

04 CHAPTER

随机变量的期望



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第4-2章 随机变量的期望

1. 随机变量的期望
2. 方差与标准差
3. 协方差
4. 相关系数
5. 动差生成函数*

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1.1 期望的概念

- 随机变量的结果值乘以该随机变量的概率分布的均值
- 无数次地进行随机试验时，观测到的随机变量值的平均

[定义 5-1] 随机变量的期望(expected value)

– 离散型

– 连续型

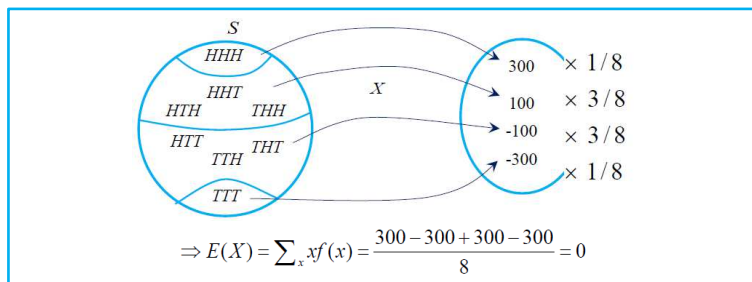
$$\mu_X = E(X) = \sum_x x f(x)$$

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

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[例 5-1] 根据三次扔硬币出现的结果(正面的个数 - 反面的个数)，每次给出或得到100元的游戏中，收益X的期望



[例 5-2] 连续型随机变量X的期望 $f(x) = 2e^{-2x}, (0 < x < \infty)$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} 2x e^{-2x} dx = \left[-x e^{-2x} \right]_0^{\infty} - \int_0^{\infty} (-e^{-2x}) dx \\ &= 0 - \frac{1}{2} (0 - 1) = \frac{1}{2} \end{aligned}$$

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1.2 随机变量函数的期望, $Y=g(X)$

- 离散型 $E(Y) = E[g(X)] = \sum_x g(x)f(x)$
- 连续型 $E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$

[例 5-3] 根据三次扔硬币时出现反面的个数 X 的平方来获得相应分红的游戏中, 分红的期望

$$f(0) = \frac{1}{8}, f(1) = \frac{3}{8}, f(2) = \frac{3}{8}, f(3) = \frac{1}{8}$$

$$\Rightarrow E(Y) = E(X^2) = \sum_x x^2 f(x) = \frac{0+3+2^2 \times 3+3^2}{8} = \frac{24}{8} = 3$$

例 5-4] 连续型随机变量, $Y=3X-3$ $f(x) = 2e^{-2x}$, $0 < x < \infty$

$$\Rightarrow E(Y) = E(3X-3) = \int_0^{\infty} (3x-3) \times 2e^{-2x} dx$$

$$= \left[-3xe^{-2x} \right]_0^{\infty} - \int_0^{\infty} (-3e^{-2x}) dx - \left[3e^{-2x} \right]_0^{\infty} = 0 - \frac{3}{2}(0-1) - 3 = -\frac{3}{2}$$

$$E(Y) = 3E(X) - 3 = \frac{3}{2} - 3 = -\frac{3}{2}$$

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1.4 期望的特征

[定理 5-1] 期望的特征

(1) $E(aX+b) = aE(X)+b$

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b)f(x)dx$$

$$= a \int_{-\infty}^{\infty} xf(x)dx + b \int_{-\infty}^{\infty} f(x)dx = aE(X)+b$$

(2) $E(X+Y) = E(X)+E(Y)$

$$E(X+Y) = \iint (x+y)f(x,y)dxdy$$

$$= \int x \int f(x,y)dydx + \int y \int f(x,y)dxdy$$

$$= \int xf_X(x)dx + \int yf_Y(y)dy = E(X)+E(Y)$$

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1.4 期望的特征

[定理 5-1] 期望的特征(接上)

$$(3) E[c_1 g_1(X) + \cdots + c_n g_n(X)] = c_1 E[g_1(X)] + \cdots + c_n E[g_n(X)]$$

$$\begin{aligned} E[c_1 g_1(X) + \cdots + c_n g_n(X)] &= \int [c_1 g_1(x) + \cdots + c_n g_n(x)] f(x) dx \\ &= c_1 \int g_1(x) f(x) dx + \cdots + c_n \int g_n(x) f(x) dx \\ &= c_1 E[g_1(X)] + \cdots + c_n E[g_n(X)] \end{aligned}$$

$$(4) X \& Y \text{ indep.} \Rightarrow E(XY) = E(X)E(Y)$$

$$\begin{aligned} E(XY) &= \iint xy f(x, y) dx dy = \iint xy f_X(x) f_Y(y) dx dy \\ &= \int x f_X(x) dx \int y f_Y(y) dy = E(X)E(Y) \end{aligned}$$

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2.1 方差的概念

[定理 5-2] 随机变量的方差(variance)

$$\text{Var}(X) \equiv \sigma_X^2 = E[(X - \mu_X)^2], \quad \mu_X \equiv E(X)$$

$$\blacksquare \text{ 离散型 } \sigma_X^2 = E[(X - \mu_X)^2] = \sum_x (x - \mu_X)^2 f(x)$$

$$\blacksquare \text{ 连续型 } \sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) dx$$

$$\begin{aligned} \blacksquare \text{ 简便式 } E[(X - \mu_X)^2] &= E[X^2 - 2\mu_X X + \mu_X^2] \\ &= E(X^2) - 2\mu_X E(X) + \mu_X^2 = E(X^2) - \mu_X^2 \end{aligned}$$

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[例 5-7] 在掷一次骰子的试验中出现的点数X的期望与方差

$$\mu_X = E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

[例 5-8] 连续型随机变量 $f(x) = 2e^{-2x}$, $0 < x < \infty$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} 2xe^{-2x}dx = \left[-xe^{-2x}\right]_0^{\infty} - \int_0^{\infty} (-e^{-2x})dx = 0 - \frac{1}{2}(0-1) = \frac{1}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} 2x^2 e^{-2x}dx = \left[-x^2 e^{-2x}\right]_0^{\infty} - \int_0^{\infty} (-2xe^{-2x})dx$$

$$= \left[-xe^{-2x}\right]_0^{\infty} - \int_0^{\infty} (-e^{-2x})dx = \int_0^{\infty} e^{-2x}dx = \frac{1}{2}$$

$$\Rightarrow \sigma_X^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

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2.2 方差的特征

$$Var(g(X)) = E[g(X)^2] - E[g(X)]^2$$

- 离散型 $Var(g(X)) = \sum_x g(x)^2 f(x) - \left[\sum_x g(x)f(x)\right]^2$
- 连续型 $Var(g(X)) = \int_{-\infty}^{\infty} g(x)^2 f(x)dx - \left[\int_{-\infty}^{\infty} g(x)f(x)dx\right]^2$

• [定理 5-2] 方差的特征

$$Var(aX+b) = a^2 Var(X)$$

$$Var(aX+b) = E\left[\{(aX+b) - (a\mu_X+b)\}^2\right]$$

$$= a^2 E\left[(x - \mu_X)^2\right] = a^2 Var(X)$$

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[例 5-9] 掷一次骰子出现的点数乘以100，然后减400后的结果Y的数学期望与方差

$$\begin{aligned} Y &= 100X - 400 & \mu_X &= \frac{7}{2}, \sigma_X^2 = \frac{35}{12} \\ \Rightarrow \mu_Y &\equiv E(Y) = 100E(X) - 400 = 350 - 400 = -50 \\ \Rightarrow \sigma_Y^2 &\equiv Var(Y) = 100^2 \sigma_X^2 = \frac{350,000}{12} \doteq 29,166.1667 \end{aligned}$$

[例 5-10] 连续型随机变量 $f(x) = 2e^{-2x}, 0 < x < \infty$

$$\begin{aligned} Y &= 20X - 10 & \mu_X &= \frac{1}{2}, \sigma_X^2 = \frac{1}{4} \\ \Rightarrow \mu_Y &\equiv E(Y) = 20E(X) - 10 = 10 - 10 = 0 \\ \Rightarrow \sigma_Y^2 &\equiv Var(Y) = 20^2 \sigma_X^2 = \frac{400}{4} = 100 \end{aligned}$$

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