On a new method towards proof of Riemann's Hypothesis

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Abstract

It is well known that a real two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, does not have zeros in its Fourier Transform, where $u(t)$ is Heaviside unit step function and $a, b > 0$ are real. We consider the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, where σ, ω are real and $-\infty \leq \omega \leq \infty$ and compute its inverse Fourier transform given by $E_p(t)$, which is expressed as an **infinite summation of two-sided decaying exponential functions** using Taylor series expansion.

We use a new method and show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2} + \sigma + i\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line and prove Riemann's hypothesis. We also use the new method **without** using Taylor series expansion and prove Riemann's hypothesis.

More importantly, the new method **does not affect** the zeros on the critical line and **does not** contradict Riemann Hypothesis and the existence of zeros on the critical line. It is shown that the new method is **not** applicable to Hurwitz zeta function and related functions which have their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$.

If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$ is the Gamma function.[5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function $\xi(s)$ also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ has no zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, critical strip $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

Preprint submitted to Zenodo.org *July 17, 2021*

In Section 2, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ and compute inverse Fourier transform of $E_{p\omega}(\omega)$ given by $E_p(t)$, which is expressed as an **infinite summation of two-sided decaying exponential functions** using Taylor series expansion and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

In Section 3, we prove Riemann's hypothesis **without** using Taylor series representation of $E_p(t)$ and show that its Fourier transform $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$.

In Section 4, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions which have their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$, because the new method requires $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and hence $E_0(t) = E_0(-t)$ is a real and even function of t and this condition is satisfied for Riemann's Zeta function.

In Appendix A to Appendix I, well known results which are used in this paper are re-derived.

We present an **outline** of the new method below. (short video presentation)

1.1. *Step 1: Inverse Fourier Transform of* $\xi(\frac{1}{2}+i\omega)$

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$, where $-\infty \le \omega \le \infty$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, where ω, t are real, as follows[3]. (Page 5 in Brian Conrey's 2003 article) This is re-derived in Appendix H.

$$E_0(t) = 2\sum_{n=1}^{\infty}[2n^4\pi^2 e^{\frac{9t}{2}} - 3n^2\pi e^{\frac{5t}{2}}]e^{-\pi n^2 e^{2t}} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}} \quad (1)$$

We see that $E_0(t) = E_0(-t)$ is a real and even function of t, given that $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ because $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at $s = \frac{1}{2}+i\omega$.

The inverse Fourier Transform of $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is given by the real function $E_p(t)$. We can write $E_p(t)$ as follows for $0 < |\sigma| < \frac{1}{2}$ and this is shown in detail in Appendix A.

$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} = f(e^{2t})e^{\frac{t}{2}}e^{-\sigma t} \quad (2)$$

We can see that $E_p(t)$ is an analytic function in the interval $|t| \le \infty$, given that the sum and product of exponential functions are analytic in the same interval and hence infinitely differentiable.

1.2. *Step 2: Taylor's series representation of* $E_p(t)$

We can substitute $z = e^{2t}$ in Eq. 2 and we can expand real analytic function $f(z)$ using Taylor series expansion around $z = 0$ as follows.

$$f(z) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 z^2 - 3\pi n^2 z]e^{-\pi n^2 z} = \sum_{n,k}(a_{nk}z^{(k+2)} - b_{nk}z^{k+1})$$

$$a_{nk} = 4\pi^2 n^4 d_{nk}, \quad b_{nk} = 6\pi n^2 d_{nk}, \quad d_{nk} = \frac{(-\pi n^2)^k}{!(k)}, \quad \sum_{n,k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}$$

(3)

Now we can substitute $z = e^{2t}$ in Eq. 3 and write the Taylor series expansion of $E_p(t)$ in Eq. 2 and use the shorthand notation as follows.

$$E_p(t) = [\sum_{n,k}(a_{nk}e^{(2k+\frac{9}{2})t} - b_{nk}e^{(2k+\frac{5}{2})t})]e^{-\sigma t} = \sum_{n,k,r} c_{nkr}e^{b_{kr}t}e^{-\sigma t}$$
$$\sum_{n,k,r} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{1}, \quad b_{kr} = (2k + \frac{5}{2} + 2r), \quad c_{nk1} = a_{nk}, \quad c_{nk0} = -b_{nk} \tag{4}$$

Given that $E_0(t) = E_0(-t)$, we can write $E_p(t) = E_0(t)e^{-\sigma t}$ as an **infinite summation** of two-sided decaying exponential functions as follows.

$$E_p(t) = [\sum_{n,k,r} c_{nkr}e^{b_{kr}t}u(-t) + \sum_{n,k,r} c_{nkr}e^{-b_{kr}t}u(t)]e^{-\sigma t} \tag{5}$$

In Appendix B, we show that we can also expand $f(z)$ using an alternate Taylor series expansion around $z = 1$.

1.3. Step 3: Two-sided decaying exponentials and zeros in their Fourier transform

We know that a real two-sided decaying exponential function $g_0(t) = e^{bt}u(-t) + e^{-at}u(t)$, where $u(t)$ is Heaviside unit step function and $a, b > 0$ and t are real, has Fourier Transform given by $G_0(\omega)$, where ω is real, as follows. (link)

$$G_0(\omega) = \int_{-\infty}^{\infty} g_0(t)e^{-i\omega t}dt = \frac{1}{b - i\omega} + \frac{1}{a + i\omega} = \frac{b + i\omega}{b^2 + \omega^2} + \frac{a - i\omega}{a^2 + \omega^2}$$
$$= [\frac{b}{b^2 + \omega^2} + \frac{a}{a^2 + \omega^2}] + i\omega[\frac{1}{b^2 + \omega^2} - \frac{1}{a^2 + \omega^2}] \tag{6}$$

We can see that the real part of $G_0(\omega)$ given by $\frac{b}{b^2+\omega^2} + \frac{a}{a^2+\omega^2}$ **does not have zeros** for any finite value of ω and hence $G_0(\omega)$ does not have zeros for any finite value of ω.

Given that the inverse Fourier Transform of Riemann Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ given by $E_p(t)$ is expressed as an **infinite summation of two-sided decaying exponential functions** in previous subsection, we will investigate if $E_{p\omega}(\omega)$ also does not have zeros for any finite real value of ω.

1.4. Step 4: On the zeros of a related function $G(\omega)$

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

In **Section 2.1**, we will show that the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ to satisfy Statement 1, where ω_1 is real and finite.

1.5. *Step 5: On the zeros of the function* $G_I(\omega)$

In **Section 2.2**, we compute the Fourier transform of the function $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_{odd}(\omega) = iG_I(\omega)$. We **require** $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, to satisfy Statement 1. Hence $S_0 = G_I(\omega_1) = 0$ and we will derive as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{7}$$

Using Taylor series representation of $E_p(t) = \sum_{n,k,r} c_{nkr}e^{b_{kr}t}e^{-\sigma t}$, and we use the fact that $E_0(t) = E_0(-t)$ and we will derive as follows.

$$S_0 = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{8}$$

1.6. *Step 6: Even order Derivatives of* $g(t)$

In **Section 2.3**, we consider the **even order derivative** of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ and compute the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and show results as follows. We will also show that **dirac delta functions vanish** in the computation of $g_{2r_{odd}}(t)$.

$$S_{2r} = \omega_1 \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)^{2r}\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^{2r}\frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{9}$$

1.7. *Step 7: New Function* $A(t_1)$

Next, we will form a new function $A(t_1) = \sum_{r=0}^{\infty} S_{2r}\frac{t_1^{2r}}{!(2r)} = 0$ for $-\infty \leq t_1 \leq \infty$ where t_1 is real and we can write

$$A(t_1) = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1} + e^{-(b_{kr}-2\sigma)t_1})}{2}\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)} + e^{-(b_{kr}t_1)})}{2}\frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{10}$$

We can write $A(t_1) = \frac{\omega_1}{2}[y(t_1) + y(-t_1)] = 0$ as follows. We know that $\omega_1 \neq 0$ and we can write

$$y(t_1) = \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1}}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)}}{(\omega_1^2 + b_{kr}^2)}] = -y(-t_1) = y_{odd}(t_1) \tag{11}$$

We can see that $y(t_1)$ is an **odd function** of variable t_1.

1.8. *Step 8: Final Step in the proof of theorem.*

We can evaluate the **odd** symmetry function $z_{odd}(t_1)$ as follows.

$$
\begin{aligned}
\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\
\sum_{n,k,r} c_{nkr}[e^{(b_{kr}-2\sigma)t_1} - e^{(b_{kr}t_1)}] &= z_{odd}(t_1) \\
\sum_{n,k,r} c_{nkr}e^{(b_{kr}t_1)}(e^{-2\sigma t_1} - 1) &= z_{odd}(t_1)
\end{aligned}
\tag{12}
$$

We know that $\sum_{n,k,r} c_{nkr}e^{(b_{kr}t_1)} = E_0(t_1)$ is an **even function** of variable t_1, hence we require $(e^{-2\sigma t_1}-1)$ to be an **odd function** of variable t_1, to satisfy Eq. 12, which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2}+\sigma+i\omega) = \xi(\frac{1}{2}-\sigma-i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence this proves Riemann hypothesis.

In **Section 3**, we will prove the same result, **without** using Taylor series expansion for $E_p(t)$.

2. Proof of Riemann's Hypothesis using Taylor Series Expansion of $E_p(t)$

Theorem 1: Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$, $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$, given that $E_0(t) = E_0(-t)$ is an even function of variable t.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2}+\sigma+i\omega) = \xi(\frac{1}{2}-\sigma-i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

The inverse Fourier Transform of the function $E_{p\omega}(\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}} > 0$ for all $0 \le t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$. We see that $E_p(t) = 0$ at $t = \pm\infty$ and its Fourier transform given by $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ does not have a zero at $\omega = 0$ and hence $\omega_0 \neq 0$.

2.1. ***On the zeros of a related function*** $G(\omega)$

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where

$h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can show that $E_p(t), h(t), g(t)$ are real L^1 integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), H(\omega), G(\omega)$ are finite for $|\omega| \leq \infty$ and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma. This is shown in detail in Appendix C.1.

If we take the Fourier transform of the equation $g(t)h(t) = E_p(t)$, we get $\frac{1}{2\pi}[G(\omega)*H(\omega)] = E_{p\omega}(\omega)$ as per convolution theorem, where $*$ denotes **convolution** operation given by $E_{p\omega}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ and $H(\omega) = [\frac{1}{\sigma-i\omega}+\frac{1}{\sigma+i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is the Fourier transform of the function $h(t)$ and $G(\omega) = G_R(\omega)+iG_I(\omega)$ is the Fourier transform of the function $g(t)$. This is shown in detail in Appendix I.1.

We can write $g(t) = g_{even}(t)+g_{odd}(t)$ where $g_{even}(t)$ is an even function and $g_{odd}(t)$ is an odd function of variable t. If Statement 1 is true, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$ where ω_1 is real and finite and can be different from ω_0 in general. We call this **Statement 2**.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_I(\omega)$ does not have at least one zero for some $\omega = \omega_1 \neq 0$, **then** the **imaginary part** of $E_{p\omega}(\omega)$ given by $E_I(\omega) = \frac{1}{2\pi}[G_I(\omega)*H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_I(\omega)$, **cannot** possibly have zeros for any non-zero finite value of ω, which goes against **Statement 1**. This is shown in detail in Lemma 1.

Lemma 1: If Riemann's Xi function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0 \neq 0$ where ω_0 is real and finite, then the **imaginary** part of the Fourier transform of the **odd function** $g_{odd}(t) = \frac{1}{2}[g(t)-g(-t)]$ given by $G_I(\omega)$ must have **at least one zero** at $\omega = \omega_1 \neq 0$, where ω_1 is real and finite, where $g(t)h(t) = E_p(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $0 < \sigma < \frac{1}{2}$.

Proof: If $E_{p\omega}(\omega)$ has a zero at finite $\omega = \omega_0 \neq 0$ to satisfy Statement 1, then its imaginary part given by $E_I(\omega)$ also has a zero at the same location $\omega = \omega_0 \neq 0$.

Let us consider the case where $G_I(\omega)$ **does not** have at least one zero for finite $\omega = \omega_1 \neq 0$ and show that $E_I(\omega)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Statement 1. Given that $H(\omega)$ is real, we can write the convolution theorem only for the imaginary parts as follows.

$$E_I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(\omega')H(\omega-\omega')d\omega' \tag{13}$$

We can show that the above integral converges for all $|\omega| \leq \infty$, given that $G(\omega)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t)$ and $h(t)$ are discontinuous at $t = 0$.(Appendix C.2)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 13 and we get

$$E_I(\omega) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' \tag{14}$$

We can write Eq. 14 as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}[\int_{-\infty}^{0} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega' + \int_{0}^{\infty} G_I(\omega')\frac{1}{(\sigma^2+(\omega-\omega')^2)}d\omega'] \tag{15}$$

We see that $G_I(-\omega) = -G_I(\omega)$ because $g(t)$ is a real function (Appendix I.2). We can substitute $\omega' = -\omega''$ in the first integral in Eq. 15 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$E_I(\omega) = \frac{\sigma}{\pi}\int_0^{\infty} G_I(\omega')[\frac{1}{(\sigma^2 + (\omega-\omega')^2)} - \frac{1}{(\sigma^2 + (\omega+\omega')^2)}]d\omega' \tag{16}$$

We can see that for $\omega' = 0$ and $\omega' = \infty$, the integrand in Eq. 16 is zero. For finite $\omega > 0$, and $0 < \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} - \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$.

Case 1: $G_I(\omega') > 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) > 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix I.2). Hence $E_I(\omega) < 0$ for all finite $\omega < 0$.

This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

Case 2: $G_I(\omega') < 0$ **for all finite** $\omega' > 0$

We see that $E_I(\omega) < 0$ for all finite $\omega > 0$. We see that $E_I(-\omega) = -E_I(\omega)$ because $E_p(t)$ is a real function (Appendix I.2). Hence $E_I(\omega) > 0$ for all finite $\omega < 0$.
This **contradicts** Statement 1 which requires $E_I(\omega)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_I(\omega')$ must have **at least one zero** at $\omega' = \omega_1 \neq 0$, where ω_1 is real and finite.

We have shown that, $G_I(\omega)$ must have **at least one zero** at finite $\omega = \omega_1 \neq 0$ to satisfy **Statement 1**. We call this **Statement 2**. We will investigate if Statement 2 leads to a contradiction for $0 < \sigma < \frac{1}{2}$.

2.2. ***On the zeros of the function*** $G_I(\omega)$

We take the Fourier transform of $g(t)$ and get $G(\omega)$ as follows.

$$g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{0} E_p(t)e^{-\sigma t}e^{-i\omega t}dt + \int_{0}^{\infty} E_p(t)e^{\sigma t}e^{-i\omega t}dt \tag{17}$$

We can substitute $t = -\tau$ in the second integral in Eq. 17 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows.

$$G(\omega) = \int_{-\infty}^{0} E_p(\tau)e^{-\sigma\tau}e^{-i\omega\tau}d\tau + \int_{-\infty}^{0} E_q(\tau)e^{-\sigma\tau}e^{i\omega\tau}d\tau = G_R(\omega) + iG_I(\omega) \tag{18}$$

Eq. 18 can be expanded as follows using Euler's formula $e^{i\omega\tau} = \cos(\omega\tau) + i\sin(\omega\tau)$ and comparing the **imaginary parts** of $G(\omega)$, we can write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_I(\omega) = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega\tau)d\tau$$

(19)

We require $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$, to satisfy **Statement 1** as shown in Section 2.1.

We can set $S_0 = G_I(\omega_1) = 0$ and write as follows.

$$S_0 = -\int_{-\infty}^{0} E_0(\tau)e^{-2\sigma\tau}\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} E_0(-\tau)\sin(\omega_1\tau)d\tau = 0 \tag{20}$$

We use Taylor series representation of $E_p(\tau) = \sum_{n,k,r} c_{nkr}e^{b_{kr}\tau}e^{-\sigma\tau}$, and we use the fact that $E_0(\tau) = E_0(-\tau)$. We can see that $b_{kr} = (2k + \frac{5}{2} + 2r) > 2\sigma$ for all k, r and $0 < \sigma < \frac{1}{2}$. We can interchange the order of integration and summation in Eq. 20 because for each term in Taylor series, integral in Eq. 20 converges.

We use the well known result $\int e^{a\tau}\sin(\omega_1\tau)d\tau = \frac{e^{a\tau}}{(\omega_1^2+a^2)}[a\sin(\omega_1\tau) - \omega_1\cos(\omega_1\tau)]$ in Eq. 20 and then evaluate the integral at $\tau = 0$ for $a = (b_{kr} - 2\sigma)$ in the first integral and $a = b_{kr}$ in the second integral.

We can see that the two integrals in Eq. 20 equal zero when evaluated at the lower limit of $\tau = -\infty$ because $b_{kr} - 2\sigma > 0$ for all k, r and $0 < \sigma < \frac{1}{2}$. Hence we can write as follows.

$$S_0 = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{21}$$

2.3. *Second Derivative of* $g(t)$

In Section 1.1, we showed that $E_p(t)$ is an **analytic** function in the interval $-\infty \leq t \leq \infty$ which is infinitely differentiable in that interval. Let us consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2g(t)}{dt^2}$ where $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$.

We can see that $g_2(t) = \frac{d^2g(t)}{dt^2}$ produces a **Dirac delta function**, which is an **even function** of variable t. Hence, when we take the **odd part** of $g_2(t)$ given by $g_{2_{odd}}(t) = \frac{1}{2}[g_2(t) - g_2(-t)]$, the dirac delta impulse function **vanishes** (Appendix D). We will compute the Fourier transform of $g_{2_{odd}}(t)$ given by $G_{2_I}(\omega)$ shortly.

First we compute the Fourier transform of $g_2(t)$ given by $G_2(\omega)$ as follows.

$$G_2(\omega) = \int_{-\infty}^{0} \frac{d^2(E_p(t)e^{-\sigma t})}{dt^2}e^{-i\omega t}dt + \int_{0}^{\infty} \frac{d^2(E_p(t)e^{\sigma t})}{dt^2}e^{-i\omega t}dt \tag{22}$$

We can substitute $t = -\tau$ in the second integral in Eq. 22 and then substitute $E_p(-\tau) = E_q(\tau)$ and we also substitute $t = \tau$ in the first integral and write as follows. We use the fact that $E_p(\tau) = E_0(\tau)e^{-\sigma\tau}$ and $E_q(\tau) = E_0(-\tau)e^{\sigma\tau}$.

$$G_2(\omega) = \int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2}e^{-i\omega\tau}d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2}e^{i\omega\tau}d\tau$$

(23)

Eq. 23 can be expanded as follows using Euler's formula $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and comparing the **imaginary parts** of $G_2(\omega) = G_{2_R}(\omega) + iG_{2_I}(\omega)$, we can write as follows.

$$G_{2_I}(\omega) = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} \sin(\omega\tau)d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2} \sin(\omega\tau)d\tau$$

(24)

We see that the Fourier transform of $g_{2_{odd}}(t)$ is given by $iG_{2_I}(\omega)$ where $G_2(\omega) = G_{2_R}(\omega) + iG_{2_I}(\omega)$ and $G_2(\omega)$ is the Fourier transform of $g_2(t)$. We see that $G_2(\omega) = -\omega^2 G(\omega) = -\omega^2[G_R(\omega) + iG_I(\omega)]$ and hence $G_{2_I}(\omega) = -\omega^2 G_I(\omega)$.

We require $G_{2_I}(\omega) = 0$ for the **same** $\omega = \omega_1$, to satisfy **Statement 1**, because we derived the result that $G_I(\omega) = 0$ for $\omega = \omega_1 \neq 0$ in Section 2.1 and $G_{2_I}(\omega) = -\omega^2 G_I(\omega)$. Hence $S_2 = G_{2_I}(\omega_1) = 0$ and is given as follows.

$$S_2 = G_{2_I}(\omega_1) = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} \frac{d^2E_0(-\tau)}{d\tau^2} \sin(\omega_1\tau)d\tau = 0$$

(25)

Using Taylor series representation of $E_0(\tau) = \sum_{n,k,r} c_{nkr}e^{b_{kr}\tau}$ and we use the fact that $E_0(\tau) = E_0(-\tau)$, we can write as follows.

$$S_2 = \omega_1 \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)^2 \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^2 \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0$$

(26)

2.4. ***Even order Derivatives of*** $g(t)$

In Section 1.1, we showed that $E_p(t)$ is an **analytic** function in the interval $-\infty \leq t \leq \infty$ which is infinitely differentiable in that interval. Let us consider the $(2r)^{th}$ derivative of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ where $r = 0, 1, ..., \infty$. Its Fourier transform is given by $G_{2r}(\omega) = \int_{-\infty}^{\infty} g_{2r}(t)e^{-i\omega t}dt$. We take the **odd part** of $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ and the dirac delta impulse function related terms **vanish** because dirac delta and its even derivatives are **even functions** of variable t. This is shown in detail in **Appendix D**.

We take the Fourier transform of $g_{2r_{odd}}(t)$ and we see that $G_{2r_I}(\omega) = 0$ for the **same** $\omega = \omega_1$ because $G_{2r}(\omega) = (-\omega^2)^r G(\omega) = (-\omega^2)^r[G_R(\omega) + iG_I(\omega)]$ and hence $G_{2r_I}(\omega) = (-\omega^2)^r G_I(\omega)$ and we derived the result that $G_I(\omega) = 0$ for $\omega = \omega_1$ in Section 2.1 . We can derive results similar to Eq. 25, Eq. 26 as follows.

$$S_{2r} = G_{2r_I}(\omega_1) = -\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1\tau)d\tau] + \int_{-\infty}^{0} \frac{d^{2r}E_0(-\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau = 0$$

(27)

Using Taylor series representation of $E_0(\tau) = \sum_{n,k,r} c_{nkr} e^{b_{kr}\tau}$ and we use the fact that $E_0(\tau) = E_0(-\tau)$, we can write as follows.

$$S_{2r} = \omega_1 \sum_{n,k,r} c_{nkr}[(b_{kr} - 2\sigma)^{2r} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - b_{kr}^{2r} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{28}$$

Now, we can form a new function $A(t_1)$ as follows, for real $-\infty \le t_1 \le \infty$.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r} \frac{t_1^{2r}}{!(2r)} = 0$$

$$A(t_1) = \omega_1 \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1} + e^{-(b_{kr}-2\sigma)t_1})}{2} \frac{1}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)} + e^{-(b_{kr}t_1)})}{2} \frac{1}{(\omega_1^2 + b_{kr}^2)}] = 0 \tag{29}$$

We see that $A(t_1) = \frac{\omega_1}{2}[y(t_1) + y(-t_1)] = 0$ where $y(t_1)$ is an **odd** function of variable t_1, because there is **at least one non-zero** value of $\omega_1 \ne 0$ as explained in Section 2.1, we write as follows.

$$y(t_1) = \sum_{n,k,r} c_{nkr}[\frac{(e^{(b_{kr}-2\sigma)t_1}}{(\omega_1^2 + (b_{kr} - 2\sigma)^2)} - \frac{(e^{(b_{kr}t_1)}}{(\omega_1^2 + b_{kr}^2)}] = -y(-t_1) = y_{odd}(t_1) \tag{30}$$

We can evaluate the **odd** symmetry function $z_{odd}(t_1)$ as follows.

$$\begin{aligned} \frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\ \sum_{n,k,r} c_{nkr}[e^{(b_{kr}-2\sigma)t_1} - e^{(b_{kr}t_1)}] &= z_{odd}(t_1) \\ \sum_{n,k,r} c_{nkr} e^{(b_{kr}t_1)}(e^{-2\sigma t_1} - 1) &= z_{odd}(t_1) \end{aligned} \tag{31}$$

We know that $\sum_{n,k,r} c_{nkr} e^{(b_{kr}t_1)} = E_0(t_1)$ is an **even function** of variable t_1 and $E_0(t_1) \ne 0$, hence we require $(e^{-2\sigma t_1} - 1)$ to be an **odd function** of variable t_1 to satisfy Eq. 31, which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

3. Proof of Riemann's Hypothesis without using Taylor series expansion of $E_p(t)$

In this section, we re-derive the results in Section 2.3 and Section 2.4 **without** using Taylor series expansion of $E_p(t)$. Results in Section 2.1 and Section 2.2 hold for the case **without** using Taylor series expansion of $E_p(t)$ as well.

We consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2 g(t)}{dt^2}$ and using procedure in Section 2.3, we can write as follows.

$$S_2 = -\int_{-\infty}^{0} \frac{d^2(E_0(\tau)e^{-2\sigma\tau})}{d\tau^2} \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} \frac{d^2 E_0(-\tau)}{d\tau^2} \sin(\omega_1\tau)d\tau = 0 \tag{32}$$

Let us consider the $(2r)^{th}$ **derivative** of the function $g(t)$ given by $g_{2r}(t) = \frac{d^{2r} g(t)}{dt^{2r}}$ and using procedure discussed in Section 2.4, we can write as follows. We use the fact that $E_0(\tau) = E_0(-\tau)$.

$$S_{2r} = -\int_{-\infty}^{0} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} \frac{d^{2r} E_0(\tau)}{d\tau^{2r}} \sin(\omega_1\tau)d\tau = 0 \tag{33}$$

We can form a new function $A(t_1)$ as follows, for real $-\infty \leq t_1 \leq \infty$. We can see that for every value of r, the integrals in the equation below converge and we can interchange the order of integration and summation as follows.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r} \frac{t_1^{2r}}{!(2r)} = -\int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [\sum_{r=0}^{\infty} \frac{d^{2r} E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}] \sin(\omega_1\tau)d\tau = 0 \tag{34}$$

For the specific case of **complex exponential** function $C(\tau) = e^{i\omega\tau}$, we define a new function $D(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r} C(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$ which can be written as $D(\tau) = \frac{1}{2}[C(\tau + t_1) + C(\tau - t_1)]$. We can show similar results for the summation terms in Eq. 34 as follows.

Let $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$. In Eq. 34 we have $f_1(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r} x(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)}$. In **Appendix F**, we show that $f_1(\tau) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ using the inverse Fourier transform representation of $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega$, given that $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ is an analytic function and is Fourier transformable. Similarly, we can show that $f_2(\tau) = \sum_{r=0}^{\infty} \frac{d^{2r} E_0(\tau)}{d\tau^{2r}} \frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)]$. Hence we can write Eq. 34 as follows.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0} [x(\tau + t_1) + x(\tau - t_1)] \sin(\omega_1\tau)d\tau + \int_{-\infty}^{0} [E_0(\tau + t_1) + E_0(\tau - t_1)] \sin(\omega_1\tau)d\tau] = 0 \tag{35}$$

We define $B(t_1) = -\int_{-\infty}[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau$ and evaluate the integral at the lower limit of $\tau = -\infty$. We can evaluate the integrals in Eq. 35 separately at the upper limit and lower limit as follows.

$$A(t_1) = \frac{1}{2}[-\int^0 [x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int^0 [E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau - B(t_1)] = 0 \tag{36}$$

We see that $B(t_1)$ equals **integration constant** K_I, added to an extra term which is **non-zero** in the **general** case.

In **Appendix G**, for the **specific case** of our function $E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4t} - 3\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}e^{-\sigma t} = E_0(t)e^{-\sigma t}$ and for $0 < |\sigma| < \frac{1}{2}$, we show that $B(t_1) = 0 + K_I$ by using integration by parts method and evaluating the integrals in Eq. 36 at the lower limit of $\tau = -\infty$.

Integration constant K_I gets **cancelled** at the upper and lower limit of the indefinite integrals in Eq. 36 given by $-\int[x(\tau+t_1)+x(\tau-t_1)]\sin(\omega_1\tau)d\tau$ and $\int[E_0(\tau+t_1)+E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau$.

3.1. ***Final Step in the proof of theorem.***

We can write $A(t_1) = y(t_1) + y(-t_1) = 0$ as follows, with integrals evaluated **only** at the upper limit and integration constant K_I **omitted** in equations below because it gets **cancelled** at the upper and lower limit of the indefinite integrals in Eq. 36.

$$y(t_1) = -\frac{1}{2}\int^0 x(\tau+t_1)\sin(\omega_1\tau)d\tau + \frac{1}{2}\int^0 E_0(\tau+t_1)\sin(\omega_1\tau)d\tau = -y(-t_1) = y_{odd}(t_1) \tag{37}$$

We can see that $y(t_1)$ is an **odd function** of variable t_1.

We can substitute $\tau + t_1 = t$ and write as follows.

$$\begin{aligned} y(t_1) = -\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ +\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] = y_{odd}(t_1) \end{aligned} \tag{38}$$

We can evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = z_{odd}(t_1)$ as follows, where $z_{odd}(t_1)$ is an **odd function** of variable t_1. In **Appendix E**, we show that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is an analytic function and the indefinite integral is evaluated only at the upper limit and we also derive in detail the equation $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$. We use $x(t_1) = E_0(t_1)e^{-2\sigma t_1}$ below.

$$\begin{aligned} \frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) &= z_{odd}(t_1) \\ \frac{\omega_1}{2}[x(t_1) - E_0(t_1)] &= z_{odd}(t_1) \\ \frac{\omega_1}{2}[E_0(t_1)e^{-2\sigma t_1} - E_0(t_1)] &= z_{odd}(t_1) \\ \frac{\omega_1}{2}E_0(t_1)[e^{-2\sigma t_1} - 1] &= z_{odd}(t_1) \end{aligned}$$

(39)

We use the fact that $\omega_1 \neq 0$. We know that $E_0(t_1) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t_1} - 3\pi n^2 e^{2t_1}]e^{-\pi n^2 e^{2t_1}} e^{\frac{t_1}{2}}$ is an **even function** of variable t_1 and $E_0(t_1) \neq 0$, hence we require $e^{-2\sigma t_1} - 1$ to be an **odd function** of variable t_1 which is possible **only** for $\sigma = 0$ corresponding to the critical line.

We have derived this result for $0 < \sigma < \frac{1}{2}$ and we use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show that the result holds for $-\frac{1}{2} < \sigma < 0$.

Therefore, the assumption in **Statement 1** that Riemann's Xi Function given by $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. This means $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

4. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related functions which have their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$, because the new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2} + i\omega) = \xi(\frac{1}{2} - i\omega)$ when evaluated at the critical line $s = \frac{1}{2} + i\omega$. This means $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ is a real and even function of t and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function and related functions satisfy a symmetry relation similar to $\xi(s) = \xi(1-s)$ and hence $E_0(t) \neq E_0(-t)$ and hence the new method is **not** applicable to Hurwitz zeta function and related functions which have their non-trivial zeros away from the critical line.

It was shown in **Appendix G** that, for the **specific case** of our function $E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} = E_0(t)e^{-\sigma t}$ and for $0 < |\sigma| < \frac{1}{2}$, we get $B(t_1) = 0 + K_I$ by using integration by parts method and evaluating the integrals in Eq. 36 at the lower limit of $\tau = -\infty$. This was required to prove Riemann's Hypothesis. This condition may not be satisfied for many other functions.

5. Conclusion

We considered the analytic continuation of Riemann's Zeta Function derived from Riemann's Xi function $\xi(s)$ given by $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ and computed its inverse Fourier transform given by $E_p(t)$, which is expressed as an **infinite summation of two-sided decaying exponential functions** using Taylor series expansion.

We used a new method and showed that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line and proved Riemann's hypothesis. We also used the new method **without** using Taylor series expansion and proved Riemann's hypothesis.

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Appendix A.

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2} + i\omega$ given by $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. This is re-derived in Appendix H.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 < |\sigma| < \frac{1}{2}$ is real.

$$\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} + i(\omega - i\sigma)) = E_{p\omega}(\omega) = E_{0\omega}(\omega - i\sigma)$$
$$E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega - i\sigma)e^{i\omega t}d\omega \tag{A.1}$$

We substitute $\omega' = \omega - i\sigma$ in Eq. A.1 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} E_{0\omega}(\omega')e^{i\omega' t}d\omega' \tag{A.2}$$

We can evaluate the above integral in the complex plane using contour integration, substituting $\omega' = z = x + iy$ and we use a rectangular contour comprised of C_1 along the line $x = [-\infty, \infty]$, C_2 along the line $z = [\infty, \infty - i\sigma]$, C_3 along the line $z = [\infty - i\sigma, -\infty - i\sigma]$ and then C_4 along the line $z = [-\infty - i\sigma, -\infty]$. We can see that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz)$ has no singularities in the region bounded by the contour because $\xi(\frac{1}{2} + iz)$ is an entire function in the Z-plane.

Given that $\xi(\frac{1}{2} + \sigma + i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 integrable function .

We use the fact that $E_{0\omega}(z) = \xi(\frac{1}{2} + iz) = \xi(\frac{1}{2} - y + ix) = \int_{-\infty}^{\infty} E_0(t)e^{-izt}dt = \int_{-\infty}^{\infty} E_0(t)e^{yt}e^{-ixt}dt$, **goes to zero** as $x \to \pm\infty$ when $-\sigma \leq y \leq 0$, given that $E_0(t)e^{yt}$ is a L^1 integrable function in the interval $-\infty \leq t \leq \infty$ as per (Riemann-Lebesgue Lemma). Hence the integral in above equation **vanishes** along the contours C_2 and C_4. We can write Eq. A.2 as follows.

$$E_p(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega')e^{i\omega' t}d\omega'$$
$$E_p(t) = E_0(t)e^{-\sigma t} = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} \tag{A.3}$$

Thus we have arrived at the desired result $E_p(t) = E_0(t)e^{-\sigma t}$.

Appendix B. Alternate Taylor's series representation of $E_p(t)$

We can substitute $z = e^{2t}$ in Eq. 2 for $E_p(t)$ reproduced below.

$$E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} = E_0(t)e^{-\sigma t} = f(e^{2t})e^{\frac{t}{2}} e^{-\sigma t}$$
$$f(z) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 z^2 - 3\pi n^2 z]e^{-\pi n^2 z} \tag{B.1}$$

We can expand the real analytic function $f(z)$ using Taylor series expansion **around** $z = 1$ as follows.

$$f(z) = [\sum_{n,k}(a_{nk}(z-1)^{(k+2)} - b_{nk}(z-1)^{(k+1)})]e^{-\pi n^2}$$
$$\sum_{n,k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}, \quad a_{nk} = 4\pi^2 n^4 d_{nk}, \quad b_{nk} = 6\pi n^2 d_{nk}, \quad d_{nk} = \frac{(-\pi n^2)^k}{!(k)} \tag{B.2}$$

Now we substitute $z = e^{2t}$ in Eq. B.2 and we can write the Taylor series expansion of $E_p(t)$ as follows and we use binomial series expansion for $(e^{2t}-1)^v = \sum_{p=0}^{v}\binom{v}{p}(-1)^p e^{2t(v-p)}$ for v is a positive integer.

$$E_p(t) = [\sum_{n,k}(a_{nk}(e^{2t}-1)^{(k+2)} - b_{nk}(e^{2t}-1)^{(k+1)})]e^{-\pi n^2} e^{\frac{t}{2}} e^{-\sigma t}$$
$$E_p(t) = [\sum_{n,k}(a_{nk}[\sum_{p=0}^{k+2}\binom{k+2}{p}(-1)^p e^{2t(k+2-p)}] - b_{nk}[\sum_{p=0}^{k+1}\binom{k+1}{p}(-1)^p e^{2t(k+1-p)}]]e^{-\pi n^2} e^{\frac{t}{2}} e^{-\sigma t} \tag{B.3}$$

This equation can be simplified as follows.

$$E_p(t) = \sum_{n,k}[\sum_{p=0}^{k+2} a'_{nkp} e^{(2k+\frac{9}{2}-2p)t} - \sum_{p=0}^{k+1} b'_{nkp} e^{(2k+\frac{5}{2}-2p)t}]e^{-\sigma t} = E_0(t)e^{-\sigma t}$$
$$a'_{nkp} = a_{nk}e^{-\pi n^2}\binom{k+2}{p}(-1)^p, b'_{nkp} = b_{nk}e^{-\pi n^2}\binom{k+1}{p}(-1)^p \tag{B.4}$$

Given that $E_0(t) = E_0(-t)$, we can write $E_p(t)$ as an **infinite summation** of two-sided decaying exponential functions as follows.

$$E_p(t) = E_0(t)e^{-\sigma t}, E_0(t) = \sum_{n,k}[\sum_{p=0}^{k+2} a'_{nkp} e^{(2k+\frac{9}{2}-2p)t} - \sum_{p=0}^{k+1} b'_{nkp} e^{(2k+\frac{5}{2}-2p)t}]$$
$$E_p(t) = [E_0(t)u(-t) + E_0(-t)u(t)]e^{-\sigma t} \tag{B.5}$$

Appendix C. Properties of Fourier Transforms Part 1

Appendix C.1. $E_p(t), h(t), g(t)$ ***are*** L^1 ***integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega) = \xi(\frac{1}{2}+\sigma+i\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t}$. We see that $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} > 0$ for all $0 \leq t < \infty$. Given that $E_0(t) = E_0(-t)$, we see that $E_0(t) > 0$ and $E_p(t) = E_0(t)e^{-\sigma t} > 0$ for all $-\infty < t < \infty$. We see that $E_p(t) = 0$ at $t = \pm\infty$ and hence $E_p(t) \geq 0$ for all $|t| \leq \infty$.

Given that $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$ is an entire function in the whole of s-plane, it is finite for $|\omega| \leq \infty$ and also for $\omega = 0$. Hence $\int_{-\infty}^{\infty} E_p(t)dt$ is finite. We see that $E_p(t) \geq 0$ for all $|t| \leq \infty$. Hence we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an L^1 **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an L^1 **integrable function** because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma-i\omega} + \frac{1}{\sigma+i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite and its Fourier transform $H(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

We can see that $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t) \geq 0$ for all $|t| \leq \infty$ because $E_p(t) \geq 0$ for all $|t| \leq \infty$. Given that $E_p(t) = E_0(t)e^{-\sigma t} = [E_0(t)u(-t) + E_0(-t)u(t)]e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$, we see that $g(t)$ goes to zero as $t \to -\infty$ with its order of decay greater than e^{2t} and $g(t)$ goes to zero as $t \to \infty$ with its order of decay greater than $e^{\frac{-5t}{2}}$, for $0 < \sigma < \frac{1}{2}$. Hence $g(t)$ is an L^1 **integrable function** and its Fourier transform $G(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma.

Appendix C.2. ***Convolution integral convergence***

Let us consider a function whose **first derivative is discontinuous** at $t = 0$, for example $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $-\frac{A_0}{\omega^2}$ and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.

We see that $E_p(t) = E_0(t)e^{-\sigma t}$ where $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

Let us consider a new function $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ where $g(t)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $0 < \sigma < \frac{1}{2}$. We can see that $g(t)h(t) = E_p(t)$ where $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$.

We can see that $G(\omega), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the **first derivatives** of $g(t), h(t)$ are **discontinuous** at $t = 0$. Also, $E_p(t), h(t), g(t)$ are L^1 integrable functions and their Fourier Transforms are finite as shown in Appendix C.1. Hence the convolution integral below converges to a finite value for $|\omega| \leq \infty$.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \frac{1}{2\pi}[G(\omega) * H(\omega)] \tag{C.1}$$

Appendix D. Dirac delta derivatives vanish when we consider even derivatives of $g(t)$ and take their odd part $g_{2r_{odd}}(t)$

Let us consider the **second derivative** of the function $g(t)$ given by $g_2(t) = \frac{d^2g(t)}{dt^2}$ where $g(t) = E_p(t)e^{-\sigma t}u(-t) + E_p(t)e^{\sigma t}u(t)$ and $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ and $g(t)h(t) = E_p(t)$. In Section 1.1, we showed that $E_p(t)$ is an analytic function in the interval $-\infty \leq t \leq \infty$ and even derivatives of $g(t)$ have dirac delta functions at $t = 0$.

We can show that **dirac delta function** $d_0(t) = \delta(t)$ which is present in $g_2(t)$ and its **even derivatives** $d_{2r-2}(t)$ which are present in $g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}}$ **vanish**, when we take the Fourier transform of the function $g_{2r_{odd}}(t) = \frac{1}{2}[g_{2r}(t) - g_{2r}(-t)]$ for positive integer r, because **dirac delta function and its even derivatives have even symmetry**, while $g_{2r_{odd}}(t)$ has **odd symmetry**.

$$\begin{aligned}
g(t) &= g_-(t)u(-t) + g_+(t)u(t) \\
g_-(t) &= E_p(t)e^{-\sigma t}, \quad g_+(t) = E_p(t)e^{\sigma t} \\
g_2(t) = \frac{d^2g(t)}{dt^2} = \frac{d^2g_-(t)}{dt^2}u(-t) + \frac{d^2g_+(t)}{dt^2}u(t) + A_0d_0(t), &\quad A_0 = [\frac{dg_+(t)}{dt} - \frac{dg_-(t)}{dt}]_{t=0} \\
g_{2r}(t) = \frac{d^{2r}g(t)}{dt^{2r}} = \frac{d^{2r}g_-(t)}{dt^{2r}}u(-t) + \frac{d^{2r}g_+(t)}{dt^{2r}}u(t) + A_{2r-2}d_0(t) &+ \sum_{k=0}^{r-2} A_{2k}\frac{d^{2r-2-2k}(d_{2k}(t))}{dt^{2r-2-2k}} \\
A_{2r-2} = [\frac{d^{2r-1}g_+(t)}{dt^{2r-1}} - \frac{d^{2r-1}g_-(t)}{dt^{2r-1}}]_{t=0}, &\quad A_{2k} = [\frac{d^{2k+1}g_+(t)}{dt^{2k+1}} - \frac{d^{2k+1}g_-(t)}{dt^{2k+1}}]_{t=0}
\end{aligned} \tag{D.1}$$

Then we take the **odd part** of the functions $g_{2r}(t)$ given by $g_{2r_{odd}}(t) = \frac{1}{2}(g_{2r}(t) - g_{2r}(-t))$ and take their Fourier transforms given by $iG_{2r_I}(\omega) = i(-\omega^2)^rG_I(\omega)$. We can see that the Fourier transform of the delta function and its even derivatives **vanish** given that **dirac delta function and its even derivatives have even symmetry** in Eq. D.1 and **do not interfere** with the results. This is shown below.

Let us consider the Fourier transform of Dirac delta function $d_0(t) = \delta(t)$ and its derivatives for $r = 0, 1, ...\infty$. We use notation $F[d_0(t)]$ to represent Fourier transform of $d_0(t)$. We use the well known property that $F[\frac{d^{2r}g(t)}{dt^{2r}}] = (-\omega^2)^rG(\omega)$ for a general Fourier transformable function $g(t)$.

$$\begin{aligned}
F[d_0(t)] &= \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = 1 \\
F[d_2(t)] &= \int_{-\infty}^{\infty} \frac{d^2d_0(t)}{dt^2}e^{-i\omega t}dt = -\omega^2 \\
F[d_{2r}(t)] &= \int_{-\infty}^{\infty} \frac{d^{2r}d_0(t)}{dt^{2r}}e^{-i\omega t}dt = (-\omega^2)^r
\end{aligned} \tag{D.2}$$

We can see that $(-\omega^2)^r$ is a real and even function of ω and hence its inverse Fourier transform given by $\frac{d^{2r}d_0(t)}{dt^{2r}}$ is also a real and even function of t.

Appendix E.

In this section, we show that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$, where $x(t)$ is an analytic function in the interval $-\infty \leq t \leq \infty$ and the indefinite integral is evaluated only at the upper limit.

If $x(\tau)$ is an analytic function, then we can express it using taylor series expansion around $\tau = 0$ as follows, where $x_n = \frac{1}{!n}[\frac{d^n(x(\tau))}{d\tau^n}]_{\tau=0}$ and K_0 is an integration constant in the indefinite integral $f(\tau) = \int x(\tau)d\tau$.

$$\begin{aligned} x(\tau) &= x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... \\ f(\tau) = \int x(\tau)d\tau &= K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ... \\ \frac{df(\tau)}{d\tau} &= x_0 + x_1\tau + x_2\tau^2 + x_3\tau^3 + ... = x(\tau) \end{aligned} \tag{E.1}$$

Now we can repeat the steps above for $f(t) = \int^t x(\tau)d\tau$ as follows.

$$\begin{aligned} f(t) = \int^t x(\tau)d\tau = [K_0 + x_0\tau + x_1\frac{\tau^2}{2} + x_2\frac{\tau^3}{3} + x_3\frac{\tau^4}{4} + ...]^t &= K_0 + x_0t + x_1\frac{t^2}{2} + x_2\frac{t^3}{3} + x_3\frac{t^4}{4} + ... \\ \frac{df(t)}{dt} &= x_0 + x_1t + x_2t^2 + x_3t^3 + ... = x(t) \end{aligned} \tag{E.2}$$

We have shown that if $f(t) = \int^t x(\tau)d\tau$, then $\frac{df(t)}{dt} = x(t)$.

Now, we start with $y(t_1)$ in Eq. 38 and derive in detail $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$ in Eq. 39 as follows.

$$\begin{aligned} y(t_1) = &-\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ &+\frac{1}{2}[\cos(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt - \sin(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] \end{aligned} \tag{E.3}$$

We take the first derivative of $y(t_1)$ as follows.

$$\begin{aligned} \frac{dy(t_1)}{dt_1} = &-\frac{\omega_1}{2}[-\sin(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt - \cos(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt] \\ &+\frac{\omega_1}{2}[-\sin(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt - \cos(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] \end{aligned} \tag{E.4}$$

We take the second derivative of $y(t_1)$ as follows.

$$\frac{d^2y(t_1)}{dt_1^2} = -\frac{\omega_1^2}{2}[-\cos(\omega_1 t_1)\int^{t_1} x(t)\sin(\omega_1 t)dt + \sin(\omega_1 t_1)\int^{t_1} x(t)\cos(\omega_1 t)dt]$$
$$+\frac{\omega_1^2}{2}[-\cos(\omega_1 t_1)\int^{t_1} E_0(t)\sin(\omega_1 t)dt + \sin(\omega_1 t_1)\int^{t_1} E_0(t)\cos(\omega_1 t)dt] + \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$$

(E.5)

Now we evaluate $\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1)$ as follows and get Eq. 39 .

$$\frac{d^2y(t_1)}{dt_1^2} + \omega_1^2 y(t_1) = \frac{\omega_1}{2}[x(t_1) - E_0(t_1)]$$

(E.6)

Appendix F.

We start with Eq. 34 as follows.

$$A(t_1) = \sum_{r=0}^{\infty} S_{2r}\frac{t_1^{2r}}{!(2r)} = -\int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}(E_0(\tau)e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau] + \int_{-\infty}^{0}[\sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]\sin(\omega_1\tau)d\tau] = 0$$

(F.1)

In Eq. F.1 we have $f(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}$ inside the first integral, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$ and we can show that $f(\tau) = \frac{1}{2}[x(\tau + t_1) + x(\tau - t_1)]$ using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$, given that $E_0(\tau)e^{-2\sigma\tau}$ is an analytic function in the interval $-\infty \leq \tau \leq \infty$ and hence infinitely differentiable and it is also Fourier transformable.

Similarly, we can show that $d(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau + t_1) + E_0(\tau - t_1)]$ inside the second integral.

We substitute $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ in the equation for $f(\tau)$ and we write as follows.

$$f(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{d^{2r}([\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega]e^{-2\sigma\tau})}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}$$

(F.2)

In Appendix C.2, we have shown that if the $(N-1)^{th}$ **derivative** of a function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$. In Section 1.1, we showed that $E_0(t)$ is an analytic function which is infinitely differentiable which produces no discontinuities in $|t| \leq \infty$. Hence its Fourier transform $E_{0\omega}(\omega)$ has a fall-off rate faster than $\frac{1}{\omega^M}$ as $M \to \infty$ and it should have a fall-off rate **at least** of the order of $e^{-A|\omega|}$ where $A > 0$.

We can interchange the order of integration and summation as follows, because for every value of r in equation below, **the integral converges**.

$$f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty} \frac{d^{2r}e^{(i\omega-2\sigma)\tau}}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)}]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)[\sum_{r=0}^{\infty}(i\omega-2\sigma)^{2r}e^{(i\omega-2\sigma)\tau}\frac{t_1^{2r}}{!(2r)}]d\omega \tag{F.3}$$

We can simplify this equation as follows.

$$\begin{aligned} f(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)t_1} + e^{-(i\omega-2\sigma)t_1}]e^{(i\omega-2\sigma)\tau}d\omega \\ f(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)\frac{1}{2}[e^{(i\omega-2\sigma)(\tau+t_1)} + e^{(i\omega-2\sigma)(\tau-t_1)}]d\omega \end{aligned} \tag{F.4}$$

We can simplify this equation as follows, using the inverse Fourier transform representation of $E_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega\tau}d\omega$ and $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$f(\tau) = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)] \tag{F.5}$$

Comparing Eq. F.2 and Eq. F.5, we can see that $f(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)]$.

Similarly, we see that $d(\tau) = \sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)]$.

$$\begin{aligned} f(\tau) &= \sum_{r=0}^{\infty}\frac{d^{2r}x(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[x(\tau+t_1) + x(\tau-t_1)] \\ d(\tau) &= \sum_{r=0}^{\infty}\frac{d^{2r}E_0(\tau)}{d\tau^{2r}}\frac{t_1^{2r}}{!(2r)} = \frac{1}{2}[E_0(\tau+t_1) + E_0(\tau-t_1)] \end{aligned} \tag{F.6}$$

Hence we can write Eq. F.1 as follows.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{F.7}$$

Appendix G.

In this section, we want to show that the inner indefinite integral in Eq. F.7 reproduced below, can be expressed as $I_0(\tau, t_1) = J_0(\tau, t_1) + K_I$ where K_I is the integration constant and we will show that $J_0(\tau, t_1) = 0$ when evaluated at the lower limit of $\tau = -\infty$, for the **specific case** of our function $E_p(t) =$

$2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t} = E_0(t)e^{-\sigma t}$. The **integration constant** K_I gets cancelled when evaluating $I_0(\tau, t_1)$ at the upper and lower limits of the integral.

$$A(t_1) = \frac{1}{2}[-\int_{-\infty}^{0}[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int_{-\infty}^{0}[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{G.1}$$

The inner indefinite integral in Eq. G.1 can be written as follows, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$.

$$I_0(\tau, t_1) = \frac{1}{2}[-\int[x(\tau+t_1) + x(\tau-t_1)]\sin(\omega_1\tau)d\tau + \int[E_0(\tau+t_1) + E_0(\tau-t_1)]\sin(\omega_1\tau)d\tau] = 0 \tag{G.2}$$

We can write $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ as follows using the shorthand notation $E_0(t) = \sum_{n,r} c_{nr} e^{-\pi n^2 e^{2t}} e^{K_r t}$ where $\sum_{n,r} = \sum_{n=1}^{\infty}\sum_{r=0}^{1}$, where $c_{n1} = a_n, c_{n0} = -b_n$, $a_n = 4\pi^2 n^4; b_n = 6\pi n^2$ and $K_r = \frac{5}{2} + 2r > 1$ for $r = 0, 1$.

Appendix G.1.

In Eq. G.2, let us evaluate the indefinite integral term $I_1(\tau) = \int x(\tau)\cos(\omega_1\tau)d\tau$ as follows, where $x(\tau) = E_0(\tau)e^{-2\sigma\tau}$. We show that the indefinite integral can be expressed as $I_1(\tau) = J_1(\tau) + K_{I_1}$ where K_{I_1} is the integration constant and we will show that $J_1(\tau) = 0$ when evaluated at the lower limit of $\tau = -\infty$,

$$I_1(\tau) = \int x(\tau)\cos(\omega_1\tau)d\tau = \int \sum_{n,r} c_{nr} e^{-\pi n^2 e^{2\tau}} e^{K_r\tau} e^{-2\sigma\tau}\cos(\omega_1\tau)d\tau \tag{G.3}$$

Using theorem of dominated convergence, we can interchange the order of integration and summation as follows, given that for every value of n and r, the integral converges.

$$I_1(\tau) = \sum_{n,r} c_{nr} \int e^{-\pi n^2 e^{2\tau}} e^{K_r\tau} e^{-2\sigma\tau}\cos(\omega_1\tau)d\tau \tag{G.4}$$

We substitute $e^{2\tau} = x, dx = 2x d\tau, \tau = \frac{\log_e(x)}{2}$ and write as follows. we use $K_2 = \frac{(K_r - 2\sigma)}{2} - 1$.

$$I_1(x) = \frac{1}{2}\sum_{n,r} c_{nr} \int e^{-\pi n^2 x} x^{K_2}\cos(\omega_1\frac{\log_e(x)}{2})dx \tag{G.5}$$

Using **integration by parts** method $\int u dv = uv - \int v du$, we substitute $u = e^{-\pi n^2 x}\cos(\omega_1\frac{\log_e(x)}{2}), dv = x^{K_2}dx$ and hence we get $v = \frac{x^{(K_2+1)}}{(K_2+1)}$ and $du = e^{-\pi n^2 x}[\cos(\omega_1\frac{\log_e(x)}{2})(-\pi n^2) - \sin(\omega_1\frac{\log_e(x)}{2})(\frac{\omega_1}{2x})]dx$ and we can write as follows using this method repeatedly.

$$
\begin{aligned}
I_1(x) = \frac{1}{2}\sum_{n,r} c_{nr} e^{-\pi n^2 x}[\frac{x^{(K_2+1)}}{(K_2+1)}\cos{(\omega_1 \frac{\log_e(x)}{2})} \\
-\frac{x^{(K_2+2)}}{(K_2+1)(K_2+2)}[\cos{(\omega_1 \frac{\log_e(x)}{2})}[(-\pi n^2)] + \sin{(\omega_1 \frac{\log_e(x)}{2})}[\frac{-\omega_1}{2x}]] \\
+\frac{x^{(K_2+3)}}{(K_2+1)(K_2+2)(K_2+3)}[\cos{(\omega_1 \frac{\log_e(x)}{2})}[(-\pi n^2)^2 - (\frac{\omega_1}{2x})^2] + \sin{(\omega_1 \frac{\log_e(x)}{2})}[2(-\pi n^2)\frac{-\omega_1}{2x}]] - ...] + K_{I_1}
\end{aligned} \tag{G.6}
$$

We can simplify this as follows.

$$
\begin{aligned}
I_1(x) = \frac{1}{2}\sum_{n,r} c_{nr} e^{-\pi n^2 x}[\frac{x^{(K_2+1)}}{(K_2+1)}\cos{(\omega_1 \frac{\log_e(x)}{2})} \\
-\frac{x^{(K_2+1)}}{(K_2+1)(K_2+2)}[\cos{(\omega_1 \frac{\log_e(x)}{2})}[(-\pi n^2)x] + \sin{(\omega_1 \frac{\log_e(x)}{2})}[\frac{-\omega_1}{2}]] \\
+\frac{x^{(K_2+1)}}{(K_2+1)(K_2+2)(K_2+3)}[\cos{(\omega_1 \frac{\log_e(x)}{2})}[(-\pi n^2)^2 x^2 - (\frac{\omega_1}{2})^2] + \sin{(\omega_1 \frac{\log_e(x)}{2})}[2(-\pi n^2)\frac{-\omega_1 x}{2}]] - ...] + K_{I_1}
\end{aligned} \tag{G.7}
$$

We want to evaluate the above indefinite integral $I_1(x)$ at the lower limit of $\tau = -\infty$ which corresponds to $x = 0$ under the substitution $e^{2\tau} = x$. We can see that $I_1(x) = 0$ at $x = 0$ plus an **integration constant** K_{I_1} which gets cancelled when evaluating the indefinite integral at the upper and lower limits. We can see that $K_2 + 1 > 0$ given that $K_2 = \frac{(K_r - 2\sigma)}{2} - 1$ and $K_r = \frac{5}{2} + 2r > 1$ for $r = 0, 1$ and $0 < |\sigma| < \frac{1}{2}$.

Similar to above method, we can evaluate the indefinite integral term $I_2(\tau) = \int x(\tau)\sin{(\omega_1 \tau)} d\tau$ in Eq. G.2 and we can show that the indefinite integral **equals zero**, when evaluated at the lower limit of $\tau = -\infty$, plus an **integration constant** which gets cancelled when evaluating the indefinite integral at the upper and lower limits.

We can use integration by parts method for the terms $x(\tau+t_1), x(\tau-t_1), E_0(\tau+t_1), E_0(\tau-t_1)$ in Eq. G.2 and show that the indefinite integral **equals zero**, when evaluated at the lower limit of $\tau = -\infty$, plus an **integration constant** which gets cancelled when evaluating the indefinite integral at the upper and lower limits.

Hence $I_0(\tau, t_1)$ in Eq. G.2, when evaluated at the lower limit of $\tau = -\infty$, equals zero plus the **integration constant** K_I.

We can see that the indefinite integral $I_1(x)$ in Eq. G.7 evaluated at the upper limit of $\tau = t$ which corresponds to $x = e^{2t}$ is a finite value plus **integration constant** K_{I_1}.

Hence $I_0(\tau, t_1)$ in Eq. G.2 is finite.

Appendix H.

Let us start with Riemann's Xi Function $\xi(\frac{1}{2} + i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega) e^{i\omega t} d\omega$.

In this section, we will re-derive the inverse Fourier Transform of Riemann's Xi function as $E_0(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. [4]

We will use the steps in Ellison's book "Prime Numbers" pages 151-152 and rederive the steps below. We start with the gamma function $\Gamma(s) = \int_0^\infty y^{s-1}e^{-y}dy$ and substitute $y = \pi n^2 x$ and rederive as follows.

$$\begin{aligned}\Gamma(\frac{s}{2}) &= \int_0^\infty y^{\frac{s}{2}-1}e^{-y}dy \\ \Gamma(\frac{s}{2})(\pi n^2)^{-\frac{s}{2}} &= \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx\end{aligned} \tag{H.1}$$

For real part of s greater than 1, we can do a summation of both sides of above equation for all positive integers n and obtain as follows. We note that $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \tag{H.2}$$

For real part of s greater than 1, we can use theorem of doominated convergence and interchange the order of summation and integration as follows. We use the fact that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$.

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}w(x)dx \tag{H.3}$$

We multiply above equation by $\frac{1}{2}s(s-1)$ and get

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}s(s-1)\int_0^\infty x^{\frac{s}{2}-1}w(x)dx \tag{H.4}$$

$\xi(s)$ is an entire function, for all values of $Re[s]$ in the complex plane. We see that $\xi(s) = \xi(1-s)$.

Given that $w(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$, we substitute $x = e^{2t}$, $\frac{dx}{x} = 2dt$ in above equation and get

$$\xi(s) = 2\frac{1}{2}s(s-1)\int_{-\infty}^{\infty} e^{st}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}dt \tag{H.5}$$

We evaluate above equation at $s = \frac{1}{2} + i\omega$ as follows.

$$\begin{aligned}\xi(\frac{1}{2}+i\omega) &= 2\frac{1}{2}(\frac{1}{2}+i\omega)(-\frac{1}{2}+i\omega)\int_{-\infty}^{\infty} e^{\frac{t}{2}}e^{i\omega t}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}dt \\ \xi(\frac{1}{2}+i\omega) &= 2\frac{1}{2}[-(\frac{1}{4}+\omega^2)\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}}e^{\frac{t}{2}}e^{i\omega t}dt\end{aligned}$$

(H.6)

We define $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and get the inverse Fourier transform of $\xi(\frac{1}{2}+i\omega)$ given by $E_0(t)$ as follows.

$$\begin{aligned}
E_0(t) &= 2\frac{1}{2}[-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\
A(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} \\
\frac{dA(t)}{dt} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{2} - 2\pi n^2 e^{2t}] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[-4\pi n^2 e^{2t} + (\frac{1}{2} - 2\pi n^2 e^{2t})^2] \\
\frac{d^2A(t)}{dt^2} &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[\frac{1}{4} + 4\pi^2 n^4 e^{4t} - 2\pi n^2 e^{2t} - 4\pi n^2 e^{2t}]
\end{aligned} \tag{H.7}$$

We have arrived at the desired result for $E_0(t)$ as follows.

$$\begin{aligned}
E_0(t) &= [-\frac{1}{4}A(t) + \frac{d^2A(t)}{dt^2}] \\
E_0(t) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]
\end{aligned} \tag{H.8}$$

Appendix H.1.

Let us start with Riemann's Xi Function $\xi(s)$ evaluated at $s = \frac{1}{2}+i\omega$ given by $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega)$. Its inverse Fourier Transform is given by $E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$.

We will show in this section that the inverse Fourier Transform of the function $\xi(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega)$, is given by $E_p(t) = E_0(t)e^{-\sigma t}$ where $0 < |\sigma| < \frac{1}{2}$ is real.

We evaluate Eq. H.5 at $s = \frac{1}{2}+\sigma+i\omega$ as follows.

$$\begin{aligned}
\xi(\frac{1}{2}+\sigma+i\omega) &= 2\frac{1}{2}(\frac{1}{2}+\sigma+i\omega)(-\frac{1}{2}+\sigma+i\omega)\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t}dt \\
\xi(\frac{1}{2}+\sigma+i\omega) &= 2\frac{1}{2}[(-\frac{1}{4}+\sigma^2-\omega^2+i\omega(2\sigma))\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t} e^{i\omega t}dt
\end{aligned} \tag{H.9}$$

We define $A(t) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{i\omega t}$ and get the inverse Fourier transform of $\xi(\frac{1}{2}+\sigma+i\omega)$ given by $E_p(t)$ as follows.

$$E_p(t) = 2\frac{1}{2}[(-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}]$$
$$A(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}$$
$$\frac{dA(t)}{dt} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[\frac{1}{2}+\sigma - 2\pi n^2 e^{2t}]$$
$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[-4\pi n^2 e^{2t} + (\frac{1}{2}+\sigma - 2\pi n^2 e^{2t})^2]$$
$$\frac{d^2A(t)}{dt^2} = \sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{\sigma t}[\frac{1}{4}+\sigma^2+\sigma+4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t} - 4\sigma\pi n^2 e^{2t}]$$
$$\tag{H.10}$$

We have arrived at the desired result for $E_p(t)$ as follows.

$$E_p(t) = [(-\frac{1}{4}+\sigma^2)A(t) - 2\sigma\frac{dA(t)}{dt} + \frac{d^2A(t)}{dt^2}]e^{-\sigma t}$$
$$E_p(t) = 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4t} - 3\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}} e^{-\sigma t}$$
$$\tag{H.11}$$

Appendix I. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

Appendix I.1. ***Convolution Theorem: Multiplication of g(t) and h(t) corresponds to convolution in Fourier transform domain***

We start with the Fourier transform equation $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ where $f(t) = g(t)h(t)$ and show that $F(\omega) = \frac{1}{2\pi}[G(\omega) * H(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega'$ obtained by the **convolution** of the functions $G(\omega)$ and $H(\omega)$ which correspond to the Fourier transforms of $g(t)$ and $h(t)$ respectively.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{I.1}$$

We use the inverse Fourier transform equation $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega'$ and interchange the order of integration in equations below.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} G(\omega')e^{i\omega' t}d\omega']h(t)e^{-i\omega t}dt$$
$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} e^{i\omega' t}h(t)e^{-i\omega t}dt]d\omega'$$
$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')[\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt]d\omega'$$

(I.2)

We substitute $\int_{-\infty}^{\infty} h(t)e^{-i(\omega-\omega')t}dt = H(\omega-\omega')$ in Eq. I.2 and arrive at the convolution theorem.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')H(\omega-\omega')d\omega' = \int_{-\infty}^{\infty} g(t)h(t)e^{-i\omega t}dt \tag{I.3}$$

Appendix I.2. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a real function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{I.4}$$

Appendix I.3. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we show that the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t)+g(-t)]$, corresponds to **real part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t)+g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega)+G(-\omega)] = G_R(\omega) \end{aligned} \tag{I.5}$$

Appendix I.4. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we show that the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t)-g(-t)]$, corresponds to **imaginary part** of its Fourier transform $G(\omega)$. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t)-g(-t)]e^{-i\omega t}dt &= \frac{1}{2}[G(\omega)-G(-\omega)] = iG_I(\omega) \end{aligned} \tag{I.6}$$