1. Alternate Proof (Author: Akhila Raman)

Step 1: In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and rederive the steps below. (link) We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and rederive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{2}$$

Let $s = \sigma' + i\omega$. For real part of s given by $\sigma' > 1$, we can use theorem of dominated convergence and **interchange** the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty}\int_0^{\infty} |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx \tag{3}$$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. 3 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$. We define $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$ and write as follows for $\sigma' = \frac{1}{2}+\sigma > 1$.

$$F(\frac{1}{2}+\sigma+i\omega) = 2\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t}dt = 2\int_{-\infty}^{\infty} A(t)e^{i\omega t}dt \tag{4}$$

Critical Strip: For $0 < \sigma' = \frac{1}{2}+\sigma < 1$, $\zeta(s)$ **diverges** and $F(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ is said to diverge.

Counter-intuitively, it is shown in Section 3 that the integral in right hand side of Eq. 4 **converges**, for $0 < \sigma' < 1$ which corresponds to the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$ and that $\int_{-\infty}^{\infty}|A(t)|dt$ is finite and hence we can **interchange** the order of summation and integration in Eq. 2, using **Fubini's theorem**.

We show this result in Section 3.1, by considering $E_p(t) = (-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$, whose Fourier transform is given by $E_{p\omega}(\omega) = A(\omega)(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))$ and **because** $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ and $\frac{1}{(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))}$ converge for all real ω, $A(\omega)$ **converges** for all real ω.

2. Step 2: Proof

We use **the well known theorem** $1+2w(x)=\frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $x>0$ is real. and get the well known result below.(link)

$$\xi(s)=\frac{1}{2}s(s-1)F(s)=\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)=\frac{1}{2}[1+s(s-1)\int_{1}^{\infty}(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{5}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s)=\xi(1-s)$.

Given that $w(x)=\sum_{n=1}^{\infty}e^{-\pi n^2 x}$, we substitute $x=e^{2t}$, $\frac{dx}{x}=2dt$ in Eq. 5 and evaluate at $s=\frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_{0}^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{6}$$

We can substitute $t=-t$ in the first term in above integral and simplify above equation as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt+\int_{0}^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt] \tag{7}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega)=\frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{8}$$

Statement A: If $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ **has a zero** at $\omega=\omega_0$ where ω_0 is real and finite, **then** $\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega_0 t}dt=-\frac{1}{2}\frac{1}{(-\frac{1}{4}+\sigma^2-\omega_0^2+i\omega_0(2\sigma))}$.

We can write the integral in Eq. 4 as follows using $t=-t$, split into two integrals and we use **the well known theorem** $1+2w(x)=\frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$ in the second integral, where $x>0$ is real and we use $x=e^{2t}$.

$$F(\frac{1}{2}+\sigma+i\omega)=2\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}dt=2\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{-\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt$$

$$F(\frac{1}{2}+\sigma+i\omega)=2[\int_{-\infty}^{0}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt+\int_{0}^{\infty}\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]$$
$$+\frac{}{2}\int_{0}^{\infty}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt-\int_{0}^{\infty}e^{-\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt$$

$$(9)$$

If **Statement A** is true, then $F(s) = \frac{\xi(s)}{\frac{1}{2}s(s-1)}$ also **has a zero** at $\omega = \omega_0$, for $s = \frac{1}{2} + \sigma + i\omega$ and $0 < |\sigma| < \frac{1}{2}$. We can compute Eq. 9 as follows. We use $\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} u(t)] e^{-\sigma t} e^{-i\omega_0 t} dt = -\frac{1}{2}\frac{1}{(-\frac{1}{4}+\sigma^2-\omega_0^2+i\omega_0(2\sigma))}$.

$$F(\frac{1}{2} + \sigma + i\omega_0) = \frac{1}{(\frac{1}{4} - \sigma^2 + \omega_0^2 - i\omega_0(2\sigma))} + \int_0^{\infty} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt - \int_0^{\infty} e^{-\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt = 0 \tag{10}$$

For $0 < |\sigma| < \frac{1}{2}$, we can write

$$\begin{aligned} \int_0^{\infty} e^{-\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt &= \frac{1}{\frac{1}{2} + \sigma + i\omega_0} \\ F(\frac{1}{2} + \sigma + i\omega_0) = \frac{1}{(\frac{1}{4} - \sigma^2 + \omega_0^2 - i\omega_0(2\sigma))} - \frac{1}{\frac{1}{2} + \sigma + i\omega_0} + \int_0^{\infty} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt &= 0 \\ F(\frac{1}{2} + \sigma + i\omega_0) = \frac{1}{\frac{1}{2} - \sigma - i\omega_0} + \int_0^{\infty} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt &= 0 \end{aligned} \tag{11}$$

We can see that the integral $\int_0^{\infty} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt$ **diverges** for $0 \leq |\sigma| < \frac{1}{2}$.

$$\int_0^{\infty} e^{\frac{t}{2}} e^{-\sigma t} e^{-i\omega_0 t} dt = \lim_{T\to\infty} [\frac{e^{t(\frac{1}{2}-\sigma-i\omega_0)}}{(\frac{1}{2} - \sigma - i\omega_0)}]_{t=0}^{t=T} = \frac{-1}{\frac{1}{2} - \sigma - i\omega_0} + \frac{1}{\frac{1}{2} - \sigma - i\omega_0} \lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma-i\omega_0)} \tag{12}$$

Substituting Eq. 12 in Eq. 11, we get

$$F(\frac{1}{2} + \sigma + i\omega_0) = \frac{1}{\frac{1}{2} - \sigma - i\omega_0} \lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma-i\omega_0)} = 0 \tag{13}$$

We can see that $\lim_{T\to\infty} e^{T(\frac{1}{2}-\sigma-i\omega_0)} \neq 0$ and hence $F(\frac{1}{2} + \sigma + i\omega_0)$ **diverges** for $0 \leq |\sigma| < \frac{1}{2}$.

We see that the assumption in **Statement A** that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ **has a zero** at $\omega = \omega_0$ where ω_0 is real and finite, leads to a **contradiction** for the critical strip $0 \leq |\sigma| < \frac{1}{2}$.

3. Integral in Eq. 4 is L^1 integrable

In Eq. 4 copied below, we replace $\omega = -\omega$. If $F(\frac{1}{2}+\sigma-i\omega)$ converges, then $F(\frac{1}{2}+\sigma+i\omega)$ also converges.

$$F(\frac{1}{2}+\sigma-i\omega) = 2\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt = 2\int_{-\infty}^{\infty} A(t) e^{-i\omega t} dt \tag{14}$$

We see that $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} \geq 0$ and finite for all $|t| \leq \infty$. As $t \to \infty$, the integral in Eq. 14 goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, the integral in Eq. 14 goes to zero, for $0 < |\sigma| < \frac{1}{2}$, due to the term $e^{\frac{t}{2}} e^{\sigma t}$.

Hence the integral in Eq. 14 is L^1 integrable and hence $\int_{-\infty}^{\infty} |A(t)| dt$ is finite and hence we can **interchange** the order of summation and integration in Eq. 2, using **Fubini's theorem**, for $0 < |\sigma| < \frac{1}{2}$.

The **series** in Eq. 14 inside the integral, **converges** for all $t > -\infty$, using Integral test, because $\int_1^{\infty} C e^{-Bu^2} du$ is finite, where $B = \pi e^{2t} > 0$, $C = e^{\frac{t}{2}} e^{\sigma t}$ and n is replaced by u. For $t = -\infty$, the integral in Eq. 14 goes to zero, due to the term $e^{\frac{t}{2}} e^{\sigma t}$, for $0 < |\sigma| < \frac{1}{2}$.

*3.1. **Convergence of** $A(\omega)$ **and Integral in Eq. 4 is** L^1 **integrable***

For every value of n in equation below, the integral converges, because $F_n(s) = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\frac{1}{n^s}$ converges.

We will show that $\int_{-\infty}^{\infty} |A(t)| dt$ is finite and hence we can **interchange** the order of summation and integration in Eq. 2, using **Fubini's theorem**, where $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$.

$$F(\frac{1}{2}+\sigma-i\omega) = 2\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t} dt = 2\int_{-\infty}^{\infty} A(t) e^{-i\omega t} dt \tag{15}$$

We will show that $A(\omega) = \int_{-\infty}^{\infty} A(t) e^{-i\omega t} dt$ **converges** for all real ω.

We start with $A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$ and show that $E_p(t) = [(-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}]$ as follows.

$$\begin{aligned}
A(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} \\
\frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[\frac{1}{2}+\sigma-2\pi n^2 e^{2t}] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2}+\sigma-2\pi n^2 e^{2t})^2] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} - 4\sigma\pi n^2 e^{2t}]
\end{aligned} \tag{16}$$

We have arrived at the desired result for $E_p(t)$ as follows.

$$
\begin{aligned}
E_p(t) &= [(-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}]e^{-\sigma t} \\
E_p(t) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}
\end{aligned}
\tag{17}
$$

The Fourier transform of $E_p(t)$ is given by $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ **converges** for real ω. Using the properties of Fourier transform, we get $E_{p\omega}(\omega) = A(\omega)(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))$ and we see that $A(\omega)$ **converges** for all real ω. **because** $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ and$\frac{1}{(-\frac{1}{4}+\sigma^2-\omega^2-i\omega(2\sigma))}$ converge for all real ω,

Hence we have shown that $A(\omega) = \int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{-i\omega t}dt$ **converges** for all real ω. We know that $A(t) = e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} \geq 0$ and finite for all $|t| \leq \infty$. Hence the integral in Eq. 14 is L^1 integrable and hence $\int_{-\infty}^{\infty} |A(t)|dt$ is finite and hence we can **interchange** the order of summation and integration in Eq. 2, using **Fubini's theorem**, for $0 < |\sigma| < \frac{1}{2}$.