1. Inherent Fourier Uncertainty Principle

Let us start from the well-known time-frequency Fourier Uncertainty relation for a **light pulse** of duration $\triangle t$ (time uncertainty) modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(derived in Section 3).

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{1}$$

*1.1. **Signal Processing methods Can get us below Fourier uncertainty Limit***

In this example, we use a **triangular pulse** in time domain of duration $2T$, where $T = 1$ microseconds within a signal duration of $T_1 = 20$ microseconds, which is modulated by a carrier frequency $f_c = 100$ MHz and sampled by $f_s = 10$ GHz. This pulse is transmitted and reflected by a target, a car for example, of length 3 meters and carrier wavelength is $\lambda = \frac{c}{f} = 3$ meters , very similar to a **Radar** system. We wish to estimate the **time of arrival** and **carrier frequency** of the received pulse and determine the Fourier uncertainty.

Uncertainty in **time of arrival** is given by $\triangle t = \frac{10}{f_s} = 1e - 9$ seconds, if we can identify the **peak** of the triangular pulse with a **precision** of 10 times sampling period, which is **possible** with signal processing.

Uncertainty in **carrier frequency** is given by $\triangle f = \frac{10 f_s}{N} = 500$ Khz, if we can identify the **peak** of the Fourier Transform of triangular pulse with a **precision** of 10 times frequency resolution given by $\frac{fs}{N}$, where $N = T_1 * fs = 20e - 6 * 1e10 = 2e5$ is the total number of samples, which is **possible** with signal processing.

Hence we can get Fourier Uncertainty in our system **below** Fourier Uncertainty Limit of $\frac{1}{4\pi}$.

$$\triangle t \triangle f = 5e - 4 < \frac{1}{4\pi} \tag{2}$$

In the next section, we will explain how this pulse can be used to get $\triangle x \triangle p = 5e - 4 * h$ **below** Heisenberg's uncertainty Limit.

Effects of sampling frequency

The Triangular pulse is given by $g(t) = (\frac{T-t}{T})u(t) + (\frac{T+t}{T})u(-t)$. Its Fourier Transform is given by $G(f) = (T)^2 * sinc^2(fT)$ and we can see that it has **zero crossings** at $f = \frac{1}{T} = 1$ MHz, which **does not** cause **interference** with carrier frequency peak detection, for our choice of sampling frequency $fs = 1$ GHz, carrier frequency $fc = 100$ Mhz. Any small drifts in carrier and sampling frequencies should not matter if transmitted signal power is sufficiently high, given that Fourier Transform of a triangular pulse falls off as $\frac{1}{f^2}$ and the analysis in above subsection still holds.

Effects of Receiver Thermal Noise

Receiver thermal noise power is given by $K * T_0 * BW$ where BW is signal bandwidth, T_0 is noise temperature and K is Boltzmann constant. We can choose to transmit the signal with **sufficiently high power** level, so that received signal power is at least 100 times higher than thermal noise power and hence signal amplitude is **at least 10 times higher** than noise amplitude. Then the analysis in above subsection still holds.

The Figure in link plots the triangular pulse in time domain and its Fourier Transform in the frequency domain.

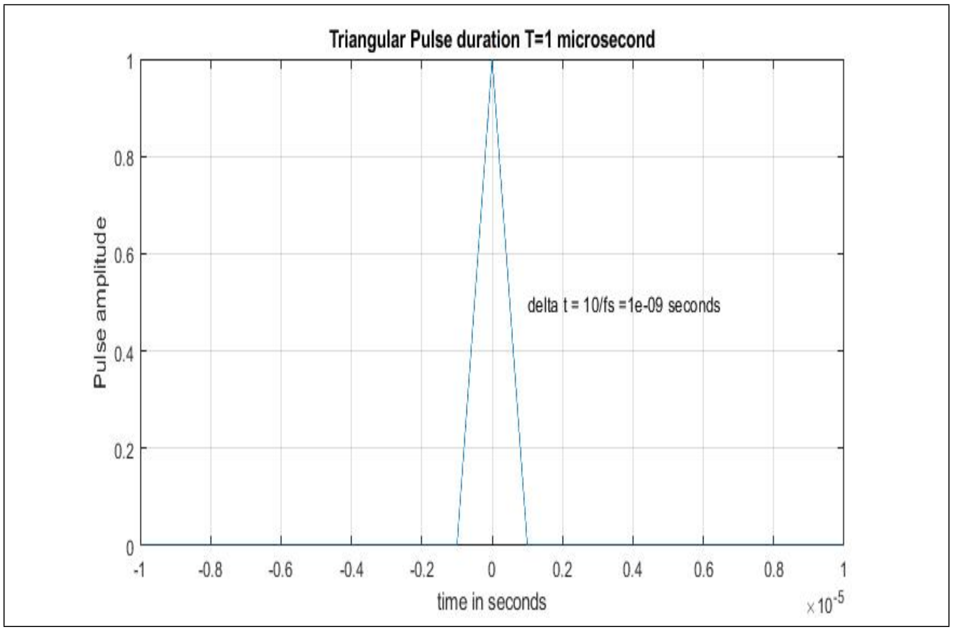

Figure 1:

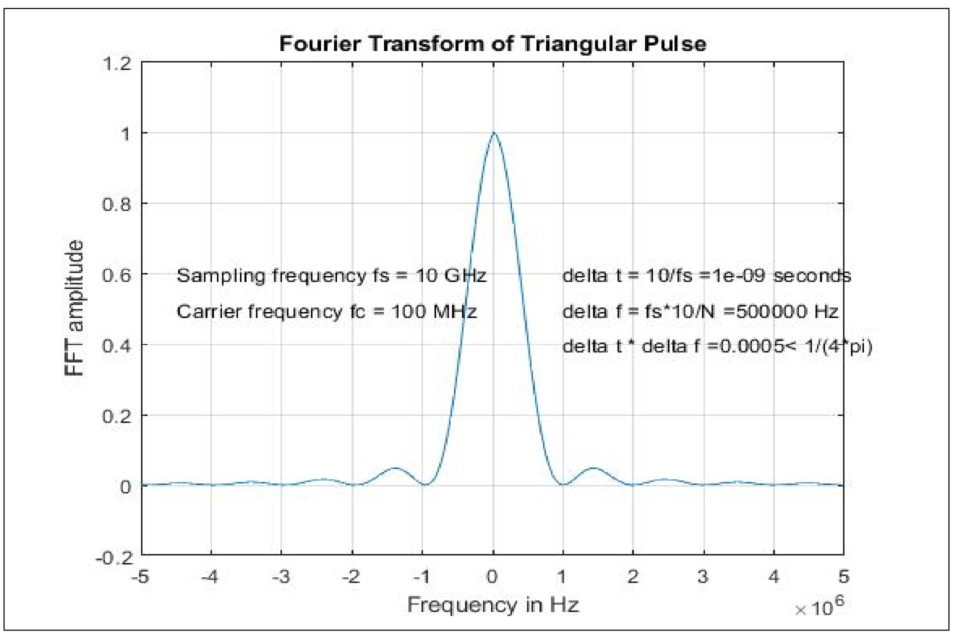

Figure 2:

2. Heisenberg's Uncertainty Principle Part 1

It is well known that Heisenberg's Uncertainty Principle says that $\triangle x \triangle p \geq \frac{h}{4\pi}$ where $\triangle x, \triangle p$ are the uncertainty in particle position and momentum. Heisenberg used the analogy of Compton scattering of an electron by photon, while deriving this result in the link. [Compton effect derivation in the link]

Consider the case where a **photon** of energy $E = hf = \frac{hc}{\lambda}$ and momentum $p_c = \frac{h}{\lambda}$ hits an **electron** moving with a momentum $p_e = m_e v = \frac{m_0 v}{\sqrt{(1-\frac{v^2}{c^2})}}$ **before** being hit by the photon, where m_0 is the rest mass of electron. If we use **photon wavelength** comparable to the **size of the electron** of $1e-11$ meters, then **uncertainty in electron position** $\triangle x \approx \lambda = 1e-11$ meters. because the photon imparts some of its momentum to the electron, **Uncertainty in electron momentum** $\triangle p \approx \frac{h}{\lambda}$ meters.

$$\begin{aligned} p_c &= \frac{h}{\lambda} \\ \triangle p &\approx \frac{h}{\lambda}, \quad \triangle x \approx \lambda \\ \triangle x \triangle p &= h \end{aligned} \tag{3}$$

Thus we get $\triangle x \triangle p \approx h$ and thus we can **approximate** version of Heisenberg's uncertainty principle.

If we want to get the **exact version** of Heisenberg's uncertainty principle, and the **factor** $\frac{1}{4\pi}$, we show below that it is derived from Inherent Fourier Uncertainty relation. Thus **measurement induced** position-momentum uncertainty is related to **Inherent Fourier Uncertainty**.

Let us start from the well-known time-frequency Fourier Uncertainty relation for a **light pulse** of duration $\triangle t$ (time uncertainty) modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(derived in Section 3).

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{4}$$

Given that such a light pulse is made up of large number of photons and that each photon has an energy $E = hf = pc$ where c is light speed and p is the momentum of the photon, we can deduce Energy uncertainty as $\triangle E = h \triangle f = c \triangle p$, position uncertainty as $\triangle x = c \triangle t$ and hence we can write

$$\begin{aligned} \triangle t \triangle f &\geq \frac{1}{4\pi} \\ \triangle t &= \frac{\triangle x}{c}, \quad \triangle f = \tfrac{c}{h} \triangle p \\ \frac{\triangle x}{c} \triangle p \frac{c}{h} &\geq \frac{1}{4\pi} \\ \triangle x \triangle p &\geq \frac{h}{4\pi} \end{aligned} \tag{5}$$

Thus we have derived Heisenberg's uncertainty principle, starting from Fourier Uncertainty Principle and we have shown that Heisenberg's **measurement induced** position-momentum uncertainty is related to **Inherent Fourier Uncertainty**.

3. Appendix A

Let us derive the well-known time-frequency Fourier Uncertainty relation for a light pulse of duration $\triangle t$ modulated by a frequency f_0 whose Fourier Transform has a frequency uncertainty of $\triangle f$(Figure 2).

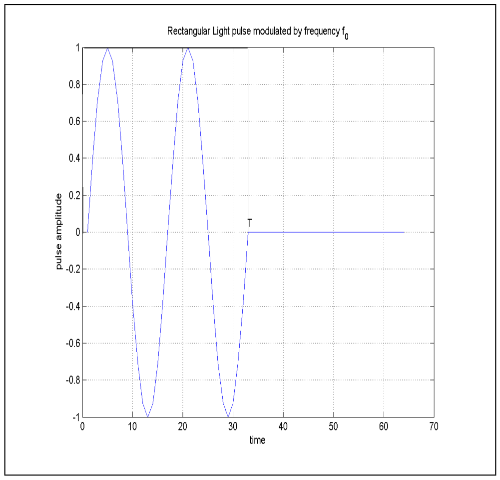

Figure 3:

Let us first consider the limiting case of a two-sided Gaussian pulse $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$. The standard deviation of the pulse $g(t)$ is given by $\sigma_t = \sqrt{\frac{1}{2\pi}}$ and the standard deviation of the transform $G(f)$ is given by $\sigma_f = \sqrt{\frac{1}{2\pi}}$. Given that the standard deviation of a signal represents a measure of uncertainty in its value, we can interpret the product of the two standard deviations as a product of time-frequency uncertainty and write

$$\sigma_t \sigma_f = \frac{1}{2\pi} \tag{6}$$

which represents the Gaussian limiting case.

4. Fourier Uncertainty relation $\triangle t \triangle f \geq \frac{1}{4\pi}$

Gabor Limit in signal processing gives Fourier Uncertainty relation $\sigma_t \sigma_f \geq \frac{1}{4\pi}$ (link)

Let us derive the Fourier Uncertainty relation for a general signal $g(t)$.[See Simon Haykin "**Communication systems** Second Edition 1978", page 102]. (link)

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{7}$$

Let us consider the following measure for an energy signal $g(t)$ whose Fourier Transform is given by $G(f)$.

$$T_{rms} = [\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}]^{\frac{1}{2}}$$
$$W_{rms} = [\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df}]^{\frac{1}{2}} \tag{8}$$

Let us show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. Define $g_1(t) = tg(t)$ and $g_2(t) = \frac{dg(t)}{dt}$ and using **Schwarz's inequality**, we have

$$\int_{-\infty}^{\infty} |g_1(t)|^2 dt \int_{-\infty}^{\infty} |g_2(t)|^2 dt \geq (\int_{-\infty}^{\infty} g_1(t) g_2(t) dt)^2$$
$$\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt \geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2 \tag{9}$$

Using **Parseval's relation** $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$, we can show that $W_{rms} T_{rms} \geq \frac{1}{4\pi}$. We also use the fact that $\int_{-\infty}^{\infty} |\frac{dg(t)}{dt}|^2 dt = (2\pi)^2 \int_{-\infty}^{\infty} f^2 |G(f)|^2 df$ using properties of fourier transform and Parseval's relation.

$$(2\pi)^2 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df \geq [\int_{-\infty}^{\infty} g_1(t) g_2(t) dt]^2 \tag{10}$$

Using **Parseval's relation** $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$, we can write

$$W_{rms} T_{rms} = \frac{(\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df)^{\frac{1}{2}}}{\int_{-\infty}^{\infty} |G(f)|^2 df} \geq \frac{1}{(2\pi)} |\int_{-\infty}^{\infty} g_1(t) g_2(t) dt| \frac{1}{\int_{-\infty}^{\infty} |G(f)|^2 df} \tag{11}$$

For **Gaussian pulse** $g(t) = e^{-\pi t^2}$ which has a Fourier Transform $G(f) = e^{-\pi f^2}$, we use the fact that $g_1(t) = tg(t), g_2(t) = \frac{dg(t)}{dt} = -2\pi t g(t)$, $\int_{-\infty}^{\infty} |G(f)|^2 df = 1$ and write as follows.

$$W_{rms} T_{rms} \geq \frac{1}{(2\pi)} (2\pi) \int_{-\infty}^{\infty} (tg(t))^2 dt$$
$$W_{rms} T_{rms} \geq \int_{-\infty}^{\infty} t^2 g^2(t) dt = \int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt \tag{12}$$

We see that the **inverse fourier transform** of $t^2 e^{-2\pi t^2}$ is given by $(\frac{1}{-i2\pi})^2 \frac{d^2(e^{-\frac{\pi}{2} f^2})}{df^2}$. We see that $\frac{d(e^{-\frac{\pi}{2} f^2})}{df} = e^{-\frac{\pi}{2} f^2}(-\pi f)$ and $\frac{d^2(e^{-\frac{\pi}{2} f^2})}{df^2} = e^{-\frac{\pi}{2} f^2}[-\pi + \pi^2 f^2]$. Hence $\int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt = (\frac{1}{-i2\pi})^2 [\frac{d^2(e^{-\frac{\pi}{2} f^2})}{df^2}]_{f=0} = (\frac{1}{-i2\pi})^2(-\pi) = \frac{1}{4\pi}$.

Hence we can write **Fourier uncertainty relations** as follows.

$$\triangle t \triangle f \geq \frac{1}{4\pi} \tag{13}$$