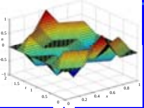

数理方程与特殊函数

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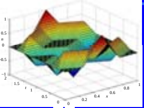
应用数学学院



《数学物理方程》

作者：李明奇、田太心

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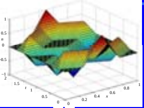


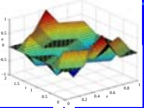
参考文献

[1] 梁昆淼，《数学物理方法》，人民教育出版社，1998

[2] 沈施，《数学物理方法》，同济大学出版社，2002

[3] 姚瑞正，梁家宝，《数学物理方法》，武汉大学出版社，1992

[4] 谢鸿证，杨枫林，《数学物理方程》，科学出版社，2001

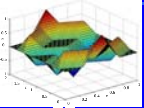


[5] 南京工学院数学教研组，《数学物理方程与特殊函数》，人民教育出版社，1983

[6] 孙振绮，《数学物理方程》，机械工业出版社，2004

[7] 胡嗣柱，倪光炯，《数学物理方法》，复旦大学出版社，1989

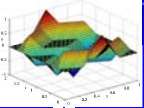
[8] 姜尚礼，陈亚浙，《数学物理方程讲义》，高等教育出版社，1996



[9] F.W.拜伦，R.w.富勒，《物理中的数学方法》，科学出版社，1982

[10] 陈恕行，洪家兴，《偏微分方程近代方法》，复旦大学出版社，1989

[11] 王元明，管平，《线性偏微分方程引论》，东南大学出版社，2002



- 课程的背景
- 课程的基本要求
- 常微分方程
- 积分方程
- 积分公式
- 常用算子

➢物理、力学、电磁学、自动化工程、生物工程等领域中, 需要研究某物理量和其它物理量之间的变化关系。这种关系在数学上称为函数关系。

➢例如, 在弹道设计中求导弹飞行过程中某时刻的飞行路程$S(t)$、飞行高度$H(t)$、速度$v(t)$。

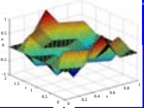
➢物理学中的定律，往往只给出这些函数和它们的各阶导数与自变量的关系。

牛顿第二定律: $F = ma$

a—物体加速度;F—合外力;m—物体质量

付里叶热传导定律: $$Q = -\kappa \frac{dT}{dx}$$

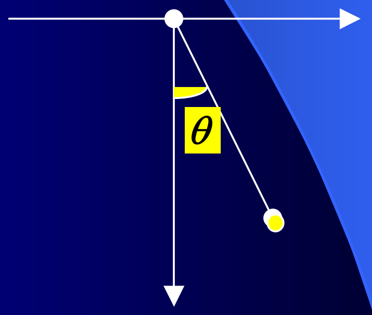
Q—热量;T—温度;κ—热导率



➢如果微分方程中涉及单因素(一个自变量），这种方程称为常微分方程；如果微分方程涉及多因素(多个自变量), 这时方程中出现的导数是偏导数, 相应的方程称为偏微分方程。

$$\frac{d^2\theta}{dt^2}+a\sin\theta=0$$

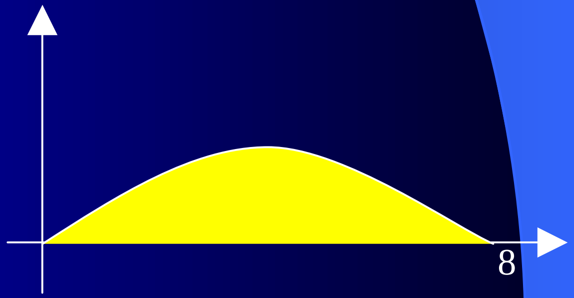
单摆:$\boldsymbol{\theta=\theta(t)}$

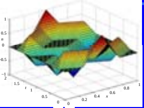


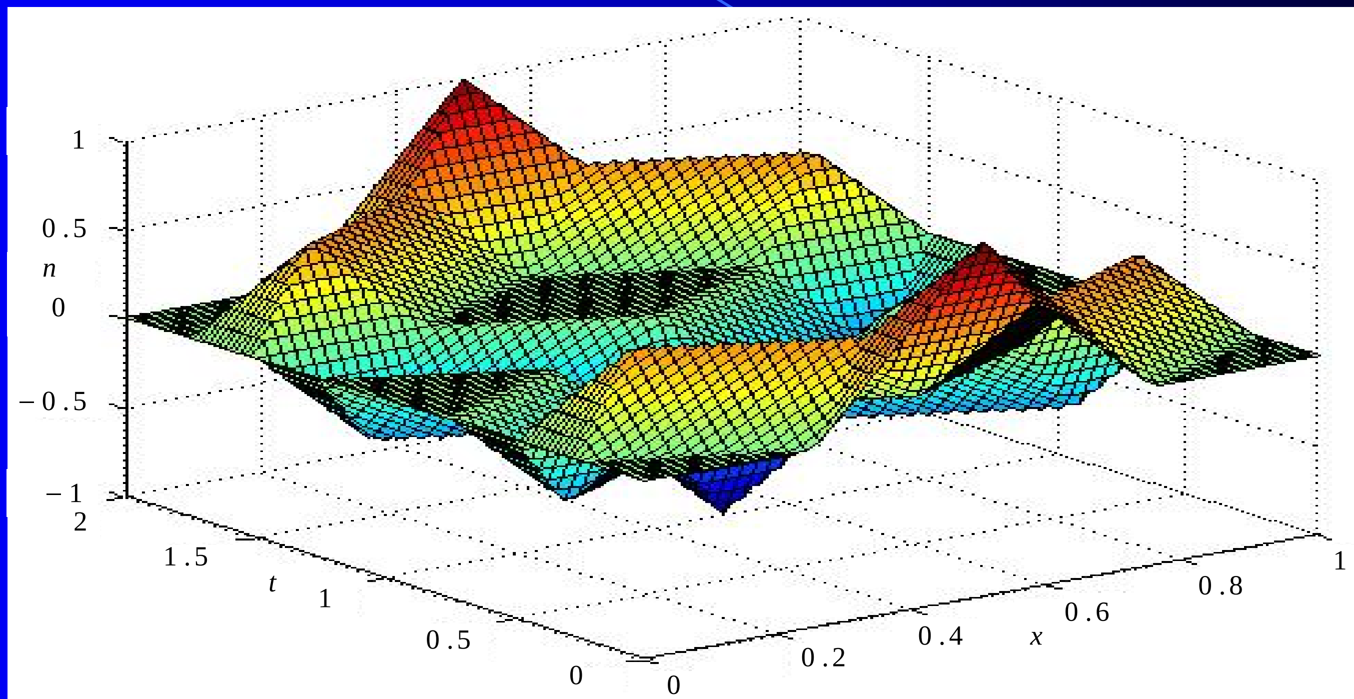
$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

弦振动:

$\boldsymbol{u=u(x,t)}$






1
0.5
n
0
−0.5
−1
2
1.5
t
1
0.5
0
0
0.2
0.4
x
0.6
0.8
1

对于n阶常微分方程的解，通解中带有n个任意常数，例如一阶常微分方程$y'=f(x)$

$$y=\int_{x_0}^{x} f(t)dt+C$$

对偏微分方程

$$\frac{\partial^2 u}{\partial x\partial y}=f(x,y)$$

解可以表示为

$$u(x,y)=\int_{y_0}^{y}\int_{x_0}^{x} f(\xi,\eta)d\xi d\eta+w(x)+v(y)$$

由于多数偏微分方程是从物理问题中导出的，所以称为数学物理方程。我们主要讨论的物理过程分为三类：

- 振动与波

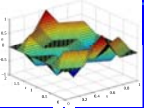
$$u_{tt} = a^2 u_{xx}$$

- 输运过程

$$u_t = a^2 u_{xx}$$

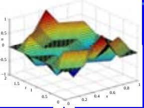
- 稳定过程

$$u_{xx} + u_{yy} + u_{zz} = 0$$



课程的基本要求:

- 理解数学物理方程中出现的基本概念
- 掌握基本理论和基本方法
- 了解数理方程的来源与有关概念的物理解释
- 通过习题对定解问题解法进行必要的训练
- 掌握二阶偏微分方程几种主要的求解方法



常微分方程

1. 可分离变量的一阶微分方程。

$$f(x)dx = g(y)dy$$

2. 齐次方程基本形式为：

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

3. 一阶线性微分方程基本形式为：

$$y' + p(x)y = q(x)$$

4. 贝努里方程：

$$y' + p(x)y = q(x)y^n$$

5. 可降阶的二阶微分方程：

$$y'' = f(x, y')$$

$$y'' = f(y, y')$$

线性微分方程

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

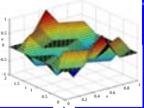
$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}y' + a_n y = 0$$

7. 二阶常系数非齐次线性微分方程的特解

$$y'' + py' + qy = p_m(x)e^{\lambda_0 x}$$

$$y = x^i q_m(x)e^{\lambda_0 x}$$



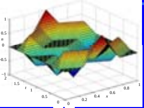
$$y'' + py' + qy = p_m(x)e^{\lambda_0 x}$$

$$y = Q_n(x)e^{\lambda_0 x}$$

$$Q_n'' + (2\lambda_0 + p)Q_n' + (\lambda_0^2 + p\lambda_0 + q)Q_n = p_m(x)$$

$$\left[(x-x_0)^k g_m(x)\right]' = k(x-x_0)^{k-1} g_m(x) + (x-x_0)^k g_m'(x)$$

$$= (x-x_0)^{k-1}\left[k g_m(x) + (x-x_0) g_m'(x)\right]$$



$$y'' + py' + qy = e^{\lambda_0 x}[p_m(x)\cos\omega_0 x + p_n(x)\sin\omega_0 x]$$

$$y = x^k e^{\lambda_0 x}[p_l(x)\cos\omega_0 x + q_l(x)\sin\omega_0 x]$$

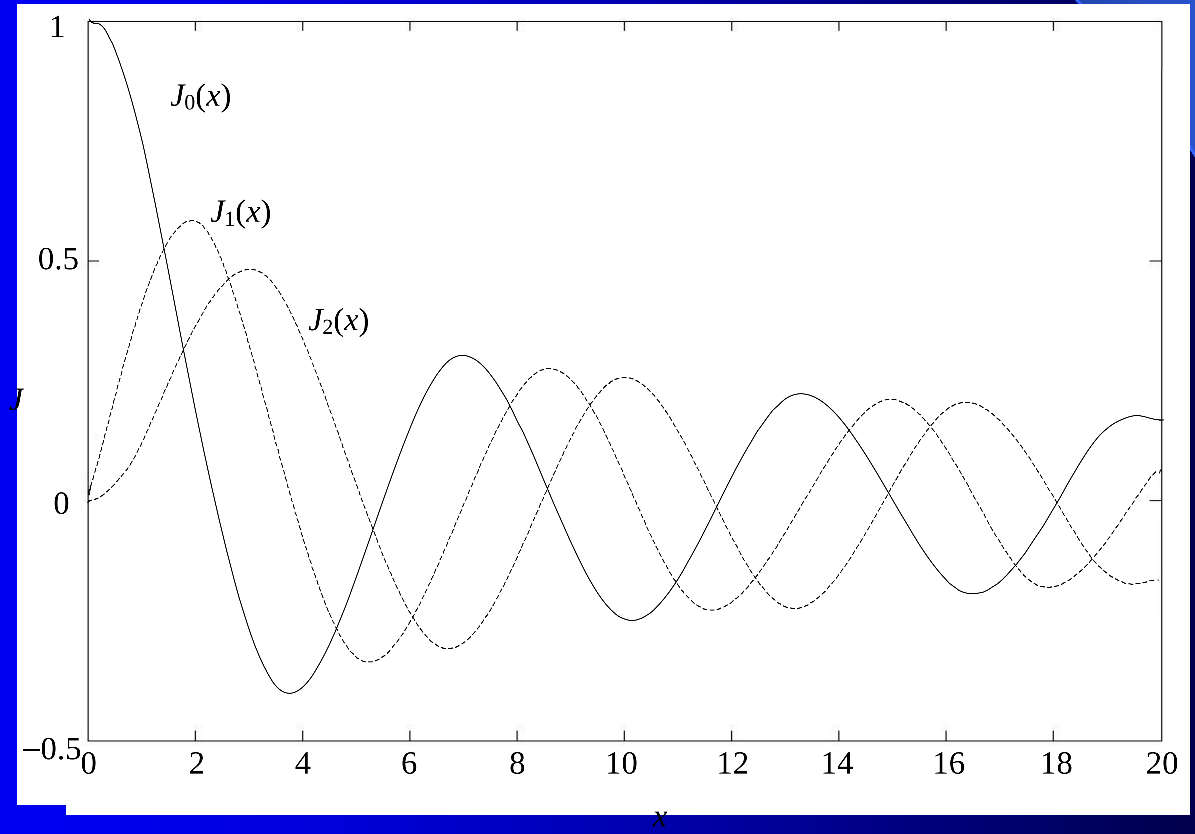
8. 欧拉（**Euler**）方程

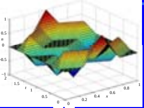
$$x^n y^{(n)} + p_1 x^{n-1} y^{(n-1)} + \cdots + p_{n-1} x y' + p_n y = f(x)$$

$$\sum_{k=0}^{n} p_{n-k} D(D-1)\cdots(D-k+1)y = f(e^t)$$

9. 贝塞尔（Bessel）方程

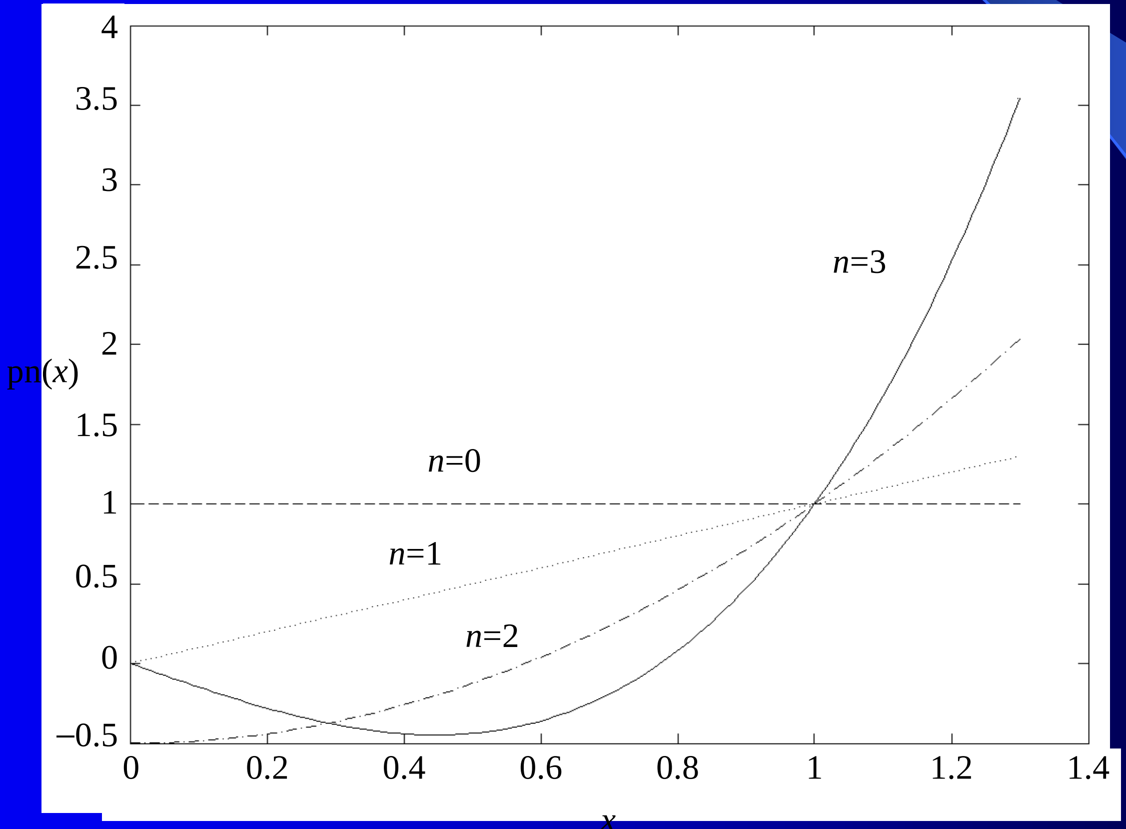
$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

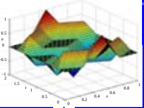




10. 勒让德方程

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, x \in [-1,1]$$





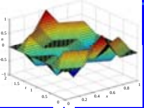
积分方程

$$f(x)=\int_a^b k(x,t)y(t)dt \quad x\in[a,b]$$

$$f(x)=y(x)-\int_a^b k(x,t)y(t)dt \quad x\in[a,b]$$

$$f(x)=\int_a^b k(x,t)f(t,y(t))dt \quad x\in[a,b]$$

$$f(x)=\int_a^b k(x,t,y(t))dt \quad x\in[a,b]$$



定义**1** 积分号下含有未知函数的方程称为积分方程。

若方程关于未知函数是线性的，则称之为线性积分方程。

定义**2** 若未知函数只出现在积分号下，称为第一类线性积分方程；

若未知函数不仅出现在积分号下，还出现在其它部分，称为第二类线性积分方程。

核分类

连续核

L^2—核 $$\int_a^b\int_a^b |k(x,t)|^2 dxdt < \infty$$

Hermite核 $$k(x,t) = \overline{k}(t,x)$$

对称核 $$k(x,t) = k(t,x)$$

斜对称核 $$k(x,t) = -k(t,x)$$

退化核 $$k(x,t) = \sum_{i=1}^{n} \alpha_i(x)\overline{\beta}_i(t)$$

线性积分方程分类

L^2—核 ，第一、二类线性积分方程称为第一、二类**Fredholm**积分方程。

$$k(x,t)\equiv 0 \quad x<t$$

,

第一、二类**Volterra**积分方程

$$k(x,t)=\frac{H(x,t)}{\left|x-t\right|^{\alpha}} \quad \alpha\in(0,\ 1/2)$$

第一、二类弱奇异性积分方程

特征值　特征函数

$$y(x)=\lambda\int_a^b k(x,t)y(t)dt$$

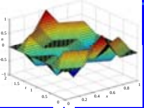
迭代法

$$f(x)=y(x)-\lambda\int_a^b k(x,t)y(t)dt \quad x\in[a,b]$$

$$y_0(x)=f(x)$$

$$y_1(x)=f(x)+\lambda\int_a^b k(x,t)y_0(t)dt=f(x)+\lambda\int_a^b k(x,t)f(t)dt$$

$$y_n(x)=\sum_{i=0}^{n}\lambda^i\varphi_i(x),\ \ \varphi_0(x)=f(x),\ \ \varphi_i(x)=\int_a^b k(x,t)\varphi_i(x)dt$$



预解方程

$$R(x,t;\lambda)=k(x,t)+\lambda_0\int_a^b k(x,\tau)R(\tau,t;\lambda_0)d\tau$$

$$R(x,t;\lambda)=k(x,t)+\lambda_0\int_a^b k(\tau,t)R(x,\tau;\lambda_0)d\tau$$

收敛性

定理 1 若 $f(x)$ 在 $x\in[a,b]$，$k(x,t)$ 在 $[a,b]\times[a,b]$ 内都连续，且 $\left|f(x)\right|\le m$，$\left|k(x,t)\right|\le M$，$\left|\lambda\right|<\dfrac{1}{M(b-a)}$。级数 $\sum\limits_{i=0}^{+\infty}\lambda^i\varphi_i(x)$ 在 $x\in[a,b]$ 一致绝对收敛，并且为方程 $y(x)=f(x)+\lambda\int_a^b k(x,t)y(t)dt$ 的唯一解。

积分公式

格林（Green）公式

$$\oint_L p(x,y)dx + q(x,y)dy = \iint_D [q_x(x,y) - p_y(x,y)]dxdy$$

斯托克斯（Strokes）公式

$$\oint_L p(x,y,z)dx + q(x,y,z)dy + r(x,y,z)dz = \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ p & q & r \end{vmatrix}$$

高斯（Gauss）公式

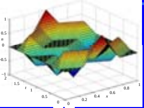
$$\oiint_S p(x,y,z)dydz + q(x,y,z)dzdx + r(x,y,z)dxdy = \iiint_V (p_x + q_y + r_z)dxdydz$$

常用算子

$$Df(x) = f'(x)$$

$$\nabla = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



$$grad\ u = \nabla u$$

$$div\ \vec{A} = \nabla \cdot \vec{A}$$

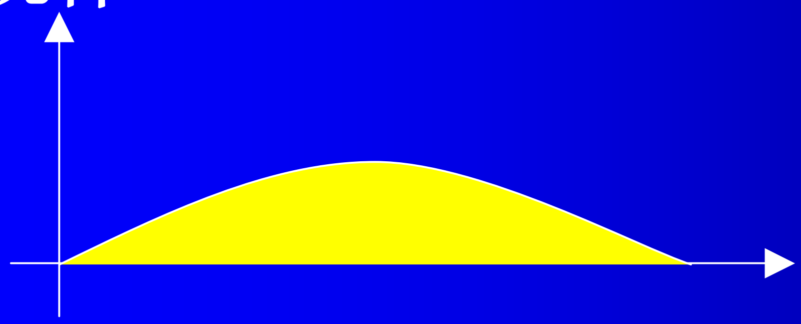
$$rot\ \vec{A} = \nabla \times \vec{A}$$

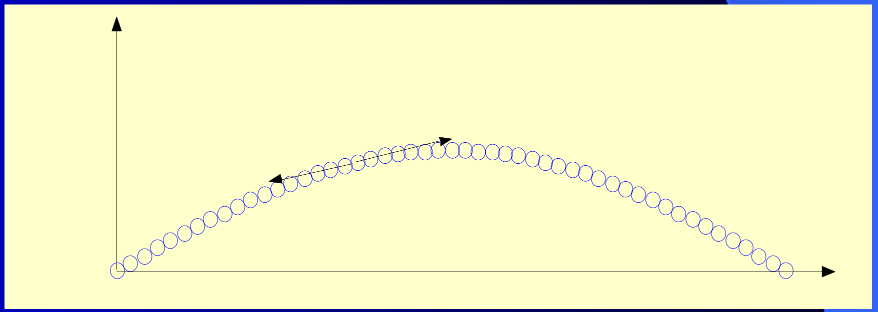
$$\nabla^2\ u = \nabla \cdot \nabla u = \nabla \cdot grad\ u = \Delta u$$

$$\nabla(uv)\ = \nabla u \cdot v + u \nabla v$$

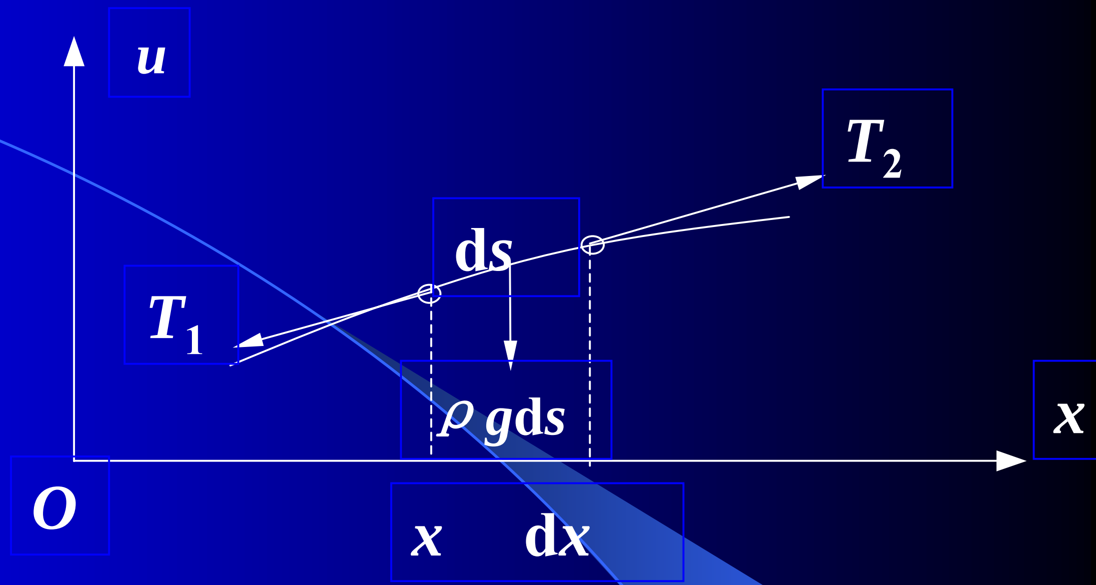
一、均匀细弦微小横振动问题

一根均匀柔软的细弦线，一端固定在坐标原点，另一端沿x轴拉紧固定在x轴上的L处，受到扰动，开始沿x轴（平衡位置）上下作微小横振动（细弦线上各点运动方向垂直于x轴）。试建立细弦线上任意点位移函数$u(x，t)$所满足的规律。





设细弦线各点线密度为 ρ，细弦线质点之间相互作用力为张力 $T(x, t)$



水平合力为零 => $T_2 \cos\alpha_2 - T_1 \cos\alpha_1 = 0$

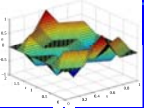
$$T_2 \approx T_1 \approx T$$

铅直合力: $F=ma$

$$T(\sin\alpha_1 - \sin\alpha_2) = \rho\,\mathrm{d}s\, u_{tt}$$

=>

$$T(\tan\alpha_1 - \tan\alpha_2) = \rho\,\mathrm{d}s\, u_{tt}$$



$$T[\,u_x(x+\mathrm{d}x,\ t)-u_x(x,\ t)] = \rho\,\mathrm{ds}\,u_{tt}$$

$$T\,u_{xx}\,(x+v\mathrm{d}x,\ t) = \rho\,\mathrm{ds}\,u_{tt}$$

$$u_{tt}= a^2\,u_{xx}$$

其中 $\dfrac{T}{\rho} = a^2$

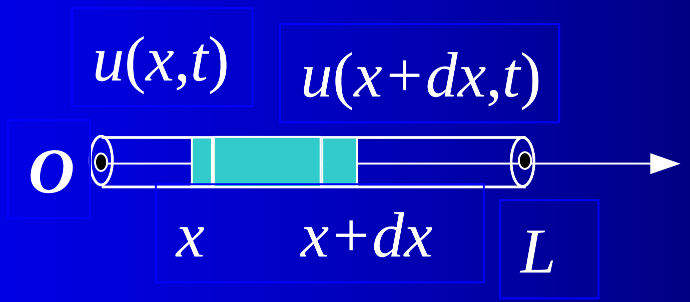
一维波动方程： $u_{tt} = a^2\,u_{xx}$

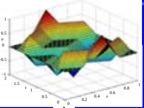
考虑有恒外力密度$F(x，t)$作用时，可以得到一维波动方程的非齐次形式

$$u_{tt} = a^2\,u_{xx} + f(x,\,t)$$

二、细杆的纵向振动问题。

设均匀细杆长为L，线密度为ρ，杨氏模量为Y，杆的一端固定在坐标原点，细杆受到沿杆长方向的扰动（沿x轴方向的振动）。试建立杆上质点位移函数$u(x，t)$的纵向振动方程。





由牛顿第二定律

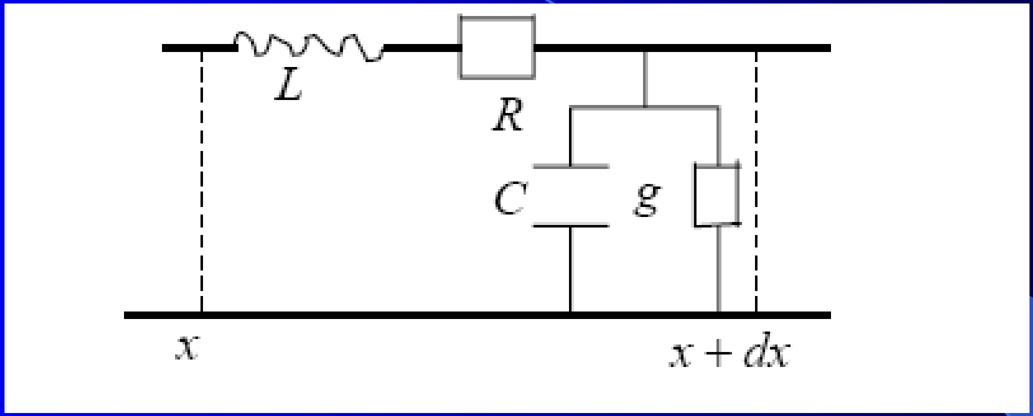
$$SY[u_x(x+\mathrm{d}x，t)-u_x(x，t)]=\rho S\mathrm{d}x u_{tt}$$

令$a^2 = Y/\rho$。化简，得

$$u_{tt} = a^2 u_{xx}$$

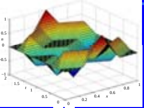
习题2.1（P.21）1，2，3， 5

高频传输方程



$$\begin{cases} u(x,t)-u(x+dx,t)=Ridx+Ldxi_t \\ i(x,t)-i(x+dx,t)=Cdxu_t+gudx \end{cases}$$

$$\begin{cases} -u_x(x+\theta_1 dx,t)dx=Ridx+Ldxi_t \\ -i_x(x+\theta_2 dx,t)dx=gudx+Cdxu_t \end{cases}$$



$$\begin{cases} -u_x = Ri + Li_t \\ -i_x = gu + Cu_t \end{cases}$$

$$u_{xx} - LCu_{tt} - (Lg + RC)u_t - Rgu = 0$$

$$i_{xx} - LCi_{tt} - (Lg + RC)i_t - Rgi = 0$$

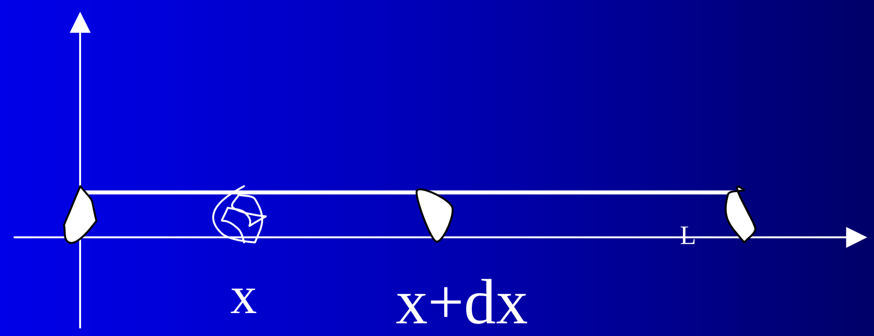
$$\begin{cases} u_{tt} = a^2 u_{xx} \\ i_{tt} = a^2 i_{xx} \end{cases}$$

三、热传导方程

1、物理模型

截面积为A的均匀细杆，侧面绝热，沿杆长方向有温差，求热量的流动。首先，我们来复习一下关热量的几个概念：

温度T，密度，时间t， 体积V ， 面积S ， 热量Q



(1)比热：单位物质，温度升高一度所需热量

$$C=\frac{Q}{(\rho V)T}$$

(2)热流密度：单位时间流过单位面积的热量

导热率–

$$q=\frac{Q}{tS}=-K\frac{\partial u}{\partial n}\quad k-\text{导热率}$$

(3)热源强度：单位时间，单位体积放出热量

$$F=\frac{Q}{tV}$$

2、任一Δx段在Δt时间热量情况：

流入x面：$Q_1 = -k\dfrac{\partial u}{\partial x}\Big|_x \cdot A\Delta t$

流出$x+\Delta x$面：$Q_2 = -k\dfrac{\partial u}{\partial x}\Big|_{x+\Delta x} \cdot A\Delta t$

热源产生：设有热源其密度为 $f(x, t)$

$Q_3 = F \cdot \Delta t$（$A\Delta x$）

升温所需：设杆比热为C，体密度为ρ

$$Q = C \cdot (\rho A\Delta x) u(x, t+\Delta t) - u(x, t)$$

根据热量守恒定律：

$$Q = Q_1 - Q_2 + Q_3$$

即 $c\rho A x\left[u(x,t+\Delta t) - u(x,t)\right]$

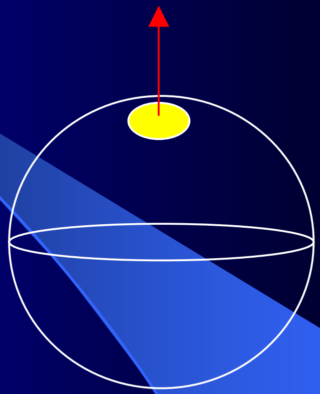
$$= k\left[u_x(x+\Delta x,t) - u_x(x,t)\right]A\Delta t + FA\Delta t\Delta x$$

$$u_t = a^2 u_{xx} + f$$

$$a^2 = \frac{k}{c\rho}$$

$$u_t = a^2 u_{xx}$$

设均匀且各向同性的导热体, 置于温度比它高的热场中, 求物体中温度$u(x, y, z)$所分布的规律。



通过微元曲面 ds 的热量:

$$dQ_1 = k\frac{\partial u}{\partial n}dsdt$$

通过曲面进入导热体的总热量:

$$Q_1 = \int_{t_1}^{t_2}[\iint_S k\frac{\partial u}{\partial n}ds]dt$$

$$Q_1 = \int_{.t_1}^{.t_2} [\iint_s k \frac{\partial u}{\partial n} ds] dt$$

$$= \int_{.t_1}^{.t_2} [\iint_s k grad u \cdot \vec{n}_0 ds] dt$$

$$= \int_{.t_1}^{.t_2} [\iiint_v k div\left(grad u\right) dv] dt$$

$$Q_2 = \iiint\limits_v c\rho\left[u(x,y,z,t_2) - u(x,y,z,t_1)\right]dv$$

$$= \iiint\limits_v c\rho\left[\int_{t_1}^{t_2} \frac{\partial u}{\partial t}dt\right]dv = \int_{t_1}^{t_2}[\iiint\limits_v c\rho\frac{\partial u}{\partial t}dv]dt$$

$$\int_{t_1}^{t_2}[\iiint\limits_v k\nabla^2 u dv]dt = \int_{t_1}^{t_2}[\iiint\limits_v c\rho\frac{\partial u}{\partial t}dv]dt$$

$$u_t = a^2\nabla^2 u = a^2(u_{xx} + u_{yy} + u_{zz})$$

四、稳态方程

静电场特点

$$\vec{E} = -\nabla u$$

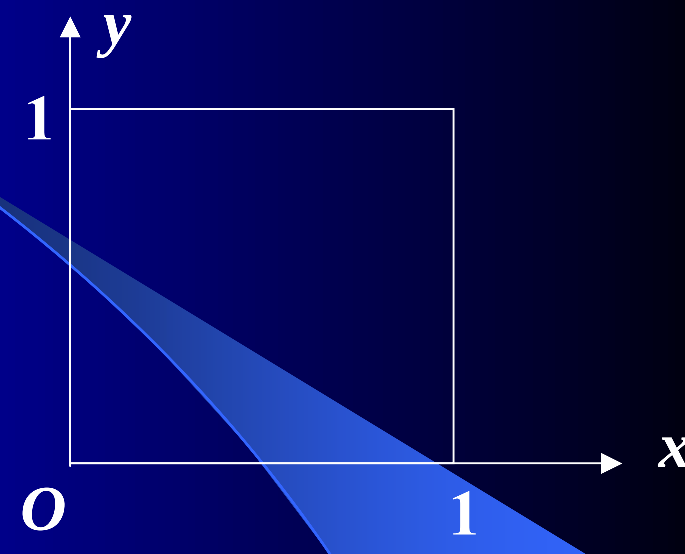
$$\nabla \cdot \left(\varepsilon \vec{E} \right) = \rho$$

$$\nabla \times \vec{E} = 0$$

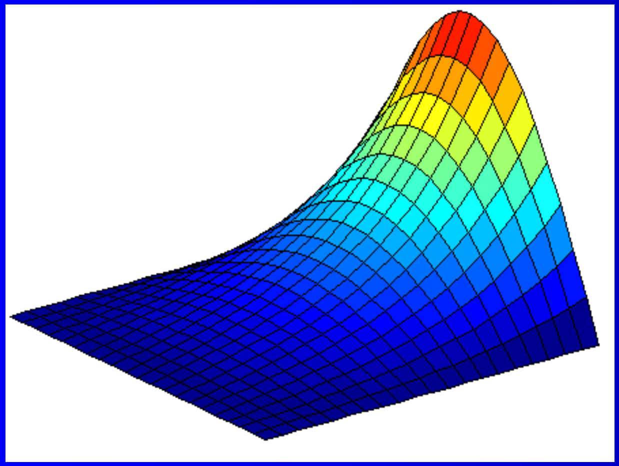
静电场方程

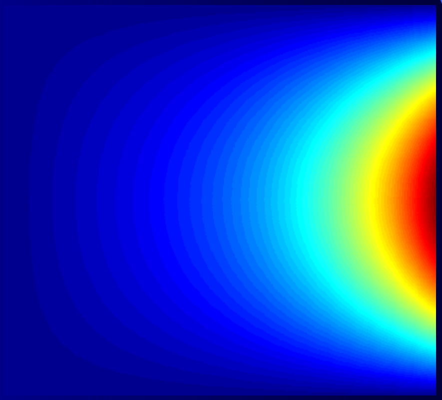
$$\nabla^2 u = \Delta u = -\frac{\rho}{\varepsilon}$$

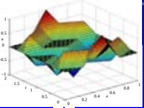
$$\begin{cases} u_{xx}+u_{yy}=0, \quad 0<x,\, y<1 \\ u(0,y)=u(x,0)=u(x,1)=0 \\ u(1,y)=\sin \pi y \end{cases}$$



准确解: $$u(x,y)=\frac{sh\,\pi x}{sh\,\pi}\sin \pi y$$





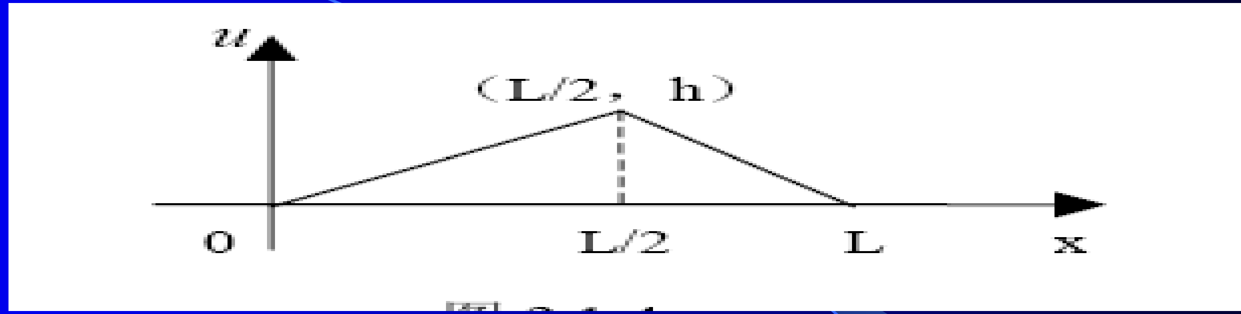


初始条件: $u(x, y, z, 0)= \varphi(x, y, z)$

I. 第一类边界条件: $u|_S = \alpha(x, y, z, t)$

II. 第二类边界条件: $\left.\frac{\partial u}{\partial n}\right|_S = \beta(x, y, z, t)$

III. 第三类边界条件: $\left.[\frac{\partial u}{\partial n} + \sigma u]\right|_S = \gamma(x, y, z, t)$



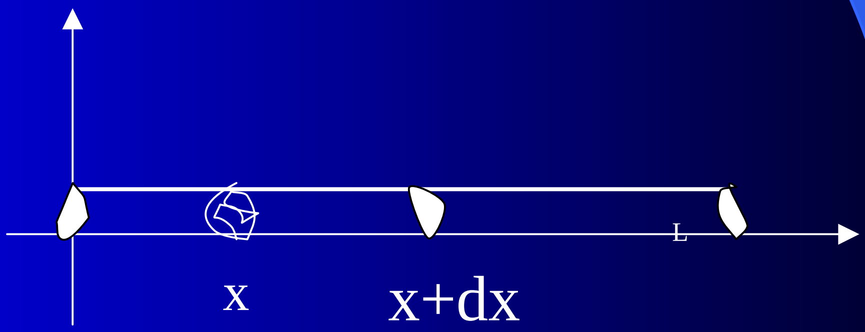
$$u \mid u_{t=0} = \begin{cases} \frac{2h}{L}x, & 0 \le x < \frac{L}{2} \\ \frac{2h}{L}(L-x), & \frac{L}{2} \le x \le L \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(x,0) = f(x), \quad u_t(x,0) = 0 \\ u(0,t) = u(l,t) = 0 \end{cases}$$

例 6 细杆在 $x=0$ 点固定，在 $x=L$ 端受外力 $F(t)$ 的作用，作微小纵振动（沿 x 轴方向的振动）。求证其边界条件为：

$$u\big|_{x=0}=0 \text{（第一类）} \qquad u_x\big|_{x=L}=\frac{F(t)}{YS} \text{（第二类）}$$

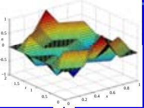
$$T=YSu_x(L-\varepsilon,t)$$



例4 面积为1的导热杆在 $x=0$ 端绝热，在 $x=L$ 端有恒定热流 q 进入（单位时间通过单位面积流入的热量 q）。试写出边界条件（设杆的密度为 ρ）。

$$ku_x(l,t)=q$$

$$\begin{cases} u_t = a^2 u_{xx} \\ u_x(0,t)=0, u_x(l,t)=q/k \end{cases}$$



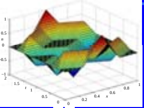
牛顿冷却定律

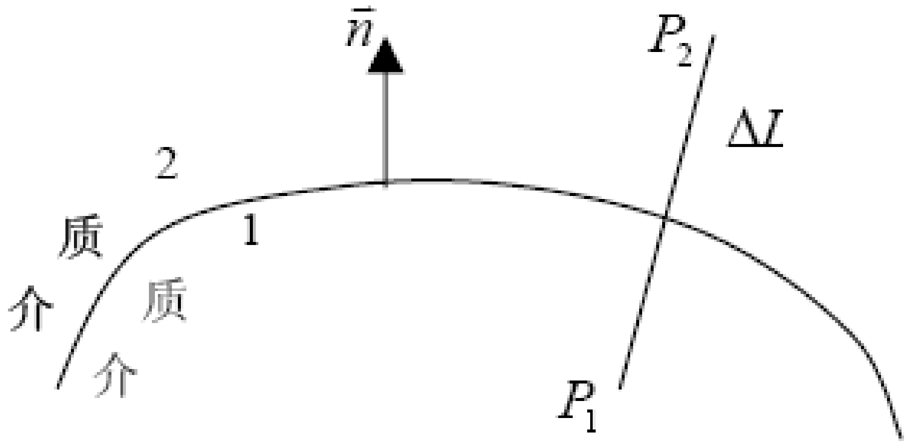
$$q(t)=k_1(u|_s-u_1)$$

(3) 一端($x=0$)温度为$u_1(t)$,另一端($x=L$)与温度为$\theta(t)$的介质有热交换。

$$ku_x(l,t)=k_1(u|_s-u_1)=k_1(\theta(t)-u(l,t))$$

$$\begin{cases} u_t=a^2u_{xx} \\ u(0,t)=u_1(t), ku_x(l,t)+k_1u(l,t)=k_1\theta(t) \end{cases}$$

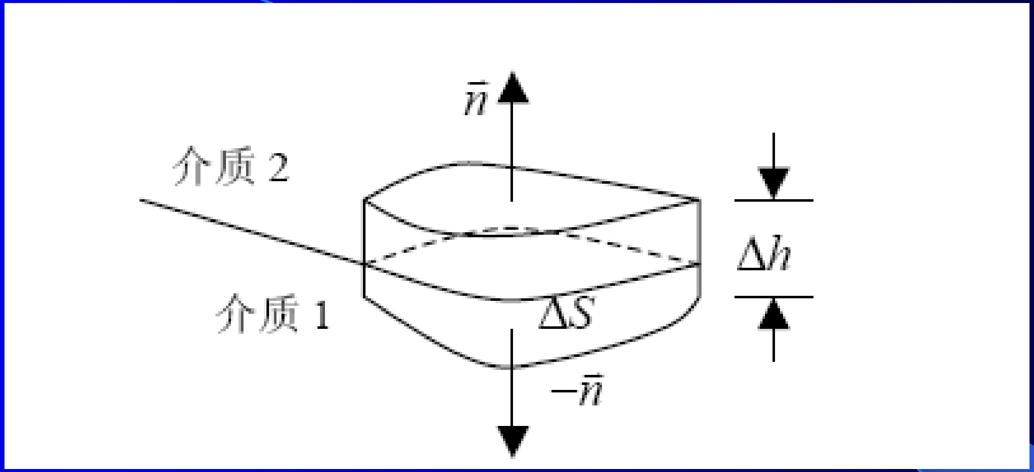




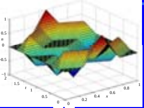
在两种介质的分界面上，静电场电势u的边值关系为

$$u_1 = u_2$$

$$\varepsilon_2 \frac{\partial u_2}{\partial n} - \varepsilon_1 \frac{\partial u_1}{\partial n} = -\sigma_f$$



$\vec{n}$
介质 2
Δh
介质 1
ΔS
$-\vec{n}$



$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{E} = -\nabla u$$

$$\iint_s \vec{D} \cdot d\vec{s}$$

$$= -\varepsilon \iint_s \frac{\partial u}{\partial n} ds$$

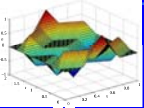
$$= -\varepsilon \iint_s \frac{\partial u}{\partial r} ds$$

$$= Q_f$$

$$\vec{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\frac{\Delta Q_f}{\Delta s}=\sigma_f$$

$$\vec{n}\cdot\left(-\varepsilon_2\nabla u_2+\varepsilon_1\nabla u_1\right)=\sigma_f$$

$$\varepsilon_2\frac{\partial u_2}{\partial n}-\varepsilon_1\frac{\partial u_1}{\partial n}=-\sigma_f$$

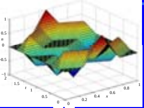


u_1 为导体的电势，u_2 为绝缘介质的电势；

$$u_1 = u_2$$

$$\varepsilon_2 \frac{\partial u_2}{\partial n} = -\sigma_f$$

$$\iint_S \varepsilon_2 \frac{\partial u_2}{\partial n} ds = -Q_f$$



例　均匀介质球的中心置一点电荷，球的介电常数为，球外为真空。试写出电势所满足的泛定方程及定解条件。

解：由于除原点外处处没有自由电荷分布，故

$$\boxed{\nabla^2 u_1 = 0 (r \neq 0)}$$

$$\boxed{\nabla^2 u_2 = 0}$$

定解条件为：

a. $\boxed{u_2\big|_{r\to\infty}=0}$

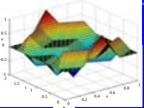
b. 界面上电势连续，设介质球半径为 $\boxed{r_0}$

$$\boxed{u_1\big|_{r=r_0}=u_2\big|_{r=r_0}}$$

c. 因为界面上没有自由电荷分布, $\sigma_f = 0$ 。

$$\varepsilon \frac{\partial u_1}{\partial r}\bigg|_{r=r_0} = \varepsilon_0 \frac{\partial u_2}{\partial r}\bigg|_{r=r_0}$$

d. $$-\iint_s \frac{\partial u_2}{\partial r} ds = \frac{Q_f}{\varepsilon_0}$$



方程的化简与分类·特征方程

1. 二阶线性方程的一般形式：

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1 u_x + b_2 u_y + cu = f$$

引入 $$L = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{12}\frac{\partial^2}{\partial x \partial y} + a_{22}\frac{\partial^2}{\partial y^2} + b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + c$$

则

$$Lu = f$$

2. 二阶线性方程分类

$$\Delta = a_{12}{}^{2} - a_{11}a_{22}$$

a. $\Delta > 0$ 双曲型

b. $\Delta = 0$ 抛物型

c. $\Delta < 0$ 椭圆型

3. 化简方法

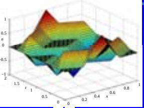
(1) 作变换

$$\begin{cases} \zeta = \varphi_1(x, y) \\ \eta = \varphi_2(x, y) \end{cases}$$

(2) 代入 得

$$\bar{a}_{11}u_{\zeta\zeta} + 2\bar{a}_{12}u_{\zeta\eta} + \bar{a}_{22}u_{\eta\eta} + \bar{b}_1u_\zeta + \bar{b}_2u_\eta + \bar{c}_1u = \bar{f}$$

$$
\begin{cases}
\bar{a}_{11} = a_{11}\zeta_x{}^2 + 2a_{12}\zeta_x\zeta_y + a_{22}\zeta_y{}^2 \\
\bar{a}_{22} = a_{11}\eta_x{}^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y{}^2 \\
\bar{a}_{12} = a_{11}\zeta_x\eta_x + a_{12}\left(\zeta_x\eta_y + \zeta_y\eta_x\right) + a_{22}\zeta_y\eta_y
\end{cases}
$$



(3)

$$\zeta = \varphi_1(x, y), \quad \eta = \varphi_2(x, y)$$

$$a_{11}\varphi_x^{\ 2} + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^{\ 2} = 0$$

(4)

$$a_{11}\frac{\varphi_x^{\ 2}}{\varphi_y^{\ 2}} + 2a_{12}\frac{\varphi_x}{\varphi_y} + a_{22} = 0$$

(5) $$\varphi(x, y) = 0$$

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = 0$$

(6) $$\frac{dy}{dx} = \frac{-2a_{12} - 2\sqrt{a_{12}{}^2 - a_{11}a_{22}}}{2a_{11}}$$

$$\Delta = a_{12}{}^2 - a_{11}a_{22}$$

(7) $\Delta > 0$

$$y + ax = c, y + bx = c$$

$$\begin{cases} \zeta = ax + y \\ \eta = bx + y \end{cases}$$

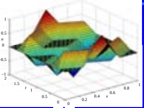
$$2\bar{a}_{12}u_{\zeta\eta} + \bar{b}_1 u_\zeta + \bar{b}_2 u_\eta + \bar{c}_1 u = \bar{f}$$

再作变换

$$\begin{cases} \zeta = s + t \\ \eta = s - t \end{cases}$$

第二标准形

$$u_{ss} - u_{tt} + du_s + eu_\zeta + hu = f_1$$

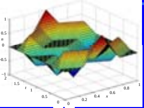


$$\Delta < 0$$

$$y + (a \pm bi)x = y + ax \pm bxi = c$$

$$\begin{cases} \zeta = ax + y \\ \eta = bx \end{cases}$$

$$\bar{a}_{11}u_{\zeta\zeta} + \bar{a}_{11}u_{\eta\eta} + \bar{b}_1 u_{\zeta} + \bar{b}_2 u_{\eta} + \bar{c}_1 u = \bar{f}$$



$$\Delta = 0 \qquad y + ax = c$$

$$\begin{cases} \zeta = ax + y \\ \eta = \phi(x, y) \end{cases}$$

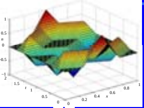
$$\bar{a}_{22} u_{\eta\eta} + \bar{b}_1 u_{\zeta} + \bar{b}_2 u_{\eta} + \bar{c}_1 u = \bar{f}$$

例1 求方程 $u_{tt}-a^2u_{xx}=0$ 的通解

解：此方程是双曲型的第二标准形，但我们要求解它可将其化成第一标准形的形式，所以先得由特征方程求特征函数：

$$\left(\frac{dx}{dt}\right)^2-a^2=0$$

$$\frac{dx}{dt}=\pm a$$



$$\begin{cases} \zeta = x + at \\ \eta = x - at \end{cases}$$

$$u_{tt} = a^2 \left(u_{\zeta\zeta} - 2u_{\zeta\eta} + u_{\eta\eta} \right)$$

$$u_{xx} = u_{\zeta\zeta} + u_{\zeta\eta} + u_{\eta\zeta} + u_{\eta\eta}$$

$$= u_{\zeta\zeta} + 2u_{\zeta\eta} + u_{\eta\eta}$$

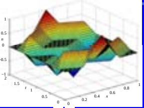
可得 $u_{\zeta\eta} = 0$

$$u_{\zeta} = g_1(\zeta)$$

$$u = \int g_1(\zeta)\,d\zeta + f_2(\eta)$$

$$= f_1(\zeta) + f_2(\eta)$$

$u = f_1(x+at) + f_2(x-at)$ 是原方程的解



二阶线性偏微分方程理论

1.叠加原理物理 意义

2. 二阶线性方程的一般形式：

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=f$$

引入 $$L=a_{11}\frac{\partial^2}{\partial x^2}+2a_{12}\frac{\partial^2}{\partial x\partial y}+a_{22}\frac{\partial^2}{\partial y^2}+b_1\frac{\partial}{\partial x}+b_2\frac{\partial}{\partial y}+c$$

则

$$Lu=f$$

一般的线性边界条件（它包括了三类边界条件）

$$Lu\big|_s=\left(\alpha+\beta\frac{\partial}{\partial n}\right)u\big|_s=\varphi$$

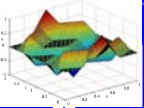
3. 算子性质:

(1) 线性算子，即满足线性条件 $Lu_i=f_i$

$$L[c_1u_1+c_2u_2]=c_1Lu_1+c_2Lu_2$$

(2) $Lu_i=f_i$

$$L\sum_{i=1}^{n}c_iu_i=\sum_{i=1}^{n}c_if_i$$



(3) $Lu_i = f_i$

$$L\sum_{i=1}^{\infty} c_i u_i = \sum_{i=1}^{\infty} c_i f_i$$

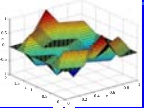
M

(4)

$$Lu = f(M, M_0)$$

其中，M表示自变量组，M_0 为参数组

$$L\int_v u(M, M_0)\,dM_0 = \int_v f(M, M_0)\,dM_0$$



例1 求泊松方程 $\Delta_2 u = x^2 + 3xy + y^2$ 的一般解。

解：先求出方程的一个特解。方程右端是一个二元二次齐次多项式

u_1 设为四次齐次多项式 $u_1 = ax^4 + bx^3 y + cy^4$

$$\Delta_2 u_1 = 12ax^2 + 6bxy + 12cy^2 = x^2 + 3xy + y^2$$

$$u_1 = \frac{1}{12}\left(x^4 + 6x^3 y + y^4\right)$$

令 $v = u - u_1$

$$v_{xx} + v_{yy} = 0$$

作代换，$x = \zeta, y = i\eta$ 得：

$$v_{\zeta\zeta} - v_{\eta\eta} = 0$$

$$v = f(\zeta - \eta) + g(\zeta + \eta)$$

$$= f(x + iy) + g(x - iy)$$

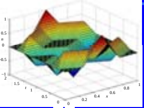
4. 齐次方程

$$\frac{\partial^2 \omega}{\partial t^2} = L\omega, (M \in R^3, t > \tau)$$

5. 齐次化原理1

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu + f(t, M), (M \in R^3, t > 0) \\ u\big|_{t=0} = 0, \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases} \longrightarrow \begin{cases} \dfrac{\partial^2 \omega}{\partial t^2} = L\omega, (M \in R^3, t > \tau) \\ \omega\big|_{t=\tau} = 0, \dfrac{\partial \omega}{\partial t}\Big|_{t=\tau} = f(\tau, M) \end{cases}$$

$$u = \int_0^t \omega(t, M; \tau) d\tau$$



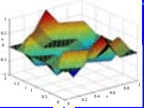
$$\frac{\partial u}{\partial t} = \int_0^t \frac{\partial \omega}{\partial t} d\tau + \omega(t, M; t) = \int_0^t \frac{\partial \omega}{\partial t} d\tau$$

$$\frac{\partial u}{\partial t}\bigg|_{t=0} = 0$$

$$\frac{\partial^2 u}{\partial t^2} = \int_0^t \frac{\partial^2 \omega}{\partial t^2} d\tau + \frac{\partial \omega(t, M; \tau)}{\partial t}\bigg|_{\tau = t}$$

$$= L\left(\int_0^t \omega d\tau\right) + f(t, M)$$

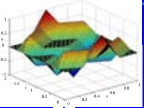
$$= Lu + f(t, M)$$



6. 齐次化原理2

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(t, M), (M \in R^3, t > 0) \\ u\big|_{t=0} = 0 \end{cases} \longrightarrow \begin{cases} \dfrac{\partial \omega}{\partial t} = L\omega, (M \in R^3, t > \tau), \\ \omega\big|_{t=\tau} = f(\tau, M), \end{cases}$$

$$u = \int_0^t \omega(t, M; \tau) d\tau$$



分离变量解法

§3.1 利用分离变量法求解齐次弦振动方程的混合问题

1.方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, (0 < x < L, t > 0) \\ u\big|_{x=0} = 0, u\big|_{x=L} = 0 \\ u\big|_{t=0} = \varphi(x), u_t\big|_{t=0} = \psi(x) \end{cases}$$

2.设方程（1）具有可以分离变量的解

$$u(x,t)=T(t)X(x)$$

则

$$\frac{T''}{a^2T}=\frac{X''}{X}=-\lambda$$

$$X''+\lambda X=0$$

$$T''+\lambda a^2T=0$$

3.代入边界条件（2），得

$$T(t)X(0)=0$$
$$T(t)X(L)=0$$

$$X(0)=X(L)=0$$

$$\begin{cases} X''+\lambda X=0 \\ X(0)=0 \\ X(l)=0 \end{cases}$$

a. 当 $\lambda < 0$ 时

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

$$X(0) = A \cdot 1 + B \cdot 1 = 0$$

$$X(L) = Ae^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = 0$$

从而

$$X(x) \equiv 0$$

b. 当 $\lambda = 0$ 时

$$X = Ax + B$$

$$A = B = 0$$

c,当 $\lambda > 0$ 时

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

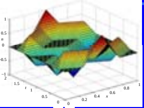
$$A = 0, B\sin\sqrt{\lambda}L = 0$$

$$\sin\sqrt{\lambda}L = 0$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$$X_n(x) = B_n \sin\frac{n\pi x}{L}$$

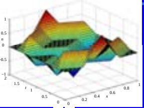
$$T'' + \lambda_n a^2 T = 0$$



$$T_n(t) = C_n \cos\frac{n\pi at}{L} + D_n \sin\frac{n\pi at}{L}$$

$$u_n(x,t) = T_n(t)X_n(x)$$

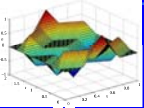
$$= (C_n \cos\frac{n\pi at}{L} + D_n \sin\frac{n\pi at}{L})\sin\frac{n\pi x}{L}$$



$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi at}{L}+D_n\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}$$

$$u(x,0)=\varphi(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{L}$$

$$u_t(x,0)=\psi(x)=\sum_{n=1}^{\infty}D_n\frac{n\pi a}{L}\sin\frac{n\pi x}{L}$$



例

$$\begin{cases} u_{tt} = a^2 u_{xx}, \left(0 < x < L, t > 0\right) \\ u\big|_{x=0} = 0, u_x\big|_{x=L} = 0 \\ u\big|_{t=0} = \varphi\left(x\right), u_t\big|_{t=0} = \psi\left(x\right) \end{cases}$$

解:

$$u(x,t) = T(t)X(x)$$

$$X'' + \lambda X = 0$$

$$T'' + \lambda a^2 T = 0$$

$$X(0) = X'(L) = 0$$

特征值

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$$

$$X_n(x) = C_n \sin\frac{(2n+1)\pi x}{2L}$$

$$T'' + \lambda_n a^2 T = 0$$

$$T_n(t) = A_n \cos\frac{(2n+1)\pi a t}{2L} + B_n \sin\frac{(2n+1)\pi a t}{2L}$$

$$u_n(x,t) = T_n(t) X_n(x)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{(2n+1)\pi a t}{2L} + B_n \sin\frac{(2n+1)\pi a t}{2L}\right)\sin\frac{(2n+1)\pi x}{2L}$$

$$\begin{cases} A_n = \dfrac{2}{L}\displaystyle\int_0^L \varphi(\xi)\sin\frac{(2n+1)\pi\xi}{2L}\,d\xi \\ B_n = \dfrac{4}{(2n+1)\pi a}\displaystyle\int_0^L \psi(\xi)\frac{(2n+1)\pi\xi}{2L}\,d\xi \end{cases}$$

例. 两端固定的弦长为，用细棒敲击弦上 $x = x_0$ 点处，亦即在点 $x = x_0$ 施加冲量，设其冲量为 I 。求解弦的振动。

$$
\begin{cases}
u_{tt} = a^2 u_{xx}, (0 < x < l, t > 0) \\
u\big|_{x=0} = 0, u_x\big|_{x=l} = 0 \\
u\big|_{t=0} = 0, u_t\big|_{t=0} = \dfrac{I}{\rho}\delta(x - x_0), (0 < x < l)
\end{cases}
$$

解:

$$u(x,t) = T(t)X(x)$$

$$X'' + \lambda X = 0$$

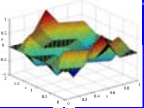
$$T'' + \lambda a^2 T = 0$$

特征值 $\lambda_n = \dfrac{n^2\pi^2}{l^2}$

$$X_n(x) = A_n \sin\frac{n\pi x}{l}$$

$$u_n(x,t) = T_n(t)X_n(x)$$

$$= (C_n \cos\frac{n\pi at}{l} + D_n \sin\frac{n\pi at}{l})\sin\frac{n\pi x}{l}$$



$$u(x,0)=\varphi(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}-=0$$

$$u_t(x,0)=\psi(x)=\sum_{n=1}^{\infty}D_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}=\frac{I}{\rho}\delta\left(x-x_0\right)$$

$$\begin{cases}C_n=0\\ D_n=\dfrac{2I}{n\pi a\rho}\sin\dfrac{n\pi x_0}{l}\end{cases}$$

热传导方程混合问题分离变量解法

例1 设有长度为的，均匀的，内部无热源的热传导细杆，侧面绝热，其左端保持零度，右端绝热，初始温度分布为已知。该定解问题应为

$$\begin{cases} u_t = a^2 u_{xx}, (0 < x < L, t > 0) \\ u\big|_{t=0} = \varphi(x) \\ u\big|_{x=0} = 0, u_x\big|_{x=L} = 0 \end{cases}$$

解：设特解形式为

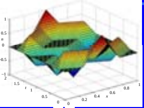
$$u(x,t) = X(x)T(t)$$

$$X(x)T'(t) = a^2 X''(x)T(t)$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$T'(t) + \lambda a^2 T(t) = 0$$

$$X''(x) + \lambda X(x) = 0$$



$$u\big|_{x=0} = X(0)T(t) = 0$$

$$u_x\big|_{x=L} = X'(L)T(t) = 0$$

$$X(0) = X'(L) = 0$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X'(L) = 0 \end{cases}$$

a. 当 $\lambda \le 0$时，特征值问题无非零解

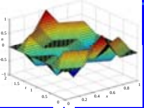
b. $\lambda > 0$

$$X(x) = C\sin\sqrt{\lambda}x + D\cos\sqrt{\lambda}x$$

$$X\left(x\right) = C\sin\sqrt{\lambda}x$$

$$\cos\sqrt{\lambda}L = 0$$

$$\sqrt{\lambda}L = \frac{\left(2n+1\right)\pi}{2}$$

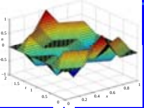


相应的特征函数为：

$$X_n(x) = C_n \sin\frac{\left(n+\frac{1}{2}\right)\pi x}{L}, n = 0,1,2,\cdots$$

$$T_n(t) = A_n e^{-\frac{\left(n+\frac{1}{2}\right)^2 \pi^2 a^2}{L^2}t}, n = 0,1,2,\cdots$$

$$u_n(x,t) = X_n(x)T_n(t) = a_n e^{-\frac{(n+\frac{1}{2})^2 \pi^2 a^2}{L^2}t} \sin\frac{(n+\frac{1}{2})\pi x}{L}$$



利用迭加原理

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}a_n e^{-\frac{(n+\frac{1}{2})^2\pi^2a^2}{L^2}t}\sin\frac{(n+\frac{1}{2})\pi x}{L}$$

$$u(x,0)=\sum_{n=0}^{\infty}a_n\sin\frac{(n+\frac{1}{2})\pi x}{L}=\varphi(x)$$

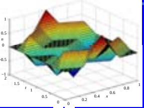
例 设有长度为 L 的，均匀的，内部无热源的热传导细杆，侧面绝热, 求温度分布:

(1)左端右端保持零度，初始温度分布为 $\varphi(x)$

(2) $\varphi(x)=x(L-x)$ (exercise)

该定解问题应为

$$\begin{cases} u_t=a^2u_{xx},(0<x<L,t>0) \\ u\big|_{t=0}=\varphi(x) \\ u\big|_{x=0}=0,u\big|_{x=L}=0 \end{cases}$$



解：设 $u(x,t)=X(x)T(t)$

$$X(x)T'(t)=a^2X''(x)T(t)$$

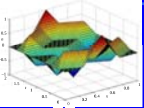
$$\frac{T'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$$T'(t)+\lambda a^2T(t)=0$$

$$X''(x)+\lambda X(x)=0$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(L)=0 \end{cases}$$

$$\lambda=\left(\frac{n\pi}{L}\right)^2$$



相应的特征函数为：

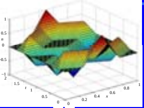
$$X_n(x)=C_n\sin\frac{n\pi x}{L},n=0,1,2,\cdots$$

$$T_n(t)=A_n e^{\frac{-n^2\pi^2a^2}{L^2}t},n=0,1,2,\cdots$$

$$u_n(x,t)=X_n(x)T_n(t)=a_n e^{-\frac{n^2\pi^2a^2}{L^2}t}\sin\frac{n\pi x}{L}$$

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}a_n e^{-\frac{n^2\pi^2a^2}{L^2}t}\sin\frac{n\pi x}{L}$$

$$u(x,0)=\sum_{n=0}^{\infty}a_n\sin\frac{n\pi x}{L}=\varphi(x)$$



$$a_n = \frac{2}{L}\int_{.0}^{.L} \varphi(x)\sin\frac{n\pi x}{L}dx, n = 0,1,2,\cdots$$

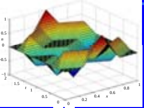
(2) $\varphi(x) = x(L-x)$

$$a_n = \frac{2}{L}\int_{.0}^{.L} \varphi(x)\sin\frac{n\pi x}{L}dx = \frac{8L^2}{(n\pi)^3}, n = 1,3,5\ldots.$$

例 设有一条长为2L、温度为零的均匀杆，其两端与侧面都绝热。现在用一个火焰集中在杆的中点烧它一下，使传给杆的热量恰好等于 $c\rho$ （设c为杆的比热, ρ 为线密度）。求杆上的温度分布。

解：问题归结为解定解问题

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \left(0 < x < 2L, t > 0\right) \\ u(0,x) = \delta(x-L) \\ u_x(t,0) = u_x(t,2L) = 0 \end{cases}$$



$$\lambda = \left(\frac{n\pi}{2L}\right)^2$$

$$X(x) = \cos\frac{n\pi x}{2L}, (n = 0, 1, 2, \cdots)$$

$$u(t, x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-\left(\frac{n\pi a}{2L}\right)^2 t} \cos\frac{n\pi}{2L} x$$

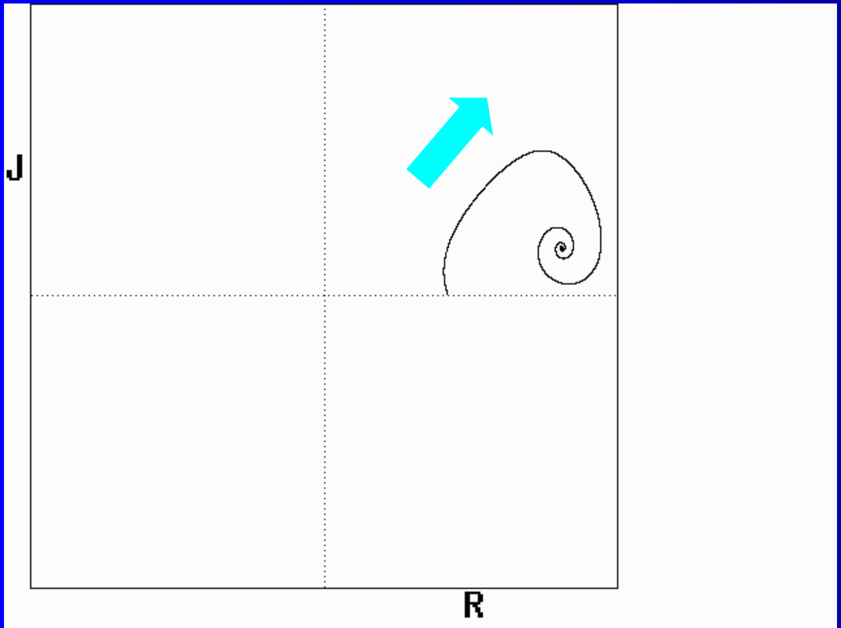
$$u(0, x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi x}{2L} = \delta(x - L)$$

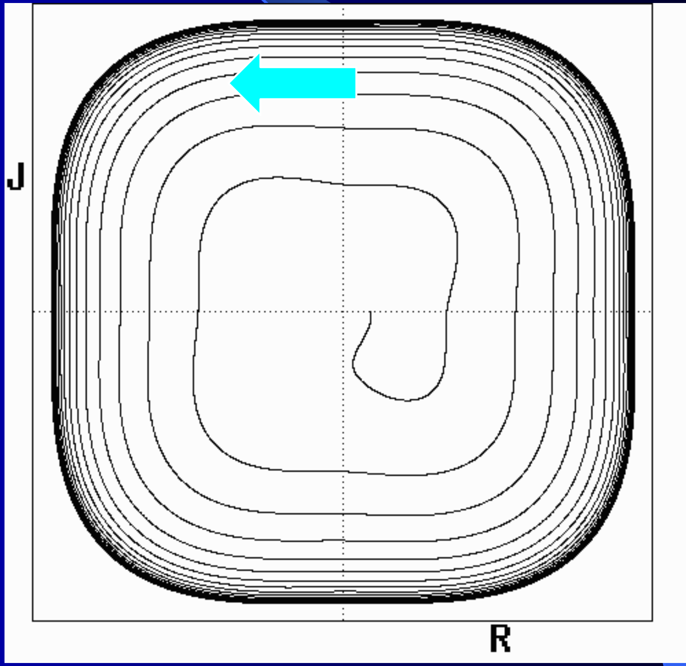
$$a_n = \frac{2}{2L}\int_{0}^{2L} \delta(x - L) \cos\frac{n\pi x}{2L} dx$$

$$= \begin{cases} 0 & n = 2k + 1 \\ (-1)^k \dfrac{1}{L} & n = 2k \end{cases}$$

$$u(x,t)=\frac{1}{2L}+\sum_{k=1}^{+\infty}(-1)^k\frac{1}{L}e^{-\left(\frac{k\pi a}{2L}\right)^2 t}\cos\frac{k\pi}{L}x$$

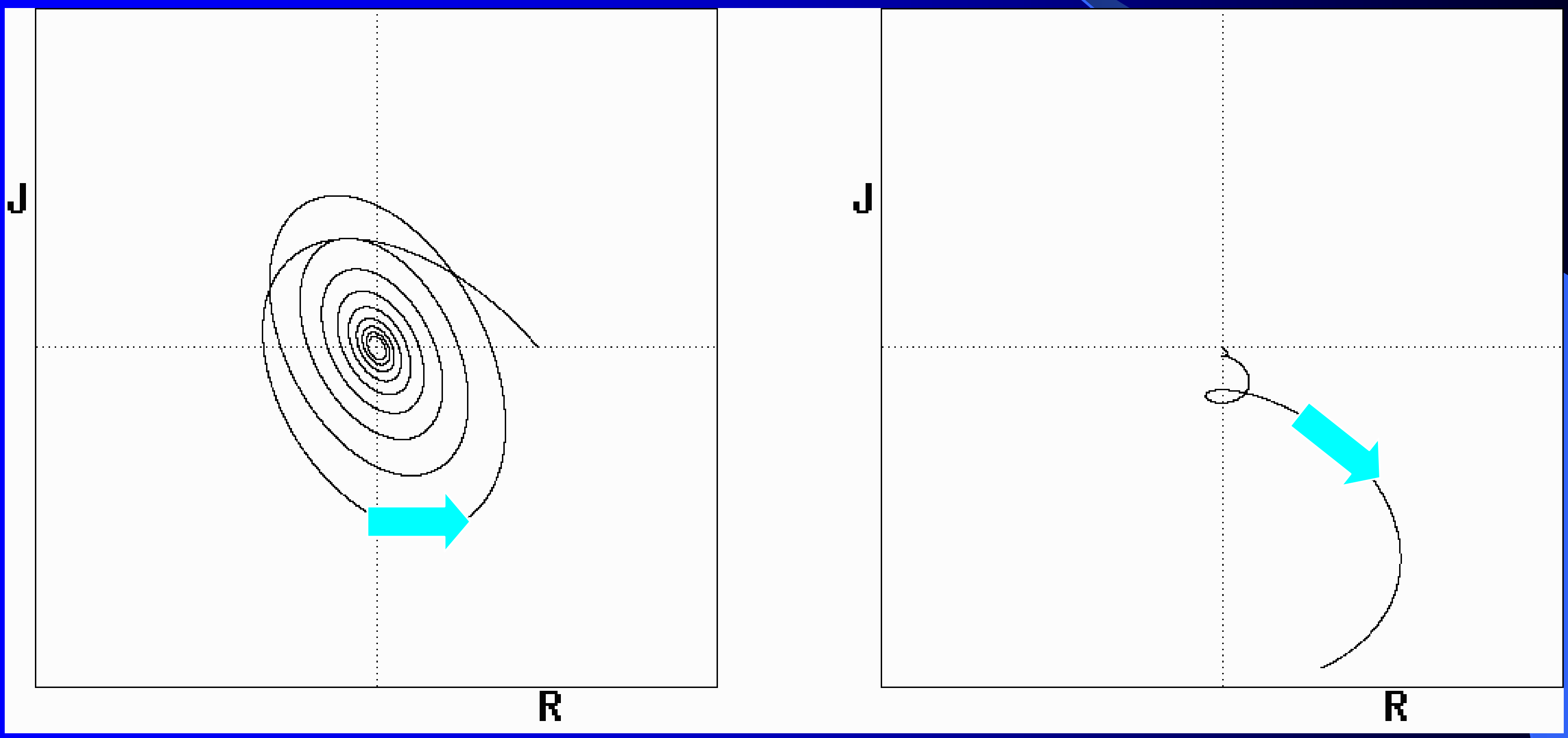
New kinds of Dynamics





New equilibrium points

Limit cycles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圆域内拉普拉斯方程分离变量解法

一个半径为 ρ_0 的薄圆盘，上下两面绝热，圆周边缘温度分布为已知，求达到稳恒状态时圆盘内的温度分布。

$$\begin{cases} \Delta u = 0 \\ u\big|_{\rho=\rho_0} = f(\theta) \end{cases}$$

$$\begin{cases} \Delta u = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial u}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, (\rho < \rho_0) \\ u(\rho_0, \theta) = f(\theta) \end{cases}$$

解:

$$u(\rho, \theta) = R(\rho)\Phi(\theta)$$

$$R''\Phi + \frac{1}{\rho}R'\Phi + \frac{1}{\rho^2}R\Phi'' = 0$$

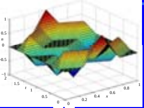
$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{\Phi''}{\Phi}$$

$$\Phi'' + \lambda\Phi = 0$$

$$\rho^2 R'' + \rho R' - \lambda R = 0$$

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(\theta + 2\pi) = \Phi(\theta) \end{cases}$$

$$\begin{cases} \rho^2 R'' + \rho R' - \lambda R = 0 \\ |R(0)| < +\infty \end{cases}$$



$$\begin{cases}\Phi'' + \lambda\Phi = 0 \\ \Phi(\theta + 2\pi) = \Phi(\theta)\end{cases}$$

当$\lambda > 0$时，取 $\lambda = \beta^2$

$$\Phi_\beta(\theta) = a'_\beta \cos\beta\theta + b'_\beta \sin\beta\theta$$

$$\beta_n^2 = n^2 = \lambda$$

$$\Phi_n(\theta) = a'_n \cos n\theta + b'_n \sin n\theta$$

$\lambda = 0$ $\Phi_0(\theta) = a'_0 \neq 0$

欧拉（Euler）方程 $R_n = c_n\rho^n + d_n\rho^{-n} = c_n\rho^n$

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho_0^n \left(c_n \cos n\theta + b_n \sin n\theta\right)$$

$$\begin{cases} a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)d\theta \\ a_n = \dfrac{1}{\rho_0^n \pi}\displaystyle\int_0^{2\pi} f(\theta)\cos n\theta d\theta \\ b_n = \dfrac{1}{\rho_0^n \pi}\displaystyle\int_0^{2\pi} f(\theta)\sin n\theta d\theta \end{cases}$$

$$u(\rho,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{\rho_0}\right)^n \cos(\theta - t)\right]dt$$

例 解下列定解问题：

$$\begin{cases} \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, (\rho < \rho_0) \\ u\big|_{\rho=\rho_0} = A\cos\theta \end{cases}$$

解：

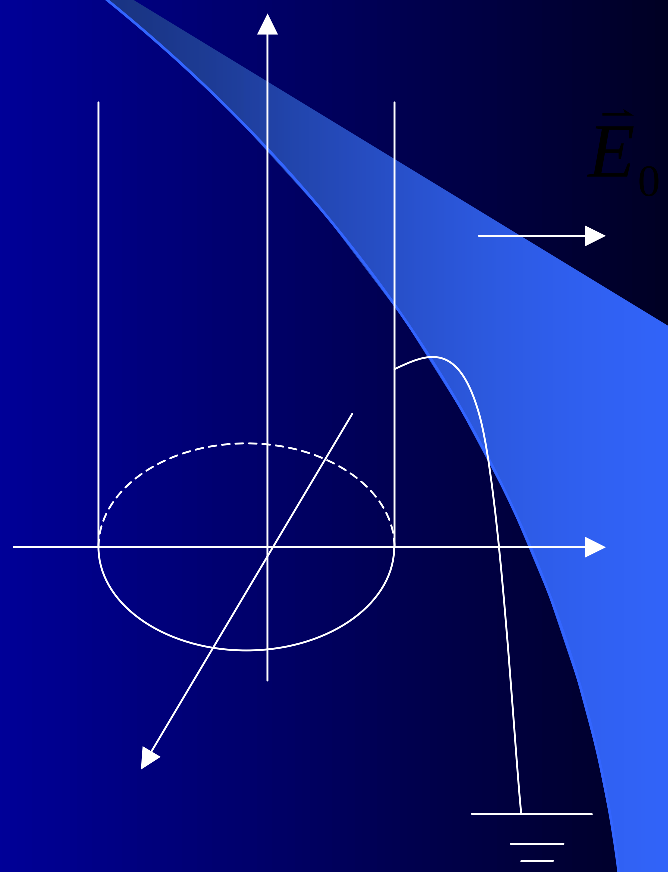
$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

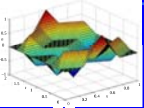
$$b_n = 0,$$

$$a_1 = \frac{A}{\rho_0}, a_n = 0 (n \neq 1)$$

例 半径为b的“无限长”圆柱形接地导体，放置在均匀外电场 $\vec{E}_0$ 中，圆柱的轴线与 $\vec{E}_0$ 方向垂直。求电势分布

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_{\rho}+\dfrac{1}{\rho^2}u_{\theta\theta}=0,(b<\rho<+\infty) \\ u\big|_{\rho=b}=0, u_{\rho\to+\infty}=-E_0\rho\cos\theta \\ u(\rho,\theta+2\pi)=u(\rho,\theta) \end{cases}$$





$$u(\rho,\theta)=R(\rho)\Theta(\theta)$$

$$\rho^2R''+\rho R'-n^2R=0$$

$$\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(\theta+2\pi)=\Theta(\theta)\end{cases}$$

$$\begin{cases}\rho^2R''+\rho R'-\lambda R=0\\ |R(0)|<+\infty\end{cases}$$

$$\Theta'' + n^2\Theta = 0 \qquad \lambda_n = n^2, n = 0,1,2,\cdots$$

$$R_0 = C_0 + D_0 Ln\rho \qquad R_n = C_n\rho^n + D_n\rho^{-n}$$

$$R\big|_{\rho=b} = 0 \qquad C_0 = -D_0 Lnb$$

$$D_n = -C_n b^{2n}$$

$$R_n(\rho) = \begin{cases} D_0 Ln\dfrac{\rho}{b}, (n=0) \\ C_n\left(\rho^n - b^{2n}\rho^{-n}\right), (n = 1,2,3,\cdots) \end{cases}$$

$$\begin{cases} u = D_0 Ln\dfrac{\rho}{b} + \sum\limits_{n=1}^{\infty}\left[C_n\cos n\theta + D_n\sin n\theta\right]\left(\rho^n - \dfrac{b^{2n}}{\rho^n}\right) \\ u\big|_{\rho\to+\infty} = -E_0\rho\cos\theta \end{cases}$$

$$D_0 = 0, C_1 = -E_0, C_n = 0 (n \neq 1), D_n = 0$$

$$u = -E_0 \rho \cos\theta + \frac{E_0 b^2}{\rho} \cos\theta$$

高维混合问题的分离变量解法

例

$$\begin{cases} u_{tt} = a^2 \Delta u, (0 < x < a, 0 < y < b, t > 0) \\ u\big|_{x=0} = u\big|_{x=a} = 0 \\ u\big|_{y=0} = u\big|_{y=b} = 0 \\ u\big|_{t=0} = \varphi(x, y) \\ u_t\big|_{t=0} = \psi(x, y) \end{cases}$$

解： 1. 时空变量的分离： $u = T(t)V(x, y)$

$$T''V = a^2(V_{xx} + V_{yy})T, \qquad \frac{T''}{a^2 T} = \frac{V_{xx} + V_{yy}}{V} = -\lambda_1$$

$$T'' + k^2 \lambda_1 T = 0$$

$$V_{xx} + V_{yy} + \lambda_1 V = 0$$

2. 空间变量的分离：

$$V = X(x)Y(y)$$

$$X''Y + XY'' + \lambda_1 XY = 0$$

$$X''Y = -XY'' - \lambda_1 XY$$

$$\frac{X''}{X} = -\frac{Y'' + \lambda_1 Y}{Y} = -\lambda_2$$

3.

$$T'' + \lambda_1 a^2 T = 0, \quad X'' + \lambda_2 X = 0, \quad Y'' + \lambda_3 Y = 0, \quad \lambda_3 = \lambda_1 - \lambda_2$$

$$\begin{cases} X'' + \lambda_2 X = 0 \\ X(0) = 0, X(a) = 0 \end{cases} \quad \begin{cases} Y'' + \lambda_3 Y = 0 \\ Y(0) = 0, Y(b) = 0 \end{cases}$$

$$\lambda_2 = \frac{n^2\pi^2}{l^2}, X_n = \sin\frac{n\pi}{a}x, (n = 1, 2, 3, \cdots)$$

$$\lambda_3 = \frac{m^2\pi^2}{b^2}, Y_m = \sin\frac{m\pi}{b}y, (m = 1, 2, 3, \cdots)$$

$$\lambda_1 = \lambda_2 + \lambda_3 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2},$$

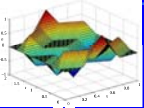
5.

$$T_{mn} = C_{mn}\cos\sqrt{\lambda_1}t + D_{mn}\sin\sqrt{\lambda_1}t$$

$$u = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}(C_{m,n}\cos\sqrt{\lambda_1}t + D_{m,n}\sin\sqrt{\lambda_1}t)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$$

$$C_{m,n} = \frac{2}{b}\int_0^b[\frac{2}{a}\int_0^a\varphi(x,y)\sin\frac{n\pi}{a}xdx]\sin\frac{m\pi}{b}ydy$$

$$D_{m,n} = \frac{4}{ab\sqrt{\lambda_1}}\int_0^a\int_0^b\psi(x,y)\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}ydxdy$$



例2 求边长分别为a,b,c的长方体中的温度分布，设物体表面温度保持零度，初始温度分布为

$$u(x,y,z,0)=\varphi(x,y,z)$$

解：定解问题为：

$$\begin{cases} u_t = k^2\Delta u, (0<x<a, 0<y<b, 0<z<c, t>0) \\ u\big|_{x=0} = u\big|_{x=a} = 0 \\ u\big|_{y=0} = u\big|_{y=b} = 0 \\ u\big|_{z=0} = u\big|_{z=c} = 0 \\ u\big|_{t=0} = \varphi(x,y,z) \end{cases}$$

1. 时空变量的分离：

$$u = T(t)V(x, y, z)$$

$$T'' + k^2\lambda_1 T = 0$$

$$V_{xx} + V_{yy} + V_{zz} + \lambda_1 V = 0$$

2. 空间变量的分离 ：

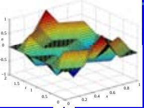
$$V = X(x)\omega(y, z)$$

$$\begin{cases} X'' + (\lambda_1 - \lambda_2)X = 0 \\ X\big|_{x=0} = 0, X\big|_{x=a} = 0 \end{cases}$$

$$\omega_{yy} + \omega_{zz} + \lambda_2 \omega = 0$$

3.

$$\omega(y, z) = Y(y)Z(z)$$



$$\begin{cases} Y'' + (\lambda_2 - \lambda_3)Y = 0 \\ Y\big|_{y=0} = 0, Y\big|_{y=b} = 0 \end{cases} \qquad \begin{cases} Z'' + \lambda_3 Z = 0 \\ Z\big|_{z=0} = 0, Z\big|_{z=c} = 0 \end{cases}$$

4. 固有值问题的固有值与固有函数

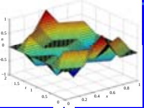
$$\lambda_3 = \frac{n^2\pi^2}{c^2}, Z_n = \sin\frac{n\pi}{c}z, (n = 1, 2, 3, \cdots)$$

$$\lambda_2 - \lambda_3 = \frac{m^2\pi^2}{b^2}, Y_m = \sin\frac{m\pi}{b}y, (m = 1, 2, 3, \cdots)$$

$$\lambda_1 - \lambda_2 = \frac{p^2\pi^2}{a^2}, X_P = \sin\frac{p\pi}{a}x, (p = 1, 2, 3, \cdots)$$

相加得

$$\lambda_1 = \lambda_{pmn} = \pi^2\left(\frac{p^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$



$$V_{pmn} = \sin\frac{P\pi}{a}x \sin\frac{m\pi}{b}y \sin\frac{n\pi}{c}z$$

$$T_{Pmn} = A_{Pmn} e^{-\lambda_{Pmn} k^2 t}$$

$$\begin{cases} u = \sum_{P=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{Pmn} e^{-\lambda_{Pmn} k^2 t} \sin\frac{P\pi}{a}x \sin\frac{m\pi}{b}y \sin\frac{n\pi}{c}z \\ u\big|_{t=0} = \varphi(x, y, z) \end{cases}$$

$$A_{Pmn} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c \varphi(x, y, z)\sin\frac{P\pi}{a}x \sin\frac{m\pi}{b}y \sin\frac{n\pi}{c}z dxdydz$$

例 求解三维静电场的边值问题：

$$\begin{cases} u_{xx}+u_{yy}+u_{zz}=0,\left(0<x<a,0<y<b,0<z<c\right) \\ u\left(0,y,z\right)=u\left(a,y,z,\right)=u\left(x,0,z\right)=u\left(x,b,z\right)=0 \\ u\left(x,y,0\right)=0,u\left(x,y,c\right)=\varphi\left(x,y\right) \end{cases}$$

解：设

$$u=X(x)Y(y)Z(z)$$

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(a)=0 \end{cases}$$

$$\begin{cases} Y''+\mu Y=0 \\ Y(0)=Y(b)=0 \end{cases}$$

$$Z''-(\lambda+\mu)Z=0$$

$$\begin{cases} \lambda_m = \left(\dfrac{m\pi}{a}\right)^2 \\ X_m = \sin\dfrac{m\pi x}{a} \end{cases}$$

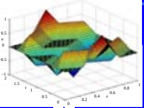
$$\begin{cases} \mu_n = \left(\dfrac{n\pi}{b}\right)^2 \\ Y_n = \sin\dfrac{n\pi y}{b} \end{cases}$$

$$\lambda + \mu = \lambda_m + \mu_n \Rightarrow Z_{mn} = A_{mn} e^{v_{mn} z} + B_{mn} e^{-v_{mn} z}$$

$$v_{mn} = \sqrt{\lambda_m + \mu_n}$$

$$u_{mn} = Z_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$u = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} (A_{mn} e^{v_{mn} z} + B_{mn} e^{-v_{mn} z}) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$



$$u(x,y,0)=\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}(A_{mn}+B_{mn})\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=0$$

$$u(x,y,c)=\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}(A_{mn}e^{v_{mn}c}+B_{mn}e^{-v_{mn}c})\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=\varphi(x,y)$$

$$\begin{cases} A_{mn}+B_{mn}=0 \\ A_{mn}e^{v_{mn}c}+B_{mn}e^{-v_{mn}c}=\varphi_{mn} \end{cases}$$

$$\varphi_{mn}=\frac{4}{ab}\int_0^a\int_0^b\varphi(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$u(x,y,z)=\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{\sqrt{\lambda_m+\mu_n}\,c}\varphi_{mn}sh\sqrt{\lambda_m+\mu_n}\,z\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

非齐次方程 的解法

$$\begin{cases} L_t u + L_x u = f(t,x), (t>0, x_1<x<x_2) \\ \alpha_1 u_x(t,x_1) - \beta_1 u(t,x_1) = 0 \\ \alpha_2 u_x(t,x_2) + \beta_2 u(t,x_2) = 0 \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \end{cases}$$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ 非负常数 $\alpha_i^2 + \beta_i^2 \neq 0 (i=1,2)$

例 解定解问题：

$$\begin{cases} u_{tt}=a^2u_{xx}+f(t,x)(t>0,0<x<L) \\ u(t,0)=u(t,L)=0 \\ u(0,x)=\varphi(x),u_t(0,x)=\psi(x) \end{cases}$$

解：先把所给的定解问题分解成两个比较简单的定解问题

$$\begin{cases} v_{tt}=a^2v_{xx} \\ v(t,0)=v(t,L)=0 \\ v(0,x)=\varphi(x),v_t(0,x)=\psi(x) \end{cases} \qquad \begin{cases} W_{tt}=a^2W_{xx}+f(t,x) \\ W(t,0)=W(t,L)=0 \\ W(0,x)=0,W_t(0,x)=0 \end{cases}$$

显然 $u=v+W$

易得相应的齐次问题 v

$$\begin{cases} W_{tt}=a^2W_{xx} \\ W(t,0)=W(t,L)=0 \end{cases}$$

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2,(n=1,2,\cdots)$$

$$X_n(x)=\sin\frac{n\pi x}{L},(n=1,2,\cdots)$$

$$W(t,x)=\sum_{n=1}^{+\infty}T_n(t)\sin\frac{n\pi x}{L}$$

由初始条件: $T_n(0)=T_n'(0)=0$

$$f(t,x)=\sum_{n=1}^{+\infty}f_n(t)\sin\frac{n\pi x}{L}\qquad f_n(t)=\frac{2}{L}\int_0^L f(t,x)\sin\frac{n\pi x}{L}dx$$

代入方程: $\sum_{n=1}^{\infty}T_n''(t)\sin\frac{n\pi x}{L}=-\sum_{n=1}^{\infty}a^2\lambda_n T_n(t)\sin\frac{n\pi x}{L}+\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi x}{L}$

系数相等: $\begin{cases}T_n''+\lambda_n a^2T_n=f_n(t)\\ T_n(0)=T_n'(0)=0\end{cases},(n=1,2,\cdots)$

$$T_n(t)=\frac{L}{n\pi a}\int_0^t f_n(\xi)\sin\frac{n\pi a}{L}(t-\xi)d\xi$$

$$W(t,x)=\sum_{n=1}^{\infty}\frac{L}{n\pi a}\int_{0}^{t}f_n(\xi)\sin\frac{n\pi a}{L}(t-\xi)d\xi\cdot\sin\frac{n\pi}{L}x$$

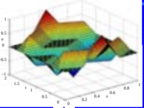
****

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x)(t > 0, 0 < x < L) \\ u(t,0) = u(t,L) = 0 \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \end{cases}$$

$$u(t,x) = v(x) + W(t,x)$$

$$\begin{cases} w_{tt} = a^2 w_{xx} + a^2 v'' + f(x) \\ w(t,0) = -v(0), w(t,L) = -v(L) \\ w(0,x) = \varphi(x) - v(x), w_t(0,x) = \psi(x) \end{cases}$$

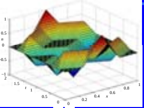
$$a^2 v'' + f(x) = 0 \longrightarrow \begin{cases} W_{tt} = a^2 W_{xx} \\ W(t,0) = W(t,L) = 0 \\ W(0,x) = \varphi(x) - v(x) \\ W_t(0,x) = \psi(x) \end{cases}$$



$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \left(0 < x < L, t > 0\right) \\ u\big|_{x=0} = u\big|_{x=L} = 0 \\ u\big|_{t=0} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial W}{\partial t} = a^2 \dfrac{\partial^2 W}{\partial x^2}, \left(0 < x < L, t > \tau > 0\right) \\ W\big|_{x=0} = W\big|_{x=L} = 0 \\ W\big|_{t=\tau} = f(x,t) \end{cases}$$

$$u(x,t) = \int_{.0}^{.t} W(t,x,\tau)\, d\tau$$



$$\begin{cases} u_t = a^2 u_{xx}, (0 < x < L, t > 0) \\ u\big|_{t=0} = \varphi(x) \\ u\big|_{x=0} = 0, u_x\big|_{x=L} = 0 \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} a_n e^{-\frac{(n+\frac{1}{2})^2 \pi^2 a^2}{L^2} t} \sin \frac{(n+\frac{1}{2})\pi x}{L}$$

例2　解定解问题：

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + A\left(1-\dfrac{x}{L}\right)e^{-ht}, (0 < x < L, t > 0) \\ u\big|_{x=0} = u\big|_{x=L} = 0 \\ u\big|_{t=0} = 0 \end{cases}$$

解：考虑相应齐次方程的定解问题

$$\begin{cases} \dfrac{\partial W}{\partial t} = a^2 \dfrac{\partial^2 W}{\partial x^2}, (0 < x < L, t > \tau > 0) \\ W\big|_{x=0} = W\big|_{x=L} = 0 \\ W\big|_{t=\tau} = A\left(1-\dfrac{x}{L}\right)e^{-h\tau} \end{cases}$$

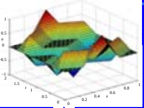
$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}a_n e^{-\frac{(n+\frac{1}{2})^2\pi^2a^2}{L^2}t}\sin\frac{(n+\frac{1}{2})\pi x}{L}$$

$$A\left(1-\frac{x}{L}\right)e^{-h\tau}=\sum_{n=0}^{\infty}u_n(x,t)=\sum_{n=0}^{\infty}a_n e^{-\frac{(n+\frac{1}{2})^2\pi^2a^2}{L^2}\tau}\sin\frac{(n+\frac{1}{2})\pi x}{L}$$

$$a_n e^{-\frac{(n+\frac{1}{2})^2\pi^2a^2}{L^2}\tau}$$

$$=\frac{2}{L}\int_0^L A\left(1-\frac{x}{L}\right)e^{-h\tau}\sin\frac{(n+\frac{1}{2})\pi x}{L}dx$$

$$,n=0,1,2,\cdots$$

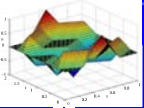


$$W(t,x,\tau)=\sum_{n=1}^{\infty}\frac{2A}{n\pi}\exp\left\{-\left(\frac{n\pi a}{L}\right)^2(t-\tau)-h\tau\right\}\sin\frac{n\pi x}{L}$$

由齐次化原理，有

$$u(x,t)=\int_0^t W(t,x,\tau)d\tau$$

$$=\frac{2AL^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n[(n\pi a)^2-L^2h]}\left\{e^{-ht}-\exp\left\{-\left(\frac{n\pi a}{L}\right)^2 t\right\}\right\}\sin\frac{n\pi x}{L}\right\}$$



非齐次边界的处理

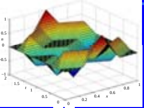
非齐解定解问题的主要步骤：

一、根据边界的形状选取适当的坐标系

二、边界非齐次, 作函数的代换化为齐次边界问题

三、非齐方程、齐次边界条件:

1. 原来初始条件的齐次方程的定解问题;
2. 齐边定解条件的非齐次方程的定解问题.



$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), (0 < x < L, t > 0) \\ u\big|_{x=0} = u_1(t), u\big|_{x=L} = u_2(t) \\ u\big|_{t=0} = \varphi(x), \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

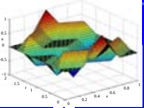
$$u(x,t) = V(x,t) + W(x,t)$$

$$V\big|_{x=0} = V\big|_{x=L} = 0$$

$$W(x,t) = A(t)x + B(t)$$

$$A(t) = \frac{1}{L}\left[u_2(t) - u_1(t)\right], B(t) = u_1(t)$$

$$u = V + \left[u_1 + \frac{u_2 - u_1}{L} x\right]$$



$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2} + f_1(x,t), (0 < x < L, t > 0) \\ V\big|_{x=0} = V\big|_{x=L} = 0 \\ V\big|_{t=0} = \varphi_1(x), \dfrac{\partial V}{\partial t}\Big|_{t=0} = \psi_1(x) \end{cases}$$

$$\begin{cases} f_1(x,t) = f(x,t) - \left[u_1''(t) + \dfrac{u_2''(t) - u_1''(t)}{L} x \right] \\ \varphi_1(x) = \varphi(x) - \left[u_1(0) + \dfrac{u_2(0) - u_1(0)}{L} x \right] \\ \psi_1(x) = \psi(x) - \left[u_1'(0) + \dfrac{u_2'(0) - u_1'(0)}{L} x \right] \end{cases}$$

$$u\big|_{x=0} = u_1(t), \qquad \frac{\partial u}{\partial x}\Big|_{x=L} = u_2(t)$$

例1 求下例定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + A, (0 < x < L, t > 0) \\ u\big|_{x=0} = 0, u\big|_{x=L} = B \\ u\big|_{t=0} = \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

解：

$$u(x,t) = V(x,t) + W(x)$$

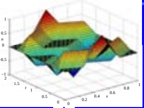
$$\frac{\partial^2 V}{\partial t^2} = a^2 \left[\frac{\partial^2 V}{\partial x^2} + W''(x) \right] + A$$

$$\begin{cases} a^2 W''(x) + A = 0 \\ W\big|_{x=0} = 0, W\big|_{x=L} = B \end{cases}$$

$$W(x) = -\frac{A}{2a^2} x^2 + \left(\frac{AL}{2a^2} + \frac{B}{L} \right) x$$

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2}, (0 < x < L, t > 0) \\ V\big|_{x=0} = V\big|_{x=L} = 0 \\ V\big|_{t=0} = -W(x), \dfrac{\partial V}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

$$V(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi a}{L} t + D_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x$$



$$V(x,t)=\sum_{n=1}^{\infty}C_n\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$$

$$-W(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{L}x$$

$$\frac{A}{2a^2}x^2-\left(\frac{AL}{2a^2}+\frac{B}{L}\right)x=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{L}x$$

$$C_n=\frac{2}{L}\int_0^L\left[\frac{A}{2a^2}x^2-\left(\frac{AL}{2a^2}+\frac{B}{L}\right)x\right]\sin\frac{n\pi}{L}xdx$$

$$=\frac{A}{a^2L}\int_0^L x^2\sin\frac{n\pi}{L}xdx-\left(\frac{A}{a^2}+\frac{2B}{L^2}\right)\int_0^L x\sin\frac{n\pi}{L}xdx$$

$$=-\frac{2AL^2}{a^2n^3\pi^3}+\frac{2}{n\pi}\left(\frac{AL^2}{a^2n^2\pi^2}+B\right)\cos n\pi$$

$$u(x,t)=-\frac{A}{2a^2}x^2+\left(\frac{AL}{2a^2}+\frac{B}{L}\right)x+\sum_{n=1}^{\infty}C_n\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$$

例 设弦的一端（=0）固定，另一端（x=L）以 $\sin\omega t\left(\omega\neq\frac{n\pi a}{L},n=1,2,\cdots\right)$ 作周期振动，且初值为零。试研究弦的自由振动。

解：依题意，得定解问题

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2},\left(0<x<L,t>0\right)\\ u(0,t)=0,u(L,t)=\sin\omega t,\left(\omega\neq\dfrac{n\pi a}{L}\right)\\ u(x,0)=0,u_t(x,0)=0\end{cases}$$

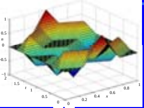
$$W(x,t)=X(x)\sin\omega t$$

$$\begin{cases} X'' + \dfrac{\omega^2}{a^2} X = 0 \\ X(0) = 0, X(L) = 1 \end{cases}$$

$$X(x) = C_1 \cos\frac{\omega x}{a} + C_2 \sin\frac{\omega x}{a}$$

$$W(x,t) = \frac{\sin\dfrac{\omega x}{a}}{\sin\dfrac{\omega L}{a}} \sin \omega t$$

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2} \\ V(t,0) = V(t,L) = 0 \\ V(0,x) = 0, V_t(0,x) = -\omega \dfrac{\sin\dfrac{\omega x}{a}}{\sin\dfrac{\omega L}{a}} \end{cases}$$



$$V(t,x) = 2\omega a L \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(\omega L)^2 - (n\pi a)^2} \sin\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}$$

例 解环形域内的定解问题：

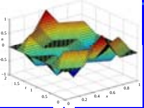
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=12\left(x^2-y^2\right) \\ u\big|_{x^2+y^2=a^2}=1, \dfrac{\partial u}{\partial n}\Big|_{x^2+y^2=b^2}=0 \end{cases}$$

$$a^2 \le x^2+y^2 \le b^2$$

解： $u=V+W$ 寻找特解V

$$V(x,y)=ax^4+by^4$$

$$V=x^4-y^4=(x^2+y^2)(x^2-y^2)=r^4\cos 2\theta$$



$$\begin{cases} \Delta W = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho\dfrac{\partial u}{\partial\rho}) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0, (a < r < b) \\ W(a,\theta) = u(a,\theta) - V(a,\theta) = 1 - a^4\cos 2\theta \\ \dfrac{\partial W}{\partial r}\Big|_{r=b} = \left(\dfrac{\partial u}{\partial r} - \dfrac{\partial V}{\partial r}\right)\Big|_{r=b} = -4b^3\cos 2\theta \end{cases}$$

$$W = A_0 + B_0 Lnr + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n\cos n\theta + D_n\sin n\theta\right)$$

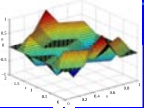
$$W(a,\theta) = 1 - a^4\cos 2\theta = A_0 + B_0 Lna + \left(A_2 a^2 + B_2 a^{-2}\right)\cos 2\theta$$

$$\frac{\partial W}{\partial r}\Big|_{r=b} = -4b^3\cos 2\theta = \frac{B_0}{b} + \left(2A_2 b - 2B_2 b^{-3}\right)\cos 2\theta$$

$$\begin{cases} \dfrac{B_0}{b} = 0 \\ A_0 + B_0 Lna = 1 \end{cases} \qquad \begin{cases} A_2 a^2 + B_2 a^{-2} = -a^4 \\ A_2 b - B_2 b^{-3} = -2b^3 \end{cases}$$

$$\begin{cases} A_0 = 1 \\ B_0 = 0 \\ A_2 = -\dfrac{a^6 + 2b^6}{a^4 + b^4} \\ B_2 = -\dfrac{a^4 b^4 \left(a^2 - 2b^2\right)}{a^4 + b^4} \end{cases}$$

$$u = V + W = 1 + \left[r^4 - \frac{a^6 + 2b^6}{a^4 + b^4} r^2 - \frac{a^4 b^4 \left(a^2 - 2b^2\right)}{a^4 + b^4} r^{-2} \right] \cos 2\theta$$



第三章 行波法与达朗贝尔公式

分离变量法:

a. 求解有限域内定解问题

b. 求解的区域很规则（边界用只含一个坐标变量的方程表示）

c. 对三种典型的方程均可运用

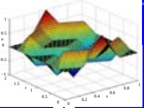
行波法(又称为特征线法):

a. 只能求解无界域内*波动方程*的定解问题——柯西问题

b. 它的解是达朗贝尔公式

c. 解法不能随意的扩大到一般的偏微分方程

d. 可以求出偏微分方程的通解

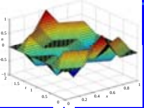


§ 4.1 一维波动方程的达朗贝尔公式

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases}$$

$$u(x,t) = f_1(x+at) + f_2(x-at)$$



$$\begin{cases} u\big|_{t=0} = \varphi(x) \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

$$\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ af_1'(x) - af_2'(x) = \psi(x) \end{cases}$$

无限长弦自由振动的达朗贝尔（DALEMBERT）公式

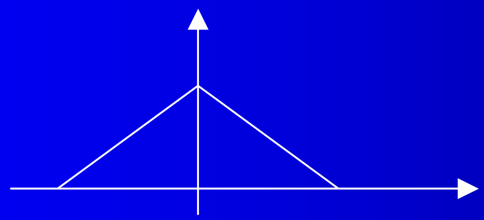
$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi$$

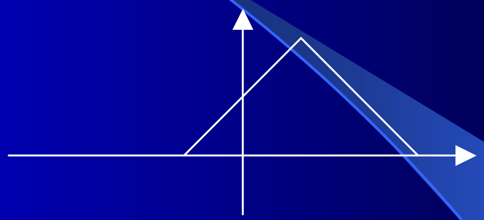
达朗贝尔公式在区间上的平均值形式

物理意义：

$$u_1 = f_1(x)$$

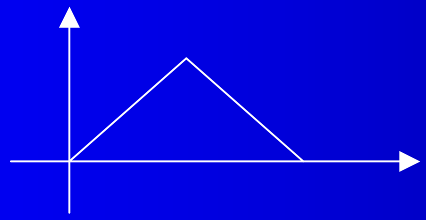
$$u_1 = f_1(x-at)$$

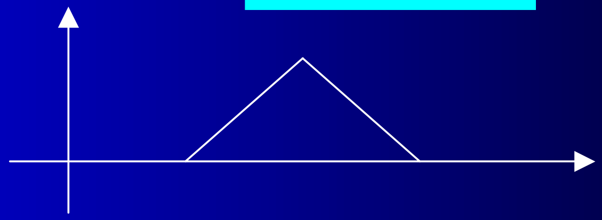




$$u_2 = f_2(x+at)$$

$$u_2 = f_2(x)$$

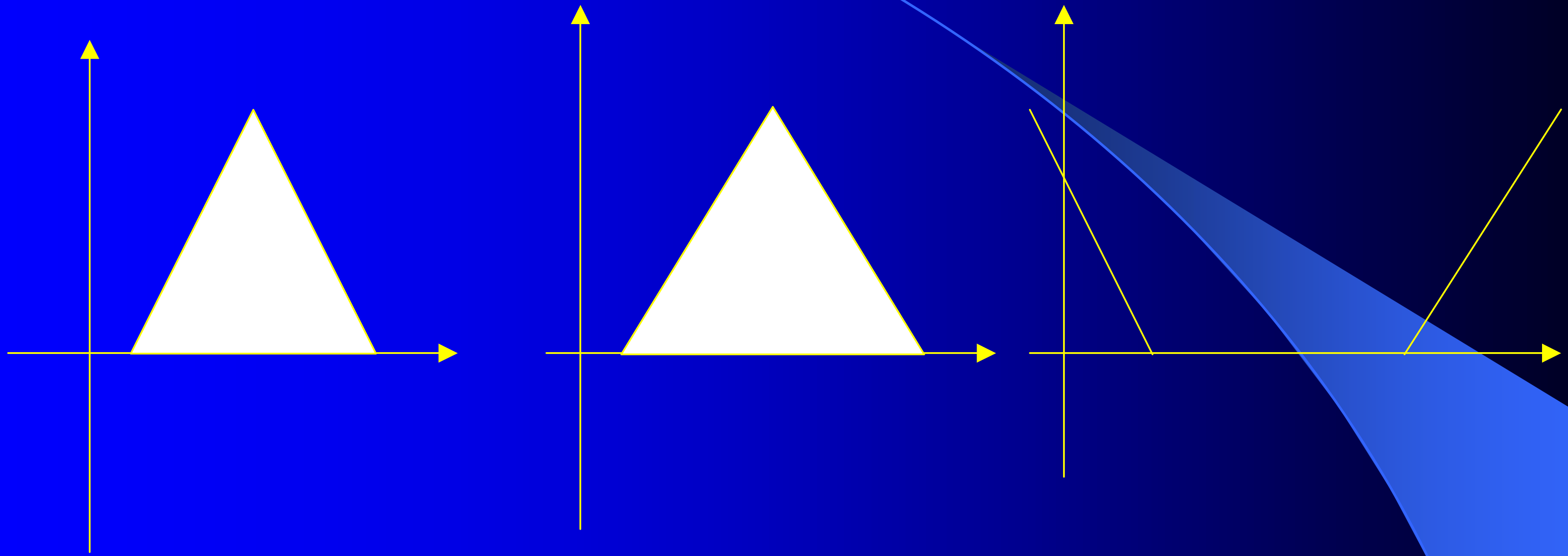




点的依赖区间： $[x-at, x+at]$

$[x_1, x_2]$ 区间的决定区域：

$[x_1, x_2]$ 影响的区域



例1）下列柯西问题的解：

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{y=0} = 3x^2, \dfrac{\partial u}{\partial y}\Big|_{y=0} = 0 \end{cases}$$

解：特征方程 $(dy)^2 - 2dxdy - 3(dx)^2 = 0$

$$3x - y = C_1, x + y = C_2$$

特征变换
$$\begin{cases} \xi = 3x - y \\ \eta = x + y \end{cases}$$

变换原方程化成标准型：$\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$

通解为：

$$u = f_2(\xi) + f_2(\eta) = f_1(3x - y) + f_2(x + y)$$

代入条件 $\begin{cases} f_1(3x)+f_2(x)=3x^2 \\ -f_1'(3x)+f_2'(x)=0 \end{cases}$

$$\begin{cases} f_1(x)=\dfrac{1}{4}x^2-C' \\ f_2(x)=\dfrac{3}{4}x^2+C' \end{cases}$$

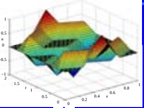
$$u(x,y)=\frac{1}{4}(3x-y)^2+\frac{3}{4}(x+y)^2=3x^2+y^2$$

无限长静止弦在点 $x = x_0$ 受到冲击，冲量 I，弦的密度为 ρ。试求解弦的振动。

动量定理

$$\frac{\rho}{I} u_t(x,0) = \delta(x - x_0)$$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, (-\infty < x < +\infty) \\ \\ u\big|_{t=0} = 0, (-\infty < x < +\infty) \\ u_t\big|_{t=0} = \dfrac{I}{\rho}\delta(x - x_0), (-\infty < x < \infty) \end{cases}$$



$$u(x,t)=\frac{1}{2a}\int_{x-at}^{x+at}\frac{I}{\rho}\delta\left(s-x_0\right)ds$$

$$=\frac{I}{2a\rho}\int_{x-x_0-at}^{x-x_0+at}\delta(\xi)d\xi=\frac{I}{2a\rho}H(\xi)\Big|_{x-x_0-at}^{x-x_0+at}$$

$$=\frac{I}{2a\rho}\left[H\left(x-x_0+at\right)-H\left(x-x_0-at\right)\right]$$

半无界弦的自由振动

端点固定

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\left(0<x<\infty\right) \\ u(x,0)=\varphi\left(x\right),u_t\left(x,0\right)=\psi\left(x\right),\left(0\le x<+\infty\right) \\ u\left(0,t\right)=0 \end{cases}$$

端点自由

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\left(0<x<\infty\right) \\ u(x,0)=\varphi\left(x\right),u_t\left(x,0\right)=\psi\left(x\right),\left(0\le x<\infty\right) \\ u_x\left(0,t\right)=0 \end{cases}$$

端点固定 $u(0,t)=0$

$$0=u(0,t)=\frac{1}{2}\left[\varphi(at)+\varphi(-at)\right]+\frac{1}{2a}\int_{-at}^{at}\psi(s)\,ds$$

$$\varphi(at)=-\varphi(-at);\qquad \int_{-at}^{at}\psi(s)\,ds=0$$

$\varphi(x)$ $\psi(x)$ 为奇函数

奇延拓

$$\Phi(x)=\begin{cases}\varphi(x), & (x\geq 0)\\ -\varphi(-x), & (x<0)\end{cases}$$

$$\Psi(x)=\begin{cases}\psi(x), & (x\geq 0)\\ -\psi(-x), & (x<0)\end{cases}$$

$$
u(x,t)=\frac{1}{2}\left[\Phi\left(x+at\right)+\Phi\left(x-at\right)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(s)ds
$$

$$
=\begin{cases}\dfrac{1}{2}\left[\varphi\left(x+at\right)+\varphi\left(x-at\right)\right]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(s)ds,\left(t\le\dfrac{x}{a}\right)\\ \dfrac{1}{2}\left(\varphi\left(x+at\right)-\varphi\left(at-x\right)\right)+\dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at}\psi(s)ds,\left(t>\dfrac{x}{a}\right)\end{cases}
$$

端点自由

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\
u(x,0) = \varphi\left(x\right), u_t\left(x,0\right) = \psi\left(x\right), \left(0 \le x < \infty\right) \\
u_x\left(0,t\right) = 0
\end{cases}
$$

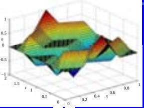
$$u_x(0,t) = 0$$

$$u_x(0,t) = \frac{1}{2}\left[\varphi'(at) + \varphi'(-at)\right] + \frac{1}{2a}\left[\psi(at) - \psi(-at)\right] = 0$$

$$\varphi'(at) = -\varphi'(-at), \psi(at) = \psi(-at)$$

$$
\Phi(x) = \begin{cases} \varphi(x), (x \ge 0) \\ \varphi(-x), (x < 0) \end{cases}
$$

$$
\Psi(x) = \begin{cases} \psi(x), (x \ge 0) \\ \psi(-x), (x < 0) \end{cases}
$$



$$u(x,t)=\frac{1}{2}\left[\Phi\left(x+at\right)+\Phi\left(x-at\right)\right]+\frac{1}{2a}\int_{.x-at}^{.x+at}\Psi(s)ds$$

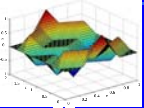
$$=\begin{cases}\frac{1}{2}\left[\varphi\left(x+at\right)+\varphi\left(x-at\right)\right]+\frac{1}{2a}\int_{.x-at}^{.x+at}\psi(s)ds,\left(t\le\frac{x}{a}\right)\\ \frac{1}{2}\left(\varphi\left(x+at\right)+\varphi\left(at-x\right)\right)+\frac{1}{2a}[\int_{.0}^{.x+at}\psi(s)ds+\int_{.0}^{.at-x}\psi(s)ds,\left(t>\frac{x}{a}\right)\end{cases}$$

非齐次方程的柯西问题

$$\begin{cases} u_{tt} = a_{xx} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = 0, u_t(x,0) = 0 \left(-\infty < x < \infty\right) \end{cases}$$

1. $$\begin{cases} w_{tt} = a^2 w_{xx}, \left(-\infty < x < \infty, t > 0\right) \\ w\big|_{t=\tau} = 0, w_t\big|_{t=\tau} = f(x,t), \left(-\infty < x < \infty\right) \end{cases}$$

$$u(x,t) = \int_{0}^{t} w(x,t,\tau)\,d\tau$$



2. $$t' = t - \tau$$

$$\begin{cases} w_{t't'} = a^2 w_{xx}, \left(t' > 0, -\infty < x < \infty\right) \\ w\big|_{t'=0} = 0, w_{t'}\big|_{t'=0} = f(x,\tau), \left(-\infty < x < \infty\right) \end{cases}$$

$$w(x,t',\tau) = \frac{1}{2a}\int_{x-at'}^{x+at'} f(\alpha,\tau)\,d\alpha$$

$$w(x,t,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\alpha,\tau)\,d\alpha$$

$$u(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\alpha,\tau)\,d\alpha\,d\tau$$

§ 4.3 高维波动方程柯西问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right), (-\infty < x, y, z < +\infty, t > 0) \\ u\big|_{t=0} = \varphi_0(x, y, z) \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \varphi_1(x, y, z) \end{cases}$$

方程等价形式

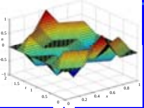
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

$$r\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial u}{\partial r} = \frac{\partial^2(ru)}{\partial r^2}$$

$$\frac{\partial^2(ru)}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2(ru)}{\partial t^2}$$

0



方程通解 $ru = f_1(r+at) + f_2(r-at)$

$$u(r,t) = \frac{f_1(r+at) + f_2(r-at)}{r}$$

平均值特解求法

$u(x,y,z,t) = u(M,t)$　　球面　　S_r^M

$$\overline{u}(r,t) = \frac{1}{4\pi r^2}\oint_{S_r^M} u(M',t)dS = \frac{1}{4\pi}\oint_{S_r^M} u(M',t)d\Omega$$

求 $\overline{u}(r,t)$ 满足的方程

$$\frac{1}{4\pi}\iiint\limits_{V_r^M}\frac{\partial^2 u}{\partial t^2}dV=\frac{a^2}{4\pi}\iiint\limits_{V_r^M}\nabla^2 u\,dVv=\frac{a^2}{4\pi}\oint_{S_r^M}\nabla u\cdot d\vec{S}$$

$$=\frac{a^2}{4\pi}\oiint\limits_{S_r^M}\frac{\partial u}{\partial r}dS=\frac{a^2r^2}{4\pi}\iint\limits_{D_M}\frac{\partial u}{\partial r}d\Omega$$

$$=a^2r^2\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\iint\limits_{D_M}u\,d\Omega\right]=a^2r^2\frac{\partial\overline{u}(r,t)}{\partial r}$$

$$\because\quad \int_{V_r^M}g(x,y,z)dV=\int_0^r d\rho\oiint\limits_{S_\rho^M}g(x,y,z)dS$$

$$\frac{1}{4\pi}\iiint\limits_{V_r^M}\frac{\partial^2 u}{\partial t^2}dV=\frac{\partial^2}{\partial t^2}\frac{1}{4\pi}\int_0^r d\rho\oiint\limits_{S_\rho^M}u(M',t)dS$$

联立上两式

$$\frac{\partial^2}{\partial t^2}\frac{1}{4\pi}\int_{0}^{r}d\rho\oiint_{S_{\rho}^{M}}u(M',t)\,dS=a^2r^2\frac{\partial\overline{u}(r,t)}{\partial r}$$

两端对r求导

$$\frac{\partial^2}{\partial t^2}\frac{1}{4\pi r^2}\oiint_{S_{r}^{M}}u(M',t)\,dS=\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{u}}{\partial r}\right)$$

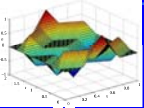
利用 $\overline{u}$ 的定义

$$\frac{\partial^2\overline{u}}{\partial t^2}=\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{u}}{\partial r}\right)$$

$$\because\ \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{u}}{\partial r}\right)=r\frac{\partial^2\overline{u}}{\partial r^2}+2\frac{\partial\overline{u}}{\partial r}=\frac{\partial^2(r\overline{u})}{\partial r^2}$$

$$\therefore\quad\frac{\partial^2(r\overline{u})}{\partial t^2}=a^2\frac{\partial^2(r\overline{u})}{\partial r^2}$$

$$\therefore r\overline{u}(r,t)=f_1(r+at)+f_2(r-at)$$



$$r = 0 \Rightarrow f_1'(at) = f_2'(-at)$$

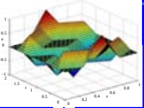
$$\frac{\partial}{\partial r}\left[r\bar{u}(r,t)\right] = f_1'(r+at) + f_2'(r-at)$$

$$\therefore \bar{u}(r,t) + r\frac{\partial \bar{u}(r,t)}{\partial r} = f_1'(r+at) + f_2'(r-at)$$

$$r = 0 \Rightarrow \bar{u}(0,t) = f_1'(at) + f_2'(-at)$$

$$\therefore u(M,t) = \bar{u}(0,t) = f_1'(at) + f_2'(-at) = 2f_1'(at)$$

下面求解 $f_i(r)$



$$\frac{1}{a}\frac{\partial}{\partial t}\left[r\bar{u}(r,t)\right]=f_1'(r+at)-f_2'(r-at)$$

$$\Rightarrow \frac{\partial}{\partial r}\left[r\bar{u}(r,t)\right]+\frac{1}{a}\left[r\bar{u}_t(r,t)\right]=2f_1'(r+at)$$

令t=0

$$2f_1'(r)=\frac{\partial}{\partial r}\left[r\bar{u}(r,0)\right]+\frac{1}{a}r\bar{u}_t(r,0)$$

$$=\frac{\partial}{\partial r}\left[\frac{r}{4\pi r^2}\oint_{S_r^M}u(r,0)dS\right]+\frac{r}{4\pi a r^2}\oint_{S_r^M}u_t(r,0)dS$$

$$=\frac{1}{4\pi}\frac{\partial}{\partial r}\oint_{S_r^M}\frac{\varphi(x',y',z')}{r}dS+\frac{1}{4\pi a}\oint_{S_r^M}\frac{\psi(x',y',z')}{r}dS$$

$$u(M,t)=2f_1'(at)=\frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\oint_{S_{at}^M}\frac{\varphi(x',y',z')}{t}dS+\oint_{S_{at}^M}\frac{\psi(x',y',z')}{t}dS\right]$$

$$u(M,t)=\frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\oiint_{S_{at}^M}\frac{\varphi(M')}{t}dS+\oiint_{S_{at}^M}\frac{\psi(M')}{t}dS\right]$$

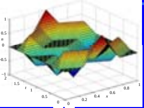
例1）设 $\varphi(x,y,z)=x+y+z,\psi(x,y,z)=0$ 已知。
求方程（1）相应柯西问题的解

解：

$$u(M,t)=\frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\oiint_{S_{at}^{M}}\frac{\varphi(M')}{t}dS+\oiint_{S_{at}^{M}}\frac{\psi(M')}{t}dS\right]$$

$$u(x,y,z,t)=\frac{1}{4\pi a}$$

$$\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}\frac{x+y+z+at(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)}{at}(at)^2\sin\theta d\varphi d\theta$$



$$= \frac{1}{4\pi a}\frac{\partial}{\partial t}[at(x+y+z)\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta$$

$$+ a^2t^2\int_0^{2\pi}(\sin\varphi+\cos\varphi)d\varphi\int_0^{\pi}\sin^2\theta d\theta$$

$$+ a^2t^2\int_0^{2\pi} d\varphi\int_0^{\pi}\sin\theta\cos\theta d\theta]$$

$$= x+y+z$$

例2）用泊松公式解如下定解问题

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz}, (t>0, -\infty < x,y,z < \infty) \\ u(x,y,z,0) = 0, u_t(x,y,z,0) = x^2, (-\infty < x,y,z < \infty) \end{cases}$$

解：

$$u(x,y,z,t) = \frac{1}{4\pi}\iint_{S_t^M}\frac{\xi^2}{t}dS = \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}(x+t\sin\theta\cos\phi)^2\sin\theta d\theta d\phi$$

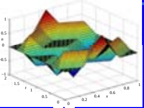
$$= x^2t + \frac{1}{3}t^3$$

例**5**）
$$\begin{cases} u_{tt}=a^2\left(u_{xx}+u_{yy}\right),\left(t>0,-\infty<x,y<\infty\right) \\ u\left(x,y,0\right)=x^2(x+y),\left(-\infty<x,y<\infty\right) \\ u_t\left(x,y,0\right)=0 \end{cases}$$

§ 4.4 非齐次波动方程解法 . 推迟势

$$
\begin{cases}
u_{tt} = a^2 \Delta u + f(x,\ y,\ z,\ t), (-\infty < x,\ y,\ z < +\infty) \\
u\big|_{t=0} = \varphi(x,\ y,\ z) \\
u_t\big|_{t=0} = \psi(x,\ y,\ z)
\end{cases}
$$

$$
\begin{cases}
u_{tt} = a^2 \Delta u + f(x,\ y,\ z,\ t) \\
u\ (x,\ y,\ z, 0) = 0 \\
u_t(x,\ y,\ z, 0) = 0
\end{cases}
\qquad
\begin{cases}
u_{tt} = a^2 \Delta u \\
u\big|_{t=0} = \varphi(x,\ y,\ z) \\
u_t\big|_{t=0} = \psi(x,\ y,\ z)
\end{cases}
$$



$$\begin{cases} U_{tt} = a^2 \Delta U \\ U(x,\ y,\ z,\ \tau) = 0 \\ U_t(x,\ y,\ z,\ \tau) = f(x,\ y,\ z,\ \tau) \end{cases}$$

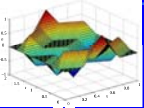
$$U(x,\ y,\ z,\ t;\ \tau) = \frac{t-\tau}{4\pi}$$

$$\int_0^{2\pi}\int_0^{\pi} f[x+\alpha_1 a(t-\tau),\ y+\alpha_2 a(t-\tau),\ z+\alpha_3 a(t-\tau),\ \tau]d\Omega,$$

$$u(x,y,z,t) = \int_0^t U(x,y,z,t;\tau)d\tau$$

$$= \frac{1}{4\pi}\int_0^t (t-\tau)$$

$$\int_0^{2\pi}\int_0^{\pi} f[x+a_1 a(t-\tau), y+a_2 a(t-\tau), z+a_3 a(t-\tau), \tau]d\Omega d\tau$$



方程特解:

$$u(x,\ y,\ z,\ t)$$

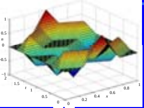
$$=\frac{1}{4\pi a^2}\int_0^{at}\int_0^{2\pi}\int_0^{\pi}\frac{f\left(x+\alpha_1 r,\ y+\alpha_2 r,\ z+\alpha_3 r,\ t-\frac{r}{a}\right)}{r}r^2\sin\theta d\theta d\varphi dr$$

$$=\frac{1}{4\pi a^2}\iiint_{r\le at}\frac{f\left(\alpha,\ \beta,\ \gamma,\ r,\ t-\frac{r}{a}\right)}{r}dV$$

$$\alpha=x+\alpha_1 r,\quad \beta=y+\alpha_2 r$$

$$\gamma=z+\alpha_3 r r=\sqrt{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2}$$

$$\left(\alpha_1^2+\alpha_2^2+\alpha_3^2=1\right)$$



原方程特解: $u = u_0(x,y,t) + u_1(x,y,t)$

$$u_0(x,y,t) = \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi} \frac{\varphi\left(x + r\cos\theta, y + r\sin\theta\right)}{\sqrt{a^2t^2 - r^2}} r dr d\theta\right.$$

$$\left. + \int_0^{at}\int_0^{2\pi} \frac{\psi\left(x + r\cos\theta, y + r\sin\theta\right)}{\sqrt{a^2t^2 - r^2}} r dr d\theta\right]$$

$$u_1(x,\ y,\ z,\ t)$$

$$= \frac{1}{4\pi a^2}\int_0^{at}\int_0^{2\pi}\int_0^{\pi} \frac{f\left(x+\alpha_1 r,\ y+\alpha_2 r,\ z+\alpha_3 r,\ t-\frac{r}{a}\right)}{r} r^2 \sin\theta d\theta d\varphi dr$$

$$= \frac{1}{4\pi a^2}\iiint\limits_{r\le at} \frac{f\left(\alpha,\ \beta,\ \gamma,\ r,\ t-\frac{r}{a}\right)}{r} dV$$

第五章 积分变换

$f(x)$ 定义在 $(-\infty,\infty)$内，且在任一有限区间 $[-L,L]$上分段光滑，则可以展开为傅氏级数

$$f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$\begin{cases} a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L}f(\xi)\cos\frac{n\pi\xi}{L}d\xi \\ b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L}f(\xi)\sin\frac{n\pi\xi}{L}d\xi \end{cases},(n=0,1,2,\cdots)$$

傅氏变换（傅里叶变换）

$$F(\lambda)=\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx$$

逆傅氏变换

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}d\lambda$$

记号 $\tilde{f}(\lambda)=F[f(x)]$ $\quad F^{-1}F[f]=f$

$$f(x)=F^{-1}[\tilde{f}]=F^{-1}[F(\lambda)]$$

$$f_1(x)*f_2(x)=\int_{-\infty}^{\infty} f_1(x-\eta)f_2(\eta)d\eta$$

傅氏变换的基本性质

性质1.（线性定理）

$$F\left[\alpha f_1+\beta f_2\right]=\alpha F[f_1]+\beta F[f_2]$$

性质2.（卷积定理）

$$F[f_1 * f_2]=F[f_1]F[f_2]$$

性质3.（乘积定理）

$$F[f_1 f_2]=\frac{1}{2\pi}F[f_1]*F[f_2]$$

性质4.（原象的导数定理）

$$F[f^{(k)}]=(i\lambda)^k F[f]$$

$$F[f']=i\lambda F[f]$$

性质5.（象的导数定理）

$$\frac{d}{d\lambda}F[f]=F[-ixf]$$

性质6.（延迟定理）

$$F[f(x-x_0)]=e^{-i\lambda x_0}F[f(x)]$$

性质7.（位移定理）

$$F[e^{i\lambda_0 x}f(x)]=\tilde{f}(\lambda-\lambda_0)$$

性质8.（积分定理）

$$F[\int_{-\infty}^{x}f(\xi)d\xi]=\frac{1}{i\lambda}F[f(x)]$$

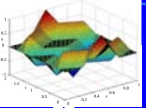
性质9.(广义函数)

$$F[\delta(x)]=\int_{-\infty}^{\infty}\delta(x)\ e^{-i\lambda x}dx=e^{-i\lambda x}\big|_{x=0}=1$$

$$F[\delta(x-\xi)]=\int_{-\infty}^{\infty}\delta(x-\xi)e^{-i\lambda x}dx=e^{-i\lambda\xi}$$

性质10.(相似定理)

$$F[f(ax)]=\frac{1}{|a|}\tilde{f}(\frac{\lambda}{a})$$



N 维傅氏变换

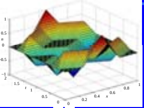
定义

$$F(\lambda_1,\lambda_2,\cdots,\lambda_n)=F[f(x_1,x_2,\cdots,x_n)]$$

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,x_2,\cdots,x_n)e^{-i(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)}dx_1dx_2\cdots dx_n$$

$$f(x_1,x_2,\cdots,x_n)$$

$$=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F(\lambda_1,\lambda_2,\cdots,\lambda_n)e^{i(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)}d\lambda_1d\lambda_2\cdots d\lambda_n$$



N 维傅氏变换具有与上面平行的八个性质：

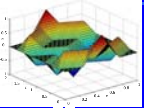
$$F[\alpha f_1 + \beta f_2] = \alpha F[f_1] + \beta F[f_2]$$

$$F[f_1 * f_2] = F[f_1]F[f_2]$$

$$F[f_1 f_2] = \frac{1}{(2\pi)^2} F[f_1] * F[f_2]$$

$$F[\frac{\partial f}{\partial x_k}] = i\lambda_k F[f], k = 1, 2, \cdots, n$$

$$\frac{\partial}{\partial \lambda_k} F[f] = F[-ix_k f], k = 1, 2, \cdots, n$$



§ 5.2 傅里叶变换的应用

傅里叶变换法求解步骤:

（1）对定解问题作傅里叶变换;

（2）求解象函数;

（3）对象函数作傅里叶逆变换得解。

应用范围:

1）求解无界区域的定解问题, 直接傅氏求解;

2）对于半无界区域的定解问题:

a. 第一类边界条件, 采用傅里叶正弦变换;

b.第二类边界条件,傅里叶余弦变换

c.将边界条件齐次化后，采用延拓法，最后用傅里叶变换法求解

简写符号

$$F[u(x,t)]=\tilde{u}(\lambda,t),F[\varphi(x)]=\tilde{\varphi}(\lambda),F[\psi(x)]=\tilde{\psi}(\lambda)$$

$$F[u_{tt}(x,t)]=\int_{-\infty}^{\infty}\frac{\partial^2 u(x,t)}{\partial t^2}e^{-i\lambda x}dx=\frac{d^2}{dt^2}\int_{-\infty}^{\infty}u(x,t)e^{-i\lambda x}dx=\frac{d^2\tilde{u}(\lambda,t)}{dt^2}$$

$$F[u_{xx}(x,t)]=\int_{-\infty}^{\infty}\frac{\partial^2 u(x,t)}{\partial x^2}e^{-i\lambda x}dx=(i\lambda)^2\tilde{u}(\lambda,t)$$

波动方程的定解问题

例1）求解无界弦振动方程的初值问题

$$\begin{cases} u_{tt}-a^2u_{xx}=0,(-\infty<x<\infty,t>0) \\ u(x,0)=\varphi(x) \\ u_t(x,0)=\psi(x) \end{cases}$$

解：

$$u_{tt} \leftrightarrow \frac{d^2\tilde{u}(\lambda,t)}{dt^2}, u_{xx} \leftrightarrow (i\lambda)^2\tilde{u}(\lambda,t)$$

$$\begin{cases} \dfrac{d^2\tilde{u}(\lambda,t)}{dt^2} + a^2\lambda^2\tilde{u}(\lambda,t) = 0 \\ \tilde{u}(\lambda,0) = \tilde{\varphi}(\lambda), \tilde{u}_t(\lambda,0) = \tilde{\psi}(\lambda) \end{cases}$$

$$\tilde{u}(\lambda,t) = C_1 e^{i\lambda at} + C_2 e^{-i\lambda at}$$

$$\tilde{u}(\lambda,t) = \frac{1}{2}\left[\tilde{\varphi}(\lambda) + \frac{1}{i\lambda a}\psi(\lambda)\right]e^{i\lambda at} + \frac{1}{2}\left[\tilde{\varphi}(\lambda) - \frac{1}{i\lambda a}\tilde{\psi}(\lambda)\right]e^{-i\lambda at}$$

$$u(x,t)=F^{-1}[\tilde{u}(\lambda,t)]$$

$$=\frac{1}{2}F^{-1}\left[\tilde{\varphi}(\lambda)e^{i\lambda at}\right]+\frac{1}{2a}F^{-1}\left[\frac{1}{i\lambda}\tilde{\psi}(\lambda)e^{i\lambda at}\right]$$

$$+\frac{1}{2}F^{-1}\left[\tilde{\varphi}(\lambda)e^{-i\lambda at}\right]-\frac{1}{2a}F^{-1}\left[\frac{1}{i\lambda}\tilde{\psi}(\lambda)e^{-i\lambda at}\right]$$

应用延迟定理 $F[\varphi(x\pm at)]=e^{\pm i\lambda at}F[\varphi(x)]=e^{\pm i\lambda at}\tilde{\varphi}(\lambda)$

$$F^{-1}[e^{\pm i\lambda at}\tilde{\varphi}(\lambda)]=\varphi(x\pm at)$$

$$F\left[\int_{-\infty}^{x\pm at}\psi(\xi)d\xi\right]=e^{\pm i\lambda at}F\left[\int_{-\infty}^{x}\psi(\xi)d\xi\right]=e^{\pm i\lambda at}\frac{1}{i\lambda}\tilde{\psi}(\lambda)$$

$$F^{-1}\left[\frac{1}{i\lambda}\tilde{\psi}(\lambda)e^{\pm i\lambda at}\right]=\int_{-\infty}^{x\pm at}\psi(\xi)d\xi$$

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\left[\int_{-\infty}^{x+at}\psi(\xi)d\xi-\int_{-\infty}^{x-at}\psi(\xi)d\xi\right]$$

$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

热传导方程定解问题

例 设有一根无限长的杆，杆上有强度为 $F(x,t)$ 的热源，杆的初始温度为 $\varphi(x)$ ，试求 $t>0$ 时杆上温度的分布规律。

解：
$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}+f(x,t),\left(-\infty<x<+\infty,t>0\right)\\ u\big|_{t=0}=\varphi(x)\end{cases}$$

$$f(x,t)=\frac{1}{\rho c}F(x,t)$$

$$\tilde{u}(\lambda,t)=\int_{-\infty}^{\infty}u(x,t)e^{i\lambda x}dx$$

$$\tilde{f}(\lambda,t)=\int_{-\infty}^{\infty}f(x,t)e^{i\lambda x}dx$$

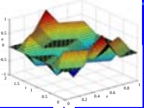
$$\begin{cases}\dfrac{d\tilde{u}}{dt}=-a^2\lambda^2\tilde{u}(\lambda,t)+\tilde{f}(\lambda,t)\\ \tilde{u}(\lambda,t)\big|_{t=0}=\tilde{\varphi}(\lambda)\end{cases}$$

$$\tilde{u}=\tilde{\varphi}e^{-a^2\lambda^2 t}+\int_0^t\tilde{f}e^{-a^2\lambda^2(t-\tau)}d\tau$$

$$F^{-1}[e^{-a^2\lambda^2 t}]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$$

$$u(x,t)=F^{-1}[\tilde{u}(\lambda,t)]$$

$$=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\frac{1}{2a\sqrt{\pi}}\int_0^t d\tau\int_{-\infty}^{\infty}\frac{f(\xi,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi$$



例　求半无界杆的热传导问题

$$\begin{cases} u_t - a^2 u_{xx} = 0, (0 < x < \infty, t > 0) \\ u\big|_{t=0} = 0 \\ u\big|_{x=0} = u_0(\text{常数}) \end{cases}$$

解：将边界条件齐次化，仿造半无界弦的波动问题作奇延拓，将问题化为无界问题

$$u\,(x,\ t) = \omega\,(x,\ t) + u_0$$

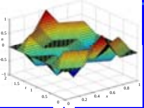
$$\begin{cases} \omega_t - a^2 \omega_{xx} = 0, (0 < x < \infty, t > 0) \\ \omega\big|_{t=0} = -u_0 \\ \omega\big|_{x=0} = 0 \end{cases}$$

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi$$

$$u(0,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{\xi^2}{4a^2t}}d\xi$$

将 $\omega(x,\ t)$ 作奇延拓

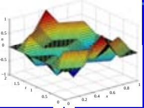
$$\begin{cases}\omega_t-a^2\omega_{xx}=0,(-\infty<x<\infty,t>0)\\ \omega\big|_{t=0}=\varphi(x)=\begin{cases}-u_0,(x>0)\\ u_0,(x<0)\end{cases}\end{cases}$$



利用上题结果

$$u(x,\ t)=u_0+\omega(x,\ t)$$

$$=u_0+\frac{u_0}{2a\sqrt{\pi t}}\left[\int_{-\infty}^{0}e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi-\int_{0}^{\infty}e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\right]$$



余弦变换：
$$\tilde{u}(t,\lambda)=\int_{.0}^{.\infty}u(t,x)\cos\lambda x dx$$

反余弦变换:
$$u(t,x)=\frac{2}{\pi}\int_{0}^{\infty}\tilde{u}(t,\lambda)\cos\lambda x d\lambda$$

例4）用余弦变换解定解问题

$$\begin{cases}u_t=a^2u_{xx},\left(x>0,t>0\right)\\u(0,x)=0,u_x(t,0)=Q,\left(Q\text{为常数}\right)\\u(t,+\infty)=u_x(t,+\infty)=0\end{cases}$$

解：以为积分变量，作余弦变换：

$$\tilde{u}(t,\lambda)=\int_{.0}^{.\infty}u(t,x)\cos\lambda x dx$$

$$\tilde{u}_{xx} = \int_{0}^{\infty} u_{xx}(t,x)\cos\lambda x dx$$

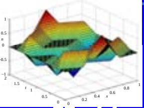
$$= u_x \cos\lambda x\Big|_0^\infty + \lambda\int_{0}^{\infty} u_x \sin\lambda x dx$$

$$= -Q + \lambda u \sin\lambda x\Big|_0^\infty - \lambda^2\int_{0}^{\infty} u\cos\lambda x dx$$

$$= -Q - \lambda^2\tilde{u}$$

$$\begin{cases} \dfrac{d\tilde{u}}{dt} + a^2\lambda^2\tilde{u} = -a^2 Q \\ \tilde{u}\left(0,\lambda\right) = 0 \end{cases}$$

$$\tilde{u}(t,\lambda) = \frac{Q}{\lambda^2}\left[\exp\left\{-a^2\lambda^2 t\right\} - 1\right] = -a^2 Q\int_{0}^{t}\exp\left\{-a^2\lambda^2\tau\right\}d\tau$$



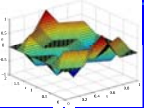
作反余弦变换

$$u(t,x)=\frac{2}{\pi}\int_{.0}^{.\infty}\tilde{u}(t,\lambda)\cos\lambda x d\lambda$$

$$=-\frac{2a^2Q}{\pi}\int_{.0}^{.t}d\tau\int_0^{\infty}\exp\left\{-a^2\lambda^2\tau\right\}\cos\lambda x d\lambda$$

$$=-\frac{aQ}{\sqrt{\pi}}\int_{.0}^{.t}\frac{1}{\sqrt{\tau}}\exp\left\{-\frac{x^2}{4a^4\tau^2}\right\}d\tau$$

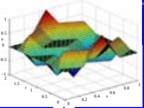
$$=-\frac{2aQ}{\sqrt{\pi}}\int_{.0}^{\sqrt{t}}\exp\left\{-\frac{x^2}{4a^4\tau^4}\right\}d\tau$$



$$\int_0^{\infty} e^{-a^2\lambda^2\tau}\cos\lambda x\,d\lambda=\frac{1}{2}\int_0^{\infty} e^{-a^2\lambda^2\tau}\left(e^{i\lambda x}+e^{-i\lambda x}\right)d\lambda$$

$$=\frac{1}{2}\int_0^{\infty} e^{-a^2\tau\left(\lambda-\frac{\lambda^2 x}{2a^2\tau}\right)^2}d\lambda\, e^{-\frac{x^2}{4a^2\tau}}+\frac{1}{2}\int_0^{\infty} e^{-a^2\tau\left(\lambda+\frac{\lambda^2 x}{2a^2\tau}\right)^2}d\lambda\, e^{-\frac{x^2}{4a^2\tau}}$$

$$=\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\,e^{-\frac{x^2}{4a^2\tau}}$$



例5）用傅里叶变换方法求 $\Delta u(x,y,z)=\delta(x,y,z)$ 三维场势方程的基本解。

解：对方程两边作傅里叶变换

$$\tilde{u}(\lambda,\mu,v)=F[u]$$

$$=\iiint\limits_{R^3} u(x,y,z)\exp\left\{-i\left(\lambda x+\mu y+vz\right)\right\}dxdydz$$

$$\because F[\Delta u]=-\left(\lambda^2+\mu^2+v^2\right)\tilde{u},F[\delta\left(x,y,z\right)]=1$$

$$\therefore\left(\lambda^2+\mu^2+v^2\right)\tilde{u}=-1$$

$$\therefore\tilde{u}=-\frac{1}{\rho^2},(\rho^2=\lambda^2+\mu^2+v^2)$$

$$u=F^{-1}[\tilde{u}]=-\frac{1}{(2\pi)^3}\iiint\limits_{R^3}\frac{1}{\rho^2}\exp\left\{i\left(\lambda x+\mu y+vz\right)\right\}d\lambda d\mu dv$$

例6）

$$\begin{cases} u_t = -a^2 u_{xxxx}, \left(x \in R, t > 0\right) \\ u(x,0) = \varphi(x), u_t(x,0) = a\psi''(x) \end{cases}$$

$$u_{tt} \leftrightarrow \frac{d^2\tilde{u}(\lambda,t)}{dt^2}, u_{xxxx} \leftrightarrow (i\lambda)^4 \tilde{u}(\lambda,t)$$

$$\begin{cases} \dfrac{d^2\tilde{u}(\lambda,t)}{dt^2} + a^2\lambda^4\tilde{u}(\lambda,t) = 0 \\ \tilde{u}(\lambda,0) = \tilde{\varphi}(\lambda), \tilde{u}_t(\lambda,0) = -a\lambda^2\widetilde{\psi}(\lambda) \end{cases}$$

$$\tilde{u}(\lambda,t) = C_1 \sin\lambda^2 at + C_2 \cos\lambda^2 at$$

$$\tilde{\varphi}(\lambda)=\tilde{u}(\lambda,0)=C_2$$

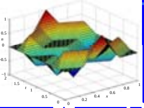
$$-a\lambda^2\tilde{\psi}(\lambda)=\tilde{u}_t(\lambda,0)=\lambda^2 aC_1$$

$$\tilde{u}(\lambda,t)=\frac{1}{2}\left[\tilde{\varphi}(\lambda)+i\widetilde{\psi}(\lambda)\right]e^{i\lambda^2 at}+\frac{1}{2}\left[\tilde{\varphi}(\lambda)-i\tilde{\psi}(\lambda)\right]e^{-i\lambda^2 at}$$

$$\tilde{u}(\lambda,t)=-\tilde{\psi}(\lambda)\sin\lambda^2 at+\tilde{\varphi}(\lambda)\cos\lambda^2 at$$

$$F^{-1}(\cos a\lambda^2 t)=\frac{1}{2\sqrt{2a\pi t}}\left(\cos\frac{x^2}{4at}+\sin\frac{x^2}{4at}\right)$$

$$F^{-1}(\sin a\lambda^2 t)=\frac{1}{2\sqrt{2a\pi t}}\left(\cos\frac{x^2}{4at}-\sin\frac{x^2}{4at}\right)$$



一根半无限长的杆,初始温度分布等于零,且在端点 $x=0$ 处以为速率输人热量 $h(t)$,求杆的温度分布。

解:

$$
\begin{cases}
u_t = a^2 u_{xx} & x \in (0,+\infty), t>0 \\
u_x(0,t) = -\dfrac{1}{k} h(t) = g(t) & \\
u(x,0) = 0 &
\end{cases}
$$

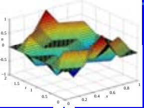
$$U_c(\lambda,t) = F(u(x,t))$$

$$\int_{.0}^{.\infty} u_{xx}(t,x)\cos\lambda x dx$$

$$= u_x \cos\lambda x \Big|_0^\infty + \lambda \int_{.0}^{.\infty} u_x \sin\lambda x dx$$

$$= -g(t) + \lambda u \sin\lambda x \Big|_0^\infty - \lambda^2 \int_{.0}^{.\infty} u \cos\lambda x dx$$

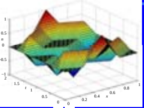
$$= -g(t) - \lambda^2 U_c$$



$$\begin{cases} \dfrac{dU_c}{dt} + a^2\lambda^2 U_c = -a^2 g(t) \\ U_c(\lambda, 0) = 0 \end{cases}$$

$$U_c(\lambda, t) = -a^2 \int_0^t g(x) e^{-a^2\lambda^2(t-x)} dx$$

$$\therefore \quad u(x,t) = \frac{-a}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} e^{-x^2/4a^2(t-\tau)} d\tau$$



Laplace变换

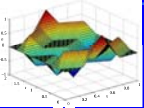
（一） Laplace变换的定义

$$L[f(t)]=\tilde{f}(s)=\int_{0}^{\infty}f(t)e^{-st}dt$$

$$L^{-1}[\tilde{f}(s)]=f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)e^{st}ds$$

（二）常用函数的Laplace变换

$$L[ce^{at}]=\int_{0}^{\infty}ce^{at}e^{-st}dt=-\frac{ce^{-(s-a)t}}{s-a}\Big|_{0}^{\infty}=\frac{c}{s-a},(\mathrm{Re}\,s>\mathrm{Re}\,a)$$



$$L[\sin bt]=\int_{.0}^{.\infty}\sin bte^{-st}dt=\frac{1}{2i}\int_{.0}^{.\infty}\left[e^{-(s-ib)t}-e^{-(s+ib)t}\right]dt$$

$$=\frac{1}{2i}\left(\frac{1}{s-ib}-\frac{1}{s+ib}\right)=\frac{b}{s^2+b^2},\left(\operatorname{Re}s>0\right)$$

$$L[\cos bt]=\frac{1}{2}\left(\frac{1}{s-ib}-\frac{1}{s+ib}\right)=\frac{b}{s^2+b^2},\left(\operatorname{Re}s>0\right)$$

$$L[t^\beta]=\int_{.0}^{.\infty}t^\beta e^{-st}dt=\frac{1}{s^{\beta+1}}\int_{.0}^{.\infty}e^{-st}\left(st\right)^\beta d\left(st\right)=\frac{\Gamma\left(\beta+1\right)}{s^{\beta+1}},\left(\operatorname{Re}s>0\right)$$

Laplace变换的性质

1.
$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)]$$
$$L^{-1}[a_1 F_1(s) + a_2 F_2(s)] = a_1 L^{-1}[F_1(s)] + a_2 L^{-1}[F_2(s)]$$

2. 延迟定理
$$L[f(t-\tau)] = e^{-s\tau} L[f(t)]$$

3. 位移定理
$$L[e^{at} f(t)] = F(s-a), \mathrm{Re}(s-a) > \sigma_0$$

4. 相似定理

若c为大于零的常数，则
$$L[f(ct)] = \frac{1}{c} F(\frac{s}{c})$$

5 . 微分定理

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$\cdots$$

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

6. 积分定理

$$L[\int_{0}^{t} f(\tau)d\tau] = \frac{1}{s} L[f(t)]$$

7. 象函数的微分定理

$$\frac{d^n}{ds^n} F(s) = L[(-t)^n f(t)]$$

8. 象函数的积分定理

$$\int_{s}^{\infty} F(\tau)d\tau = L[\frac{f(t)}{t}]$$

9.卷积定理

$$L[f_1(t) * f_2(t)] = L[f_1(t)] \cdot L[f_2(t)]$$

1. 约当引理

2. 展开定理

$$f(t) = \sum_k \mathrm{Res}[L(\tau)e^{\tau t}, s_k]$$

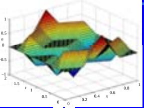
例　已知 $F(s)=\dfrac{s}{(s+\alpha)(s+\beta)^2}$ 求 $L^{-1}[F(s)]$

$$L^{-1}[f(s)]=L^{-1}\left[\frac{s}{(s+\alpha)(s+\beta)^2}\right]$$

$$=\sum_k \mathrm{Res}\left[\frac{se^{st}}{(s+\alpha)(s+\beta)^2},\ \ s_k\right]$$

$$=\lim_{s\to-\alpha}(s+\alpha)\cdot\frac{se^{st}}{(p+\alpha)(p+\beta)^2}+\lim_{p\to-\beta}\left[(s+\beta)^2\frac{se^{st}}{(s+\alpha)(s+\beta)^2}\right]_p^{/}$$

$$=\frac{\alpha-\beta(\alpha-\beta)t}{(\beta-\alpha)^2}e^{-\beta t}-\frac{\alpha}{(\beta-\alpha)^2}e^{-\alpha t}$$



例　已知 $F(s)=\dfrac{2s^2-5s-5}{(s+1)(s-1)(s-2)}$ 求 $L^{-1}[F(s)]$

例　已知 $F(s)=\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ 求 $L^{-1}[F(s)]$

例　求解常微分方程：

$$\begin{cases} y''-3y'+2y=2e^{3t} \\ y\big|_{t=0}=0,\ y'\big|_{t=0}=0 \end{cases}$$

$$L[y]=\tilde{y},\ L[2e^{3t}]=\frac{2}{s-3}$$

$$s^2\tilde{y}-3s\tilde{y}+2\tilde{y}=\frac{2}{s-3}$$

$$\tilde{y}=\frac{2}{(s-3)(s-2)(s-1)}$$

$$y = L^{-1}[\tilde{y}] = e^{3t} - 2e^{2t} + e^{t}$$

例 求解积分方程 $f(t) = at + \int_0^t \sin(t-\tau) f(\tau) d\tau$

解：由卷积定义，将方程写成：$f(t) = at + f(t) * \sin t$

$$\tilde{f} = \frac{a}{p^2} + \frac{1}{p^2+1}\tilde{f}$$

$$\tilde{f} = \frac{a}{p^2} + \frac{a}{p^4}$$

$$f(t) = a(t + \frac{t^3}{6})$$

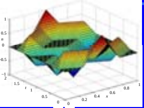
Laplace变换解数理方程

例 求解硅片的恒定表面浓度扩散问题，在恒定表面浓度扩散中，包围硅片的气体中含有大量杂质原子，它们源源不断穿过硅片表面向硅片内部扩散。由于气体中杂质原子供应充分，硅片表面浓度得以保持某个常数 u_0，这里所求的是半无限空间x＞0中定解问题

$$\begin{cases} u_t = a^2 u_{xx}, (x > 0, t > 0) \\ u\big|_{x=0} = N_0 \\ u\big|_{t=0} = 0 \end{cases}$$

解：对自变量作Laplace变换

$$\begin{cases} a^2 \dfrac{d^2\tilde{u}}{dx^2} - s\tilde{u} = 0 \\ \tilde{u}\big|_{x=0} = N_0 / s \end{cases}$$



$$\tilde{u} = Ae^{-\frac{\sqrt{s}}{a}x} + Be^{\frac{\sqrt{s}}{a}x}$$

$$\lim_{x\to\infty} \tilde{u}$$

$$B = 0$$

$$A = N_0 / s$$

$$\tilde{u} = N_0 \frac{1}{s} e^{-\frac{\sqrt{s}}{a}x}$$

$$L^{-1}[\frac{1}{s} e^{-\frac{\sqrt{s}}{a}x}] = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2} dy$$

$$u = N_0 erfc(\frac{x}{2a\sqrt{t}}) = N_0 \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2} dy$$

例 一条半无限长的杆，端点的温度变化为已知，杆的初始温度为零。求杆上的温度分布规律。

解：所提问题归结为解定解问题

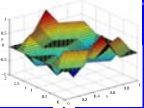
$$\begin{cases} u_t = a^2 u_{xx}, (x>0, t>0) \\ u|_{t=0} = 0, u|_{x=0} = f(t) \end{cases}$$

$$\begin{cases} \dfrac{d^2\tilde{u}}{dx^2} - \dfrac{s}{a^2}\tilde{u} = 0 \\ \tilde{u}|_{x=0} = \tilde{f} \end{cases}$$

$$\tilde{u} = \tilde{f} e^{-\frac{\sqrt{s}}{a}x}$$

$$u = L^{-1}[\tilde{f} e^{-\frac{\sqrt{s}}{a}x}] = L^{-1}[\tilde{f}] * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}]$$

$$= f(t) * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}]$$



$$L^{-1}[\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}] = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2}dy$$

$$L[f'(t)] = s\tilde{f} - f(0)$$

$$L^{-1}[e^{-\frac{x}{a}\sqrt{s}}] = L^{-1}[s\cdot\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}] = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2}dy\right]$$

$$= \frac{x}{2a\sqrt{\pi}t^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2t}}$$

$$u(x,t)=L^{-1}[F(s)e^{-\frac{x}{a}\sqrt{s}}]=\frac{x}{2a\sqrt{\pi}}\int_0^t f(\tau)(t-\tau)^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau$$

例 求解半无界弦的强迫振动定解问题为：

$$\begin{cases} u_{tt}=a^2u_{xx}+\cos\omega t,(0<x<+\infty,t>0) \\ u\big|_{x=0}=0,u_x\big|_{x\to\infty}=0 \\ u\big|_{t=0}=0,u_t\big|_{t=0}=0 \end{cases}$$

解：对自变量取Laplace变换

$$\begin{cases} s^2\tilde{u}=a^2\dfrac{d^2\tilde{u}}{dx^2}+\dfrac{s}{\omega^2+s^2} \\ \tilde{u}\big|_{x=0}=0,\tilde{u}_x\big|_{x\to\infty}=0 \end{cases}$$

$$\tilde{u}=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}+\frac{1}{s(\omega^2+s^2)}$$

$$A=0$$

$$B=-\frac{1}{s(\omega^2+s^2)}$$

$$\tilde{u} = \frac{1}{s(\omega^2 + s^2)}[1 - e^{-\frac{s}{a}x}]$$

$$L^{-1}[\tilde{u}] = L^{-1}[\frac{1}{s(\omega^2 + s^2)}] - L^{-1}[\frac{1}{s(\omega^2 + s^2)} e^{-\frac{s}{a}x}]$$

$$= \sum_k \mathrm{Res}[\frac{e^{st}}{s(\omega^2 + s^2)}, s_k] - \sum_k \mathrm{Res}[\frac{e^{s(t-\frac{x}{a})}}{s(\omega^2 + s^2)}, s_k]$$

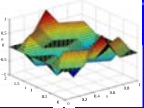
$$= \frac{1}{\omega^2} - \frac{1}{\omega^2}(\frac{e^{i\omega t} + e^{-i\omega t}}{2})$$

$$= \frac{1}{\omega^2}(1 - \cos \omega t)$$

$$= \frac{2}{\omega^2} \sin^2 \frac{\omega}{2} t$$

$$\sum_k \mathrm{Res}[\frac{e^{s(t-\frac{x}{a})}}{s(s^2+\omega^2)}, s_k] = \begin{cases} \frac{2}{\omega^2}\sin^2\frac{\omega}{2}(t-\frac{x}{a}), (t>\frac{x}{a}) \\ 0, (t \le \frac{x}{a}) \end{cases}$$

$$u = L^{-1}(\tilde{u}) = \begin{cases} \frac{2}{\omega^2}\sin^2\frac{\omega}{2}t - \frac{2}{\omega^2}\sin^2\frac{\omega}{2}(t-\frac{x}{a}), (t>\frac{x}{a}) \\ \frac{2}{\omega^2}\sin^2\frac{\omega}{2}t, (t \le \frac{x}{a}) \end{cases}$$



求解
$$\begin{cases} u_{tt} - a^2 u_{xx} = -g, (0 < x < \infty, t > 0) \\ u\big|_{x=0} = 0, u\big|_{x\to\infty} = 0 \\ u\big|_{t=0} = 0, u_t\big|_{t=0} = 0 \end{cases}$$

解：
$$\begin{cases} a^2 \dfrac{d^2\tilde{u}}{dx^2} - s^2\tilde{u} = g/s \\ \tilde{u}\big|_{x=0} = 0, \tilde{u}_x\big|_{x\to\infty} = 0 \end{cases}$$

$$\tilde{u} = -g\frac{1}{s^3}(1 - e^{-\frac{s}{a}x})$$

$$L^{-1}[\tilde{u}] = -gL^{-1}[\frac{1}{s^3}(1-e^{-\frac{s}{a}x})] = \begin{cases} -\dfrac{1}{2}gt^2 & \mathrm{t} < \dfrac{x}{a} \\ -\dfrac{1}{2a^2}gx(2at-x) & \mathrm{t} > \dfrac{x}{a} \end{cases}$$

用Laplace变换求解电报方程的定解问题：

$$\begin{cases} u_{xx} = au_{tt} + bu_t + cu, (x > 0, t > 0) \\ u\big|_{t=0} = u_t\big|_{t=0} = 0 \\ u\big|_{x=0} = \varphi(t) \qquad\qquad b = 4ac \end{cases}$$

解：

$$\begin{cases} \dfrac{d^2\tilde{u}}{dx^2} = (as^2 + bs + c)\tilde{u} = (\sqrt{a}s + \dfrac{b}{2\sqrt{a}})^2 \tilde{u} \\ \tilde{u}\big|_{x=0} = \tilde{\varphi} \end{cases}$$

$$\tilde{u} = \tilde{\varphi} e^{-(\sqrt{a}s + \frac{b}{2\sqrt{a}})x}$$

$$L^{-1}[\tilde{u}] = L^{-1}[\tilde{\varphi} e^{-(\sqrt{a}s + \frac{b}{2\sqrt{a}})x}]$$

$$= e^{-\frac{b}{2\sqrt{a}}x} L^{-1}[\tilde{\varphi} e^{-\sqrt{a}sx}]$$

$$= \begin{cases} e^{-\frac{b}{2\sqrt{a}}x} \varphi(t - \sqrt{a}x) & \mathrm{t} > \sqrt{a}x \\ 0 & \mathrm{t} < \sqrt{a}x \end{cases}$$

第六章 Green函数法

从物理上看，一个数理方程表示一种特定的场和产生这种场的源之间的关系

a. 热传导方程表示温度场和热源的关系;

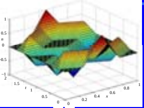
b. Poisson方程表示静电场和电荷分布的关系

Green函数意义

a. Green函数则代表一个点源所产生的场。

b. 知道了一个点源的场，就可以用叠加的方法算出任意源的场。

线性系统理论: 脉冲响应与系统输入------卷积



Laplace方程的基本解

Laplace方程： $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$

Poisson方程： $\Delta u = u_{xx} + u_{yy} + u_{zz} = -f(x, y, z)$

调和函数 $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$

Dirichlet问题（第一类边值问题）

Neumann问题（第二类边值问题） **Robin问题（第三类边值问题）**

基本解: 对于 $\Delta_3 u = f(M)$，我们把满足方程 $\Delta_3 u = \delta(M - M_0)$ 或 $\Delta_3 u = 0$ 的解称为方程的基本解 。

Laplace方程的球面坐标及柱面坐标下的形式如下：

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=0$$

空间Laplace方程的基本解（无界域上的Green函数）：

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)=0 \qquad U=\frac{C_1}{r}+C_2$$

C_1, C_2 为任意常数

$$C_1=1, C_2=0 \qquad U=\frac{1}{r} \qquad \Delta\left(\frac{1}{r}\right)=0, (r\neq 0)$$

$U=\frac{1}{r}$ 为空间Laplace方程的基本解

平面Laplace方程的基本解（或无界域上的Green函数）：

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU}{d\rho}\right)=0$$

$$U=C_1 Ln\rho+C_2$$

C_1, C_2 为任意常数

$$C_1=-1, C_2=0 \qquad U=Ln\frac{1}{\rho} \qquad \Delta\left(Ln\frac{1}{\rho}\right)=0,(\rho\neq 0)$$

$$U=Ln\frac{1}{\rho}$$ 平面Laplace方程的基本解

例 用Fourier变换法求基本解

$$\begin{cases} u_t = a^2 u_{xx}, (-\infty < x < +\infty, t > 0) \\ u\big|_{t=0} = \varphi(x) \end{cases}$$

解：此即求

$$\begin{cases} V_t = a^2 V_{xx} \\ V\big|_{t=0} = \delta(x) \end{cases}$$

$$\begin{cases} \dfrac{d\tilde{V}}{dt} = -a^2\lambda^2\tilde{V} \\ \tilde{V}\big|_{t=0} = 1 \end{cases}$$

基本解： $\tilde{V} = e^{-a^2\lambda^2 t}$

$$V = F^{-1}[\tilde{V}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t} e^{-i\lambda x} d\lambda = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

方程解(卷积):

$$u = \varphi(x) * V(x) = \int_{-\infty}^{+\infty} \varphi(\zeta) \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-\zeta)^2}{4a^2 t}} d\zeta$$

Green公式及调和函数的性质

Gauss公式

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \oiint_S \left[P\cos(n,x) + Q\cos(n,y) + R\cos(n,t) \right] d\sigma$$

第一Green公式 $\vec{A} = u\nabla v$

$$\iint_\Gamma u\nabla v \cdot d\vec{\sigma} = \iiint_\Omega \nabla \cdot (u\nabla v) dV = \iiint_\Omega u\Delta v dV + \iiint_\Omega \nabla u \cdot \nabla v dV$$

$$\iint_\Gamma v\nabla u \cdot d\vec{\sigma} = \iiint_\Omega \nabla \cdot (v\nabla u) dV = \iiint_\Omega v\Delta u dV + \iiint_\Omega \nabla v \cdot \nabla u dV$$

第二Green公式

$$\iint_{\Gamma}\left(u\nabla v - v\nabla u\right)\cdot d\vec{\sigma} = \iiint_{\Omega}\left(u\Delta v - v\Delta u\right)dV$$

第三Green公式

$$u(M_0) = \frac{1}{4\pi}\oint\!\!\!\oint_{\Gamma}\left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma - \frac{1}{4\pi}\iiint_{\Omega}\left(\frac{1}{r}\Delta u\right)dV$$

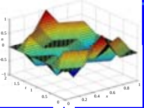
Proof $v(M)=1/r_{MM_0}$

球面 Γ_ε 以 M_0 为中心，为 ε 半径

$$\iiint\limits_{\Omega-K_\varepsilon}\left[u(M)\Delta\left(\frac{1}{r_{MM_0}}\right)-\frac{1}{r_{MM_0}}\Delta u(M)\right]dV_M$$

$$=\oiint\limits_{\Gamma}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right)-\frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right]d\sigma_M$$

$$+\oiint\limits_{\Gamma_\varepsilon}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right)-\frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right]d\sigma_M$$



$$\iiint\limits_{\Omega-K_\varepsilon} -\frac{1}{r_{MM_0}}\Delta_M u(M)\,dV_M$$

$$\iint\limits_{\Gamma_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\sigma = -\iint\limits_{\Gamma_\varepsilon} u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)d\sigma = \iint\limits_{\Gamma_\varepsilon} u\frac{1}{r^2}d\sigma$$

$$= \frac{1}{\varepsilon^2}\iint\limits_{\Gamma_\varepsilon} u\,d\sigma = 4\pi\bar{u} \xrightarrow[(\varepsilon\to 0)]{} 4\pi u\left(M_0\right)$$

$$-\iint\limits_{\Gamma_\varepsilon}\frac{1}{r}\frac{\partial u}{\partial n}d\sigma = -\frac{1}{\varepsilon}\iint\limits_{\Gamma_\varepsilon}\frac{\partial u}{\partial r}d\sigma = -4\pi\varepsilon\frac{\partial\bar{u}}{\partial r} \to 0,(\varepsilon\to 0)$$

$$\iiint\limits_{\Omega}\left(-\frac{1}{r}\Delta u\right)dV = \oiint\limits_{\Gamma}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]d\sigma + 4\pi u(M_0)$$

$$u(M_0) = \frac{1}{4\pi}\oiint\limits_{\Gamma}\left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma - \frac{1}{4\pi}\iiint\limits_{\Omega}\left(\frac{1}{r}\Delta u\right)dV$$

（二）调和函数性质　$\vec{n}$ 为 Γ 外法线方向

性质1　设$u(x,y,z)$是区域 Ω 上的调和函数，则有

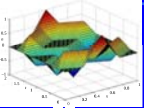
$$\iint_{\Gamma} \frac{\partial u}{\partial n} d\sigma = 0$$

证明：第二Green公式

$$\iint_{\Gamma} (u\nabla v - v\nabla u)\cdot d\vec{\sigma} = \iiint_{\Omega} (u\Delta v - v\Delta u)dV$$

取 $v \equiv 1$

$$\Delta u = 0, \Delta(1) = 0, \frac{\partial(1)}{\partial n} = 0$$



推论

$$\begin{cases} \Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma} = \varphi \end{cases}$$

有解的必要条件为:

$$\iint_{\Gamma} \varphi d\sigma = 0$$

性质2 设 $u(x,y,z)$ 是区域 Ω 上的调和函数，则有

$$u(M_0)=\frac{1}{4\pi}\iint_{\Gamma}\left[\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma$$

性质3 设 $u(x,y,z)$ 是区域 Ω 上的调和函数，则在球心的值等于它在球面上的算术平均值，即

$$u(M_0)=\frac{1}{4\pi R^2}\iint_{\Gamma_R}u(M)d\sigma$$

Γ_R 以 M_0 为球心 R为半径的球面

NOTE:

$$u(M_0) = \frac{1}{4\pi} \oiint_{\Gamma_R} \left[\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] d\sigma$$

$$= \frac{1}{4\pi} \oiint_{\Gamma_R} \frac{1}{r} \frac{\partial u}{\partial n} d\sigma - \frac{1}{4\pi} \oiint_{\Gamma_R} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) d\sigma$$

$$= \frac{1}{4\pi R} \oiint_{\Gamma_R} \frac{\partial u}{\partial n} d\sigma + \frac{1}{4\pi R^2} \oiint_{\Gamma_R} u d\sigma = \frac{1}{4\pi R^2} \oiint_{\Gamma_R} u d\sigma$$

最大值原理：

性质**4** 设 $u(x,y,z)$ 是区域 Ω 上的调和函数，$\Omega+\Gamma$ 上连续，则 $u(x,y,z)$ 的最大值和最小值只能在边界面上达到

证明

Γ 上的最大值为 M^*

$M_0=u(x_0,y_0,z_0)$ 表示 $u(x,y,z)$ 在 Ω 内的最大值

反证法

$$v(x,y,z)=u(x,y,z)+\frac{M_0-M^*}{8R^2}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]$$

$$v(x_0,y_0,z_0)=u(x_0,y_0,z_0)=M_0$$

$$v(x,y,z)\le M^*+\frac{M_0-M^*}{2}=\frac{M_0+M^*}{2}<M_0$$

$$v_{xx}+v_{yy}+v_{zz}=u_{xx}+u_{yy}+u_{zz}+\frac{3(M_0-M^*)}{4R^2}=\frac{3(M_0-M^*)}{4R^2}>0$$

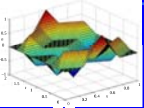
$v(x,y)$ 达到最大值时有 $v_{xx}\le 0, v_{yy}\le 0, v_{zz}\le 0$

$v_{xx}+v_{yy}+v_{zz}\le 0$ 矛盾

推论1 设在有界区域 Ω 内的调和函数，在闭区域 $\Omega+\Gamma$ 上为连续，如果还在的边界面 Γ 上为常数K，则它在内各点的值也等于常数K。

推论2 设在有界区域 Ω 内的调和函数，在闭区域 $\Omega+\Gamma$ 上为连续，如果还在的边界面 Γ 上恒为零，则它在内各点处的值都等于零。

推论3 设在有界区域 Ω 内的两个调和函数，在闭区域上 $\Omega+\Gamma$ 为连续，如果它们还在区域的边界面 Γ 上取相等的值，则它们在内所取的值也彼此相等。



（三）Laplace方程Dirichlet问题解的唯一性和稳定性

定理1（唯一性定理） **Dirichlet**内问题的解如果存在，必是唯一的。

$$\begin{cases} \Delta u_1 = 0, (\text{在}\Omega\text{内}) \\ u_1\big|_\Gamma = f \end{cases} \qquad \begin{cases} \Delta u_2 = 0, (\text{在}\Omega\text{内}) \\ u_2\big|_\Gamma = f \end{cases}$$

$$v = u_1 - u_2$$

$$v = u_1 - u_2 = f - f = 0$$

$$v = u_1 - u_2 \equiv 0$$

定理2（稳定性定理） Dirichlet内问题的解连续地依赖于所给的边界条件。

$$\begin{cases}\Delta u_1 = 0, (\text{在}\Omega\text{内}) \\ u_1|_\Gamma = f_1\end{cases}$$

$$\begin{cases}\Delta u_2 = 0, (\text{在}\Omega\text{内}) \\ u_2|_\Gamma = f_2\end{cases}$$

$$v = u_1 - u_2$$

$$\begin{cases}\Delta v = 0, (\text{在}\Omega\text{内}) \\ v|_\Gamma = f_1 - f_2\end{cases}$$

$$|f_1 - f_2| < \varepsilon$$

$$-\varepsilon < v_{\min} \le v \le v_{\max} < \varepsilon$$

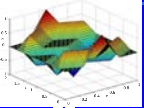
$$|u_1 - u_2| < \varepsilon$$

定理2 方程的Neumann问题的解，若不管任意常数的差别，也仍然是唯一的。

$$\begin{cases} \Delta u_1 = 0 \\ \dfrac{\partial u_1}{\partial n}\Big|_S = \varphi \end{cases} \qquad \begin{cases} \Delta u_2 = 0 \\ \dfrac{\partial u_2}{\partial n}\Big|_S = \varphi \end{cases}$$

$$u = u_1 - u_2$$

$$\begin{cases} \Delta u = \Delta(u_1 - u_2) = 0 \\ \dfrac{\partial u}{\partial n}\Big|_S = \dfrac{\partial (u_1 - u_2)}{\partial n}\Big|_S = 0 \end{cases}$$



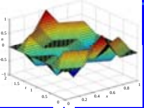
取 $u = v = u_1 - u_2$ **第一Green公式**

$$\iint\limits_{\Gamma} u\nabla v \cdot d\vec{\sigma} = \iiint\limits_{\Omega} \nabla \cdot (u\nabla v) dV = \iiint\limits_{\Omega} u\Delta v dV + \iiint\limits_{\Omega} \nabla u \cdot \nabla v dV$$

$$\iiint\limits_{V} [(\frac{\partial(u_1 - u_2)}{\partial x})^2 + (\frac{\partial(u_1 - u_2)}{\partial y})^2 + (\frac{\partial(u_1 - u_2)}{\partial z})^2] dV = 0$$

$$\frac{\partial(u_1 - u_2)}{\partial x} = \frac{\partial(u_1 - u_2)}{\partial y} = \frac{\partial(u_1 - u_2)}{\partial z}$$

$$u = u_1 - u_2 \equiv c$$



Laplace方程和Poisson方程边值问题

（一） $Lu=0$ 型方程的基本解

$$\left(L=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+2\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+c\right)$$

$Lu=\delta(M)$ 的解，称为方程 $Lu=f(M)$ 的基本解

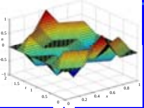
$$U(r)=-\frac{1}{4\pi r}$$

若U是一个基本解，是相应齐次方程的任一解u，则U+u仍是基本解

定理1 $Lu = f(M)$ 的解 为 $U * f = \iiint\limits_{R^3} U(M - M_0) f(M_0) dM_0$

$$L(U * f) = \iiint\limits_{R^3} LU(M - M_0) f(M_0) dM_0$$

$$= \iiint\limits_{R^3} \delta(M - M_0) f(M_0) dM_0 = f(M)$$

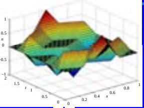


$$I_1:\begin{cases}\Delta u=-f(x,y,z),((x,y,z)\in V)\\ u\big|_S=\varphi(x,y,z),\dfrac{\partial u}{\partial n}\Big|_S=\psi(x,y,z)\end{cases}$$

$$u(M_0)=\frac{1}{4\pi}\oiint_{\Gamma}\left[\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma-\frac{1}{4\pi}\iiint_{\Omega}\left(\frac{1}{r}\Delta u\right)dV$$

$$I_1:\begin{cases}\Delta u=0,((x,y,z)\in V)\\ u\big|_S=\varphi(x,y,z),\dfrac{\partial u}{\partial n}\Big|_S=\psi(x,y,z)\end{cases}$$

$$u(M_0)=\frac{1}{4\pi}\oiint_{\Gamma}\left[\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma$$



$$I_2 : \begin{cases} \Delta G = -\delta\left(x-\xi, y-\eta, z-\zeta\right), \left((x, y, z) \in V\right) \\ G\big|_S = 0 \end{cases}$$

$$\Delta G = -\delta\left(x-\xi, y-\eta, z-\zeta\right), \left((x, y, z) \in V\right)$$

$$G = u_1 + u_2$$

$$u_1 = \frac{1}{4\pi r(M, M_0)}$$

$$r(M, M_0) = \overline{MM_0} = \sqrt{\left(x-\xi\right)^2 + \left(y-\eta\right)^2 + \left(z-\zeta\right)^2}$$

$$I_3 : \begin{cases} \Delta u_2 = 0 \\ u_2\big|_S = (G - u_1)\big|_S = -\dfrac{1}{4\pi r(M, M_0)}\Big|_{M \in S} \end{cases}$$

定理2 Green函数具有对称性(物理上称为互易性)，即

$$G(M_1;M_2)=G(M_2;M_1)$$

证：

$$\begin{cases}\Delta G(M;M_1)=-\delta\left(M-M_1\right),(M\in V)\\ G(M;M_1)=0,(M\in S)\end{cases}$$

$$\begin{cases}\Delta G(M;M_2)=-\delta\left(M-M_2\right),(M\in V)\\ G(M;M_2)=0,(M\in S)\end{cases}$$

$$\iiint\limits_V\left[G(M;M_1)\Delta G(M;M_2)-G(M;M_2)\Delta G(M;M_1)\right]dV$$
$$=\iint\limits_S\left[G(M;M_1)\frac{\partial G(M;M_2)}{\partial n}-G(M;M_2)\frac{\partial G(M;M_1)}{\partial n}\right]dS$$
$$=0,$$

$$\iiint_V G(M;M_1)\delta(M-M_2)dV=\iiint_V G(M;M_2)\delta(M-M_1)dV$$

$$G(M_2;M_1)=G(M_1;M_2)$$

定理3 $I_1:\begin{cases}\Delta u=-f(x,y,z),((x,y,z)\in V)\\ u|_S=\varphi(x,y,z)\end{cases}$ 的解的积分表达式为

$$u(M)=-\iint_S \varphi(M_0)\frac{\partial G}{\partial n}dS+\iiint_V Gf(M_0)dM_0$$

证：求边值问题

$$\begin{cases}\Delta u=0\\ u|_S=\varphi(M)\end{cases}$$

$$\begin{cases}\Delta u=-f(x,y,z),((x,y,z)\in V)\\ u|_S=0\end{cases}$$

$$\iint_{\Gamma} \left(u \nabla v - v \nabla u\right) \cdot d\vec{\sigma} = \iiint_{\Omega} \left(u \Delta v - v \Delta u\right) dV$$

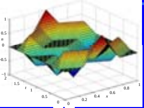
$$u(M) = \iiint_{V} \delta\left(M - M_0\right) u(M_0) dM_0$$

$$= -\iint_{S} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS - \iiint_{V} G \Delta u dM_0$$

$$u\big|_S = \varphi\left(M_0\right)$$

$$G(M; M_0)\big|_{M_0 \in S} = G(M_0; M)\big|_{M_0 \in S} = 0$$

$$u(M) = -\iint_{S} \varphi\left(M_0\right) \frac{\partial G}{\partial n} dS$$



$$\omega = \iiint\limits_V G f(M_0) dM_0$$

$$\Delta\omega = \Delta \iiint\limits_V G(M;M_0) f(M_0) dM_0$$

$$\omega(M)\big|_S = \iiint\limits_V G(M;M_0)\big|_{M \in S} f(M_0) dM_0 = 0$$

$$u(M_0) = -\iint\limits_S \varphi(M) \frac{\partial G}{\partial n} dS + \iiint\limits_V G f(M) dM$$

（三）用镜像法求Green函数

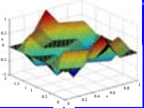
半空间的Green函数

$$\begin{cases} \Delta G = -\delta(x-\xi, y-\eta, z-\zeta), (z>0) \\ G\big|_{z=0} = 0 \end{cases}$$

$$u_1 = \frac{1}{4\pi r(M, M_0)}$$

$$u_2 = -\frac{1}{4\pi r(M, M_1)}$$

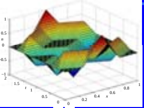
$$G = u_1 + u_2 = \frac{1}{4\pi}\left[\frac{1}{r(M, M_0)} - \frac{1}{r(M, M_1)}\right]$$



$$G\Big|_{z=0} = \frac{1}{4\pi}\left[\frac{1}{r(M,M_0)} - \frac{1}{r(M,M_1)}\right]\Bigg|_{z=0}$$

$$= \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}} - \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}\right]$$

$$= 0$$



$$I_1:\begin{cases}\Delta u=0,((x,y,z)\in V)\\ u\big|_S=\varphi(x,y,z)\end{cases}$$

$$u(M_0)=-\iint\limits_S\varphi(M)\frac{\partial G}{\partial n}dS+\iiint\limits_V Gf(M)dM$$

$$\frac{\partial G}{\partial n}=-\frac{\partial G}{\partial z}=\frac{1}{4\pi}\left[\frac{z-\zeta}{r^3(M,M_0)}-\frac{z+\zeta}{r^3(M,M_1)}\right]$$

$$\frac{\partial G}{\partial n}\Big|_{z=0}=-\frac{\zeta}{2\pi}\left[(x-\xi)^2+(y-\eta)^2+\zeta^2\right]^{-\frac{3}{2}}$$

$$I_1:\begin{cases}\Delta u=0,((x,y,z)\in V)\\ u\big|_{z=0}=\varphi(x,y,z)\end{cases}$$

$$u(\xi,\eta,\zeta)=-\iint\limits_{z=0}\varphi\frac{\partial G}{\partial n}dS$$

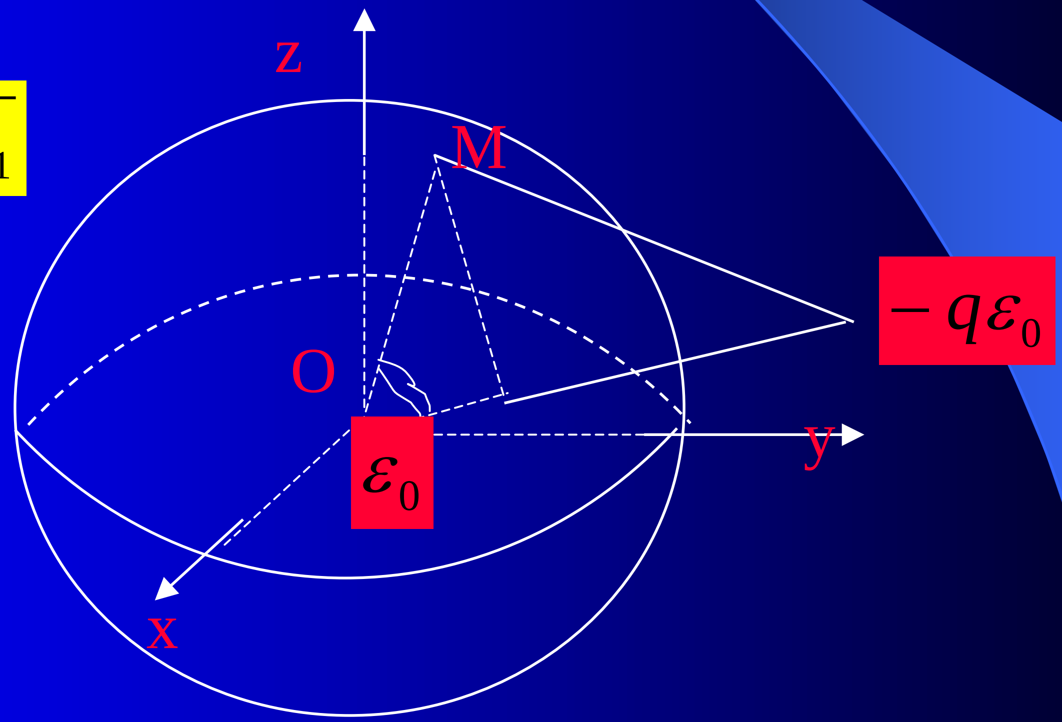
$$=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\varphi(x,y)\zeta}{\left[(x-\xi)^2+(y-\eta)^2+\zeta^2\right]^{\frac{3}{2}}}dxdy$$

2，球形域上的Green函数

$$
\begin{cases}
\Delta G = -\delta(x-\xi, y-\eta, z-\zeta), (x^2+y^2+z^2<R^2) \\
G\big|_S = 0
\end{cases}
$$

$$\rho_0 = \overline{OM}_0, \rho_1 = \overline{OM}_1$$

$$\rho_0 \cdot \rho_1 = R^2$$



$$\Delta G(M)=\frac{1}{4\pi}\left[\Delta\left(\frac{1}{r(M,M_0)}\right)-\Delta\left(\frac{q}{r(M,M_1)}\right)\right]$$
$$=-\delta\left(M-M_0\right)+q\delta\left(M-M_1\right)$$
$$=-\delta\left(M-M_0\right)$$

$$\frac{r(M,M_1)}{r(M,M_0)}=\frac{R}{\rho_0}$$

$$G(M)=\frac{1}{4\pi}\left[\frac{1}{r(M,M_0)}-\frac{q}{r(M,M_1)}\right]=0$$

$$q=\frac{R}{\rho_0}$$

$$G(M)=\frac{1}{4\pi}\left[\frac{1}{r(M,M_0)}-\frac{R}{\rho_0}\frac{1}{r(M,M_1)}\right]$$

利用Green函数求球内Dirichlet问题

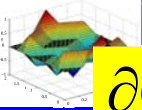
$$\begin{cases} \Delta u = 0, (x^2 + y^2 + z^2 < R^2) \\ u\big|_S = \varphi(x, y, z) \end{cases}$$

$$r(M, M_0) = \sqrt{\rho_0^2 + r^2 - 2\rho_0 r \cos\phi}$$

$$r(M, M_1) = \sqrt{\rho_1^2 + r^2 - 2\rho_1 r \cos\phi}$$

$$\rho_1 = \frac{R^2}{\rho_0}$$

$$G(M; M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho_0^2 + r^2 - 2\rho_0 r \cos\phi}} - \frac{R}{\sqrt{r^2\rho_0^2 + R^4 - 2R^2\rho_0 r \cos\phi}}\right]$$



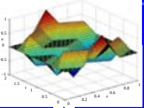
$$\frac{\partial G}{\partial n}\Big|_S=\frac{\partial G}{\partial r}\Big|_S$$

$$=-\frac{1}{4\pi}\left\{\frac{r-\rho_0\cos\phi}{\left(r^2+\rho_0^2-2\rho_0 r\cos\phi\right)^{\frac{3}{2}}}-\frac{\left(\rho_0^2 r-R^2\rho_0\cos\phi\right)R}{\left(r^2\rho_0^2+R^4-2R^2\rho_0 r\cos\phi\right)^{\frac{3}{2}}}\right\}\Big|_{r=R}$$

$$=-\frac{R^2-\rho_0^2}{4\pi R\left(R^2+\rho_0^2-2R\rho_0\cos\phi\right)^{\frac{3}{2}}}$$

定理3 $I_1:\begin{cases}\Delta u=-f(x,y,z),((x,y,z)\in V)\\ u|_S=\varphi(x,y,z)\end{cases}$ 的解的积分表达式为

$$u(M)=-\iint_S\varphi(M_0)\frac{\partial G}{\partial n}dS+\iiint_V Gf(M_0)dM_0$$



$$u(M_0) = -\iint\limits_S \frac{\partial G(M;M_0)}{\partial n}\varphi(M)dS$$

$$= \frac{1}{4\pi R}\iint\limits_S \frac{R^2-\rho_0^2}{\left(R^2+\rho_0^2-2R\rho_0\cos\phi\right)^{\frac{3}{2}}}\varphi(M)dS$$

$$u(\rho_0,\theta_0,\varphi_0)=\frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{R^2-\rho_0^2}{\left(R^2+\rho_0^2-2R\rho_0\cos\phi\right)^{\frac{3}{2}}}\varphi\left(R,\theta,\varphi\right)\sin\theta d\theta d\varphi$$

圆上的Green函数

$$\begin{cases} \Delta G = \delta(M - M_0), (x^2 + y^2 < R^2) \\ G\big|_S = 0 \end{cases}$$

$$v = Ln\frac{1}{r_{MM_0}} - Ln\frac{1}{r_{MM_1}} = Ln\frac{r_{MM_1}}{r_{MM_0}}$$

$$v\big|_L = Ln\frac{r_{MM_1}}{r_{MM_0}}\bigg|_L = Ln\frac{R}{r_0}$$

$$G(M, M_0) = \frac{1}{2\pi}\left(Ln\frac{1}{r_{MM_0}} - Ln\frac{1}{r_{MM_1}} - Ln\frac{R}{r_0}\right) = \frac{1}{2\pi}\left[Ln\frac{1}{r_{MM_0}} - Ln\left(\frac{R}{r_0}\frac{1}{r_{MM_1}}\right)\right]$$

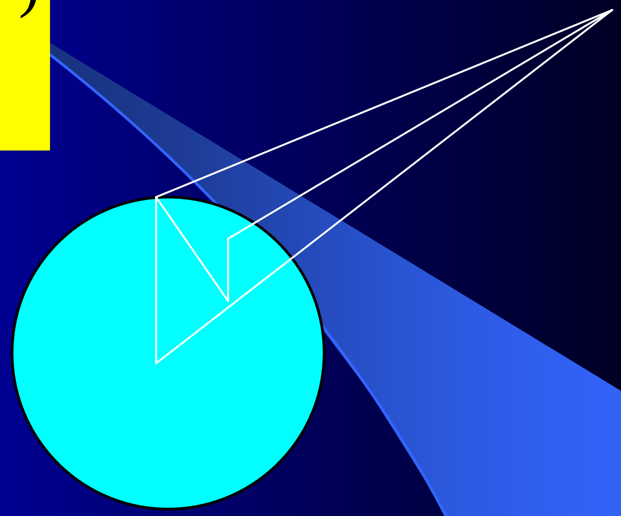
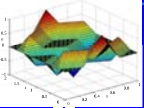


图 5.5.3



$$\begin{cases} \Delta u = 0, (x^2 + y^2 < R^2) \\ u\big|_S = \varphi(x, y) \end{cases}$$

$$\frac{1}{r_{MM_0}} = \frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\gamma}}$$

$$\frac{1}{r_{MM_1}} = \frac{1}{\sqrt{r_1^2 + r^2 - 2r_1 r\cos\gamma}}$$

$$\frac{\partial G}{\partial n}\Big|_L = \frac{\partial G}{\partial r}\Big|_{r=R} = \frac{1}{2\pi}\frac{\partial}{\partial r}\left\{ Ln\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\gamma}} - Ln\frac{R}{\sqrt{r_0^2 r^2 - 2R^2 r_0 r\cos\gamma + R^4}} \right\}_{r=R}$$

$$= \frac{-1}{2\pi}\left\{ \frac{r - r_0\cos\gamma}{r^2 + r_0^2 - 2rr_0\cos\gamma} - \frac{r_0^2 r - R^2 r_0\cos\gamma}{r_0^2 r^2 - 2R^2 r_0 r\cos\gamma + R^4} \right\}_{r=R}$$

$$= -\frac{1}{2\pi R}\frac{R^2 - r_0^2}{R^2 - 2Rr_0\cos\gamma + r_0^2}$$

$$u(M_0)=\frac{1}{2\pi R}\int_{x^2+y^2=R^2}\frac{R^2-r_0^2}{R^2-2Rr_0\cos\gamma+r_0^2}f(M)ds$$

$$u(r_0,\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r_0^2)f(\theta)}{R^2-2Rr_0\cos(\theta-\theta_0)+r_0^2}d\theta$$

半平面上的Green函数

$$G(M, M_0) = Ln\frac{1}{r_{MM_0}} - Ln\frac{1}{r_{MM_1}}$$

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial y}$$

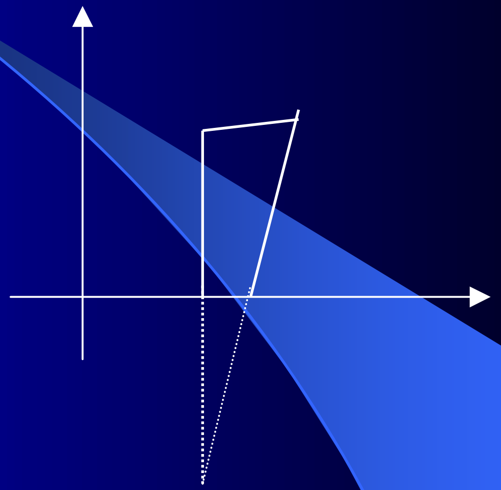
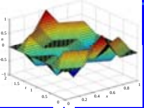


图 5.5.4

$$\frac{\partial G}{\partial n}\Big|_L = -\frac{\partial}{\partial y}\left[Ln\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} - Ln\frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}}\right]\Bigg|_{y=0}$$

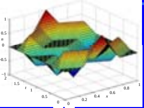
$$= -\frac{2y_0}{(x-x_0)^2+y_0^2}$$



$$\begin{cases} \Delta u = 0, (z > 0) \\ u\big|_{z=0} = \varphi(x, y) \end{cases}$$

$$u(M_0) = -\int_L \varphi(M) \frac{\partial}{\partial n} G(M, M_0) dS$$

$$u(x_0, y_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y_0}{(x - x_0)^2 + y_0^2} f(x) dx$$



热传导方程Cauchy问题的基本解解法

$$II_1:\begin{cases}\dfrac{\partial u}{\partial t}=Lu,(t>0,-\infty<x,y,z<+\infty)\\ u(0,x,y,z)=\delta(x,y,z)\end{cases}$$

的解 $\boxed{U(t,x,y,z)}$ 称为Cauchy问题

$$II_2:\begin{cases}\dfrac{\partial u}{\partial t}=Lu+f(t,x,y,z)\\ u(0,x,y,z)=\varphi(x,y,z)\end{cases}$$

的基本解 ,也称为热传导方程Cauchy问题的Green函数

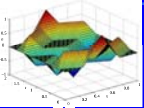
定理： II_2 的解为

$$u(t,x,y,z)=U(t,M)*\varphi(M)+\int_0^t U(t-\tau,M)*f(\tau,M)d\tau$$

$$=\iiint\limits_{R^3} U(t,x-\xi,y-\eta,z-\zeta)\varphi\left(\xi,\eta,\zeta\right)d\xi d\eta d\zeta$$

$$+\int_0^t d\tau\iiint\limits_{R^3} U(t-\tau,x-\xi,y-\eta,z-\zeta)f(\tau,\xi,\eta,\zeta)d\xi d\eta d\zeta$$

证： II_2 分解为



（1）齐次方程定解问题：

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu \\ u(0,x,y,z) = \varphi(x,y,z) \end{cases}$$

（2）齐次初始条件定解问题：

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(t,x,y,z) \\ u(0,x,y,z) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = L\omega, \quad （t > \tau） \\ \omega\big|_{t=\tau} = f（\tau，M） \end{cases}$$

先证 $v=U(t,M)*\varphi(M)$

$$\begin{cases}\dfrac{\partial v}{\partial t}=Lv\\ v\big|_{t=0}=\varphi(M)\end{cases}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}[U(t,M)*\varphi(M)]=[\frac{\partial}{\partial t}U(t,M)]*\varphi(M)$$

$$=LU(t,M)*\varphi(M)=L[U(t,M)*\varphi(M)]$$

$$=Lv$$

$$v\big|_{t=0}=U(0,M)*\varphi(M)=\delta(M)*\varphi(M)$$

$$=\varphi(M)$$

再证 $\omega(t,\ M)=U\left(t-\tau,\ m\right)*f\left(\tau,\ M\right)$ 满足

$$\begin{cases}\dfrac{\partial\omega}{\partial t}=L\omega, & (t>\tau)\\ \omega\big|_{t=\tau}=f(\tau,\ M)\end{cases}$$

$$\begin{cases}\dfrac{\partial u}{\partial t}=Lu+f(t,M)\\ u\big|_{t=0}=0\end{cases}$$

$$u=\int_0^t U(t-\tau,M)*f(\tau,M)\,d\tau$$

例 用Fourier变换法求基本解

$$\begin{cases} u_t = a^2 u_{xx}, (-\infty < x < +\infty, t > 0) \\ u|_{t=0} = \varphi(x) \end{cases}$$

解：此即求

$$\begin{cases} V_t = a^2 V_{xx} \\ V|_{t=0} = \delta(x) \end{cases}$$

$$\begin{cases} \dfrac{d\tilde{V}}{dt} = -a^2\lambda^2\tilde{V} \\ \tilde{V}|_{t=0} = 1 \end{cases}$$

基本解：

$$\tilde{V} = e^{-a^2\lambda^2 t}$$

$$V = F^{-1}[\tilde{V}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t} e^{-i\lambda x} d\lambda = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

方程解(卷积):

$$u = \varphi(x) * V(x) = \int_{-\infty}^{+\infty} \varphi(\zeta) \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-\zeta)^2}{4a^2 t}} d\zeta$$

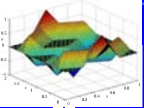
例2 求三维热传导方程Cauchy问题的基本解，即解定解问题

$$\begin{cases} U_t = a^2 \Delta U, (t > 0, -\infty < x, y, z < +\infty) \\ U(0, x, y, z) = \delta(x, y, z) \end{cases}$$

解：对空间坐标作**Fourier**变换

$$\begin{cases} \dfrac{d\tilde{U}}{dt} = a^2[(i\lambda)^2 + (i\mu)^2 + (i\nu)^2]\tilde{U} = -a^2\rho^2\tilde{U} \\ \tilde{U}\big|_{t=0} = 1 \end{cases}$$

$$\tilde{U}(t, \lambda, \mu, \nu) = \exp\{-a^2\rho^2 t\}$$



$$U(t,x,y,z) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\exp\{-a^2\lambda^2 t + i\lambda x\}d\lambda \cdot\int_{-\infty}^{+\infty}\exp\{-a^2\mu^2 t + i\mu y\}d\mu\cdot\int_{-\infty}^{+\infty}\exp\{-a^2\nu^2 t + i\nu z\}d\nu$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{-a^2\lambda^2 t + i\lambda x\}d\lambda = \frac{1}{2a\sqrt{\pi t}}\exp\{-\frac{x^2}{4a^2 t}\}$$

$$U(t,x,y,z) = \left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \exp\left\{-\frac{x^2+y^2+z^2}{4a^2 t}\right\}$$

波动方程Cauchy问题的基本解

定义：定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu, (-\infty < x, y, z < +\infty, t > 0) \\ u(0, M) = 0, u_t(0, M) = \delta(M) \end{cases}$$

$$III_1 : \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu + f(t, M), \left(t > 0, M \in R^3\right) \\ u(0, M) = \varphi(M), u_t(0, M) = \phi(M) \end{cases}$$

基本解

定理：$\boxed{III_1}$ 的解为

$$u(t,M)=\frac{\partial}{\partial t}\left[U(t,M)*\varphi(M)\right]+U(t,M)*\phi(M)+\int_0^t U(t-\tau,M)*f(\tau,M)\,d\tau$$

（1）齐次方程定解问题：

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=Lu\\ u(0,M)=\varphi(M),\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=0\end{cases}$$

（2）齐次初始条件定解问题：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu \\ u(0,M) = 0, \dfrac{\partial u}{\partial t}\Big|_{t=0} = \phi\left(M\right) \end{cases}$$

（3）齐次初始条件定解问题：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu + f\left(t, M\right) \\ u\Big|_{t=0} = 0, \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

例　求三维波动方程Cauchy问题的基本解，即解定解问题

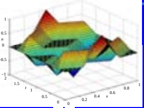
$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} = a^2 \Delta U, \quad -\infty < x, y, z < +\infty, t > 0 \\ U(0, x, y, z) = 0 \\ U_t(0, x, y, z) = \delta(x, y, z) \end{cases}$$

解：作Fourier变换

$$\begin{cases} \dfrac{d^2 \tilde{U}}{dt^2} = -a^2 \rho^2 \tilde{U}, (\rho^2 = \lambda^2 + \mu^2 + \nu^2) \\ \tilde{U}(0, \lambda, \mu, \nu) = 0, \tilde{U}_t(0, \lambda, \mu, \nu) = 1 \end{cases}$$

$$\tilde{U} = \frac{\sin a\rho t}{a\rho}$$

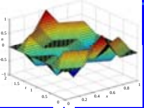
$$U(t, x, y, z) = \frac{1}{(2\pi)^3} \iiint_{R^3} \tilde{U} \exp\{i(\lambda x + \mu y + \nu z)\} d\lambda d\mu d\nu$$



$$= \left(\frac{1}{2\pi}\right)^3 \iiint_{R^3} \frac{\sin a\rho t}{a\rho} \exp\{i(\lambda x + \mu y + \nu z)\} d\lambda d\mu d\nu$$

$$\begin{cases} \lambda = \rho \sin\theta \cos\varphi \\ \mu = \rho \sin\theta \sin\varphi \\ \nu = \rho \cos\theta \end{cases}$$

$$U(t,x,y,z) = \left(\frac{1}{2\pi}\right)^3 \int_0^{+\infty} \frac{\sin a\rho t}{a\rho} \rho^2 d\rho \int_0^{2\pi} d\varphi \int_0^{\pi} \exp\{i\rho r \cos\theta\} \sin\theta d\theta$$



$$=\frac{1}{8\pi^2 ar}\int_{-\infty}^{+\infty}[\cos\rho(r-at)-\cos\rho(r+at)]d\rho$$

$$=\frac{1}{8\pi^2 ar}\int_{-\infty}^{+\infty}[\frac{1}{2}e^{\rho(r-at)j}+\frac{1}{2}e^{-\rho(r-at)j}-\frac{1}{2}e^{\rho(r+at)j}-\frac{1}{2}e^{-\rho(r+at)j}]d\rho$$

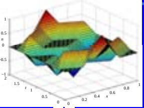
$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\lambda x}d\lambda$$

$$U(t,x,y,z)=\frac{1}{4\pi ar}[\delta(r-at)-\delta(r+at)]$$

第六章 Bessel函数

例 设有半径为的薄圆盘，其侧面绝缘，若圆盘边界上的温度恒保持为零度，且初始温度为已知。求圆盘内的瞬时温度分布规律

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \left(x^2 + y^2 < R^2\right) \\
u\big|_{t=0} = \varphi\left(x, y\right) \\
u\big|_{x^2+y^2=R^2} = 0
\end{cases}
$$



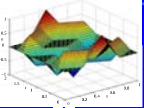
$$u(x,y,t)=V(x,y)T(t)$$

$$T'(t)+a^2\lambda T(t)=0$$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\lambda V=0$$

$$T(t)=Ae^{-a^2\lambda t}$$

$$V\Big|_{x^2+y^2=R^2}=0$$



$$\begin{cases} \dfrac{\partial^2 V}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial V}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial \theta^2} + \lambda V = 0, (\rho < R) \\ V\Big|_{\rho=R} = 0 \end{cases}$$

$$V(\rho,\theta) = P(\rho)\Theta(\theta)$$

$$\Theta''(\theta) + \mu\Theta(\theta) = 0$$

$$\rho^2 P''(\rho) + \rho P'(\rho) + (\lambda\rho^2 - \mu)P(\rho) = 0$$

$$\Theta(\theta) = \Theta(2\pi + \theta)$$

$$\mu_n = n^2$$

$$\Theta_0(\theta) = \frac{a_0}{2}(\text{为常数})$$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, (n = 1, 2, \cdots)$$

$$\rho^2 P''(\rho) + \rho P'(\rho) + (\lambda\rho^2 - n^2)P(\rho) = 0$$

$$\begin{cases} P(R) = 0 \\ |P(0)| < +\infty \end{cases}$$

$$r = \sqrt{\lambda}\rho$$

$$r^2 F''(r) + rF'(r) + (r^2 - n^2)F(r) = 0$$

例2）在圆柱内传播的电磁波问题。设沿方向均匀的电磁波在底半径为1的圆柱域内传播，在侧面沿法方向导数为零，从静止状态开始传播，初速为$1-\rho^2$。求其传播规律（假设对极角 θ 对称）

$$\begin{cases} u_{tt}=a^2\left(u_{\rho\rho}+\dfrac{1}{\rho}u_\rho\right),(t>0,0<\rho<1) \\ u_\rho\big|_{\rho=1}=0,u\big|_{\rho=0}<+\infty \\ u\big|_{t=0}=0,u_t\big|_{t=0}=1-\rho^2 \end{cases}$$

$$u=R(\rho)T(t)$$

$$T''(t)+\lambda a^2T(t)=0$$

$$\begin{cases} \rho^2R''+\rho R'+\lambda\rho^2R=0 \\ R'\big|_{\rho=1}=0,R\big|_{\rho=0}<+\infty \end{cases}$$

（二）Bessel方程的求解

n 阶Bessel方程为

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

n为任意实数和复数

在讨论时，不妨暂先假定 $n \ge 0$

设方程有一个级数解形式为

$$y=x^c(a_0+a_1x+a_2x^2+\cdots a_kx^k+\cdots)$$

$$=\sum_{k=0}^{\infty}a_kx^{c+k} \qquad a_0\neq 0$$

代入得 $$\sum_{k=0}^{\infty}\left\{\left[(c+k)(c+k-1)+(c+k)+\left(x^2-n^2\right)\right]a_k x^{c+k}\right\}=0$$

化简后写成

$$\left(c^2-n^2\right)a_0x^c+\left[(c+1)^2-n^2\right]a_1x^{c+1}+\sum_{k=2}^{\infty}\left\{\left[(c+k)^2-n^2\right]a_k+a_{k-2}\right\}x^{c+k}=0$$

$$a_0(c^2-n^2)=0$$

从而得下列各式：

$$a_1\left[(c+1)^2-n^2\right]=0$$

$$\left[(c+k)^2-n^2\right]a_k+a_{k-2}=0,(k=2,3,\cdots)$$

得 $c = \pm n$ $a_1 = 0$

暂取 $c = n$

$$a_k = \frac{-a_{k-2}}{k(2n+k)}$$

$$a_1 = a_3 = a_5 = a_7 = \cdots = 0$$

$$a_2 = \frac{-a_0}{2(2n+2)}, a_4 = \frac{a_0}{2\cdot 4(2n+2)(2n+4)}, a_6 = \frac{-a_0}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)}$$

. . ..

$$a_{2m} = (-1)^m \frac{a_0}{2\cdot 4\cdot 6\cdots 2m(2n+2)(2n+4)\cdots(2n+2m)}$$

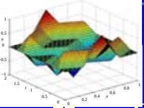
$$=\frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$$

一般项为 $$(-1)^m \frac{a_0 x^{n+2m}}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$$

选取 $$a_0=\frac{1}{2^n\Gamma(n+1)}$$

$$a_{2m}=(-1)^m\frac{1}{2^{n+2m} m!\Gamma(n+m+1)}$$

$$y_1=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m} m!\Gamma(n+m+1)},(n\geq 0)$$



n阶第一类Bessel函数

$$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}m!\Gamma(n+m+1)},(n\geq 0)$$

$$\Gamma(n+m+1)=(n+m)!,n\in Z^+$$

$$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}m!(n+m)!},(n=0,1,2,\cdots),n\in Z^+$$

取时 $c=-n$，用同样方法可得另一特解

$$J_{-n}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{-n+2m}}{2^{-n+2m}m!\,\Gamma(-n+m+1)},(n\neq 1,2,\cdots)$$

1, 当 n 不为整数时

$$J_n(x)\approx\frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n\to 0$$

$$J_{-n}(x)\approx\frac{1}{\Gamma(-n+1)}\left(\frac{x}{2}\right)^{-n}\to\infty$$

$$y=AJ_n(x)+BJ_{-n}(x)$$

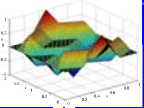
例3）试证半奇阶Bessel函数

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{\frac{1}{2}}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{\frac{1}{2}+2m}}{2^{\frac{1}{2}+2m}m!\,\Gamma(\frac{1}{2}+m+1)}$$

$$\Gamma(\frac{3}{2}+m)=\frac{1\cdot3\cdot5\cdot\cdots\cdot(2m+1)}{2^{m+1}}\sqrt{\pi}$$

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}=\sqrt{\frac{2}{\pi x}}\sin x$$



2， 当 n 为整数时

$$J_0(x)=1-(\frac{x}{2})^2+\frac{1}{(2!)^2}(\frac{x}{2})^4-\frac{1}{(3!)^2}(\frac{x}{2})^6+\cdots$$

$$J_1(x)=\frac{x}{2}-\frac{1}{2!}(\frac{x}{2})^3+\frac{1}{2!3!}(\frac{x}{2})^5-\cdots$$

$$J_{-n}(x)=(\frac{x}{2})^{-n}\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{x}{2})^{2m}}{m!\Gamma(m-n+1)}=(\frac{x}{2})^{-n}\sum_{m=0}^{\infty}(-1)^{n+l}\frac{(\frac{x}{2})^{2l+2n}}{(n+l)!l!}$$

$$=(-1)^n(\frac{x}{2})^n\sum_{l=0}^{\infty}(-1)^l\frac{(\frac{x}{2})^{2l}}{l!\Gamma(m+l)!}=(-1)^nJ_n(x)$$

定义第二类Bessel函数为

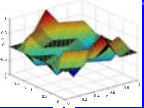
$$Y_n(x)=\lim_{\alpha\to n}\frac{J_\alpha(x)\cos\alpha\pi-J_{-\alpha}(x)}{\sin\alpha\pi}$$

它既满足Bessel方程，又与 $J_n(x)$ 线性无关

$$Y_0(x)=\frac{2}{\pi}J_0(x)(Ln\frac{x}{2}+c)-\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{2m}}{(m!)^2}\sum_{k=0}^{m-1}\frac{1}{k+1}$$

$$Y_n(x)=\frac{2}{\pi}J_n(x)(Ln\frac{x}{2}+c)-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}(\frac{x}{2})^{-n+2m}$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{n+2m}}{m!(n+m)!}(\sum_{k=0}^{n+m-1}\frac{1}{k+1}+\sum_{k=0}^{m-1}\frac{1}{k+1})$$



$$c = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) = 0.5772\cdots$$

不论n是否为整数，Bessel方程的通解都可表示为

$$y = AJ_n(x) + BY_n(x)$$

n

§6.2 Bessel函数的母函数及递推公式

（一）Bessel函数的母函数（生成函数）

$$e^{\frac{x}{2}z} = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^k}{k!} z^k$$

$$e^{-\frac{x}{2}z^{-1}} = \sum_{l=0}^{\infty} \frac{(\frac{x}{2})^l}{l!} (-z)^{-l}$$

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l}{k!\,l!} (\frac{x}{2})^{k+l} z^{k-l}$$

$$= \sum_{n=-\infty}^{\infty} [\sum_{l=0}^{\infty} \frac{(-1)^l}{(n+l)!\,l!} (\frac{x}{2})^{2l+n}] z^n = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

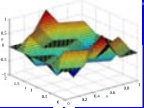
Bessel函数的母函数 $e^{\frac{x}{2}(z-\frac{1}{z})}$

令 $z = ie^{i\theta}$

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x) i^n e^{in\theta}$$

$$= J_0(x) + \sum_{n=1}^{\infty}[J_n(x) i^n e^{in\theta} + J_{-n}(x) i^{-n} e^{-in\theta}]$$

$$= J_0(x) + 2\sum_{n=1}^{\infty} i^n J_n(x) \cos n\theta$$



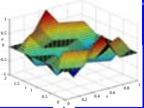
$$\cos(x\cos\theta) = J_0(x) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(x)\cos 2m\theta$$

$$\sin(x\cos\theta) = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(x)\cos(2m+1)\theta$$

$$(x^n J_n(x))' = x^n J_{n-1}(x)$$

例1） 用母函数证明：

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x) J_{n-k}(y)$$



把母函数x换成 x+y

$$\sum_{n=-\infty}^{+\infty} J_n(x+y)\zeta^n = \exp\left\{\frac{x+y}{2}(\zeta-\zeta^{-1})\right\}$$

$$= \sum_{k=-\infty}^{+\infty} J_k(x)\zeta^k \cdot \sum_{m=-\infty}^{+\infty} J_m(y)\zeta^m$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_k(x)J_m(y)\zeta^{k+m}$$

$$= \sum_{n=-\infty}^{+\infty}\left\{\sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)\right\}\zeta^n$$

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)$$

（二）Bessel函数的积分表达式

罗朗展式的系数公式 $$J_n(x)=\frac{1}{2\pi i}\int_{C}\frac{e^{\frac{x}{2}(\zeta-\frac{1}{\zeta})}}{\zeta^{n+1}}d\zeta$$

取 C 为单位圆

$$J_n(x)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}e^{ix\sin\theta}(e^{i\theta})^{-n-1}ie^{i\theta}d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-n\theta)}d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta-n\theta)d\theta,(n=0,\pm1,\pm2,\cdots)$$

（三）Bessel函数的递推公式

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots + (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \cdot 2!} + \frac{x^5}{2^5 \cdot 2!3!} - \frac{x^7}{2^7 \cdot 3!4!} + \cdots + (-1)^k \frac{x^{2k+1}}{2^{2k+1} k!(k+1)!} + \cdots$$

$$\frac{d}{dx}(-1)^{k+1} \frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2} = -(-1)^k \frac{(2k+2)x^{2k+1}}{2^{2k+2}[(k+1)!]^2} = -(-1)^k \frac{x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

$$\frac{d}{dx}[xJ_1(x)] = \frac{d}{dx}[\frac{x^2}{2} - \frac{x^4}{2^3 \cdot 2!} + \cdots + (-1)^k \frac{x^{2k+2}}{2^{2k+1}k!(k+1)!} + \cdots]$$

$$= x - \frac{x^3}{2^2} + \cdots + (-1)^k \frac{x^{2k+1}}{2^{2k}(k!)^2} + \cdots$$

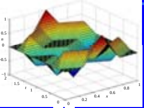
$$= x[1 - \frac{x^2}{2^2} + \cdots + (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} + \cdots]$$

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$$

$$\frac{d}{dx}[x^n J_n(x)] = \frac{d}{dx}\sum_{m=0}^{\infty}(-1)^m \frac{x^{2n+2m}}{2^{n+2m}m!\Gamma(n+m+1)}$$

$$= x^n \sum_{m=0}^{\infty}(-1)^m \frac{x^{n+2m-1}}{2^{n+2m-1}m!\Gamma(n+m)}$$

$$= x^n J_{n-1}(x)$$



$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

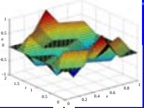
$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x), \quad xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x),$$

Bessel函数的递推公式

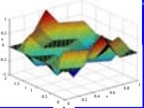
$$J_{n-1}(x) + J_{n+1}(x) = \frac{2}{x} nJ_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$



第二类Bessel函数递推公式

$$
\begin{cases}
\dfrac{d}{dx}[x^n Y_n(x)] = x^n Y_{n-1}(x) \\
\dfrac{d}{dx}[x^{-n} Y_n(x)] = -x^{-n} Y_{n+1}(x) \\
Y_{n-1}(x) + Y_{n+1}(x) = \dfrac{2n}{x} Y_n(x) \\
Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x)
\end{cases}
$$



例 利用递推公式求: $J_{\frac{3}{2}}(x)$

$$J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)-J_{-\frac{1}{2}}(x)$$

$$=-\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\sin x}{x}\right)$$

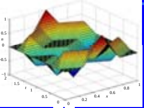
例3）计算：$\int x^3J_0(x)dx$ $\qquad (x^nJ_n(x))'=x^nJ_{n-1}(x)$

$$\int x^3J_0(x)dx=\int x^2(xJ_0(x))dx=x^3J_1(x)-2\int x^2J_1(x)dx$$

$$=x^3J_1(x)-2x^2J_2(x)+C$$

Note:

1. 27日考试;
2. 作业上交:
3. 答疑时间:
4. 认真备考.



Thank You !