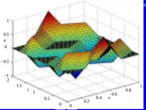


数理方程与特殊函数

李明奇

Email: 2725923590@qq.com

Office: 科A210

数学科学学院

第四章 行波法

分离变量法:

a. 求解有限域内定解问题

b. 求解的区域很规则（边界用只含一个坐标变量的方程表示）

c. 对三种典型的方程均可运用

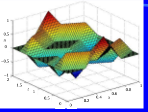
行波法(又称为特征线法):

a. 只能求解无界域内*波动方程*的定解问题——Cauchy问题

b. 它的解是达朗贝尔公式

c. 解法不能随意的扩大到一般的偏微分方程

d. 可以求出偏微分方程的通解

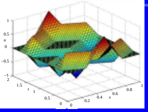


§ 4.1 一维波动方程的达朗贝尔公式

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, (-\infty < x < +\infty) \\ u\big|_{t=0} = \varphi(x), \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases}$$



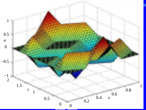
$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}=\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial^2 u}{\partial \xi \partial \eta}+\frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta}=0$$

$$u(x,t)=\int f(\xi)d\xi+f_2(\eta)=f_1(x+at)+f_2(x-at)$$

$$\begin{cases} u\big|_{t=0}=\varphi(x) \\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=\psi(x) \end{cases} \qquad \begin{cases} f_1(x)+f_2(x)=\varphi(x) \\ af_1'(x)-af_2'(x)=\psi(x) \end{cases}$$



$$f_1(x)-f_2(x)=\frac{1}{a}\int_{0}^{x}\psi(\xi)d\xi+C$$

$$f_1(x)=\frac{1}{2}\varphi(x)+\frac{1}{2a}\int_{0}^{x}\psi(\xi)d\xi+\frac{C}{2}$$

$$f_2(x)=\frac{1}{2}\varphi(x)-\frac{1}{2a}\int_{0}^{x}\psi(\xi)d\xi-\frac{C}{2}$$

无限长弦自由振动的达朗贝尔（**DALEMBERT**）公式

$$u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\varphi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

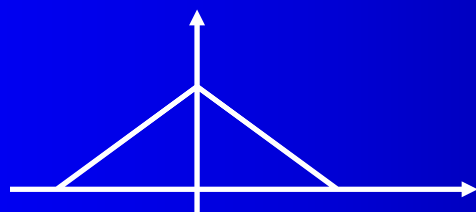
达朗贝尔公式在区间上的平均值形式

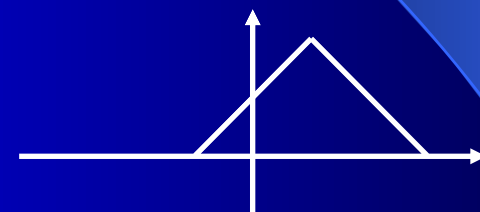
$$u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\varphi(x-at)\right]+t\frac{1}{2at}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

物理意义

$$u_1 = f_1(x)$$

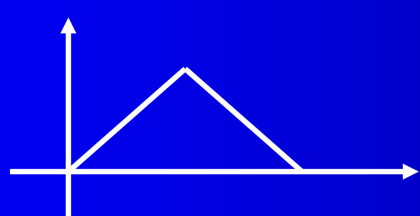
$$u_1 = f_1(x-at)$$

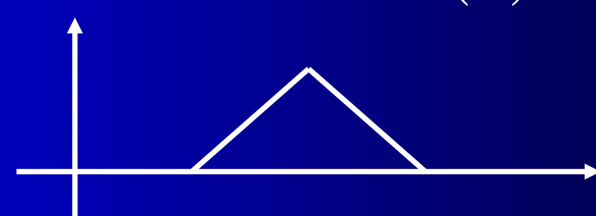




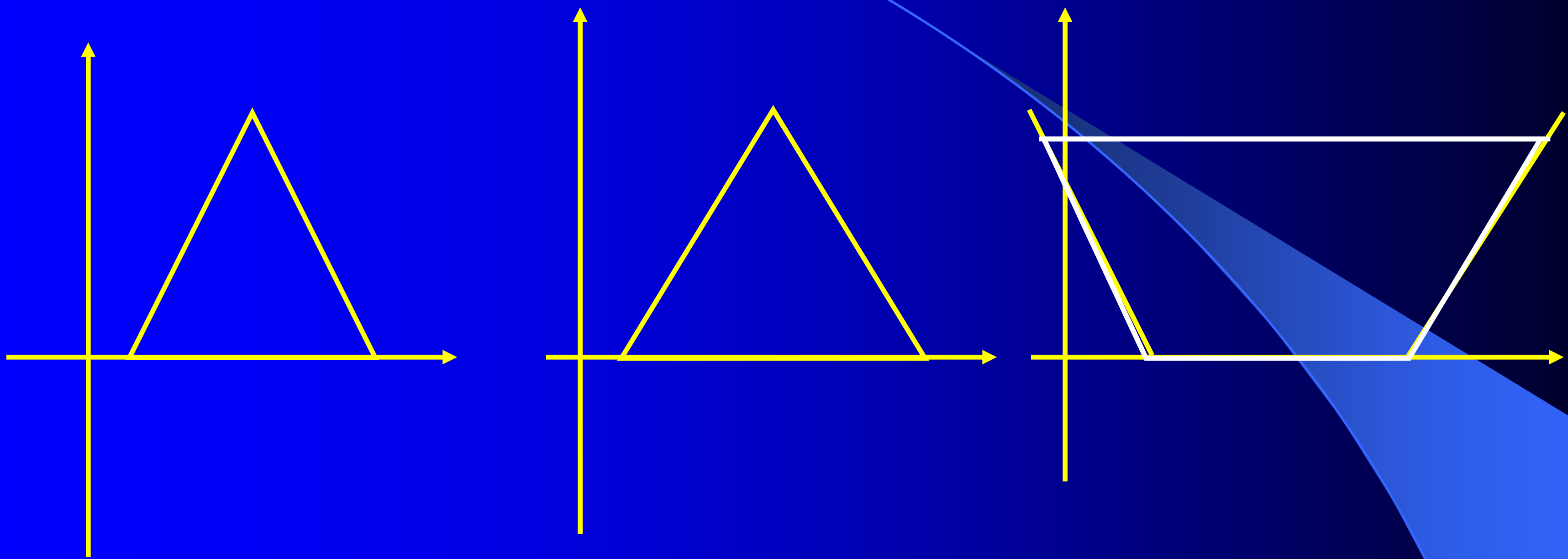
$$u_2 = f_2(x+at)$$

$$u_2 = f_2(x)$$



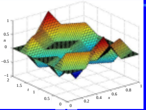


点的依赖区间 $[x-at, x+at]$



$[x_1, x_2]$ 区间的决定区域

$[x_1, x_2]$ 影响的区域



例 求Cauchy问题的解
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{y=0} = 3x^2, \dfrac{\partial u}{\partial y}\Big|_{y=0} = 0 \end{cases}$$

解：特征方程 $(dy)^2 - 2dxdy - 3(dx)^2 = 0$

$3x - y = C_1, x + y = C_2$

特征变换 $\begin{cases} \xi = 3x - y \\ \eta = x + y \end{cases}$

变换原方程化成标准型： $u_{\xi\eta} = 0$

通解为：

$$u = f_1(\xi) + f_2(\eta) = f_1(3x - y) + f_2(x + y)$$

代入条件

$$\begin{cases} f_1(3x) + f_2(x) = 3x^2 \\ -f_1'(3x) + f_2'(x) = 0 \end{cases}$$

$$\begin{cases} f_1(x) = \dfrac{1}{4}x^2 - C' \\ f_2(x) = \dfrac{3}{4}x^2 + C' \end{cases}$$

$$u(x,y) = \frac{1}{4}(3x-y)^2 + \frac{3}{4}(x+y)^2 = 3x^2 + y^2$$

例 无限长静止弦在点$x=x_0$受到冲击，冲量I，弦的密度为ρ。试求解弦的振动。

弦所受的总冲量为：

$$\int_{-\infty}^{+\infty} \rho u_t(x,0)\,dx = I \qquad \int_{-\infty}^{+\infty} \frac{\rho}{I} u_t(x,0)\,dx = 1$$

动量定理

$$\frac{\rho}{I} u_t(x,0) = \delta(x-x_0)$$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, (-\infty < x < +\infty) \\ u\big|_{t=0} = 0, u_t\big|_{t=0} = \frac{I}{\rho}\delta(x-x_0) \end{cases}$$

$$u(x,t)=\frac{1}{2a}\int_{x-at}^{x+at}\frac{I}{\rho}\delta\left(s-x_0\right)ds$$

$$=\frac{I}{2a\rho}\int_{x-x_0-at}^{x-x_0+at}\delta(\xi)d\xi=\frac{I}{2a\rho}H(\xi)\Big|_{x-x_0-at}^{x-x_0+at}$$

$$=\frac{I}{2a\rho}\left[H\left(x-x_0+at\right)-H\left(x-x_0-at\right)\right]$$

非齐次方程的Cauchy问题

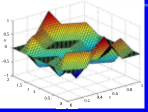
1. 求解

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = \varphi(\mathrm{x}), u_t(x,0) = \psi(\mathrm{x}) \left(-\infty < x < \infty\right) \end{cases}$$

等价于求解

$$\begin{cases} u_{tt} = a^2 u_{xx}, \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = \varphi(\mathrm{x}), u_t(x,0) = \psi(\mathrm{x}), \left(-\infty < x < \infty\right) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = 0, u_t(x,0) = 0, \left(-\infty < x < \infty\right) \end{cases}$$



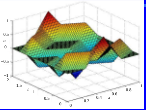
2. 求解

$$\begin{cases} u_{tt} = a^2 \Delta u + f(x,y,z,t), \left(-\infty < x,y,z < +\infty, t > 0\right) \\ u(x,0) = \varphi(x,y,z), u_t(x,0) = \psi(x,y,z) \end{cases}$$

等价于求解

$$\begin{cases} u_{tt} = a^2 \Delta u, \left(-\infty < x,y,z < +\infty, t > 0\right) \\ u(x,0) = \varphi(x,y,z), u_t(x,0) = \psi(x,y,z) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 \Delta u + f(x,y,z,t), \left(-\infty < x,y,z < +\infty, t > 0\right) \\ u(x,0) = 0, u_t(x,0) = 0 \end{cases}$$



求解
$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = 0, u_t(x,0) = 0 \left(-\infty < x < \infty\right) \end{cases}$$

1. 齐次化原理

$$\begin{cases} w_{tt} = a^2 w_{xx}, \left(-\infty < x < \infty, t > 0\right) \\ w\big|_{t=\tau} = 0, w_t\big|_{t=\tau} = f(x,\tau), \left(-\infty < x < \infty\right) \end{cases}$$

$$u(x,t) = \int_0^t w(x,t,\tau)\, d\tau$$

2. 自变量替换 $t' = t - \tau$

$$\begin{cases} w_{t't'} = a^2 w_{xx}, \left(t' > 0, -\infty < x < \infty\right) \\ w\big|_{t'=0} = 0, w_{t'}\big|_{t'=0} = f(x,\tau), \left(-\infty < x < \infty\right) \end{cases}$$

$$w(x,t',\tau) = \frac{1}{2a}\int_{x-at'}^{x+at'} f(\alpha,\tau)\,d\alpha$$

$$w(x,t,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\alpha,\tau)\,d\alpha$$

$$\boldsymbol{u(x,t) = \frac{1}{2a}\int_{0}^{t}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\alpha,\tau)\,d\alpha\, d\tau}$$

半无界问题:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = 0, u_t\left(x,0\right) = 0, \left(0 \le x < +\infty\right) \\ u\left(0,t\right) = A\sin bt \end{cases}$$

方法1：通解法

$$u(x,t) = f_1(x+at) + f_2(x-at)$$

$$0 = f_1(x) + f_2(x)$$

$$0 = f_1'(x) - f_2'(x)$$

$$f_1(x) = -f_2(x) = c, x \ge 0$$

$$f_1(at) + f_2(-at) = c + f_2(-at) = A\sin bt$$

$$f_2(x) = -A\sin\frac{bx}{a} - c, x < 0$$

$$u(x,t) = \begin{cases} 0, & x > at \\ A\sin b(t - \frac{x}{a}), & x < at \end{cases}$$

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\
u(x,0) = 0, u_t\left(x,0\right) = 0, \left(0 \le x < +\infty\right) \\
u_x\left(0,t\right) = h(t)
\end{cases}
$$

$$u(x,t) = f_1(x+at) + f_2(x-at)$$

$$u_x\left(x,t\right) = f_2'(x-at)$$

$$u_x\left(0,t\right) = f_2'(-at) = h(t)$$

$$f_2'(t) = h(-t/a)$$

$$f_2(t) = \int_0^t h(-s/a)ds + f_2(0) = \int_0^t h(-s/a)ds - c$$

例 $$\begin{cases} u_{tt} - a^2 u_{xx} = e^x, -\infty < x < +\infty \\ u(x,0) = 5, u_t\left(x,0\right) = x^2 \end{cases}$$

$$u(x,t) = v(x,t) + w(x)$$

$$\begin{cases} v_{tt} - a^2\left(v_{xx} + w''\right) = e^x \\ v(x,0) = 5 - w(x), u_t\left(x,0\right) = x^2 \end{cases}$$

$$\begin{cases} v_{tt} - a^2 v_{xx} - a^2 w'' - e^x = 0 \\ v(x,0) = 5 - w(x), u_t\left(x,0\right) = x^2 \end{cases}$$

$$a^2 w'' + e^x = 0, w = -e^x / a^2$$

$$u(x,t) = v(x,t) - e^x / a^2$$

$$\begin{cases} v_{tt} - a^2 v_{xx} = 0 \\ v(x,0) = 5 + e^x / a^2, u_t\left(x,0\right) = x^2 \end{cases}$$

$$\boldsymbol{u(x,t) = \frac{1}{2}\left[\varphi\left(x + at\right) + \varphi\left(x - at\right)\right] + \frac{1}{2a}\int_{.x-at}^{.x+at} \psi\left(\xi\right) d\xi}$$

例

$$\begin{cases} u_{tt} - a^2 u_{xx} = xe^t, -\infty < x < +\infty \\ u(x,0) = \sin x, u_t\left(x,0\right) = 0 \end{cases}$$

$$u(x,t) = v(x,t) + xw(t)$$

$$\begin{cases} v_{tt} + xw''(t) = a^2 v_{xx} + xe^t, -\infty < x < +\infty \\ v(x,0) = \sin x - xw(0), v_t\left(x,0\right) = -xw'(0) \end{cases}$$

$$xw''(t) = xe^t \qquad w = e^t$$

$$u(x,t) = v(x,t) + xe^t$$

$$\begin{cases} v_{tt} - a^2 v_{xx} = 0 \\ u(x,0) = \sin x - x, u_t\left(x,0\right) = -x \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\varphi\left(x+at\right) + \varphi\left(x-at\right)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi\left(\xi\right) d\xi$$

例

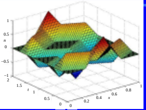
$$\begin{cases} u_{tt} = u_{xx} + \cos x, -\infty < x < +\infty \\ u(x,0) = \cos x, u_t\left(x,0\right) = xe^x \end{cases}$$

$$u(x,\ t) = v(x,\ t) + w(x)$$

$$v_{tt} = [v_{xx} + \quad w'' \ ] + \cos x$$

$$w(x) = \cos x$$

$$\begin{cases} v_{tt} = v_{xx}, -\infty < x < +\infty \\ v(x,0) = 0, v_t\left(x,0\right) = xe^x \end{cases}$$



$$v(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\xi e^{\xi}d\xi=\frac{1}{2}[(x+t)e^{x+t}-(x-t)e^{x-t}-e^{x+t}+e^{x-t}]$$

$$=\frac{1}{2}[(x+t-1)e^{x+t}-(x-t+1)e^{x-t}]$$

$$u(x,t)=\cos x+\frac{1}{2}[(x+t-1)e^{x+t}-(x-t+1)e^{x-t}]$$

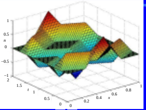
半无界弦的自由振动

端点固定

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = \varphi\left(x\right), u_t\left(x,0\right) = \psi\left(x\right), \left(0 \le x < +\infty\right) \\ u\left(0,t\right) = 0 \end{cases}$$

端点自由

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = \varphi\left(x\right), u_t\left(x,0\right) = \psi\left(x\right), \left(0 \le x < \infty\right) \\ u_x\left(0,t\right) = 0 \end{cases}$$



$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(-\infty < x < +\infty\right) \\ u\big|_{t=0} = \varphi(x), \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\varphi\left(x+at\right)+\varphi\left(x-at\right)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi\left(\xi\right)d\xi$$

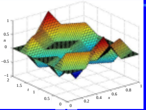
求解思路:

1.利用已有公式;

2.有界变无界;

3.解的截取

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = \varphi\left(x\right), u_t\left(x,0\right) = \psi\left(x\right), \left(0 \le x < +\infty\right) \\ u\left(0,t\right) = 0 \end{cases}$$



端点固定 $u(0,t)=0$

$$u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\varphi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$0=u(0,t)=\frac{1}{2}\left[\varphi(at)+\varphi(-at)\right]+\frac{1}{2a}\int_{-at}^{at}\psi(s)ds$$

$$\varphi(at)=-\varphi(-at); \qquad \int_{-at}^{at}\psi(s)ds=0$$

φ(x),ψ(x)为奇函数

奇延拓

$$\Phi(x)=\begin{cases}\varphi(x),(x\geq 0)\\ -\varphi(-x),(x<0)\end{cases}$$

$$\Psi(x)=\begin{cases}\psi(x),(x\geq 0)\\ -\psi(-x),(x<0)\end{cases}$$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, (-\infty < x < \infty) \\ u(x,0) = \Phi(x), u_t(x,0) = \Psi(x), (-\infty < x < \infty) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{.x-at}^{.x+at} \psi(\xi)\,d\xi$$

$$u(x,t) = \frac{1}{2}\left[\Phi(x+at) + \Phi(x-at)\right] + \frac{1}{2a}\int_{.x-at}^{.x+at} \Psi(s)\,ds$$

$$= \begin{cases} \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{.x-at}^{.x+at} \psi(s)\,ds, \left(t \le \frac{x}{a}\right) \\ \frac{1}{2}\left(\varphi(x+at) - \varphi(at-x)\right) + \frac{1}{2a}\int_{.at-x}^{.x+at} \psi(s)\,ds, \left(t > \frac{x}{a}\right) \end{cases}$$

端点自由

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = \varphi\left(x\right), u_t\left(x,0\right) = \psi\left(x\right), \left(0 \le x < \infty\right) \\ u_x\left(0,t\right) = 0 \end{cases}$$

$$u_x(0,t)=0$$

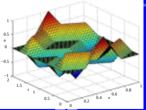
$$u(x,t)=\frac{1}{2}\left[\varphi\left(x+at\right)+\varphi\left(x-at\right)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi\left(\xi\right)d\xi$$

$$u_x(0,t)=\frac{1}{2}\left[\varphi'\left(at\right)+\varphi'\left(-at\right)\right]+\frac{1}{2a}\left[\psi\left(at\right)-\psi\left(-at\right)\right]=0$$

$$\varphi'\left(at\right)=-\varphi'\left(-at\right),\psi\left(at\right)=\psi\left(-at\right)$$

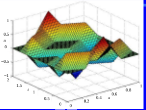
$$\Phi\left(x\right)=\begin{cases}\varphi\left(x\right),\left(x\geq 0\right)\\ \varphi\left(-x\right),\left(x<0\right)\end{cases},\Psi\left(x\right)=\begin{cases}\psi\left(x\right),\left(x\geq 0\right)\\ \psi\left(-x\right),\left(x<0\right)\end{cases}$$

$$\begin{cases}u_{tt}-a^2u_{xx}=0,\left(-\infty<x<\infty\right)\\ u(x,0)=\Phi(x),u_t(x,0)=\Psi(x),\left(-\infty<x<\infty\right)\end{cases}$$



$$u(x,t)=\frac{1}{2}\left[\Phi\left(x+at\right)+\Phi\left(x-at\right)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(s)ds$$

$$=\begin{cases}\dfrac{1}{2}\left[\varphi\left(x+at\right)+\varphi\left(x-at\right)\right]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(s)ds,\left(t\le\dfrac{x}{a}\right)\\ \dfrac{1}{2}\left(\varphi\left(x+at\right)+\varphi\left(at-x\right)\right)+\dfrac{1}{2a}[\displaystyle\int_{0}^{x+at}\psi(s)ds+\int_{0}^{at-x}\psi(s)ds,\left(t>\dfrac{x}{a}\right)\end{cases}$$



半无界问题:
$$\begin{cases} u_{tt}-a^2u_{xx}=0,\left(0<x<\infty\right) \\ u(x,0)=0,u_t\left(x,0\right)=0,\left(0\le x<+\infty\right) \\ u\left(0,t\right)=A\sin bt \end{cases}$$

方法1：通解法

$$u(x,t)=f_1(x+at)+f_2(x-at)$$

$$f_1(x)=-f_2(x)=c,x\ge 0$$

$$f_1(at)+f_2(-at)=c+f_2(-at)=A\sin bt$$

$$f_2(x)=-A\sin\frac{bx}{a}-c,x<0$$

$$u(x,t)=\begin{cases} 0, & x>at \\ A\sin b(t-\frac{x}{a}), & x<at \end{cases}$$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(0 < x < \infty\right) \\ u(x,0) = 0, u_t\left(x,0\right) = 0, \left(0 \le x < +\infty\right) \\ u\left(0,t\right) = A\sin bt \end{cases}$$

方法2

$$u(x,t) = \frac{1}{2}\left[\Phi\left(x+at\right) + \Phi\left(x-at\right)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(s)ds$$

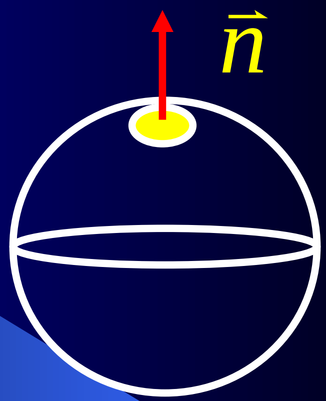
$$u(0,t) = \frac{1}{2}\left[\Phi\left(at\right) + \Phi\left(-at\right)\right] + \frac{1}{2a}\int_{-at}^{at} \Psi(s)ds = A\sin bt$$

$$\Phi\left(x\right) = 0$$

$$\frac{1}{2a}\int_{-at}^{0} \Psi(s)ds = A\sin bt$$

$$-\frac{1}{2}\Psi(-at) = A\cos bt, \Psi(x) = 2A\cos\frac{bx}{a}$$

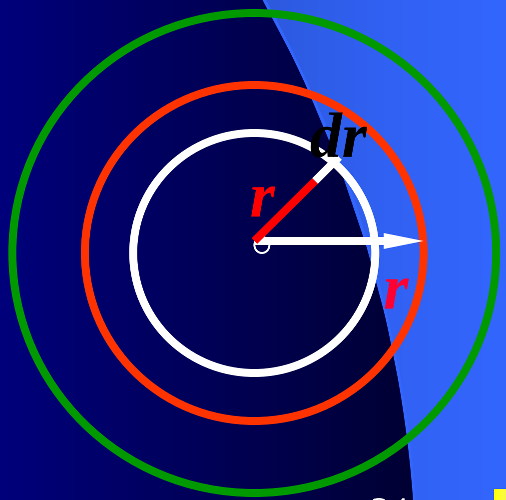
导出匀质且在每一同心球上等温的孤立球体的热传导方程。

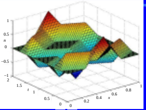


$$\frac{\partial u}{\partial n} = u_r$$

$$r = \left(x^2 + y^2 + z^2\right)^{1/2},\ \theta = \arctan\frac{y}{x},\ \varphi = \arccos\frac{z}{r}$$

$$\begin{cases} r = x r_x + y r_y + z r_z \\ u_x = u_r r_x + u_\theta \theta_x + u_\varphi \varphi_x \\ u_y = u_r r_y + u_\theta \theta_y + u_\varphi \varphi_y \\ u_z = u_r r_z + u_\theta \theta_z + u_\varphi \varphi_z \end{cases}$$





$$\frac{\partial u}{\partial n} = \nabla u \cdot \frac{1}{r}\left\{x, y, z\right\} = \frac{1}{r}\{u_x, u_y, u_z\}\cdot\{x, y, z\} = xu_x + yu_y + zu_z$$

$$= \frac{1}{r}u_r(xr_x + yr_y + zr_z) + \frac{1}{r}u_\theta(x\theta_x + y\theta_y + z\theta_z) + \frac{1}{r}u_\varphi\left(x\varphi_x + y\varphi_y + z\varphi_z\right)$$

$$= u_r + \frac{u_\theta}{r}\left(\frac{-y/x}{1+\left(y/x\right)^2} + \frac{y/x}{1+\left(y/x\right)^2}\right) - \frac{u_\varphi}{r}\left(\frac{-xzr_x/r^2}{\sqrt{1-\left(z/r\right)^2}} + \frac{-yzr_y/r^2}{\sqrt{1-\left(z/r\right)^2}} + \frac{-z^2r_z/r^2 + z/r}{\sqrt{1-\left(z/r\right)^2}}\right)$$

$$= u_r - \frac{u_\varphi}{r}\left(1-\left(z/r\right)^2\right)^{-1/2}\left[\frac{-z}{r^2}(xr_x + yr_y + zr_z) + \frac{z}{r}\right]$$

$$= u_r$$

$$\boldsymbol{kdt[u_r(r+dr,t)s\left(r+dr\right)-u_r(r,t)s(r)]=c\rho drs\frac{\partial u}{\partial t}dt}$$

$$k[u_r(r+dr,t)s\left(r+dr\right)-u_r(r,t)s(r)]=c\rho s u_t dr$$

$$\frac{\partial\left(u_r(r,t)s(r)\right)}{\partial r}=u_{rr}(r,t)s(r)+u_r(r,t)s'(r)$$

$$k[u_{rr}(r+\theta dr,t)s\left(r+\theta dr\right)+u_r(r+\theta dr,t)s'(r+\theta dr)]dr=c\rho s u_t dr$$

$$a^2[u_{rr}s+u_r s']=su_t$$

$$a^2[u_{rr}r+2u_r]=ru_t$$

$$v=ru$$

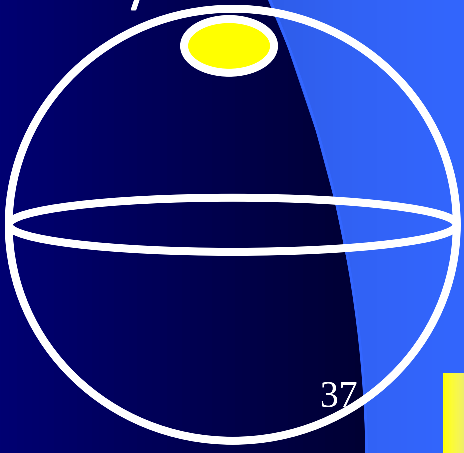
$$v_t=a^2v_{rr}$$

三维波动方程Cauchy问题

求解
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \Delta u = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right), \left(-\infty < x, y, z < +\infty, t > 0\right) \\ u\big|_{t=0} = \varphi(x, y, z), \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x, y, z) \end{cases}$$

方程球坐标系等价形式

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$



方程球坐标系等价形式

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

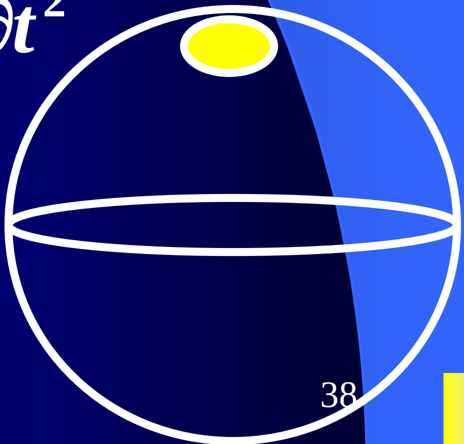
球对称形式

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

$$r\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}=\frac{r}{a^2}\frac{\partial^2 u}{\partial t^2}$$

$$r\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}=\frac{\partial^2 (ru)}{\partial r^2}$$

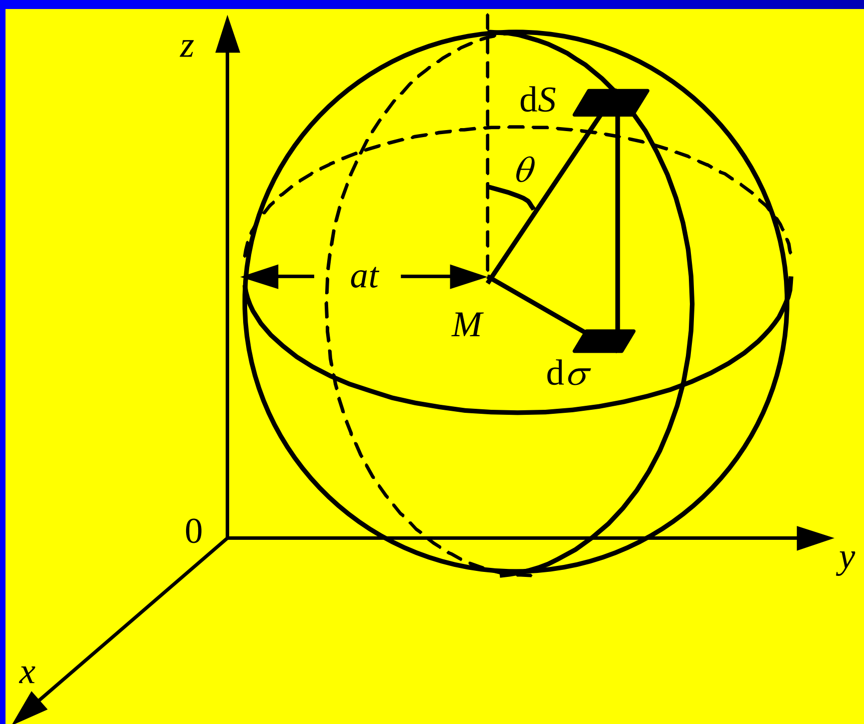
$$\frac{\partial^2 (ru)}{\partial r^2}=\frac{1}{a^2}\frac{\partial^2 (ru)}{\partial t^2}$$



方程通解

$$ru = f_1(r+at) + f_2(r-at)$$

$$u(r,t) = \frac{f_1(r+at) + f_2(r-at)}{r}$$



平均值与函数值

$$u(x,y,z,t)=u(M,t)$$

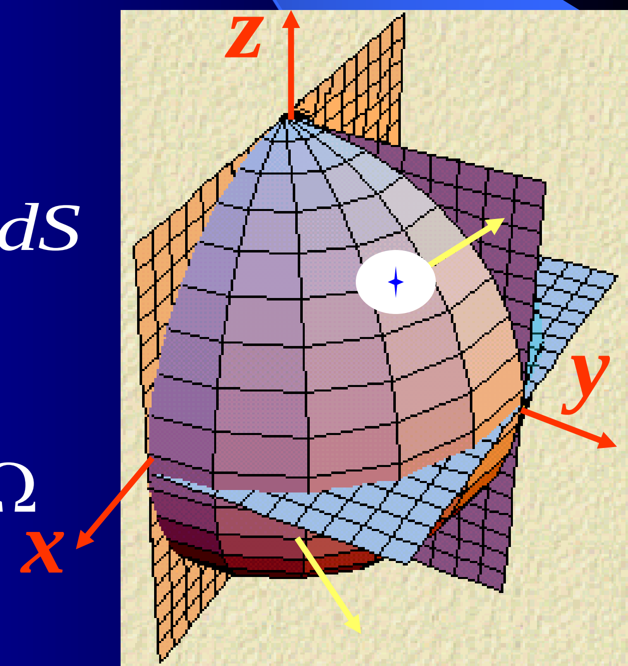
$$\bar{u}(r,t)=\frac{1}{4\pi r^2}\oint\!\!\oint_{S_r^M} u(M',t)\,dS=\frac{1}{4\pi}\oint\!\!\oint_{S_r^M} u(M',t)\,d\Omega$$

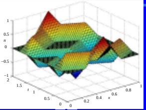
$$u(M,t)=\lim_{r\to 0}\bar{u}(r,t)=\bar{u}(0,t)$$

$$\iiint_{V_r^M} g(x,y,z)\,dV=\int_0^r d\rho\oint\!\!\oint_{S_\rho^M} g(x,y,z)\,dS$$

$$dS=r^2\sin\theta\, d\theta\, d\varphi=r^2 d\Omega$$

球面S_r^M





求满足的方程的特解 $\overline{u}(r,t)$ 　　右端

$$\frac{\partial^2 u}{\partial t^2}=a^2\Delta u=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\iiint\limits_{V_{.r}^{.M}}\frac{\partial^2 u}{\partial t^2}dV=a^2\iiint\limits_{V_{.r}^{.M}}\Delta u dV$$

$$=a^2\iiint\limits_{V_{.r}^{.M}}\nabla\cdot\nabla u dV=a^2\oiint\limits_{S_{.r}^{.M}}\nabla u\cdot d\vec{S}=a^2\oiint\limits_{S_{.r}^{.M}}\nabla u\cdot\vec{n}dS=a^2\oiint\limits_{S_{.r}^{.M}}\frac{\partial u}{\partial\vec{n}}dS$$

$$=a^2\oiint\limits_{S_{.r}^{.M}}\frac{\partial u}{\partial r}dS=a^2r^2\oiint\limits_{D_M}\frac{\partial u}{\partial r}d\Omega, dS=r^2\sin\theta d\theta d\varphi=r^2 d\Omega$$

$$=4\pi a^2r^2\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\oiint\limits_{D_M}u d\Omega\right]$$

$$=4\pi a^2r^2\frac{\partial}{\partial r}\left[\frac{1}{4\pi r^2}\oiint\limits_{D_M}u r^2 d\Omega\right]=4\pi a^2r^2\frac{\partial\overline{u}(r,t)}{\partial r}$$

左端

$$\iiint\limits_{V_r^M} g(x,y,z)\,dV = \int_0^r d\rho \oiint\limits_{S_\rho^M} g(x,y,z)\,dS$$

$$\frac{1}{4\pi}\iiint\limits_{V_r^M}\frac{\partial^2 u}{\partial t^2}\,dV = \frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi}\int_0^r d\rho \oiint\limits_{S_\rho^M} u(M',t)\,dS\right]$$

联立左右两式 $$\frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi}\int_0^r d\rho \oiint\limits_{S_\rho^M} u(M',t)\,dS\right] = a^2 r^2 \frac{\partial \bar{u}(r,t)}{\partial r}$$

两端对**r**求导 $$\frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi}\oiint\limits_{S_r^M} u(M',t)\,dS\right] = a^2\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{u}(r,t)}{\partial r}\right)$$

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi r^2}\oiint_{S_{r}^{M}} u(M',t)\,dS\right]=\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{u}}{\partial r}\right)$$

$$\boldsymbol{\frac{\partial^2 \bar{u}}{\partial t^2}=\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{u}}{\partial r}\right)}$$

$$r\frac{\partial^2 \bar{u}}{\partial t^2}=\frac{a^2}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{u}}{\partial r}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{u}}{\partial r}\right)=r\frac{\partial^2 \bar{u}}{\partial r^2}+2\frac{\partial \bar{u}}{\partial r}=\frac{\partial^2(r\bar{u})}{\partial r^2}$$

$$\frac{\partial^2(r\bar{u})}{\partial t^2}=a^2\frac{\partial^2(r\bar{u})}{\partial r^2}$$

$$r\bar{u}(r,t)=f_1(r+at)+f_2(r-at)$$

$$0=f_1(at)+f_2(-at)$$

$$f_1'(at)=f_2'(-at)$$

下面求解$f_i(r)$，关于r求导

$$\frac{\partial}{\partial r}\left[r\bar{u}\left(r,t\right)\right]=f_1'\left(r+at\right)+f_2'\left(r-at\right)$$

$$\bar{u}\left(r,t\right)+r\frac{\partial\bar{u}\left(r,t\right)}{\partial r}=f_1'\left(r+at\right)+f_2'\left(r-at\right)$$

$$r=0\Rightarrow\bar{u}\left(0,t\right)=f_1'(at)+f_2'\left(-at\right)$$

$$u(M,t)=\bar{u}(0,t)=f_1'(at)+f_2'(-at)$$

$$=2f_1'(at)$$

关于t求导

$$\frac{1}{a}\frac{\partial}{\partial t}\left[r\bar{u}\left(r,t\right)\right]=f_1'\left(r+at\right)-f_2'\left(r-at\right)$$

$$\Rightarrow\frac{\partial}{\partial r}\left[r\bar{u}(r,t)\right]+\frac{1}{a}\left[r\bar{u}_t(r,t)\right]=2f_1'\left(r+at\right)$$

令t=0

$$2f_1'\left(r\right)=\frac{\partial}{\partial r}\left[r\bar{u}(r,0)\right]+\frac{1}{a}r\bar{u}_t(r,0)$$

$$=\frac{\partial}{\partial r}\left[\frac{r}{4\pi r^2}\iint\limits_{S_{.r}^{.M}}u\left(M',0\right)dS\right]+\frac{r}{4\pi a r^2}\iint\limits_{S_{.r}^{.M}}u_t\left(M',0\right)dS$$

$$=\frac{1}{4\pi}\frac{\partial}{\partial r}\iint\limits_{S_{.r}^{.M}}\frac{\varphi(x',y',z')}{r}dS+\frac{1}{4\pi a}\iint\limits_{S_{.r}^{.M}}\frac{\psi(x',y',z')}{r}dS$$

$$2f_1'(r)=\frac{\partial}{\partial r}\left[r\bar{u}(r,0)\right]+\frac{1}{a}r\bar{u}_t(r,0)$$

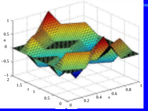
$$=\frac{\partial}{\partial r}\left[\frac{r}{4\pi r^2}\oiint\limits_{S_{.r}^{.M}}u\left(M',0\right)dS\right]+\frac{r}{4\pi a r^2}\oiint\limits_{S_{.r}^{.M}}u_t\left(M',0\right)dS$$

$$=\frac{1}{4\pi}\frac{\partial}{\partial r}\oiint\limits_{S_{.r}^{.M}}\frac{\varphi(x',y',z')}{r}dS+\frac{1}{4\pi a}\oiint\limits_{S_{.r}^{.M}}\frac{\psi(x',y',z')}{r}dS$$

$$\boldsymbol{f(x)\to f'(x)\to f'(at)}$$

$$\boldsymbol{f(at)\to af'(at)}$$

$$u(M,t)=2f_1'(at)=\frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\oiint\limits_{S_{.at}^{.M}}\frac{\varphi(x',y',z')}{t}dS+\oiint\limits_{S_{.at}^{.M}}\frac{\psi(x',y',z')}{t}dS\right]$$



Poisson公式

$$u(M,t)=\frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\iint\limits_{S_{at}^{M}}\frac{\varphi(M')}{t}dS+\iint\limits_{S_{at}^{M}}\frac{\psi(M')}{t}dS\right]$$

$$u(M,t)=\frac{\partial}{\partial t}\left[t\frac{1}{4\pi a^2 t^2}\iint\limits_{S_{at}^{M}}\varphi(M')dS\right]+t\frac{1}{4\pi a^2 t^2}\iint\limits_{S_{at}^{M}}\psi(M')dS$$

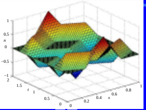
第五章 积分变换

f(x)定义在 (-∞, +∞) 内，且在任一有限区间 [-L, L]上分段光滑，则可以展开为**Fourier**级数

$$f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$\begin{cases} a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(s)\cos\dfrac{n\pi s}{L}ds \\ b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(s)\sin\dfrac{n\pi s}{L}ds \end{cases},(n=0,1,2,\cdots)$$

Fourier级数的推广



Fourier变换

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx$$

Fourier逆变换

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{j\omega x}d\omega$$

记号 $\hat{f}(\omega)=F[f(x)]$ $\quad F^{-1}F[f]=f$

$$f(x)=F^{-1}[\hat{f}(\omega)]=F^{-1}[F(f(x))]$$

$$f_1(x)*f_2(x)=\int_{-\infty}^{\infty} f_1(x-s)f_2(s)ds$$

例　求证

$$F^{-1}\left[\mathrm{e}^{-a^2\omega^2 t}\right]=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}$$

$$F^{-1}\left[\mathrm{e}^{-a^2\omega^2 t}\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-a^2\omega^2 t}\mathrm{e}^{\mathrm{j}\omega x}\mathrm{d}\omega$$

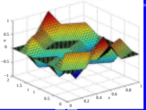
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-a^2t(\omega^2-\frac{\mathrm{j}\omega x}{a^2t})}\mathrm{d}\omega$$

$$=\frac{1}{2\pi}\mathrm{e}^{-\frac{x^2}{4a^2t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-a^2t(\omega-\frac{\mathrm{j}x}{2a^2t})^2}\mathrm{d}\omega$$

$$F^{-1}\left[e^{-a^2\omega^2 t}\right]=\frac{1}{2\pi}e^{-\frac{x^2}{4a^2t}}\int_{-\infty}^{+\infty}e^{-a^2t(\omega-\frac{jx}{2a^2t})^2}d\omega$$

$$=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{2ta}}}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\left(\frac{1}{\sqrt{2ta}}\right)^{-2}\left(\omega-\frac{jx}{2a^2t}\right)^2\right)d\omega$$

$$=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$$



Fourier变换的基本性质

性质1（线性定理）$F\left[\alpha f_1+\beta f_2\right]=\alpha F[f_1]+\beta F[f_2]$

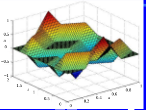
性质2（卷积定理）$F[f_1 * f_2]=F[f_1]F[f_2]$

性质3（乘积定理）$F[f_1 f_2]=\dfrac{1}{2\pi}F[f_1]*F[f_2]$

性质4（原象的导数定理）$F[f']=j\omega F[f]$

$$F[f^{(k)}]=(j\omega)^k F[f]$$

性质5（象的导数定理）$\dfrac{d}{d\omega}F[f]=F[-jxf]$



性质6（延迟定理） $F[f(x-x_0)]=e^{-j\omega x_0}F[f(x)]$

性质7（位移定理） $F[e^{j\omega_0 x}f(x)]=\hat{f}(\omega-\omega_0)$

性质8（积分定理） $F[\int_{-\infty}^{x}f(\xi)d\xi]=\frac{1}{j\omega}F[f(x)]$

性质9 (广义函数) $F[\delta(x)]=\int_{-\infty}^{\infty}\delta(x)\ e^{-j\omega x}dx=e^{-j\omega x}\Big|_{x=0}=1$

$$F[\delta(x-\xi)]=\int_{-\infty}^{\infty}\delta(x-\xi)e^{-j\omega x}dx=e^{-j\omega\xi}$$

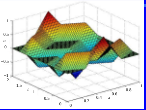
性质10.(相似定理) $F[f(ax)]=\frac{1}{|a|}\hat{f}(\frac{\omega}{a})$

性质11（对偶性） $\mathrm{F}\big(g(x)\big)=f(\omega)\Rightarrow\mathrm{F}\big(f(x)\big)=2\pi g(-\omega)$

$$f(\omega)=\hat{g}(\omega),\, g(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(s)\mathrm{e}^{\mathrm{j}sx}\mathrm{d}s$$

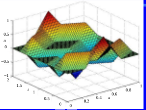
$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(s)\mathrm{e}^{\mathrm{j}\omega s}\mathrm{d}s,\, g(-\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(s)\mathrm{e}^{-\mathrm{j}\omega s}\mathrm{d}s$$

$$\mathrm{F}\big(f(x)\big)=\int_{-\infty}^{+\infty} f(s)\mathrm{e}^{-\mathrm{j}\omega s}\mathrm{d}s$$



性质12（Parseval 公式） $\int_{-\infty}^{+\infty} f^2(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\hat{f}(\omega)\right|^2 \mathrm{d}\omega$

Parseval定理指出，一个信号所含有的能量（功率）恒等于此信号在完备正交函数集中各分量能量（功率）之和。它表明信号在时域的总能量等于信号在频域的总能量，即信号经傅里叶变换后其总能量保持不变，符合能量守恒定律。



n 维Fourier变换

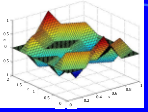
定义

$$F(\omega_1,\omega_2,\cdots,\omega_n)=F[f(x_1,x_2,\cdots,x_n)]$$

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,x_2,\cdots,x_n)e^{-j(\omega_1 x_1+\omega_2 x_2+\cdots+\omega_n x_n)}dx_1 dx_2\cdots dx_n$$

$$f(x_1,x_2,\cdots,x_n)=$$

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\omega_1,\omega_2,\cdots,\omega_n)e^{j(\omega_1 x_1+\omega_2 x_2+\cdots+\omega_n x_n)}d\omega_1 d\omega_2\cdots d\omega_n$$



n 维Fourier变换具有与上面平行的性质：

$$F[\alpha f_1+\beta f_2]=\alpha F[f_1]+\beta F[f_2]$$

$$F[f_1 * f_2]=F[f_1]F[f_2]$$

$$F(f_1 f_2)=\frac{1}{(2\pi)^n}F(f_1)*F(f_2)$$

$$F\left(\frac{\partial f}{\partial x_k}\right)=j\omega_k F(f),k=1,2,\cdots,n$$

$$\frac{\partial}{\partial \omega_k}F(f)=F(-jx_k f),k=1,2,\cdots,n$$

定义2 Fourier余弦变换

$$\hat{f}_c(\omega)=\int_0^{+\infty} f(x)\cos\omega x\mathrm{d}x$$

定义3 Fourier逆余弦变换

$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\hat{f}_c(\omega)\cos\omega x\mathrm{d}\omega$$

定义4 Fourier正弦变换

$$\hat{f}_s(\omega)=\int_0^{+\infty} f(x)\sin\omega x\mathrm{d}x$$

定义5 Fourier逆正弦变换

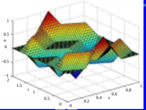
$$f(x)=\frac{2}{\pi}\int_0^{+\infty}\hat{f}_s(\omega)\sin\omega x\mathrm{d}\omega$$

例　求证：$\mathbf{e}^{-a^2\omega^2\tau}$　的余弦变换为 $\dfrac{1}{2a}\sqrt{\dfrac{\pi}{\tau}}\mathbf{e}^{-\frac{x^2}{4a^2\tau}}$

证明1：

$$\int_0^{+\infty}\mathrm{e}^{-a^2\omega^2\tau}\cos(\omega x)\mathrm{d}\omega=\frac{1}{2}\int_0^{+\infty}\mathrm{e}^{-a^2\omega^2\tau}(\mathrm{e}^{\mathrm{j}\omega x}+\mathrm{e}^{-\mathrm{j}\omega x})\mathrm{d}\omega$$

$$=\left[\frac{1}{2}\int_0^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}(\omega-\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega+\frac{1}{2}\int_0^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}(\omega+\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega\right]\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$



$$=\left[\frac{1}{\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_0^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}\left(\omega-\frac{\mathrm{j}x}{2a^2\tau}\right)^2}\mathrm{d}\omega+\frac{1}{\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_0^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}\left(\omega+\frac{\mathrm{j}x}{2a^2\tau}\right)^2}\mathrm{d}\omega\right]\frac{1}{2}\sqrt{2\pi\frac{1}{2a^2\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$=\left[\frac{1}{\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_{-\frac{\mathrm{j}x}{2a^2\tau}}^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}\omega^2}\mathrm{d}\omega+\frac{1}{2a\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_{\frac{\mathrm{j}x}{2a^2\tau}}^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}\omega^2}\mathrm{d}\omega\right]\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$=\left[\frac{1}{\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_{-\frac{\mathrm{j}x}{2a^2\tau}}^{+\infty}\mathrm{e}^{\frac{-2a^2\tau}{2}\omega^2}\mathrm{d}\omega+\frac{1}{2a\sqrt{2\pi\frac{1}{2a^2\tau}}}\int_{-\infty}^{-\frac{\mathrm{j}x}{2a^2\tau}}\mathrm{e}^{\frac{-2a^2\tau}{2}\omega^2}\mathrm{d}\omega\right]\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$=\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$\int_0^{+\infty}\mathrm{e}^{-a^2\omega^2\tau}\cos\omega x\,\mathrm{d}\omega=\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

证明2: $\displaystyle\int_0^{+\infty} \mathrm{e}^{-a^2\omega^2\tau}\cos(\omega x)\mathrm{d}\omega = \frac{1}{2}\int_0^{+\infty} \mathrm{e}^{-a^2\omega^2\tau}(\mathrm{e}^{\mathrm{j}\omega x}+\mathrm{e}^{-\mathrm{j}\omega x})\mathrm{d}\omega$

$$=\left[\frac{1}{2}\int_0^{+\infty} \mathrm{e}^{\frac{-2a^2\tau}{2}(\omega-\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega+\frac{1}{2}\int_0^{+\infty} \mathrm{e}^{\frac{-2a^2\tau}{2}(\omega+\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega\right]\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$=\left[\frac{1}{2}\int_0^{+\infty} \mathrm{e}^{\frac{-2a^2\tau}{2}(\omega-\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega+\frac{1}{2}\int_{-\infty}^{0} \mathrm{e}^{\frac{-2a^2\tau}{2}(\omega-\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega\right]\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

$$=\frac{1}{2}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}\int_{-\infty}^{+\infty} \mathrm{e}^{\frac{-2a^2\tau}{2}(\omega-\frac{\mathrm{j}x}{2a^2\tau})^2}\mathrm{d}\omega=\frac{1}{2a}\sqrt{\frac{\pi}{\tau}}\mathrm{e}^{-\frac{x^2}{4a^2\tau}}$$

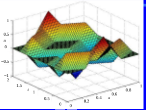
§5.2 Fourier变换的应用

Fourier变换法求解步骤：

（1）对定解问题作Fourier变换；

（2）求解象函数；

（3）对象函数作Fourier逆变换得解。



应用范围:

1）求解无界区域的定解问题, 直接Fourier求解;

2）对于半无界区域的定解问题:

a. 第一类边界条件, 采用Fourier正弦变换;

b.第二类边界条件, Fourier余弦变换

c.将边界条件齐次化后，采用延拓法，最后用Fourier变换法求解

简写符号

$$F[u(x,t)]=\hat{u}(\omega,t),$$

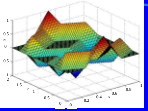
$$F[\varphi(x)]=\hat{\varphi}(\omega),F[\psi(x)]=\hat{\psi}(\omega)$$

$$F[u_{tt}(x,t)]=\int_{-\infty}^{+\infty}\frac{\partial^2 u(x,t)}{\partial t^2}e^{-j\omega x}dx$$

$$=\frac{d^2}{dt^2}\int_{-\infty}^{+\infty}u(x,t)e^{-j\omega x}dx$$

$$=\frac{d^2\hat{u}(\omega,t)}{dt^2}$$

$$F[u_{xx}(x,t)]=\int_{-\infty}^{+\infty}\frac{\partial^2 u(x,t)}{\partial x^2}e^{-j\omega x}dx=(j\omega)^2\hat{u}(\omega,t)$$



波动方程的定解问题

例 求解无界弦振动方程的初值问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \left(-\infty < x < \infty, t > 0\right) \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

解:

$$u_{tt} \leftrightarrow \frac{d^2\hat{u}(\omega,t)}{dt^2}, u_{xx} \leftrightarrow (j\omega)^2\hat{u}(\omega,t)$$

$$\begin{cases} \dfrac{d^2\hat{u}(\omega,t)}{dt^2} + a^2\omega^2\hat{u}(\omega,t) = 0 \\ \hat{u}(\omega,0) = \hat{\varphi}(\omega), \hat{u}_t(\omega,0) = \hat{\psi}(\omega) \end{cases}$$

$$\hat{u}(\omega,t) = C_1 e^{j\omega at} + C_2 e^{-j\omega at}$$

$$\hat{\varphi}(\omega) = \hat{u}(\omega,0) = C_1 + C_2$$

$$\hat{\psi}(\omega) = \hat{u}_t(\omega,0) = j\omega a(C_1 - C_2)$$

$$\hat{u}(\omega,t) = \frac{1}{2}\left[\hat{\varphi}(\omega) + \frac{1}{j\omega a}\hat{\psi}(\omega)\right]e^{j\omega at} + \frac{1}{2}\left[\hat{\varphi}(\omega) - \frac{1}{j\omega a}\hat{\psi}(\omega)\right]e^{-j\omega at}$$

$$u(x,t) = F^{-1}[\hat{u}(\omega,t)] =$$

$$\frac{1}{2}F^{-1}\left[\hat{\varphi}(\omega)e^{j\omega at}\right] + \frac{1}{2a}F^{-1}\left[\frac{1}{j\omega}\hat{\psi}(\omega)e^{j\omega at}\right]$$

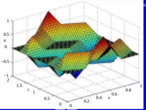
$$+\frac{1}{2}F^{-1}\left[\hat{\varphi}(\omega)e^{-j\omega at}\right] - \frac{1}{2a}F^{-1}\left[\frac{1}{j\omega}\hat{\psi}(\omega)e^{-j\omega at}\right]$$

应用延迟定理 $F[\varphi(x \pm at)] = e^{\pm j\omega at}F[\varphi(x)] = e^{\pm j\omega at}\hat{\varphi}(\omega)$

$$F^{-1}[e^{\pm j\omega at}\hat{\varphi}(\omega)] = \varphi(x \pm at)$$

$$F\left[\int_{-\infty}^{x \pm at}\psi(s)ds\right] = e^{\pm j\omega at}F\left[\int_{-\infty}^{x}\psi(s)ds\right] = e^{\pm j\omega at}\frac{1}{j\omega}\hat{\psi}(\omega)$$

$$F^{-1}\left[\frac{1}{j\omega}\hat{\psi}(\omega)e^{\pm j\omega at}\right] = \int_{-\infty}^{x \pm at}\psi(s)ds$$



$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\left[\int_{-\infty}^{x+at}\psi(s)ds-\int_{-\infty}^{x-at}\psi(s)ds\right]$$

$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$$

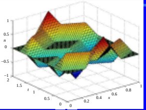
热传导方程定解问题

例设有一根无限长的杆，杆上有强度为F(x,t)的热源，杆的初始温度为φ(x)，试求t>0时杆上温度的分布规律。

解：

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u\big|_{t=0} = \varphi(x) \end{cases}$$

$$f(x,t) = \frac{1}{\rho c} F(x,t)$$



$$\hat{u}(\omega,t)=\int_{-\infty}^{\infty}u(x,t)e^{-j\omega x}dx,\ \hat{f}(\omega,t)=\int_{-\infty}^{\infty}f(x,t)e^{-j\omega x}dx$$

$$\begin{cases}\dfrac{d\hat{u}}{dt}=-a^2\omega^2\hat{u}(\omega,t)+\hat{f}(\omega,t)\\ \hat{u}(\omega,0)=\hat{\varphi}(\omega)\end{cases}\qquad \hat{u}=\hat{\varphi}e^{-a^2\omega^2 t}+\int_0^t \hat{f}e^{-a^2\omega^2(t-\tau)}d\tau$$

$$F^{-1}[e^{-a^2\omega^2 t}]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$$

$$u(x,t)=F^{-1}[\hat{u}(\omega,t)]$$

$$=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(s)e^{-\frac{(x-s)^2}{4a^2t}}ds+\frac{1}{2a\sqrt{\pi}}\int_0^t d\tau\int_{-\infty}^{\infty}\frac{f(s,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-s)^2}{4a^2(t-\tau)}}ds$$

例　求半无界杆的热传导问题

$$\begin{cases} u_t - a^2 u_{xx} = 0, (0 < x < \infty, t > 0) \\ u(x,0) = 0 \\ u(0,t) = u_0 (\text{常数}) \end{cases}$$

解：将边界条件齐次化，仿照半无界弦的波动问题作奇延拓，将问题化为无界问题

$$u(x,t) = w(x,t) + u_0$$

$$\begin{cases} w_t - a^2 w_{xx} = 0, (0 < x < \infty, t > 0) \\ w(x,0) = -u_0 \\ w(0,t) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \left(-\infty < x < +\infty, t > 0\right) \\ u(x,0) = \varphi(x) \end{cases}$$

的解为

$$u(x,t) = F^{-1}[\hat{u}(\omega,t)]$$

$$= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(s) e^{-\frac{(x-s)^2}{4a^2 t}} ds + \frac{1}{2a\sqrt{\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} \frac{f(s,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-s)^2}{4a^2(t-\tau)}} ds$$

本题

$$\begin{cases} w_t = a^2 w_{xx}, (0 < x < \infty, t > 0) \\ w(x,0) = -u_0 \\ w(0,t) = 0 \end{cases}$$

本题中 $f(x,t)=0$

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(s)e^{-\frac{(x-s)^2}{4a^2t}}ds$$

$$u(0,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(s)e^{-\frac{s^2}{4a^2t}}ds=0$$

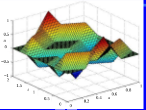
$\varphi(x)$ 作奇延拓

$$\begin{cases} w_t - a^2 w_{xx} = 0, (-\infty < x < \infty, t > 0) \\ w\big|_{t=0} = \varphi(x) = \begin{cases} -u_0, (x > 0) \\ u_0, (x < 0) \end{cases} \end{cases}$$

利用上题结果

$$u(x,t) = u_0 + w(x,t)$$

$$= u_0 + \frac{u_0}{2a\sqrt{\pi t}}\left[\int_{-\infty}^{0} e^{-\frac{(x-s)^2}{4a^2 t}} ds - \int_{0}^{\infty} e^{-\frac{(x-s)^2}{4a^2 t}} ds\right]$$

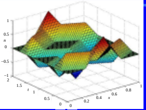


$$u(x,\ t)=u_0+\frac{u_0}{2a\sqrt{\pi t}}\left[\int_{-\infty}^{-x}\mathrm{e}^{-\frac{s^2}{4a^2t}}\mathrm{d}s-\int_{-x}^{+\infty}\mathrm{e}^{-\frac{s^2}{4a^2t}}\mathrm{d}s\right]$$

$$u(x,\ t)=u_0+\frac{u_0}{2a\sqrt{\pi t}}\left[\int_{x}^{+\infty}\mathrm{e}^{-\frac{s^2}{4a^2t}}\mathrm{d}s-\int_{-x}^{+\infty}\mathrm{e}^{-\frac{s^2}{4a^2t}}\mathrm{d}s\right]$$

$$=u_0-\frac{u_0}{a\sqrt{\pi t}}\int_0^x\mathrm{e}^{-\frac{s^2}{4a^2t}}\mathrm{d}s$$

$$=u_0\mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right)$$



Laplace变换

（一）Laplace变换的定义

$$L[f(t)] = \tilde{f}(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

$$L^{-1}[\tilde{f}(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) e^{st} ds$$

（二）常用函数的Laplace变换

$$\boldsymbol{f(t) = ce^{at}}$$

$$L[ce^{at}] = \int_{0}^{\infty} ce^{at} e^{-st} dt = -\frac{ce^{-(s-a)t}}{s-a}\Big|_{0}^{\infty} = \frac{c}{s-a}, (\mathrm{Re}\, s > \mathrm{Re}\, a)$$

$$f(t)=\sin bt$$

$$L[\sin bt]=\int_{.0}^{.\infty}\sin bt e^{-st}dt=\frac{1}{2j}\int_{.0}^{.\infty}\left[e^{-(s-jb)t}-e^{-(s+jb)t}\right]dt$$

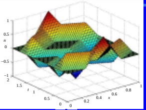
$$=\frac{1}{2j}\left(\frac{1}{s-jb}-\frac{1}{s+jb}\right)=\frac{b}{s^2+b^2},(\operatorname{Re}s>0)$$

$$L[\cos bt]=\frac{1}{2}\left(\frac{1}{s-jb}+\frac{1}{s+jb}\right)=\frac{s}{s^2+b^2},(\operatorname{Re}s>0)$$

$$f(t)=t^{\beta},(\operatorname{Re}\beta>-1)$$

$$L[t^{\beta}]=\int_{.0}^{.\infty}t^{\beta}e^{-st}dt=\frac{1}{s^{\beta+1}}\int_{.0}^{.\infty}e^{-st}(st)^{\beta}d(st)=\frac{\Gamma(\beta+1)}{s^{\beta+1}},(\operatorname{Re}s>0)$$

$$\Gamma(n+1)=n!$$



Laplace变换的性质

1.

$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)]$$

$$L^{-1}[a_1 F_1(s) + a_2 F_2(s)] = a_1 L^{-1}[F_1(s)] + a_2 L^{-1}[F_2(s)]$$

2. 延迟定理

$$L[f(t-\tau)] = e^{-s\tau} L[f(t)]$$

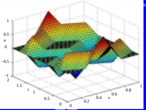
3. 位移定理

$$L[e^{at} f(t)] = F(s-a), \mathrm{Re}(s-a) > \sigma_0$$

4. 相似定理

若c为大于零的常数，则

$$L[f(ct)] = \frac{1}{c} F(\frac{s}{c})$$



5. 微分定理

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$\cdots$$

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

6. 积分定理

$$L[\int_0^t f(\tau)d\tau] = \frac{1}{s} L[f(t)]$$

7. 象函数的微分定理

$$\frac{d^n}{ds^n} F(s) = L[(-t)^n f(t)]$$

8. 象函数的积分定理

$$\int_s^{\infty} F(\tau)d\tau = L[\frac{f(t)}{t}]$$

9.卷积定理

$$L[f_1(t)*f_2(t)]=L[f_1(t)]\cdot L[f_2(t)]$$

1. 约当引理

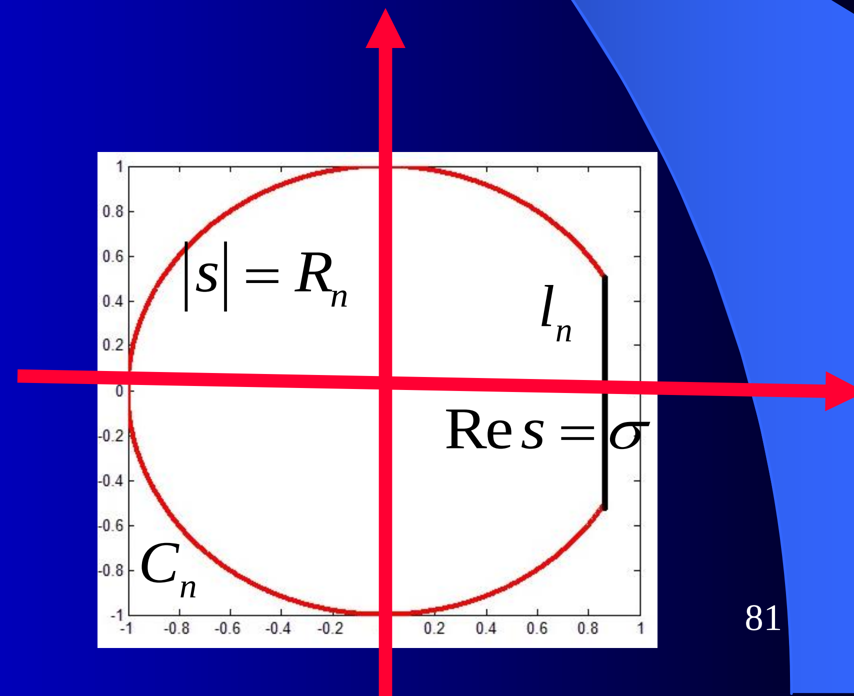
2. 展开定理

$$f(t)=\sum_k \mathrm{Res}[L(f(t))e^{st},s_k]$$

Jordan引理

设L为平行于虚轴的固定直线，C_n 为一族以原点为中心并在L左边的圆弧，C_n 的半径随而趋于无穷。若在 C_n 上，满足 $\lim\limits_{n\to+\infty} g(s)\Big|_{s\in C_n}=0$ ，则对任一正数x，均有

$$\lim_{n\to+\infty}\int_{C_n} g(s)\mathrm{e}^{sx}\mathrm{d}s=0$$



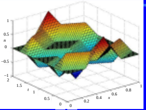
例　已知 $F(s)=\dfrac{s}{(s+\alpha)(s+\beta)^2}$ 求 $L^{-1}[F(s)]$

$$L^{-1}[F(s)]=L^{-1}\left[\frac{s}{(s+\alpha)(s+\beta)^2}\right]$$

$$=\sum_{k}\mathrm{Res}\left[\frac{se^{st}}{(s+\alpha)(s+\beta)^2},s_k\right]$$

$$=\lim_{s\to-\alpha}(s+\alpha)\cdot\frac{se^{st}}{(s+\alpha)(s+\beta)^2}+\lim_{s\to-\beta}\left[(s+\beta)^2\frac{se^{st}}{(s+\alpha)(s+\beta)^2}\right]'_{p}$$

$$=\frac{\alpha-\beta(\alpha-\beta)t}{(\beta-\alpha)^2}e^{-\beta t}-\frac{\alpha}{(\beta-\alpha)^2}e^{-\alpha t}$$



例　已知 $F(s)=\dfrac{2s^2-5s-5}{(s+1)(s-1)(s-2)}$ 求 $L^{-1}[F(s)]$

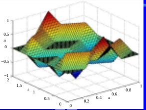
例　已知 $F(s)=\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ 求 $L^{-1}[F(s)]$

例　求解常微分方程 $\begin{cases} y''-3y'+2y=2e^{3t} \\ y\big|_{t=0}=0,\ y'\big|_{t=0}=0 \end{cases}$

$$L[y]=\tilde{y},\ L[2e^{3t}]=\frac{2}{s-3}$$

$$s^2\tilde{y}-3s\tilde{y}+2\tilde{y}=\frac{2}{s-3}$$

$$\tilde{y}=\frac{2}{(s-3)(s-2)(s-1)}$$



$$y=L^{-1}[\tilde{y}]=\lim_{s\to 3}[(s-3)\frac{2e^{st}}{(s-3)(s-2)(s-1)}]+$$

$$\lim_{s\to 2}[(s-2)\frac{2e^{st}}{(s-3)(s-2)(s-1)}]+\lim_{s\to 1}[(s-1)\frac{2e^{st}}{(s-3)(s-2)(s-1)}]$$

$$=e^{3t}-2e^{2t}+e^{t}$$

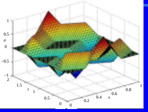
例　求解积分方程　$f(t)=at+\int_{-\infty}^{+\infty}\sin(t-\tau)f(\tau)d\tau$

解：由卷积定义，将方程写成 $f(t)=at+f(t)*\sin$

$$\tilde{f}=\frac{a}{s^2}+\frac{1}{s^2+1}\tilde{f}$$

$$\tilde{f}=\frac{a}{s^2}+\frac{a}{s^4}$$

$$f(t)=a(t+\frac{t^3}{6}$$



Laplace变换解数理方程

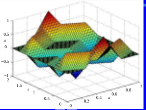
例 求解硅片的恒定表面浓度扩散问题，在恒定表面浓度扩散中，包围硅片的气体中含有大量杂质原子，它们源源不断穿过硅片表面向硅片内部扩散。由于气体中杂质原子供应充分，硅片表面浓度得以保持某个常数N_0。

解：这里所求的是半无限空间x＞0中的定解问题

$$\begin{cases} u_t = a^2 u_{xx}, (x > 0, t > 0) \\ u\big|_{x=0} = N_0 \\ u\big|_{t=0} = 0 \end{cases}$$

对自变量作Laplace变换

$$\begin{cases} a^2 \dfrac{d^2\tilde{u}}{dx^2} - s\tilde{u} = 0 \\ \tilde{u}\big|_{x=0} = N_0 / s \end{cases}$$



$$\tilde{u} = Ae^{-\frac{\sqrt{s}}{a}x} + Be^{\frac{\sqrt{s}}{a}x}$$

$\lim\limits_{x\to\infty}\tilde{u}$ 不应为无穷大 $B=0$ $A = N_0 / s$

$$\tilde{u} = N_0 \frac{1}{s} e^{-\frac{\sqrt{s}}{a}x}$$

$$L^{-1}[\frac{1}{s}e^{-\frac{\sqrt{s}}{a}x}] = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2}dy$$

$$u = N_0 erfc(\frac{x}{2a\sqrt{t}}) = N_0 \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2}dy$$

例　一条半无限长的杆，端点的温度变化为已知，杆的初始温度为零。求杆上的温度分布规律。

解：所提问题归结为

$$\begin{cases} u_t = a^2 u_{xx}, (x>0, t>0) \\ u\big|_{t=0} = 0 \\ u\big|_{x=0} = f(t) \end{cases}$$

$$\begin{cases} \dfrac{d^2\tilde{u}}{dx^2} - \dfrac{s}{a^2}\tilde{u} = 0 \\ \tilde{u}\big|_{x=0} = \tilde{f} \end{cases}$$

$$\tilde{u} = \tilde{f} e^{-\frac{\sqrt{s}}{a}x}$$

$$u = L^{-1}[\tilde{f} e^{-\frac{\sqrt{s}}{a}x}] = L^{-1}[\tilde{f}] * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}]$$

$$= f(t) * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}]$$

$$L^{-1}[\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}]=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-y^2}dy$$

$$\lim_{t\to+0}\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-y^2}dy=0$$

$$L[f'(t)]=s\tilde{f}-f(0)$$

$$L^{-1}[e^{-\frac{x}{a}\sqrt{s}}]=L^{-1}[s\cdot\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}]=\frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty}e^{-y^2}dy\right]$$

$$=\frac{x}{2a\sqrt{\pi}t^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2t}}$$

$$u(x,t)=L^{-1}[F(s)e^{-\frac{x}{a}\sqrt{s}}]=\frac{x}{2a\sqrt{\pi}}\int_0^t f(\tau)(t-\tau)^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau$$

例 求解半无界弦的强迫振动定解问题为:

$$\begin{cases} u_{tt}=a^2u_{xx}+\cos\omega.t,(0<x<+\infty,t>0) \\ u\big|_{x=0}=0,u_x\big|_{x\to\infty}=0 \\ u\big|_{t=0}=0,u_t\big|_{t=0}=0 \end{cases}$$

解：对自变量取Laplace变换

$$\begin{cases} s^2\tilde{u}=a^2\dfrac{d^2\tilde{u}}{dx^2}+\dfrac{s}{\omega^2+s^2} \\ \tilde{u}\big|_{x=0}=0,\tilde{u}_x\big|_{x\to\infty}=0 \end{cases}$$

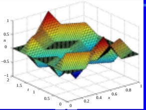
$$\tilde{u}=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}+\frac{1}{s(\omega^2+s^2)}$$

$$A=0, B=-\frac{1}{s(\omega^2+s^2)}$$

$$\tilde{u}=\frac{1}{s(\omega^2+s^2)}[1-e^{-\frac{s}{a}x}]$$

$$L^{-1}[\tilde{u}]=L^{-1}[\frac{1}{s(\omega^2+s^2)}]-L^{-1}[\frac{1}{s(\omega^2+s^2)}e^{-\frac{s}{a}x}]$$

$$=\sum_k \mathrm{Res}[\frac{e^{st}}{s(\omega^2+s^2)},s_k]-\sum_k \mathrm{Res}[\frac{e^{s(t-\frac{x}{a})}}{s(\omega^2+s^2)},s_k]$$



$$\sum_k \mathrm{Res}[\frac{e^{st}}{s(s^2+\omega^2)}, s_k]$$

$$= \lim_{s\to 0} s\cdot\frac{e^{st}}{s(s^2+\omega^2)} + \lim_{s\to j\omega}(s-j\omega)\frac{e^{st}}{s(s^2+\omega^2)}$$

$$+ \lim_{s\to -j\omega}(s+j\omega)\frac{e^{st}}{s(s^2+\omega^2)}$$

$$= \frac{1}{\omega^2} - \frac{1}{\omega^2}(\frac{e^{j\omega t}+e^{-j\omega t}}{2})$$

$$= \frac{1}{\omega^2}(1-\cos\omega t) = \frac{2}{\omega^2}\sin^2\frac{\omega}{2}t$$

$$t > \frac{x}{a} \qquad \sum_k \mathrm{Res}\left[\frac{e^{s(t-\frac{x}{a})}}{s(s^2+\omega^2)}, s_k\right] = \frac{1}{\omega^2} - \frac{e^{\omega j(t-\frac{x}{a})}}{2\omega^2} - \frac{e^{-\omega j(t-\frac{x}{a})}}{2\omega^2}$$

$$= \frac{1}{\omega^2} - \frac{e^{\omega j(t-\frac{x}{a})} + e^{-\omega j(t-\frac{x}{a})}}{2\omega^2}$$

$$= \frac{1-\cos\omega(t-\frac{x}{a})}{\omega^2}$$

$$= \frac{2}{\omega^2}\sin^2\frac{\omega}{2}(t-\frac{x}{a})$$

$$t \le \frac{x}{a} \qquad \sum_k \mathrm{Res}\left[\frac{e^{s(t-\frac{x}{a})}}{s(s^2+\omega^2)}, s_k\right] = \sum_{k=1}^{4} \mathrm{Res}\left[\frac{1}{s(s^2+\omega^2)e^{s(\frac{x}{a}-t)}}, s_k\right] = 0$$

$$\sum_{k}\mathrm{Res}[\frac{e^{s(t-\frac{x}{a})}}{s(s^{2}+\omega^{2})},s_{k}]=\begin{cases}\dfrac{2}{\omega^{2}}\sin^{2}\dfrac{\omega}{2}(t-\dfrac{x}{a}),(t>\dfrac{x}{a})\\ 0,(t\le\dfrac{x}{a})\end{cases}$$

$$\boldsymbol{u=L^{-1}(\tilde{u})=\begin{cases}\dfrac{2}{\omega^{2}}\sin^{2}\dfrac{\omega}{2}t-\dfrac{2}{\omega^{2}}\sin^{2}\dfrac{\omega}{2}(t-\dfrac{x}{a}),(t>\dfrac{x}{a})\\ \dfrac{2}{\omega^{2}}\sin^{2}\dfrac{\omega}{2}t,(t\le\dfrac{x}{a})\end{cases}}$$