1. Heisenberg's Uncertainty Principle Part 1

It is well known that Heisenberg's Uncertainty Principle says that $\triangle x \triangle p \geq \frac{h}{4\pi}$ where $\triangle x, \triangle p$ are the uncertainty in particle position and momentum. Heisenberg used the analogy of Compton scattering of an electron by photon, while deriving this result in the link. [Compton effect derivation in the link]

Consider the case where a **photon** of energy $E = hf = \frac{hc}{\lambda}$ and momentum $p_c = \frac{h}{\lambda}$ hits an **electron** moving with a momentum $p_e = m_e v = \frac{m_0 v}{\sqrt{(1-\frac{v^2}{c^2})}}$ **before** being hit by the photon, where m_0 is the rest mass of electron. If we use **photon wavelength** comparable to the **size of the electron** of $1e-11$ meters, then **uncertainty in electron position** $\triangle x \approx \lambda = 1e-11$ meters. because the photon imparts some of its momentum to the electron, **Uncertainty in electron momentum** $\triangle p \approx \frac{h}{\lambda}$ meters.

$$
\begin{aligned}
p_c &= \frac{h}{\lambda} \\
\triangle p &\approx \frac{h}{\lambda}, \quad \triangle x \approx \lambda \\
\triangle x \triangle p &= h
\end{aligned}
\tag{1}
$$

Thus we get $\triangle x \triangle p \approx h$ and thus we can **approximate** version of Heisenberg's uncertainty principle.

If we want to get the **exact version** of Heisenberg's uncertainty principle, and the **factor** $\frac{1}{4\pi}$, we will show in subsection below that it is derived from Inherent Fourier Uncertainty relation. Thus **measurement induced** position-momentum uncertainty is related to **Inherent Fourier Uncertainty**.

1.1. Inherent Fourier Uncertainty Principle

Let us start from the well-known time-frequency Fourier Uncertainty relation for a **rectangular light pulse** of duration $\triangle t$ (time uncertainty) modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(derived in **Appendix A**).

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{2}$$

Given that such a light pulse is made up of large number of photons and that each photon has an energy $E = hf = pc$ where c is light speed and p is the momentum of the photon, we can deduce Energy uncertainty as $\triangle E = h \triangle f = c \triangle p$, position uncertainty as $\triangle x = c \triangle t$ and hence we can write

$$
\begin{aligned}
\triangle t \triangle f &\geq \frac{1}{4\pi} \\
\triangle t &= \frac{\triangle t}{c}, \quad \triangle f = \tfrac{c}{h} \triangle p \\
\frac{\triangle x}{c} \triangle p \frac{c}{h} &\geq \frac{1}{4\pi} \\
\triangle x \triangle p &\geq \frac{h}{4\pi}
\end{aligned}
\tag{3}
$$

Thus we have derived Heisenberg's uncertainty principle, starting from Fourier Uncertainty Principle and we have shown that Heisenberg's **measurement induced** position-momentum uncertainty is related to **Inherent Fourier Uncertainty**.

1.2. Signal Processing methods Can Get us below Fourier uncertainty Limit

In this example, we use a **triangular pulse** in time domain of duration $2T$, where $T = 1$ microseconds within a signal duration of 20 microseconds, which is modulated by a carrier frequency $f_c = 100$ MHz and sampled by $f_s = 10$ GHz. This pulse is transmitted and reflected by a target, a car for example, of length 3 meters and carrier wavelength is $\lambda = \frac{c}{f} = 3$ meters , very similar to a **Radar** system. We wish to estimate the **time of arrival** and **carrier frequency** of the received pulse and determine the Fourier uncertainty.

Uncertainty in **time of arrival** is given by $\triangle t = \frac{10}{f_s} = 1e - 9$ seconds, if we can identify the **peak** of the triangular pulse with a **precision** of 10 times sampling period, which is **possible** with signal processing.

Uncertainty in **carrier frequency** is given by $\triangle f = \frac{10 f_s}{N} = 500$ Khz, if we can identify the **peak** of the FFT of triangular pulse with a **precision** of 10 times frequency resolution given by $\frac{fs}{N}$, where N is the total number of samples, which is **possible** with signal processing.

Hence we can get Fourier Uncertainty in our system **below** Fourier Uncertainty Limit of $\frac{1}{4\pi}$.

$$
\triangle t \triangle f = 5e - 4 < \frac{1}{4\pi}
\tag{4}
$$

In the next section, we will explain how this pulse can be used to get $\triangle x \triangle p = 5e - 4 * h$ **below** Heisenberg's uncertainty Limit.

Effects of sampling frequency

The Triangular pulse given by $g(t) = (\frac{T-t}{T})u(t) + (\frac{T+t}{T})u(-t)$. Its Fourier Transform is given by $G(f) = (T)^2 * sinc^2(fT)$ and we can see that it has **zero crossings** at $f = \frac{1}{T} = 1$ MHz, which **does not** cause **interference** with carrier frequency peak detection, for our choice of sampling frequency $fs = 1$ GHz, carrier frequency $fc = 100$ Mhz. Any small drifts in carrier and sampling frequencies should not matter if transmitted signal power is sufficiently high, given that Fourier Transform of a triangular pulse falls off as $\frac{1}{f^2}$ and the analysis in above subsection still holds.

Effects of Receiver Thermal Noise

Receiver thermal noise power is given by $K * T_0 * BW$ where BW is signal bandwidth, T_0 is noise temperature and K is Boltzmann constant. We can choose to transmit the signal with **sufficiently high power** level, so that received signal power is at least 100 times higher than thermal noise power and hence signal amplitude is **at least 10 times higher** than noise amplitude. Then the analysis in above subsection still holds.

The Figure in link plots the triangular pulse in time domain and its Fourier Transform in the frequency domain.

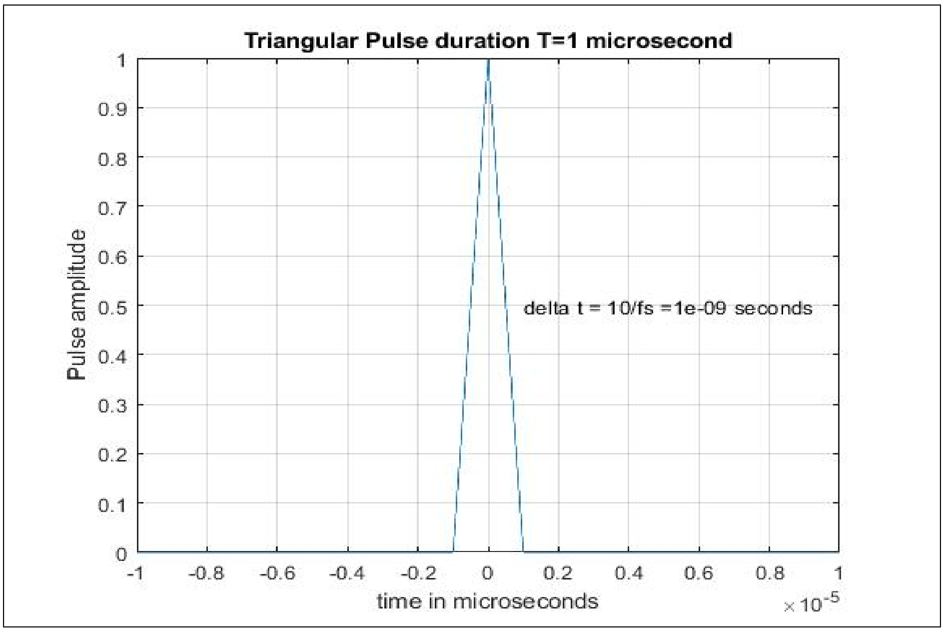

Figure 1:

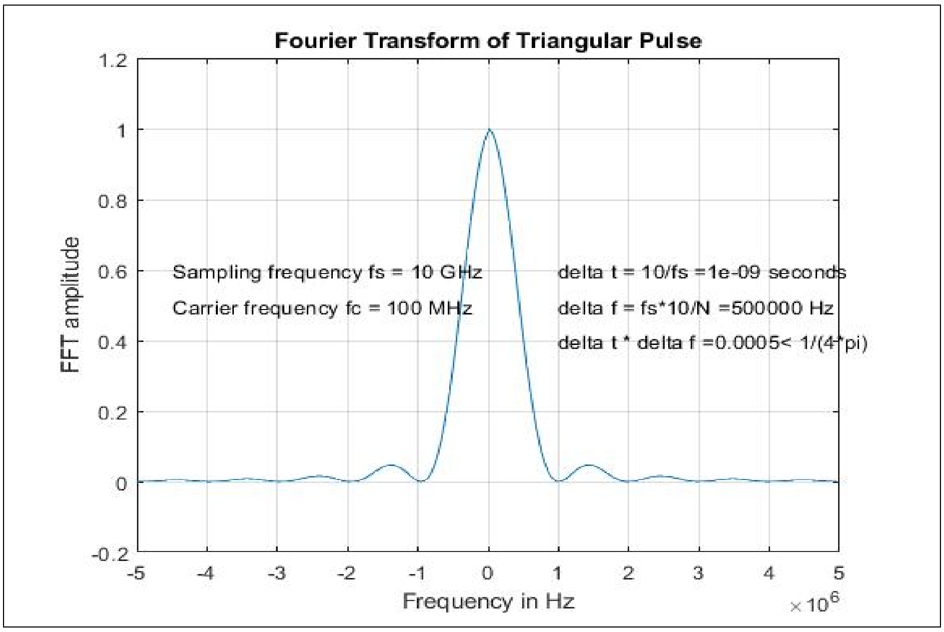
Fourier Transform of Triangular Pulse
FFT amplitude
Frequency in Hz
Sampling frequency fs = 10 GHz
Carrier frequency fc = 100 MHz
delta t = 10/fs =1e-09 seconds
delta f = fs*10/N =500000 Hz
delta t * delta f =0.0005< 1/(4*pi)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-5
-4
-3
-2
-1
0
1
2
3
4
5
× 10^6

Figure 2:

2. Heisenberg Uncertainty Principle with Compton Effect

Compton effect in the link has been used to argue that when a photon hits an electron, some of photon energy is transferred to electron with rest mass m_e and it starts moving from **rest** with a velocity v and this results in **measurement induced uncertainty** in electron velocity and momentum which is related to **inherent Fourier uncertainty** $\triangle t \triangle f \geq \frac{1}{4\pi}$.

We can use the result from Compton effect $(p_e'c)^2 = (hf + hf^{'})^2$ for $\theta = \pi$ where $f, f^{'}$ are the frequencies of photon before and after collision and p_e' is the electron momentum after collision, m_e is electron mass, c, h are speed of light and Planck's constant respectively, as derived follows. we use $\triangle p = d(p_e'), \triangle f = d(f')$. In the last step, if $f^{'} \approx f$, then $\triangle x \triangle p \geq \frac{h}{4\pi}$.

$$\begin{aligned} (p_e^{'}c)^2 = (hf)^2 + (hf^{'})^2 - 2h^2 ff^{'} \cos\theta = (hf + hf^{'})^2 \\ p_e' = m_e^{'} v = \frac{h}{c}(f + f^{'}) \\ \frac{d(p_e')}{df} = \frac{h}{c} \end{aligned} \tag{5}$$

Replacing $d(p_e')$ by $\triangle p$ and df by $\triangle f$, and using $\triangle x = c \triangle t$ where we use a time limited pulse like a rectangualr or triangular pulse to hit the electron and we have

$$\begin{aligned} \triangle x \triangle p = c \triangle t \triangle p = c \triangle t \triangle f \frac{h}{c} \\ \triangle t \triangle f \geq \frac{1}{4\pi} \\ \triangle x \triangle p \geq \frac{h}{4\pi} \end{aligned} \tag{6}$$

2.1. Signal Processing methods Can Get us below Heisenberg's uncertainty Limit

As explained in Section 1, we can use a triangular pulse of duration $T = 1$ microseconds and carrier frequency $f_c = 100$ MHz, to hit the electron and it is reflected back and using **signal processing** methods, we can estimate the **peak** of the triangular pulse in time domain and the **peak** of carrier frequency in frequency domain more accurately and thus we can get below the Uncertainty Limit.

$$\begin{aligned} \triangle t \triangle f = 5e - 4 < \frac{1}{4\pi} \\ \triangle x \triangle p = 5e - 4 * h < \frac{h}{4\pi} \end{aligned} \tag{7}$$

3. Heisenberg Uncertainty Principle with Compton Effect and Non-zero electron velocity before collision

We can repeat above section for the case when electron has **non-zero velocity** and momentum p_e **before** collision with photon.(link) This is derived in **Appendix B**. We get extra terms as below.

$$
\begin{aligned}
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\
\frac{h}{m_ec}[1-\cos\theta] &= (\lambda^{'} - \lambda)\sqrt{1 + (\frac{p_e}{m_ec})^2} - \frac{1}{ff^{'}m_e}p_e[f\cos\theta_1 - f^{'}\cos\theta_2] \\
\triangle x \triangle p &\geq \frac{h}{4\pi} + \frac{c}{4\pi}\frac{d(Y)}{df} \\
Y(f, f^{'}, p_e) &= \frac{h}{c}(f + f^{'})A(f, f^{'}, p_e) \\
A(f, f^{'}, p_e) &= [\frac{1}{2}Z(f, f^{'}, p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f, f^{'}, p_e)^2 + ...] \\
Z(f, f^{'}, p_e) &= \frac{p_e}{c}(p_ec + 2h(f + f^{'}))
\end{aligned}
\tag{8}
$$

We can see that the extra term $\frac{c}{4\pi}\frac{d(Y)}{df}$ in above equation makes Heisenberg's Uncertainty Limit larger $\triangle x \triangle p > \frac{h}{4\pi}$.

3.1. *Signal Processing methods Can Get us below Heisenberg's uncertainty Limit*

As explained in Section 1, we can use a triangular pulse of duration $T = 1$ microseconds and carrier frequency $f_c = 100$ MHz, to hit the electron and it is reflected back and using **signal processing** methods, we can estimate the **peak** of the triangular pulse in time domain and the **peak** of carrier frequency in frequency domain more accurately and thus we can get below the Uncertainty Limit.

$$
\begin{aligned}
\triangle t \triangle f &= 5e-4 < \frac{1}{4\pi} \\
\triangle x \triangle p &= 5e-4 * h < \frac{h}{4\pi}
\end{aligned}
\tag{9}
$$

3.2. *More Examples: Signal Processing methods Can Get us below Heisenberg's uncertainty Limit*

3a) For the **specific case** of photon and electron aligned before collision in same direction and photon gets reflected back at an angle 180 degrees after collision with **electron at rest** and imparting some of its momentum to the electron, we can write $p_\gamma = p_e' + p_{\gamma'}$ given p_e is zero. We can estimate $p_\gamma, p_{\gamma'}$ accurately using signal processing techniques and hence we can estimate p_e'

accurately.

For the theoretical case of a system with only one photon and one electron, with electron at rest before collision, we can estimate p_e' accurately and **get below** Heisenberg Uncertainty Limit.

3b) Let us take a **specific example** of electron mass $m_e = 9.1e-31$ Kgs, electron velocity $v_e = 1e6$ meters per second **before** collision, hence uncertainty in electron momentum $\triangle p = m_e * v_e$ if electron is **in motion before collision** with photon. Let uncertainty in electron position be $\triangle x = \lambda = 1e-11$ meters. $\triangle p_e \approx 1e-24$.

Let us differentiate $p_e^{'}$ with respect to p_e, to estimate **uncertainty** in $p_e^{'}$ due to **not knowing** value of p_e. Given that our estimation accuracy of f, f' does not depend on p_e, for $\theta = \theta_2 = \pi, \theta_1 = 0$, we can write

$$\begin{aligned}
(p_e^{'})^2c^2 = [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\
2p_e^{'}\frac{dp_e^{'}}{dp_e} = 2p_e + 2\frac{h}{c}(f + f') \\
\triangle p_e' = \triangle p_e\frac{(p_e + 2\frac{h}{c}(f + f'))}{p_e'}
\end{aligned} \tag{10}$$

Given that $2\frac{h}{c}(f + f') << p_e$, for the case $\frac{p_e}{p_e'} \approx 1$, we have $\triangle p_e' \approx \triangle p_e \approx 1e-24$.

$$\triangle x \triangle p_e' \approx \triangle x \triangle p_e = \lambda * m_e * v_e = 1e-11 * 9.1e-31 * 1e6 = 9e-36 < \frac{h}{4\pi} \tag{11}$$

We can see that, **even if do not know** electron momentum before collision, by estimating accurately f, f', we can **get below** Heisenberg Uncertainty Limit.

For the theoretical case of a system with only one photon and one electron, with electron in motion before collision, we can estimate p_e' accurately and **get below** Heisenberg Uncertainty Limit.

4. Appendix A

Let us derive the well-known time-frequency Fourier Uncertainty relation for a rectangular light pulse of duration $\triangle t$ modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(Figure 2).

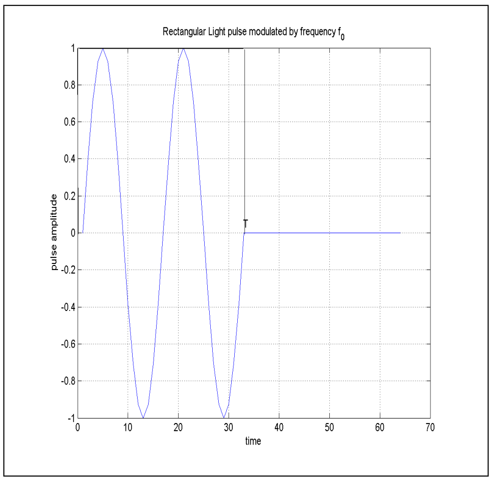

Figure 3:

Let us first consider the limiting case of a two-sided Gaussian pulse $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$. The standard deviation of the pulse $g(t)$ is given by $\sigma_t = \sqrt{\frac{1}{2\pi}}$ and the standard deviation of the transform $G(f)$ is given by $\sigma_f = \sqrt{\frac{1}{2\pi}}$. Given that the standard deviation of a signal represents a measure of uncertainty in its value, we can interpret the product of the two standard deviations as a product of time-frequency uncertainty and write

$$\sigma_t \sigma_f = \frac{1}{2\pi} \tag{12}$$

which represents the Gaussian limiting case.

5. Fourier Uncertainty relation $\triangle t \triangle f \geq \frac{1}{4\pi}$

Gabor Limit in signal processing gives Fourier Uncertainty relation $\sigma_t \sigma_f \geq \frac{1}{4\pi}$ (link)

Let us derive the Fourier Uncertainty relation for a general signal $g(t)$.[See Simon Haykin "**Communication systems** Second Edition 1978", page 102]. (link)

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{13}$$

Let us consider the following measure for an energy signal $g(t)$ whose Fourier Transform is given by $G(f)$.

$$T_{rms} = [\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}]^{\frac{1}{2}}$$
$$W_{rms} = [\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df}]^{\frac{1}{2}} \tag{14}$$

Let us show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. Define $g_1(t) = tg(t)$ and $g_2(t) = \frac{dg(t)}{dt}$ and using **Schwarz's inequality**, we have

$$\int_{-\infty}^{\infty} |g_1(t)|^2 dt \int_{-\infty}^{\infty} |g_2(t)|^2 dt \geq (\int_{-\infty}^{\infty} g_1(t) g_2(t) dt)^2$$
$$\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt \geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2 \tag{15}$$

Using **Parseval's relation** $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$, we can show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. We also use the fact that $\int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt = (2\pi)^2 \int_{-\infty}^{\infty} f^2 |G(f)|^2 df$ using properties of fourier transform and Parseval's relation.

$$(2\pi)^2 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df \geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2 \tag{16}$$

Using **Parseval's relation** $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$, we can write

$$W_{rms} T_{rms} = \frac{(\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df)^{\frac{1}{2}}}{\int_{-\infty}^{\infty} |G(f)|^2 df} \geq \frac{1}{(2\pi)} | \int_{-\infty}^{\infty} g_1(t) g_2(t) dt | \frac{1}{\int_{-\infty}^{\infty} |G(f)|^2 df} \tag{17}$$

For **Gaussian pulse** $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$, we use the fact that $g_1(t) = tg(t), g_2(t) = \frac{dg(t)}{dt} = -2\pi t g(t)$, $\int_{-\infty}^{\infty} |G(f)|^2 df = 1$ and write as follows.

$$W_{rms} T_{rms} \geq \frac{1}{(2\pi)} (2\pi) \int_{-\infty}^{\infty} (tg(t))^2 dt$$
$$W_{rms} T_{rms} \geq \int_{-\infty}^{\infty} t^2 g^2(t) dt = \int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt \tag{18}$$

We see that the **inverse fourier transform** of $t^2 e^{-2\pi t^2}$ is given by $(\frac{1}{-i2\pi})^2 \frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2}$. We see that $\frac{d(e^{-\frac{\pi}{2}f^2})}{df} = e^{-\frac{\pi}{2}f^2}(-\pi f)$ and $\frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2} = e^{-\frac{\pi}{2}f^2}[-\pi + \pi^2 f^2]$. Hence $\int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt = (\frac{1}{-i2\pi})^2 [\frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2}]_{f=0} = (\frac{1}{-i2\pi})^2(-\pi) = \frac{1}{4\pi}$.

Hence we can write **Fourier uncertainty relations** as follows.

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{19}$$

6. Appendix B: Compton Effect Rederived with Non-zero electron velocity before collision with photon

Let $f, f^{'}$ be the frequency of the light before and after collision with an electron with rest mass m_e. Let $v, v^{'}$ be the velocity of electron before and after collision. Let $p_e = \frac{m_e v}{\sqrt{1-\frac{v^2}{c^2}}}, p_e^{'} = \frac{m_e v^{'}}{\sqrt{1-\frac{(v^{'})^2}{c^2}}}$ be the momentum of electron before and after collision. Let $K = \sqrt{1 - \frac{v^2}{c^2}}$ and $K^{'} = \sqrt{1 - \frac{(v^{'})^2}{c^2}}$.

Energy of electron before and after collision is given by $E_e = m_T c^2$ and $E_e^{'} = m_T^{'} c^2$ and Energy of photon before and after collision is given by $E_\gamma = hf$ and $E_\gamma^{'} = hf^{'}$. Using Conservation of Energy we have as follows.

$$\begin{aligned}
E_\gamma + E_e &= E_\gamma^{'} + E_e^{'} \\
hf + \sqrt{(m_e c^2)^2 + (p_e c)^2} &= hf^{'} + \sqrt{(m_e c^2)^2 + (p_e^{'} c)^2} \\
(m_e c^2)^2 + (p_e^{'} c)^2 &= (hf - hf^{'} + \sqrt{(m_e c^2)^2 + (p_e c)^2})^2 \\
(p_e^{'} c)^2 &= -(m_e c^2)^2 + (hf)^2 + (hf^{'})^2 - 2h^2 ff^{'} + (m_e c^2)^2 + (p_e c)^2 + 2h(f - f^{'})\sqrt{(m_e c^2)^2 + (p_e c)^2} \\
(p_e^{'} c)^2 &= (hf)^2 + (hf^{'})^2 - 2h^2 ff^{'} + (p_e c)^2 + 2h(f - f^{'})\sqrt{(m_e c^2)^2 + (p_e c)^2}
\end{aligned} \tag{20}$$

Using Conservation of Momentum we have as follows.

$$\begin{aligned}
\vec{p_\gamma} + \vec{p_e} &= \vec{p_\gamma^{'}} + \vec{p_e^{'}} \\
\vec{p_e^{'}} &= \vec{p_\gamma} + \vec{p_e} - \vec{p_\gamma^{'}} \\
(p_e^{'})^2 = \vec{p_e^{'}} \cdot \vec{p_e^{'}} &= (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \cdot (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \\
(p_e^{'})^2 &= [(p_\gamma)^2 + (p_\gamma^{'})^2 - 2p_\gamma p_\gamma^{'} \cos\theta] + [p_e^2 + 2p_e p_\gamma \cos\theta_1 - 2p_e p_\gamma^{'} \cos\theta_2]
\end{aligned} \tag{21}$$

We multiply both sides of above equation by c^2 and use $p_\gamma = \frac{hf}{c}, p_\gamma^{'} = \frac{hf^{'}}{c}$ and write as follows.

$$(p_e^{'})^2 c^2 = [(hf)^2 + (hf^{'})^2 - 2h^2 ff^{'} \cos\theta] + [p_e^2 c^2 + 2p_e hfc \cos\theta_1 - 2p_e hf^{'} c \cos\theta_2] \tag{22}$$

Equating Eq. 20 and Eq. 22 and cancelling common terms, we have

$$2h^2 ff^{'}[1 - \cos\theta] = 2h(f - f^{'})\sqrt{(m_e c^2)^2 + (p_e c)^2} - 2hcp_e[f\cos\theta_1 - f^{'}\cos\theta_2] \tag{23}$$

Dividing both sides of above equation by the term $2hff^{'}m_ec$, we use $p_e = \frac{m_e v}{\sqrt{(1-\frac{v^2}{c^2})}}$, $\lambda = \frac{c}{f}, \lambda^{'} = \frac{c}{f^{'}}$ we have

$$\frac{h}{m_e c}[1-\cos\theta] = (\lambda^{'}-\lambda)\sqrt{1+(\frac{p_e}{m_e c})^2} - \frac{1}{ff^{'}m_e}p_e[f\cos\theta_1 - f^{'}\cos\theta_2] \tag{24}$$

If **electron is at rest** before collision, $p_e = 0$ and we get the familiar **Compton effect equation** as follows.

$$\frac{h}{m_e c}[1-\cos\theta] = (\lambda^{'}-\lambda) \tag{25}$$

Thus we can see that Eq. 24 has **extra terms** when electron has **non-zero velocity** before collision with photon.

Now we substitute $\theta = \pi, \theta_1 = 0, \theta_2 = \pi$ in Eq. 22 , assuming the case where electron direction is the same before and after collision and is aligned with photon direction before collision and photon is reflected back at angle π after collision.

$$\begin{aligned}
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 + 2h^2ff^{'}] + [p_e^2c^2 + 2p_ehfc + 2p_ehf^{'}c] \\
(p_e^{'})^2c^2 &= h^2(f+f^{'})^2 + p_ec[p_ec + 2h(f+f^{'})] \\
(p_e^{'})^2 &= \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_ec + 2h(f+f^{'})]
\end{aligned} \tag{26}$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle **more complicated**, compared to Eq. 5 where $p_e = 0$.

$$\begin{aligned}
(p_e^{'})^2 &= \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_ec + 2h(f+f^{'})] = \frac{h^2}{c^2}(f+f^{'})^2X(f,f^{'},p_e) \\
X(f,f^{'},p_e) &= 1 + \frac{p_e}{c}(p_ec + 2h(f+f^{'})) = [1 + Z(f,f^{'},p_e)] \\
A(f,f^{'},p_e) &= [\sqrt{X(f,f^{'},p_e)} - 1] = ([1 + \frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] - 1) \\
&= [\frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] \\
Y(f,f^{'},p_e) &= \frac{h}{c}(f+f^{'})A(f,f^{'},p_e)
\end{aligned} \tag{27}$$

We can see that $Y(f, f^{'}, p_e) > 0$ for all f.

$$
\begin{aligned}
p_e' = m_e^{'} v = \frac{h}{c}(f + f^{'}) + Y(f, f^{'}, p_e) \\
\frac{d(p_e')}{df} = \frac{h}{c} + \frac{d(Y)}{df}
\end{aligned}
\tag{28}
$$

Replacing $d(p_e')$ by $\triangle p$, we have

$$
\begin{aligned}
\triangle x \triangle p = c \triangle t \triangle p = c \triangle t \triangle f[\frac{h}{c} + \frac{d(Y)}{df}] \\
\triangle t \triangle f \geq \frac{1}{4\pi} \\
\triangle x \triangle p \geq \frac{h}{4\pi} + \frac{c}{4\pi}\frac{d(Y)}{df}
\end{aligned}
\tag{29}
$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle **more complicated**, compared to Eq. 5 where $p_e = 0$.