习题6.4

2. 解 Green函数G的计算可以采取点源法，在M_0放置一个电量为ε（ε为介电常数）的点电荷激发的电势规律就可以满足方程，该电势一般不满足边界条件，需要在边界面放置负电荷使边界为0

3. 证明
$$\begin{cases} \Delta u = 0 \quad x \in (0,a) \quad y \in (0,b) \\ u(0,y) = u(a,y) = 0 \\ u(x,b) = 0 \quad u(x,0) = -f(x-\xi) \end{cases}$$

$$G(x,y) = \frac{-2}{a}\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{a}\xi\right)\left(ch\left(\frac{n\pi y}{a}\right) - cth\left(\frac{n\pi b}{a}\right)sh\left(\frac{n\pi y}{a}\right)\right)\right]si$$

$$u = -\int_0^a f(\xi)G(x-\xi, y)\,d\xi$$

$$= \frac{2}{a}\sum_{n=1}\left[\int_0^a f(\xi)\sin\frac{n\pi}{a}\xi\right)d\xi\left[ch\left(\frac{n\pi y}{a}\right) - cth\left(\frac{n\pi b}{a}\right)sh\left(\frac{n\pi}{a}\right.\right.$$

$$\sin$$

习题6.5

(1) $G(M,M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_0}} - \ln\frac{1}{r_{MM_1}} - \ln\frac{R}{r_0}\right) - \frac{1}{2\pi}\left(\ln\frac{1}{r_{M}}\right.$

$\left. - \ln\frac{1}{r_{MM_1'}} - \ln\frac{R}{r_0}\right)$

$G(M,M_0) = \frac{1}{2\pi}\left(\ln\frac{r_0 r_{MM_1}}{R r_{MM_0}} - \ln\frac{r_0 r_{MM_1'}}{R r_{MM_0'}}\right.$

(2) $G(M,M_0) = \frac{1}{4\pi}\left(\frac{1}{r_{MM_0}} - \frac{R}{r_0}\frac{1}{r_{MM_1}}\right) - \left(\frac{1}{r_{MM_0'}} - \frac{R}{r_0}\frac{1}{r_{MM}}\right.$

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2.(1) $u(M_0)=\frac{1}{2\pi R}\int_{x^2+y^2=R^2}\frac{R^2-r_0^2}{R^2-2Rr_0\cos\gamma+r_0^2}\varphi(M)\,ds$

极坐标的形式为

$$u(r_0,\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r_0^2)\varphi(\theta)}{R^2-2Rr_0\cos(\theta-\theta_0)+r_0^2}d\theta$$

$$u(r_0,\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r_0)^2a\cos\theta}{1-2r_0\cos(\theta-\theta_0)}$$

$$=ar_0\cos\theta_0$$

(2) $u(r_0,\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r_0)^2(b+a\cos\theta)}{1-2r_0\cos(\theta-\theta_0)+r_0^2}d\theta$

$$=b+ar_0\cos\theta_0$$

5. $\begin{cases}\Delta u=0 & r<R\\ u|_{x^2+y^2+z^2=R^2}=\varphi(M)\end{cases}$

$$u(M_0)=\frac{1}{4\pi R}\oint\!\!\oint_S\varphi(M)\frac{R_0^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\gamma)^{\frac{3}{2}}}ds$$

$$u(M_0)=\frac{1}{4\pi}\oint\!\!\oint_S(3\cos2\theta+1)\frac{1-\rho_0^2}{(1+\rho_0^2-2\rho_0\cos\gamma)^{\frac{3}{2}}}ds$$

$$u=3\rho^2\cos2\theta+\rho^2$$

习题 6.6

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2. $u(M,t)=\frac{\partial}{\partial t}\int_0^L \varphi(s)U(x-s,t)ds+\int_0^L \psi(s)U(x-s,t)ds$
$+\int_0^t d\tau\int_0^L f(s,\tau)U(x-s,t-\tau)ds$

$u(M,t)=\frac{\partial}{\partial t}\int_0^L \varphi(s)U(x-s,t)ds+\int_0^L \psi(s)U(x-s,t)ds$
$+\int_0^t d\tau\int_0^L f(s,\tau)U(x-s,t-\tau)ds$
$=A\sum_{n=2}^{\infty}\left(\frac{1}{2na\pi}\int_0^t\left(\cos\frac{n\pi a(t-\tau)-L\omega\tau}{L}-\cos\frac{n\pi a(t-\tau)+L\omega\tau}{L}\right)d\tau\right.$

$=\frac{AL^2}{2a\pi}\sum_{n=2}^{\infty}\left(\frac{1}{n}\left(\frac{1}{n\pi a+L\omega}\left(\sin\frac{n\pi at}{L}+\sin\omega t\right)+\frac{1}{n\pi a-L\omega}\right.\right.$
$\left.\left.\left(\sin\omega t-\sin\frac{n\pi at}{L}\right)\right)\times\left(\frac{1-(-1)^{n+1}}{(n+1)\pi}+\frac{1-(-1)^{n+1}}{(n-1)\pi}\right)\sin\frac{n\pi x}{L}\right.$

3.解 $G\Leftrightarrow\hat{G}$ $a^2G_{xx}\Leftrightarrow -a^2\omega^2\hat{G}$

$$\begin{cases}\frac{d^2\hat{G}}{dt^2}+a^2\omega^2\hat{G}=0\\ \hat{G}|_{t=0}=0\quad \hat{G}_t|_{t=0}=e^{j\omega x_0}\end{cases}$$

$F^{-1}(\hat{G})=G=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{G}e^{-j\omega x}$
$=\frac{1}{2a}\int_{-at}^{at}\left(\frac{e^{j\omega x_0}}{2\pi}\int_{-at}^{at}e^{-j\omega\beta}d\beta\right)e^{-j\omega x}d\omega$

$$=\begin{cases}\frac{1}{2a},\ (|x-x_0|\le at)\\ 0,\ (|x-x_0|>at)\end{cases}$$

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4. $\begin{cases} V_t = a^2 V_{xx} \\ V|_{t=0} = \delta(x) \end{cases}$

$\begin{cases} \hat{V}_t = -a^2 \omega^2 \hat{V} \\ \hat{V}|_{t=0} = 1 \end{cases}$

$\hat{V} = e^{-a^2\omega^2 t}$

$V = F^{-1}[\hat{V}] = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}$

$u = \varphi(x) * V = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(s) e^{-\frac{(s-x)^2}{4a^2t}}$

6. 解 $U(x-s, t-\tau) = \sum_{n=1}^{+\infty} \frac{2}{L} \exp\left(-\left(\frac{n\pi}{L} a\right)^2 (t-\tau)\right) \sin\frac{n\pi s}{L} \sin\frac{n\pi}{L} x$

$u(x,t) = \int_0^L U(x-s,t)\varphi(s)\,ds + \int_0^t d\tau \int_0^L U(x-s,t-s) f(s,\tau)\,ds$

$u(x,t) = \int_0^t d\tau \int_0^L U(x-s,t-\tau) f(s,\tau)\,ds$

$= -\frac{2AL^2}{a^2\pi^3} \sum_{n=1}^{+\infty} \left(\frac{1}{n^3}\left(1-\exp\left(-\left(\frac{n\pi}{L}a\right)^2 t\right)\right) \sin\frac{n\pi x}{L}\right)$

习题6.7

2. $u(M,t) = \frac{\partial}{\partial t} \int_0^L \varphi(s) U(x-s,t)\,ds + \int_0^L \psi(s) U(x-s,t)\,d$

$+ \int_0^t d\tau \int_0^L f(s,\tau) U(x-s,t-\tau)\,ds$

$u(M,t) = \frac{\partial}{\partial t} \int_0^L \varphi(s) U(x-s,t)\,ds + \int_0^L \psi(s) U(x-s,t)$

$+ \int_0^t d\tau \int_0^L f(s,\tau) U(x-s,t-\tau)\,ds$

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$$=A\int_0^t d\tau\int_0^L \sin\omega\tau\cos\frac{\pi s}{L}\sum_{n=1}^{\infty}\frac{2}{na\pi}\sin\frac{n\pi s}{L}\sin\frac{n\pi a(t-\tau)}{L}\sin\frac{n\pi x}{L}dx$$

$$=A\sum_{n=2}^{\infty}\left(\frac{1}{2na\pi}\int_0^t\left(\cos\frac{n\pi a(t-\tau)-L\omega\tau}{L}-\cos\frac{n\pi a(t-\tau)+L\omega\tau}{L}\right)d\tau\right.$$
$$\left.\times\left(\frac{L(1+(-1)^{n+1})}{(n+1)\pi}+\frac{L(1+(-1)^{n+1})}{(n-1)\pi}\right)\sin\frac{n\pi x}{L}\right)$$

3.解 $F^{-1}(\hat{G})=G=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{G}e^{-j\omega x}d\omega$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{e^{j\omega x_0}}{2a}\int_{-at}^{at}\frac{e^{j\omega x_0}}{2\pi}\int_{-\infty}^{+\infty}e^{-j\omega(x-x_0+\beta)}d\omega\right)d\beta$$

$$=\frac{1}{2a}\int_{-at}^{at}\delta(x-x_0+\beta)d\beta=\frac{1}{2a}\int_{x-x_0-at}^{x-x_0+at}\delta(\xi)d\xi$$

$$=\begin{cases}\frac{1}{2a} & (|x-x_0|\le at)\\ 0 & (|x-x_0|>at)\end{cases}$$