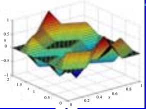


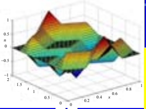
第八章 Legendre多项式

$$\Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

$$u(r,\ \theta,\ \varphi)=R(r)\Theta(\theta)\Phi(\varphi)$$

$$\Theta\Phi\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+R\Phi\frac{1}{r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+R\Theta\frac{1}{r^2\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=0$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{1}{\Phi\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}$$



$$\begin{cases} \dfrac{1}{R}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^2\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)=n(n+1)\\ \dfrac{1}{\Theta\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\dfrac{1}{\Phi\sin^2\theta}\dfrac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=-n(n+1)\end{cases}$$

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-n(n+1)R=0$$

$$R(r)=A_1r^n+A_2r^{-(n+1)}$$

$$\frac{1}{\Theta\sin^{-1}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+n(n+1)\sin^2\theta+\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=0$$

$$\begin{cases} \dfrac{1}{\Theta \sin^{-1}\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+n(n+1)\sin^2\theta = m^2 \\ \dfrac{1}{\Phi}\dfrac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -m^2 \end{cases}$$

$$\Phi(\varphi) = B_1 \cos m\varphi + B_2 \sin m\varphi$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + n(n+1)\Theta = 0$$

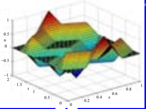
连带的Legendre方程

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}+\cot\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}=\frac{\mathrm{d}P}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta\frac{\mathrm{d}P}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}=-\cos\theta\frac{\mathrm{d}P}{\mathrm{d}x}+\sin^2\theta\frac{\mathrm{d}^2P}{\mathrm{d}x^2}$$

$$(1-x^2)\frac{\mathrm{d}^2P}{\mathrm{d}x^2}-2x\frac{\mathrm{d}P}{\mathrm{d}x}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]P=0$$



Legendre方程

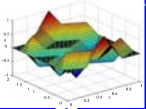
$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

$$c(c-1)a_0x^{c-2}+c(c+1)a_1x^{c-1}+\sum_{k=0}^{\infty}\{(k+c+2)(k+c+1)a_{k+2}$$

$$-[(k+c)(k+c+1)-n(n+1)]a_k\}x^{k+c}=0$$

$$\begin{cases} c(c-1)a_0 = 0 \\ c(c+1)a_1 = 0 \\ (k+c+2)(k+c+1)a_{k+2} - [(k+c)(k+c+1) - n(n+1)]a_k = 0 \end{cases}$$

$$\begin{cases} c = 0,1 \\ c = 0,-1, a_1 = 0 \\ a_{k+2} = \dfrac{(k+c)(k+c+1) - n(n+1)}{(k+c+1)(k+c+2)} a_k, (k = 0,1,2,\cdots) \end{cases}$$



$$a_{k+2} = \frac{(k-n)(k+n+1)}{(k+2)(k+1)} a_k ,(k = 0,1,2,\cdots)$$

$$a_2 = \frac{-n(n+1)}{2!} a_0$$

$$a_4 = (-1)^2 \frac{n(n-2)(n+1)(n+3)}{4!} a_0$$

……………………

$$a_{2i} = (-1)^i \frac{n(n-2)\cdots(n-2i+2)(n+1)(n+3)\cdots(n+2i-1)}{(2i)!} a_0$$

……………………

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_5 = (-1)^2\frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{2i+1} = (-1)^i\frac{(n-1)(n-3)\cdots(n-2i+1)(n+2)(n+4)\cdots(n+2i)}{(2i+1)!}a_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y = a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4+\cdots\right]$$

$$+a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5+\cdots\right]$$

$$y_1 = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots\right]$$

$$y_2 = a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots\right]$$

通解

$$y = C_1 y_1 + C_2 y_2,\ x \in [-1,\ 1]$$

当n是整数时

$$a_k = -\frac{(k+2)(k+1)}{(n-k)(k+n+1)}a_{k+2}, (k \le n-2)$$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}a_n$$

$$a_{n-6} = -\frac{(n-4)(n-5)}{6(2n-5)}a_{n-4} = -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)}a_n$$

……

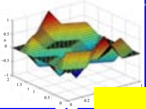
当

$$a_n=\frac{(2n)!}{2^n(n!)^2}=\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{n!},n-1,2,\cdots$$

$$a_{n-2}=-\frac{n(n-1)}{2(2n-1)}\cdot\frac{(2n)!}{2^n(n!)^2}$$

$$=-\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!n(n-1)(n-2)!}$$

$$=-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}$$



$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}\left(-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}\right)$$

$$a_{n-6} = -\frac{(2n-6)!}{2^n 3!(n-3)!(n-6)!}$$

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

整理得

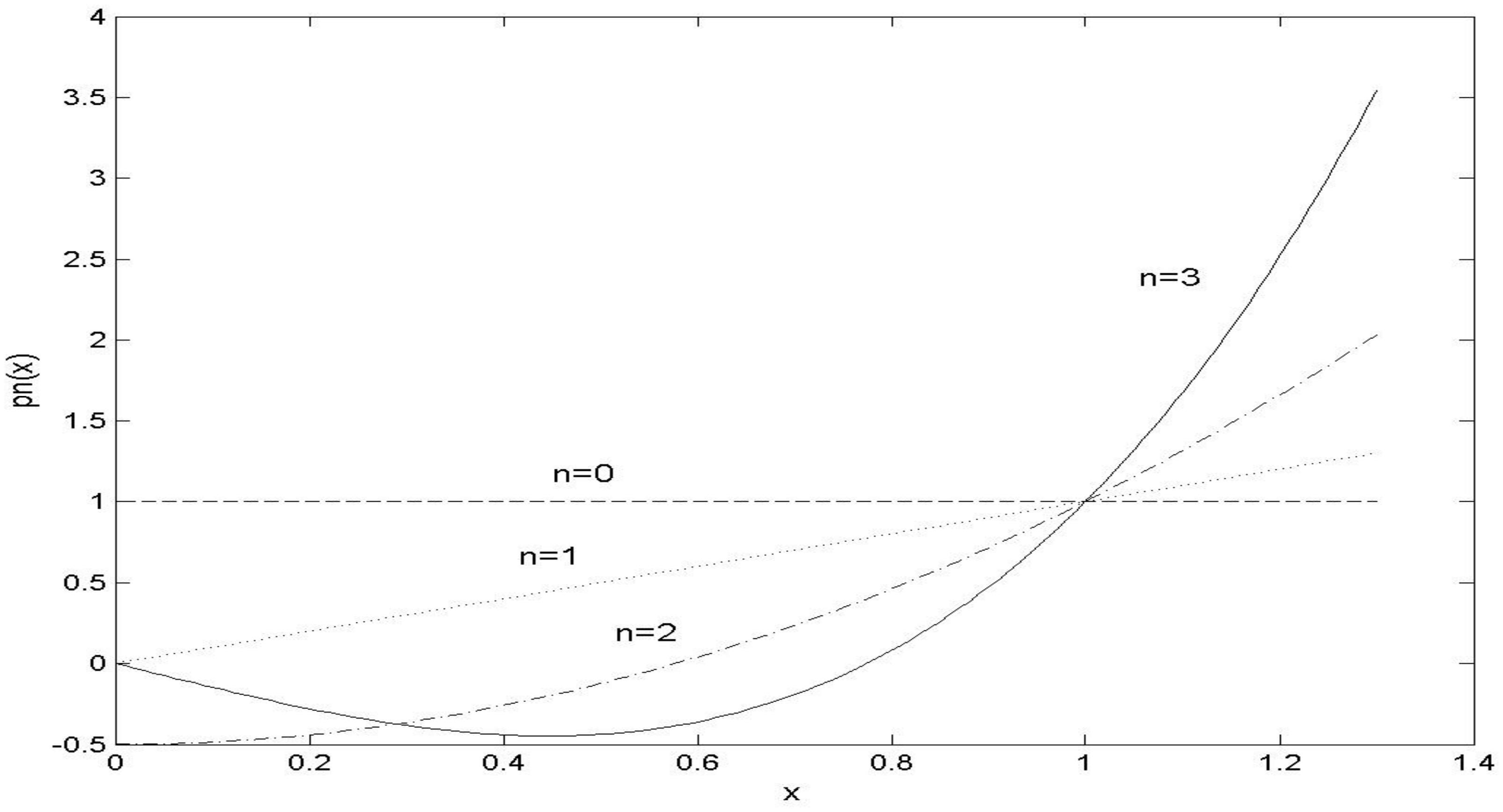
$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

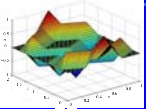
当n是整数时通解

$$y=C_1P_n(x)+C_2Q_n(x),\ x\in[-1,\ 1]$$

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3),\ P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$$


4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
pn(x)
n=3
n=0
n=1
n=2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x



定理1 Rodrigues公式

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

证明:

$$(x^2-1)^n=\sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}$$

$$\begin{aligned}
\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n &= \frac{1}{2^n n!}\sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{2n-2m} \\
&= \frac{1}{2^n n!}\sum_{m=0}^{[n/2]}\frac{(-1)^m n!}{m!(n-m)!}(2n-2m)(2n-2m-1)\cdots(n-2m+1)x^{n-2m} \\
&= \sum_{m=0}^{[n/2]}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} \\
&= P_n(x)
\end{aligned}$$

定理2 Legendre多项式的积分表示

$$P_n(z)=\frac{1}{2\pi j}\int_C \frac{(\zeta^2-1)^n}{2^n(\zeta-z)^{n+1}}d\zeta$$

证明：

$$f(z)=(z^2-1)^n$$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^2-1)^n=\frac{n!}{2\pi \mathrm{j}}\int_C \frac{(\zeta^2-1)^n}{(\zeta-z)^{n+1}}\mathrm{d}\zeta$$

$$\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^2-1)^n=\frac{1}{2\pi \mathrm{j}}\int_C \frac{(\zeta^2-1)^n}{2^n(\zeta-z)^{n+1}}\mathrm{d}\zeta$$

圆心 $z = x \ (x \neq \pm 1)$

圆周 $\zeta = x + \sqrt{x^2 - 1}\mathrm{e}^{\mathrm{j}\varphi}$

$$P_n(x) = \frac{1}{2\pi j}\int_C \frac{(\zeta^2 - 1)^n}{2^n(\zeta - z)^{n+1}} d\zeta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [x + \sqrt{x^2 - 1}\frac{\mathrm{e}^{-\mathrm{j}\varphi} + \mathrm{e}^{\mathrm{j}\varphi}}{2}]^n \mathrm{d}\varphi$$

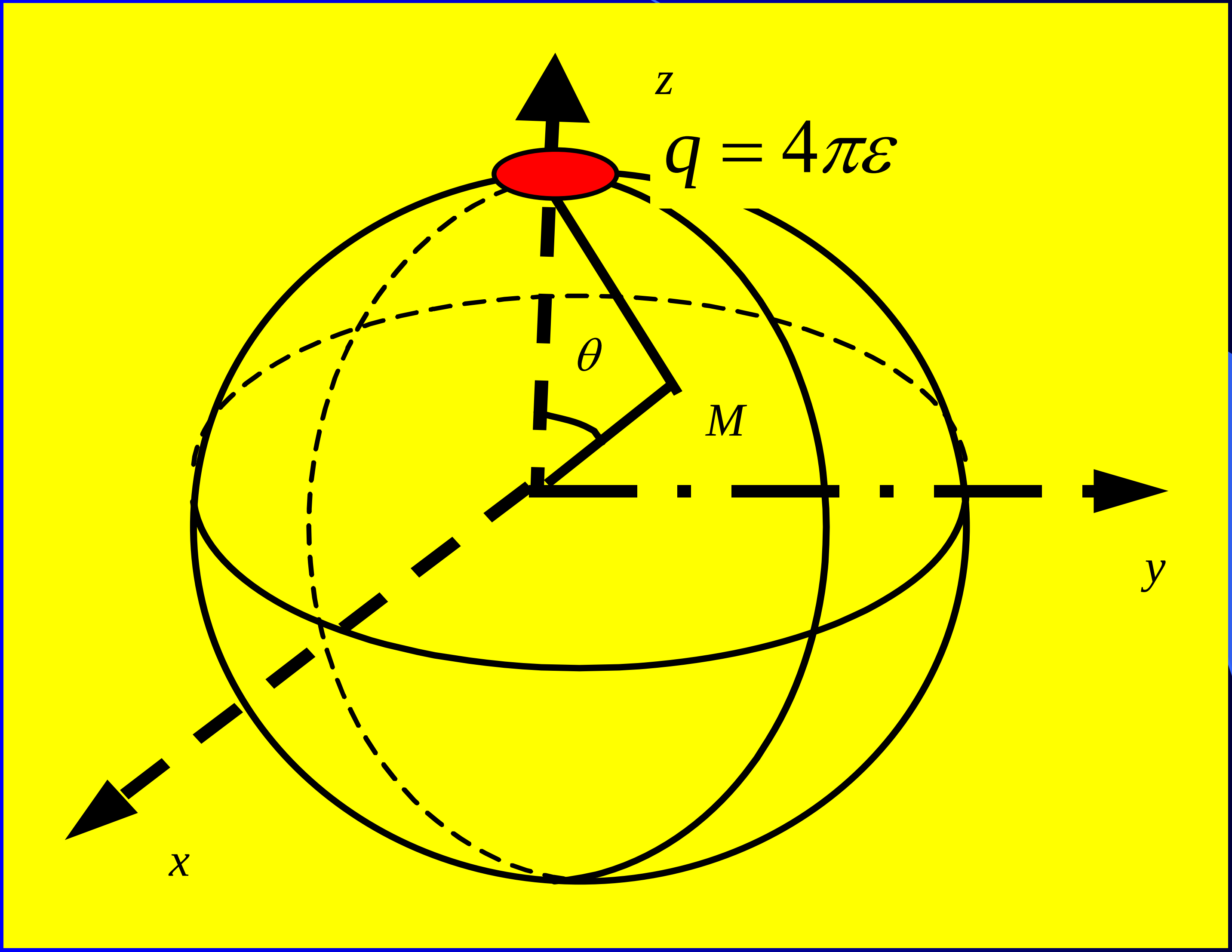
$$= \frac{1}{\pi}\int_0^{\pi} [x + \sqrt{x^2 - 1}\cos\varphi]^n \mathrm{d}\varphi$$

$$x = \cos\theta (0 < \theta < \pi)$$

Laplace积分

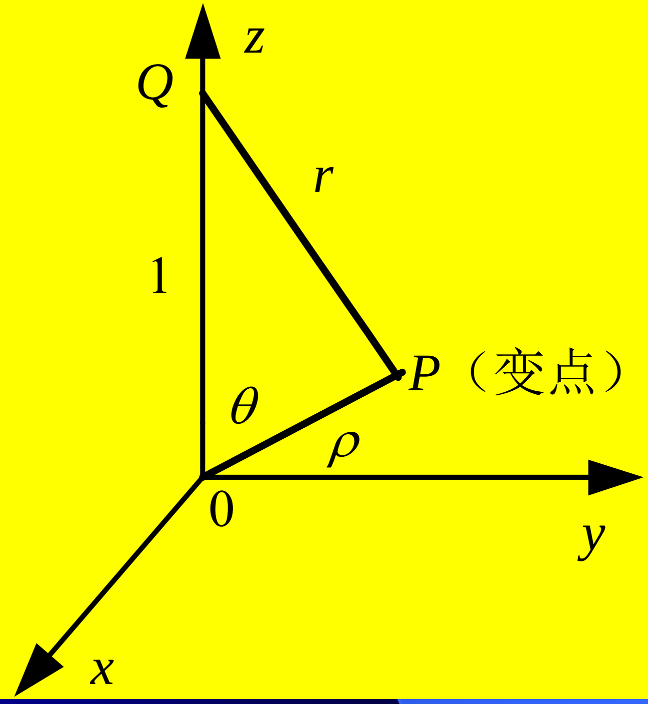
$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi} [\cos\theta + \mathrm{j}\sin\theta\cos\varphi]^n \mathrm{d}\varphi$$

Legendre多项式的母函数

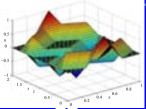


$$r^2 = 1 - 2\rho\cos\theta + \rho^2$$

$$\frac{1}{r_{QP}} = \frac{1}{\sqrt{1 - 2x\rho + \rho^2}}$$



$$G(x, z) = \frac{1}{\sqrt{1 - 2xz + z^2}}, |x| \le 1, |z| < 1$$



$$G(x,\ z)=\frac{1}{\sqrt{1-2xz+z^2}},|\,x\,|\le 1,|\,z\,|<1$$

$$G(x,\ z)=(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}c_n(x)z^n$$

$$\mathrm{z}=\frac{2(u-x)}{u^2-1},\ \mathrm{dz}=2\frac{2xu-1-u^2}{(u^2-1)^2}\mathrm{d}u,\ 1-\mathrm{z}u=\frac{2xu-1-u^2}{u^2-1}$$

$$
\begin{aligned}
c_n(x) &= \frac{1}{2\pi \mathrm{j}}\int_{C'} \left(\frac{2x\zeta-1-\zeta^2}{\zeta^{/2}-1}\right)^{-1} 2^{-(n+1)}\left(\frac{\zeta-x}{\zeta^2-1}\right)^{-(n+1)} 2\frac{2x\zeta-1-\zeta^2}{(\zeta^2-1)^2}\mathrm{d}\zeta \\
&= \frac{1}{2\pi \mathrm{j}}\int_{C'} \frac{(\zeta^2-1)^n}{2^n(\zeta-x)^{n+1}}\mathrm{d}\zeta \\
&= P_n(x)
\end{aligned}
$$

$$G(1,\ z)=\frac{1}{\sqrt{1-2z+z^2}}=\sum_{n=0}^{\infty}P_n(1)z^n=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$$

$$P_n(1)=1$$

$$G(-1,\ z)=\frac{1}{\sqrt{1+2z+z^2}}=\sum_{n=0}^{\infty}P_n(-1)z^n=\frac{1}{1+z}=\sum_{n=0}^{\infty}(-z)^n$$

$$P_n(-1)=(-1)^n$$

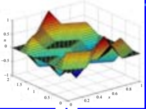
Legendre多项式的递推公式

$$(2n+1)xP_n(x) - nP_{n-1}(x) = (n+1)P_{n+1}(x)$$

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x)$$

$$nP_{n-1}(x) + xP'_{n-1}(x) = P'_n(x)$$

$$n = 1,\ 2,\ 3,\ \cdots$$



连带Legendre方程

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y=0$$

$$y(x)=(1-x^2)^{\frac{m}{2}}v(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(1-x^2)^{\frac{m}{2}}v'-mx(1-x^2)^{\frac{m}{2}-1}v$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=(1-x^2)^{\frac{m}{2}}v''-2mx(1-x^2)^{\frac{m}{2}-1}v'+(1-x^2)^{\frac{m}{2}-1}\left[\frac{m(m-2)x^2}{1-x^2}-m\right]v$$

$$(1-x^2)v'' - 2(m+1)xv' + [n(n+1) - m(m+1)]v = 0$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

$$(1-x^2)P_n^{(m+2)} - 2(m+1)xP_n^{(m+1)} + [n(n+1) - m(m+1)]P_n^{(m)} = 0$$

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} P_n^{(m)}(x)$$

$$\int_{-1}^{1} P_k^m(x) P_n^m(x) \mathrm{d}x = 0$$

$$\int_{-1}^{1} [P_n^m(x)]^2 \mathrm{d}x = \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}$$

Laplace方程在球形区域上的Dirichlet问题

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\left(\dfrac{\partial^2 u}{\partial \theta^2}+\dfrac{\cos\theta}{\sin\theta}\dfrac{\partial u}{\partial \theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u}{\partial \varphi^2}=0 \\
u\big|_{r=R}=f(\theta,\ \varphi),\quad r<R,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi
\end{cases}
$$

$$u(r,\ \theta,\ \varphi)=R(r)\Phi(\varphi)\Theta(\theta)$$

$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-\lambda R=0$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+\mu\Phi=0,\ \mu=m^2,m=0,\ 1,\ 2,\ \cdots$$

$$(1-t^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2}-2t\frac{\mathrm{d}\Theta}{\mathrm{d}t}+\left(\lambda-\frac{\mu}{1-t^2}\right)\Theta=0,t=\cos\theta$$