On a new method towards proof of Riemann's Hypothesis

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix F, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3]

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider a new function $g(t)=e^{\sigma t_0}E_p(t+t_0)e^{-\sigma t}u(-t)+e^{\sigma t_0}E_p(t+t_0)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t)=e^{\sigma t_0}E_p(t+t_0)$ where $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t)=\frac{1}{2}[g(t)+g(-t)]$ given by $G_{even}(\omega)=G_R(\omega)$ must have **at least one zero** at $\omega=\omega_2(t_0)\neq 0$, for every value of t_0, to satisfy Statement 1, where $\omega_2(t_0)$ is real and finite.

1.3. *Step 3: On the zeros of the function* $G_R(\omega)$

In Section 2.2, we compute the Fourier transform of the function $g_{even}(t)$ given by $G_R(\omega)$. We require $G_R(\omega)=0$ for $\omega=\omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**. In general, $\omega_2(t_0)\neq\omega_0$.

It is shown that $R(t_0)=G_R(\omega_2(t_0),t_0)=0$ for all t_0 as follows.

$$\begin{aligned}R(t_0)=e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau+\sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0}E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]\\ +[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0}E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau-\sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0}E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]=0\\ R(t_0)=\int_{-\infty}^{0}[E_0(\tau+t_0)e^{-2\sigma\tau}+E_0(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau=0\end{aligned} \tag{3}$$

In Section 2.3, it is shown that $\omega_2(t_0)=\omega_2(-t_0)$.

1.4. ***Step 4: First derivative of*** $R(t_0)$

In Section 3.2, we derive the first derivative of $R(t_0)$ at $t_0 = 0$ as follows, where $\omega_{20} = [\omega_2(t_0)]_{t_0=0}$. $m_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_{20}\tau)d\tau$, $n_0 = \int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_{20}\tau)d\tau$, $m_{0p} = \int_{-\infty}^{0} E_0(\tau)\cos(\omega_{20}\tau)d\tau$, $n_{0p} = \int_{-\infty}^{0} E_0(\tau)\sin(\omega_{20}\tau)d\tau$.

$$
\begin{aligned}
[R(t_0)]_{t_0=0} &= R_0 = m_0 + m_{0p} = 0 \\
(\frac{dR(t_0)}{dt_0})_{t_0=0} &= R_1 = \omega_{20}[n_0 - n_{0p}] + 2\sigma m_0 = 0
\end{aligned}
\tag{4}
$$

1.5. ***Step 5: Next Step***

In Section 3.3, we replace $E_p(t)$ by $E_p^{'}(t) = e^{\sigma t_2}E_p(t+t_2)$, for $|t_2| \leq \infty$ and derive as follows.

$$
\begin{aligned}
R_0^{'}(t_2) &= m_0^{'}(t_2) + m_{0p}^{'}(t_2) = 0 \\
m_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau &+ \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau] \\
m_{0p}^{'}(t_2) = \cos(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2(t_2)\tau)d\tau &- \sin(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2(t_2)\tau)d\tau
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
R_1{'}(t_2) &= \omega_2(t_2)[n_0{'}(t_2) - n_{0p}{'}(t_2)] + 2\sigma m_0{'}(t_2) = 0 \\
n_0^{'}(t_2) = e^{2\sigma t_2}[\cos(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_2)\tau)d\tau &- \sin(\omega_2(t_2)t_2)\int_{-\infty}^{t_2} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_2)\tau)d\tau] \\
n_{0p}^{'}(t_2) = \cos(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\sin(\omega_2(t_2)\tau)d\tau &+ \sin(\omega_2(t_2)t_2)\int_{-\infty}^{-t_2} E_0(\tau)\cos(\omega_2(t_2)\tau)d\tau
\end{aligned}
\tag{6}
$$

1.6. ***Step 6: Asymptotic Case and Final result***

In Section **??**, we consider the asymptotic case $\lim_{t_2\to-\infty}$ and show that $\lim_{t_2\to-\infty}\omega_2(t_2) = \omega_z$ is a constant where ω_z is a zero on the critical line and derive as follows.

$$
\begin{aligned}
\lim_{t_2\to-\infty} m_{0p}^{'}(t_2) &= 0 \\
\lim_{t_2\to-\infty} n_{0p}^{'}(t_2) &= 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{i(\omega_z t)}dt &= 0
\end{aligned}
\tag{7}
$$

Then we consider the asymptotic case $\lim_{t_2\to+\infty}$ and derive as follows.

$$
\begin{aligned}
\lim_{t_2\to\infty} m_0^{'}(t_2) &= 0 \\
\lim_{t_2\to\infty} n_0^{'}(t_2) &= 0 \\
\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}e^{i(\omega_z t)}dt &= 0
\end{aligned}
$$

We started with **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ which means that $\int_{-\infty}^{\infty} E_0(\tau)e^{-\sigma\tau}e^{-i\omega_0\tau}d\tau = 0$ and we derived the result that $\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}e^{-i\omega_z\tau}d\tau = 0$.

We repeat above steps N times till $(2^{N+1}\sigma) > \frac{1}{2}$ and get the result $\int_{-\infty}^{\infty} E_0(\tau)e^{-(2^{N+1}\sigma)\tau}e^{-i\omega_{(zN)}\tau}d\tau = 0$. In each iteration $n = 1, ...N$, we use $h(t) = e^{(2^n\sigma)t}u(-t)+e^{-(2^n\sigma)t}u(t)$. We know that the Fourier Transform of $E_0(t)e^{-(2^{N+1}\sigma)t}$ **does not** have a real zero for $(2^{N+1}\sigma) > \frac{1}{2}$, corresponding to $Re[s] > 1$ and we show a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$.

2. An Approach towards Riemann's Hypothesis: Method 3

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix C.1.

2.1. ***New function*** $g(t)$

Let us consider the function $f(t) = e^{\sigma t_0}E_p(t + t_0)$ where $|t_0| \leq \infty$ and we can see that the Fourier Transform of this function $F(\omega) = e^{\sigma t_0}E_{p\omega}(\omega)e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can show that $E_p(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

We can see that $g(t)$ is a real L^1 integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma]. This is explained in detail in Appendix C.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}(\omega)e^{\sigma t_0}e^{i\omega t_0}) = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_2(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ where $\omega_2(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_2(t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ for **every value** of t_0, where $\omega_2(t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{\sigma t_0} E_p(t + t_0)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}(\omega)e^{\sigma t_0}e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_2(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega')H(\omega - \omega')d\omega' \tag{9}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 9 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{10}$$

We can split the integral in Eq. 10 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{11}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 11 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega')[\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{12}$$

In Appendix C.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 12 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_2(t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**.

2.2. ***On the zeros of a related function*** $G(\omega)$

We can compute the fourier transform of the function $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$. We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for **every value** of t_0, to satisfy **Statement 1**. In general, $\omega_2(t_0) \neq \omega_0$.

First we compute the fourier transform of the function $g(t)$ given by $G(\omega) = G_R(\omega) + iG_I(\omega)$. We use $g(t) = e^{\sigma t_0} E_p(t+t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p(t+t_0) e^{\sigma t} u(t)$.

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{0} g_-(t) e^{-i\omega t} dt + \int_{0}^{\infty} g_+(t) e^{-i\omega t} dt \\
G(\omega) &= \int_{-\infty}^{0} e^{\sigma t_0} E_p(t+t_0) e^{-\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{\sigma t_0} E_p(t+t_0) e^{\sigma t} e^{-i\omega t} dt
\end{aligned}
\tag{13}
$$

We use $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_p(t+t_0) = E_0(t+t_0)e^{-\sigma t}e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 13, we have

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} E_0(t+t_0) e^{-i\omega t} dt \\
G(\omega) &= \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_0(-t+t_0) e^{i\omega t} dt
\end{aligned}
\tag{14}
$$

We define $E_{0m}(t) = E_0(-t)$ and get $E_0(-t+t_0) = E_{0m}(t-t_0)$ and write Eq. 14 as follows.

$$
G(\omega) = \int_{-\infty}^{0} E_0(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_{0m}(t-t_0) e^{i\omega t} dt = G_R(\omega) + iG_I(\omega)
\tag{15}
$$

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **real parts** of $G(\omega)$, we have

$$G_R(\omega) = \int_{-\infty}^{0} E_0(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0m}(t-t_0)\cos(\omega t)dt \tag{16}$$

We require $G_R(\omega) = 0$ for $\omega = \omega_2(t_0)$ for every value of t_0, to satisfy **Statement 1**. Hence we can see that $R(t_0) = G_R(\omega_2(t_0)) = 0$ and we can write as follows using $t = \tau$.

$$R(t_0) = \int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau = 0 \tag{17}$$

We can rewrite Eq. 17 as follows, using the substitution $\tau + t_0 = \tau'$ in the first integral and $\tau - t_0 = \tau''$ in the second integral and substituting back $\tau' = \tau$ and $\tau'' = \tau$.

$$\begin{aligned} R(t_0) = e^{2\sigma t_0}[&\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\ &+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_{0m}(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_{0m}(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0 \end{aligned} \tag{18}$$

Now we replace t_0 by $-t_0$ in $f(t)$ and consider the function $f_2(t) = e^{-\sigma t_0}E_p(t-t_0)$ where $|t_0| \leq \infty$ and use the procedure in above section and we can write as follows.

$$\begin{aligned} R(-t_0) &= \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)]\cos(\omega_2(-t_0)\tau)d\tau = 0 \\ R(t_0) + R(-t_0) &= \int_{-\infty}^{0} [E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau-t_0)]\cos(\omega_2(t_0)\tau)d\tau \\ &+ \int_{-\infty}^{0} [E_0(\tau-t_0)e^{-2\sigma\tau} + E_{0m}(\tau+t_0)]\cos(\omega_2(-t_0)\tau)d\tau = 0 \end{aligned} \tag{19}$$

2.3. $\omega_2(t_0)$ *is an even function of variable* t_0

Now we consider the function $f_T(t) = f(t) + f_2(t) = e^{\sigma t_0}E_p(t+t_0) + e^{-\sigma t_0}E_p(t-t_0)$ where $|t_0| \leq \infty$ and $g_T(t)h(t) = f_T(t)$ where $g_T(t) = f_T(t)e^{-\sigma t}u(-t) + f_T(t)e^{\sigma t}u(t)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g_T(t)$ and compute its real part using the procedure in above section, similar to Eq. 16 and we can write as follows. We use $E_0(-\tau) = E_0(\tau)$.

$$\begin{aligned} G_{T_R}(\omega) &= G_1(\omega, t_0) + G_1(\omega, -t_0) \\ G_1(\omega, t_0) &= \int_{-\infty}^{0} E_0(t+t_0)e^{-2\sigma t}\cos(\omega t)dt + \int_{-\infty}^{0} E_{0m}(t-t_0)\cos(\omega t)dt \end{aligned} \tag{20}$$

We require $G_{T_R}(\omega) = 0$ for $\omega = \omega_0(t_0)$ for every value of t_0, to satisfy **Statement 1**. In general $\omega_0(t_0) \neq \omega_2(t_0)$. Hence we can see that $P(t_0) = G_{T_R}(\omega_0(t_0)) = 0$ and we can rewrite as follows using the substitution $t = \tau$.

$$
\begin{aligned}
P(t_0) = &\int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_0(t_0)\tau)} d\tau \\
&+ \int_{-\infty}^{0} [E_0(\tau - t_0)e^{-2\sigma\tau} + E_{0m}(\tau + t_0)] \cos{(\omega_0(t_0)\tau)} d\tau = 0
\end{aligned}
\tag{21}
$$

We see that $f_T(t) = e^{\sigma t_0} E_p(t+t_0) + e^{-\sigma t_0} E_p(t-t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and hence $\omega_0(t_0)$ is an **even** function of variable t_0. Hence we can rewrite the second integral in Eq. 21 as follows using $\omega_0(t_0) = \omega_0(-t_0)$.

$$
\int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_0(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau - t_0)e^{-2\sigma\tau} + E_{0m}(\tau + t_0)] \cos{(\omega_0(-t_0)\tau)} d\tau = 0
\tag{22}
$$

We compare Eq. 22 and Eq. 19 as follows.

$$
\begin{aligned}
&\int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_0(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau - t_0)e^{-2\sigma\tau} + E_{0m}(\tau + t_0)] \cos{(\omega_0(-t_0)\tau)} d\tau = 0 \\
&\int_{-\infty}^{0} [E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos{(\omega_2(t_0)\tau)} d\tau + \int_{-\infty}^{0} [E_0(\tau - t_0)e^{-2\sigma\tau} + E_{0m}(\tau + t_0)] \cos{(\omega_2(-t_0)\tau)} d\tau = 0
\end{aligned}
\tag{23}
$$

We can see that there must be **at least one** common solution where $\omega_2(t_0) = \omega_0(t_0)$ to satisfy Eq. 23. Because $\omega_0(t_0)$ is an **even** function of variable t_0, we see that $\omega_2(t_0) = \omega_0(t_0)$ is also an **even** function of variable t_0.

2.4. *Method 5*

We define $A(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]$ and $B(t_0) = \cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau$ and we write $R(t_0) = A(t_0) + B(-t_0)$ in Eq. 18 as follows. We use $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$.

$$\begin{aligned}
R(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = A(t_0) + B(-t_0) = 0
\end{aligned} \tag{24}$$

We use $\int_{-\infty}^{t_0} = \int_{-\infty}^{-t_0} + \int_{-t_0}^{t_0}$ in the first term and $\int_{-\infty}^{-t_0} = \int_{-\infty}^{t_0} - \int_{-t_0}^{t_0}$ in the second term in Eq. 29 and write as follows.

$$\begin{aligned}
R(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&+e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
&-[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0
\end{aligned} \tag{25}$$

Next, we define $X(t_0) = X_1(t_0) + X_2(t_0)$ as follows. We have replaced the integral $\int_{-\infty}^{t_0}$ in Eq. 29 with the integral $\int_{-t_0}^{\infty}$ and we replace the integral $\int_{-\infty}^{-t_0}$ in Eq. 29 with the integral $\int_{t_0}^{\infty}$ as follows.

$$\begin{aligned}
X(t_0) &= X_1(t_0) + X_2(t_0) \\
X_1(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-t_0}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
X_2(t_0) &= [\cos(\omega_2(t_0)t_0)\int_{t_0}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{t_0}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned} \tag{26}$$

The integrals in Eq. 31 converge because $\int_{-\infty}^{\infty}|E_0(\tau)e^{-2\sigma\tau}|d\tau$ converges for $|\sigma| < \frac{1}{2}$(Appendix C.1).

We add $R(t_0) = 0$ in Eq. 30 with $X(t_0)$ in Eq. 31 as follows. We use $\int_{-\infty}^{-t_0} + \int_{-t_0}^{\infty} = \int_{-\infty}^{\infty}$ and $\int_{-\infty}^{t_0} + \int_{t_0}^{\infty} = \int_{-\infty}^{\infty}$.

$$\begin{aligned}
R(t_0) + X(t_0) &= X(t_0) \\
X(t_0) &= e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&+e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
&-[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned}$$

(27)

We use $e^{-2\sigma\tau} = \cosh(2\sigma\tau) - \sinh(2\sigma\tau)$ in the integrals $\int_{-t_0}^{t_0}$ in Eq. 32 and note that $\int_{-t_0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau = 0$ and $\int_{-t_0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ because the integrands are **odd** functions of variable τ.

Given that $E_0(\tau) = E_0(-\tau)$, we see that $E_0(\tau)\sin(\omega_2(t_0)\tau)$ is an **odd** function of variable τ and $\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ and $\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$. Given that $E_0(\tau)\cos(\omega_2(t_0)\tau)$ is an **even** function of variable τ, we can write as follows.

$$\begin{aligned}
X(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+e^{2\sigma t_0}[2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau - 2\sin(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
+2\cos(\omega_2(t_0)t_0)\int_{0}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - 2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau
\end{aligned} \tag{28}$$

3. Alternate Method 6

We define $A(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau]$ and $B(t_0) = \cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau$ and we write $R(t_0) = A(t_0) + B(-t_0) = 0$ in Eq. 18 as follows. We use $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$.

$$\begin{aligned}
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = A(t_0) + B(-t_0) = 0
\end{aligned} \tag{29}$$

We use $\int_{-\infty}^{t_0} = \int_{-\infty}^{-t_0} + \int_{-t_0}^{t_0}$ in the first term and $\int_{-\infty}^{-t_0} = \int_{-\infty}^{t_0} - \int_{-t_0}^{t_0}$ in the second term in Eq. 29 and write as follows.

$$\begin{aligned}
R(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
-[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] = 0
\end{aligned} \tag{30}$$

Next, we define $X(t_0) = X_1(t_0) + X_2(t_0)$ as follows. We replace the integral $\int_{-\infty}^{t_0}$ in Eq. 29 with the integral $\int_{-t_0}^{\infty}$ and we replace the integral $\int_{-\infty}^{-t_0}$ in Eq. 29 with the integral $\int_{t_0}^{\infty}$ as follows.

$$\begin{aligned}
X(t_0) = X_1(t_0) + X_2(t_0) \\
X_1(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-t_0}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
X_2(t_0) = [\cos(\omega_2(t_0)t_0)\int_{t_0}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{t_0}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned}$$

(31)

The integrals in Eq. 31 converge because $\int_{-\infty}^{\infty} |E_0(\tau)e^{-2\sigma\tau}|d\tau$ converges for $|\sigma| < \frac{1}{2}$(Appendix C.1).

We add $R(t_0) = 0$ in Eq. 30 with $X(t_0)$ in Eq. 31 as follows. We use $\int_{-\infty}^{-t_0} + \int_{-t_0}^{\infty} = \int_{-\infty}^{\infty}$ and $\int_{-\infty}^{t_0} + \int_{t_0}^{\infty} = \int_{-\infty}^{\infty}$.

$$
\begin{aligned}
R(t_0) + X(t_0) &= X(t_0) \\
X(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &+ \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &+ \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau &- \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
-[\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau &- \sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned}
\tag{32}
$$

We use $e^{-2\sigma\tau} = \cosh(2\sigma\tau) - \sinh(2\sigma\tau)$ in the integrals $\int_{-t_0}^{t_0}$ in Eq. 32 and note that $\int_{-t_0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau = 0$ and $\int_{-t_0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ because the integrands are **odd** functions of variable τ.

Given that $E_0(\tau) = E_0(-\tau)$, we see that $E_0(\tau)\sin(\omega_2(t_0)\tau)$ is an **odd** function of variable τ and $\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ and $\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$.

Given that $E_0(\tau)\cos(\omega_2(t_0)\tau)$ is an **even** function of variable τ, we can write $\int_{-\infty}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau = 2\int_{-\infty}^{0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ and $\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau = 2\int_{0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ as follows.

$$
\begin{aligned}
X(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &+ \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+e^{2\sigma t_0}[2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau &- 2\sin(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
+2\cos(\omega_2(t_0)t_0)\int_{-\infty}^{0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau &- 2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{33}
$$

We compute $X_3(t_0) = X(t_0)e^{-2\sigma t_0}$ as follows.

$$
\begin{aligned}
X_3(t_0) = X(t_0)e^{-2\sigma t_0} = [\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau &+ \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+[2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau &- 2\sin(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
+2\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{-\infty}^{0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau &- 2\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{34}
$$

3.1. ***Another expression for*** $X(t_0)$

We use $\int_{-t_0}^{\infty} = \int_{-\infty}^{\infty} - \int_{-\infty}^{-t_0}$ in the integrals in $X_1(t_0)$ and $\int_{t_0}^{\infty} = \int_{-\infty}^{\infty} - \int_{-\infty}^{t_0}$ in the integrals in $X_2(t_0)$ in Eq. 31 as follows.

$$X(t_0) = X_1(t_0) + X_2(t_0)$$

$$\begin{aligned}
X(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
-e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\
-[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned} \tag{35}$$

We compute $X_4(t_0) = X(t_0)e^{-2\sigma t_0}$ and use $\int_{-\infty}^{\infty} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ because the integrand is an **odd** function of variable t_0, and we use $\int_{-\infty}^{\infty} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau = 2\int_{-\infty}^{0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ because the integrand is an **even** function of variable t_0, as follows.

$$\begin{aligned}
X_4(t_0) = X(t_0)e^{-2\sigma t_0} = [\cos(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{\infty} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
-[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
+e^{-2\sigma t_0}[2\cos(\omega_2(t_0)t_0)\int_{-\infty}^{0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau] \\
-e^{-2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned} \tag{36}$$

We compare Eq. 34 and Eq. 36 and cancel out common terms as follows.

$$\begin{aligned}
X_5(t_0) = 2\cos(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau - 2\sin(\omega_2(t_0)t_0)\int_{0}^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau \\
-2\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
= -[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
-e^{-2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned} \tag{37}$$

We compute the **even part** of $X_5(t_0)$ given by $X_6(t_0) = \frac{1}{2}[X_5(t_0) + X_5(-t_0)]$ as follows. The term $2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau$ is an **odd** function of variable t_0 and is omitted. We note that $-2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau$ is an **even** function of variable t_0 and the term $-2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ is an **odd** function of variable t_0. Hence the even part of $-2\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ is given by $+2\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$. We use $\omega_2(-t_0) = \omega_2(t_0)$.

$$
\begin{aligned}
X_6(t_0) = &-2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau + 2\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
= &-\frac{1}{2}\cos(\omega_2(t_0)t_0)[\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau] \\
&-\frac{1}{2}\sin(\omega_2(t_0)t_0)[\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau - \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\
&-\frac{1}{2}\cos(\omega_2(t_0)t_0)[e^{-2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + e^{2\sigma t_0}\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau] \\
&+\frac{1}{2}\sin(\omega_2(t_0)t_0)[e^{-2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau - e^{2\sigma t_0}\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau]
\end{aligned}
\tag{38}
$$

We use $\int_{-\infty}^{t_0} = \int_{-\infty}^{-t_0} + \int_{-t_0}^{t_0}$ in the right hand side of Eq. 38 and cancel common terms as follows.

$$
\begin{aligned}
X_6(t_0) = &-2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau + 2\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
= &-\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau - \frac{1}{2}\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau \\
&+\frac{1}{2}\sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau \\
&-\cos(\omega_2(t_0)t_0)\cosh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \frac{1}{2}\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
&-\sin(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + \frac{1}{2}\sin(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{39}
$$

We use $\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ and $\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau = 2\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ and see that the term $\frac{1}{2}\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau = \frac{1}{2}\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau$ is an **odd** function of variable t_0 which is **omitted** because we are considering only the even terms in $X_6(t_0)$.
We note that the term $\frac{1}{2}\sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau = -\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau$.

$$
\begin{aligned}
X_6(t_0) = &-2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau + 2\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
= &-\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau \\
&-\cos(\omega_2(t_0)t_0)\cosh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\
&-\sin(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau
\end{aligned}
\tag{40}
$$

We can simplify as follows, consolidating the terms $\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau$ on both sides of Eq. 40.

$$X_6(t_0) = -\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau + 2\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$$
$$= -\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau$$
$$-\cos(\omega_2(t_0)t_0)\cosh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau - \cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$$
$$-\sin(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \tag{41}$$

In Section 3.2, it is shown that there **exists** a value of $t_0 = t_1 > 0$ for which $\omega_2(t_1)t_1 = \frac{\pi}{2}$. We **set** $t_0 = t_1$ and $\omega_2(t_1)t_1 = \frac{\pi}{2}$ in Eq. 41 and we get $\cos(\omega_2(t_1)t_1) = 0$ and $\sin(\omega_2(t_1)t_1) = 1$.

$$\int_0^{t_1} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_1)\tau)d\tau = \sinh(2\sigma t_1)\int_{-\infty}^{-t_1} E_0(\tau)\cos(\omega_2(t_1)\tau)d\tau \tag{42}$$

In Section 3.3, it is shown that the integral $\int_{-\infty}^{-t_1} E_0(\tau)\cos(\omega_2(t_1)\tau)d\tau = \int_{t_1}^{\infty} E_0(\tau)\cos(\omega_2(t_1)\tau)d\tau < 0$ and we see that $\sinh(2\sigma t_1) > 0$ for $t_1 > 0$. Hence the integral in the right hand side of Eq. 42 is **less than zero** for $0 < \sigma < \frac{1}{2}$.

For $0 < \sigma < \frac{1}{2}$, we see that the **each** of the terms in the integrand in the left hand side of Eq. 42 are **greater than or equal to zero**, in the interval $\tau = [0, t_1]$ where $t_1 > 0$. For $\omega_2(t_1)t_1 = \frac{\pi}{2}$, $\omega_2(t_1)\tau = \frac{\pi}{2t_1}\tau$ lies in the range $[0, \frac{\pi}{2}]$ and hence $\sin(\omega_2(t_1)\tau) \geq 0$ in that interval $\tau = [0, t_1]$. Hence the integral in the left hand side of Eq. 42 is **greater than zero** for $0 < \sigma < \frac{1}{2}$.

Hence Eq. 42 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$. For $\sigma = 0$, Eq. 46 **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3.2. ***Existence of*** $\omega_2(t_0)t_0 = \pi$

We consider the case $\omega_2(t_0)t_0 \neq \pi$. In this case, we consider the **"t" scaled** function $E_{pr}(t) = E_p(Rt)$ whose Fourier transform $E_{pr\omega}(\omega) = \frac{1}{R}E_{p\omega}(\frac{\omega}{R})$ has a real zero at $\omega = R\omega_0$ and $\sigma_r = R\sigma$. We consider $f_r(t) = E_{pr}(t + t_0)$ and see that $G_{rR}(\omega)$ has a real zero at $\omega = \omega_{2r}(t_0) = R\omega_2(t_0)$ and we can derive similar results in earlier sections as follows. We use $E_{0m}(\tau) = E_0(-\tau) = E_0(\tau)$. ((link))

$$\begin{aligned}R_r(t_0) = e^{2\sigma_r t_0}[\sin(\omega_{2r}(t_0)t_0)\int_{-\infty}^{t_0} E_{0r}(\tau)e^{-2\sigma_r\tau}\cos(\omega_{2r}(t_0)\tau)d\tau - \cos(\omega_{2r}(t_0)t_0)\int_{-\infty}^{t_0} E_{0r}(\tau)e^{-2\sigma_r\tau}\sin(\omega_{2r}(t_0)\tau)d\tau] \\ -[-\sin(\omega_{2r}(t_0)t_0)\int_{-\infty}^{-t_0} E_{0r}(\tau)\cos(\omega_{2r}(t_0)\tau)d\tau - \cos(\omega_{2r}(t_0)t_0)\int_{-\infty}^{-t_0} E_{0r}(\tau)\sin(\omega_{2r}(t_0)\tau)d\tau] = 0 \\ R_r(t_0) = e^{2R\sigma t_0}[\sin(R\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(R\tau)e^{-2R\sigma\tau}\cos(R\omega_2(t_0)\tau)d\tau - \cos(R\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(R\tau)e^{-2R\sigma\tau}\sin(R\omega_2(t_0)\tau)d\tau] \\ -[-\sin(R\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(R\tau)\cos(R\omega_2(t_0)\tau)d\tau - \cos(R\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(R\tau)\sin(R\omega_2(t_0)\tau)d\tau] = 0\end{aligned} \tag{43}$$

In this case, we can choose R such that $\omega_{2r}(t_0) = R\omega_2(t_0)t_0 = \pi$ in Eq. 47 and get $\cos(R\omega_2(t_0)t_0) = -1$ and $\sin(R\omega_2(t_0)t_0) = 0$ and derive the results in previous section.

$$R_r(t_0) = e^{2\sigma_r t_0}\int_{-\infty}^{t_0} E_{0r}(\tau)e^{-2\sigma_r\tau}\cos(\omega_{2r}(t_0)\tau)d\tau + \int_{-\infty}^{-t_0} E_{0r}(\tau)\cos(\omega_{2r}(t_0)\tau)d\tau = 0 \tag{44}$$

3.3. ***Show that*** $\int_{-\infty}^{-t_1} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_1)\tau)d\tau < 0$

It is well known that $E_0(t) = \Phi(t)$ is positive for $t > 0$ and its first derivative is negative for $t > 0$ and hence $E_0(t)$ is a **strictly decreasing** function for $t > 0$.(link and link) We rederive this result below.

We start with the equation for $E_0(t)$ and substitute $x = e^{2t}$ and take the first derivative with respect to x, as follows.

$$
\begin{aligned}
E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{2t} - 6\pi n^2]e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} = D(e^{2t})e^{\frac{5t}{2}} \\
D(x) = \sum_{n=1}^{\infty}[4\pi^2 n^4 x - 6\pi n^2]e^{-\pi n^2 x} = \sum_{n=1}^{\infty} 2\pi n^2[2\pi n^2 x - 3]e^{-\pi n^2 x} \\
\frac{dD(x)}{dx} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 x}[2\pi n^2 - \pi n^2(2\pi n^2 x - 3)] = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 x}[5\pi n^2 - 2\pi^2 n^4 x] = \sum_{n=1}^{\infty} 2\pi^2 n^4 e^{-\pi n^2 x}[5 - 2\pi n^2 x]
\end{aligned}
\tag{45}
$$

We see that the term $(2\pi n^2 x - 3) > 0$ and hence $D(x) > 0$, for $x \geq 1$ and for all $n = 1, 2, ...$

We see that the term $(5 - 2\pi n^2 x) < 0$ and hence $\frac{dD(x)}{dx} < 0$, for $x \geq 1$ and for all $n = 1, 2, ...$

If $x = e^{2t}$, then $\frac{dx}{dt} = 2e^{2t} > 0$ for $x \geq 1$. Using chain rule with derivatives, we see that $\frac{dD(e^{2t})}{dt} = \frac{dD(x)}{dx}\frac{dx}{dt} < 0$ for all $t \geq 0$.

For $t < 0$, we see that $\frac{dD(e^{2t})}{dt} > 0$ and $D(e^{2t})$ is a **strictly increasing** function for $t < 0$. Now we consider $E_a(t) = E_0(t)e^{-2\sigma t} = D(e^{2t})e^{(\frac{5}{2}-2\sigma)t}$ and we see that, for $0 < \sigma < \frac{1}{2}$, $E_a(t)$ is a **strictly increasing** function for $t < 0$.

Now we consider $\int_{-\infty}^{-t_1} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_1)\tau)d\tau$ for $\omega_2(t_1)t_1 = \pi$, we can show that this integral is less than zero. Let $F(t) = D(e^{2t})$.

$$
\int_{-\infty}^{-t_1} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_1)\tau)d\tau = \int_{-\infty}^{-t_1} F(\tau)e^{\frac{5}{2}\tau}e^{-2\sigma\tau}\sin(\omega_2(t_1)\tau)d\tau = -\int_{t_1}^{\infty} F(-\tau)e^{-(\frac{5}{2}-2\sigma)\tau}\sin(\omega_2(t_1)\tau)d\tau
\tag{46}
$$

$$
\begin{aligned}
E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{2t} - 6\pi n^2]e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} = D(e^{2t}) \\
D(x) = \sum_{n=1}^{\infty}[4\pi^2 n^4 x - 6\pi n^2]e^{-\pi n^2 x} x^{\frac{5}{4}} = \sum_{n=1}^{\infty} 2\pi n^2[2\pi n^2 x - 3]e^{-\pi n^2 x} x^{\frac{5}{4}} \\
\frac{dD(x)}{dx} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 x} x^{\frac{5}{4}}[2\pi n^2 + (2\pi n^2 x - 3)(-\pi n^2 + \frac{5}{4}x^{\frac{1}{4}})] \\
= \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 x} x^{\frac{5}{4}}[5\pi n^2 - 2\pi^2 n^4 x + 2\pi n^2 \frac{5}{4} x^{\frac{5}{4}} - 3\frac{5}{4}x^{\frac{1}{4}}] \\
\sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 x} x^{\frac{5}{4}}[5\pi n^2 + 2\pi n^2 \frac{5}{4} x^{\frac{5}{4}} - 2\pi^2 n^4 x - 3\frac{5}{4}x^{\frac{1}{4}}]
\end{aligned}
\tag{47}
$$

For $x = 1$ and for all $n = 1$, we see that the term $5\pi n^2 + 2\pi n^2 \frac{5}{4} x^{\frac{5}{4}} - 2\pi^2 n^4 x - 3\frac{5}{4}x^{\frac{1}{4}} = 5\pi + \pi\frac{5}{2} - 2\pi^2 - \frac{15}{4} < 0$. Now we consider the first derivative of $E_0(t)e^{-2\sigma t}$ as follows.

$$
\begin{aligned}
E_a(t) = E_0(t)e^{-2\sigma t} = D(e^{2t})e^{(\frac{5}{2}-2\sigma)t} \\
\frac{dE_a(t)}{dt} = e^{(\frac{5}{2}-2\sigma)t}[\frac{dD(e^{2t})}{dt} + D(e^{2t})(\frac{5}{2} - 2\sigma)]
\end{aligned}
\tag{48}
$$

We have shown that $D(e^{2t}) > 0$ and $\frac{dD(e^{2t})}{dt} < 0$ for $t > 0$ and we see that, for $0 < \sigma < \frac{1}{2}$, $D(e^{2t})(\frac{5}{2} - 2\sigma) > 0$. Hence $\frac{dE_a(t)}{dt} > 0$ for $t > 0$.

4. Odd Part

We compare Eq. 34 and Eq. 36 and cancel out common terms as follows.

$$\begin{aligned}X_5(t_0) = -2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau - 2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau \\ +2\sin(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\ = -[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau - \sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau] \\ +e^{-2\sigma t_0}[\cos(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau + \sin(\omega_2(t_0)t_0)\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau]\end{aligned} \tag{49}$$

We compute the **odd part** of $X_5(t_0)$ given by $X_6(t_0) = \frac{1}{2}[X_5(t_0) + X_5(-t_0)]$ as follows. The term $2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\cosh(2\sigma\tau)\cos(\omega_2(t_0)\tau)d\tau$ is an **even** function of variable t_0 and cancels out. We note that $-2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau$ is an **odd** function of variable t_0 and the term $2\sin(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau$ is an **even** function of variable t_0. We use $\omega_2(-t_0) = \omega_2(t_0)$.

$$\begin{aligned}X_6(t_0) = -2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau - 2\sin(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\ = -\frac{1}{2}\cos(\omega_2(t_0)t_0)[\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau - \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau] \\ +\frac{1}{2}\sin(\omega_2(t_0)t_0)[\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \int_{-\infty}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau] \\ +\frac{1}{2}\cos(\omega_2(t_0)t_0)[e^{-2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau - e^{2\sigma t_0}\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau] \\ +\frac{1}{2}\sin(\omega_2(t_0)t_0)[e^{-2\sigma t_0}\int_{-\infty}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + e^{2\sigma t_0}\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau]\end{aligned} \tag{50}$$

We use $\int_{-\infty}^{t_0} = \int_{-\infty}^{-t_0} + \int_{-t_0}^{t_0}$ and cancel common terms as follows.

$$\begin{aligned}X_6(t_0) = -2\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau - 2\sin(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_0^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau \\ = \frac{1}{2}\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau \\ +\sin(\omega_2(t_0)t_0)\int_{-\infty}^{-t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau + \frac{1}{2}\sin(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\cos(\omega_2(t_0)\tau)d\tau \\ -\cos(\omega_2(t_0)t_0)\sinh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau + \frac{1}{2}\cos(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau \\ +\sin(\omega_2(t_0)t_0)\cosh(2\sigma t_0)\int_{-\infty}^{-t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau + \frac{1}{2}\sin(\omega_2(t_0)t_0)e^{-2\sigma t_0}\int_{-t_0}^{t_0} E_0(\tau)\cos(\omega_2(t_0)\tau)d\tau]\end{aligned} \tag{51}$$

We use $\int_{-t_0}^{t_0} E_0(\tau)\sin(\omega_2(t_0)\tau)d\tau = 0$ and see that the term $\frac{1}{2}\cos(\omega_2(t_0)t_0)\int_{-t_0}^{t_0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_2(t_0)\tau)d\tau = -\cos(\omega_2(t_0)t_0)\int_0^{t_0} E_0(\tau)\sinh(2\sigma\tau)\sin(\omega_2(t_0)\tau)d\tau$ as follows.

We note that the term $\frac{1}{2}\sin\left(\omega_2(t_0)t_0\right)\int_{-t_0}^{t_0}E_0(\tau)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau = 2\sin\left(\omega_2(t_0)t_0\right)\int_{-t_0}^{t_0}E_0(\tau)\cosh\left(2\sigma\tau\right)\cos\left(\omega_2(t_0)\tau\right)d\tau$ is an **even** function of variable t_0 which is **omitted** because we are considering only the odd terms.

$$
\begin{aligned}
X_6(t_0) &= -2\cos\left(\omega_2(t_0)t_0\right)\int_0^{t_0}E_0(\tau)\sinh\left(2\sigma\tau\right)\sin\left(\omega_2(t_0)\tau\right)d\tau - 2\sin\left(\omega_2(t_0)t_0\right)\sinh\left(2\sigma t_0\right)\int_0^{t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau \\
&= -\cos\left(\omega_2(t_0)t_0\right)\int_0^{t_0}E_0(\tau)\sinh\left(2\sigma\tau\right)\sin\left(\omega_2(t_0)\tau\right)d\tau + \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)e^{-2\sigma\tau}\cos\left(\omega_2(t_0)\tau\right)d\tau \\
&\qquad - \cos\left(\omega_2(t_0)t_0\right)\sinh\left(2\sigma t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\sin\left(\omega_2(t_0)\tau\right)d\tau \\
&\quad + \sin\left(\omega_2(t_0)t_0\right)\cosh\left(2\sigma t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau + \frac{1}{2}\sin\left(\omega_2(t_0)t_0\right)e^{-2\sigma t_0}\int_{-t_0}^{t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau
\end{aligned}
\tag{52}
$$

We consider the case $\sigma = 0$ in Eq. 57.

$$
\begin{aligned}
X_6(t_0) = \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau + \sin\left(\omega_2(t_0)t_0\right)\int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau \\
+\frac{1}{2}\sin\left(\omega_2(t_0)t_0\right)\int_{-t_0}^{t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau = 0
\end{aligned}
\tag{53}
$$

For $\sin\left(\omega_2(t_0)t_0\right) \neq 0$, we can write as follows.

$$
\begin{aligned}
X_6(t_0) = \int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau + \int_{-\infty}^{-t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau \\
+\int_0^{t_0}E_0(\tau)\cos\left(\omega_2(t_0)\tau\right)d\tau = 0
\end{aligned}
\tag{54}
$$

5. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty}\frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}$ diverges for real part of $s\leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x)=1+2\sum_{n=1}^{\infty}e^{-\pi n^2 x}=\frac{1}{\sqrt{x}}(1+2\sum_{n=1}^{\infty}e^{-\pi\frac{n^2}{x}})$, where $x>0$ is real and then derive $E_0(t)=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ (Appendix F). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0\leq|\sigma|<\frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t)=E_0(t)e^{-\sigma t}$ is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

References

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[4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152

[5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)

[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ (link). This is re-derived in Appendix F.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$, is given by $E_p(t)=E_0(t)e^{-\sigma t}$ where $0\leq|\sigma|<\frac{1}{2}$ is real.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega)&=\xi(\frac{1}{2}+i(\omega-i\sigma))=E_{p\omega}(\omega)=E_{0\omega}(\omega-i\sigma)\\E_p(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{p\omega}(\omega)e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega-i\sigma)e^{i\omega t}d\omega\end{aligned}\tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix C.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \leq t \leq \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt$$
$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega'$$
$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'$$

$$\tag{B.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.3}$$

Appendix B.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{B.4}$$

Appendix B.3. ***Even part of g(t) corresponds to real part of Fourier transform** $G(\omega)$*

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \tag{B.5}$$

Appendix B.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform** $G(\omega)$*

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt &= \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \end{aligned} \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. $E_p(t), h(t), g(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix C.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix C.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$, as $|\omega| \to \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \to \infty$, where $A, B > 0$ are real.

Appendix C.4. ***Payley-Weiner theorem and Exponential Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix C.3) and hence it should have **exponential fall-off** rates as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix D. $\omega_2(t_0)$ is a continuous function around $t_0 = 0$

This result is shown as follows.

• $G_R(\omega) = G_R(\omega, t_0)$ in Eq. 16 is copied below, which is a **continuous** function of ω which is differentiable **at least** once with respect to ω. (Eq. D.2 and Appendix D.3)

$$G_R(\omega) = G_R(\omega, t_0) = \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \tag{D.1}$$

Given that $E_0(t) \geq 0$ for $|t_0| \leq \infty$ (Appendix C.1), we see that $G_R(\omega) > 0$ at $\omega = 0$. **Set** $t_0 = 0$ and $G_R(\omega, t_0)$ passes through its **first zero** at $\omega = \omega_2(t_0) = \omega_2(0)$. In the rest of this section, we consider the **interval** $[-\delta t_0, \delta t_0]$

around $t_0 = 0$, in $\omega_2(t_0)$. There are 3 possibilities.

Case 1: $G_R(\omega) < 0$ for $\omega = \omega_2(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_2(0) - dw$ for infinitesimal dw (example plot)

In this case, we will show in Appendix D.1 that $\omega_2(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$.

Case 2: $G_R(\omega) > 0$ for $\omega = \omega_2(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_2(0) - dw$ (example plot)

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_2(0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_2(0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_2(0) + dw$.

$$\frac{dG_R(\omega)}{d\omega} = -\int_{-\infty}^{0} t[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]\sin(\omega t)dt \tag{D.2}$$

In this case, we will show Appendix D.2 that $\omega_2(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$.

Case 3: $G_R(\omega) = 0$ for $\omega = \omega_2(0)$ and $\omega = \omega_2(0) + dw$.

This is **not** possible because $G_R(\omega, t_0)$ in Eq. D.1 is an **analytic** function and infinitely differentiable with respect to ω(Appendix D.3). We know that analytic functions have **isolated** zeros. (link). Hence we cannot have $G_R(\omega) = 0$ for $\omega = \omega_2(0)$ and $\omega = \omega_2(0) + dw$ as $dw \to 0$.

Appendix D.1. ***Case 1:*** $G_R(\omega) < 0$ ***for*** $\omega = \omega_2(0) + dw$***,*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) - dw$

• Consider the **segment** S in $G_R(\omega, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega,t_0)}{d\omega} < 0$. (Segment S is the portion between the green lines in example plot)

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.1 is a **continuous** function of ω, for **each** value of t_0. Hence $G_R(\omega, t_0 - \delta t_0)$ and $G_R(\omega, t_0 + \delta t_0)$ are **continuous** functions of ω, which are differentiable **at least** once, and $G_R(\omega, t_0 \pm \delta t_0)$ tends to $G_R(\omega, t_0)$, as infinitesimal $\delta t_0 \to 0$.

$$\begin{aligned} G_R(\omega, t_0) &= \int_{-\infty}^{0} [E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]\cos(\omega t)dt \\ G_R(\omega, t_0 + \delta t_0) &= \int_{-\infty}^{0} [E_0(t+t_0+\delta t_0)e^{-2\sigma t} + E_{0m}(t-t_0-\delta t_0)]\cos(\omega t)dt \end{aligned} \tag{D.3}$$

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.3 is a **continuous** function of ω, for **each** value of t_0 and $\frac{dG_R(\omega,t_0)}{d\omega} < 0$ in the neighborhood around the **first zero crossing**. If we fix the X-coordinate ω, $G_R(\omega, t_0)$ is a **continuous** function of t_0, for **each** value of ω. Hence, for **each** value of ω, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0)$ moves towards $G_R(\omega, t_0 + \delta t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Every point in the segment S, moves continuously, as we change t_0 by an infinitesimal δt_0.

This also applies to the first **zero crossing** in $G_R(\omega, t_0)$ in the segment S, which corresponds to $\omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$ where $G_R(\omega, t_0) = 0$ in Eq. D.3. The zero crossing moves continuously, as we change t_0 by an infinitesimal δt_0. This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega, t_0)$ with zero crossing at point P_0, **Green** plot corresponds to $G_R(\omega, t_0 + \delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to

$G_R(\omega, t_0 - \delta t_0)$ with zero crossing at point P_{21}.

We **define** the **point** P_{12} in $G_R(\omega, t_0 + \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_2(0)$. We **define** the **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_2(0)$.

We **define** the **point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 + \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{12} in $G_R(\omega, t_0+\delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_2(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_2(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

Similarly, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 - \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_2(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_2(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

$$
\begin{aligned}
G_R(\omega, t_0) &= \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 + \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 - \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 - \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 + \delta t_0)] \cos(\omega t) dt \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 + \delta t_0) &= G_R(\omega, t_0) \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 - \delta t_0) &= G_R(\omega, t_0)
\end{aligned}
\tag{D.4}
$$

• Hence in the **segment** S, $\omega_2(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$.

$$
\begin{aligned}
G_R(\omega_2(t_0), t_0) &= \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0m}(t - t_0)] \cos(\omega_2(t_0)t) dt = 0 \\
G_R(\omega_2(t_0 + \delta t_0), t_0 + \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0m}(t - t_0 - \delta t_0)] \cos((\omega_2(t_0 + \delta t_0)t) dt = 0
\end{aligned}
\tag{D.5}
$$

Appendix D.2. ***Case 2:*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) + dw$**,** $G_R(\omega) > 0$ ***for*** $\omega = \omega_2(0) - dw$

• In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_2(t_0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_2(t_0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_2(t_0) + dw$.

• Consider the **segment** S' in $\frac{dG_R(\omega,t_0)}{d\omega}$ in the neighborhood around the first zero crossing where $\frac{d^2G_R(\omega,t_0)}{d\omega^2} > 0$. (Segment S' is the portion between the green lines in example plot) In this segment S', $\frac{dG_R(\omega,t_0)}{d\omega}$ is a **continuous** function of ω which is differentiable **at least** once.(Appendix D.3)

• In the **segment** S', $\frac{dG_R(\omega,t_0)}{d\omega} = 0$ at the **same** $\omega = \omega_2(t_0)$. The arguments in Appendix D.1 can be applied here, with $G_R(\omega, t_0)$ replaced by $\frac{dG_R(\omega,t_0)}{d\omega}$.

Hence $\omega_2(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_2(t_0) = \omega_2(0)$ at $t_0 = 0$ in the **segment** S'.

Appendix D.3. ***Integral convergence in*** $\frac{dG_R(\omega)}{d\omega}$

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have exponential fall-off rates as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence the integrand $A_r(t) = \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]\sin(\omega t)$ in Eq. D.2 copied below, is an absolutely **integrable function** and $\int_{-\infty}^{0} |A_r(t)|dt = \int_{-\infty}^{0} \frac{|t^r|}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]dt$ is **finite**, for $r = 0, 1, ..,$ given the **exponential** fall-off rate of $E_0(t)e^{-2\sigma t}$ and $E_0(t)$.

$$\frac{1}{!(r)}\frac{d^r G_R(\omega)}{d\omega^r} = (-1)^{\frac{r+1}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]\sin(\omega t)dt, \quad r = odd$$
$$\frac{1}{!(r)}\frac{d^r G_R(\omega)}{d\omega^r} = (-1)^{\frac{r}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0m}(t-t_0)]\cos(\omega t)dt, \quad r = even$$

(D.6)

Appendix E. $\omega_2(t_0)$ is differentiable at least once around $t_0 = 0$.

In Appendix D, we showed that $\omega_2(t_0)$ is a continuous function around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$. In this section, we show that $\omega_2(t_0)$ is differentiable **at least** once, in that interval. Thus we **rule out** the case of $\omega_2(t_0)$ as a **Weierstrass** type of function, which is continuous everywhere, but differentiable nowhere.

We take the first derivative of $R(t_0)$ in Eq. 17 as follows where $E_0'(\tau, t_0) = E_0(\tau+t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)$.

$$R(t_0) = \int_{-\infty}^{0} E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0}\int_{-\infty}^{0} E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = \lim_{\delta t_0 \to 0}\int_{-\infty}^{0} \frac{1}{\delta t_0}[E_0'(\tau, t_0+\delta t_0)\cos(\omega_2(t_0+\delta t_0)\tau) - E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)]d\tau = 0$$

(E.1)

The integrands in Eq. E.1 are continuous functions which are well defined and bounded and the integral converges. Hence we can use Leibnitz integral rule (link) and **interchange** the order of integration and differentiation and write as follows. (Details in Appendix E.1)

$$\frac{dR(t_0)}{dt_0} = \int_{-\infty}^{0} \frac{\partial}{\partial t_0}[E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)]d\tau = 0$$
$$\frac{dR(t_0)}{dt_0} = -\frac{d\omega_2(t_0)}{dt_0}\int_{-\infty}^{0} \tau E_0'(\tau, t_0)\sin(\omega_2(t_0)\tau)d\tau + \int_{-\infty}^{0} \frac{\partial}{\partial t_0}E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau = 0$$
$$\frac{d\omega_2(t_0)}{dt_0}P(t_0) = Q(t_0), \quad P(t_0) = \int_{-\infty}^{0} \tau E_0'(\tau, t_0)\sin(\omega_2(t_0)\tau)d\tau, \quad Q(t_0) = \int_{-\infty}^{0} \frac{\partial}{\partial t_0}E_0'(\tau, t_0)\cos(\omega_2(t_0)\tau)d\tau$$

(E.2)

• We see that both integrals $P(t_0), Q(t_0)$ in Eq. E.2 are **continuous** functions, because integral of a well defined continuous function, is continuous. If we assume that $\omega_2(t_0)$ is a Weierstrass type of function which is **differentiable nowhere (Statement 5)**, $\frac{d\omega_2(t_0)}{dt_0}$ is not well defined in this case and we require $P(t_0) = Q(t_0) = 0$ for all $|t_0| \leq \infty$ to satisfy Eq. E.2. We will show that this leads to a contradiction and thus **rule out** this pathological case of $\omega_2(t_0)$ as follows.

$$Q(t_0) = \int_{-\infty}^{0} \frac{\partial}{\partial t_0} E_0^{'}(\tau, t_0) \cos{(\omega_2(t_0)\tau)} d\tau = 0 \tag{E.3}$$

Using the procedure outlined in Section 3.2, we use the fact that $\frac{\partial}{\partial t_0} E_0(\tau + t_0) = \frac{\partial}{\partial \tau} E_0(\tau + t_0)$ and $\frac{\partial}{\partial t_0} E_0(\tau - t_0) = -\frac{\partial}{\partial \tau} E_0(\tau - t_0)$ and write as follows. We use $E_{0m}(t) = E_0(-t) = E_0(t)$ and $E_0^{'}(\tau, t_0) = E_0(\tau + t_0)e^{-2\sigma\tau} + E_{0m}(\tau - t_0)$.

$$Q(t_0) = \int_{-\infty}^{0} [\frac{\partial}{\partial \tau} E_0(\tau + t_0)e^{-2\sigma\tau} - \frac{\partial}{\partial \tau} E_0(\tau - t_0)] \cos{(\omega_2(t_0)\tau)} d\tau = 0 \tag{E.4}$$

We use the fact that $\int_{-\infty}^{0} \frac{\partial}{\partial \tau} (E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)}) d\tau = \int_{-\infty}^{0} \frac{\partial}{\partial \tau} E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)} d\tau$ $+ \int_{-\infty}^{0} E_0(\tau + t_0) \frac{\partial}{\partial \tau} (e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)}) d\tau$ and write the first term in Eq. E.4 as follows.

$$\begin{aligned}\int_{-\infty}^{0} \frac{\partial}{\partial \tau} E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)} d\tau &= [E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)}]_{-\infty}^{0} \\ &\quad - \int_{-\infty}^{0} E_0(\tau + t_0) \frac{\partial}{\partial \tau} (e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)}) d\tau \\ = E_0(t_0) + \omega_2(t_0) \int_{-\infty}^{0} E_0(\tau + t_0)e^{-2\sigma\tau} \sin{(\omega_2(t_0)\tau)} d\tau &+ 2\sigma \int_{-\infty}^{0} E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)} d\tau\end{aligned} \tag{E.5}$$

Similarly we can write write the second term in Eq. E.4 as follows.

$$\begin{aligned}\int_{-\infty}^{0} \frac{\partial}{\partial \tau} E_0(\tau - t_0) \cos{(\omega_2(t_0)\tau)} d\tau = [E_0(\tau - t_0) \cos{(\omega_2(t_0)\tau)}]_{-\infty}^{0} &- \int_{-\infty}^{0} E_0(\tau - t_0) \frac{\partial}{\partial \tau} (\cos{(\omega_2(t_0)\tau)}) d\tau \\ &= E_0(-t_0) + \omega_2(t_0) \int_{-\infty}^{0} E_0(\tau - t_0) \sin{(\omega_2(t_0)\tau)} d\tau\end{aligned} \tag{E.6}$$

Now we evaluate $Q(t_0)$ in Eq. E.4, using Eq. E.5 and Eq. E.6 as follows. We see that $E_0(t_0) = E_0(-t_0)$ and hence those terms cancel.

$$\begin{aligned}Q(t_0) = \omega_2(t_0)[\int_{-\infty}^{0} E_0(\tau + t_0)e^{-2\sigma\tau} \sin{(\omega_2(t_0)\tau)} d\tau &- \int_{-\infty}^{0} E_0(\tau - t_0) \sin{(\omega_2(t_0)\tau)} d\tau] \\ &+2\sigma \int_{-\infty}^{0} E_0(\tau + t_0)e^{-2\sigma\tau} \cos{(\omega_2(t_0)\tau)} = 0\end{aligned} \tag{E.7}$$

We can substitute $\tau + t_0 = \tau'$ and $\tau - t_0 = \tau''$ in Eq. E.7 and expand it and we get the same equation as Eq. **??** as follows, with t_2 replaced by t_0.

$$R_1{}'(t_0) = \omega_2(t_0)[n_0{}'(t_0) - n_{0p}{}'(t_0)] + 2\sigma m_0{}'(t_0) = 0$$
$$n_0^{'}(t_0) = \int_{-\infty}^{0} E_0(\tau + t_0) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau, \quad n_{0p}^{'}(t_0) = \int_{-\infty}^{0} E_0(\tau - t_0) \sin(\omega_2(t_0)\tau) d\tau$$
$$m_0{}'(t_0) = \int_{-\infty}^{0} E_0(\tau + t_0) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau)$$
$$n_0^{'}(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau - \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau]$$
$$n_{0p}^{'}(t_0) = \cos(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \sin(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{-t_0} E_0(\tau) \cos(\omega_2(t_0)\tau) d\tau$$
$$m_0{}'(t_0) = e^{2\sigma t_0}[\cos(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \cos(\omega_2(t_0)\tau) d\tau + \sin(\omega_2(t_0)t_0) \int_{-\infty}^{t_0} E_0(\tau) e^{-2\sigma\tau} \sin(\omega_2(t_0)\tau) d\tau] \tag{E.8}$$

In Section **??**, we have shown that the above equations which are the same as Eq. **??**, lead to a **contradiction** for the asymptotic case $t_0 \to \pm\infty$, **if** Statement 1 is true. This suggests one of the following:

a) Statement 1 is true and above result **contradicts** Statement 5 and hence we can **rule out** pathological case for $\omega_2(t_0)$ **or**

b) Statement 5 is true and **Statement 1 is false** and we **complete the proof** of Theorem 1 at this point. We **do not** require to show that $\omega_2(t_0)$ is **not** pathological, for this case.

Hence the **pathological** case where $\omega_2(t_0)$ is a Weierstrass type of function, which is continuous everywhere but **differentiable nowhere**, leads to a **contradiction**, thus **ruling out** this pathological case.

In Section 2.3, it is shown that $\omega_2(t_0) = \omega_2(-t_0)$ is an **even** function of variable t_0. Hence $\frac{d\omega_2(t_0)}{dt_0} = 0$ at $t_0 = 0$.

We have shown that $\omega_2(t_0)$ is differentiable **at least** once around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$.

Appendix E.1. ***Interchanging order of differentiation and integration***

We consider $R(t_0)$ in Eq. 17 as follows.

$$R(t_0) = \int_{-\infty}^{0} [E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau = 0 \tag{E.9}$$

We see that the integrand in Eq. E.9 is a continuous function which is well defined and bounded. We take the first derivative of $R(t_0)$ as follows. We define $R(t_0, a) = \int_{-a}^{0} [E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau$ and see that $\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0} \lim_{a\to\infty} R(t_0, a)$. We define $R_{abs}(t_0, a) = \int_{-a}^{0} |[E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau)| d\tau$ and see that $R_{abs}(t_0, a) \leq \int_{-a}^{0} |E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0)| d\tau = \int_{-a}^{0} E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0) d\tau$ is **finite**, as $a \to \infty$, because $E_0(t) \geq 0$ for $|t_0| \leq \infty$ (Appendix C.1). We see that $|R(t_0, a)| \leq R_{abs}(t_0, a)$ is **finite**, as $a \to \infty$, because magnitude of the integral is less than or equal to the integral of the magnitude of the integrand.

Given that $R(t_0)$ converges for $|t_0| \leq \infty$, we see that $\lim_{a\to\infty} R(t_0, a) < \infty$. Given that $|R(t_0, a)| < \infty$, as $a \to \infty$, we see that $\frac{d}{dt_0} R(t_0, a) < \infty$ and $\lim_{a\to\infty} \frac{d}{dt_0} R(t_0, a) < \infty$ for $|t_0| \leq \infty$. Hence $\frac{dR(t_0)}{dt_0} = \frac{d}{dt_0} \lim_{a\to\infty} R(t_0, a) = \lim_{a\to\infty} \frac{d}{dt_0} R(t_0, a)$ and we can interchange the order of integration and differentiation using Leibnitz integral rule (link) and write as follows.

$$\frac{dR(t_0)}{dt_0} = \int_{-\infty}^{0} [\frac{\partial}{\partial t_0} E_0(\tau + t_0) e^{-2\sigma\tau} + \frac{\partial}{\partial t_0} E_{0m}(\tau - t_0)] \cos(\omega_2(t_0)\tau) d\tau$$
$$- \frac{d\omega_2(t_0)}{dt_0} \int_{-\infty}^{0} \tau [E_0(\tau + t_0) e^{-2\sigma\tau} + E_{0m}(\tau - t_0)] \sin(\omega_2(t_0)\tau) d\tau = 0$$

(E.10)

Appendix F. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}] e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1} e^{-y} dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} dy$$
$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

(F.1)

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

(F.2)

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx$$

(F.3)

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. F.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0, 1]$ and $[1, \infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0, 1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1)}{2} dx$$

(F.4)

Hence we can simplify Eq. F.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_{1}^{\infty} x^{\frac{s}{2}-1}w(x)dx + \int_{1}^{\infty} x^{\frac{-(s+1)}{2}}w(x)dx \tag{F.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_{1}^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{F.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix F.1. ***Derivation of*** $E_p(t)$ ***and*** $E_0(t)$

Given that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. F.6 and evaluate at $s = \frac{1}{2} + \sigma + i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1 + 2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_{0}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t} + e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{F.7}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt \\ + \int_{0}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{F.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2} + (-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{F.9}$$

We define $A(t) = [\sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}u(-t) + \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}E_p(t) &= \frac{1}{2}\delta(t) + (-\frac{1}{4}+\sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}u(t)]e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t) + \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t} + (-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\ &+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned}$$

$$(F.10)$$

We can simplify above equation as follows.

$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$
$$\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[\frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t) + \delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2)]$$

(F.11)

We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$ (Appendix F.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$
$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+ \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$

(F.12)

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$

(F.13)

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1 - s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$

(F.14)

Appendix F.2. **Derivation of** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$
\begin{aligned}
F(x) = 1 + 2w(x) &= \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x})) \\
F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}) \\
\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} &= \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}}
\end{aligned}
\tag{F.15}
$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$
\begin{aligned}
[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} &= \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2}) \\
\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) &= -\frac{1}{2}
\end{aligned}
\tag{F.16}
$$