On a new method towards proof of Riemann's Hypothesis

Akhila Raman

University of California at Berkeley. Email: akhila.raman@berkeley.edu.

Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from **Riemann's Xi function** $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ **does not have zeros** for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line and prove Riemann's hypothesis.

More importantly, the new method **does not** contradict the existence of non-trivial zeros on the critical line with real part of $s = \frac{1}{2}$ and **does not** contradict Riemann Hypothesis. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Keywords: Riemann, Hypothesis, Zeta, Xi, exponential functions

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.[4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Appendix A to Appendix E, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below.

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s=\frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega)=\Xi(\omega)=E_{0\omega}(\omega)$, where $-\infty\leq\omega\leq\infty$. Its inverse Fourier Transform is given by $E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω,t are real, as follows (link).[3]

$$E_0(t)=\Phi(t)=2\sum_{n=1}^{\infty}[2n^4\pi^2e^{\frac{9t}{2}}-3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2e^{2t}}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} \tag{1}$$

We see that $E_0(t)=E_0(-t)$ is a real and **even** function of t, given that $E_{0\omega}(\omega)=E_{0\omega}(-\omega)$ because $\xi(s)=\xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega)=\xi(\frac{1}{2}-i\omega)$ when evaluated at $s=\frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0<|\sigma|<\frac{1}{2}$ and this is shown in detail in Appendix A using contour integration.

$$E_p(t)=E_0(t)e^{-\sigma t}=2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t}-3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} \tag{2}$$

We can see that $E_p(t)$ is an analytic function in the interval $|t|\leq\infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable in that interval.

1.2. *Step 2: On the zeros of a related function* $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega)=E_{p\omega}(\omega)$ has a zero at $\omega=\omega_0$ where ω_0 is real and finite and $0<|\sigma|<\frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0<\sigma<\frac{1}{2}$ at first. Let us consider a new function $g(t)=e^{\sigma t_0}E_p'(t+t_0)e^{-\sigma t}u(-t)+e^{\sigma t_0}E_p'(t+t_0)e^{\sigma t}u(t)$, where $E_p'(t)=e^{\sigma t_2}E_p(t+t_2)$ and $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t)=e^{\sigma t_0}E_p'(t+t_0)$ where $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **odd function** $g_{odd}(t)=\frac{1}{2}[g(t)-g(-t)]$ given by $G_{odd}(\omega)=G_I(\omega)$ must have **at least one zero** at $\omega=\omega_z(t_2,t_0)\neq 0$, for every value of t_0, to satisfy Statement 1, where $\omega_z(t_2,t_0)$ is real and finite.

1.3. *Step 3: On the zeros of the function* $G_R(\omega)$

In Section 2.4.1, we compute the Fourier transform of the function $g_{odd}(t)$ given by $G_I(\omega)$. We require $G_I(\omega)=0$ for $\omega=\omega_z(t_2,t_0)$ for every value of t_0, to satisfy **Statement 1**. In general, $\omega_z(t_2,t_0)\neq\omega_0$.

It is shown that $R(t_2,t_0)=-G_I(\omega_z(t_2,t_0),t_0)=0$ for all t_0 as follows. We use $E_0'(t)=E_0(t+t_2)$ and $E_{0n}'(t)=E_0(t-t_2)$.

$$\begin{aligned}R(t_2,t_0)=e^{2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0)&\int_{-\infty}^{t_0}E_0'(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau\\&-\sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0}E_0'(\tau)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau]\\-[\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{-t_0}E_{0n}'(\tau)\sin(\omega_z(t_2,t_0)\tau)d\tau&+\sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{-t_0}E_{0n}'(\tau)\cos(\omega_z(t_2,t_0)\tau)d\tau]=0\\R(t_0)=\int_{-\infty}^{0}[E_0'(\tau+t_0)e^{-2\sigma\tau}&-E_{0n}'(\tau-t_0)]\sin(\omega_z(t_2,t_0)\tau)d\tau=0\end{aligned} \tag{3}$$

1.4. *Step 4: $\omega_z(t_2, t_0)$ is an even function of variable t_0*

In Section 2.5, we show the result in Eq. 4 and that $\omega_0(t_2, t_0) = \omega_z(t_2, t_0) = \omega_z(t_2, -t_0)$.

$$
\begin{aligned}
P(t_0) = &\int_{-\infty}^{0} [E_0'(\tau + t_0)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0)] \cos(\omega_0(t_2, t_0)\tau) d\tau \\
&+ \int_{-\infty}^{0} [E_0'(\tau - t_0)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0)] \cos(\omega_0(t_2, t_0)\tau) d\tau = 0
\end{aligned}
\tag{4}
$$

1.5. *Step 5: Final Step*

We set $t_0 = t_1$ and $t_2 = t_{2c} = Kt_1$, for positive integer K, such that $\omega_z(t_{2c}, t_1)t_1 = \pi$ and substitute in the equation for $R(t_2, t_0)$ in Eq. 3 and show that this leads to the result in Eq. 5.

$$
\int_{0}^{t_1} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_1) - \cosh(2\sigma\tau)) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau = 0
\tag{5}
$$

We show that the **each** of the terms in the integrand in Eq. 5 are **greater than zero**, in the interval $\tau = [0, t_1]$ where $t_1 > 0$. For $\omega_z(t_{2c}, t_1)t_1 = \pi$, we see that $\omega_z(t_{2c}, t_1)\tau = \frac{\pi}{t_1}\tau$ lies in the range $[0, \pi]$ and hence $\sin(\omega_z(t_{2c}, t_1)\tau) > 0$ in that interval $\tau = [0, t_1]$.

Hence the result in Eq. 5 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

2. An Approach towards Riemann's Hypothesis

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 < |\sigma| < \frac{1}{2}$. [3] This is shown in detail in first two paragraphs in Appendix C.1.

2.1. ***New function*** $g(t)$

Let us consider the function $E_p^{'}(t) = e^{\sigma t_2} E_p(t+t_2) = E_0(t+t_2)e^{-\sigma t} = E_0^{'}(t)e^{-\sigma t}$, where t_2 is finite and real, and $E_0^{'}(t) = E_0(t+t_2)$. Its Fourier transform is given by $E_{p\omega}^{'}(\omega) = E_{p\omega}(\omega)e^{\sigma t_2}e^{i\omega t_2}$ which has a zero at the **same** $\omega = \omega_0$.

Let us consider the function $f(t) = e^{\sigma t_0} E_p^{'}(t+t_0)$ where $|t_0| \leq \infty$ and we can see that the Fourier Transform of this function $F(\omega) = e^{\sigma t_0} E_{p\omega}^{'}(\omega)e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$.

Let us consider a new function $g(t) = g_-(t)u(-t) + g_+(t)u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t) = f(t)e^{-\sigma t}$ and $g_+(t) = f(t)e^{\sigma t}$. We can see that $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can show that $E_p(t), E_p^{'}(t), h(t), g(t)$ are real absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}^{'}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). This is shown in detail in Appendix C.1.

We can see that $g(t)$ is a real L^1 integrable function , its Fourier transform $G(\omega)$ is finite for $|\omega| < \infty$ and goes to zero as $\omega \to \pm\infty$, as per **Riemann-Lebesgue Lemma**[Riemann Lebesgue Lemma]. This is explained in detail in Appendix C.1.

If we take the Fourier transform of the equation $g(t)h(t) = f(t)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, we get $\frac{1}{2\pi}[G(\omega) * H(\omega)] = F(\omega) = E_{p\omega}^{'}(\omega)e^{\sigma t_0}e^{i\omega t_0} = F_R(\omega) + iF_I(\omega)$ as per convolution theorem (link), where $*$ denotes convolution operation given by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega) + iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix B.1.

For a given fixed value of t_2, for **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **odd** function $g_{odd}(t)$ given by $G_I(\omega)$ must have **at least one real zero** at $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, for a given fixed value of t_2. Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ does not have real zeros, if $G_I(\omega)$ does not have real zeros, then $F_I(\omega) = G_I(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_I(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ where $\omega_z(t_2, t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_I(\omega)$ does not have at least one zero for some $\omega = \omega_z(t_2, t_0) \neq 0$, **then** the **imaginary part** of $F(\omega)$ given by $F_I(\omega) = \frac{1}{2\pi}[G_I(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

2.2. ***Lemma 1: On the zeros of a related function*** $G_I(\omega)$

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$ for **every value** of t_0, for a given fixed value of t_2, where $\omega_z(t_2, t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{\sigma t_0} E_p^{'}(t+t_0)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}^{'}(\omega)e^{\sigma t_0}e^{i\omega t_0} = E_{p\omega}(\omega)e^{\sigma t_2}e^{i\omega t_2}e^{\sigma t_0}e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$ and its imaginary part given by $F_I(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_I(\omega)$ **does not** have at least one zero for finite $\omega = \omega_z(t_2, t_0) \neq 0$ and show that $F_I(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the imaginary parts as follows.

$$F_I(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_I(\omega') H(\omega - \omega') d\omega' \tag{6}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 6 and we get

$$F_I(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{7}$$

We can split the integral in Eq. 7 as follows.

$$F_I(\omega) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_I(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{8}$$

We see that $G_I(-\omega) = -G_I(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 8 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_I(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_I(\omega') [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} - \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{9}$$

In Appendix C.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 9 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} - \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_I(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_I(\omega) > 0$ for all finite $\omega > 0$. We see that $F_I(-\omega) = -F_I(\omega)$ because $E_p(t)$ is a real function (Appendix B.2). Hence $F_I(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_I(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

- **Case 2:** $G_I(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_I(\omega) < 0$ for all finite $\omega > 0$. We see that $F_I(-\omega) = -F_I(\omega)$ because $E_p(t)$ is a real function (Appendix B.2). Hence $F_I(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) \neq 0$, where $\omega_z(t_2, t_0)$ is real and finite.

We have shown that, $G_I(\omega)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**. We will investigate if Statement 2 leads to a contradiction for $0 < \sigma < \frac{1}{2}$.

We will show a similar result below, for the even function $G_R(\omega)$.

2.3. *Lemma 2: On the zeros of a related function* $G_R(\omega)$

For **every value** of t_0, we require the Fourier transform of the function $f(t)$ given by $F(\omega)$ to have a zero at $\omega = \omega_0$. This implies that the Fourier transform of the **even** function $g(t)$ given by $G(\omega) = G_R(\omega)$ must have **at least one real zero** at $\omega = \omega_2(t_0)$ for **every value** of t_0. Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ does not have real zeros, if $G_R(\omega)$ does not have real zeros, then $F_R(\omega) = G_R(\omega) * H_R(\omega)$ obtained by the convolution of $H_R(\omega)$ and $G_R(\omega)$, cannot possibly have real zeros, which goes against **Statement 1**.

We can write $g(t) = g_{even}(t) + g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_2(t_0) \neq 0$ where $\omega_2(t_0)$ is real and finite and can be different from ω_0 in general. We call this **Statement 3**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega)$ does not have at least one zero for some $\omega = \omega_2(t_0) \neq 0$, **then** the **real part** of $F(\omega)$ given by $F_R(\omega) = \frac{1}{2\pi}[G_R(\omega) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 2.

Lemma 2: If Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **real** part of the Fourier transform of the **even function** $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ given by $G_R(\omega)$ must have **at least one zero** at $\omega = \omega_z'(t_2, t_0) \neq 0$ for **every value** of t_0, for a given fixed value of t_2, where $\omega_z'(t_2, t_0)$ is real and finite, where $g(t)h(t) = f(t) = e^{\sigma t_0} E_p'(t + t_0)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then $F(\omega) = E_{p\omega}(\omega)e^{\sigma t_0}e^{i\omega t_0}$ also has a zero at $\omega = \omega_0$ and its real part given by $F_R(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_R(\omega)$ **does not** have at least one zero for finite $\omega = \omega_2(t_0) \neq 0$ and show that $F_R(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the real parts as follows.

$$F_R(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega')H(\omega - \omega')d\omega' \tag{10}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 10 and we get

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{11}$$

We can split the integral in Eq. 11 as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega') \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{12}$$

We see that $G_R(-\omega) = G_R(\omega)$ because $g(t)$ is a real function (Appendix B.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 12 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega') [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{13}$$

In Appendix C.1 last paragraph, it is shown that $G(\omega)$ is finite for $|\omega| \leq \infty$ and goes to zero as $|\omega| \to \infty$. We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 13 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

- **Case 1:** $G_R(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) > 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) > 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$ because we showed that $\omega_0 \neq 0$ in **Section 2** paragraph 5. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

- **Case 2:** $G_R(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $F_R(\omega) < 0$ for all finite $\omega > 0$. We see that $F_R(-\omega) = F_R(\omega)$ because $f(t)$ is a real function (Appendix B.2). Hence $F_R(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $F_R(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega')$ must have **at least one zero** at $\omega' = \omega_2(t_0) \neq 0$, where $\omega_2(t_0)$ is real and finite.

We have shown that, $G_R(\omega)$ must have **at least one zero** at finite $\omega = \omega_2(t_0) \neq 0$ to satisfy **Statement 1**. We call this **Statement 3**.

2.4. ***On the zeros of a related function*** $G_I(\omega)$

We can compute the fourier transform of the function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$. We require $G_I(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for **every value** of t_0, to satisfy **Statement 1**. In general, $\omega_z(t_2, t_0) \neq \omega_0$.

First we compute the fourier transform of the function $g(t)$ given by $G(\omega) = G_R(\omega) + iG_I(\omega)$. We use $g(t) = e^{\sigma t_0} E_p'(t+t_0) e^{-\sigma t} u(-t) + e^{\sigma t_0} E_p'(t+t_0) e^{\sigma t} u(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{0} g_-(t) e^{-i\omega t} dt + \int_{0}^{\infty} g_+(t) e^{-i\omega t} dt$$
$$G(\omega) = \int_{-\infty}^{0} e^{\sigma t_0} E_p'(t+t_0) e^{-\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{\sigma t_0} E_p'(t+t_0) e^{\sigma t} e^{-i\omega t} dt \tag{14}$$

We use $E_p'(t) = E_0'(t) e^{-\sigma t}$ where $E_0'(t) = E_0(t+t_2)$ and $E_p'(t+t_0) = E_0'(t+t_0) e^{-\sigma t} e^{-\sigma t_0}$. Substituting $t = -t$ in the second integral in Eq. 14, we have

$$G(\omega) = \int_{-\infty}^{0} E_0'(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{0}^{\infty} E_0'(t+t_0) e^{-i\omega t} dt$$
$$G(\omega) = \int_{-\infty}^{0} E_0'(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_0'(-t+t_0) e^{i\omega t} dt \tag{15}$$

We define $E_{0n}'(t) = E_0'(-t)$ and get $E_0'(-t+t_0) = E_{0n}'(t-t_0)$ and write Eq. 15 as follows.

$$G(\omega) = \int_{-\infty}^{0} E_0'(t+t_0) e^{-2\sigma t} e^{-i\omega t} dt + \int_{-\infty}^{0} E_{0n}'(t-t_0) e^{i\omega t} dt = G_R(\omega) + iG_I(\omega)$$

(16)

The above equations can be expanded as follows using the identity $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Comparing the **imaginary parts** of $G(\omega)$, we have

$$G_I(\omega) = -\int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}\sin{(\omega t)}dt + \int_{-\infty}^{0} E_{0n}^{'}(t-t_0)\sin{(\omega t)}dt \tag{17}$$

We require $G_I(\omega) = 0$ for $\omega = \omega_z(t_2, t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. Hence we can see that $R(t_2, t_0) = -G_I(\omega_z(t_2, t_0)) = 0$ and we can write as follows using $t = \tau$.

$$R(t_2, t_0) = \int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} - E_{0n}^{'}(\tau-t_0)]\sin{(\omega_z(t_2,t_0)\tau)}d\tau = 0 \tag{18}$$

We can rewrite Eq. 18 as follows, using the substitution $\tau + t_0 = \tau'$ in the first integral and $\tau - t_0 = \tau''$ in the second integral and substituting back $\tau' = \tau$ and $\tau'' = \tau$.

$$\begin{aligned}
R(t_2, t_0) = e^{2\sigma t_0}[\cos{(\omega_z(t_2,t_0)t_0)} &\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin{(\omega_z(t_2,t_0)\tau)}d\tau \\
- \sin{(\omega_z(t_2,t_0)t_0)} &\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos{(\omega_z(t_2,t_0)\tau)}d\tau] \\
-[\cos{(\omega_z(t_2,t_0)t_0)}\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\sin{(\omega_z(t_2,t_0)\tau)}d\tau + \sin{(\omega_z(t_2,t_0)t_0)} &\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\cos{(\omega_z(t_2,t_0)\tau)}d\tau] = 0
\end{aligned} \tag{19}$$

Now we replace t_0 by $-t_0$ in $f(t)$ and consider the function $f_2(t) = e^{-\sigma t_0}E_p^{'}(t-t_0)$ where $|t_0| \leq \infty$ and use the procedure in above section and we can write as follows.

$$\begin{aligned}
R(t_2, -t_0) &= \int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} - E_{0n}^{'}(\tau+t_0)]\sin{(\omega_z(t_2,-t_0)\tau)}d\tau = 0 \\
R(t_2, t_0) + R(t_2, -t_0) &= \int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} - E_{0n}^{'}(\tau-t_0)]\sin{(\omega_z(t_2,t_0)\tau)}d\tau \\
&+ \int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} - E_{0n}^{'}(\tau+t_0)]\sin{(\omega_z(t_2,-t_0)\tau)}d\tau = 0
\end{aligned} \tag{20}$$

2.4.1. ***On the zeros of a related function*** $G_R(\omega)$

Comparing the **real parts** of $G(\omega)$ in Eq. 16, we have

$$G_R(\omega) = \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}\cos{(\omega t)}dt + \int_{-\infty}^{0} E_{0n}^{'}(t-t_0)\cos{(\omega t)}dt \tag{21}$$

We require $G_R(\omega) = 0$ for $\omega = \omega_z'(t_2, t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. Hence we can see that $R'(t_2, t_0) = G_R(\omega_z'(t_2, t_0)) = 0$ and we can write as follows using $t = \tau$.

$$R'(t_2,t_0)=\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0)]\cos{(\omega_z^{\prime}(t_2,t_0)\tau)}d\tau=0 \tag{22}$$

We can rewrite Eq. 22 as follows, using the substitution $\tau+t_0=\tau'$ in the first integral and $\tau-t_0=\tau''$ in the second integral and substituting back $\tau'=\tau$ and $\tau''=\tau$.

$$\begin{aligned}R'(t_2,t_0)=e^{2\sigma t_0}[\cos{(\omega_z^{\prime}(t_2,t_0)t_0)}\int_{-\infty}^{t_0}E_0^{'}(\tau)e^{-2\sigma\tau}\cos{(\omega_z^{\prime}(t_2,t_0)\tau)}d\tau\\+\sin{(\omega_z^{\prime}(t_2,t_0)t_0)}\int_{-\infty}^{t_0}E_0^{'}(\tau)e^{-2\sigma\tau}\sin{(\omega_z^{\prime}(t_2,t_0)\tau)}d\tau]\\+[\cos{(\omega_z^{\prime}(t_2,t_0)t_0)}\int_{-\infty}^{-t_0}E_{0n}^{'}(\tau)\cos{(\omega_z^{\prime}(t_2,t_0)\tau)}d\tau-\sin{(\omega_z^{\prime}(t_2,t_0)t_0)}\int_{-\infty}^{-t_0}E_{0n}^{'}(\tau)\cos{(\omega_z^{\prime}(t_2,t_0)\tau)}d\tau]=0\end{aligned} \tag{23}$$

Now we replace t_0 by $-t_0$ in $f(t)$ and consider the function $f_2(t)=e^{-\sigma t_0}E_p^{'}(t-t_0)$ where $|t_0|\leq\infty$ and use the procedure in above section and we can write as follows.

$$\begin{aligned}R'(t_2,-t_0)=\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0)]\cos{(\omega_z(t_2,-t_0)\tau)}d\tau=0\\R'(t_2,t_0)+R'(t_2,-t_0)=\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0)]\cos{(\omega_z(t_2,t_0)\tau)}d\tau\\+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0)]\cos{(\omega_z(t_2,-t_0)\tau)}d\tau=0\end{aligned} \tag{24}$$

2.5. $\omega_z(t_2,t_0)$ *is an even function of variable* t_0

Now we consider the function $f_T(t)=f(t)+f_2(t)=e^{\sigma t_0}E_p^{'}(t+t_0)+e^{-\sigma t_0}E_p^{'}(t-t_0)$ where $|t_0|\leq\infty$ and $g_T(t)h(t)=f_T(t)$ where $g_T(t)=f_T(t)e^{-\sigma t}u(-t)+f_T(t)e^{\sigma t}u(t)$ and $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g_T(t)$ and compute its imaginary part using the procedure in above section, similar to Eq. 17 and we can write as follows.

$$\begin{aligned}G_{T_I}(\omega,t_0)=G_1(\omega,t_0)+G_1(\omega,-t_0)\\G_1(\omega,t_0)=-\int_{-\infty}^{0}E_0^{'}(t+t_0)e^{-2\sigma t}\sin{(\omega t)}dt+\int_{-\infty}^{0}E_{0n}^{'}(t-t_0)\sin{(\omega t)}dt\\G_{T_I}(\omega,t_0)=-[\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau-t_0)]\sin{(\omega\tau)}d\tau\\+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau+t_0)]\sin{(\omega\tau)}d\tau]\end{aligned} \tag{25}$$

We require $G_{T_I}(\omega,t_0)=0$ for $\omega=\omega_0(t_2,t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_0(t_2,t_0)\neq\omega_z(t_2,t_0)$. Hence we can see that $P(t_2,t_0)=-G_{T_I}(\omega_0(t_2,t_0))=0$ and we can rewrite as follows using the substitution $t=\tau$.

$$P(t_2,t_0)=\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau-t_0)]\sin{(\omega_0(t_2,t_0)\tau)}d\tau$$
$$+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau+t_0)]\sin{(\omega_0(t_2,t_0)\tau)}d\tau=0 \tag{26}$$

We see that $f_T(t)=e^{\sigma t_0}E_p^{'}(t+t_0)+e^{-\sigma t_0}E_p^{'}(t-t_0)$ is **unchanged** by the substitution $t_0=-t_0$ and hence $\omega_0(t_2,t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2. Hence we can rewrite the second integral in Eq. 26 as follows using $\omega_0(t_2,t_0)=\omega_0(t_2,-t_0)$.

$$\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau-t_0)]\sin{(\omega_0(t_2,t_0)\tau)}d\tau$$
$$+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau+t_0)]\sin{(\omega_0(t_2,-t_0)\tau)}d\tau=0 \tag{27}$$

We compare Eq. 27 and Eq. 20 as follows.

$$\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau-t_0)]\sin{(\omega_0(t_2,t_0)\tau)}d\tau$$
$$+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau+t_0)]\sin{(\omega_0(t_2,-t_0)\tau)}d\tau=0$$
$$\int_{-\infty}^{0}[E_0^{'}(\tau+t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau-t_0)]\sin{(\omega_z(t_2,t_0)\tau)}d\tau$$
$$+\int_{-\infty}^{0}[E_0^{'}(\tau-t_0)e^{-2\sigma\tau}-E_{0n}^{'}(\tau+t_0)]\sin{(\omega_z(t_2,-t_0)\tau)}d\tau=0 \tag{28}$$

We can see that there must be **at least one** common solution where $\omega_z(t_2,t_0)=\omega_0(t_2,t_0)$ to satisfy Eq. 28. Because $\omega_0(t_2,t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2, we see that $\omega_z(t_2,t_0)=\omega_0(t_2,t_0)$ is also an **even** function of variable t_0, for **every fixed value** of t_2.

Given that $E_p^{'}(t)=e^{\sigma t_2}E_p(t+t_2)$, we see that $f(t)=e^{\sigma t_0}E_p^{'}(t+t_0)=e^{\sigma t_0}e^{\sigma t_2}E_p(t+t_2+t_0)$ is **unchanged** if we interchange the variables t_2 and t_0 and hence the location of the zeros in Fourier transform of $g(t,t_0,t_2)$ represented by $\omega_z(t_2,t_0)$ **remain the same**. Hence $\omega_z(t_2,t_0)=\omega_z(t_0,t_2)$. Given that, for **every value** of t_2, $\omega_z(t_2,t_0)=\omega_z(t_2,-t_0)$, we see that $\omega_z(t_0,t_2)=\omega_z(t_0,-t_2)$.

The results in this section apply **only** for the case $0<\sigma<\frac{1}{2}$. For $\sigma=0$, $g_T(t)=E_0^{'}(t+t_0)+E_0^{'}(t-t_0)$ is an even function of variable t_0 and $2g_{T_{odd}}(t)=g_T(t)-g_T(-t)=0$ and hence $G_{T_I}(\omega)=0$ for all $|\omega|\leq\infty$.

2.6. $\omega_z^{'}(t_2,t_0)$ *is an even function of variable* t_0

Now we consider the function $f_T(t)=f(t)+f_2(t)=e^{\sigma t_0}E_p^{'}(t+t_0)+e^{-\sigma t_0}E_p^{'}(t-t_0)$ where $|t_0|\leq\infty$ and $g_T(t)h(t)=f_T(t)$ where $g_T(t)=f_T(t)e^{-\sigma t}u(-t)+f_T(t)e^{\sigma t}u(t)$ and $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$ and compute the Fourier transform of the function $g_T(t)$ and compute its **real part** using the procedure in above section, similar to Eq. 21 and we can write as follows.

$$
\begin{aligned}
G_{T_R}(\omega,t_0) &= G_2(\omega,t_0) + G_2(\omega,-t_0)\\
G_2(\omega,t_0) = \int_{-\infty}^{0} E_0^{'}(t+t_0)e^{-2\sigma t}\cos{(\omega t)}dt &+ \int_{-\infty}^{0} E_{0n}^{'}(t-t_0)\cos{(\omega t)}dt\\
G_{T_R}(\omega,t_0) = [\int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} &+ E_{0n}^{'}(\tau-t_0)]\cos{(\omega\tau)}d\tau\\
+\int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} &+ E_{0n}^{'}(\tau+t_0)]\cos{(\omega\tau)}d\tau]
\end{aligned}
\tag{29}
$$

We require $G_{T_R}(\omega,t_0)=0$ for $\omega=\omega_0'(t_2,t_0)$ for every value of t_0, for **every given fixed value** of t_2, to satisfy **Statement 1**. In general $\omega_0'(t_2,t_0)\neq\omega_z'(t_2,t_0)$. Hence we can see that $P'(t_2,t_0)=G_{T_R}(\omega_0'(t_2,t_0))=0$ and we can rewrite as follows using the substitution $t=\tau$.

$$
\begin{aligned}
P'(t_2,t_0) = \int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)]\cos{(\omega_0'(t_2,t_0)\tau)}d\tau\\
+\int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0)]\cos{(\omega_0'(t_2,t_0)\tau)}d\tau = 0
\end{aligned}
\tag{30}
$$

We see that $f_T(t)=e^{\sigma t_0}E_p^{'}(t+t_0)+e^{-\sigma t_0}E_p^{'}(t-t_0)$ is **unchanged** by the substitution $t_0=-t_0$ and hence $\omega_0'(t_2,t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2. Hence we can rewrite the second integral in Eq. 30 as follows using $\omega_0'(t_2,t_0)=\omega_0'(t_2,-t_0)$.

$$
\begin{aligned}
\int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)]\cos{(\omega_0'(t_2,t_0)\tau)}d\tau\\
+\int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0)]\cos{(\omega_0'(t_2,-t_0)\tau)}d\tau = 0
\end{aligned}
\tag{31}
$$

We compare Eq. 31 and Eq. 24 as follows.

$$
\begin{aligned}
\int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)]\cos{(\omega_0'(t_2,t_0)\tau)}d\tau\\
+\int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0)]\cos{(\omega_0'(t_2,-t_0)\tau)}d\tau = 0\\
\int_{-\infty}^{0} [E_0^{'}(\tau+t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0)]\cos{(\omega_z'(t_2,t_0)\tau)}d\tau\\
+\int_{-\infty}^{0} [E_0^{'}(\tau-t_0)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0)]\cos{(\omega_z'(t_2,-t_0)\tau)}d\tau = 0
\end{aligned}
\tag{32}
$$

We can see that there must be **at least one** common solution where $\omega_z'(t_2,t_0)=\omega_0'(t_2,t_0)$ to satisfy Eq. 32. Because $\omega_0'(t_2,t_0)$ is an **even** function of variable t_0, for **every fixed value** of t_2, we see that $\omega_z'(t_2,t_0)=\omega_0'(t_2,t_0)$ is also an **even** function of variable t_0, for **every fixed value** of t_2.

3. $\omega_z(t_2, t_0)$ is a continuous function of t_0

In Section 2.1, $\omega_z(t_0)$ is shown to be **finite** for all $|t_0| \leq \infty$. Hence we can see that $\omega_z(t_0)$ is **finite** for all $|t_0| \leq \infty$, for **every value** of t_2. This means there are **no** Dirac delta functions present in $\omega_z(t_0)$.

In Appendix D, it is shown that $\omega_z(t_0)$ is a continuous function of t_0 around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$. Hence we see that $\omega_z(t_2, t_0)$ is a continuous function around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$, for **every given fixed value** of t_2.

In this section, we show that $\omega_z(t_2, t_0)$ is a continuous function of t_0 in the interval $|t_0| < \infty$, using **proof by contradiction**.

First, we assume that $\omega_z(t_2, t_0)$ has **one or more** discontinuities in the interval $0 < \omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$ (**Statement A**). We will show that this assumption leads to a **contradiction**.

We write $R(t_2, t_0) = 0$ in Eq. 19 as follows. We use $E_{0n}^{'}(\tau) = E_0^{'}(-\tau)$ and $E_0^{'}(\tau) = E_0(\tau + t_2)$.

$$
\begin{aligned}
R(t_2, t_0) &= \cos(\omega_z(t_2, t_0)t_0) I_a(t_0) - \sin(\omega_z(t_2, t_0)t_0) I_b(t_0) = 0 \\
I_a(t_0) &= e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau - \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau \\
I_b(t_0) &= e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau + \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau \\
R(t_2, t_0) &= e^{2\sigma t_0} [\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau) d\tau \\
&\quad - \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau] \\
&\quad -[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau) \sin(\omega_z(t_2, t_0)\tau) d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau] = 0
\end{aligned}
\tag{33}
$$

We can write Eq. 33 as follows, which holds for $0 < \omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$.

$$
\frac{I_a(t_0)}{I_b(t_0)} = \frac{\sin(\omega_z(t_2, t_0)t_0)}{\cos(\omega_z(t_2, t_0)t_0)} = \tan(\omega_z(t_2, t_0)t_0)
\tag{34}
$$

If **Statement A** is true, then the integrands in $I_a(t_0), I_b(t_0)$ have **one or more** discontinuities. But the integral of a discontinuous function is a continuous function and hence $I_a(t_0), I_b(t_0)$ are continuous functions. If $I_b(t_0)$ is **NOT** an all-zero function in the interval $0 < \omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$, then $\frac{I_a(t_0)}{I_b(t_0)}$ is a continuous function of t_0 in the interval $0 < \omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$, which **contradicts** Statement A. If **Statement A** is true, then we **require** $I_b(t_0)$ to be an all-zero function in the interval $|t_0| < \infty$ (**Statement B**), to satisfy Eq. 34.

3.1. Statement B: $I_b(t_0)$ is an all-zero function

In this section, we **assume** that $I_b(t_0)$ is an all-zero function and show that this assumption leads to a **contradiction**.

$$
I_b(t_0) = e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0^{'}(\tau) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau) d\tau + \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau) \cos(\omega_z(t_2, t_0)\tau) d\tau = 0
$$

(35)

Next, we derive the corresponding result for $R(t_2, t_0$ by considering the **real part** of $G(\omega)$ in Section 2.4.1, by **replacing** replacing $\omega_z(t_2, t_0)$ with $\omega_z'(t_2, t_0)$, replacing$\cos(\omega_z(t_2, t_0)t_0)$ with $\sin(\omega_z'(t_2, t_0)t_0)$ and replacing $\sin(\omega_z(t_2, t_0)t_0)$ with $-\cos(\omega_z'(t_2, t_0)t_0)$, as follows.

$$
\begin{aligned}
R'(t_2, t_0) &= \cos(\omega_z'(t_2, t_0)t_0)I_b'(t_0) + \sin(\omega_z'(t_2, t_0)t_0)I_a'(t_0) = 0 \\
I_a'(t_0) &= e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z'(t_2, t_0)\tau)d\tau - \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\sin(\omega_z'(t_2, t_0)\tau)d\tau \\
I_b'(t_0) &= e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_z'(t_2, t_0)\tau)d\tau + \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\cos(\omega_z'(t_2, t_0)\tau)d\tau \\
R'(t_2, t_0) &= e^{2\sigma t_0}[\cos(\omega_z'(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_z'(t_2, t_0)\tau)d\tau \\
&\quad + \sin(\omega_z'(t_2, t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z'(t_2, t_0)\tau)d\tau] \\
&+[\cos(\omega_z'(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\cos(\omega_z'(t_2, t_0)\tau)d\tau - \sin(\omega_z'(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\sin(\omega_z'(t_2, t_0)\tau)d\tau] = 0
\end{aligned}
$$

(36)

We can write Eq. 36 as follows, which holds for $0 < \omega_z'(t_2, t_0)t_0 < \frac{\pi}{2}$.

$$
\frac{I_a'(t_0)}{I_b'(t_0)} = -\frac{\cos(\omega_z'(t_2, t_0)t_0)}{\sin(\omega_z'(t_2, t_0)t_0)} = -\cot(\omega_z'(t_2, t_0)t_0)
$$

(37)

If **Statement A** is true, then the integrands in $I_a'(t_0), I_b'(t_0)$ have **one or more** discontinuities. But the integral of a discontinuous function is a continuous function and hence $I_a'(t_0), I_b'(t_0)$ are continuous functions. If $I_b'(t_0)$ is **NOT** an all-zero function in the interval $0 < \omega_z'(t_2, t_0)t_0 < \frac{\pi}{2}$, then $\frac{I_a'(t_0)}{I_b'(t_0)}$ is a continuous function of t_0 in the interval $0 < \omega_z'(t_2, t_0)t_0 < \frac{\pi}{2}$, which **contradicts** Statement A. If **Statement A** is true, then we **require** $I_b'(t_0)$ to be an all-zero function in the interval $0 < \omega_z'(t_2, t_0)t_0 < \frac{\pi}{2}$ (**Statement C**), to satisfy Eq. 37.

3.2. ***Statement C: $I_b'(t_0)$ is an all-zero function***

We **assume** that $I_b'(t_0)$ is an all-zero function and show that this assumption leads to a **contradiction**.

$$
I_b'(t_0) = e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\cos(\omega_z'(t_2, t_0)\tau)d\tau + \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\cos(\omega_z'(t_2, t_0)\tau)d\tau = 0
$$

(38)

We compare Eq. 35 and Eq. 38 and see that there must be **at least one** common solution where $\omega_z'(t_2, t_0) = \omega_z(t_2, t_0)$ to satisfy the two equations.

In Eq. 33, we see that $\omega_z(t_2, t_0) = 0$ is one solution at $t_0 = 0$, given that it corresponds to the zero of an **odd** function $G_I(\omega)$.

$$
I_a(t_0) = e^{2\sigma t_0}\int_{-\infty}^{t_0} E_0^{'}(\tau)e^{-2\sigma\tau}\sin(\omega_z(t_2, t_0)\tau)d\tau - \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau)\sin(\omega_z(t_2, t_0)\tau)d\tau = 0, \quad at \quad t_0 = 0
$$

(39)

This means that the **same solution** must apply for $\omega'_z(t_2,t_0) = \omega_z(t_2,t_0) = 0$ at $t_0 = 0$, which **cannot** be the case, because $\omega'_z(t_2,t_0) = 0$ at $t_0 = 0$, which corresponds to zero of an the even function $G_R(\omega)$, is **NOT** possible. In Eq. 36, we set $t_0 = 0$ and get $I'_b(t_0) = 0$ as follows.

$$
\begin{aligned}
I'_b(t_0) = e^{2\sigma t_0}\int_{-\infty}^{t_0} E'_0(\tau)e^{-2\sigma\tau}\cos\left(\omega'_z(t_2,t_0)\tau\right)d\tau + \int_{-\infty}^{-t_0} E'_{0n}(\tau)\cos\left(\omega'_z(t_2,t_0)\tau\right)d\tau = 0, \quad at \quad t_0 = 0 \\
I'_b(0) = \int_{-\infty}^{0} E'_0(\tau)e^{-2\sigma\tau}d\tau + \int_{-\infty}^{0} E'_{0n}(\tau)d\tau = 0
\end{aligned}
\tag{40}
$$

Given that $E_0(\tau) > 0$ for all $|\tau| \leq \infty$, (Appendix C), we see that $E'_0(\tau) = E_0(\tau + t_2) > 0$ and $E'_{0n}(\tau) = E_0(\tau - t_2) > 0$, for all $|\tau| \leq \infty$ and hence $I'_b(0)$ in Eq. 40 **cannot** equal zero.

Hence the assumption that $I_b(t_0), I'_b(t_0)$ are all-zero functions, lead to a **contradiction** and hence **Statement B** and **Statement C** are **false**. If **Statement A** is true, then we **require Statement B** and **Statement C** to be true, to satisfy Eq. 34 and Eq. 37. We have shown that **Statement B** and **Statement C** are **false** and hence **Statement A** is false.

Hence $\omega_z(t_2,t_0)$ is a **continuous** function of t_0, for every value of t_2, in the interval $0 < \omega_z(t_2,t_0)t_0 < \frac{\pi}{2}$.

Using the method described in this section, we can show that $\omega_z(t_2,t_0)$ is a **continuous** function of t_0, for every value of t_2, in the interval $|t_0| < \infty$, **except** at $\omega_z(t_2,t_0)t_0 = \frac{(2k+1)\pi}{2}$ and $\omega_z(t_2,t_0)t_0 = k\pi$, for integer k.

Using first principles of continuous functions, we can see that $\omega_z(t_2,t_0)$ remains a **continuous** function of t_0, at $\omega_z(t_2,t_0)t_0 = \frac{(2k+1)\pi}{2}$ and $\omega_z(t_2,t_0)t_0 = k\pi$, for integer k, to satisfy Eq. 33 and Eq. 36.

We consider $t_0 = t_1$ and $t_2 = t_{2c}$ such that $\omega_z(t_{2c},t_1)t_1 = k\pi$. As t_0 approaches t_1 from the left limit, $\omega_z(t_{2c},t_0)$ approaches $\omega_z(t_{2c},t_1)$ in a continuous manner and $\omega_z(t_{2c},t_0)t_0$ approaches $k\pi$. Similarly, as t_0 approaches t_1 from the right limit, $\omega_z(t_{2c},t_0)$ approaches $\omega_z(t_{2c},t_1)$ in a continuous manner and $\omega_z(t_{2c},t_0)t_0$ approaches $k\pi$. At $\omega_z(t_2,t_0)t_0 = k\pi$, $t_0 = t_1$ and $\omega_z(t_{2c},t_1) = \frac{k\pi}{t_1}$. Hence we see that $\omega_z(t_2,t_0)$ remains continuous of t_0, at $\omega_z(t_2,t_0)t_0 = k\pi$. We can show that $\omega_z(t_2,t_0)$ remains a **continuous** function of t_0, at $\omega_z(t_2,t_0)t_0 = \frac{(2k+1)\pi}{2}$, using similar argument.

Hence $\omega_z(t_2,t_0)$ is a **continuous** function of t_0, for every value of t_2, in the interval $|t_0| < \infty$.

4. Final Proof

We write $R(t_2, t_0) = 0$ in Eq. 19 as follows. We use $E^{'}_{0n}(\tau) = E^{'}_0(-\tau)$ and $E^{'}_0(\tau) = E_0(\tau + t_2)$.

$$
\begin{aligned}
R(t_2, t_0) = e^{2\sigma t_0}[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_2, t_0)\tau)d\tau \\
- \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{t_0} E^{'}_0(\tau)e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)\tau)d\tau] \\
-[\cos(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E^{'}_{0n}(\tau) \sin(\omega_z(t_2, t_0)\tau)d\tau + \sin(\omega_z(t_2, t_0)t_0) \int_{-\infty}^{-t_0} E^{'}_{0n}(\tau) \cos(\omega_z(t_2, t_0)\tau)d\tau] = 0
\end{aligned}
\tag{41}
$$

In 3, it is shown that $\omega_z(t_0)$ is a continuous function of t_0 in the interval $|t_0| < \infty$. Hence we see that $\omega_z(t_2, t_0)$ is a **continuous** function around $t_0 = 0$ in the interval $[-\delta t_0, \delta t_0]$, for **every given fixed value** of t_2.

In Section 5, it is shown that $E_0(t)$ is **strictly decreasing** for $t \geq t_d = \frac{1}{8}$ and that the **minimum** value $Min(E_0(t)) = \frac{1}{5} = E_{min}$ in the interval $-t_d \leq t \leq t_d$.

Given $\omega_z(t_2, t_0)$ is a **continuous** function of both t_0 and t_2, we can **make sure** that $\omega_z(t_{2c}, t_1)t_1 = \pi$, by finding a **suitable** value of $t_0 = t_1$ and $t_2 = t_{2c} = Kt_1$, where K is a positive integer, **such that** $E_0(t) < E_{min}$ for $t \geq t_{2c}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0, for every value of t_2, and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, as t_0 is increased from zero to ∞, we see that $\omega_z(Kt_1, t_1)t_1$ increases from zero towards ∞ in a continuous manner and will **certainly pass through** π. More details of the algorithm to ensure that $\omega_z(t_{2c}, t_1)t_1 = \pi$ is in Section 5.5.

We set $t_0 = t_1$ and $t_2 = t_{2c} = Kt_1$ such that $\omega_z(t_{2c}, t_1)t_1 = \pi$ in Eq. 41 as follows.

$$
e^{2\sigma t_1} \int_{-\infty}^{t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau = \int_{-\infty}^{-t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau
\tag{42}
$$

We split the integral in the left hand side of Eq. 42 and write as follows.

$$
\begin{aligned}
\int_{-\infty}^{t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau = e^{-2\sigma t_1} \int_{-\infty}^{-t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau \\
\int_{-\infty}^{-t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau + \int_{-t_1}^{t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau \\
= e^{-2\sigma t_1} \int_{-\infty}^{-t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau
\end{aligned}
\tag{43}
$$

In Eq. 41, we substitute $t_0 = -t_1$ for which $\omega_z(t_{2c}, t_1)(-t_1) = -\pi$ as follows. We use $\omega_z(t_{2c}, -t_1) = \omega_z(t_{2c}, t_1)$.

$$
\begin{aligned}
e^{-2\sigma t_1} \int_{-\infty}^{-t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau = \int_{-\infty}^{t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau \\
\int_{-\infty}^{-t_1} E^{'}_0(\tau)e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_1)\tau)d\tau = e^{2\sigma t_1} \int_{-\infty}^{t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau \\
= e^{2\sigma t_1}[\int_{-\infty}^{-t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau + \int_{-t_1}^{t_1} E^{'}_{0n}(\tau) \sin(\omega_z(t_{2c}, t_1)\tau)d\tau]
\end{aligned}
$$

(44)

We substitute Eq. 44 in Eq. 43 as follows and we substitute $E_0^{'}(\tau) = E_0(\tau+t_{2c})$ and $E_{0n}^{'}(\tau) = E_0^{'}(-\tau) = E_0(\tau-t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$.

$$\int_{-t_1}^{t_1} E_0(\tau + t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c},t_1)\tau)d\tau = e^{-2\sigma t_1}\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$-e^{2\sigma t_1}[\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau + \int_{-t_1}^{t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau]$$

(45)

We can rearrange the terms in Eq. 45 as follows.

$$\int_{-t_1}^{t_1}[E_0(\tau + t_{2c})e^{-2\sigma\tau} + E_0(\tau - t_{2c})e^{2\sigma t_1}]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$= -2\sinh(2\sigma t_1)\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$

(46)

We can split the integral in Eq. 46 using $\int_{-t_1}^{t_1} = \int_{-t_1}^{0} + \int_0^{t_1}$ and substitute $\tau = -\tau$ in the first integral as follows. We use $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ and $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$, given that $E_0(\tau) = E_0(-\tau)$.

$$\int_{t_1}^{0}[E_0(-\tau + t_{2c})e^{2\sigma\tau} + E_0(-\tau - t_{2c})e^{2\sigma t_1}]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$+\int_0^{t_1}[E_0(\tau + t_{2c})e^{-2\sigma\tau} + E_0(\tau - t_{2c})e^{2\sigma t_1}]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$= -2\sinh(2\sigma t_1)\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$\int_{t_1}^{0}[E_0(\tau - t_{2c})e^{2\sigma\tau} + E_0(\tau + t_{2c})e^{2\sigma t_1}]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$+\int_0^{t_1}[E_0(\tau + t_{2c})e^{-2\sigma\tau} + E_0(\tau - t_{2c})e^{2\sigma t_1}]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$= -2\sinh(2\sigma t_1)\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$

(47)

Given that $\int_{t_1}^{0} = -\int_0^{t_1}$, we can simplify as follows.

$$\int_0^{t_1}[E_0(\tau + t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_1}) + E_0(\tau - t_{2c})(e^{2\sigma t_1} - e^{2\sigma\tau})]\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$= -2\sinh(2\sigma t_1)\int_{-\infty}^{-t_1} E_0(\tau - t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$

(48)

We substitute $\tau = -\tau$ in the right hand side of Eq. 48 as follows. We use $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$.

$$\int_0^{t_1} E_0(\tau + t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_1})\sin(\omega_z(t_{2c},t_1)\tau)d\tau + \int_0^{t_1} E_0(\tau - t_{2c})(e^{2\sigma t_1} - e^{2\sigma\tau})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$
$$= 2\sinh(2\sigma t_1)\int_{t_1}^{\infty} E_0(\tau + t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau$$

$$ \tag{49} $$

We split the integral on the right hand side in Eq. 49 as follows.

$$
\begin{aligned}
\int_0^{t_1} E_0(\tau+t_{2c})(e^{-2\sigma\tau}-e^{2\sigma t_1})\sin(\omega_z(t_{2c},t_1)\tau)d\tau + \int_0^{t_1} E_0(\tau-t_{2c})(e^{2\sigma t_1}-e^{2\sigma\tau})\sin(\omega_z(t_{2c},t_1)\tau)d\tau \\
= 2\sinh(2\sigma t_1)[\int_0^{\infty} E_0(\tau+t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau - \int_0^{t_1} E_0(\tau+t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau]
\end{aligned}
\tag{50}
$$

We consolidate the integrals with the term $\int_0^{t_1} E_0(\tau+t_{2c})$ on both sides of Eq. 50 as follows. We use $2\sinh(2\sigma t_1) = e^{2\sigma t_1} - e^{-2\sigma t_1}$.

$$
\begin{aligned}
\int_0^{t_1} E_0(\tau+t_{2c})(e^{-2\sigma\tau}-e^{-2\sigma t_1})\sin(\omega_z(t_{2c},t_1)\tau)d\tau + \int_0^{t_1} E_0(\tau-t_{2c})(e^{2\sigma t_1}-e^{2\sigma\tau})\sin(\omega_z(t_{2c},t_1)\tau)d\tau \\
= 2\sinh(2\sigma t_1)\int_0^{\infty} E_0(\tau+t_{2c})\sin(\omega_z(t_{2c},t_1)\tau)d\tau
\end{aligned}
\tag{51}
$$

In Section 2.5, we showed that $\omega_z(t_2,t_0) = \omega_z(t_0,t_2)$ and $\omega_z(t_0,t_2) = \omega_z(t_0,-t_2)$, for **every value** of t_2 and t_0. Hence we see that $\omega_z(t_1,t_{2c}) = \omega_z(t_1,-t_{2c})$ and $\omega_z(t_{2c},t_1) = \omega_z(-t_{2c},t_1)$ for our choice of $t_0 = t_1$ and $t_2 = t_{2c}$.

We substitute $\tau + t_{2c} = \tau'$ in the right hand side of Eq. 51 and then substitute $\tau' = \tau$ as follows.

$$
\begin{aligned}
\int_0^{t_1} E_0(\tau+t_{2c})(e^{-2\sigma\tau}-e^{-2\sigma t_1})\sin(\omega_2(t_{2c},t_1)\tau)d\tau + \int_0^{t_1} E_0(\tau-t_{2c})(e^{2\sigma t_1}-e^{2\sigma\tau})\sin(\omega_2(t_{2c},t_1)\tau)d\tau \\
= 2\sinh(2\sigma t_1)\int_{t_{2c}}^{\infty} E_0(\tau)\sin(\omega_2(t_{2c},t_1)(\tau-t_{2c}))d\tau \\
= 2\sinh(2\sigma t_1)[\cos(\omega_2(t_{2c},t_1)t_{2c})\int_{t_{2c}}^{\infty} E_0(\tau)\sin(\omega_2(t_{2c},t_1)\tau)d\tau - \sin(\omega_2(t_{2c},t_1)t_{2c})\int_{t_{2c}}^{\infty} E_0(\tau)\cos(\omega_2(t_{2c},t_1)\tau)d\tau]
\end{aligned}
\tag{52}
$$

In Eq. 52, given that $\omega_z(t_{2c},t_1)t_1 = \pi$ and $t_{2c} = Kt_1$ and hence $\omega_z(t_{2c},t_1)t_{2c} = K\pi$ and $\sin(\omega_2(t_{2c},t_1)t_{2c}) = 0$ and $\cos(\omega_2(t_{2c},t_1)t_{2c}) = -1$ for odd integer K. Hence we write Eq. 52 as follows.

$$
\begin{aligned}
\int_0^{t_1} E_0(\tau+t_{2c})(e^{-2\sigma\tau}-e^{-2\sigma t_1})\sin(\omega_2(t_{2c},t_1)\tau)d\tau + \int_0^{t_1} E_0(\tau-t_{2c})(e^{2\sigma t_1}-e^{2\sigma\tau})\sin(\omega_2(t_{2c},t_1)\tau)d\tau \\
= -2\sinh(2\sigma t_1)\int_{t_{2c}}^{\infty} E_0(\tau)\sin(\omega_2(t_{2c},t_1)\tau)d\tau
\end{aligned}
\tag{53}
$$

In Section 2.5, we showed that, for **every value** of t_{2c}, $\omega_2(t_{2c},t_0) = \omega_2(t_{2c},-t_0)$ and given that $\omega_z(t_{2c},t_0) = \omega_z(t_0,t_{2c})$, we see that $\omega_2(t_0,t_{2c}) = \omega_2(t_0,-t_{2c})$.

Hence we substitute t_{2c} by $-t_{2c}$ in Eq. 53 as follows. We use $\int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin(\omega_2(t_{2c},t_1)\tau)d\tau = 0$, because $E_0(\tau)\sin(\omega_2(t_{2c},t_1)\tau)$ is an **odd** function of variable τ, given that $E_0(\tau) = E_0(-\tau)$.

$$
\begin{aligned}
\int_0^{t_1} E_0(\tau - t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_1}) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau + \int_0^{t_1} E_0(\tau + t_{2c})(e^{2\sigma t_1} - e^{2\sigma\tau}) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau \\
= -2\sinh(2\sigma t_1) \int_{-t_{2c}}^{\infty} E_0(\tau) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau = -2\sinh(2\sigma t_1) [\int_{-t_{2c}}^{t_{2c}} E_0(\tau) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau \\
+ \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau] \\
= -2\sinh(2\sigma t_1) \int_{t_{2c}}^{\infty} E_0(\tau) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau
\end{aligned} \tag{54}
$$

Now we subtract Eq. 53 from Eq. 54 as follows.

$$
\begin{aligned}
\int_0^{t_1} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_1} - e^{2\sigma t_1} + e^{2\sigma\tau}) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau = 0 \\
2\int_0^{t_1} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_1)) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau = 0
\end{aligned} \tag{55}
$$

We can divide Eq. 55 by a factor -2 as follows.

$$
\int_0^{t_1} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma t_1) - \cosh(2\sigma\tau)) \sin(\omega_2(t_{2c}, t_1)\tau) d\tau = 0 \tag{56}
$$

In Eq. 56, given that $\omega_z(t_{2c}, t_1)t_1 = \pi$, as τ varies over the interval $[0, t_1]$, $\omega_z(t_{2c}, t_1)\tau = \frac{\pi\tau}{t_1}$ varies from $[0, \pi]$ and hence the sinusoidal function varies over a **half cycle** and is > 0, in the interval $0 < \tau < t_1$, for $t_1 > 0$.

In Eq. 56, we see that in the interval $\tau = [0, t_1]$, the integral on the left hand side is > 0 for $t_1 > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_1$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$, we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Section 5.4) . The term $(\cosh(2\sigma t_1) - \cosh(2\sigma\tau))$ is > 0 and the integrand is zero at $\tau = 0$ and $\tau = t_1$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 56. Hence this leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 56 is zero and **does not** lead to a contradiction. It should be noted that the results from Section 2.5 to Section 4 are valid only for $\sigma \neq 0$.

We have shown this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

5. Strictly decreasing $E_0(t)$ for $t \geq \frac{1}{8}$

It is well known that $E_0(t) = \Phi(t)$ is positive for $t > 0$ and its first derivative is negative for $t > 0$ and hence $E_0(t)$ is a **strictly decreasing** function for $t > 0$.(link and link) In this section, we derive the loose bound that $\frac{dE_0(t))}{dt} \leq 0$ for $t \geq \frac{1}{8}$.

Let us consider $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$. (link)

$$
\begin{aligned}
E_0(t) = \Phi(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} &= \sum_{n=1}^{\infty}[4\pi^2n^4e^{4t} - 6\pi n^2e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \\
E_0(t) &= \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[2\pi n^2 e^{4t} - 3e^{2t}] \\
\frac{dE_0(t))}{dt} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[8\pi n^2e^{4t} - 6e^{2t} &+ (2\pi n^2e^{4t} - 3e^{2t})(\frac{1}{2} - 2\pi n^2e^{2t})] \\
\frac{dE_0(t))}{dt} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[8\pi n^2e^{4t} - 6e^{2t} + (\pi n^2e^{4t} - \frac{3}{2}e^{2t} &- 4\pi^2n^4e^{6t} + 6\pi n^2e^{4t})] \\
\frac{dE_0(t))}{dt} &= \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}[-4\pi^2n^4e^{6t} + 15\pi n^2e^{4t} - \frac{15}{2}e^{2t}] \\
\frac{dE_0(t))}{dt} &= \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{2t}[-4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2}]
\end{aligned}
\tag{57}
$$

5.1. *Numerical results*

For $n = 1$ and $t = 0$, the term $S_1 = -4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2} = -4\pi^2 + 15\pi - \frac{15}{2} = 0.14547$ and the summand in Eq. 57 is **positive** for $n = 1$.

For $n = 1$ and $t = 0.0025$, the term $S_1 = -4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2} = -0.015$ and the summand in Eq. 57 is **negative** and for all $t \geq 0.0025$.

5.2. *Mathematical results*

For $n > 1$ and $t \geq 0$, the term $S_1 = -4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2}$ and the summand in Eq. 57 is **negative**.

For $n = 2, t = 0$, the term $S_1 = -4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2} = -4\pi^2 * 16 + 15\pi * 4 - \frac{15}{2} = 4\pi(15 - 16\pi) - \frac{15}{2} < 0$ because $(15 - 16\pi) < 0$ and $\pi > 3$. Similar arguments for $n > 1$ and $t \geq 0$.

We can show that for $n = 1$ and $t > \frac{1}{8}$ (loose bound), the summand S_1 in Eq. 57 is **negative** as follows.

$$
\begin{aligned}
S_1 = -4\pi^2n^4e^{4t} + 15\pi n^2e^{2t} - \frac{15}{2} &= -\pi n^2e^{2t}(4\pi n^2e^{2t} - 15) - \frac{15}{2} \\
S_2 = 4\pi n^2e^{2t} - 15 &\geq 4\pi n^2(1 + 2t) - 15 = 4\pi n^2 - 15 + 8\pi n^2 t \\
n = 1, \quad S_2 \geq 4\pi + 8\pi t - 15 > 0 \quad if \quad 8\pi t > 15 - 4\pi, \quad t &> \frac{(15 - 4\pi)}{8\pi}
\end{aligned}
\tag{58}
$$

We see that the term $S_2 > 0$ if $t > \frac{(15-4\pi)}{8\pi} = t_m$ and hence the summand S_1 in Eq. 58 is **negative**.

We can get a **loose bound** for $t_m = \frac{(15-4\pi)}{8\pi} = \frac{15}{8\pi} - \frac{1}{2}$ as follows. We see that $\pi > 3$, hence the **maximum value** of t_m is given by $\frac{5}{8} - \frac{4}{8} = \frac{1}{8}$. Hence $\frac{dE_0(t))}{dt} \leq 0$ for $t \geq \frac{1}{8}$.

5.3. *Minimum value of* $E_0(t)$

In this section, it is shown that the $E_0(t) \geq \frac{1}{5} = E_{min}$ in the interval $-t_d \leq t \leq t_d$ where $t_d = \frac{1}{8}$ and E_{min} is the **minimum** value of $E_0(t)$ in that interval.

$$E_0(t) = \Phi(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} = \sum_{n=1}^{\infty} 2\pi n^2 e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}}[2\pi n^2 e^{2t} - 3] \tag{59}$$

We want to find the **minimum** value of $E_0(t)$ in the interval $-t_d \leq t \leq t_d$, where $t_d = \frac{1}{8}$. We set $n = 1$ and compute $E_0(t_d, n)$ at $n = 1$.

$$E_0(t_d, 1) = 2\pi e^{-\pi e^{2*\frac{1}{8}}} e^{\frac{5}{2*8}}[2\pi e^{2*\frac{1}{8}} - 3] = 2\pi e^{-\pi e^{\frac{1}{4}}} e^{\frac{5}{16}}[2\pi e^{\frac{1}{4}} - 3] \tag{60}$$

Given that $\frac{5}{16} > \frac{4}{16} = \frac{1}{4}$ and $\pi > 3$ and $e^{\frac{1}{4}} > 2^{\frac{1}{4}} > 1$, we see that $2\pi e^{\frac{1}{4}} - 3 > 2\pi - 3 > 3$ and $e^{-\pi} > 3^{-4}$, we can write as follows.

$$\begin{aligned} E_0(t_d, 1) > 6\pi e^{-\pi} > 6\pi 3^{-\pi} > 6\pi 3^{-4} > \frac{6\pi}{81} \\ > \frac{6*3}{81} > \frac{6}{27} > \frac{6}{30} > \frac{1}{5} \end{aligned} \tag{61}$$

Hence we have shown that $E_0(t_d, 1) > \frac{1}{5}$, where $t_d = \frac{1}{8}$.

We set $n = 1$ and at $t = 0$, we get $E_0(t, n) = E_0(0, 1) = 2\pi e^{-\pi}[2\pi - 3] > 6\pi e^{-\pi} > \frac{1}{5}$.

The **minimum** value of $E_0(t, n)$ in the interval $-t_d \leq t \leq t_d$, for $n = 1$ is given by $2\pi e^{-\pi e^{2*t_d}}[2\pi - 3] > \frac{1}{5}$, using procedure above. Hence we see that in the interval $-t_d \leq t \leq t_d$, $E_0(t, n) = E_0(t, 1) > \frac{1}{5}$.

For $n > 1$, $E_0(t, n) > 0$. Hence we see that $E_0(t) \geq \frac{1}{5}$ in the interval $-t_d \leq t \leq t_d$.

Hence we have shown that $E_0(t) \geq \frac{1}{5} = E_{min}$ in the interval $-t_d \leq t \leq t_d$ where $t_d = \frac{1}{8}$.

5.4. *Result* $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

In this section,it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t \leq t_1$ and $t_{2c} = Kt_1$ in Eq. 56, where $t_1 = \frac{1}{8}$.

In Section 5, we showed that $E_0(t)$ is a **strictly decreasing** function for $t \geq t_d = \frac{1}{8}$. In 5.3, we showed that the **minimum** value $E_{min} = \frac{1}{5}$ in the interval $-t_d \leq t \leq t_d$ where $t_d = \frac{1}{8}$ and $t_{2c} > t_d$ is chosen such that $E_0(t) < E_{min}$ for $t \geq t_{2c}$.

We see that $E_0(t)$ is an **even** function of variable t. We see that $E_0(t + t_{2c}) < E_{min} = \frac{1}{5}$ in the interval $t \geq 0$ by our **specific** choice of t_{2c}.

Given that t_{2c} is chosen such that $E_0(t) < E_{min}$ for $t \geq t_{2c}$, we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$ in the interval $0 < t \leq 2t_{2c}$. Further, for $t > 2t_{2c}$, we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$ given that $E_0(t)$ is a **strictly decreasing** function for $t \geq t_d = \frac{1}{8}$.

Given that $E_0(t)$ is a **strictly decreasing** function for $t \geq \frac{1}{8}$ and $E_0(t)$ is an **even** function of variable t, and $t_{2c} = Kt_1 > t_d$ for positive integer K, is chosen such that $E_0(t) < E_{min}$ for $t \geq t_{2c}$, we see that, in the interval $0 < t \leq t_1$, $E_0(t+t_{2c}) = E_0(t+Kt_1)$ ranges from $E_0(Kt_1)$ to $E_0((K+1)t_1)$, which is **less than** $E_0(t-t_{2c}) = E_0(t-Kt_1)$ which ranges from $E_0(-Kt_1)$ to $E_0((1-K)t_1)$ respectively. Hence we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$, in the interval $0 < t \leq t_1$. At $t = 0$, $E_0(t-t_{2c}) = E_0(t+t_{2c})$.

Hence $E_0(t-t_{2c}) - E_0(t+t_{2c}) > 0$ for $0 < t \leq t_1$ in Eq. 56.

5.5. ***Algorithm to find*** $\omega_z(t_{2c}, t_1)t_1 = \pi$

Given $\omega_z(t_2, t_0)$ is a **continuous** function of both t_0 and t_2, we can **make sure** that $\omega_z(t_{2c}, t_1)t_1 = \pi$, by finding a **suitable** value of $t_0 = t_1$ and $t_2 = t_{2c} = Kt_1$, where K is a positive integer, **such that** $E_0(t) < E_{min}$ for $t \geq t_{2c}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0, for every value of t_2, and t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function as well. Given that $0 < \omega_z(t_2, t_0) < \infty$, as t_0 is increased from zero to ∞, we see that $\omega_z(Kt_1, t_1)t_1$ increases from zero towards ∞ in a continuous manner and will **certainly pass through** π.

In Section 2.1, $\omega_z(t_0)$ is shown to be **finite** for all $|t_0| \leq \infty$. Hence we can see that $\omega_z(t_2, t_0)$ is **finite** for all $|t_0| \leq \infty$, for **every value** of t_2.

- Let ω_{max} be the **maximum** value of $\omega_z(t_2, t_0)$ for all t_0, t_2. We start with $t_0 = 0$ and increase t_0 to t_{00} such that $\omega_{max} t_{00} = \pi$. We see that $\omega_z(t_2, t_{00})t_{00} \leq \pi$ for all t_2.

- In Section 5, it is shown that $E_0(t)$ is **strictly decreasing** for $t \geq t_d = \frac{1}{8}$ and that the **minimum** value $Min(E_0(t)) = \frac{1}{5} = E_{min}$ in the interval $-t_d \leq t \leq t_d$.

- We **set** K such that $t_{20} = Kt_{00}$, where K is a positive integer, **such that** $E_0(t) < E_{min}$ for $t \geq t_{20}$. For this choice of K, we see that $\omega_z(Kt_{00}, t_{00})t_{00} \leq \pi$.

- If $\omega_z(Kt_{00}, t_{00})t_{00} = \pi$, then we set $t_0 = t_1 = t_{00}$ and $t_2 = t_{2c} = Kt_{00}$ and exit.

- If $\omega_z(Kt_{00}, t_{00})t_{00} < \pi$, then we increase t_0 from t_{00} to t_{01} such that $\omega_z(Kt_{01}, t_{01})t_{01} = \pi$ for the same choice of K. Given $\omega_z(t_2, t_0)$ is a **continuous** function of both t_0 and t_2 and given that $0 < \omega_z(t_2, t_0) < \infty$, as t_0 is increased from zero to ∞, we see that $\omega_z(Kt_0, t_0)t_0$ increases from zero towards ∞ in a continuous manner and will **certainly pass through** π. We set $t_0 = t_1 = t_{01}$ and $t_2 = t_{2c} = Kt_{01}$ and exit.

- Thus we have **ensured** that $\omega_z(t_{2c}, t_1)t_1 = \pi$ and $\omega_z(t_{2c}, t_1)t_{2c} = K\pi$.

6. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s,a) = \sum\limits_{m=0}^{\infty}\frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied[6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty}\frac{1}{m^s}$ diverges for real part of $s \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\displaystyle\sum_{n=1}^{\infty}e^{-\pi n^2x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty}e^{-\pi\frac{n^2}{x}})$, where $x > 0$ is real and then derive $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$ (Appendix E). In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function. These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

If the proof presented in this paper is internally consistent and does not have mistakes and gaps, then it should be considered correct, **regardless** of whether it contradicts any previously known external theorems, because it is possible that those previously known external theorems may be incorrect.

References

[1] Bernhard Riemann, On the Number of Prime Numbers less than a Given Quantity.(Ueber die Anzahl der Primzahlen untereiner gegebenen Grosse.) Monatsberichte der Berliner Akademie,November 1859. (Link to Riemann's 1859 paper)

[2] Hardy, G.H., Littlewood, J.E. The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift volume 10, pp.283 to 317 (1921).

[3] E. C. Titchmarsh, The Theory of the Riemann Zeta Function. (1986) pp.254 to 255

[4] Fern Ellison and William J. Ellison, Prime Numbers (1985).pp147 to 152

[5] J. Brian Conrey, The Riemann Hypothesis (2003). (Link to Brian Conrey's 2003 article)

[6] Mathworld article on Hurwitz Zeta functions. (Link)

[7] Wikipedia article on Dirichlet L-functions. (Link)

Appendix A. Derivation of $E_p(t)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}$ (link). This is re-derived in Appendix E.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 \le |\sigma| < \frac{1}{2}$ is real.

$$
\begin{aligned}
\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma) \\
E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega
\end{aligned}
\tag{A.1}
$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$
E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega'
\tag{A.2}
$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $y = [\infty, \infty - i\sigma]$, C_3 along the line $x = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $y = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

In **Appendix C.1**, we show that $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, for $0 \le |\sigma| < \frac{1}{2}$.

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \le y \le 0$, as per Riemann-Lebesgue Lemma (link), because $E_0(t)e^{yt}$ is a absolutely integrable function in the interval $-\infty \le t \le \infty$. Hence the integral in Eq. A.2 **vanishes** along the contours C_2 and C_4. Using Cauchy's Integral theroem, we can write Eq. A.2 as follows.

$$
\begin{aligned}
E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \\
E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}
\end{aligned}
\tag{A.3}
$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix B.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and we interchange the order of integration in equations below using Fubini's theorem (link).

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt \\ F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega' \\ F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega' \end{aligned} \tag{B.2}$$

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. B.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{B.3}$$

Appendix B.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{B.4}$$

Appendix B.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega) + G(-\omega)] = G_R(\omega) \end{aligned} \tag{B.5}$$

Appendix B.4. *Odd part of g(t) corresponds to imaginary part of Fourier transform* $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{1}{2}[G(\omega) - G(-\omega)] = iG_I(\omega) \tag{B.6}$$

Appendix C. Properties of Fourier Transforms Part 2

Appendix C.1. $E_p(t), h(t), g(t)$ *are absolutely integrable functions and their Fourier Transforms are finite.*

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$.

As $t \to \infty$, $E_p(t)$ goes to zero, due to the term $e^{-\pi n^2 e^{2t}}$. As $t \to -\infty$, $E_p(t)$ goes to zero, because for every value of n, the term $e^{-\pi n^2 e^{2t}} e^{\frac{5t}{2}} e^{-\sigma t}$ goes to zero, for $0 \le |\sigma| < \frac{1}{2}$. Hence $E_p(t) = E_0(t)e^{-\sigma t} = 0$ at $t = \pm\infty$ and we showed that $E_p(t) > 0$ for all $-\infty < t < \infty$. Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \ne 0$.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \le \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \ge 0$ for all $|t| \le \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence $g(t) = E_0(t)e^{-2\sigma t}u(-t) + E_0(t)u(t)$ is an absolutely **integrable function** and $\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} g(t)dt$ is finite and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Appendix C.2. *Convolution integral convergence*

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose **first derivative is discontinuous** at $t = 0$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $h(t), g(t)$ are absolutely integrable functions and their Fourier Transforms are finite as shown in Appendix C.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix C.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N-1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_N(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N-1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$, as $|\omega| \to \infty$ and it should have a fall-off rate **at least** of the order of $\omega^A e^{-B|\omega|}$ as $|\omega| \to \infty$, where $A, B > 0$ are real.

Appendix C.4. ***Payley-Weiner theorem and Exponential Fall off rate of analytic functions.***

We know that Payley-Weiner theorem relates analytic functions and exponential decay rate of their Fourier transforms (link). Using similar arguments, we will show that the functions $E_0(t), E_p(t)$ and $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$ for $0 < \sigma < \frac{1}{2}$.

We know that the order of Riemann's Xi function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = \Xi(\omega)$ is given by $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ where A is a constant[3] (link). Hence both $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega) = E_{0\omega}(\omega - i\sigma)$ have **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$ and they are absolutely integrable and Fourier transformable, given that they are derived from an entire function $\xi(s)$.

Given that $\xi(s)$ is an entire function in the s-plane, we see that $E_{0\omega}(\omega)$ and $E_{p\omega}(\omega)$ are **analytic** functions which are infinitely differentiable which produce no discontinuities for all $|\omega| \leq \infty$ and $0 < \sigma < \frac{1}{2}$. Hence their respective **inverse Fourier transforms** $E_0(t), E_p(t)$ have fall-off rates faster than $\frac{1}{t^M}$ as $M \to \infty$, as $|t| \to \infty$ (Appendix C.3) and hence it should have **exponential fall-off** rates as $|t| \to \infty$.

We can use similar arguments to show that $x(t) = E_0(t)e^{-2\sigma t}$ and $\frac{d^{2r}x(t)}{dt^{2r}}$ have fall-off rates **at least** $\frac{1}{t^2}$ as $|t| \to \infty$, because their Fourier transforms are **analytic** functions for all $|\omega| \leq \infty$ with **exponential fall-off** rate $O(\omega^A e^{-\frac{|\omega|\pi}{4}})$ as $|\omega| \to \infty$.

Appendix D. $\omega_z(t_0)$ is a continuous function around $t_0 = 0$

This result is shown as follows.

• $G_R(\omega) = G_R(\omega, t_0)$ in Eq. 17 is copied below, which is a **continuous** function of ω which is differentiable **at least** once with respect to ω. (Eq. D.2 and Appendix D.3)

$$G_R(\omega) = G_R(\omega, t_0) = \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0n}(t - t_0)] \cos(\omega t) dt \tag{D.1}$$

Given that $E_0(t) \geq 0$ for $|t_0| \leq \infty$ (Appendix C.1), we see that $G_R(\omega) > 0$ at $\omega = 0$. **Set** $t_0 = 0$ and $G_R(\omega, t_0)$ passes through its **first zero** at $\omega = \omega_z(t_0) = \omega_z(0)$. In the rest of this section, we consider the **interval** $[-\delta t_0, \delta t_0]$ around $t_0 = 0$, in $\omega_z(t_0)$. There are 3 possibilities.

Case 1: $G_R(\omega) < 0$ for $\omega = \omega_z(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(0) - dw$ for infinitesimal dw (example plot)

In this case, we will show in Appendix D.1 that $\omega_z(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_0) = \omega_z(0)$.

Case 2: $G_R(\omega) > 0$ for $\omega = \omega_z(0) + dw$, $G_R(\omega) > 0$ for $\omega = \omega_z(0) - dw$ (example plot)

In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(0) + dw$.

$$\frac{dG_R(\omega)}{d\omega} = -\int_{-\infty}^{0} t[E_0(t + t_0)e^{-2\sigma t} + E_{0n}(t - t_0)] \sin(\omega t) dt \tag{D.2}$$

In this case, we will show Appendix D.2 that $\omega_z(t_0)$ is a continuous function of t_0 in the interval $[-\delta t_0, \delta t_0]$, in the neighborhood around the first zero crossing at $\omega = \omega_z(t_0) = \omega_z(0)$.

Case 3: $G_R(\omega) = 0$ for $\omega = \omega_z(0)$ and $\omega = \omega_z(0) + dw$.

This is **not** possible because $G_R(\omega, t_0)$ in Eq. D.1 is an **analytic** function and infinitely differentiable with respect to ω(Appendix D.3). We know that analytic functions have **isolated** zeros. (link). Hence we cannot have $G_R(\omega) = 0$ for $\omega = \omega_z(0)$ and $\omega = \omega_z(0) + dw$ as $dw \to 0$.

Appendix D.1. ***Case 1:*** $G_R(\omega) < 0$ ***for*** $\omega = \omega_z(0) + dw$***,*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_z(0) - dw$

• Consider the **segment** S in $G_R(\omega, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega,t_0)}{d\omega} < 0$. (Segment S is the portion between the green lines in example plot)

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.1 is a **continuous** function of ω, for **each** value of t_0. Hence $G_R(\omega, t_0 - \delta t_0)$ and $G_R(\omega, t_0 + \delta t_0)$ are **continuous** functions of ω, which are differentiable **at least** once, and $G_R(\omega, t_0 \pm \delta t_0)$ tends to $G_R(\omega, t_0)$, as infinitesimal $\delta t_0 \to 0$.

$$G_R(\omega, t_0) = \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0n}(t - t_0)] \cos(\omega t) dt$$

$$G_R(\omega, t_0 + \delta t_0) = \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0n}(t - t_0 - \delta t_0)] \cos(\omega t) dt$$

(D.3)

• In the **segment** S, $G_R(\omega, t_0)$ in Eq. D.3 is a **continuous** function of ω, for **each** value of t_0 and $\frac{dG_R(\omega,t_0)}{d\omega} < 0$ in the neighborhood around the **first zero crossing**. If we fix the X-coordinate ω, $G_R(\omega, t_0)$ is a **continuous** function of t_0, for **each** value of ω. Hence, for **each** value of ω, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0)$ moves towards $G_R(\omega, t_0 + \delta t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Every point in the segment S, moves continuously, as we change t_0 by an infinitesimal δt_0.

This also applies to the first **zero crossing** in $G_R(\omega, t_0)$ in the segment S, which corresponds to $\omega_z(t_0) = \omega_z(0)$ at $t_0 = 0$ where $G_R(\omega, t_0) = 0$ in Eq. D.3. The zero crossing moves continuously, as we change t_0 by an infinitesimal δt_0. This is explained below.

• **Explanation:** This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega, t_0)$ with zero crossing at point P_0, **Green** plot corresponds to $G_R(\omega, t_0 + \delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to $G_R(\omega, t_0 - \delta t_0)$ with zero crossing at point P_{21}.

We **define** the **point** P_{12} in $G_R(\omega, t_0 + \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(0)$. We **define** the **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(0)$.

We **define** the **point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

As we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 + \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{12} in $G_R(\omega, t_0 + \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_z(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{11} in $G_R(\omega, t_0 + \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

Similarly, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_0 - \delta t_0)$ in Eq. D.4 moves towards $G_R(\omega, t_0)$ in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega, t_0 - \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(0)$, moves towards corresponding point P_0 in $G_R(\omega, t_0)$, for the **same** $\omega = \omega_z(0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega, t_0 - \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

$$\begin{aligned}
G_R(\omega, t_0) &= \int_{-\infty}^{0} [E_0(t + t_0)e^{-2\sigma t} + E_{0n}(t - t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 + \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 + \delta t_0)e^{-2\sigma t} + E_{0n}(t - t_0 - \delta t_0)] \cos(\omega t) dt \\
G_R(\omega, t_0 - \delta t_0) &= \int_{-\infty}^{0} [E_0(t + t_0 - \delta t_0)e^{-2\sigma t} + E_{0n}(t - t_0 + \delta t_0)] \cos(\omega t) dt \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 + \delta t_0) &= G_R(\omega, t_0) \\
\lim_{\delta t_0 \to 0} G_R(\omega, t_0 - \delta t_0) &= G_R(\omega, t_0)
\end{aligned} \tag{D.4}$$

• Hence in the **segment** S, $\omega_z(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_0) = \omega_z(0)$ at $t_0 = 0$.

$$G_R(\omega_z(t_0),t_0) = \int_{-\infty}^{0} [E_0(t+t_0)e^{-2\sigma t} + E_{0n}(t-t_0)]\cos(\omega_z(t_0)t)dt = 0$$

$$G_R(\omega_z(t_0+\delta t_0),t_0+\delta t_0) = \int_{-\infty}^{0} [E_0(t+t_0+\delta t_0)e^{-2\sigma t} + E_{0n}(t-t_0-\delta t_0)]\cos((\omega_z(t_0+\delta t_0)t)dt = 0 \tag{D.5}$$

Appendix D.2. ***Case 2:*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_z(0) + dw$***,*** $G_R(\omega) > 0$ ***for*** $\omega = \omega_z(0) - dw$

- In this case, $\frac{dG_R(\omega)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_0)$ because $\frac{dG_R(\omega)}{d\omega} < 0$ at $\omega = \omega_z(t_0) - dw$ and $\frac{dG_R(\omega)}{d\omega} > 0$ at $\omega = \omega_z(t_0) + dw$.

- Consider the **segment** S' in $\frac{dG_R(\omega,t_0)}{d\omega}$ in the neighborhood around the first zero crossing where $\frac{d^2G_R(\omega,t_0)}{d\omega^2} > 0$. (Segment S' is the portion between the green lines in example plot) In this segment S', $\frac{dG_R(\omega,t_0)}{d\omega}$ is a **continuous** function of ω which is differentiable **at least** once.(Appendix D.3)

- In the **segment** S', $\frac{dG_R(\omega,t_0)}{d\omega} = 0$ at the **same** $\omega = \omega_z(t_0)$. The arguments in Appendix D.1 can be applied here, with $G_R(\omega,t_0)$ replaced by $\frac{dG_R(\omega,t_0)}{d\omega}$.

Hence $\omega_z(t_0)$ is a **continuous** function of t_0 in the neighborhood $[-\delta t_0, \delta t_0]$ around the first zero crossing at $\omega = \omega_z(t_0) = \omega_z(0)$ at $t_0 = 0$ in the **segment** S'.

Appendix D.3. ***Integral convergence in*** $\frac{dG_R(\omega)}{d\omega}$

It is shown in Appendix C.4 that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ have exponential fall-off rates as $|t| \to \infty$ and hence are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$. Hence the integrand $A_r(t) = \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0n}(t-t_0)]\sin(\omega t)$ in Eq. D.2 copied below, is an absolutely **integrable function** and $\int_{-\infty}^{0} |A_r(t)|dt = \int_{-\infty}^{0} \frac{|t^r|}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0n}(t-t_0)]dt$ is **finite**, for $r = 0, 1, ..$, given the **exponential** fall-off rate of $E_0(t)e^{-2\sigma t}$ and $E_0(t)$.

$$\begin{aligned}\frac{1}{!(r)}\frac{d^rG_R(\omega)}{d\omega^r} &= (-1)^{\frac{r+1}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0n}(t-t_0)]\sin(\omega t)dt, \quad r = odd \\ \frac{1}{!(r)}\frac{d^rG_R(\omega)}{d\omega^r} &= (-1)^{\frac{r}{2}} \int_{-\infty}^{0} \frac{t^r}{!(r)}[E_0(t+t_0)e^{-2\sigma t} + E_{0n}(t-t_0)]\cos(\omega t)dt, \quad r = even\end{aligned} \tag{D.6}$$

Appendix E. Derivation of entire function $\xi(s)$

In this section, we will re-derive Riemann's Xi function $\xi(s)$ and the inverse Fourier Transform of $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$ and show the result $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$.

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and re-derive the steps below[4] (link). We start with the gamma function $\Gamma(s) = \int_0^{\infty} y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and derive as follows.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} y^{\frac{s}{2}-1}e^{-y}dy$$

$$\Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

(E.1)

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \tag{E.2}$$

For real part of s (σ') greater than 1, we can use theorem of dominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $\sum_{n=1}^{\infty} \int_0^{\infty} |x^{\frac{s}{2}-1} e^{-\pi n^2 x}| dx = \Gamma(\frac{\sigma'}{2})\pi^{-\frac{\sigma'}{2}}\zeta(\sigma')$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \tag{E.3}$$

For real part of s less than or equal to 1, $\zeta(s)$ **diverges**. Hence we do the following. In Eq. E.3, first we consider real part of s greater than 1 and we divide the range of integration into two parts: $(0, 1]$ and $[1, \infty)$ and make the substitution $x \to \frac{1}{x}$ in the first interval $(0, 1]$. We use **the well known theorem** $1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $x > 0$ is real.[4]

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} \frac{x^{-(\frac{s}{2}-1)}}{x^2} \frac{(1 + 2w(x))\sqrt{x} - 1)}{2} dx \tag{E.4}$$

Hence we can simplify Eq. E.4 as follows.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{\frac{-(s+1)}{2}} w(x) dx \tag{E.5}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}[1 + s(s-1)\int_1^{\infty}(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})w(x)\frac{dx}{x}] \tag{E.6}$$

We see that $\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane and hence we get an analytic continuation of $\xi(s)$ over the entire complex plane. We see that $\xi(s) = \xi(1-s)$ [4].

Appendix E.1. ***Derivation of*** $E_p(t)$ ***and*** $E_0(t)$

Given that $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in Eq. E.6 and evaluate at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}[1+2(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}(e^{\frac{t}{2}}e^{\sigma t}e^{i\omega t}+e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t})dt] \tag{E.7}$$

We can substitute $t = -t$ in the first term in above integral and simplify above equation as follows.

$$\begin{aligned}\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))[\int_{-\infty}^{0}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}e^{-i\omega t}dt \\ +\int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt]\end{aligned} \tag{E.8}$$

We can write this as follows.

$$\xi(\frac{1}{2}+\sigma+i\omega) = \frac{1}{2}+(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}[\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}e^{-i\omega t}dt \tag{E.9}$$

We define $A(t) = [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t}$ and get the **inverse Fourier transform** of $\xi(\frac{1}{2}+\sigma+i\omega)$ in above equation given by $E_p(t)$ as follows. We use dirac delta function $\delta(t)$.

$$\begin{aligned}E_p(t) &= \frac{1}{2}\delta(t)+(-\frac{1}{4}+\sigma^2)A(t)+2\sigma\frac{dA(t)}{dt}+\frac{d^2A(t)}{dt^2} \\ A(t) &= [\sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}u(t)]e^{-\sigma t} \\ \frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t}]u(-t)+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{2}-\sigma-2\pi n^2 e^{2t}]u(t) \\ \frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[-4\pi n^2 e^{-2t}+(-\frac{1}{2}-\sigma+2\pi n^2 e^{-2t})^2]u(-t) \\ &+\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[-4\pi n^2 e^{2t}+(\frac{1}{2}-\sigma-2\pi n^2 e^{2t})^2]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned} \tag{E.10}$$

We can simplify above equation as follows.

$$\begin{aligned}\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}}e^{\frac{-t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{-4t}-6\pi n^2 e^{-2t}-4\sigma\pi n^2 e^{-2t}]u(-t) \\ &\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t}[\frac{1}{4}+\sigma^2-\sigma+4\pi^2 n^4 e^{4t}-6\pi n^2 e^{2t}+4\sigma\pi n^2 e^{2t}]u(t)+\delta(t)[\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2)]\end{aligned} \tag{E.11}$$

We use the fact that $F(x) = 1+2w(x) = \frac{1}{\sqrt{x}}(1+2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$. We see that $\sum_{n=1}^{\infty} e^{-\pi n^2}(1-4\pi n^2) = -\frac{1}{2}$ (Appendix E.2) and hence **dirac delta terms cancel each other** in equation below.

$$E_p(t) = \frac{1}{2}\delta(t) + (-\frac{1}{4} + \sigma^2)A(t) + 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}$$

$$E_p(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(\frac{1}{2} - \sigma - 2\pi n^2 e^{2t}) + \frac{1}{4} + \sigma^2 - \sigma + 4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} + 4\sigma\pi n^2 e^{2t}]u(t)$$
$$+ \sum_{n=1}^{\infty} e^{-\pi n^2 e^{-2t}} e^{\frac{-t}{2}} e^{-\sigma t}[-\frac{1}{4} + \sigma^2 + 2\sigma(-\frac{1}{2} - \sigma + 2\pi n^2 e^{-2t})$$
$$+ \frac{1}{4} + \sigma^2 + \sigma + 4\pi^2 n^4 e^{-4t} - 6\pi n^2 e^{-2t} - 4\sigma\pi n^2 e^{-2t}]u(-t)$$
$$\text{(E.12)}$$

We can simplify above equation as follows.

$$E_p(t) = [E_0(-t)u(-t) + E_0(t)u(t)]e^{-\sigma t}$$
$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$\text{(E.13)}$$

We use the fact that $E_0(t) = E_0(-t)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ and we arrive at the desired result for $E_p(t)$ as follows.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$
$$\text{(E.14)}$$

Appendix E.2. ***Derivation of*** $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$

In this section, we derive $\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}$. We use the fact that $F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$, where $w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and $x > 0$ is real[4], and we take the first derivative of $F(x)$ and evaluate it at $x = 1$.

$$F(x) = 1 + 2w(x) = \frac{1}{\sqrt{x}}(1 + 2w(\frac{1}{x}))$$
$$F(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})$$
$$\frac{dF(x)}{dx} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2 \frac{1}{x}}(\frac{1}{x^2}) + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}})(\frac{-1}{2})\frac{1}{x^{\frac{3}{2}}}$$
$$\text{(E.15)}$$

We evaluate the above equation at $x = 1$ and we simplify as follows.

$$
\begin{aligned}
[\frac{dF(x)}{dx}]_{x=1} = 2\sum_{n=1}^{\infty}(-\pi n^2)e^{-\pi n^2} = \sum_{n=1}^{\infty}(2\pi n^2)e^{-\pi n^2} + (1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2})(\frac{-1}{2}) \\
\sum_{n=1}^{\infty} e^{-\pi n^2}(1 - 4\pi n^2) = -\frac{1}{2}
\end{aligned}
\tag{E.16}
$$