omega^2) N_0(\omega)$ where $N_0(\omega_z) \neq 0$. We see that $M_0(\omega_z) = 0$ (**Result 0.a**) and $M_0'(\omega) = \frac{dM_0(\omega)}{d\omega} = (\omega_z^2 - \omega^2)\frac{dN_0(\omega)}{d\omega} + N_0(\omega)(-2\omega)$. At $\omega = \omega_z$, we see that $M_0'(\omega_z) = N_0(\omega_z)(-2\omega_z)$. Given that $\omega_z \neq 0$ and $N_0(\omega_z) \neq 0$, we get $M_0'(\omega_z) \neq 0$ and hence $\frac{dM_0(\omega)}{d\omega} \neq 0$ at $\omega = \omega_z$ (**Result 0.b**).

4.8.1. ***r=1 and*** $s = 0, 1, 2, 3$

For $r = 1$, we see that $M_1(\omega) = (\omega_z^2 - \omega^2)^3 N_1(\omega)$ where $N_1(\omega) \neq 0$ at $\omega = \omega_z$. We will compute $\frac{d^s M_r(\omega)}{d\omega^s}$ and show that $\frac{d^s M_r(\omega)}{d\omega^s} = \sum_{r'=2r+1-s}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{s,r',r}(\omega)$, for $r = 1$ and $s = 0, 1, ...(2r+1)$. Hence we write $M_1(\omega) = \sum_{r'=3}^{3} (\omega_z^2 - \omega^2)^{r'} A_{0,r',1}(\omega)$ where $A_{0,3,1}(\omega) = N_1(\omega)$, for $s = 0$.

We define $K_{p,r} = 2(2r+2-p) \neq 0$ where $p \leq s$ and $s \leq 2r+1$ and compute $A_{s,r',r}(\omega)$ for $r' = 2r+1-s$, as a **recursive product** and will show that $A_{s,2r+1-s,r}(\omega_z) = (-1)^s \prod_{p=1}^{s} K_{p,r} \omega_z^s N_r(\omega_z) \neq 0$, for $s = 0, 1, ..2r + 1$, for a given $r = 1$ in Eq. 62 to Eq. 64.

We compute the first derivative of $M_1(\omega)$, using $s = 1$. We combine the two terms in the first line in Eq. 62 and write concisely in the second line using $\frac{dM_1(\omega)}{d\omega} = \sum_{r'=2}^{3}(\omega_z^2 - \omega^2)^{r'} A_{1,r',1}(\omega)$, as follows.

$$\frac{dM_1(\omega)}{d\omega} = (\omega_z^2 - \omega^2)^3 \frac{dN_1(\omega)}{d\omega} + N_1(\omega)(3(\omega_z^2 - \omega^2)^2)(-2\omega)$$
$$\frac{dM_1(\omega)}{d\omega} = \sum_{r'=2}^{3}(\omega_z^2 - \omega^2)^{r'} A_{1,r',1}(\omega), \quad A_{1,2,1}(\omega) = -6\omega N_1(\omega) = -6\omega A_{0,3,1}(\omega) = -\prod_{p=1}^{1} K_{p,1}\omega^1 N_1(\omega) \tag{62}$$

We see that $K_{p,r} = 2(2r + 2 - p)$ and $K_{1,1} = 6$ for $p = 1, r = 1$ and $A_{1,3,1}(\omega) = \frac{dN_1(\omega)}{d\omega}$. We see that $2r + 1 - s = 2$ for $r = 1, s = 1$ and hence $A_{s,2r+1-s,r}(\omega_z) = A_{1,2,1}(\omega_z) = -6\omega_z N_1(\omega_z) \neq 0$ given that $\omega_z \neq 0$ and $N_1(\omega_z) \neq 0$.(**Result 4.6.1**)

We take the derivative of $\frac{dM_1(\omega)}{d\omega}$ in Eq. 62, using $s = 2$. The second term $(\omega_z^2 - \omega^2)^{r'-1} = (\omega_z^2 - \omega^2)^1$ for $r' = 2$, in the summation in the first line in Eq. 63 and hence we combine the two terms in the first line, by **including** $r' = 1$ in the summation in the second line and write concisely as follows.

$$\frac{d^2M_1(\omega)}{d\omega^2} = \sum_{r'=2}^{3}(\omega_z^2 - \omega^2)^{r'} \frac{dA_{1,r',1}(\omega)}{d\omega} + A_{1,r',1}(\omega) r'(\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^2M_1(\omega)}{d\omega^2} = \sum_{r'=1}^{3}(\omega_z^2 - \omega^2)^{r'} A_{2,r',1}(\omega), \quad A_{2,1,1}(\omega) = -4\omega A_{1,2,1}(\omega) = 24\omega^2 N_1(\omega) = \prod_{p=1}^{2} K_{p,1}\omega^2 N_1(\omega) \tag{63}$$

We see that $K_{2,1} = 2(2r + 2 - p) = 4$ for $p = 2, r = 1$ and $A_{2,2,1}(\omega) = \frac{dA_{1,2,1}(\omega)}{d\omega} - 6\omega A_{1,3,1}(\omega)$ and $A_{2,3,1}(\omega) = \frac{dA_{1,3,1}(\omega)}{d\omega}$. We see that $2r+1-s = 1$ for $r = 1, s = 2$ and hence $A_{s,2r+1-s,r}(\omega) = A_{2,1,1}(\omega) = -4\omega A_{1,2,1}(\omega) = 24\omega^2 N_1(\omega)$ using Eq. 62 and Result 4.6.1 and $A_{2,1,1}(\omega_z) = 24\omega_z^2 N_1(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_1(\omega_z) \neq 0$ (**Result 4.6.2**)

We take the next derivative of $\frac{d^2M_1(\omega)}{d\omega^2}$ in Eq. 63 and combine the two terms as follows, using $s = 3$.

$$\frac{d^3M_1(\omega)}{d\omega^3} = \sum_{r'=1}^{3}(\omega_z^2 - \omega^2)^{r'} \frac{dA_{2,r',1}(\omega)}{d\omega} + A_{2,r',1}(\omega) r'(\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^3M_1(\omega)}{d\omega^3} = \sum_{r'=0}^{3}(\omega_z^2 - \omega^2)^{r'} A_{3,r',1}(\omega), \quad A_{3,0,1}(\omega) = -2\omega A_{2,1,1}(\omega) = -48\omega^3 N_1(\omega) = -\prod_{p=1}^{3} K_{p,1}\omega^3 N_1(\omega) \tag{64}$$

We see that $K_{3,1} = 2(2r + 2 - p) = 2$ for $p = 3, r = 1$ and $A_{3,1,1}(\omega) = \frac{dA_{2,1,1}(\omega)}{d\omega} - 4\omega A_{2,2,1}(\omega)$, $A_{3,2,1}(\omega) = \frac{dA_{2,2,1}(\omega)}{d\omega} - 6\omega A_{2,3,1}(\omega)$ and $A_{3,3,1}(\omega) = \frac{dA_{2,3,1}(\omega)}{d\omega}$. We see that $2r+1-s = 0$ for $r = 1, s = 3$ and hence $A_{s,2r+1-s,r}(\omega) = A_{3,0,1}(\omega) = -2\omega A_{2,1,1}(\omega) = -48\omega^3 N_1(\omega)$ using Eq. 63 and Result 4.6.2

and $A_{3,0,1}(\omega_z) = -48\omega_z^3 N_1(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_1(\omega_z) \neq 0$.(**Result 4.6.3**)

We see that $\frac{d^2M_1(\omega)}{d\omega^2} = 0$ at $\omega = \omega_z$ in Eq. 63 (**Result 1.a**). We evaluate $B_3(\omega) = \frac{d^3M_1(\omega)}{d\omega^3}$ at $\omega = \omega_z$ and see that all terms become zero except the term with $r' = 0$ in Eq. 64. Hence $B_3(\omega_z) = A_{3,0,1}(\omega_z) \neq 0$ using Result 4.6.3 and hence $\frac{d^3M_1(\omega)}{d\omega^3} \neq 0$ at $\omega = \omega_z$ (**Result 1.b**).

4.8.2. ***r=2 and*** $s = 0, 1, 2, 3, 4, 5$

For $r = 2$, we see that $M_2(\omega) = (\omega_z^2 - \omega^2)^5 N_2(\omega)$ where $N_2(\omega) \neq 0$ at $\omega = \omega_z$. We will compute $\frac{d^sM_r(\omega)}{d\omega^s}$ and show that $\frac{d^sM_r(\omega)}{d\omega^s} = \sum_{r'=2r+1-s}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{s,r',r}(\omega)$, for $r = 2$ and $s = 0, 1, ...(2r+1)$. Hence we write $M_2(\omega) = \sum_{r'=5}^{5} (\omega_z^2 - \omega^2)^{r'} A_{0,r',2}(\omega)$ where $A_{0,5,2}(\omega) = N_2(\omega)$, for $s = 0$.

We define $K_{p,r} = 2(2r + 2 - p) \neq 0$ where $p \leq s$ and $s \leq 2r + 1$. We compute $A_{s,r',r}(\omega)$ for $r' = 2r+1-s$, as a **recursive product** and will show that $A_{s,2r+1-s,r}(\omega_z) = (-1)^s \prod_{p=1}^{s} K_{p,r}\omega_z^s N_r(\omega_z) \neq 0$ for $s = 0, 1, ..2r + 1$, for a given $r = 2$ in Eq. 65 to Eq. 69. We compute the first derivative of $M_2(\omega)$ and combine the two terms as follows, using $s = 1$.

$$\frac{dM_2(\omega)}{d\omega} = (\omega_z^2 - \omega^2)^5 \frac{dN_2(\omega)}{d\omega} + N_2(\omega)(5(\omega_z^2 - \omega^2)^4)(-2\omega)$$
$$\frac{dM_2(\omega)}{d\omega} = \sum_{r'=4}^{5} (\omega_z^2 - \omega^2)^{r'} A_{1,r',2}(\omega), \quad A_{1,4,2}(\omega) = -10\omega N_2(\omega) = -10\omega A_{0,5,2}(\omega) = -\prod_{p=1}^{1} K_{p,2}\omega^1 N_2(\omega) \tag{65}$$

We see that $K_{p,r} = 2(2r + 2 - p) = 10$ for $p = 1, r = 2$ and $A_{1,5,2}(\omega) = \frac{dN_2(\omega)}{d\omega}$. We see that $2r + 1 - s = 4$ for $r = 2, s = 1$ and hence $A_{s,2r+1-s,r}(\omega_z) = A_{1,4,2}(\omega_z) = -10\omega_z N_2(\omega_z) \neq 0$ given that $\omega_z \neq 0$ and $N_2(\omega_z) \neq 0$.(**Result 4.6.4**)

We take the next derivative of $\frac{dM_2(\omega)}{d\omega}$ in Eq. 65, using $s = 2$. The second term $(\omega_z^2 - \omega^2)^{r'-1} = (\omega_z^2 - \omega^2)^3$ for $r' = 4$, in the summation in the first line in Eq. 66 and hence we combine the two terms in the first line, by **including** $r' = 3$ in the summation in the second line and write concisely as follows.

$$\frac{d^2M_2(\omega)}{d\omega^2} = \sum_{r'=4}^{5} (\omega_z^2 - \omega^2)^{r'} \frac{dA_{1,r',2}(\omega)}{d\omega} + A_{1,r',2}(\omega) r' (\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^2M_2(\omega)}{d\omega^2} = \sum_{r'=3}^{5} (\omega_z^2 - \omega^2)^{r'} A_{2,r',2}(\omega), \quad A_{2,3,2}(\omega) = -8\omega A_{1,4,2}(\omega) = 80\omega^2 N_2(\omega) = \prod_{p=1}^{2} K_{p,2}\omega^2 N_2(\omega) \tag{66}$$

We see that $K_{2,2} = 2(2r + 2 - p) = 8$ for $p = 2, r = 2$ and $A_{2,4,2}(\omega) = \frac{dA_{1,4,2}(\omega)}{d\omega} - 10\omega A_{1,5,2}(\omega)$ and $A_{2,5,2}(\omega) = \frac{dA_{1,5,2}(\omega)}{d\omega}$. We see that $2r+1-s = 3$ for $r = 2, s = 2$ and hence $A_{s,2r+1-s,r}(\omega) = A_{2,3,2}(\omega) = -8\omega A_{1,4,2}(\omega) = 80\omega^2 N_2(\omega)$ using Eq. 65 and Result 4.6.4 and $A_{2,3,2}(\omega_z) = 80\omega_z^2 N_2(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_2(\omega_z) \neq 0$ (**Result 4.6.5**)

We take the next derivative of $\frac{d^2 M_2(\omega)}{d\omega^2}$ in Eq. 66 and combine the two terms as follows, using $s = 3$.

$$\frac{d^3 M_2(\omega)}{d\omega^3} = \sum_{r'=3}^{5} (\omega_z^2 - \omega^2)^{r'} \frac{dA_{2,r',2}(\omega)}{d\omega} + A_{2,r',2}(\omega) r' (\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^3 M_2(\omega)}{d\omega^3} = \sum_{r'=2}^{5} (\omega_z^2 - \omega^2)^{r'} A_{3,r',2}(\omega), \quad A_{3,2,2}(\omega) = -6\omega A_{2,3,2}(\omega) = -480\omega^3 N_2(\omega) = -\prod_{p=1}^{3} K_{p,2}\omega^3 N_2(\omega) \tag{67}$$

We see that $K_{3,2} = 2(2r + 2 - p) = 6$ for $p = 3, r = 2$ and $A_{3,3,2}(\omega) = \frac{dA_{2,3,2}(\omega)}{d\omega} - 8\omega A_{2,4,2}(\omega)$, $A_{3,4,2}(\omega) = \frac{dA_{2,4,2}(\omega)}{d\omega} - 10\omega A_{2,5,2}(\omega)$ and $A_{3,5,2}(\omega) = \frac{dA_{2,5,2}(\omega)}{d\omega}$. We see that $2r+1-s = 2$ for $r = 2, s = 3$ and hence $A_{s,2r+1-s,r}(\omega) = A_{3,2,2}(\omega) = -6\omega A_{2,3,2}(\omega) = -480\omega^3 N_2(\omega)$ using Eq. 66 and Result 4.6.5 and $A_{3,2,2}(\omega_z) = -480\omega_z^3 N_2(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_2(\omega_z) \neq 0$.(**Result 4.6.6**)

We take the next derivative of $\frac{d^3 M_2(\omega)}{d\omega^3}$ in Eq. 67 and combine the two terms as follows, using $s = 4$.

$$\frac{d^4 M_2(\omega)}{d\omega^4} = \sum_{r'=2}^{5} (\omega_z^2 - \omega^2)^{r'} \frac{dA_{3,r',2}(\omega)}{d\omega} + A_{3,r',2}(\omega) r' (\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^4 M_2(\omega)}{d\omega^4} = \sum_{r'=1}^{5} (\omega_z^2 - \omega^2)^{r'} A_{4,r',2}(\omega), \quad A_{4,1,2}(\omega) = -4\omega A_{3,2,2}(\omega) = 480 * 4\omega^4 N_2(\omega) = \prod_{p=1}^{4} K_{p,2}\omega^4 N_2(\omega) \tag{68}$$

We see that $K_{4,2} = 2(2r + 2 - p) = 4$ for $p = 4, r = 2$. We see that $2r + 1 - s = 1$ for $r = 2, s = 4$ and hence $A_{s,2r+1-s,r}(\omega) = A_{4,1,2}(\omega) = -4\omega A_{3,2,2}(\omega) = 480 * 4\omega^4 N_2(\omega)$ using Result 4.6.6 and $A_{4,1,2}(\omega_z) = 480 * 4\omega_z^4 N_2(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_2(\omega_z) \neq 0$.(**Result 4.6.7**)

We take the next derivative of $\frac{d^4 M_2(\omega)}{d\omega^4}$ in Eq. 68 and combine the two terms as follows, using $s = 5$.

$$\frac{d^5 M_2(\omega)}{d\omega^5} = \sum_{r'=1}^{5} (\omega_z^2 - \omega^2)^{r'} \frac{dA_{4,r',2}(\omega)}{d\omega} + A_{4,r',2}(\omega) r' (\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^5 M_2(\omega)}{d\omega^5} = \sum_{r'=0}^{5} (\omega_z^2 - \omega^2)^{r'} A_{5,r',2}(\omega), \quad A_{5,0,2}(\omega) = -2\omega A_{4,1,2}(\omega) = -480 * 4 * 2\omega^5 N_2(\omega) = -\prod_{p=1}^{5} K_{p,2}\omega^5 N_2(\omega) \tag{69}$$

We see that $K_{5,2} = 2(2r + 2 - p) = 2$ for $p = 5, r = 2$. We see that $2r + 1 - s = 0$ for $r = 2, s = 5$ and hence $A_{s,2r+1-s,r}(\omega) = A_{5,0,2}(\omega) = -2\omega A_{4,1,2}(\omega) = -480 * 4 * 2\omega^5 N_2(\omega)$ using Result 4.6.7 and $A_{5,0,2}(\omega_z) = -480 * 4 * 2\omega_z^5 N_2(\omega_z) \neq 0$, given that $\omega_z \neq 0$ and $N_2(\omega_z) \neq 0$.(**Result 4.6.8**)

We see that $\frac{d^4 M_2(\omega)}{d\omega^4} = 0$ at $\omega = \omega_z$ in Eq. 68 (**Result 2.a**). We evaluate $B_5(\omega) = \frac{d^5 M_2(\omega)}{d\omega^5}$ at $\omega = \omega_z$ and see that all terms become zero except the term with $r' = 0$ in Eq. 69. Hence $B_5(\omega_z) = A_{5,0,2}(\omega_z) \neq 0$ using Result 4.6.8 and hence $\frac{d^5 M_2(\omega)}{d\omega^5} \neq 0$ at $\omega = \omega_z$ (**Result 2.b**).

4.8.3. *Induction Proof for each* $r \in W$

For a general $r \in W$, we see that $M_r(\omega) = (\omega_z^2 - \omega^2)^{2r+1} N_r(\omega)$ where $N_r(\omega_z) \neq 0$. Using the equations for $r = 1$ in Section 4.8.1 and $r = 2$ in Section 4.8.2 , we build the equation used in Induction hypothesis for $\frac{d^s M_r(\omega)}{d\omega^s}$, for $s = 0, 1, ...(2r+1)$, for **each** $r \in W$, as follows. (Set $r = 1$, $s = 2$ in Eq. 70 and we get Eq. 63 and Result 4.6.2. Set $r = 2$, $s = 5$ in Eq. 70 and we get Eq. 69 and Result 4.6.8.)

$$\frac{d^s M_r(\omega)}{d\omega^s} = \sum_{r'=2r+1-s}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{s,r',r}(\omega), \quad A_{s,2r+1-s,r}(\omega) = A_{s-1,2r+2-s,r}(\omega)(-2\omega)(2r+2-s)$$
$$A_{s,2r+1-s,r}(\omega_z) = (-1)^s \prod_{p=1}^{s} K_{p,r} \omega_z^s N_r(\omega_z) \neq 0, \quad K_{p,r} = 2(2r+2-p) \neq 0 \tag{70}$$

It is noted that we only need the coefficient $A_{s,r',r}(\omega)$ corresponding to $r' = 2r+1-s$ because the terms for $r' \neq 0$ in the equation for $\frac{d^s M_r(\omega)}{d\omega^s}$ for $s = 2r+1$ vanish at $\omega = \omega_z$, as shown in Eq. 74.

- **Induction Hypothesis:** We assume that Eq. 70 holds for $s = S$, for $S < 2r+1$.

$$\frac{d^S M_r(\omega)}{d\omega^S} = \sum_{r'=2r+1-S}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{S,r',r}(\omega), \quad A_{S,2r+1-S,r}(\omega) = A_{S-1,2r+2-S,r}(\omega)(-2\omega)(2r+2-S)$$
$$A_{S,2r+1-S,r}(\omega_z) = (-1)^S \prod_{p=1}^{S} K_{p,r} \omega_z^S N_r(\omega_z) \neq 0, \quad K_{p,r} = 2(2r+2-p) \neq 0 \tag{71}$$

- **Induction Step:** We take the first derivative of Eq. 71 given by $\frac{d}{d\omega}\frac{d^S M_r(\omega)}{d\omega^S} = \frac{d^{S+1} M_r(\omega)}{d\omega^{S+1}}$. The second term $(\omega_z^2 - \omega^2)^{r'-1} = (\omega_z^2 - \omega^2)^{2r-S}$ for $r' = 2r+1-S$, in the summation in the first line in Eq. 72 and hence we combine the two terms in the first line, by **including** $r' = 2r - S$ in the summation in the second line and write concisely as follows.

$$\frac{d^{S+1} M_r(\omega)}{d\omega^{S+1}} = \sum_{r'=2r+1-S}^{2r+1} (\omega_z^2 - \omega^2)^{r'} \frac{dA_{S,r',r}(\omega)}{d\omega} + A_{S,r',r}(\omega) r' (\omega_z^2 - \omega^2)^{r'-1}(-2\omega)$$
$$\frac{d^{S+1} M_r(\omega)}{d\omega^{S+1}} = \sum_{r'=2r-S}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{S+1,r',r}(\omega), \quad A_{S+1,2r-S,r}(\omega) = A_{S,2r+1-S,r}(\omega)(-2\omega)(2r+1-S)$$
$$A_{S+1,2r-S,r}(\omega_z) = -A_{S,2r+1-S,r}(\omega_z)(\omega_z)2(2r+1-S) = (-1)^{S+1} \prod_{p=1}^{S+1} K_{p,r} \omega_z^{S+1} N_r(\omega_z) \neq 0 \tag{72}$$

We see that $K_{S+1,r} = 2(2r+1-S) \neq 0$ for $S < 2r+1$ and we use $A_{S,2r+1-S,r}(\omega_z)$ in Eq. 71 to get $A_{S+1,2r-S,r}(\omega_z)$ in Eq. 72.

We see that Eq. 72 is **exactly the same** as the equation we get, if we set $s = S + 1$ in Eq. 70. Thus we have proved Eq. 70 by **principle of mathematical induction**.

• We set $s = 2r$ in Eq. 70 and get

$$\frac{d^{2r} M_r(\omega)}{d\omega^{2r}} = \sum_{r'=1}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{2r,r',r}(\omega), \quad A_{2r,1,r}(\omega) = A_{2r-1,2,r}(\omega)(-4\omega)$$
$$A_{2r,1,r}(\omega_z) = (-1)^{2r} \prod_{p=1}^{2r} K_{p,r}\omega_z^{2r} N_r(\omega_z) \neq 0 \tag{73}$$

We see that all the terms in $\frac{d^{2r} M_r(\omega)}{d\omega^{2r}}$ in Eq. 73 become zero at $\omega = \omega_z$ and hence $\frac{d^{2r} M_r(\omega)}{d\omega^{2r}} = 0$ at $\omega = \omega_z$. (**Result r.a**)

• We set $s = 2r + 1$ in Eq. 70 and get

$$\frac{d^{2r+1} M_r(\omega)}{d\omega^{2r+1}} = \sum_{r'=0}^{2r+1} (\omega_z^2 - \omega^2)^{r'} A_{2r+1,r',r}(\omega), \quad A_{2r+1,0,r}(\omega) = A_{2r,1,r}(\omega)(-2\omega)$$
$$A_{2r+1,0,r}(\omega_z) = (-1)^{2r+1} \prod_{p=1}^{2r+1} K_{p,r}\omega_z^{2r+1} N_r(\omega_z) \neq 0 \tag{74}$$

We see that all the terms in $\frac{d^{2r+1} M_r(\omega)}{d\omega^{2r+1}}$ in Eq. 74 become zero at $\omega = \omega_z$ except the term for $r' = 0$ and $A_{2r+1,0,r}(\omega_z) \neq 0$ and hence $\frac{d^{2r+1} M_r(\omega)}{d\omega^{2r+1}} \neq 0$ at $\omega = \omega_z$. (**Result r.b**)

Corollary: The Induction proof presented in this section and Result r.a and Result r.b are valid for **each** $r \in W$. Hence we see that $\frac{d^{2r} M_r(\omega)}{d\omega^{2r}} = 0$ at $\omega = \omega_z$ and $\frac{d^{2r+1} M_r(\omega)}{d\omega^{2r+1}} \neq 0$ at $\omega = \omega_z$, for each $r \in W$, where $M_r(\omega) = (\omega_z^2 - \omega^2)^{2r+1} N_r(\omega)$, where $N_r(\omega_z) \neq 0$, and $(2r + 1)$ is the highest order of the zero of $M_r(\omega)$ at $\omega = \omega_z$.

Given that $G_R(\omega, t_2, t_0) = M_r(\omega)$ for some value of $r \in W$ and fixed choice of t_0, t_2, we see that $\frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} = 0$ at $\omega = \pm\omega_z$ and $\frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z$, given that $M(\omega) = G_R(\omega, t_2, t_0)$ is a real and even function of ω, where $(2r + 1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$. This induction proof continues to hold for **each** fixed choice of positive $t_0, t_2 \in \Re$.

5. $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached for specific t_0, t_2

It is noted that we **do not** use $\lim_{t_0 \to \infty}$ in this section. Instead we consider real $t_0 > 0$ which increases to a larger and larger finite value without bounds. We use $0 < \sigma < \frac{1}{2}$ below.

We write $P_{odd}(t_2, t_0)$ in Eq. 20 derived assuming Statement 1, concisely as follows.

$$P_{odd}(t_2,t_0) = \int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau + e^{2\sigma t_0}\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau$$
$$P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0 \tag{75}$$

We note that $E_0^{'}(\tau,t_2) = E_0(\tau-t_2) - E_0(\tau+t_2)$ and $E_{0n}^{'}(\tau,t_2) = E_0^{'}(-\tau,t_2) = -E_0^{'}(\tau,t_2) = E_0(\tau+t_2) - E_0(\tau-t_2)$ (using Result 3.1 in Section 3). We choose $t_2 = 2t_0$ and we choose t_1 such that $E_0(t)$ approximates zero for $|t| > t_1$, given that $E_0(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-1.5|t|}]$ (Appendix C.5). We choose $t_0 >> t_1$ and hence $E_0(\tau-t_2) = E_0(\tau-2t_0)$ approximates zero in the interval $(-\infty,t_0]$. Hence in the interval $(-\infty,t_0]$, we see that $E_0^{'}(\tau,t_2) \approx -E_0(\tau+t_2)$ and $E_{0n}^{'}(\tau,t_2) \approx E_0(\tau+t_2)$, for sufficiently large t_0. We can write Eq. 75 as follows. We use $\omega_z(t_2,-t_0) = \omega_z(t_2,t_0)$ (Section 2.4). We **note that** $t_2 = 2t_0$ in the rest of this section and we continue to use the notation $\omega_z(t_2,t_0)$ where $t_2 = 2t_0$.

$$\begin{aligned}
P_{odd}(t_2,t_0) &\approx -\int_{-\infty}^{t_0} E_0(\tau+2t_0)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau \\
&\quad + e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau+2t_0)\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau \\
P_{odd}(t_2,-t_0) &= \int_{-\infty}^{-t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)(\tau+t_0))d\tau \\
&\quad + e^{-2\sigma t_0}\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)(\tau+t_0))d\tau
\end{aligned} \tag{76}$$

We see that the term $P_{odd}(t_2,-t_0)$ in Eq. 76 approaches a value very close to zero, as real t_0 increases to a larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals $\int_{-\infty}^{-t_0}$, given $0 < \sigma < \frac{1}{2}$ and $t_0 > 0$ and given that the integrands are absolutely integrable and finite because the terms $E_0^{'}(\tau,t_2)e^{-2\sigma\tau}$ and $E_{0n}^{'}(\tau,t_2) = -E_0^{'}(\tau,t_2)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Section 4.2) Hence we can ignore $P_{odd}(t_2,-t_0)$ for sufficiently large t_0 and write Eq. 75, using Eq. 76 and $t_2 = 2t_0$.

$$\begin{aligned}
Q(t_0) = P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) &\approx -\int_{-\infty}^{t_0} E_0(\tau+2t_0)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau \\
&\quad + e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau+2t_0)\cos(\omega_z(t_2,t_0)(\tau-t_0))d\tau \approx 0
\end{aligned} \tag{77}$$

We substitute $\tau + 2t_0 = t$, $\tau = t - 2t_0$ and $d\tau = dt$ in Eq. 77 and write as follows.

$$\begin{aligned}
Q(t_0) &\approx -e^{4\sigma t_0}\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)(t-3t_0))dt \\
&\quad + e^{2\sigma t_0}\int_{-\infty}^{3t_0} E_0(t)\cos(\omega_z(t_2,t_0)(t-3t_0))dt \approx 0
\end{aligned}$$

$$(78)$$

We multiply Eq. 78 by $e^{-3\sigma t_0}$ and ignore the last integral for sufficiently large t_0, given that $e^{2\sigma t_0}e^{-3\sigma t_0} = e^{-\sigma t_0}$ and $|\int_{-\infty}^{3t_0} E_0(t)\cos(\omega_z(t_2,t_0)(t-3t_0))dt| \leq \int_{-\infty}^{3t_0} |E_0(t)|dt < \int_{-\infty}^{\infty} |E_0(t)|dt$ is finite. (link and Appendix C.1)

$$S(t_0) = Q(t_0)e^{-3\sigma t_0} \approx -e^{\sigma t_0}\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)(t-3t_0))dt = -e^{\sigma t_0}R(t_0) \approx 0$$

$$R(t_0) = \cos(\omega_z(t_2,t_0)3t_0)\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt + \sin(\omega_z(t_2,t_0)3t_0)\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\sin(\omega_z(t_2,t_0)t)dt \tag{79}$$

In Section 2.1, it is shown that $0 < \omega_z(t_2,t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2. For $t_0 > 0$, we see that $\omega_z(t_2,t_0)t_0 > 0$. In Section 4, it is shown that $\omega_z(t_2,t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. Hence $\omega_z(t_2,t_0)t_0$ is a positive continuous function.

We **require** $\omega_z(t_2,t_0)t_0 = \frac{\pi}{2}$ in Section 3 for a specific $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$. To show that $\omega_z(t_2,t_0)t_0 = \frac{\pi}{2}$ can be reached, we **assume the opposite** case that $\omega_z(t_2,t_0)t_0 < \frac{\pi}{2}$ for all $0 < t_0 < \infty$ and $t_2 = 2t_0$ (**Statement C**) and show that this leads to a **contradiction**.

Let $\omega_z(t_2,t_0)t_0 = KF(t_2,t_0)$, where $0 < K < \frac{\pi}{2}$ and $0 < F(t_2,t_0) \leq 1$ is a positive continuous function for $0 < t_0 < \infty$ and $t_2 = 2t_0$, such that $\omega_z(t_2,t_0)t_0 < \frac{\pi}{2}$. Hence $\omega_z(t_2,t_0) = \frac{KF(t_2,t_0)}{t_0}$.

We choose t_3 such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$ (**Result 5.a**), given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Appendix C.5).We choose $t_0 >> t_3$ and note that t_3 is **independent** of t_0. As t_0 increase without bounds, in the interval $|t| \leq t_3$, we see that the term $\cos(\omega_z(t_2,t_0)t) \approx 1$ and $\sin(\omega_z(t_2,t_0)t) \approx \omega_z(t_2,t_0)t \approx 0$ (**Result 5.b**), given that $\omega_z(t_2,t_0)t = \frac{KF(t_2,t_0)t}{t_0} \leq \frac{KF(t_2,t_0)t_3}{t_0} << 1$, because $t_0 >> t_3$ and $F(t_2,t_0) \leq 1$. Hence we write Eq. 79 as follows, using Result 5.a and Result 5.b.

$$R(t_0) \approx \cos(\omega_z(t_2,t_0)3t_0)\int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt \approx \cos(3KF(t_2,t_0))\int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt \tag{80}$$

For sufficiently large t_0, the integral $R(t_0) \approx \cos(3KF(t_2,t_0))\int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt$ remains finite, because $\cos(\omega_z(t_2,t_0)3t_0)$ oscillates in the interval $[-1,1]$ and $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}dt > 0$ (Appendix C.1) and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds. This is explained in detail in Section 5.1.

The term $e^{\sigma t_0}$ in $S(t_0) = -e^{\sigma t_0}R(t_0)$ in Eq. 79 increases to a larger and larger finite value **exponentially** as t_0 increases, and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ in Eq. 78 and $P_{odd}(t_2,t_0)+P_{odd}(t_2,-t_0)$ in Eq. 75 **cannot** equal zero, to satisfy Statement 1, in this case.

Hence **Statement C** is **false** and hence $\omega_z(t_2,t_0)t_0 = \frac{\pi}{2}$ can be reached for specific values of t_0 and $t_2 = 2t_0$, as finite t_0 increases without bounds, given that $\omega_z(t_2,t_0)t_0$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

5.1. $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$ *does not have exponential fall off rate*

We compute the **minimum** value of the integral $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$ in Eq. 79 , for sufficiently large t_3 and $t_0 >> t_3$ and $0 < \sigma < \frac{1}{2}$. We note that $t_2 = 2t_0$ and note that t_3 is **independent** of t_0 below. We split $A(t_0)$ as follows.

$$
\begin{aligned}
A(t_0) &= B(t_3,t_0) + C(t_3,t_0) + D(t_3,t_0)\\
B(t_3,t_0) = \int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt, \quad C(t_3,t_0) &= \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt\\
D(t_3,t_0) &= \int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt
\end{aligned}
\tag{81}
$$

We see that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ and $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function (Appendix C.1) and hence $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt > 0$ (**Result 5.1.1**).

Given that $\omega_z(t_2,t_0) = \frac{KF(t_2,t_0)}{t_0}$ where $0 < K < \frac{\pi}{2}$ and $0 < F(t_2,t_0) \le 1$ in previous subsection and $t_0 >> t_3$, we see that $\omega_z(t_2,t_0)t = \frac{KF(t_2,t_0)t}{t_0} \le \frac{KF(t_2,t_0)t_3}{t_0} << 1$ in the interval $|t| \le t_3$ and hence $\cos(\omega_z(t_2,t_0)t) \approx 1$ and $\cos(\omega_z(t_2,t_0)t) > \frac{1}{2}$ in the interval $|t| \le t_3$. Hence we can write $C(t_3,t_0) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} > 0$, using Result 5.1.1. (**Result 5.1.2**).

We see that $|B(t_3,t_0)| = |\int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt| \le \int_{-\infty}^{-t_3} |E_0(t)e^{-2\sigma t}|dt \approx 0$ (link) and $|D(t_3,t_0)| = |\int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt| \le \int_{t_3}^{3t_0} |E_0(t)e^{-2\sigma t}|dt \approx 0$, for sufficiently large t_3 and $t_0 >> t_3$, given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Appendix C.5) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Appendix C.1).

As we increase t_3 to t_3' and t_0 to $t_0' >> t_3'$, we see that $C(t_3',t_0') > C(t_3,t_0) > 0$, using Result 5.1.1 and Result 5.1.2, given that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (**Result 5.1.3**).

As we increase t_3 to t_3' and t_0 to $t_0' >> t_3'$, we see that $|B(t_3',t_0')| < |B(t_3,t_0)|$ and $|D(t_3',t_0')| < |D(t_3,t_0)|$ approach zero (**Result 5.1.4**), given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Appendix C.5) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Appendix C.1).

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} - |B(t_3,t_0)| - |D(t_3,t_0)| \approx \frac{C_0(t_3)}{2} > 0$ using Result 5.1.2, Result 5.1.3 and Result 5.1.4.

For example, we choose $t_3 = 10$ such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. Given that $E_0(t) > 0$ for $|t| < \infty$ (Appendix C.7) and the term $e^{-2\sigma t}$ has a minimum value of $e^{-|t|}$ for $0 < \sigma < \frac{1}{2}$, we see that the integral $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt > 2\int_0^{t_3} E_0(t)e^{-|t|}dt > C_{00} = 0.42$ where C_{00} is computed by considering the first 5 terms $n = 1,2,3,4,5$ in $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$. Hence $C_0(t_3) > 0.42$. (Matlab simulation)

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} - |B(t_3,t_0)| - |D(t_3,t_0)| \approx 0.21$. As t_0 increases without bounds, we see that $A(t_0)$ **does not** have exponential fall off rate.

6. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_0(t) = \Phi(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ in Eq. 1, whose Fourier Transform is given by the entire function $E_{0\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. It is known that $\Phi(t)$ is positive for $|t| < \infty$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t > 0$. (link). This is shown below. We take the term $2\pi n^2$ out of the brackets.

$$E_0(t) = \Phi(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi n^2 e^{4t} - 3e^{2t}] \tag{82}$$

We show that $X(t) = \frac{E_0(t)}{2}$ is a **strictly decreasing** function for $t > 0$ as follows.

- In Section 6.1, it is shown that the first derivative of $X(t)$, given by $\frac{dX(t)}{dt} < 0$ for $t > t_z$ where $t_z = \frac{1}{2}\log\frac{y_z}{\pi}$ and $y_z = 3.16$.

- In Section 6.2, it is shown that, $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$.

Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$ and hence $X(t)$ is strictly decreasing for all $t > 0$ and $E_0(t) = 2X(t)$ is **strictly decreasing** for all $t > 0$.

6.1. $\frac{dX(t)}{dt} < 0$ ***for*** $t > t_z$

We consider $X(t) = \frac{E_0(t)}{2} = \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi n^2 e^{4t} - 3e^{2t}]$ in Eq. 82 and take the first derivative of $X(t)$. We note that $E_0(t)$ and $X(t)$ are analytic functions for real t and infinitely differentiable in that interval. We compute $\frac{dX(t)}{dt}$ below and take the term e^{2t} out, in the last line below.

$$\begin{aligned}
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[8\pi n^2 e^{4t} - 6e^{2t} + (2\pi n^2 e^{4t} - 3e^{2t})(\frac{1}{2} - 2\pi n^2 e^{2t})] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[8\pi n^2 e^{4t} - 6e^{2t} + (\pi n^2 e^{4t} - \frac{3}{2}e^{2t} - 4\pi^2 n^4 e^{6t} + 6\pi n^2 e^{4t})] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi^2 n^4 e^{6t} + 15\pi n^2 e^{4t} - \frac{15}{2}e^{2t}] \\
\frac{dX(t)}{dt} &= \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{2t}[-4\pi^2 n^4 e^{4t} + 15\pi n^2 e^{2t} - \frac{15}{2}]
\end{aligned} \tag{83}$$

We substitute $y = \pi e^{2t}$ in Eq. 83 and define $A(y)$ such that $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$. [8]

$$A(y) = \sum_{n=1}^{\infty} n^2 e^{-n^2 y}[-4n^4 y^2 + 15n^2 y - \frac{15}{2}] \tag{84}$$

We see that $A(y) = 0$ at $y = \pi$ which corresponds to $t = 0$ given $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, given that $\frac{dX(t)}{dt} = 0$ at $t = 0$. Because $X(t) = \frac{E_0(t)}{2}$ is an even function of variable t(Appendix C.8) and hence $\frac{dX(t)}{dt}$ is an **odd** function of variable t.

The quadratic expression $B(y,n) = (-4n^4y^2+15n^2y-\frac{15}{2})$ in Eq. 84 has roots at $y = \frac{-15n^2 \pm\sqrt{225n^4-120n^4}}{-8n^4} = \frac{(15\pm\sqrt{105})}{8n^2}$. We see that the first derivative of $B(y,n)$ is given by $\frac{dB(y,n)}{dy} = -8n^4y + 15n^2$ is zero at $y = \frac{15}{8n^2}$. The second derivative of $B(y,n)$ given by $\frac{d^2B(y,n)}{dy^2} = -8n^4$, is negative for all y and $n \geq 1$ and hence $B(y,n)$ is a **concave down** function for each n, which reaches a maximum at $y = \frac{15}{8n^2}$ and given the dominant term $-4n^4y^2$ in Eq. 84 , we see that $B(y,n) < 0$, for $y > \frac{(15+\sqrt{105})}{8} > 3.16 = y_z$, for $n \geq 1$ and hence $A(y) < 0$ for $y > y_z$. Using $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$, we see that $\frac{dX(t)}{dt} < 0$ for $t > \frac{1}{2}\log\frac{y_z}{\pi} = t_z$(**Result 1**). (concave down function)

We show in the next section that $\frac{dX(t)}{dt} < 0$ for $0 < t \leq t_z$. It suffices to show that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq y_z = 3.16$ and hence $A(y) < 0$ for $\pi < y \leq y_z = 3.16$, given that $A(y) = 0$ at $y = \pi$. [We use $y = \pi e^{2t}$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}} A(y)$ and $\frac{dX(t)}{dt} = 0$ at $t = 0$.]

6.2. $\frac{dX(t)}{dt} < 0$ *for* $0 < t \leq t_z$

It is shown in this section that $\frac{dA(y)}{dy} < 0$ for $\pi \leq y \leq 3.16$ and hence $A(y) < 0$ for $\pi < y \leq 3.16$ [8] , given that $A(y) = 0$ at $y = \pi$. We take the derivative of $A(y)$ in Eq. 84 and take the factor n^2 out of the brackets in the last line below.

$$\frac{dA(y)}{dy} = \sum_{n=1}^{\infty} n^2 e^{-n^2y}[-8n^4y + 15n^2 + (-4n^4y^2 + 15n^2y - \frac{15}{2})(-n^2)]$$

$$\frac{dA(y)}{dy} = \sum_{n=1}^{\infty} n^4 e^{-n^2y}[-8n^2y + 15 + 4n^4y^2 - 15n^2y + \frac{15}{2}] = \sum_{n=1}^{\infty} n^4 e^{-n^2y}[4n^4y^2 - 23n^2y + \frac{45}{2}] \tag{85}$$

We examine the term $C(y,n) = n^4e^{-n^2y}(4n^4y^2-23n^2y+\frac{45}{2})$ in Eq. 85 in the interval $\pi \leq y \leq 3.16$ and show that $\frac{dA(y)}{dy} = C(y,1)+\sum_{n=2}^{\infty} C(y,n) < 0$, as follows. We want the maximum value of $C(y,n)$ and we consider the maximum value of positive terms and minimum value of absolute value of negative terms in the paragraphs below.

For $n = 1$, we see that $C(y,1) = e^{-y}(4y^2-23y+\frac{45}{2}) = 4y^2e^{-y}-23ye^{-y}+\frac{45}{2}e^{-y} < 0$ in the interval $\pi \leq y \leq 3.16$ as follows. Given that $3.16^2 < 10$ and $\pi > 3.14$, in the interval $\pi \leq y \leq 3.16$, we see that $C(y,1) < 4*10e^{-3.14} - 23*3.14e^{-3.16} + \frac{45}{2}e^{-3.14} = -0.3588 < -6e^{-3} = C_{max}(1)$ where $C_{max}(1)$ is the maximum value of $C(y,1)$ in the interval $\pi \leq y \leq 3.16$.

$$C(y,1) = e^{-y}(4y^2 - 23y + \frac{45}{2}) < -6e^{-3}, \quad \pi \leq y \leq 3.16 \tag{86}$$

For $n > 1$, in the interval $\pi \leq y \leq 3.16$, we can write $C(y,n)$ as follows, given that $\pi > 3.14$ and $3.16^2 < 10$ and the term $-23n^2y < 0$ is omitted below, given that we want the maximum value of $C(y,n)$. We write the term $\frac{45}{2} < 4n^4 * 0.5$ and $e^{-0.14n^2} * 10.5 < 10$ for $n \geq 2$.

$$C(y,n) = n^4 e^{-n^2 y}(4n^4y^2 - 23n^2y + \frac{45}{2}) < n^4 e^{-\pi n^2}(4n^4((3.16)^2 + 0.5)) < 4n^8 e^{-3n^2} e^{-0.14n^2} * 10.5 < 40n^8 e^{-3n^2} \tag{87}$$

We want to show that $\frac{dA(y)}{dy} = C(y,1) + \sum_{n=2}^{\infty} C(y,n) < 0$ in the interval $\pi \leq y \leq 3.16$. Using Eq. 86 and Eq. 87, we write as follows. We multiply both sides by e^3 in the second line below.

$$\begin{aligned} \frac{dA(y)}{dy} = C(y,1) + \sum_{n=2}^{\infty} C(y,n) < -6e^{-3} + \sum_{n=2}^{\infty} 40n^8 e^{-3n^2} \\ e^3 \frac{dA(y)}{dy} < -6 + \sum_{n=2}^{\infty} 40n^8 e^{3-3n^2} \end{aligned} \tag{88}$$

We want to show that $e^3 \frac{dA(y)}{dy} < 0$ in the interval $\pi \leq y \leq 3.16$. We compute $\log{(n^8 e^{3-3n^2})}$ as follows. We note that $f(x) = \log x$ is a **concave down** function whose second derivative given by $-\frac{1}{x^2} < 0$ for $|x| < \infty$ and we can write $f(x) = \log x \leq f(x_0) + f^{'}(x_0)(x - x_0)$ using its **tangent line** equation. We see that $f^{'}(x) = \frac{1}{x}$. We set $x = n$ and $x_0 = 2$ and get $\log n \leq \log 2 + \frac{1}{2}(n-2)$ below.

$$\begin{aligned} \log{(n^8 e^{3-3n^2})} = 8\log n + (3 - 3n^2) \leq 8(\log 2 + \frac{1}{2}(n-2)) + (3 - 3n^2) \\ \log{(n^8 e^{3-3n^2})} \leq 8\log 2 + 4n - 5 - 3n^2 \end{aligned} \tag{89}$$

We note that $g(x) = 4x - 5 - 3x^2$ in Eq. 89 is a **concave down** function (concave down function), whose second derivative given by $-6 < 0$ for all x and we can write $g(x) \leq g(x_0) + g^{'}(x_0)(x - x_0)$ using its **tangent line** equation. We see that $g^{'}(x) = 4 - 6x$. We set $x = n$ and $x_0 = 2$ and get $g(n) \leq g(2) + [4 - 6x]_{x=2}(n-2) = -9 - 8(n-2)$ and write Eq. 89 as follows. We take the exponent e on both sides in the second line below.

$$\begin{aligned} \log{(n^8 e^{3-3n^2})} \leq 8\log 2 - 9 - 8(n-2) \leq 8\log 2 - 1 + 8(1-n) \\ n^8 e^{3-3n^2} \leq e^{8\log 2 - 1 + 8(1-n)} = 2^8 e^{-1} e^{8(1-n)} \end{aligned} \tag{90}$$

We substitute the result in Eq. 90 in Eq. 88 and simplify as follows.

$$\begin{aligned} e^3\frac{dA(y)}{dy} &< -6+40*2^8e^{-1}\sum_{n=2}^{\infty}e^{8(1-n)} \\ e^3\frac{dA(y)}{dy} &< -6+40*2^8e^{-1}*e^8\sum_{n=2}^{\infty}e^{-8n} \\ e^3\frac{dA(y)}{dy} &< -6+40*2^8e^{-1}*e^8\frac{e^{-8*2}}{1-e^{-8}} \\ e^3\frac{dA(y)}{dy} &< -6+40*2^8e^{-1}*\frac{e^{-8}}{1-e^{-8}} \\ e^3\frac{dA(y)}{dy} &< -6+40*2^8e^{-1}*\frac{1}{e^8-1} \end{aligned} \tag{91}$$

We multiply Eq. 91 by $\frac{(e^8-1)}{6}$ and write as follows.

$$e^3\frac{dA(y)}{dy}\frac{(e^8-1)}{6} < -e^8+1+40e^{-1}*\frac{256}{6} \approx -2352 \tag{92}$$

We see that $e^3\frac{dA(y)}{dy}\frac{(e^8-1)}{6} < 0$ in Eq. 92, given that $e > 2$ and hence $\frac{dA(y)}{dy} < 0$, in the interval $\pi \le y \le 3.16$, given that $e^3\frac{(e^8-1)}{6} > 0$. Given that $A(y) = 0$ at $y = \pi$, we see that $A(y) < 0$ in Eq. 84 , for $\pi < y \le 3.16$ and $\frac{dX(t)}{dt} = \pi e^{\frac{5t}{2}}A(y) < 0$ in the interval $0 < t \le t_z$.(**Result 2**)

In Section 6.1, it is shown that $\frac{dX(t)}{dt} < 0$ for $t > t_z$ (from Result 1). In this section, we have shown that $\frac{dX(t)}{dt} < 0$ for $0 < t \le t_z$. Hence $\frac{dX(t)}{dt} < 0$ for all $t > 0$.

Hence $E_0(t) = 2X(t)$ is a **strictly decreasing function** for $t > 0$.

7. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line given by $Re[s] = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ (Appendix C.8) where $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied [6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them. This proof does not need or use Euler product.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for $Re[s] \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real [4](link) and then derive $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function of variable t and $E_0(t) > 0$ for $|t| < \infty$ (Appendix C.7) and $E_0(t)$ is **strictly decreasing** function for $t > 0$ (Section 6). These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

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Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ using Eq. 1.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 < |\sigma| < \frac{1}{2}$ is real. We use $E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$ below.

$$\begin{aligned}\xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)\\ E_p(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega\end{aligned} \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows. We get $\omega = \omega' + i\sigma$ and $d\omega = d\omega'$.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $z = [-\infty, \infty]$, C_2 along the line $z = [\infty, \infty - i\sigma]$, C_3 along the line $z = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $z = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \le y \le 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function for real t(Appendix A.1). Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned}E_p(t) &= e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'\\ E_p(t) = E_0(t)e^{-\sigma t} &= \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}\end{aligned} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix A.1. $E_y(t) = E_0(t)e^{yt}$ ***is an absolutely integrable function***

We see that $E_0(t) > 0$ and finite for $-\infty < t < \infty$ (Appendix C.7). Hence $E_y(t) = E_0(t)e^{yt} > 0$ and finite for all $-\infty < t < \infty$, for $-\sigma \le y \le 0$ and $0 \le |\sigma| < \frac{1}{2}$ (**Result 11**).

$E_0(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-1.5|t|}]$ (Appendix C.5) and hence $E_y(t) = E_0(t)e^{yt}$ has an asymptotic **exponential** fall-off rate of $o[e^{-(1.5+y)|t|}] > o[e^{-|t|}]$, for $-\sigma \le y \le 0$ and $0 \le |\sigma| < \frac{1}{2}$. Hence $E_y(t) = E_0(t)e^{yt}$ decays exponentially, at $t \to \pm\infty$.(**Result 12**)

Using Result 11 and 12, we can write $\int_{-\infty}^{\infty} |E_y(t)|dt$ is finite and $E_y(t)$ is an absolutely **integrable function** (Appendix C.6) and its Fourier transform $E_{y\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix B. Derivation of entire function $\xi(s)$

In this section, we will start with Riemann's Xi function $\xi(s)$ and take the inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the equation for $\xi(s)$ derived in Ellison's book "Prime Numbers" pages 151-152 which uses **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real.[4] (link).

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_{1}^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{B.1}$$

We see that $\xi(s)$ is an entire function, for all values of s in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

*Appendix B.1. **Derivation of** $E_p(t)$ **and** $E_0(t)$*

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. B.1 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_{0}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t} + e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{B.2}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma))[&\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \\ &+ \int_{0}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt]\end{aligned} \tag{B.3}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)]e^{-\sigma t} e^{-i\omega t} dt \tag{B.4}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned} E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4}+\sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)]e^{-\sigma t} \end{aligned} \tag{B.5}$$

We compute the derivatives of $A(t)$ as follows.

$$\begin{aligned} \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-4\pi n^2 e^{-2t} + (-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t) + A_0\delta(t) \end{aligned} \tag{B.6}$$

We use $A_0 = [\frac{dA(t)}{dt}]_{t=0+} - [\frac{dA(t)}{dt}]_{t=0-} = \sum_{n=1}^{\infty} e^{-\pi n^2}(\frac{1}{2}-\sigma-2\pi n^2-(-\frac{1}{2}-\sigma+2\pi n^2)) = \sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)$. We can simplify above equation as follows.

$$\begin{aligned} \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)] \end{aligned} \tag{B.7}$$

We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real [4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$ (Appendix B.2) and hence **dirac delta terms cancel each other** in Eq. B.5 written as follows.

$$
\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}) \\
&\quad + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} D(t,n)u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} C(t,n)u(t)
\end{aligned}
\tag{B.8}
$$

We cancel the common terms in Eq. B.8 and simplify above equation as follows.

$$
\begin{aligned}
C(t,n) &= -\frac{1}{4} + \sigma^2 + \sigma - 2\sigma^2 - 4\sigma\pi n^2 e^{2t} + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t} \\
C(t,n) &= 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} \\
D(t,n) &= -\frac{1}{4} + \sigma^2 - \sigma - 2\sigma^2 + 4\sigma\pi n^2 e^{-2t} + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t} \\
D(t,n) &= 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t}
\end{aligned}
\tag{B.9}
$$

We see that $D(t,n) = C(-t,n)$. Hence we can write as follows.

$$
\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= \sum_{n=1}^{\infty} C(t,n) e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned}
\tag{B.10}
$$

We use the fact that $E_0(t) = E_0(-t)$ (Appendix C.8) we arrive at the desired result for $E_p(t)$ as follows.

$$
\begin{aligned}
E_0(t) &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) = E_0(t)e^{-\sigma t} &= \sum_{n=1}^{\infty} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{B.11}
$$

Appendix B.2. ***Derivation of*** $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum\limits_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real [4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$\begin{aligned} F(x) = 1+2w(x) &= \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x})) \\ F(x) = 1+2\sum_{n=1}^{\infty} e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}(1+2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\ \frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1+2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}} \end{aligned} \tag{B.12}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$\begin{aligned} [\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} &= \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1+2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2}) \\ \sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) &= -\frac{1}{2} \end{aligned} \tag{B.13}$$

Appendix C. Properties of Fourier Transforms

Appendix C.1. $E_p(t), h(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. In Eq. 1, we see that $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ and finite for all $-\infty < t < \infty$(Appendix C.7). Hence $E_p(t) = E_0(t)e^{-\sigma t} > 0$ and finite for all $-\infty < t < \infty$.

It is shown in Appendix C.5 that $E_0(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-1.5|t|}]$ and hence $E_p(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-(1.5-\sigma)|t|}] > o[e^{-|t|}]$, for $0 \le |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t}$ goes to zero, at $t \to \pm\infty$ and we showed that $E_p(t) > 0$ and finite for all $-\infty < t < \infty$ in the last paragraph.(**Result 21**) Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $E_{p\omega}(0) = \int_{-\infty}^{\infty} E_p(t)dt$ is finite. Using Result 21, we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Using the arguments in above paragraph, we replace σ in $E_p(t)$ by 0 and 2σ respectively and see that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$.

Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, its shifted versions are absolutely integrable and we see that $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = (E_0(t-t_2) - E_0(t+t_2))e^{-\sigma t}$ in Eq. 6 is an absolutely integrable function, for a finite shift of t_2. (We substitute $t - t_2 = \tau$ and $dt = d\tau$ and get $\int_{-\infty}^{\infty} |E_p(t - t_2)|dt = \int_{-\infty}^{\infty} |E_p(\tau)|d\tau$ and hence $E_p(t - t_2)$ is an absolutely integrable function, given that $E_p(t)$ is absolutely integrable. Same argument holds for $E_p(t + t_2)$.)

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $h(t) > 0$ for real t and $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix C.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose first derivative given by $\frac{dh(t)}{dt} = \sigma e^{\sigma t}u(-t) - \sigma e^{-\sigma t}u(t)$ and $A_0 = [\frac{dh(t)}{dt}]_{t=0+} - [\frac{dh(t)}{dt}]_{t=0-} = -2\sigma$ and hence $\frac{dh(t)}{dt}$ is **discontinuous** at $t = 0$, for $0 < \sigma < \frac{1}{2}$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $\frac{A_0}{(i\omega)^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega) d\omega$ converges.(**Result B.2**)

Let us consider the function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ in Eq. 6 and its first derivative given by $\frac{dg(t,t_2,t_0)}{dt} = [-\sigma e^{-\sigma t} f(t, t_2, t_0) + e^{-\sigma t} \frac{df(t,t_2,t_0)}{dt}]u(-t) + [\sigma e^{\sigma t} f(t, t_2, t_0) + e^{\sigma t} \frac{df(t,t_2,t_0)}{dt}]u(t)$. We get $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0-} = -\sigma f(0, t_2, t_0) + [\frac{df(t,t_2,t_0)}{dt}]_{t=0-}$ and $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0+} = \sigma f(0, t_2, t_0) + [\frac{df(t,t_2,t_0)}{dt}]_{t=0+}$(**Result B.2.1**).

We note that $f(t, t_2, t_0)$ is a continuous function in Eq. 6 and get $[\frac{df(t,t_2,t_0)}{dt}]_{t=0+} = [\frac{df(t,t_2,t_0)}{dt}]_{t=0-}$ and get $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0+} - [\frac{dg(t,t_2,t_0)}{dt}]_{t=0-} = 2\sigma f(0, t_2, t_0)$ using Result B.2.1. Hence $\frac{dg(t,t_2,t_0)}{dt}$ is **discontinuous** at $t = 0$, for $0 < \sigma < \frac{1}{2}$, if $f(0, t_2, t_0) \neq 0$.

We can see that the **first derivatives** of $g(t, t_2, t_0), h(t)$ are **discontinuous** at $t = 0$ and hence $G(\omega, t_2, t_0), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$, using Result B.2. Hence the convolution integral below converges to a finite value for real ω, for the case $f(0, t_2, t_0) \neq 0$.

$$F(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega', t_2, t_0) H(\omega - \omega') d\omega' = \frac{1}{2\pi}[G(\omega, t_2, t_0) * H(\omega)] \tag{C.1}$$

If $f(0, t_2, t_0) = 0$, and if the N^{th} **derivative** of $g(t, t_2, t_0)$ is **discontinuous** at $t = 0$ where $N > 1$, we see that $G(\omega, t_2, t_0)$ has **fall-off rate** of $\frac{1}{\omega^{(N+1)}}$ as $|\omega| \to \infty$(Appendix C.3). $G(\omega, t_2, t_0)$ has a minimum **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ for this case. Hence the convolution integral in Eq. C.1 converges to a finite value for real ω.

Appendix C.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_{N\omega}(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P_\omega(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P_\omega(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

Appendix C.4. ***Exponential Fall off rate of analytic functions.***

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant [3] (Titchmarsh pp256-257 and Titchmarsh pp28-31).

We consider $x(t) = E_0(t)e^{-2\sigma t}$ and its Fourier transform is given by $X(\omega) = \int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{-i\omega t}dt = \int_{-\infty}^{\infty} E_0(t)e^{-i(\omega - i2\sigma)t}dt = E_{0\omega}(\omega - i2\sigma) = \xi(\frac{1}{2} + i(\omega - i2\sigma)) = \xi(\frac{1}{2} + 2\sigma + i\omega) = E_{0\omega}(\omega - i2\sigma)$. Hence both $E_{0\omega}(\omega)$ and $X(\omega) = E_{0\omega}(\omega - i2\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable (Appendix C.6) and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $X(\omega)$ is an **analytic** function which is infinitely differentiable which produces no discontinuities for real ω and $0 < \sigma < \frac{1}{2}$. Hence its **inverse Fourier transform** $x(t)$ has fall-off rate faster than $\lim_{M\to\infty} \frac{1}{t^M}$, as $|t| \to \infty$ (Appendix C.3) and hence $x(t) = E_0(t)e^{-2\sigma t}$ should have **exponential fall-off** rate of $e^{-B|t|}$, as $|t| \to \infty$, where $B > 0$ is real.

Appendix C.5. ***Exponential Fall off rate of*** $x(t) = E_0(t)e^{-2\sigma t}$

We can write $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ in Eq. 1 as follows. We take the term $2\pi n^2 e^{2t}$ out of the brackets below. In the term $e^{-\pi n^2 e^{2t}}$, we use Taylor series expansion around $t = 0$ for $e^{2t} = \displaystyle\sum_{r=0}^{\infty} \frac{(2t)^r}{!r}$, given that e^{2t} is an analytic function for real t.

$$E_0(t) = \sum_{n=1}^{\infty} 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$= \sum_{n=1}^{\infty} 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2(1+2t)} e^{-\pi n^2(\frac{(2t)^2}{!2} + \frac{(2t)^3}{!3} ....)} e^{\frac{t}{2}} \tag{C.2}$$

We take the term $e^{-2\pi t}$ out of the summation, corresponding to $n = 1$ and then take the term $2\pi e^{4t} e^{\frac{t}{2}} = 2\pi e^{\frac{9t}{2}}$ out and write Eq. C.2 as follows.

$$E_0(t) = 2\pi e^{-2\pi t} e^{\frac{9t}{2}} \sum_{n=1}^{\infty} n^2[2\pi n^2 - 3e^{-2t}]e^{-\pi n^2} e^{-2\pi(n^2-1)t} e^{-\pi n^2(\frac{(2t)^2}{!2} + \frac{(2t)^3}{!3} ....)} \tag{C.3}$$

For $t > 0$, we see that the term corresponding to $n = 1$ in Eq. C.3 has an asymptotic fall-off rate of $o[e^{-(2\pi - \frac{9}{2})t}] > o[e^{-1.5t}]$. The terms corresponding to $n > 1$ have higher fall-off rates, due to the term $e^{-2\pi(n^2-1)t}$.

Hence we see that $E_0(t)$ has an asymptotic fall-off rate of $o[e^{-1.5t}]$, for $t > 0$. Given that $E_0(t) = E_0(-t)$(Appendix C.8), we see that $E_0(t)$ has an **exponential** asymptotic fall-off rate of $o[e^{-1.5|t|}]$.

Similarly, $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-(1.5-2\sigma)|t|}] > o[e^{-0.5|t|}]$, for $0 \leq |\sigma| < \frac{1}{2}$.

The above results which show **exponential** fall-off rates for above mentioned functions, continue to hold, as $|t|$ increases to a larger and larger finite value, without bounds.

Appendix C.6. ***Absolutely integrable functions***

We see that a real function $y(t)$ which is finite for all t and has an asymptotic falloff rate of $O[\frac{1}{t^2}]$ is an absolutely integrable function, given that $\int_{-\infty}^{\infty} |y(t)|dt = \int_{-\infty}^{-T} |y(t)|dt + \int_{-T}^{T} |y(t)|dt + \int_{T}^{\infty} |y(t)|dt$ is finite, for non-zero and finite T, because when we integrate the integrand $|y(t)|$ with order $O[\frac{1}{t^2}]$

, we get the result $O[\frac{1}{t}]$, which is finite at the limit $t = \pm T$ and the result $O[\frac{1}{t}]$ is zero at the limit $t \to \pm\infty$. If $y(t)$ has an exponential asymptotic falloff rate, when we integrate the integrand $|y(t)|$ with order $O[e^{-A|t|}]$ for real $A > 0$, we get the result $O[\frac{1}{A}e^{-A|t|}]$, which is finite at the limit $t = \pm T$ and the result is zero at the limit $t \to \pm\infty$ and hence $y(t)$ is an absolutely integrable function.

Appendix C.7. $E_0(t) > 0$ *for* $-\infty < t < \infty$

For $0 \le t < \infty$, we can show that $E_0(t) = \sum_{n=1}^{\infty} f(t,n) > 0$ where $f(t,n) = [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ as follows.

The sum is positive because each summand $f(t,n)$ is positive for finite n, and each summand is positive because the term $2\pi n^2 e^{2t} - 3 > 0$ for all $t \ge 0$ and $n \ge 1$, given that $\pi > 3$ and $2\pi n^2 e^{2t} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for $0 \le t < \infty$ and finite $n \ge 1$.(**Result B.7.1**)

For $t = 0$ and $n = 1$, we see that $f(0,1) = 2\pi[2\pi - 3]e^{-\pi} > 0$.

For $t = 0$ and for **each finite** $n \ge 1$, we see that $f(0,n) = 2\pi n^2[2\pi n^2 - 3]e^{-\pi n^2} > 0$.

For $0 < t < \infty$ and for **each finite** $n \ge 1$, we see that $f(t,n) = 2\pi n^2 e^{2t}[2\pi n^2 e^{2t} - 3]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$, using Result B.7.1.

As $n \to \infty$, $f(t,n)$ tends to zero, for $0 \le t < \infty$ due to the term $e^{-\pi n^2 e^{2t}}$. We do summation over n and see that the sum of the terms $\sum_{n=1}^{\infty} f(t,n) > 0$.

Hence $E_0(t) = \sum_{n=1}^{\infty} f(t,n) > 0$ for $0 \le t < \infty$.

Given that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for real ω and also for $\omega = 0$. Hence $E_{0\omega}(0) = \int_{-\infty}^{\infty} E_0(t)dt$ is finite. We see that $E_0(t)$ is an analytic function for real t. Hence $E_0(t) = \sum_{n=1}^{\infty} f(t,n) > 0$ is finite for $0 \le t < \infty$.

Given that $E_0(t) = E_0(-t)$(Appendix C.8), we see that $E_0(t) > 0$ and finite for all $-\infty < t < \infty$.

Appendix C.8. $E_0(t)$ ***is real and even***

We see that $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ (**Result 13**) because $\xi(s) = \xi(1-s)$ (link) and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

We take the Inverse Fourier transform of $E_{0\omega}(\omega)$ and use $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ from Result 13 and then substitute $\omega = -\omega'$ in the integrand, as follows.

$$\begin{aligned} E_0(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(-\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{-i\omega' t}d\omega' = E_0(-t) \end{aligned} \tag{C.4}$$

We see that $E_0(t)$ in Eq. 1 is real and $E_0(t)$ in Eq. C.4 is even and hence we have derived the result that $E_0(t)$ is a **real and even** function of variable t.

Appendix D. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix D.1. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a **real** function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$. We use the fact that $g(t)$ is real and $\cos(\omega t)$ is an **even** function of ω and $\sin(\omega t)$ is an **odd** function of ω below.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\
G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega)
\end{aligned}
\tag{D.1}
$$

Appendix D.2. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we take the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ and show that its Fourier transform is given by the **real part** of $G(\omega)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt &= \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt
\end{aligned}
\tag{D.2}
$$

We substitute $t = -t$ in the second integral in Eq. D.2. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix D.1)

$$
\begin{aligned}
\int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt &= \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} + \frac{G(-\omega)}{2} \\
= \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(-\omega) + iG_I(-\omega)] &= \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(\omega) - iG_I(\omega)] = G_R(\omega)
\end{aligned}
\tag{D.3}
$$

Appendix D.3. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we take the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ and show that its Fourier transform is given by the **imaginary part** of $G(\omega)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt \tag{D.4}$$

We substitute $t = -t$ in the second integral in Eq. D.4. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix D.1)

$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} - \frac{G(-\omega)}{2}$$
$$= \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(-\omega) - iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(\omega) + iG_I(\omega)] = iG_I(\omega) \tag{D.5}$$

Appendix D.4. ***Fourier transform of a real and even function*** $g(t)$

In this section, we show that the Fourier transform of a **real and even** function $g(t)$, given by $G(\omega)$ is also **real and even**. We use the fact that $\int_{-\infty}^{\infty} g(t)\sin\omega t dt = 0$ because $g(t)$ is even and the integrand is an **odd function** of variable t.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)\cos\omega t dt - i\int_{-\infty}^{\infty} g(t)\sin\omega t dt$$
$$G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt \tag{D.6}$$

We see that $G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt$ is **real** function of ω, given that $g(t)$ and the integrand are real functions. We see that $G(\omega)$ is an **even** function of ω because $\cos\omega t$ is a **even** function of ω.