

On a new method towards proof of Riemann's Hypothesis

Akhila Raman

University of California at Berkeley. Email: akhila.raman@berkeley.edu.

Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.^{[4] [5]} We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.^[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.^[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < \text{Re}[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix D, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. Step 1: Inverse Fourier Transform of $\xi(\frac{1}{2} + i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = \Xi(\omega) = E_{0s\omega}(\omega)$, where $-\infty \leq \omega \leq \infty$. Its inverse Fourier Transform is given by $E_{0s}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$, where ω, t are real, as follows (link).^[3]

$$E_{0s}(t) = \Phi(t) = 2 \sum_{n=1}^{\infty} [2n^4 \pi^2 e^{\frac{9t}{2}} - 3n^2 \pi e^{\frac{5t}{2}}] e^{-\pi n^2 e^{2t}} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (1)$$

We see that $E_{0s}(t) = E_{0s}(-t)$ is a real and **even** function of t , given that $E_{0s\omega}(\omega) = E_{0s\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2} + \sigma + i\omega) = E_{ps\omega}(\omega)$ is given by the real function $E_{ps}(t)$. We can write $E_{ps}(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_{ps}(t) = E_{0s}(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \quad (2)$$

We can see that $E_{ps}(t)$ is an analytic function in the interval $|t| \leq \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. Step 2: Modified Function $E_0(t)$

The proof presented in following sections **requires** $\Phi(t) = E_{0s}(t)$ to be a **strictly decreasing** function for $t > 0$. It is stated in Conrey's article (link) that $\Phi(t)$ is positive for $t > 0$ and its first derivative is negative for $t > 0$ and hence $\Phi(t)$ is a **strictly decreasing** function for $t > 0$. The proof to show that $\Phi(t)$ is **strictly decreasing** function for $t > 0$ is not available in textbooks.

Hence we consider $\xi'(s) = -\frac{\xi(s)}{s(s-1)}$ evaluated at $s = \frac{1}{2} + i\omega$ given by $E_{0\omega}(\omega) = \frac{E_{0s\omega}(\omega)}{(\frac{1}{4} + \omega^2)}$, whose Inverse Fourier transform is given by the **modified** function $E_0(t)$. We will show that this modified function $E_0(t)$ is a **strictly decreasing** function for $t > 0$ in Section 4. We form $E_p(t) = E_0(t) e^{-\sigma t}$ whose Fourier transform is given by $E_{p\omega}(\omega) = \frac{E_{ps\omega}(\omega)}{(\frac{1}{4} - \sigma^2 + \omega^2 - i2\sigma\omega)}$. **If** Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{ps\omega}(\omega)$ has a zero at $\omega = \omega_0$, **then** we see that $E_{p\omega}(\omega)$ also has a zero at $\omega = \omega_0$.

We start with the well known equation for Riemann's Xi function derived in Eq. D.6 as follows.

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{2} [1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}}) w(x) \frac{dx}{x}] \quad (3)$$

We substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ and $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ in Eq. 3 and evaluate at $s = \frac{1}{2} + i\omega$ as follows. We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)]$ whose Fourier transform is given by $A(\omega)$. This is re-derived in Eq. D.9 in Appendix D.

$$\begin{aligned} \xi(\frac{1}{2} + i\omega) = E_{0s\omega}(\omega) &= \frac{1}{2} + (-\frac{1}{4} - \omega^2) \int_{-\infty}^{\infty} A(t) e^{-i\omega t} dt = \frac{1}{2} + (-\frac{1}{4} - \omega^2) A(\omega) \\ E_{0\omega}(\omega) &= \frac{E_{0s\omega}(\omega)}{(\frac{1}{4} + \omega^2)} = \frac{1}{2(\frac{1}{4} + \omega^2)} - A(\omega) \end{aligned} \quad (4)$$

The Inverse Fourier Transform of $E_{0\omega}(\omega)$ is given by $E_0(t)$ as follows.

$$\begin{aligned}
E_0(t) &= \frac{1}{2}(e^{\frac{t}{2}}u(-t) + e^{-\frac{t}{2}}u(t)) - A(t) \\
E_0(t) &= [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t) \\
E_p(t) &= E_0(t)e^{-\sigma t}
\end{aligned} \tag{5}$$

We use these modified functions $E_0(t)$ and $E_p(t)$ in the subsequent sections.

1.3. Step 2: On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{4}$ at first. Let us consider a new function $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$, where $f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t)$ and $f_1(t) = e^{\sigma t_0}E'_p(t+t_0)$ and $f_2(t) = e^{-\sigma t_0}E'_p(t-t_0)$ and $E'_p(t) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2)$ and t_0, t_2 are real and $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, for every value of t_0 , to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

1.4. Step 3: On the zeros of the function $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g(t)$ and compute its real part given by $G_R(\omega) = G_R(\omega, t_2, t_0)$ and we can write as follows.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0)] \cos(\omega\tau) d\tau
\end{aligned} \tag{6}$$

We require $G_R(\omega, t_2, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **each fixed value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$.

1.5. Step 4: Zero Crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

In Section 2.3, we show the result in Eq. 7 and that $\omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$. It is shown that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = P_{\text{odd}}(t_2, t_0) + P_{\text{odd}}(t_2, -t_0)$ is an **odd** function of t_0 , for all t_0 , for a given value of t_2 as follows.

$$\begin{aligned}
P_{\text{odd}}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&\quad + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned} \tag{7}$$

1.6. Step 5: Final Step

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = Kt_{0c}$, for positive even integer K , such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2, t_0)$ in Eq. 7 and show that this leads to the result in Eq. 8. We use $E'_0(t) = E_0(t - t_2) - E_0(t + t_2)$ and $E'_{0n}(t) = E'_0(-t)$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \quad (8)$$

We show that the **each** of the terms in the integrand in Eq. 8 are **greater than zero**, in the interval $\tau = [0, t_{0c}]$ where $t_{0c} > 0$. For $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, we see that $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi}{2t_{0c}}\tau$ lies in the range $[0, \frac{\pi}{2}]$ and hence $\sin(\omega_{c1}\tau) > 0$ in that interval $\tau = [0, t_{0c}]$.

Hence the result in Eq. 8 leads to a **contradiction** for $0 < \sigma < \frac{1}{4}$.

We have shown this result for $0 < \sigma < \frac{1}{4}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{4} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{4}$.

In Section 3.2, the outline of the proof for the case of $0 < |\sigma| < \frac{1}{2}$ is presented.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{ps\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t , where $E_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ and $E_{p\omega}(\omega) = \frac{E_{ps\omega}(\omega)}{(\frac{1}{4} - \sigma^2 + \omega^2 - i2\sigma\omega)}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3]

2.1. New function $g(t)$

In this section, we consider $0 < \sigma < \frac{1}{4}$ first and then in Section 3.2, we outline a method to prove Theorem 1 for the region $\frac{1}{4} \leq \sigma < \frac{1}{2}$.

Let us consider the function $E'_p(t) = E'_p(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E'_0(t)e^{-\sigma t}$, where t_2 is finite and real, and $E'_0(t) = E'_0(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Its Fourier transform is given by $E'_{p\omega}(\omega) = E'_{p\omega}(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = f_1(t, t_2, t_0) = e^{\sigma t_0} E'_p(t + t_0)$ and $f_2(t) = f_2(t, t_2, t_0) = e^{-\sigma t_0} E'_p(t - t_0)$ where t_0 is finite and real and we can see that the Fourier Transform of this function $F(\omega) = F(\omega, t_2, t_0) = E'_{p\omega}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$.

Let us consider a new function $g(t) = g(t, t_2, t_0) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), F(\omega)$ and $G(\omega)$ which are also functions of variables t_2, t_0 . Similarly we use the shorthand notation for the functions $E'_p(t), E'_0(t)$ and $E'_{p\omega}(\omega)$ which are also functions of variable t_2 .

We see that $E_0(t) = [\frac{1}{2}e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}]u(-t) + [\frac{1}{2}e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}]u(t)$ goes to zero, as $|t| \rightarrow \infty$. In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$. This implies that $E_0(t) > 0$ for all $|t| < \infty$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $|t| < \infty$.

We can see that $g(t)$ is a real L^1 integrable function as follows. $E_p(t)$ has a fall off rate of $e^{-t(\frac{1}{2}+\sigma)}$ as $t \rightarrow \infty$ and a fall off rate of $e^{t(\frac{1}{2}-\sigma)}$ as $t \rightarrow -\infty$. Hence $g(t)$ has a fall off rate of $e^{-\frac{t}{2}}$ as $t \rightarrow \infty$ and a fall off rate of $e^{t(\frac{1}{2}-2\sigma)}$ as $t \rightarrow -\infty$. Hence **for** $0 < \sigma < \frac{1}{4}$, we see that $g(t)$ is a real L^1 integrable function. Its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \rightarrow \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma].

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E'_{p\omega}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0 , we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_z(t_0)$ for **every value** of t_0 . Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{\text{even}}(t) + g_{\text{odd}}(t)$ where $g_{\text{even}}(t)$ is an even function and $g_{\text{odd}}(t)$ is an odd function of variable t . If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ is real and does not have zeros for any finite value of ω , **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω , which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0 , where $\omega_z(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{-2\sigma t_0} f_1(t) + e^{2\sigma t_0} f_2(t)$ where $f_1(t) = e^{\sigma t_0} E'_p(t + t_0)$ and $f_2(t) = e^{-\sigma t_0} E'_p(t - t_0)$, $E'_p(t) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$, and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{4}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E'_{p\omega}(\omega)(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0}) = E_{p\omega}(\omega)(e^{-\sigma t_2} e^{-i\omega t_2} - e^{\sigma t_2} e^{i\omega t_2})(e^{-\sigma t_0} e^{i\omega t_0} + e^{\sigma t_0} e^{-i\omega t_0})$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega') H(\omega - \omega') d\omega' \quad (9)$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$. (Appendix C.1)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2 + \omega^2)}$ in Eq. 9 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \quad (10)$$

We can split the integral in Eq. 10 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \left[\int_{-\infty}^0 G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_0^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \right] \quad (11)$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 11 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_0^{\infty} G_R(\omega') \left[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} \right] d\omega' \quad (12)$$

We showed earlier in this section that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \rightarrow \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 12 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)} > 0$.

• **Case 1:** $G_R(\omega') > 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

• **Case 2:** $G_R(\omega') < 0$ for all finite $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

2.2. On the zeros of a related function $G(\omega)$

We can compute the fourier transform of the function $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0 , for a given value of t_2 , to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the Fourier transform of the function $g_1(t)$ given by $G_1(\omega) = G_{1R}(\omega) + iG_{1I}(\omega)$. We use $g_1(t) = f_1(t)e^{-\sigma t}u(-t) + f_1(t)e^{\sigma t}u(t) = e^{\sigma t_0}E'_p(t+t_0)e^{-\sigma t}u(-t) + e^{\sigma t_0}E'_p(t+t_0)e^{\sigma t}u(t)$.

We **note** that we use the **shorthand** notation for the functions $f(t), g(t), f_1(t), f_2(t), g_1(t), G(\omega)$ and $G_1(\omega)$ which are also functions of variables t_2, t_0 . Similarly we use the shorthand notation for the functions $E'_p(t), E'_0(t)$ and $E'_{0n}(t)$ which are also functions of variable t_2 .

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt = \int_{-\infty}^0 g_1(t)e^{-i\omega t}dt + \int_0^{\infty} g_1(t)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 e^{\sigma t_0}E'_p(t+t_0)e^{-\sigma t}e^{-i\omega t}dt + \int_0^{\infty} e^{\sigma t_0}E'_p(t+t_0)e^{\sigma t}e^{-i\omega t}dt \end{aligned} \quad (13)$$

We use $E'_p(t) = E'_0(t)e^{-\sigma t}$ where $E'_0(t) = E_0(t-t_2) - E_0(t+t_2)$ and $E'_p(t+t_0) = E'_0(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 13, we have

$$\begin{aligned} G_1(\omega) &= \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_0^{\infty} E'_0(t+t_0)e^{-i\omega t}dt \\ G_1(\omega) &= \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_0(-t+t_0)e^{i\omega t}dt \end{aligned} \quad (14)$$

We define $E'_{0n}(t) = E'_0(-t)$ and get $E'_0(-t+t_0) = E'_{0n}(t-t_0)$ and write Eq. 14 as follows.

$$G_1(\omega) = \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t}e^{-i\omega t}dt + \int_{-\infty}^0 E'_{0n}(t-t_0)e^{i\omega t}dt = G_R(\omega) + iG_I(\omega) \quad (15)$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$G_{1R}(\omega) = G_{1R}(\omega, t_0) = \int_{-\infty}^0 E'_0(t+t_0)e^{-2\sigma t} \cos(\omega t)dt + \int_{-\infty}^0 E'_{0n}(t-t_0) \cos(\omega t)dt \quad (16)$$

2.3. Zero crossing function $\omega_z(t_2, t_0)$ is an even function of variable t_0

Now we consider the function $f(t) = e^{-2\sigma t_0}f_1(t) + e^{2\sigma t_0}f_2(t) = e^{-\sigma t_0}E'_p(t+t_0) + e^{\sigma t_0}E'_p(t-t_0)$ where $f_1(t) = e^{\sigma t_0}E'_p(t+t_0)$ and $f_2(t) = f_1(t, -t_0) = e^{-\sigma t_0}E'_p(t-t_0)$ and $g(t)h(t) = f(t)$ where $g(t) = f(t)e^{-\sigma t}u(-t) + f(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g(t)$ and compute its real part using the procedure in above section, similar to Eq. 16 and we can write as follows. We substitute $t = \tau$.

$$\begin{aligned}
G_R(\omega, t_0) &= e^{-2\sigma t_0} G_{1R}(\omega, t_0) + e^{2\sigma t_0} G_{1R}(\omega, -t_0) \\
G_{1R}(\omega, t_0) &= \int_{-\infty}^0 [E'_0(\tau + t_0)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0)] \cos(\omega\tau) d\tau
\end{aligned} \tag{17}$$

We require $G_R(\omega, t_0) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0 , for **every given fixed value** of t_2 , to satisfy **Statement 1**. In general $\omega_z(t_2, t_0) \neq \omega_0$. Hence we can see that $P(t_2, t_0) = G_R(\omega_z(t_2, t_0), t_0) = 0$ and we can rearrange the terms as follows.

$$\begin{aligned}
P(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \int_{-\infty}^0 [e^{2\sigma t_0} E'_0(\tau - t_0)e^{-2\sigma\tau} + e^{-2\sigma t_0} E'_{0n}(\tau - t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0
\end{aligned} \tag{18}$$

We can write as follows, where $P_{odd}(t_2, t_0)$ is an **odd** function of variable t_0 .

$$\begin{aligned}
P(t_2, t_0) &= P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0 \\
P_{odd}(t_2, t_0) &= \int_{-\infty}^0 [e^{-2\sigma t_0} E'_0(\tau + t_0)e^{-2\sigma\tau} + e^{2\sigma t_0} E'_{0n}(\tau + t_0)] \cos(\omega_z(t_2, t_0)\tau) d\tau
\end{aligned} \tag{19}$$

We see that $f(t, t_0) = e^{-\sigma t_0} E'_p(t + t_0) + e^{\sigma t_0} E'_p(t - t_0) = f(t, -t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_z(t_2, t_0)$ is an **even** function of variable t_0 , for **every fixed value** of t_2 .

3. Final Step

We expand $P_{odd}(t_2, t_0)$ in Eq. 19 as follows, using the substitution $\tau + t_0 = \tau'$ and substituting back $\tau' = \tau$. We use $E'_{0n}(\tau) = E'_0(-\tau)$ and $E'_0(\tau) = E_0(\tau - t_2) - E_0(\tau + t_2)$.

We **note** that we use the **shorthand** notation for the functions $E'_0(t)$ and $E'_{0n}(t)$ which are also functions of variable t_2 .

$$\begin{aligned}
P_{odd}(t_2, t_0) &= [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau] \\
&\quad + e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E'_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau]
\end{aligned} \tag{20}$$

In Section 2.1, $\omega_z(t_2, t_0)$ is shown to be **finite** for all $|t_0| < \infty$, for a given value of t_2 . This means there are **no** Dirac delta functions present in $\omega_z(t_2, t_0)$.

In Section 5, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 for all $|t_0| < \infty$, for **every given fixed value** of t_2 .

In Section 4, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2 , we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 , for every value of t_2 , and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, we see that $\omega_z(t_2, t_0)t_0$ will **certainly pass through** π , as t_0 is increased from zero to ∞ .

We use $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ as follows. We set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 20 as follows. We use the fact that $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.3.

$$\begin{aligned} & \int_{-\infty}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-\infty}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & - \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \end{aligned} \quad (21)$$

We split the integral in the left hand side of Eq. 21 and write as follows.

$$\begin{aligned} & \left[\int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \\ & + e^{2\sigma t_{0c}} \left[\int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \\ & - \int_{-\infty}^{-t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \end{aligned} \quad (22)$$

We combine the terms with common integrals and cancel common terms in Eq. 22 as follows.

$$\begin{aligned} & \int_{-t_{0c}}^{t_{0c}} E'_0(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (23)$$

We can rearrange the terms in Eq. 23 as follows.

$$\begin{aligned} & \int_{-t_{0c}}^{t_{0c}} [E'_0(\tau) e^{-2\sigma\tau} + E'_{0n}(\tau) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = -2 \sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E'_{0n}(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (24)$$

We denote the right hand side of Eq. 24 as RHS . We can split the integral in Eq. 24 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^0 + \int_0^{t_{0c}}$ as follows.

$$\begin{aligned} & \int_{-t_{0c}}^0 [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \int_0^{t_{0c}} [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned} \quad (25)$$

We substitute $\tau = -\tau$ in the first integral in Eq. 25 as follows. We use $E'_0(-\tau) = E'_{0n}(\tau)$ and $E'_{0n}(-\tau) = E'_0(\tau)$.

$$\begin{aligned} & \int_{t_{0c}}^0 [E'_{0n}(\tau)e^{2\sigma\tau} + E'_0(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \int_0^{t_{0c}} [E'_0(\tau)e^{-2\sigma\tau} + E'_{0n}(\tau)e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \end{aligned} \quad (26)$$

Given that $\int_{t_{0c}}^0 = -\int_0^{t_{0c}}$, we can simplify as follows.

$$\int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \quad (27)$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 24 as follows. We use $E'_{0n}(-\tau) = E'_0(\tau)$.

$$RHS = 2 \sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \quad (28)$$

We split the integral on the right hand side in Eq. 28 as follows.

$$RHS = 2 \sinh(2\sigma t_{0c}) \left[\int_0^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \quad (29)$$

We consolidate the integrals with the term $\int_0^{t_{0c}} E'_0(\tau)$ in Eq. 27 and Eq. 29 as follows. We use $2 \sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned} & \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = 2 \sinh(2\sigma t_{0c}) \int_0^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (30)$$

We cancel common terms in Eq. 30 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} [E'_0(\tau)(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E'_{0n}(\tau)(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & = 2 \sinh(2\sigma t_{0c}) \int_0^{\infty} E'_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned}$$

(31)

We substitute $E'_0(\tau) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ and $E'_{0n}(\tau) = E'_0(-\tau) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$. We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$. Hence we see that $E'_{0n}(\tau) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E'_0(\tau)$. We can write Eq. 31 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &= 2 \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (32)$$

We substitute $2 \cosh(2\sigma\tau) = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2 \cosh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 32 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &= \sinh(2\sigma t_{0c}) \int_0^\infty (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \end{aligned} \quad (33)$$

Next Step:

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 33 and then substitute $\tau' = \tau$. Similarly we substitute $\tau - t_{2c} = \tau'$ as follows.

$$\begin{aligned} RHS = & \sinh(2\sigma t_{0c}) [\cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & + \sin(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{-t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ & - \cos(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \sin(\omega_z(t_{2c}, t_{0c}))t_{2c} \int_{t_{2c}}^\infty E_0(\tau) \cos(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \end{aligned} \quad (34)$$

In Eq. 34, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin(\omega_z(t_{2c}, t_{0c})t_{2c}) = 0$ and $\cos(\omega_z(t_{2c}, t_{0c}))t_{2c} = -1$. Hence we cancel common terms and write Eq. 34 and Eq. 33 as follows.

$$\begin{aligned} & \int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ &= -\sinh(2\sigma t_{0c}) \left[\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \right] \end{aligned} \quad (35)$$

We use $\int_{-t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ and cancel the common term $\int_{t_{2c}}^\infty E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 35 as follows. Given that $E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau)$ is an **odd** function of variable τ , we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0$.

$$\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c})) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \quad (36)$$

We can multiply Eq. 36 by a factor of -1 as follows.

$$\int_0^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh 2\sigma t_{0c} - \cosh(2\sigma\tau) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0 \quad (37)$$

In Eq. 37, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $[0, t_{0c}]$, $\omega_z(t_{2c}, t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $[0, \frac{\pi}{2}]$ and hence the sinusoidal function varies over a **half cycle** and is > 0 , in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 37, we see that in the interval $0 < \tau < t_{0c}$, the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0 , in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t \geq \frac{1}{8}$, we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 3.1). The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 37. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{4}$.

For $\sigma = 0$, both sides of Eq. 37 is zero and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{4}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{4} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{4}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{4}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3.1. Result $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 4 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 37.

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t , and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(2t_{0c})$ to $E_0(3t_{0c})$, which is **less than** $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c})$ to $E_0(-t_{0c})$. Hence we see that $E_0(t - t_{2c}) > E_0(t + t_{2c})$, in the interval $0 < t < t_{2c} = 2t_{0c}$. At $t = 0$, $E_0(t - t_{2c}) = E_0(t + t_{2c})$. At $t = t_{0c}$, $E_0(t - t_{2c}) > E_0(t + t_{2c})$ because $E_0(-t_{0c}) > E_0(3t_{0c})$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 37, for $t_{0c} > 0$.

3.2. Case: $\frac{1}{4} \leq |\sigma| < \frac{1}{2}$

This will be shown in a later paper.

4. Strictly decreasing $E_0(t)$ for $t > 0$

Let us consider $E_{0s}(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ whose Fourier Transform is given by the entire function $E_{0s\omega}(\omega) = \xi(\frac{1}{2} + i\omega)$. (link)

We consider $\xi'(s) = -\frac{\xi(s)}{s(s-1)}$ evaluated at $s = \frac{1}{2} + i\omega$ given by $E_{0\omega}(\omega) = \frac{E_{0s\omega}(\omega)}{(\frac{1}{4} + \omega^2)}$, which is a **holomorphic** function in the entire s plane, except at $s = 0$ and $s = 1$, whose Inverse Fourier Transform is given by $E_0(t)$ as follows. (Section 1.2)

$$E_0(t) = [\frac{1}{2} e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}] u(t) + [\frac{1}{2} e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}] u(-t) \quad (38)$$

We define $X(t) = E_0(t)$ and consider **only** $t \geq 0$ in the rest of this section. For $t > 0$, we see that the first two derivatives of $X(t)$ are continuous functions. At $t = 0$, if the second derivative of $X(t)$ has a dirac delta function, it does not affect the arguments below.

$$X(t) = E_0(t) = \frac{1}{2} e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = A_0(t) - B_0(t), \quad t \geq 0$$

$$A_0(t) = \frac{1}{2} e^{\frac{-t}{2}}, \quad B_0(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \quad (39)$$

We show that $X(t) = E_0(t)$ is a **strictly decreasing** function for $t \geq 0$ as follows.

- In Section 4.1 it is shown that the second derivative of $B_0(t)$ in Eq. 39, given by $B_2(t)$ is a **strictly decreasing** function for $t \geq t_{min}$.
- In Section 4.2, it is shown that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. (example plot)
- In Section 4.3 it is shown that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$.

4.1. Second derivative of $B_0(t)$ given by $B_2(t)$ is a strictly decreasing function for $t > t'_{max}$ and $t \geq t_{min}$

We take the first derivative of $B_0(t)$ for $t \geq 0$ as follows.

$$B_1(t) = \frac{dB_0(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [\frac{1}{2} - 2\pi n^2 e^{2t}] \quad (40)$$

We take the second derivative of $B_0(t)$ for $t \geq 0$ as follows.

$$B_2(t) = \frac{dB_1(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi n^2 e^{2t} + (\frac{1}{2} - 2\pi n^2 e^{2t})^2] = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [-4\pi n^2 e^{2t} + \frac{1}{4} - 2\pi n^2 e^{2t} + 4\pi^2 n^4 e^{4t}]$$

$$B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}]$$

We see that $B_0(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ is a **strictly decreasing** function of t , given that its first derivative given by $B_1(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [\frac{1}{2} - 2\pi n^2 e^{2t}]$, is **negative** for $t \geq 0$ for each $n = 1, 2, 3, \dots$ and hence for **all** $t \geq 0$. (example plot)

We see that $B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}]$ in Eq. 41, is **positive** for for each $n = 1, 2, 3, \dots$ and hence for **all** $t \geq 0$. This implies that, as t increases from zero, $B_1(t)$ starts from a **negative** value and increases towards zero.

We define $B'_2(t) = B_2(t)e^{\frac{t}{2}}$ and compute its **first derivative** as follows.

$$\begin{aligned} B'_3(t) &= \frac{dB'_2(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t [16\pi^2 n^4 e^{4t} - 12\pi n^2 e^{2t} + (4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4})(1 - 2\pi n^2 e^{2t})] \\ B'_3(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t [16\pi^2 n^4 e^{4t} - 12\pi n^2 e^{2t} + (4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4} - 8\pi^3 n^6 e^{6t} + 12\pi^2 n^4 e^{4t} - \frac{1}{2}\pi n^2 e^{2t})] \\ B'_3(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^t Y'(t), \quad Y'(t) = -8\pi^3 n^6 e^{6t} + 32\pi^2 n^4 e^{4t} - \frac{37}{2}\pi n^2 e^{2t} + \frac{1}{4} \end{aligned} \quad (42)$$

We examine the term $Y'(t) = -8\pi^3 n^6 e^{6t} + 32\pi^2 n^4 e^{4t} - \frac{37}{2}\pi n^2 e^{2t} + \frac{1}{4}$ in Eq. 42. We see that, for $n = 2$ and $t = 0$, $Y'(t) < -8\pi^3 * 64 + 32\pi^2 * 16 + \frac{1}{4} < 8 * 64 * \pi^2(1 - \pi) + \frac{1}{4} < 0$. We see that, **for all** $n > 1$ and $t \geq 0$, $Y'(t) < 0$.

For $n = 1$, we want to find the **minimum value** of t for which $Y'(t) < 0$. We choose a **loose bound** $t_a = 1$ and for $t \geq t_a$ and $n = 1$, we see that $Y'(t) \leq 8\pi^2 e^{4t_a}(4 - \pi e^{2t_a}) + \frac{1}{4}$. Given that $\pi > 3$ and $e^{2t_a} = e^2 > 2^2 = 4$, we see that $Y'(t) \leq 8\pi^2 e^4(4 - 12) + \frac{1}{4}$ which is **negative**. Hence $B'_3(t) < 0$ for $t > t_a = 1$ and $B'_2(t)$ is **strictly decreasing** for all $t > t_a$.

At $t = t'_{max}$ (**point B'**), which lies **somewhere** in the interval $0 \leq t \leq t_a$, $B'_2(t)$ reaches a **maximum** and then starts decreasing for $t > t'_{max}$. (example plot) We require $B'_3(t) = 0$ at $t = t'_{max}$ and the partial term in $B'_3(t)$ corresponding to $n = 1$ **equals** a positive value B'_{30} which **cancels** the negative value $-B'_{30}$ due to the partial terms corresponding to $n > 1$, to make $B'_3(t) = 0$.

For $t > t'_{max}$, the partial term in $B'_3(t)$ corresponding to $n = 1$ continues to **decrease below** B'_{30} due to the dominant term $-8\pi^3 n^6 e^{6t}$, while the partial terms corresponding to $n > 1$ become **more negative** than $-B'_{30}$. Hence $B'_3(t) < 0$ for $t > t'_{max}$. Hence $B'_2(t)$ is **strictly decreasing** for all $t > t'_{max}$. (**Result 1**)

Given that $B_2(t) = B'_2(t)e^{-\frac{t}{2}}$, we see that $B_2(t)$ is **strictly decreasing** for all $t > t'_{max}$. (**Result 2**) In Section 4.3, it is shown that $t'_{max} < t_{min}$ and hence $B_2(t)$ is a **strictly decreasing** function for $t \geq t_{min}$. (example plot)

4.2. First and Second derivatives of $X(t)$

We take the first derivative of $X(t)$ for $t \geq 0$ in Eq. 39 and substitute $B_1(t)$ in Eq. 40 as follows.

$$\begin{aligned} X(t) &= \frac{1}{2}e^{-\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = A_0(t) - B_0(t) \\ \frac{dX(t)}{dt} &= -\frac{1}{4}e^{-\frac{t}{2}} - B_1(t) = -\frac{1}{4}e^{-\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [\frac{1}{2} - 2\pi n^2 e^{2t}] \end{aligned} \quad (43)$$

We take the second derivative of $X(t)$ for $t \geq 0$ and substitute $B_2(t)$ in Eq. 41 as follows.

$$\frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t) = \frac{1}{8}e^{-\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}] \quad (44)$$

• In the section below, **we will show** that, as t increases from zero, $\frac{dX(t)}{dt}$ starts from zero and reaches a **negative minimum** value at $t = t_{min}$ and then starts increasing towards zero, for $t > t_{min}$. Thus we will show that $X(t)$ is a **strictly decreasing** function of t . We will also show that $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. (example plot)

• We see that in $\frac{d^2 X(t)}{dt^2}$ in Eq. 44, partial term corresponding to $t = 0$ and $n = 1$, equals $a_1 = \frac{1}{8} - a_2 < 0$, $a_2 = e^{-\pi}(4\pi^2 - 6\pi + \frac{1}{4})$. Because $3 < \pi < \frac{22}{7} = 3.1429$ (link) and $e^{-\pi} \geq e^{-0.1429}(e^{-\frac{1}{2}})^6$ and $e^{-0.1429} \geq 1 - 0.1429 = 0.8571$ and $e^{-\frac{1}{2}} \geq 1 - \frac{1}{2} = \frac{1}{2}$, we can write $a_1 = 0.125 - a_2$, $a_2 \geq \frac{0.8571}{26}(\pi(4\pi - 6) + \frac{1}{4})$. We see that the **minimum** value of a_2 is given by $a_2 \geq \frac{0.8571}{64} * (3(4 * 3 - 6)) = \frac{0.8571 * 18}{64} \geq \frac{0.8571 * 16}{64} = \frac{0.8571}{4} \geq 0.21$ and the **maximum** value of a_1 is given by $a_1 = 0.125 - 0.21 = -0.085$. For $n \geq 1$ and $t = 0$, partial terms in $B_2(t)$ are positive and the **minimum** value of $B_2(t)$ is given by 0.21.

• Hence the **maximum** value of $\frac{d^2 X(t)}{dt^2}$ at $t = 0$ is given by $0.125 - 0.21 = -0.085$, which is **negative**. Hence $\frac{d^2 X(t)}{dt^2}$ is **negative** at $t = 0$ and remains **negative** for small values of positive t close to zero.

• We know that $\frac{dX(t)}{dt} = 0$ at $t = 0$, given that $X(t) = E_0(t)$ is an **even** function. This implies that $\frac{dX(t)}{dt}$ is **negative** for small values of positive t close to zero.

• This implies that $X(t)$ is a **strictly decreasing** function of t , for small values of positive t close to zero.

• As t increases from zero (point A), $\frac{dX(t)}{dt}$ reaches a **negative minimum** value at $t = t_{min}$ (point C) and then starts increasing towards zero, for $t > t_{min}$. This means that $\frac{d^2 X(t)}{dt^2}$ reaches a **zero** at $t = t_{min}$, at which point $B_2(t) = \frac{1}{8}e^{-\frac{t_{min}}{2}}$ and then becomes **positive** for $t > t_{min}$, given that $B_2(t)$ decreases towards zero for $t > t_{min}$ (**from Result 1**). (example plot)

• Then $\frac{d^2 X(t)}{dt^2}$ reaches a **positive maximum** at $t = t_{max2}$ (point D) and then starts decreasing to zero, as $t \rightarrow \infty$. This means that $\frac{dX(t)}{dt}$ **increases** towards zero, as $t \rightarrow \infty$. (example plot)

4.3. $\frac{dX(t)}{dt}$ **does not become positive for any** $t > 0$

• $\frac{dX(t)}{dt}$ **does not** become positive for any $t > 0$. We can **rule out** the case of $\frac{dX(t)}{dt}$ becoming positive, as $t \rightarrow \infty$, as follows. Let us assume that $\frac{dX(t)}{dt}$ becomes positive at $t = t_p$ as t increases. (**Statement A**). In this case, we **require** $\frac{dX(t)}{dt}$ to reach a maximum at $t = t_m$ and then fall back towards zero, as $t \rightarrow \infty$, given that $X(t)$ and $\frac{dX(t)}{dt}$ approach zero, as $t \rightarrow \infty$. (example plot)

This means $\frac{d^2 X(t)}{dt^2}$ becomes **negative** for $t > t_m$. This is **not** possible, because $B_2(t)$ is decreasing towards zero for $t > t_{min}$ (shown in Eq. 46) and hence $\frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t) > 0$ for $t > t_{min}$. Hence $\frac{dX(t)}{dt}$ remains **negative** for all $t > 0$ and hence $X(t)$ is a **strictly decreasing** function of t , as $t \rightarrow \infty$.

$$B_2(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad \frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t)$$

$$B'_2(t) = B_2(t)e^{\frac{t}{2}}$$

In Eq. 43 and Eq. 45, we see that $B_2(t)$ at $t = 0$ (point A) equals $B_2(0) > e^{-\pi}(4\pi^2 - 6\pi) > \pi(4 * 3 - 6) * \frac{0.8571}{26} = \frac{0.8571 * 18}{64} \geq \frac{0.8571 * 16}{64} \geq 0.21$, which is **greater** than $\frac{1}{8} = 0.125$ and hence $|B_2(0)| > \frac{1}{8}$ and $|B'_2(0)| > \frac{1}{8}$ (**Result 3**) and $\frac{d^2 X(t)}{dt^2} < 0$ at $t = 0$.

As t increases from zero, $B_2(t)$ increases and reaches a **maximum** inflection point at $t = t_{max}$ (point B) if $t_{max} > 0$ and then $B_2(t)$ starts **decreasing** and reaches $B_2(t) = \frac{1}{8}e^{-\frac{t}{2}}$ at $t = t_{min}$ (point C) and hence $\frac{d^2 X(t)}{dt^2} = 0$ at $t = t_{min}$ (point C). We see that $t_{max} < t_{min}$ and t_{max} can be zero as well. (example plot) We see that $\frac{d^2 X(t)}{dt^2} < 0$ in $0 \leq t \leq t_{min}$ and **cannot** equal zero in the segment $0 \leq t \leq t_{max}$ or in the segment $t_{max} \leq t \leq t_{min}$. We see that $\frac{d^2 X(t)}{dt^2} = 0$ at $t = t_{min}$. (example plot)

Similarly, as t increases from zero, $B'_2(t)$ increases and reaches a **maximum** inflection point at $t = t'_{max}$ if $t'_{max} > 0$ and then $B'_2(t)$ starts **decreasing** and reaches $B'_2(t) = \frac{1}{8}$ at $t = t_{min}$ (point C). We see that $t'_{max} < t_{min}$. (**Result 4**) We **note** that t'_{max} can be zero as well.

We see that $B_2(t) = \frac{1}{8}e^{-\frac{t}{2}}$ and $\frac{d^2 X(t)}{dt^2} = 0$ at $t = t_{min}$ (point C) and $\frac{d^2 X(t)}{dt^2}$ reaches a **positive maximum** at $t = t_{max2}$ (point D). For $t_{min} \leq t \leq t_{max2}$, in the segment between points C and D, we see that $\frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t) > 0$, given that $\frac{dX(t)}{dt}$ is negative and increasing towards zero. (example plot) This means $B_2(t) < \frac{1}{8}e^{-\frac{t}{2}}$ in Eq. 45 in the segment $t_{min} \leq t \leq t_{max2}$ (**Result 5**), which continues in the segment $t > t_{max2}$. Given the fact that $B_2(t)$ is **strictly decreasing** for all $t > t'_{max}$ (given **Result 2**), and $t'_{max} < t_{min}$ (given **Result 4**), $B_2(t)$ **cannot** increase suddenly to greater than $\frac{1}{8}e^{-\frac{t}{2}}$, shown as follows.

At $t = t_{min}$, we can write as follows.

$$\begin{aligned} \frac{1}{8}e^{-\frac{t}{2}} = B_2(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad t = t_{min} \\ \frac{1}{8} = B_2(t)e^{\frac{t}{2}} = B'_2(t) &= \sum_{n=2}^{\infty} e^{-\pi n^2 e^{2t}} e^t [4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + \frac{1}{4}], \quad t = t_{min} \\ &, \quad \frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t) \end{aligned}$$

Given **Result 5** in the segment $t_{min} \leq t \leq t_{max2}$, we see that $B_2(t) < \frac{1}{8}e^{-\frac{t}{2}}$. For $t > t_{max2}$, $B'_2(t) = B_2(t)e^{\frac{t}{2}}$ is positive and decreasing further towards zero (from **Result 1 and 4**) and the right hand side of Eq. 46 is **decreasing lower** than $\frac{1}{8}$ and $B'_2(t) < \frac{1}{8}$.

Given the fact that $B'_2(t)$ is **strictly decreasing** for all $t > t'_{max}$ (given **Result 1**), and $t'_{max} < t_{min}$ (given **Result 4**), $B'_2(t)$ is **strictly decreasing** for all $t > t_{min}$ as well and $B'_2(t) < \frac{1}{8}$ in the segment $t > t_{min}$ and $B'_2(t)$ **cannot** suddenly increase to a value larger than $\frac{1}{8}$, in order to make $B_2(t)e^{\frac{t}{2}} > \frac{1}{8}$, which is required to make $\frac{d^2 X(t)}{dt^2} < 0$ for some value of $t > t_{min}$. Hence $\frac{d^2 X(t)}{dt^2} > 0$ for $t > t_{min}$ and **does not** become negative.

We **note** that $B'_2(t) = B_2(t)e^{\frac{t}{2}}$ reaches a maximum at $t = t'_{max}$ where $|B'_2(t)| > \frac{1}{8}$ (given **Result 3**) and hence $B'_2(t) - \frac{1}{8} > 0$ at $t = t'_{max}$. Then $B'_2(t)$ starts decreasing and equals $\frac{1}{8}$ at $t = t_{min}$. (example plot) Hence $t_{min} > t'_{max}$ and $B'_2(t)$ is **strictly decreasing** for all $t > t_{min}$ as well and $B'_2(t) < \frac{1}{8}$ and $B'_2(t)$ **cannot** suddenly increase to a value larger than $\frac{1}{8}$. Hence $\frac{d^2 X(t)}{dt^2} = \frac{1}{8}e^{-\frac{t}{2}} - B_2(t) = (\frac{1}{8} - B'_2(t))e^{-\frac{t}{2}} > 0$ for $t > t_{min}$ and **does not** become negative.

Thus we have **ruled out** the case of $\frac{dX(t)}{dt}$ becoming positive, as $t \rightarrow \infty$. This implies that $E_0(t) = X(t)$ is a **strictly decreasing** function of t for all $t > 0$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t)$ is a **strictly decreasing** function of t for all $t < 0$, going from $t = 0$ towards $t = -\infty$.

5. $\omega_z(t_2, t_0)$ is a continuous function of t_0

It is shown in this section that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$, for **each** fixed value of t_2 .

• $G_R(\omega) = G_R(\omega, t_2, t_0)$ in Eq. 17 is copied below, which is a **continuous** function of ω which is differentiable **at least** once with respect to ω . (Eq. 48).

$$\begin{aligned} G_R(\omega) = G_R(\omega, t_2, t_0) = & e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\ & + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \end{aligned} \quad (47)$$

Given that $E_0(\tau) > 0$ for $|\tau| < \infty$ and $\lim_{\tau \rightarrow \pm\infty} E_0(\tau) = 0$ (??), we see that $G_R(\omega) > 0$ at $\omega = 0$. **Set** $t_0 = 0$ and $G_R(\omega)$ passes through its **first zero** at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$. In the rest of this section, we consider the **interval** $[-\delta t_0, \delta t_0]$ around $t_0 = 0$, in $\omega_z(t_2, t_0)$. There are 3 possibilities.

Case 1: $G_R(\omega) < 0$ for $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) - dw$ for infinitesimal dw (example plot)

In this case, we will show in Section 5.1 that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

Case 2: $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(t_2, 0) - dw$ (example plot)

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, 0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, 0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(t_2, 0) + dw$.

$$\begin{aligned} \frac{dG_R(\omega, t_2, t_0)}{d\omega} = & -[e^{-2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\ & + e^{2\sigma t_0} \int_{-\infty}^0 \tau [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \end{aligned} \quad (48)$$

In this case, we will show in Section 5.2 that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

Case 3: $G_R(\omega) = 0$ for $\omega = \omega_z(t_2, 0)$ and $\omega = \omega_z(t_2, 0) + dw$.

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, 0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, 0) - dw$ and $\frac{dG_R(\omega)}{d\omega} = 0$ at $\omega = \omega_z(t_2, 0)$. The arguments are similar to that of Case 2 presented in Section 5.2 where it is shown that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$.

5.1. **Case 1:** $G_R(\omega) < 0$ **for** $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ **for** $\omega = \omega_z(t_2, 0) - dw$

• Consider the **segment S** in $G_R(\omega, t_2, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega, t_2, t_0)}{d\omega} < 0$. (Segment S is the portion between the green lines in example plot)

• In the **segment S**, $G_R(\omega, t_2, t_0)$ in Eq. 47 is a **continuous** function of ω , for **each** value of t_0 and t_2 . Hence $G_R(\omega, t_2, t_0 - \delta t_0)$ and $G_R(\omega, t_2, t_0 + \delta t_0)$ are **continuous** functions of ω , which are differentiable **at least** once, and $G_R(\omega, t_2, t_0 \pm \delta t_0)$ tends to $G_R(\omega, t_2, t_0)$, as infinitesimal $\delta t_0 \rightarrow 0$.

$$\begin{aligned}
G_R(\omega, t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_2, t_0 + \delta t_0) &= e^{-2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 - \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 + \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
G_R(\omega, t_2, t_0 - \delta t_0) &= e^{-2\sigma(t_0 - \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 + \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma(t_0 - \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 - \delta t_0, t_2)] \cos(\omega\tau) d\tau \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_2, t_0 + \delta t_0) &= G_R(\omega, t_2, t_0) \\
\lim_{\delta t_0 \rightarrow 0} G_R(\omega, t_2, t_0 - \delta t_0) &= G_R(\omega, t_2, t_0)
\end{aligned} \tag{49}$$

• In the **segment S**, $G_R(\omega, t_2, t_0)$ in Eq. 49 is a **continuous** function of ω , for **each** value of t_0 and t_2 and $\frac{dG_R(\omega, t_2, t_0)}{d\omega} < 0$ in the neighborhood around the **first zero crossing**. If we **fix** the X-coordinate ω and t_2 , $G_R(\omega, t_2, t_0)$ is a **continuous** function of t_0 , for **each** fixed value of ω . Hence, for **each** fixed value of ω , as we change t_0 by an infinitesimal δt_0 , $G_R(\omega, t_2, t_0 - \delta t_0)$ and $G_R(\omega, t_2, t_0 + \delta t_0)$ in Eq. 49, move towards $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \rightarrow 0$. Every point in the segment S, moves continuously, as we change t_0 by an infinitesimal δt_0 .

This also applies to the first **zero crossing** in $G_R(\omega, t_2, t_0)$ in the segment S, which corresponds to $\omega_z(t_2, t_0) = \omega_z(t_2, 0)$ at $t_0 = 0$ where $G_R(\omega, t_2, t_0) = 0$ in Eq. 49. The **zero crossing** moves **continuously**, as we change t_0 by an infinitesimal δt_0 . This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega, t_2, t_0)$ with zero crossing at point P_0 , **Green** plot corresponds to $G_R(\omega, t_2, t_0 + \delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to $G_R(\omega, t_2, t_0 - \delta t_0)$ with zero crossing at point P_{21} .

We **define** the **point** P_{12} in $G_R(\omega, t_2, t_0 + \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(t_2, 0)$. We **define** the **point** P_{22} in $G_R(\omega, t_2, t_0 - \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(t_2, 0)$.

We **define** the **point** P_{11} in $G_R(\omega, t_2, t_0 + \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega, t_2, t_0 - \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change t_0 by an infinitesimal δt_0 , $G_R(\omega, t_2, t_0 + \delta t_0)$ in Eq. 49 moves towards $G_R(\omega, t_2, t_0)$ in a **continuous** manner, for **each fixed** value of ω and t_2 , including the zero crossing point, as follows. The **point** P_{12} in $G_R(\omega, t_2, t_0 + \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(t_2, 0)$, moves towards corresponding point P_0 in $G_R(\omega, t_2, t_0)$, for the **same** $\omega = \omega_z(t_2, 0)$ in a **continuous** manner, as $\delta t_0 \rightarrow 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_2, t_0)$, this is equivalent to the **Zero crossing point** P_{11} in $G_R(\omega, t_2, t_0 + \delta t_0)$ moving towards corresponding **zero crossing point** P_0 in $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \rightarrow 0$.

Similarly, as we change t_0 by an infinitesimal δt_0 , $G_R(\omega, t_2, t_0 - \delta t_0)$ in Eq. 49 moves towards $G_R(\omega, t_2, t_0)$ in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega, t_2, t_0 - \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(t_2, 0)$, moves towards corresponding point P_0 in $G_R(\omega, t_2, t_0)$, for the **same** $\omega = \omega_z(t_2, 0)$ in a **continuous** manner, as $\delta t_0 \rightarrow 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_2, t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega, t_2, t_0 - \delta t_0)$ moving towards corresponding **zero crossing point** P_0 in $G_R(\omega, t_2, t_0)$ in a **contin-**

uous manner, as $\delta t_0 \rightarrow 0$.

- Hence in the **segment** S, $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ at $t_0 = 0$.

$$\begin{aligned}
G_R(\omega_z(t_2, t_0), t_2, t_0) &= e^{-2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau + t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^0 [E'_0(\tau - t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \\
G_R(\omega_z(t_2, t_0 + \delta t_0), t_2, t_0 + \delta t_0) &= \\
e^{-2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau + t_0 + \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau - t_0 - \delta t_0, t_2)] \cos(\omega_z(t_2, t_0 + \delta t_0)\tau) d\tau \\
&\quad + e^{2\sigma(t_0 + \delta t_0)} \int_{-\infty}^0 [E'_0(\tau - t_0 - \delta t_0, t_2)e^{-2\sigma\tau} + E'_{0n}(\tau + t_0 + \delta t_0, t_2)] \cos(\omega_z(t_2, t_0 + \delta t_0)\tau) d\tau = 0
\end{aligned} \tag{50}$$

5.2. **Case 2:** $G_R(\omega) > 0$ **for** $\omega = \omega_z(t_2, 0) + dw$, $G_R(\omega) > 0$ **for** $\omega = \omega_z(t_2, 0) - dw$

- In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, t_0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_2, t_0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(t_2, t_0) + dw$.

- Consider the **segment** S' in $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$ in the neighborhood around the first zero crossing where $\frac{d^2G_R(\omega, t_2, t_0)}{d\omega^2} > 0$. (Segment S' is the portion between the green lines in example plot) In this segment S', $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$ is a **continuous** function of ω which is differentiable **at least** once. (Section ??)

- In the **segment** S', $\frac{dG_R(\omega, t_2, t_0)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_2, t_0)$. The arguments in Section 5.1 can be applied here, with $G_R(\omega, t_2, t_0)$ replaced by $\frac{dG_R(\omega, t_2, t_0)}{d\omega}$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_2, t_0) = \omega_z(t_2, 0)$ at $t_0 = 0$ in the **segment** S'.

We can use similar arguments and see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$, for **each** fixed value of t_2 .

5.3. Further Points

- Using arguments in previous subsections, we see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2 in the neighbourhood $[t_2 - \delta t_2, t_2 + \delta t_2]$ for all $|t_2| < \infty$, for **each** fixed value of t_0 .

- We **set** $t_2 = Kt_0$ for even positive integer K . Using arguments in previous subsections, we see that $\omega_z(Kt_0, t_0)$ is a **continuous** function of t_0 in the neighbourhood $[t_0 - \delta t_0, t_0 + \delta t_0]$ for all $|t_0| < \infty$.

6. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied^[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi) \xi(1-s, \bar{\chi})$ ^[7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = [\frac{1}{2} e^{\frac{t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}] u(-t) + [\frac{1}{2} e^{\frac{-t}{2}} - \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}] u(t)$ (Appendix D). In the case of **Hurwitz zeta function** and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t) e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function which is **strictly decreasing** function for $t \geq \frac{1}{8}$. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

References

- [1] Bernhard Riemann, On the Number of Prime Numbers less than a Given Quantity.(Ueber die Anzahl der Primzahlen untereiner gegebenen Grosse.) Monatsberichte der Berliner Akademie, November 1859. (Link to Riemann's 1859 paper)
- [2] Hardy, G.H., Littlewood, J.E. The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift volume 10, pp.283 to 317 (1921).
- [3] E. C. Titchmarsh, The Theory of the Riemann Zeta Function. (1986) pp.254 to 255
- [4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152
- [5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)
- [6] Mathworld article on Hurwitz Zeta functions. (Link)
- [7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ (link). This is re-derived in Appendix D.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t) e^{-\sigma t}$ where $0 \leq |\sigma| < \frac{1}{2}$ is real.

$$\begin{aligned} \xi(\frac{1}{2} + \sigma + i\omega) &= \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\ E_p(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{p\omega}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma) e^{i\omega t} d\omega \end{aligned} \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In ??, we show that $\int_{-\infty}^{\infty} |E_p(t)| dt$ is finite and $E_p(t) = E_0(t) e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t) e^{-izt} dt = \int_{-\infty}^{\infty} E_0(t) e^{yt} e^{-ixt} dt$, **goes to zero** as $x \rightarrow \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t) e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4 . Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$\begin{aligned} E_p(t) &= e^{-\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{0\omega}(\omega') e^{i\omega' t} d\omega' \\ E_p(t) &= E_0(t) e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \end{aligned} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t) e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. Convolution Theorem: Multiplication of $g(t)$ and $h(t)$ corresponds to convolution in Fourier transform domain

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \quad (\text{B.1})$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega' \right] h(t)e^{-i\omega t}dt \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[\int_{-\infty}^{\infty} e^{i\omega' t} h(t)e^{-i\omega t}dt \right] d\omega' \\ F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') \left[\int_{-\infty}^{\infty} h(t)e^{-i(\omega - \omega')t}dt \right] d\omega' \end{aligned} \quad (\text{B.2})$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega - \omega')t}dt = H(\omega - \omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \quad (\text{B.3})$$

Appendix B.2. Fourier transform of Real $g(t)$

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t) \cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= - \int_{-\infty}^{\infty} g(t) \sin(\omega t)dt = -G_I(-\omega) \end{aligned} \quad (\text{B.4})$$

Appendix B.3. Even part of $g(t)$ corresponds to real part of Fourier transform $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{\text{even}}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{\text{even}}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \quad (\text{B.5})$$

Appendix B.4. Odd part of $g(t)$ corresponds to imaginary part of Fourier transform $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. Convolution integral convergence

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{4}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \rightarrow \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi} [G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.2. Fall off rate of Fourier Transform of functions

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0 , corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \rightarrow \infty$.

Appendix D. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below^[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right) &= \int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} dy \\ \Gamma\left(\frac{s}{2}\right) (\pi n^2)^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \end{aligned} \tag{D.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \tag{D.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and

$$\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma\left(\frac{\sigma'}{2}\right) \pi^{-\frac{\sigma'}{2}} \zeta(\sigma').$$

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \tag{D.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. D.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0, 1]$ and $[1, \infty)$ and make the substitution $x \rightarrow \frac{1}{x}$ in the first interval $(0, 1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}} (1 + 2w(\frac{1}{x}))$, where $x > 0$ is real.^[4]

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1}{2} dx \tag{D.4}$$

Hence we can simplify Eq. D.4 as follows.

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}} w(x) dx \tag{D.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1) \int_1^{\infty} (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}]$$

(D.6)

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix D.1. Derivation of $E_p(t)$ and $E_0(t)$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. D.6 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi\left(\frac{1}{2} + \sigma + i\omega\right) = \frac{1}{2} \left[1 + 2\left(\frac{1}{2} + \sigma + i\omega\right) \left(-\frac{1}{2} + \sigma + i\omega\right) \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} \left(e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t} + e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} \right) dt \right] \quad (D.7)$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned} \xi\left(\frac{1}{2} + \sigma + i\omega\right) &= \frac{1}{2} + \left(-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)\right) \left[\int_{-\infty}^0 \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right. \\ &\quad \left. + \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega t} dt \right] \end{aligned} \quad (D.8)$$

We can write this as follows.

$$\xi\left(\frac{1}{2} + \sigma + i\omega\right) = \frac{1}{2} + \left(-\frac{1}{4} + \sigma^2 - \omega^2 + i\omega(2\sigma)\right) \int_{-\infty}^{\infty} \left[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t} e^{-i\omega t} dt \quad (D.9)$$

We define $A(t) = \left[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2} + \sigma + i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned} E_p(t) &= \frac{1}{2} \delta(t) + \left(-\frac{1}{4} + \sigma^2\right) A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2 A(t)}{dt^2} \\ A(t) &= \left[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t) \right] e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t} \right] u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[\frac{1}{2} - \sigma - 2\pi n^2 e^{2t} \right] u(t) \\ \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[-4\pi n^2 e^{-2t} + \left(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}\right)^2 \right] u(-t) \\ &\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[-4\pi n^2 e^{2t} + \left(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}\right)^2 \right] u(t) + \delta(t) \left[\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) \right] \end{aligned} \quad (D.10)$$

We can simplify above equation as follows.

$$\begin{aligned} \frac{d^2 A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} \left[\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t} \right] u(-t) \\ &\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \left[\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t} \right] u(t) + \delta(t) \left[\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) \right] \end{aligned}$$

(D.11)

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real^[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix D.2) and hence **dirac delta terms cancel each other** in equation below.

$$\begin{aligned}
E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma \frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\
E_p(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} [-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}] u(t) \\
&\quad + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t} [-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t}) \\
&\quad + \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}] u(-t)
\end{aligned}
\tag{D.12}$$

We can simplify above equation as follows.

$$\begin{aligned}
E_p(t) &= [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t} \\
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}
\end{aligned}
\tag{D.13}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$\begin{aligned}
E_0(t) &= 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
E_p(t) &= E_0(t)e^{-\sigma t} = 2 \sum_{n=1}^{\infty} [2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{D.14}$$

Appendix D.2. Derivation of $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real^[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$\begin{aligned}
F(x) &= 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2 \frac{1}{x}} (\frac{1}{x^2}) + (1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) (-\frac{1}{2}) \frac{1}{x^{\frac{3}{2}}}
\end{aligned}$$

(D.15)

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$\begin{aligned} \left[\frac{dF(x)}{dx}\right]_{x=1} &= 2 \sum_{n=1}^{\infty} (-\pi n^2) e^{-\pi n^2} = \sum_{n=1}^{\infty} (2\pi n^2) e^{-\pi n^2} + \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}\right) \left(\frac{-1}{2}\right) \\ &\quad \sum_{n=1}^{\infty} e^{-\pi n^2} (1 - 4\pi n^2) = -\frac{1}{2} \end{aligned}$$

(D.16)