On the Uncertainty Principle

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Abstract

It is well known that the Uncertainty Principle(UP) lays the foundation of Quantum Mechanics and protects it. In this paper, we will re-examine this principle for various scenarios and see if we can get past it. We will also examine a related principle which states that we need to use smaller wavelengths to observe an object as its size becomes smaller.

Keywords:

1. Introduction

It is well known that the Uncertainty Principle(UP) lays the foundation of Quantum Mechanics and protects it. In this paper, we will re-examine this principle for various scenarios and see if we can get past it. We will also examine a related principle which states that we need to use smaller wavelengths to observe an object as its size becomes smaller.

Let us rederive the well-known position-momentum relation by various methods.

$$\triangle x \triangle p \geq \frac{h}{4\pi} \tag{1}$$

where h is Planck's Constant.

2. Inherent Fourier Uncertainty

Let us start from the well-known time-frequency Fourier Uncertainty relation for a rectangular light pulse of duration $\triangle t$ modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(derived in Appendix A).

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{2}$$

Given that such a light pulse is made up of large number of photons and that each photon has an energy $E = hf = pc$ where c is light speed and p is the momentum of the photon, we can deduce Energy uncertainty as $\triangle E = h \triangle f = c \triangle p$, position uncertainty as $\triangle x = c \triangle t$ and hence we can write

$$\begin{aligned} \triangle t \triangle f &\geq \frac{1}{4\pi} \\ \frac{\triangle x}{c} \triangle p \frac{c}{h} &\geq \frac{1}{4\pi} \\ \triangle x \triangle p &\geq \frac{h}{4\pi} \end{aligned} \tag{3}$$

3. Two-sided Radar using light pulse

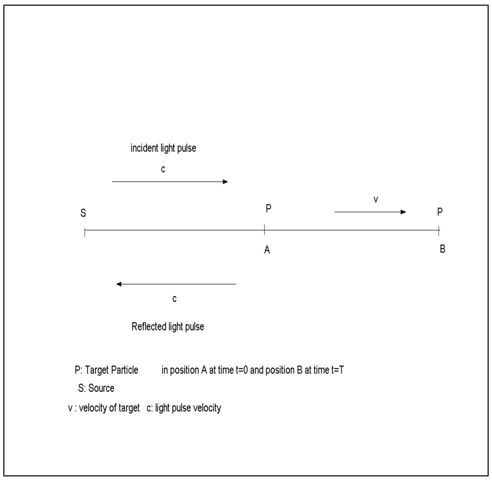

Figure 1:

Let us start from the well-known time-frequency Fourier Uncertainty relation for a rectangular light pulse of duration $\triangle t$ modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(derived in Appendix A).

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{4}$$

If we use such a light pulse to measure the position and momentum of a target particle P, we send out the pulse of duration T modulated by carrier frequency f_0 as in Figure 1 at time $t = 0$, when the particle is at a distance of d_0 from the source moving away from the source at a velocity v. This pulse gets reflected by P and is received back at the source at time $t = 2t_0$. The received carrier frequency f_r will be doppler shifted by f_d and is related to the velocity of the target as follows. Given that $d_0 + vt_0 = ct_0$ and $t_0 = \frac{d_0}{c-v}$ and that the end of the pulse at time $t = T$ is received back at time $t = T + 2t_1$ where $d_0 + v(T + t_1) = ct_1$ and $t_1 = \frac{d_0+vT}{c-v}$, received pulse length $T' = T + 2(t_1 - t_0) = T + 2\frac{vT}{c-v}$ and $f_d = \frac{1}{T} - \frac{1}{T'} = f_0 \frac{2v}{(c+v)}$, we can write

$$f_d = f_0 \frac{2v}{c+v} \tag{5}$$

Any uncertainty in the estimate of this doppler frequency will result in an uncertainty in the target velocity as follows.

$$\frac{\triangle f_d}{\triangle v} = \frac{2f_0}{c(1+\frac{v}{c})^2} \tag{6}$$

Given that the momentum of the target is given by $p = mv = v\frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}}$, the equivalent uncertainty in momentum is given by $\triangle p = \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v$. Given that position uncertainty $\triangle x = c \triangle t$ where time uncertainty of this pulse is $\triangle t = T$, we can write

$$\begin{aligned} \triangle t \triangle f &\geq \frac{1}{4\pi} \\ \frac{\triangle x}{c} \triangle v \frac{2f_0}{c(1+\frac{v}{c})^2} &\geq \frac{1}{4\pi} \\ \triangle x \triangle p &\geq \frac{m_0 c^2 (1+\frac{v}{c})^2}{8\pi f_0 (1-\frac{v^2}{c^2})^{\frac{3}{2}}} \\ \triangle x \triangle p &\geq \frac{h(m_0 c^2)(1+\frac{v}{c})^2}{8\pi (h f_0 (1-\frac{v^2}{c^2})^{\frac{1}{2}})(1-\frac{v^2}{c^2})} \end{aligned} \tag{7}$$

If we assume that the target's mass-energy $E = mc^2 = \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}} c^2$ equals the energy of the photon in the incident light pulse $E = hf_0$ and assume $v << c$, then we can write

$$\triangle x \triangle p \geq \frac{h(1+\frac{v}{c})}{8\pi(1-\frac{v}{c})} \approx \frac{h}{8\pi} \tag{8}$$

Thus we have derived the position-momentum uncertainty relation for the case of two-sided radar detection. Note that this derivation uses only **inherent Fourier uncertainty** of the light pulse and **does not use** the fact that energy of the incident light pulse affects the momentum of the target, as argued by Heisenberg in his microscope thought experiment.

4. Radar Relay Method

Radar relay Method

Let the reference frame at rest be F. Let the frame of moving vehicle V be F'. In the reference frame, receiver is located at point O and an array of receivers spaced by δ_x are located from point O to A to B. When the vehicle reaches point A, it transmits a **a light pulse** of duration δ_t to corresponding receiver in frame F at the same point A, which in turn sends another message to receiver in point O with corresponding position and time of V measured in frame F. The delay of message transmission from A to O in Frame F can be pre-calibrated. Similarly When the vehicle reaches point B, it transmits a **a light pulse** to corresponding receiver in frame F at the same point B, which in turn sends another message to receiver in point O with corresponding position and time of V measured in frame F.

When the vehicle reaches point A, let its position and time measured in Frame F by corresponding receiver in Frame F at point A be X_a and T_a. Corresponding position and time measured in Frame F' by the vehicle are X'_a and T'_a according to Lorentz Transformation. Given that we are sending encoded digital messages from A and B to O in Frame F, receiver in O has an accurate reading of X_a, T_a and X_b, T_b, accurate to the duration of the smallest digital symbol(bit) it can decode, let us call it δ_t and the smallest δ_x it can measure. As technology advances, δ_t and δ_x become smaller and smaller. If distance of vehicle is measured by sending a message with digital symbols/bits of duration δ_t from point A in Frame F' to point A in frame F, using an array of receivers spaced at δ_x, then uncertainty in position is given by δ_x.

Now the receiver at point O can compute the velocity of vehicle as $v = \frac{(X_b - X_a)}{T_b - T_a} = \frac{X_{ab}}{T_{ab}}; v + dv = \frac{X_{ab}+\delta_x}{T_{ab}+\delta_t}; dv = \frac{X_{ab}+\delta_x}{T_{ab}+\delta_t} - \frac{X_{ab}}{T_{ab}} = \frac{T_{ab}\delta_x - X_{ab}\delta_t}{T_{ab}(T_{ab}+\delta_t)} = \frac{\delta_x - v\delta_t}{(T_{ab}+\delta_t)}$. Hence we can write uncertainty relation as follows for this relay method.

$$\begin{aligned}\triangle x \triangle p &= \delta_x * m * \frac{(\delta_x - v\delta_t)}{T_{ab} + \delta_t} \\ \triangle x \triangle p &= m * \delta_x * \frac{(\delta_x - v\delta_t)}{T_{ab} + \delta_t}\end{aligned} \tag{9}$$

We can see that as technology advances, δ_t and δ_x become smaller and smaller and $\triangle x \triangle p$ can become smaller than $\frac{h}{4\pi}$.

If distance in frame F is measured by sending a light pulse of duration δ_t from point O to point A and back, $\delta_x = c * \delta_t$. Uncertainty in velocity is given by $\triangle v = \frac{(c-v)\delta_t}{T_{ab}+\delta_t}$.

$$\begin{aligned}\triangle x \triangle p &= m * c * \frac{(c - v)\delta_t^2}{T_{ab} + \delta_t} \\ \triangle x \triangle p &= m * c^2 * \frac{(1 - \frac{v}{c})\delta_t^2}{T_{ab} + \delta_t}\end{aligned} \tag{10}$$

We can see that as technology advances, δ_t become smaller and smaller and $\triangle x \triangle p$ can become smaller than $\frac{h}{4\pi}$.

Alternate Method

Let the reference frame at rest be F. Let the frame of moving vehicle V be F'. In the reference frame, receiver is located at point O. When the vehicle reaches point A, it transmits a **message** with its current position and time measured in frame F' to corresponding receiver in frame F at the point O. The delay of message transmission from A to O in Frame F can be pre-calibrated. Similarly when the vehicle reaches point B, it transmits a **message** with its current position and time measured in frame F' to corresponding receiver in frame F at the point O. Receiver at O can translate the position of V measured in F' to F. Vehicle V can also measure its current position at B by receiving a message transmitted from transmitter in position A in frame F.

When the vehicle reaches point A, let its position and time measured in Frame F' be X'_a and T'_a. Corresponding position and time measured in Frame F are X_a and T_a according to Lorentz Transformation. Given that we are sending encoded digital messages from A and B to O in Frame F, receiver in O has an accurate reading of X_a, T_a and X_b, T_b, accurate to the duration of the smallest digital symbol(bit) it can decode, let us call it δ_t and the smallest δ_x it can measure. As technology advances, δ_t and δ_x become smaller and smaller. If distance of vehicle is measured by sending a message with digital symbols/bits of duration δ_t from point A in Frame F' to point A in frame F, using an array of receivers spaced at δ_x, then uncertainty in position is given by δ_x.

5. Heisenberg's Microscope Thought Experiment

Werner Heisenberg derived the same position-momentum uncertainty relation using his famous Microscope Thought Experiment. Heisenberg asked what would happen if one would attempt to locate the position and momentum of an electron by viewing it with a microscope. The smaller the size of the electron, we need to use smaller wavelength radiation that would give the clearest indication of position which would also give the greatest impetus to the electron, and the ratio between the frequency of the radiation and the energy it delivers namely, $E = hf$ turns out to be an ineradicable factor in the uncertainty of measurements of position and momentum. In discussing this experiment, Heisenberg says, "From this photon the electron receives a Compton recoil of order of magnitude $\frac{h}{\lambda}$" where h is Planck's constant and λ is the wavelength of the photon. Given that position uncertainty $\triangle x \approx \lambda$ and momentum uncertainty of the electron is given by $\triangle p \approx \frac{h}{\lambda}$, we can derive the familiar position-momentum uncertainty relation approximately as follows.

$$\triangle x \triangle p \approx h \tag{11}$$

In Compton scattering effect, the wavelength shift of scattered radiation is given by $\lambda' - \lambda = \delta\lambda = \frac{h}{mc}(1 - \cos(\theta))$ where m is the mass of electron, c is speed of light in vaccum, h is Planck's constant and θ is the angle of scattered radiation. By the law of conservation of momentum, we know that for an electron at rest, $\mathbf{p}_{e'} = \mathbf{p}_\gamma - \mathbf{p}_{\gamma'}$ where $\mathbf{p}_{e'}$ is the momentum of electron after scattering, $\mathbf{p}_\gamma$ and $\mathbf{p}_{\gamma'}$ represent the momentum of the photon before and after scattering. Given that $\mathbf{p}_\gamma = \frac{h}{\lambda}$, $\mathbf{p}_{e'} = \frac{h(\lambda'-\lambda)}{\lambda\lambda'} = \frac{h}{\lambda\lambda'}\frac{2h}{mc} = \frac{h}{\lambda}K$ represents momentum-uncertainty of the electron after scattering, for a scattering angle of π where $K = \frac{2h}{mc\lambda'} = \frac{(\lambda' - \lambda)}{\lambda'}$. For an electron, $K \approx 1$ given that $\lambda' >> \lambda$. Let us compute the momentum uncertainty of the particle $\triangle p$ for the cases when the particle is an electron, proton, hydrogen atom and also a large object such as a car in radar application. Let us take the size of the proton to be same as size of the electron for case 1 and 100 times the size of an electron for case 2. Let us consider a large object such as a car of mass 1 Kg where we would like to have a position accuracy of 1 m and use a radiation of $\lambda = 1m$ (frequency=300 MHz) and consider the mass of electron for computing Compton recoil factor K. In general, $\triangle x \triangle p = \lambda \frac{h}{\lambda} K = hK$. We expect $K \approx 1$ for all cases, but that may not be the case as shown in Table 1.

Table 1: Position-Momentum uncertainty for various particles using heisenberg's method

Particle	Diameter m	Mass Kgs	$K = \frac{2h}{mc\lambda'} = \frac{\frac{2h}{mc}}{\lambda + \frac{2h}{mc}}$	$\triangle x \triangle p = hK$
Electron	5.6×10^{-15}	9.11×10^{-31}	0.99884712251579	$0.99884712251579 \times h$
Proton 1	5.6×10^{-15}	1.67×10^{-27}	0.32094103979088	$0.32094103979088 \times h$
Proton 2	5.6×10^{-13}	1.67×10^{-27}	0.00470402928844	$0.00470402928844 \times h$
Hydrogen Atom	1×10^{-10}	1.671×10^{-27}	0.0000264519	$0.0000264519 \times h$
Car	1	$1\ [9.11 \times 10^{-31}]$	4.852×10^{-12}	$4.852 \times 10^{-12} \times h$

Thus we can see that Heisenberg's method of deriving position-momentum uncertainty relation using the argument of incident photon affecting the momentum of the particle according to Compton effect works well for the case of the electron, while yielding much lower values for the case of hydrogen atom and a car. Hence it should be clear that position-momentum uncertainty relation can be derived by using only **inherent Fourier uncertainty** of the light pulse and is **distinct** from Heisenberg's method of **measurement induced uncertainty**. While the two methods yield similar results for the case of an electron, they will yield different results for larger objects.

Case 1: $\triangle x \triangle p < \frac{h}{4\pi}$

In Compton scattering effect, the wavelength shift of scattered radiation is given by $\lambda' - \lambda = \delta\lambda = \frac{h}{mc}(1 - \cos(\theta))$ where m is the mass of electron, c is speed of light in vaccum, h is Planck's constant and θ is the angle of scattered radiation. By the law of conservation of momentum, we know that for an electron at rest, $\mathbf{p}_{e'} = \mathbf{p}_{\gamma} - \mathbf{p}_{\gamma'}$ where $\mathbf{p}_{e'}$ is the momentum of electron after scattering, $\mathbf{p}_{\gamma}$ and $\mathbf{p}_{\gamma'}$ represent the momentum of the photon before and after scattering. Given that $\mathbf{p}_{\gamma} = \frac{h}{\lambda}$, $\mathbf{p}_{e'} = \frac{h(\lambda' - \lambda)}{\lambda\lambda'} = \frac{h}{\lambda\lambda'}\frac{2h}{mc} = \frac{h}{\lambda}K$ represents momentum-uncertainty of the electron after scattering, for a scattering angle of π where $K = \frac{2h}{mc\lambda'} = \frac{(\lambda' - \lambda)}{\lambda'}$. Let us consider the case of scattering in one dimension and photon is scattered back at angle π.

$$
\begin{aligned}
p_e^{'} = p_\gamma - p_{\gamma'} == \frac{h}{\lambda} + \frac{h}{\lambda'} = \frac{h(\lambda^{'}+\lambda)}{\lambda\lambda'} = \frac{h}{\lambda\lambda'}(2\lambda) + \frac{h}{\lambda\lambda'}\frac{2h}{mc} &= \frac{2h}{\lambda'} + \frac{1}{\lambda\lambda'}\frac{2h^2}{mc} \\
p_e^{'} &= \frac{2h}{\lambda'} + \frac{2h^2}{mc\lambda\lambda'} \\
\frac{\triangle p_e^{'}}{\triangle\lambda'} &= \frac{-2h}{(\lambda')^2} + \frac{-2h^2}{mc\lambda(\lambda')^2} \\
\triangle x &\approx \triangle\lambda^{'} \\
\triangle p_e^{'} \triangle x = \triangle p_e^{'} \triangle \lambda^{'} &= \frac{-2h^2}{mc\lambda}(\frac{\triangle\lambda^{'}}{\lambda'})^2 - 2h(\frac{\triangle\lambda^{'}}{\lambda'})^2 \\
|(\frac{\triangle\lambda^{'}}{\lambda'})^2| &>= \frac{1}{4\pi} \\
| \triangle p_e^{'} \triangle x| &>= \frac{2h^2}{mc\lambda 4\pi} - \frac{h}{2\pi}
\end{aligned}
\tag{12}
$$

We can see that if $m\lambda > \frac{2h}{c} = 4.417e-42$, then $| \triangle p_e^{'} \triangle x| > \frac{h}{2\pi}$.

For an electron, $m = 9.11 \times 10^{-31}$ and $\lambda = 5.6 \times 10^{-15}$ comparable to size of the electron and $m\lambda = 5.101600000000001e-45$.

For a proton, $m = 1.67 \times 10^{-27}$ and $\lambda = 5.6 \times 10^{-15}$ comparable to size of the electron and $m\lambda = 9.352000000000002e-42$ and $m\lambda > \frac{2h}{c} = 4.417e-42$. Hence $| \triangle p_e^{'} \triangle x| > \frac{h}{2\pi}$

For any particle of larger size and larger mass(which is expected), using a wavelength comparable to size of particle, we can achieve $m\lambda > \frac{2h}{c} = 4.417e-42$ and hence $| \triangle p_e^{'} \triangle x| > \frac{h}{2\pi}$.

We can derive the result $|(\frac{\triangle\lambda^{'}}{\lambda'})^2| >= \frac{1}{4\pi}$ as follows, starting from Fourier Uncertainty relation.

$$
\begin{aligned}
\triangle f \triangle t &>= \frac{1}{4\pi} \\
f &= \frac{c}{\lambda} \\
\triangle f &= \frac{-c}{\lambda^2} \triangle \lambda \\
\triangle x = c \triangle t &\approx \triangle\lambda \\
\frac{-c}{\lambda^2} \triangle \lambda \frac{\triangle\lambda}{c} &>= \frac{1}{4\pi} \\
|(\frac{\triangle\lambda}{\lambda})^2| &>= \frac{1}{4\pi} \\
|(\frac{\triangle\lambda^{'}}{\lambda'})^2| &>= \frac{1}{4\pi}
\end{aligned}
\tag{13}
$$

Case 2: $\triangle t \triangle f' < \frac{1}{4\pi}$

If we use signal processing techniques and are able to oversample the photon pulse of period $dT = 2 * 10^{-11}$ seconds and take the Fourier Transform of the pulse and locate the peak of the Fourier Transform to a high accuracy, then we can get $\triangle t \triangle f' < \frac{1}{4\pi}$ and hence

$$\triangle x \triangle p < \frac{h}{4\pi} \tag{14}$$

6. Compton Effect Rederived

Compton effect derivation in the link assumes electron is at rest before collision with a photon which is **questioned** in section below. It also assumes $E = hf = pc$ for a photon and $p = \frac{h}{\lambda}$. These equations are **questioned** in a separate paper $E\_hf.pdf$. It also assumes the truth of Fourier Uncertainty relation $\triangle t \triangle f \geq \frac{1}{2\pi}$ which is **questioned** in earlier sections.

6.1. Compton Effect Rederived with zero electron velocity before collision with photon

Heisenberg used **Compton effect** to argue that when a photon hits an electron, some of photon energy is transferred to electron with rest mass m_e and it starts moving from **rest** with a velocity v and this results in **measurement induced uncertainty** in electron velocity and momentum which is related to **inherent Fourier uncertainty** $\triangle t \triangle f \geq \frac{1}{2\pi}$ which is derived for Gaussian function. He used the result from Compton effect $(p'_e c)^2 = (hf + hf')^2$ for $\theta = \pi$ where f, f' are the frequencies of photon before and after collision and p'_e is the electron momentum after collision, m_e is electron mass, c, h are speed of light and Planck's constant respectively, as derived follows. we use $\triangle p = d(p'_e), \triangle f = d(f')$. In the last step, if $f' \approx f$, then $\triangle x \triangle p \geq \frac{h}{2\pi}$.

$$\begin{aligned} (p'_e c)^2 = (hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta &= (hf + hf')^2 \\ p'_e = m'_e v &= \frac{h}{c}(f + f') \\ \frac{d(p'_e)}{df} &= \frac{h}{c} \end{aligned} \tag{15}$$

Replacing $d(p'_e)$ by $\triangle p$, we have

$$\begin{aligned} \triangle x \triangle p = c \triangle t \triangle p &= c \triangle t \triangle f \frac{h}{c} \\ \triangle t \triangle f &\geq \frac{1}{4\pi} \\ \triangle x \triangle p &\geq \frac{h}{4\pi} \end{aligned} \tag{16}$$

6.2. ***Compton Effect Rederived with Non-zero electron velocity before collision with photon***

Let $f, f^{'}$ be the frequency of the light before and after collision with an electron with rest mass m_e. Let $v, v^{'}$ be the velocity of electron before and after collision. Let $p_e = \frac{m_e v}{\sqrt{1-\frac{v^2}{c^2}}}, p_e^{'} = \frac{m_e v^{'}}{\sqrt{1-\frac{(v^{'})^2}{c^2}}}$ be the momentum of electron before and after collision. Let $K = \sqrt{1 - \frac{v^2}{c^2}}$ and $K^{'} = \sqrt{1 - \frac{(v^{'})^2}{c^2}}$.

Energy of electron before and after collision is given by $E_e = m_T c^2$ and $E_e^{'} = m_T^{'} c^2$ and Energy of photon before and after collision is given by $E_\gamma = hf$ and $E_\gamma^{'} = hf^{'}$. Using Conservation of Energy we have as follows.

$$\begin{aligned} E_\gamma + E_e &= E_\gamma^{'} + E_e^{'} \\ hf + \sqrt{(m_e c^2)^2 + (p_e c)^2} &= hf^{'} + \sqrt{(m_e c^2)^2 + (p_e^{'} c)^2} \\ (m_e c^2)^2 + (p_e^{'} c)^2 &= (hf - hf^{'} + \sqrt{(m_e c^2)^2 + (p_e c)^2})^2 \\ (p_e^{'} c)^2 = -(m_e c^2)^2 + (hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} &+ (m_e c^2)^2 + (p_e c)^2 + 2h(f - f^{'}) \sqrt{(m_e c^2)^2 + (p_e c)^2} \\ (p_e^{'} c)^2 &= (hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} + (p_e c)^2 + 2h(f - f^{'}) \sqrt{(m_e c^2)^2 + (p_e c)^2} \end{aligned} \tag{17}$$

Using Conservation of Momentum we have as follows.

$$\begin{aligned} \vec{p_\gamma} + \vec{p_e} &= \vec{p_\gamma^{'}} + \vec{p_e^{'}} \\ \vec{p_e^{'}} &= \vec{p_\gamma} + \vec{p_e} - \vec{p_\gamma^{'}} \\ (p_e^{'})^2 &= \vec{p_e^{'}} \cdot \vec{p_e^{'}} = (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \cdot (\vec{p_\gamma} - \vec{p_\gamma^{'}} + \vec{p_e}) \\ (p_e^{'})^2 &= [(p_\gamma)^2 + (p_\gamma^{'})^2 - 2p_\gamma p_\gamma^{'} \cos\theta] + [p_e^2 + 2p_e p_\gamma \cos\theta_1 - 2p_e p_\gamma^{'} \cos\theta_2] \end{aligned} \tag{18}$$

We multiply both sides of above equation by c^2 and use $p_\gamma = \frac{hf}{c}, p_\gamma^{'} = \frac{hf^{'}}{c}$ and write as follows.

$$(p_e^{'})^2 c^2 = [(hf)^2 + (hf^{'})^2 - 2h^2 f f^{'} \cos\theta] + [p_e^2 c^2 + 2p_e hfc \cos\theta_1 - 2p_e hf^{'} c \cos\theta_2] \tag{19}$$

Equating Eq. 17 and Eq. 19 and cancelling common terms, we have

$$2h^2 f f^{'} [1 - \cos\theta] = 2h(f - f^{'}) \sqrt{(m_e c^2)^2 + (p_e c)^2} - 2hcp_e[f \cos\theta_1 - f^{'} \cos\theta_2] \tag{20}$$

Dividing both sides of above equation by the term $2hff^{'}m_ec$, we use $p_e = \frac{m_ev}{\sqrt{(1-\frac{v^2}{c^2})}}$, $\lambda = \frac{c}{f}, \lambda^{'} = \frac{c}{f'}$ we have

$$\frac{h}{m_ec}[1-\cos\theta] = (\lambda^{'}-\lambda)\sqrt{1+(\frac{p_e}{m_ec})^2} - \frac{1}{ff'm_e}p_e[f\cos\theta_1 - f^{'}\cos\theta_2] \tag{21}$$

If **electron is at rest** before collision, $p_e = 0$ and we get the familiar **Compton effect equation** as follows.

$$\frac{h}{m_ec}[1-\cos\theta] = (\lambda^{'}-\lambda) \tag{22}$$

Thus we can see that Eq. 21 has **extra terms** when electron has **non-zero velocity** before collision with photon.

Now we substitute $\theta = \pi, \theta_1 = 0, \theta_2 = \pi$ in Eq. 19 , assuming the case where electron direction is the same before and after collision and is aligned with photon direction before collision and photon is reflected back at angle π after collision.

$$\begin{aligned}
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\
(p_e^{'})^2c^2 &= [(hf)^2 + (hf^{'})^2 + 2h^2ff^{'}] + [p_e^2c^2 + 2p_ehfc + 2p_ehf^{'}c] \\
(p_e^{'})^2c^2 &= h^2(f+f^{'})^2 + p_ec[p_ec + 2h(f+f^{'})] \\
(p_e^{'})^2 &= \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_ec + 2h(f+f^{'})]
\end{aligned} \tag{23}$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle **more complicated**, compared to Eq. 15 where $p_e = 0$.

$$\begin{aligned}
(p_e^{'})^2 &= \frac{h^2}{c^2}(f+f^{'})^2 + \frac{p_e}{c}[p_ec + 2h(f+f^{'})] = \frac{h^2}{c^2}(f+f^{'})^2X(f,f^{'},p_e) \\
X(f,f^{'},p_e) &= 1 + \frac{p_e}{c}(p_ec + 2h(f+f^{'})) = [1 + Z(f,f^{'},p_e)] \\
A(f,f^{'},p_e) &= [\sqrt{X(f,f^{'},p_e)} - 1] = ([1 + \frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] - 1) \\
&= [\frac{1}{2}Z(f,f^{'},p_e) + \frac{\frac{1}{2}C_2}{!2}Z(f,f^{'},p_e)^2 + ...] \\
Y(f,f^{'},p_e) &= \frac{h}{c}(f+f^{'})A(f,f^{'},p_e)
\end{aligned} \tag{24}$$

We can see that $Y(f, f', p_e) > 0$ for all f.

$$p'_e = m'_e v = \frac{h}{c}(f + f') + Y(f, f', p_e)$$
$$\frac{d(p'_e)}{df} = \frac{h}{c} + \frac{d(Y)}{df} \tag{25}$$

Replacing $d(p'_e)$ by $\triangle p$, we have

$$\triangle x \triangle p = c \triangle t \triangle p = c \triangle t \triangle f[\frac{h}{c} + \frac{d(Y)}{df}]$$
$$\triangle t \triangle f \geq \frac{1}{4\pi}$$
$$\triangle x \triangle p \geq \frac{h}{4\pi} + \frac{c}{4\pi}\frac{d(Y)}{df} \tag{26}$$

We can see that the second term in above equation is an **extra term,** which makes derivation of Heisenberg's uncertainty principle **more complicated**, compared to Eq. 15 where $p_e = 0$.

Compton Effect Problems:

It is well known that Heisenberg's Uncertainty Principle says that $\triangle x \triangle p \geq \frac{h}{4\pi}$ where $\triangle x, \triangle p$ are the uncertainty in particle position and momentum. Heisenberg used **particle size and wavelength Rule** and the analogy of Compton scattering of an electron by photon, while deriving this result in the link. [Compton effect derivation in the link]

Consider the case where a **photon** of energy $E = hf = \frac{hc}{\lambda}$ and momentum $p_c = \frac{h}{\lambda}$ hits an **electron** moving with a momentum $p_e = m_e v = \frac{m_0 v}{\sqrt(1-\frac{v^2}{c^2})}$ **before** being hit by the photon, where m_0 is the rest mass of electron. If we use **photon wavelength** comparable to the **size of the electron** of $1e-11$ meters, then **uncertainty in electron position** $\triangle x \approx \lambda = 1e-11$ meters. Let us consider example values $c = 3e8, v = c * 0.99$ meters per second, $m_0 = 9.1e-31$ Kgs and $h = 6.626e-34 m^2 Kg/sec$.

$$p_c = \frac{h}{\lambda} = 6.626e-23$$
$$p_e = m_e v = \frac{m_0 v}{\sqrt(1 - \frac{v^2}{c^2})} = 1.9159e-21$$
$$\triangle x \approx \lambda$$
$$\triangle p \approx p_c = 6.626e-23 \triangle x \triangle p = h \tag{27}$$

We can see that photon momentum is **28 times smaller** than electron momentum. The photon imparts a **very small** uncertainty to electron momentum of the order of $\triangle p = 6.626e - 23$ which is the maximum momentum of the photon.

ONLY IF we assume that **entire** photon momentum is transferred to the electron, **then** we get $\triangle x \triangle p = h$. Typically this is **not** the case. Only part of the photon momentum is transferred to electron according to **Compton effect** equations. $(p_e^{'})^2c^2 = (hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta$.

2. If we can use **a short pulse of duration** $\triangle t = T = \frac{1}{f} = 3.33e - 20$ seconds, we get $\triangle x = c \triangle t = \lambda = 1e - 11$ meters and $\triangle x \triangle p = h$, for the same choice of $\lambda = 1e - 11$ meters.

$$\begin{aligned} \triangle p \approx p_c = \frac{h}{\lambda} = 6.626e - 23 \\ \triangle x = c \triangle t = \lambda = 1e - 11 \\ \triangle x \triangle p = h \end{aligned} \tag{28}$$

Note that $f = \frac{c}{\lambda} = \frac{3e8}{1e-11} = 3e19$ Hz and $T = \frac{1}{f} = 3.33e - 20$ seconds.

2a) For the **specific case** of photon and electron aligned before collision in same direction and photon gets reflected back at an angle 180 degrees after collision with **electron at rest** and imparting some of its momentum to the electron, we can write $p_\gamma = p'_e + p_{\gamma'}$ given p_e is zero. We can estimate $p_\gamma, p_{\gamma'}$ accurately using signal processing techniques and hence we can estimate p'_e accurately.

For the theoretical case of a system with only one photon and one electron, with electron at rest before collision, we can estimate p'_e accurately and **get below** Heisenberg Uncertainty Limit.

2b) Let us take a **specific example** of electron mass $m_e = 9.1e - 31$ Kgs, electron velocity $v_e = 1e6$ meters per second **before** collision, hence uncertainty in electron momentum $\triangle p = m_e * v_e$ if electron is **in motion before collision** with photon. Let uncertainty in electron position be $\triangle x = \lambda = 1e - 11$ meters. $\triangle p_e \approx 1e - 24$.

Let us differentiate $p_e^{'}$ with respect to p_e, to estimate **uncertainty** in $p_e^{'}$ due to **not knowing** value of p_e. Given that our estimation accuracy of f, f' does not depend on p_e, for $\theta = \theta_2 = \pi, \theta_1 = 0$, we can write

$$\begin{aligned} (p_e^{'})^2c^2 = [(hf)^2 + (hf^{'})^2 - 2h^2ff^{'}\cos\theta] + [p_e^2c^2 + 2p_ehfc\cos\theta_1 - 2p_ehf^{'}c\cos\theta_2] \\ 2p_e^{'}\frac{dp_e^{'}}{dp_e} = 2p_e + 2\frac{h}{c}(f + f') \\ \triangle p'_e = \triangle p_e\frac{(p_e + 2\frac{h}{c}(f + f'))}{p'_e} \end{aligned} \tag{29}$$

Given that $2\frac{h}{c}(f + f') << p_e$, for the case $\frac{p_e}{p'_e} \approx 1$, we have $\triangle p'_e \approx \triangle p_e \approx 1e - 24$.

$$\triangle x \triangle p'_e \approx \triangle x \triangle p_e = \lambda * m_e * v_e = 1e - 11 * 9.1e - 31 * 1e6 = 9e - 36 < \frac{h}{4\pi} \tag{30}$$

We can see that, **even if do not know** electron momentum before collision, by estimating accurately f, f', we can **get below** Heisenberg Uncertainty Limit.

For the theoretical case of a system with only one photon and one electron, with electron in motion before collision, we can estimate p'_e accurately and **get below** Heisenberg Uncertainty Limit.

2c) Given **conservation of momentum**, for the case of electron **at rest before** collision with photon, we use Compton effect equation $\frac{h}{m_e c}[1 - \cos\theta] = (\lambda' - \lambda) = 4.85e - 12$ meters for $\theta = \pi$, and $(p_e')^2c^2 = (hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta$ we can estimate both f, f' accurately using signal processing techniques and hence we know momentum of photon before and after collision **accurately** and can **estimate the momentum** imparted to the electron at rest. We can get **below** $p_c = \frac{h}{\lambda} = 6.626e - 23$ limit and we can **get below** Heisenberg Uncertainty Limit $\triangle x \triangle p < \frac{1}{4\pi}$. **This holds** for small particles and large objects.

2d) For the case of electron **moving before** collision with photon, if we know $m_e, \theta, \theta_1, \theta_2$, we can estimate both f, f' accurately using signal processing techniques and hence estimate p_e, p'_e accurately using equations $\frac{h}{m_e c}[1 - \cos\theta] = (\lambda' - \lambda)\sqrt{1 + (\frac{p_e}{m_e c})^2} - \frac{1}{ff' m_e} p_e[f \cos\theta_1 - f' \cos\theta_2]$ and $(p_e')^2c^2 = [(hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta] + [p_e^2c^2 + 2p_e hfc\cos\theta_1 - 2p_e hf' c\cos\theta_2]$. We can get **below** $p_c = \frac{h}{\lambda} = 6.626e - 23$ limit and we can **get below** Heisenberg Uncertainty Limit $\triangle x \triangle p < \frac{1}{4\pi}$. **This holds** for small particles and large objects.

$$\begin{aligned}
\frac{h}{m_e c}[1 - \cos\theta] &= (\lambda' - \lambda)\sqrt{1 + (\frac{p_e}{m_e c})^2} - \frac{1}{ff' m_e} p_e[f \cos\theta_1 - f' \cos\theta_2] \\
p_e(f \cos\theta_1 - f' \cos\theta_2) &= ff' m_e[(\lambda' - \lambda)\sqrt{1 + (\frac{p_e}{m_e c})^2} - \frac{h}{m_e c}(1 - \cos\theta)]
\end{aligned} \tag{31}$$

We can estimate P_e from above equation given known $m_e, \theta, \theta_1, \theta_2$ and then we can substitute above equation in equation below.

$$\begin{aligned}
(p_e')^2c^2 &= [(hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta] + [p_e^2c^2 + 2p_e hfc\cos\theta_1 - 2p_e hf' c\cos\theta_2] \\
(p_e')^2c^2 &= [(hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta] + [p_e^2c^2 + 2hcp_e(f\cos\theta_1 - f' \cos\theta_2)] \\
(p_e')^2c^2 &= (hf)^2 + (hf')^2 - 2h^2 ff' \cos\theta + p_e^2c^2 \\
&+ 2hcff' m_e[(\lambda' - \lambda)\sqrt{1 + (\frac{p_e}{m_e c})^2} - \frac{h}{m_e c}(1 - \cos\theta)]
\end{aligned}$$

(32)

We can estimate P'_e from above equation, given known P_e, m_e, θ.

3) **If** we use Heisenberg derivation of Compton effect which uses **Fourier uncertainty relation** $\triangle t \triangle f \geq \frac{1}{2\pi}$ based on specific case of Gaussian pulse of duration $\triangle t$ and frequency spread $\triangle f$ and hence we can get around this limit by using signal processing techniques to finely estimate frequency or by repeated short pulses and using duration between pulses and NOT using the frequency of the photon in calculations.

4) **Compton effect derivation** in the link shows that electron momentum **after** collision with photon is given by $(p_e^{'})^2c^2 = (hf)^2 + (hf^{'})^2 - 2h^2 ff^{'} \cos\theta$ which translates to $p_e^{'} = \frac{h}{c}(f + f^{'})$ for the case of $\theta = \pi$ where photon is reflected back at an angle π.

But photon's momentum **before** collision is given by $p_c = \frac{hf}{c}$ which is **less than** electron momentum after collision $p_e^{'} = \frac{h}{c}(f + f^{'})$ which is **NOT possible** because electron was at rest before collision and $0 < f^{'} < f_0$!

5) **Conservation of Energy and Momentum:** In a **2-body system**, energy conservation law $E_e + E_\gamma = E'_e + E_{\gamma'}$ and $\vec{p_\gamma} + \vec{p_e} = \vec{p_\gamma^{'}} + \vec{p_e^{'}}$ hold, for every point in the trajectory of the two bodies. But in real life, matter consists of large number of atoms and particles and light consists of large number of photons, so this two body model used to derive Compton effect is NOT applicable.

For the theoretical case of a system with only one photon and one electron, with electron at rest or in motion before collision, we can estimate p'_e accurately and **get below** Heisenberg Uncertainty Limit.

6) **Equivalence of Particle and Wave Energy:** Let us consider a particle P of rest mass m_0 travelling with a velocity v. According to Einstein's mass-energy equivalence principle,its energy is given by $E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ where m is the relativistic mass of the particle. According to De-Broglie, every particle has a wave associated with it of wavelength $\lambda = \frac{h}{mv}$ where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

Given that both the particle and the wave **must travel at the same velocity**, it turns out that wave group velocity equals particle velocity $v_g = v$. We can write $\lambda = \frac{v_p}{f}$ where v_p is the wave phase velocity given by $v_p = \frac{c^2}{v}$. [See De-Broglie relations.]

$$\lambda = \frac{h}{mv} = \frac{v_p}{f} = \frac{c^2}{vf}$$
$$hf = mc^2$$
$$m = \frac{m_0}{\sqrt{(1 - \frac{v^2}{c^2})}} \tag{33}$$

This means total relativistic energy of the particle given by $E = mc^2$ should equal **energy of a single photon** hf ?

As the particle velocity v **increases** towards velocity c, particle mass "m" approaches **close to infinity**. Above equation $hf = mc^2$ means that **frequency** of associated particle wave f must increase towards **infinity** and $\lambda = \frac{v_p}{f}$ approaches zero! This means **wavelength of the particle wave**, which is given by the distance travelled by the wave in a single wave cycle must **approach zero**!

7. Alternative Explanation for Compton Effect and Pound Rebka Experiment

If we model the crystal or matter or system as a collection of atoms which are exerting a gravitational force on photons in the light beam, then the **photons in first peak** of light wave leaving the medium at farthest distance from the medium **compared to** the next peak which is nearer to the medium, **are subjected to lesser gravitational force**, hence travel at a **faster velocity** and hence the wave is lengthened, with longer wavelengths.

8. Latest: Problem in T and T'! Two-sided Radar using light pulse train

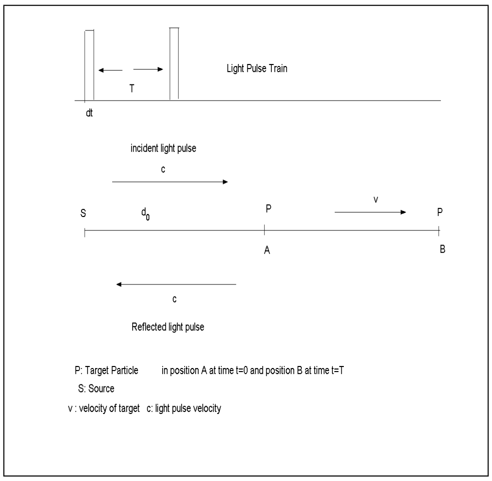

Figure 2:

Let us use a different light pulse train as shown in Figure 2 to estimate the position and momentum of a target and see if we can get below the limit $\triangle x \triangle p \geq \frac{h}{4\pi}$. The pulse train consists of a train of pulses of duration $\triangle t$ separated by time T. Thus position uncertainty of the target is estimated as $\triangle x = c \triangle t$ while momentum uncertainty of the target is related to the estimate of pulse interval T.

If we use such a light pulse train to measure the position and momentum of a target particle P, we send out the pulse 1 of duration $\triangle t$ at time $t = 0$, when the particle is at a distance of d_0 from the source moving away from the source at a velocity v. This pulse gets reflected by P and is received back at the source at time $t = 2t_0$. Given that $d_0 + vt_0 = ct_0$ and $t_0 = \frac{d_0}{c-v}$ and that the second pulse at time $t = T$ is received back at time $t = T + 2t_1$ where $d_0 + v(T + t_1) = ct_1$ and $t_1 = \frac{d_0+vT}{c-v}$, received pulse interval is given by $T' = T + 2(t_1 - t_0) = T + 2\frac{vT}{c-v} = T\frac{c+v}{c-v}$ and $\frac{\triangle T'}{\triangle v} = \frac{2T}{c(1-\frac{v}{c})^2}$ and given that $\triangle p = \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v$ and given that $d_0 + vt_0 = d_1 = ct_0$, $\triangle d_1 = c \triangle t_0$ and $\triangle x = | \triangle d_1| = c \triangle t_0$, we can write

$$\begin{aligned}
\triangle x \triangle p &= c \triangle t_0 \frac{m_0}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v \\
\triangle x \triangle p &= c \triangle t_0 \frac{m_0}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \triangle T' \frac{c(1 - \frac{v}{c})^2}{2T} \\
&= m_0 c^2 \frac{(1 - \frac{v}{c})^{\frac{1}{2}}}{(1 + \frac{v}{c})^{\frac{3}{2}}} \triangle T' \triangle t_0 \frac{1}{2T}
\end{aligned} \tag{34}$$

where $\triangle t_0$ and $\triangle T'$ are the uncertainty or accuracy of the local time measurement system. Let us examine how we can drive the product $\triangle x \triangle p < \frac{h}{4\pi}$. Firstly, it is clear that as $v \to c$, the product $\triangle x \triangle p$ tends to 0 **for any choice** of other variables and thus we can **get below** Heisenberg Uncertainty Principle Limit. [In this example, given the target is a large object, we are **ignoring** uncertainty in momentum $\triangle p = \frac{h}{\lambda}$ imparted to the target by the impinging light]

[For example, we can consider a typical Radar application where mass of the target $m_0 = 0.25$ Kg, choose $\lambda = 1$ meters **comparable to size of the target** and $\triangle x = c \triangle t_0 = \lambda = 1$ meter and choose $T = 1$ seconds. **We can use signal processing to accurately estimate** T' such that error is very low in $\triangle T' = 1e - 10$ seconds , $1 - \frac{v}{c} = 10^{-60}$, then $\triangle x \triangle p < \frac{h}{4\pi}$.

Secondly, for $v << c$, we can choose for example $m_0 = \frac{1}{9}$ Kg,$T = 1$ and as technology advances in future, accuracy in time measurements becomes smaller and smaller and We can use **signal processing to accurately estimate** T' such that error is very low in $\triangle T' = 1e-40$ seconds which gives $\triangle x \triangle p < \frac{h}{4\pi}$.]

Thus we have derived the position-momentum uncertainty relation for the case of two-sided radar detection using a different pulse train and arrived at the product of position-momentum uncertainty well below the limit $\frac{h}{4\pi}$.

We could also consider the case when duration of the pulse is T and prove a similar result.

9. Older: Two-sided Radar using light pulse train Draft 1

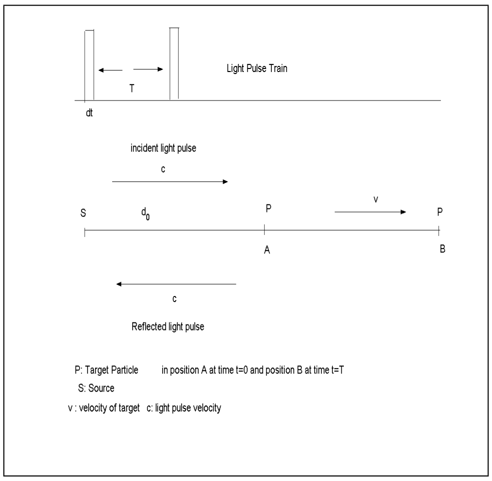

Figure 3:

Let us use a different light pulse train as shown in Figure 2 to estimate the position and momentum of a target and see if we can get below the limit $\triangle x \triangle p \geq \frac{h}{4\pi}$. The pulse train consists of a train of pulses of duration $\triangle t$ separated by time T. Thus position uncertainty of the target is estimated as $\triangle x = c \triangle t$ while momentum uncertainty of the target is related to the estimate of pulse interval T.

If we use such a light pulse train to measure the position and momentum of a target particle P, we send out the pulse 1 of duration $\triangle t$ at time $t = 0$, when the particle is at a distance of d_0 from the source moving away from the source at a velocity v. This pulse gets reflected by P and is received back at the source at time $t = 2t_0$. Given that $d_0 + vt_0 = ct_0$ and $t_0 = \frac{d_0}{c-v}$ and that the second pulse at time $t = T$ is received back at time $t = T + 2t_1$ where $d_0 + v(T + t_1) = ct_1$ and $t_1 = \frac{d_0+vT}{c-v}$, received pulse interval is given by $T' = T + 2(t_1 - t_0) = T + 2\frac{vT}{c-v} = T\frac{c+v}{c-v}$ and $\frac{\triangle T'}{\triangle v} = \frac{2T}{c(1-\frac{v}{c})^2}$ and given that $\triangle p = \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v$ and given that $d_0 + vt_0 = ct_0$, $\triangle d_0 = -t_0 \triangle v$ and $\triangle x = | \triangle d_0| = c \triangle T' \frac{t_0(1-\frac{v}{c})^2}{2T}$, we can write

$$
\begin{aligned}
\triangle x \triangle p &= c \triangle T' \frac{t_0(1-\frac{v}{c})^2}{2T} \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v \\
\triangle x \triangle p &= ct_0 \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle T'^2 \frac{c(1-\frac{v}{c})^4}{4T^2} \\
&= m_0 c^2 \frac{(1-\frac{v}{c})^{\frac{5}{2}}}{(1+\frac{v}{c})^{\frac{3}{2}}} \triangle T'^2 \frac{t_0}{4T^2} \\
&= m_0 c \frac{(1-\frac{v}{c})^{\frac{3}{2}}}{(1+\frac{v}{c})^{\frac{3}{2}}} \triangle T'^2 \frac{d_0}{4T^2}
\end{aligned} \tag{35}
$$

where $t_0 = \frac{d_0}{c(1-\frac{v}{c})}$. Let us examine how we can drive the product $\triangle x \triangle p < \frac{h}{4\pi}$. Firstly, it is clear that as $v \to c$, the product tends to 0 for any choice of other variables. Secondly, for $v << c$, we can choose for example $m_0 = 1Kg$, $d_0 = 1m$,$\triangle T' = 10^{-10}sec$, $T = 10^{11}seconds$ which gives $\triangle x \triangle p < \frac{h}{4\pi}$.

Thus we have derived the position-momentum uncertainty relation for the case of two-sided radar detection using a different pulse train and arrived at the product of position-momentum uncertainty well below the limit $\frac{h}{4\pi}$.

10. Section A: Target affected by photon. Large Target

If we use such a light pulse train to measure the position and momentum of a target particle P, we send out the pulse 1 of duration $\triangle t$ at time $t = 0$, when the particle is at a distance of d_0 from the source moving away from the source at a velocity v. This pulse gets reflected by P and is received back at the source at time $t = 2t_0$. This pulse changes the velocity of target by $v + dv$. Given that $d_0 + vt_0 = ct_0$ and $t_0 = \frac{d_0}{c-v}$ and that the second pulse at time $t = T$ is received back at time $t = T + 2t_1$ where $d_0 + vt_0 + (v + dv)(T + t_1 - t_0) = ct_1$ and $t_1 = \frac{d_0+vT+dv(T-t_0)}{c-(v+dv)}$, received pulse interval is given by $T' = T + 2(t_1 - t_0)$ we can write

$$
\begin{aligned}
d_0 + vt_0 = ct_0 = d_1; &\qquad t_0 = \frac{d_0}{c-v} \\
d_0 + vt_0 + (v+dv)(T + t_1 - t_0) = ct_1 = d_2; &\qquad t_1 = \frac{d_0 + vT + dv(T - t_0)}{(c-(v+dv))} \\
T^{'} = T + 2(t_1 - t_0) = T + 2\frac{(d_0 + vT + dv(T - t_0))}{(c-(v+dv))} - \frac{2d_0}{(c-v)} & \\
\frac{\triangle T'}{\triangle v} = 2\frac{(c-(v+dv))(T - dv\frac{d_0}{(c-v)^2}) + (d_0 + vT + dv(T - \frac{d_0}{c-v}))}{(c-(v+dv))^2} - \frac{2d_0}{(c-v)^2} & \\
\frac{\triangle T'}{\triangle v} = 2\frac{(d_0 + cT - dv(\frac{d_0}{c-v})) - (c-(v+dv))(dv\frac{d_0}{(c-v)^2}))}{(c-(v+dv))^2} - \frac{2d_0}{(c-v)^2} = f_1 &
\end{aligned}
\tag{36}
$$

We also know that

$$
\begin{aligned}
\triangle p &= \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v \\
\triangle x \triangle p &= c \triangle t \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \frac{\triangle T'}{f_1}
\end{aligned}
\tag{37}
$$

If $dv << v << c$, then $f_1 \approx \frac{2cT}{(c-v)^2}$. At low frequencies corresponding to $\lambda = 1$ meter, momentum of the photon $p = \frac{h}{\lambda} = 6.67e - 34$ and induces a change in velocity of $dv = \frac{p}{m}$ in the target of mass 1 Kg and we see that $dv \approx 6.67e - 34\frac{m}{sec}$. Hence

$$
\begin{aligned}
\triangle p &= \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v \\
\triangle x \triangle p &\approx c \triangle t \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle T' \frac{c^2(1-\frac{v}{c})^2}{2cT} \\
\triangle x \triangle p &\approx \frac{m_0 c^2}{(1+\frac{v}{c})^{\frac{3}{2}}} \triangle t \triangle T' \frac{(1-\frac{v}{c})^{\frac{1}{2}}}{2T}
\end{aligned}
\tag{38}
$$

We can make the pulse with a single photon. Use a wavelength of $\lambda = 1$ and pulse amplitude of $A = 1$ and duration dT, we require the energy of the pulse $E = A^2 dT = h * f = \frac{h*c}{\lambda}$ for a single photon pulse. Hence $dT = \frac{h*c}{\lambda A^2} = \frac{6.67e-34*3e8}{1} = 2*(10)^{-25}$.

At low velocities $v << c$, choose $T = 100$ second and $\triangle T'$ of the order of dT, choose $m_0 = 1$ Kg and we get $\triangle x \triangle p \approx 10^{-35}$ which is far less than $\frac{h}{2\pi}$.

If we use 100 photons in the pulse instead of a single photon, then $dT = \frac{100*h*c}{\lambda A^2} = \frac{6.67e-34*3e8*100}{1} = 2*(10)^{-23}$. These 100 photons induce a change in target velocity of $100 * dv = 6.67e-32$ which is much smaller than target velocity of v and if we use pulse interval of $T = 100 * 100 * 100$ seconds, we get similar results and we get $\triangle x \triangle p \approx 10^{-35}$ which is far less than $\frac{h}{2\pi}$.

11. Section A: Target affected by photon. Small Target. Latest Version.

Let us consider a small target like an electron with mass $m_e = 9.10938356 * (10)^{-31}$ Kgs and size 10^{-14} meter. We can make the pulse with a single photon. Use a wavelength of $\lambda = 10^{-14}$ meter and pulse amplitude of $A = 1$ and duration dT, we require the energy of the pulse $E = A^2 dT = h * f = \frac{h*c}{\lambda}$ for a single photon pulse. Hence $dT = \frac{h*c}{\lambda A^2} = \frac{6.67e-34*3e8}{10^{-14}} = 2 * (10)^{-11}$ seconds. Frequency of the photon is $f = \frac{c}{\lambda} = \frac{3e8}{10^{-14}} = 3e22$ Hz.

If we use such a light pulse train to measure the position and momentum of a target particle P, we send out the pulse 1 of duration $\triangle t$ at time $t = 0$, when the particle is at a distance of d_0 from the source moving away from the source at a velocity v. This pulse gets reflected by P and is received back at the source at time $t = 2t_0$. This pulse changes the velocity of target by $v + dv$. Given that $d_0 + vt_0 = ct_0$ and $t_0 = \frac{d_0}{c-v}$ and that the second pulse at time $t = T$ is received back at time $t = T + 2t_1$ where $d_0 + vt_0 + (v + dv)(T + t_1 - t_0) = ct_1$ and $t_1 = \frac{d_0 + vT + dv(T - t_0)}{c-(v+dv)}$, received pulse interval is given by $T' = T + 2(t_1 - t_0)$ we can write

$$
\begin{aligned}
d_0 + vt_0 = ct_0 = d_1; &\qquad t_0 = \tfrac{d_0}{c-v} \\
d_0 + vt_0 + (v + dv)(T + t_1 - t_0) = ct_1 = d_2; &\quad t_1 = \tfrac{d_0 + vT + dv(T - t_0)}{(c-(v+dv))} \\
ct_0 + (v + dv)(T + t_1 - t_0) = ct_1 & \\
v' = (v + dv) = c\frac{(t_1 - t_0)}{(T + t_1 - t_0)}; &\quad (t_1 - t_0) = \tfrac{v' T}{(c-v')} \\
T' = T + 2(t_1 - t_0) = T + 2\frac{v' T}{(c - v')} = T[1 + \frac{2v'}{(c - v')}] = T[\frac{(c + v')}{(c - v')}] & \\
\frac{\triangle T'}{\triangle v'} = T\frac{(c - v') + (c + v')}{(c - v')^2} = \frac{2cT}{(c - v')^2} = \frac{2T}{c(1 - \frac{v'}{c})^2} = f_2 &
\end{aligned}
\tag{39}
$$

We also know that

$$
\begin{aligned}
\triangle p = \frac{m_e}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v' & \\
\triangle x \triangle p = c \triangle t \frac{m_e}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \frac{\triangle T'}{f_2} = c \triangle t \triangle T' \frac{m_e}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \frac{c}{2T}(1 - \frac{v'}{c})^2 & \\
\triangle x \triangle p = \triangle t \triangle T' m_e c^2 \frac{1}{2T} \frac{(1 - \frac{v'}{c})^{\frac{1}{2}}}{(1 + \frac{v'}{c})^{\frac{3}{2}}} &
\end{aligned}
\tag{40}
$$

Let us consider a small target like an electron with mass $m_e = 9.10938356 * (10)^{-31}$ Kgs and $T = 1$ second.

Case 1: $v << c$

At low electron velocities, if we can achieve $\triangle t, \triangle T'$ of the order of 10^{-12}, $\triangle x \triangle p \approx 10^{-37} << \frac{h}{4\pi}$.

Case 1: $v \approx c$
As the velocity of electron approaches light speed c, the term $(1-\frac{v^{'}}{c})^{\frac{1}{2}} \rightarrow 0$ and hence $\triangle x \triangle p \rightarrow 0$ and $\triangle x \triangle p << \frac{h}{4\pi}$.

12. Section A: Target affected by photon. Small Target. Uses Compton Effect

Let us consider a small target like an electron with mass $m_e = 1e-21$ Kgs and size 10^{-14} meter. We can make the pulse with a single photon. Use a wavelength of $\lambda = 10^{-14}$ meter and pulse amplitude of $A = 1$ and duration dT, we require the energy of the pulse $E = A^2 dT = h * f = \frac{h*c}{\lambda}$ for a single photon pulse. Hence $dT = \frac{h*c}{\lambda A^2} = \frac{6.67e-34*3e8}{10^{-14}} = 2*(10)^{-11}$ seconds. Frequency of the photon is $f = \frac{c}{\lambda} = \frac{3e8}{10^{-14}} = 3e22$ Hz.

Now, we will use Compton Effect equations, to describe conservation of energy before and after the pulse hits the electron.$p_e^{'} = \frac{m_e}{\sqrt{1-\frac{v^2}{c^2}}} v$ is the momentum of the electron after collision. $K = \sqrt{1-\frac{v^2}{c^2}}$

$$
\begin{aligned}
hf + m_e c^2 &= hf^{'} + \sqrt{(m_e c^2)^2 + (p_e^{'} c)^2} \\
hf - hf^{'} &= \sqrt{(m_e c^2)^2 + (p_e^{'} c)^2} - m_e c^2 \\
hf - hf^{'} = m_e c^2 [(1 + \frac{v^2}{c^2(1-\frac{v^2}{c^2})})^{\frac{1}{2}} - 1] = m_e c^2 [(\frac{1}{K^2})^{\frac{1}{2}} - 1] &= m_e c^2 [\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1] \\
\frac{\triangle f'}{\triangle v} = \frac{-m_e c^2}{2hc^2} \frac{2v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} &= \frac{-m_e}{h} \frac{v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} = f_2
\end{aligned}
\tag{41}
$$

We also know that

$$
\begin{aligned}
\triangle p &= \frac{m_e}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \triangle v \\
\triangle x \triangle p = c \triangle t \frac{m_e}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \frac{\triangle f'}{f_2} &= c \triangle t \triangle f' m_e \frac{h}{m_e v} = \triangle t \triangle f' \frac{ch}{v}
\end{aligned}
\tag{42}
$$

Case 1: $\triangle t \triangle f' > \frac{1}{4\pi}$

If we consider the Fourier Uncertainty Principle, we get

$$\triangle x \triangle p > \frac{h}{4\pi}\frac{c}{v} \tag{43}$$

Case 2: $\triangle t \triangle f' < \frac{1}{4\pi}$

If we use signal processing techniques and are able to oversample the photon pulse of period $dT = 2 * 10^{-11}$ seconds and take the Fourier Transform of the pulse and locate the peak of the Fourier Transform to a high accuracy, then we can get $\triangle t \triangle f' < \frac{1}{4\pi}$ and hence

$$\triangle x \triangle p < \frac{h}{4\pi}\frac{c}{v} \tag{44}$$

13. Section 10: Compton Effect Revisited

Let $f, f^{'}$ be the frequency of the light before and after collision with an electron with rest mass m_0. Let $v, v^{'}$ be the velocity of electron before and after collision. Let $p_e = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}, p_e^{'} = \frac{m_0 v^{'}}{\sqrt{1-\frac{(v')^2}{c^2}}}$ be the velocity of electron before and after collision. Let $K = \sqrt{1-\frac{v^2}{c^2}}$ and $K^{'} = \sqrt{1-\frac{(v')^2}{c^2}}$. Using Conservation of Energy we have

$$\begin{aligned} hf + \sqrt{(m_0c^2)^2 + (p_ec)^2} &= hf^{'} + \sqrt{(m_0c^2)^2 + (p_e^{'}c)^2} \\ hf - hf^{'} &= m_0c^2[\sqrt{1 + \frac{(v')^2}{c^2(K')^2}} - \sqrt{1 + \frac{v^2}{c^2K^2}} -] \\ hf - hf^{'} &= m_0c^2[\frac{1}{K'} - \frac{1}{K}] \end{aligned} \tag{45}$$

Case 1: $v = 0$

We can derive $\triangle\lambda = \lambda^{'} - \lambda = \frac{h}{m_0c}(1-\cos\theta)$ where θ is the angle of incidence of photon on electron.

$$\begin{aligned} hf - hf^{'} &= m_0c^2[\frac{1}{K'} - 1] \\ \frac{h \triangle \lambda}{m_0c\lambda(\lambda + \triangle\lambda)} &= [\frac{1}{K'} - 1] \end{aligned}$$

(46)

Using values for an electron, $m_0 = 9.10938356 * (10)^{-31}$ Kgs and size $(10)^{-15}$ meters and using light wavelength comparable to electron size $\lambda = (10)^{-15}$ meters, $h = 6.62607004 * (10)^{-34}\frac{m^2kg}{s}$, $c = 299792458\frac{m}{s}$ and $\theta = \pi$, we have

$$\begin{aligned}
\triangle\lambda &= \frac{h}{m_0c}(1-\cos\theta) = 4.8528 * (10)^{-12} \\
\frac{h\triangle\lambda}{m_0c\lambda(\lambda+\triangle\lambda)} &= [\frac{1}{K'}-1] = 2425.9 \\
K' = \sqrt{1-\frac{(v')^2}{c^2}} &= \frac{1}{(1+2425.9)} = 4.1205 * (10)^{-4} \\
\frac{v'}{c} = \sqrt{1-(4.1205*(10)^{-4})^2} &= 0.999999915107395
\end{aligned} \tag{47}$$

Case 2: $v \neq 0$

Using Conservation of momentum, we have

$$\begin{aligned}
\vec{p_e'} - \vec{p_e} &= \vec{p_\gamma} - \vec{p_\gamma'} \\
(\vec{p_e'} - \vec{p_e}) \cdot (\vec{p_e'} - \vec{p_e}) &= (\vec{p_\gamma} - \vec{p_\gamma'}) \cdot (\vec{p_\gamma} - \vec{p_\gamma'}) \\
(p_e')^2 + p_e^2 - 2p_ep_e'\cos\theta_e &= (p_\gamma')^2 + p_\gamma^2 - 2p_\gamma p_\gamma'\cos\theta_\gamma
\end{aligned} \tag{48}$$

Using $p_\gamma = \frac{hf}{c}$ and $p_\gamma' = \frac{hf'}{c}$ and $p_e = \frac{m_0v}{K}, p_e' = \frac{m_0v'}{K'}$ and multiplying both sides by c^2, we have

$$m_0^2c^2[(\frac{v'}{K'})^2 + (\frac{v}{K})^2 - 2\frac{vv'}{KK'}\cos\theta_e] = (hf')^2 + (hf)^2 - 2h^2ff'\cos\theta_\gamma \tag{49}$$

Taking the Square of Equation 45 and subtracting 49, we have

$$\begin{aligned}
(hf-hf')^2 = (hf')^2 + (hf)^2 - 2h^2ff' = (m_0c^2)^2(\frac{1}{K'}-\frac{1}{K})^2 &= m_0^2c^4[(\frac{1}{K'})^2) + (\frac{1}{K})^2 - \frac{2}{KK'}] \\
(hf')^2 + (hf)^2 - 2h^2ff'\cos\theta_\gamma &= m_0^2c^4[\frac{(v')^2}{c^2(K')^2} + \frac{v^2}{c^2K^2} - 2\frac{vv'}{c^2KK'}\cos\theta_e] \\
-2h^2ff'(1-\cos\theta_\gamma) = m_0^2c^4[1+1-\frac{2(1-\frac{vv'\cos\theta_e}{c^2})}{KK'}] &= 2m_0^2c^4[1-\frac{1}{KK'}+\frac{\frac{vv'\cos\theta_e}{c^2}}{KK'}] = 2m_0^2c^4f_2
\end{aligned} \tag{50}$$

Extracting expressions for $f^{'}$ from 45 and 50, and rewriting them with only 2 unknowns $v^{'}, K^{'}$ we have

$$f^{'} = f + \frac{m_0c^2}{Kh} - \frac{m_0c^2}{hK^{'}} = A_1 + \frac{A_2}{K^{'}}$$

$$f^{'} = \frac{m_0^2c^4}{h^2f(1-\cos\theta_\gamma)}[-1 + \frac{1}{KK^{'}} - \frac{\frac{vv^{'}\cos\theta_e}{c^2}}{KK^{'}}] = -A_5 + \frac{A_3}{K^{'}} - \frac{A_4v^{'}}{K^{'}}$$

$$A_1 = f + \frac{m_0c^2}{Kh}; \quad A_2 = -\frac{m_0c^2}{h}; \quad A_3 = \frac{m_0^2c^4}{Kh^2f(1-\cos\theta_\gamma)};$$

$$A_4 = \frac{m_0^2c^4v\cos\theta_e}{c^2Kh^2f(1-\cos\theta_\gamma)}; \quad A_5 = \frac{m_0^2c^4}{h^2f(1-\cos\theta_\gamma)}$$

;

(51)

Equating expressions for $f^{'}$, we have

$$A_1 + \frac{A_2}{K^{'}} = -A_5 + \frac{A_3}{K^{'}} - \frac{A_4v^{'}}{K^{'}}$$

$$\frac{1}{K^{'}}[A_2 - A_3 + A_4v^{'}] = -A_1 - A_5 = A_7$$

$$\frac{1}{K^{'}}[A_6 + A_4v^{'}] = A_7; \quad A_6 = A_2 - A_3$$

;

(52)

Squaring the above expression, using $K^{'} = \sqrt{1 - \frac{(v^{'})^2}{c^2}}$, we can solve for quadratic equation in variable $v^{'}$.

$$A_6^2 + A_4^2(v^{'})^2 + 2A_6A_4v^{'} = A_7^2(1 - \frac{(v^{'})^2}{c^2})$$

$$A(v^{'})^2 + Bv^{'} + C = 0$$

$$v^{'} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

$$A = A_4^2 + \frac{A_7^2}{c^2}; B = 2A_6A_4; C = A_6^2 - A_7^2$$

;

(53)

14. Inherent Fourier Uncertainty Revisited

Let us revisit the well-known time-frequency Fourier Uncertainty relation for a rectangular light pulse $g(t) = rect(\frac{t}{T})cos(\omega_0 t)$ of duration T modulated by a frequency f_0 whose Fourier Transform is given by $G(f) = \frac{T}{2}[sinc((f - f_0)T) + sinc((f + f_0)T)]$.(Figure 3 and 4).

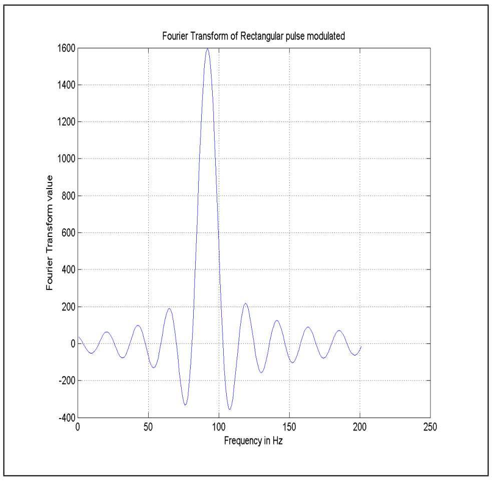

Figure 4:

While inherent Fourier uncertainty relation $\triangle t \triangle f \geq \frac{1}{2\pi}$ may hold in general for this light pulse, we can get below this limit, **if we know the modulating frequency in advance** like the case of a Radar application. In Radar application, the transmitter transmits a known sinusoidal frequency f_0 for the pulse duration T after which the pulse is switched off, which is exactly the pulse $g(t)$ delayed by duration $\frac{T}{2}$. In this specific case, we can take the Fourier transform of this pulse whose magnitude equals $|G(f)|$ which is the well known **sinc** function which reaches a maximum at $f = f_0$ as shown in the Figure. We can locate the peak of $|G(f)|$ with any desired precision which makes $\triangle t \triangle f$ well below the limit. For example, in Radar, let us consider $\triangle t = T = 10ns$, $f_0 = 1GHz$, $|G(f)|$ reaches a maximum at $f_0 = 1GHz$ and has the first zero crossing after this peak at $f = f_0 + \frac{1}{T} = 1.1GHz$. This pulse gets reflected by the target and is received at the receiver. Given the apriori knowledge of the transmitted frequency in Radar application, if we can identify this peak in received frequency with a precision of $0.01GHz$, frequency uncertainty of this pulse is $\triangle f = 0.01GHz$ and we get $\triangle t \triangle f = 10^{-9} \times 0.01^{9} = 0.1 < \frac{1}{2\pi}$ assuming that the channel is a low-pass channel with roughly unity frequency response around $f = f_0$.

If we use this pulse in the case of Two-sided Radar using Light Pulse as in section 3 and use this apriori knowledge of transmitted frequency,we can get $\triangle x \triangle p < \frac{h}{4\pi}$.

15. Object size-wavelength relation Revisited

Let us revisit the well-known principle that states that the smaller the size of the object $\triangle x$, we need to use a radiation with wavelength λ comparable to $\triangle x$. While this principle seems to apply in cases where we observe objects under the microscope and atoms under an electron microscope, let us examine the cases such as Radar applications where this limit may be crossed.

Case 1: Consider 2 objects of diameter $1m$ each, separated by a distance of $L_2 = 1m$. Size-wavelength relation states that we cannot use a $\lambda > 2m$ to resolve these 2 objects. Let us go ahead and use $\lambda = 10m$ which is far larger than this relation, but use 2 directional radar guns

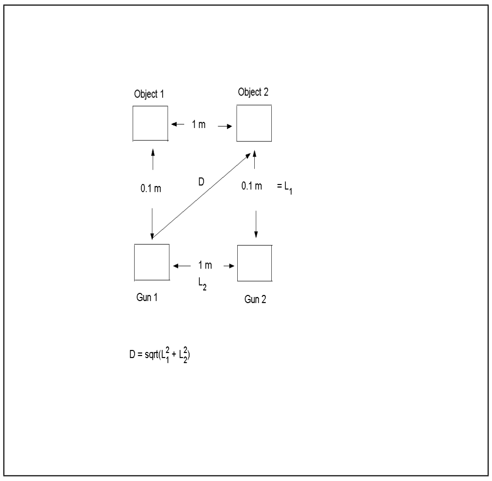

Figure 5:

located along the line of sight of the 2 objects, as shown in the Figure. Each Radar gun has a receiving antenna of cross-section $\lambda = 1m$. While quarter wavelength antennas $\lambda = 2.5m$ give the optimum signal strength, $\lambda = 1m$ cross-section antennas should do fine with lower signal strength. Each radar gun uses a modulating frequency $f = \frac{c}{\lambda} = 30MHz$ and pulsed signal of duration $T = 3.33ns$. This makes the uncertainty in the position of the objects $\triangle x = cT = 1m$. Though the radar guns are directional, the respective beams can get scattered in all directions, upon meeting the object and hence Gun 1 can receive direct reflected beam from Object 1, along with scattered beam from Object 2. Let both guns be located at $L_1 = 0.1m$ from their respective objects. If both guns transmit pulses to their respective object at $t = 0$ to $t = 3.33ns$, Gun 1 receives direct reflected beam from Object 1 from $t_1 = 2*L_1/c = 0.66ns$ to $0.66+3.33 = 4ns$, along with scattered beam from Object 2 from $t_2 = 2 * \sqrt{L_1^2 + L_2^2}/c = 6.67ns$ to $6.67 + 3.33 = 10ns$ which are at mutually exclusive times and discernible. If the two objects were moving in addition with unknown velocity v_1 and v_2 which had to be estimated, we could use the method of using light pulse trains as in Section 5.

Case 2: Consider the next case same as above with the difference that Radar Guns are omnidirectional. If we choose the same numbers as above case, it should still work, given that both guns transmit pulses to their respective object at $t = 0$ to $t = 3.33ns$, Gun 1 receives direct reflected beam from Object 1 from $t_1 = 2 * L_1/c = 0.66ns$ to $0.66 + 3.33 = 4ns$, along with scattered beam from Object 2 from $t_2 = 2 * \sqrt{L_1^2 + L_2^2}/c = 6.67ns$ to $6.67 + 3.33 = 10ns$ which are at mutually exclusive times and discernible.

16. Misc ideas Revisited

In Section 3, the familiar position-momentum uncertainty principle was derived by assuming that $E = hf_0 = mc^2 = \frac{m_0}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}}c^2$ for a particle of rest mass m_0 travelling at velocity v which has a wave associated with it of frequency f_0 according to particle-wave duality and hence we need to use a similar frequency radiation to detect its presence.Let us call this Equation A.

We also have the size-wavelength principle which states that in order to detect a particle of size $\triangle x$, we need to use a radiation of wavelength $\lambda \approx \triangle x$. Let us call this Equation B. Which means, as the size of the object gets **larger**, say from an electron to an atom, its mass becomes larger (assuming low relativistic velocities for the electron and atom) and according to Equation A, f_0 is larger and $\lambda = \frac{c}{f_0}$ for the probing radiation becomes **smaller**, which runs contrary to Equation B!

Besides, particle-wave duality states that a particle of mass m travelling at velocity v has a wave associated with it of wavelength $\lambda = \frac{h}{mv}$. Let us call this Equation C. Given that this $\lambda = \frac{v}{f}$ for the wave asociated with this particle, equating the two formulae gives $\frac{h}{mv} = \frac{v}{f}$ which gives $hf = mv^2$ which does not sit well with Equation A namely $hf = mc^2$ unless $v = c$, meaning that the particle velocity v must equal light speed in vaccum c!

17. Conclusion

18. Appendix A

Let us derive the well-known time-frequency Fourier Uncertainty relation for a rectangular light pulse of duration $\triangle t$ modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(Figure 2).

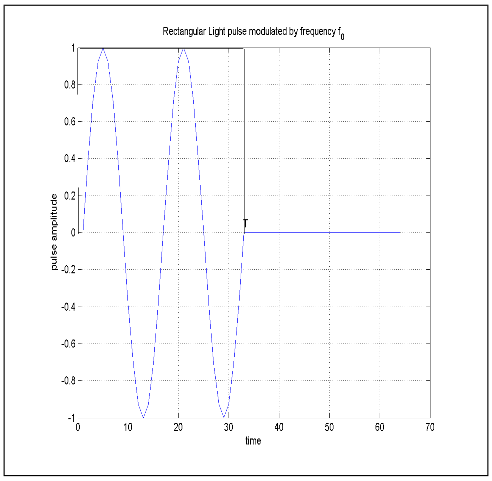

Figure 6:

Let us first consider the limiting case of a two-sided Gaussian pulse $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$. The standard deviation of the pulse $g(t)$ is given by $\sigma_t = \sqrt{\frac{1}{2\pi}}$ and the standard deviation of the transform $G(f)$ is given by $\sigma_f = \sqrt{\frac{1}{2\pi}}$. Given that the standard deviation of a signal represents a measure of uncertainty in its value, we can interpret the product of the two standard deviations as a product of time-frequency uncertainty and write

$$\sigma_t \sigma_f = \frac{1}{2\pi} \tag{54}$$

which represents the Gaussian limiting case.

19. Fourier Uncertainty relation $\triangle t \triangle f \geq \frac{1}{4\pi}$

Gabor Limit in signal processing gives Fourier Uncertainty relation $\sigma_t \sigma_f \geq \frac{1}{4\pi}$ (link)

Let us derive the Fourier Uncertainty relation for a general signal $g(t)$.[See Simon Haykin "**Communication systems** Second Edition 1978", page 102]. (link)

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{55}$$

Let us consider the following measure for an energy signal $g(t)$ whose Fourier Transform is given by $G(f)$.

$$\begin{aligned} T_{rms} &= [\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}]^{\frac{1}{2}} \\ W_{rms} &= [\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df}]^{\frac{1}{2}} \end{aligned} \tag{56}$$

Let us show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. Define $g_1(t) = tg(t)$ and $g_2(t) = \frac{dg(t)}{dt}$ and using **Schwarz's inequality**, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |g_1(t)|^2 dt \int_{-\infty}^{\infty} |g_2(t)|^2 dt &\geq (\int_{-\infty}^{\infty} g_1(t) g_2(t) dt)^2 \\ \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt &\geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2 \end{aligned} \tag{57}$$

Using **Parseval's relation** $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$, we can show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. We also use the fact that $\int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt = (2\pi)^2 \int_{-\infty}^{\infty} f^2 |G(f)|^2 df$ using properties of fourier transform and Parseval's relation.

$$(2\pi)^2 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df \geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2$$

(58)

Using **Parseval's relation** $\int_{-\infty}^{\infty}|g(t)|^2dt = \int_{-\infty}^{\infty}|G(f)|^2df$, we can write

$$W_{rms}T_{rms} = \frac{(\int_{-\infty}^{\infty}t^2|g(t)|^2dt\int_{-\infty}^{\infty}f^2|G(f)|^2df)^{\frac{1}{2}}}{\int_{-\infty}^{\infty}|G(f)|^2df} \geq \frac{1}{(2\pi)}|\int_{-\infty}^{\infty}g_1(t)g_2(t)dt|\frac{1}{\int_{-\infty}^{\infty}|G(f)|^2df} \tag{59}$$

For **Gaussian pulse** $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$, we use the fact that $g_1(t) = tg(t), g_2(t) = \frac{dg(t)}{dt} = -2\pi tg(t)$, $\int_{-\infty}^{\infty}|G(f)|^2df = 1$ and write as follows.

$$W_{rms}T_{rms} \geq \frac{1}{(2\pi)}(2\pi)\int_{-\infty}^{\infty}(tg(t))^2dt$$
$$W_{rms}T_{rms} \geq \int_{-\infty}^{\infty}t^2g^2(t)dt = \int_{-\infty}^{\infty}t^2e^{-2\pi t^2}dt \tag{60}$$

We see that the **inverse fourier transform** of $t^2e^{-2\pi t^2}$ is given by $(\frac{1}{-i2\pi})^2\frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2}$. We see that $\frac{d(e^{-\frac{\pi}{2}f^2})}{df} = e^{-\frac{\pi}{2}f^2}(-\pi f)$ and $\frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2} = e^{-\frac{\pi}{2}f^2}[-\pi+\pi^2f^2]$. Hence $\int_{-\infty}^{\infty}t^2e^{-2\pi t^2}dt = (\frac{1}{-i2\pi})^2[\frac{d^2(e^{-\frac{\pi}{2}f^2})}{df^2}]_{f=0} = (\frac{1}{-i2\pi})^2(-\pi) = \frac{1}{4\pi}$.

Hence we can write **Fourier uncertainty relations** as follows.

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{61}$$

20. Appendix B

Case 1: $\triangle x \triangle p < \frac{h}{4\pi}$

In Compton scattering effect, the wavelength shift of scattered radiation is given by $\lambda^{'} - \lambda = \delta\lambda = \frac{h}{mc}(1 - \cos(\theta))$ where m is the mass of electron, c is speed of light in vaccum, h is Planck's constant and θ is the angle of scattered radiation. By the law of conservation of momentum, we know that for an electron at rest, $\mathbf{p}_{e'} = \mathbf{p}_{\gamma} - \mathbf{p}_{\gamma'}$ where $\mathbf{p}_{e'}$ is the momentum of electron after scattering, $\mathbf{p}_{\gamma}$ and $\mathbf{p}_{\gamma'}$ represent the momentum of the photon before and after scattering. Given that $\mathbf{p}_{\gamma} = \frac{h}{\lambda}$, $\mathbf{p}_{e'} = \frac{h(\lambda^{'} - \lambda)}{\lambda\lambda'} = \frac{h}{\lambda\lambda'}\frac{2h}{mc} = \frac{h}{\lambda}K$ represents momentum-uncertainty of the electron after scattering, for a scattering angle of π where $K = \frac{2h}{mc\lambda'} = \frac{(\lambda^{'} - \lambda)}{\lambda'}$. Let us consider the case of scattering in one dimension and photon is scattered back at angle π.

$$\begin{aligned}
p_e^{'} = p_{\gamma} - p_{\gamma'} = \frac{h(\lambda^{'} - \lambda)}{\lambda\lambda'} &= \frac{h}{\lambda\lambda'}\frac{2h}{mc} \\
p_e^{'} &= \frac{2h^2}{mc\lambda\lambda'} \\
\frac{\triangle p_e^{'}}{\triangle\lambda'} &= \frac{-2h^2}{mc\lambda(\lambda')^2} \\
\triangle x &\approx \triangle\lambda^{'} \\
\triangle p_e^{'} \triangle x = \triangle p_e^{'} \triangle \lambda^{'} &= \frac{-2h^2}{mc\lambda}(\frac{\triangle\lambda^{'}}{\lambda'})^2 \\
|(\frac{\triangle\lambda^{'}}{\lambda'})^2| &>= \frac{1}{4\pi} \\
|\triangle p_e^{'} \triangle x| &>= \frac{2h^2}{mc\lambda 4\pi}
\end{aligned} \tag{62}$$

We can see that if $m\lambda > \frac{2h}{c} = 4.417e - 42$, then $|\triangle p_e^{'} \triangle x| < \frac{h}{4\pi}$.

For an electron, $m = 9.11 \times 10^{-31}$ and $\lambda = 5.6 \times 10^{-15}$ comparable to size of the electron and $m\lambda = 5.101600000000001e - 45$.

For a proton, $m = 1.67 \times 10^{-27}$ and $\lambda = 5.6 \times 10^{-15}$ comparable to size of the electron and $m\lambda = 9.352000000000002e - 42$ and $m\lambda > \frac{2h}{c} = 4.417e - 42$. Hence $|\triangle p_e^{'} \triangle x| < \frac{h}{4\pi}$

For any particle of larger size and larger mass(which is expected), using a wavelength comparable to size of particle, we can achieve $m\lambda > \frac{2h}{c} = 4.417e - 42$ and hence $|\triangle p_e^{'} \triangle x| < \frac{h}{4\pi}$.

We can derive the result $|(\frac{\triangle\lambda^{'}}{\lambda'})^2| >= \frac{1}{4\pi}$ as follows, starting from Fourier Uncertainty relation.

$$
\begin{aligned}
\triangle f \triangle t &>= \frac{1}{4\pi} \\
f &= \frac{c}{\lambda} \\
\triangle f &= \frac{-c}{\lambda^2} \triangle \lambda \\
\triangle x = c \triangle t &\approx \triangle\lambda \\
\frac{-c}{\lambda^2} \triangle \lambda \frac{\triangle\lambda}{c} &>= \frac{1}{4\pi} \\
|(\frac{\triangle\lambda}{\lambda})^2| &>= \frac{1}{4\pi} \\
|(\frac{\triangle\lambda^{'}}{\lambda^{'}})^2| &>= \frac{1}{4\pi}
\end{aligned} \tag{63}
$$

For electron

Electron has a mass of $m_e = 9.11 * 10^{-31}$ and size of the order of 10^{-14} meters and requires an incident wavelength of $\lambda = 10^{-14}$ to have effective scattering cross-section. Hence photon momentum is given by $p_\gamma = \frac{h}{\lambda}$ which is said to affect the momentum of the electron $p_e^{'}$ upon incidence. $p_e^{'}$ is said to be of the order of p_γ.

$$
\begin{aligned}
p_\gamma = \frac{h}{\lambda} = \frac{6.626 * 10^{-34}}{10^{-14}} &= 6.626 * 10^{-20} \\
p_e^{'} = \frac{m_e v_e^{'}}{\sqrt{1 - \frac{(v_e^{'})^2}{c^2}}} &\approx p_\gamma = 6.626 * 10^{-20} \\
\frac{v_e^{'}}{\sqrt{1 - \frac{(v_e^{'})^2}{c^2}}} \approx \frac{6.626 * 10^{-20}}{m_e} = \frac{6.626 * 10^{-20}}{9.11 * 10^{-31}} &\approx 10^{11} \\
(v_e^{'})^2 = (1 - \frac{(v_e^{'})^2}{c^2}) &* (10)^{22} \\
(v_e^{'})^2(1 + \frac{(10)^{22}}{c^2}) &= (10)^{22} \\
v_e^{'} = \frac{(10)^{11}}{\sqrt{(1 + \frac{(10)^{22}}{3e8*3e8})}} &\approx c
\end{aligned} \tag{64}
$$

21. References

[1] NASA Factsheet for Sun:
http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html

NASA Planetary Fact Sheet
http://nssdc.gsfc.nasa.gov/planetary/factsheet/
NASA AstroDynamic constants
http://ssd.jpl.nasa.gov/?constants

[2] The Pioneer Anomaly: seeking an explanation in newly recovered data. Toth and Turyshev. 2006 NRC Canada.