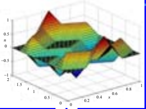


Bessel方程的求解

n 阶Bessel方程为

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

n为任意实数和复数

不妨暂先假定 n>=0

设方程有一个级数解形式为

$$y=x^c(a_0+a_1x+a_2x^2+\cdots a_kx^k+\cdots)$$

$$=\sum_{k=0}^{\infty}a_kx^{c+k} \qquad a_0\neq 0$$

取 $c=n$

$$a_1=a_3=a_5=a_7=\cdots=0$$

$$a_k=\frac{-a_{k-2}}{k(2n+k)}$$

$$a_2=\frac{-a_0}{2(2n+2)},a_4=\frac{a_0}{2\cdot 4(2n+2)(2n+4)},$$

$$a_6=\frac{-a_0}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)}$$

$\cdots$

$$a_{2m}=(-1)^m\frac{a_0}{2\cdot 4\cdot 6\cdots 2m(2n+2)(2n+4)\cdots(2n+2m)}$$

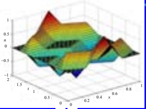
$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$$

一般项为

$$(-1)^m \frac{a_0 x^{n+2m}}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$$

$$a_{2m} = (-1)^m \frac{1}{2^{n+2m} m!\Gamma(n+m+1)}$$

$$y_1 = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} m!\Gamma(n+m+1)}, (n \geq 0)$$



n阶第一类Bessel函数

$$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}m!\Gamma(n+m+1)},(n\geq 0)$$

$$\Gamma(n+m+1)=(n+m)!,n\in Z^+$$

取时c=－n，用同样方法可得另一特解

$$J_{-n}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{-n+2m}}{2^{-n+2m}m!\Gamma(-n+m+1)},(n\neq 1,2,\cdots)$$

1. 当 n 不为整数时

$$J_n(x)\approx\frac{1}{\Gamma(n+1)}(\frac{x}{2})^n\to 0$$

$$J_{-n}(x)\approx\frac{1}{\Gamma(-n+1)}(\frac{x}{2})^{-n}\to\infty$$

$$y=AJ_n(x)+BJ_{-n}(x)$$

例3 试证半奇阶Bessel函数 $J_{\frac{1}{2}}(x)=\sqrt{\dfrac{2}{\pi x}}\sin x$

证明：

$$J_{1/2}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{1/2+2m}}{2^{1/2+2m}m!\Gamma(1/2+m+1)}$$

$$\Gamma(x+1)=x\Gamma(x)$$

$$\Gamma(\frac{3}{2}+m)=\frac{1\cdot3\cdot5\cdots\cdots(2m+1)}{2^{m+1}}\sqrt{\pi}$$

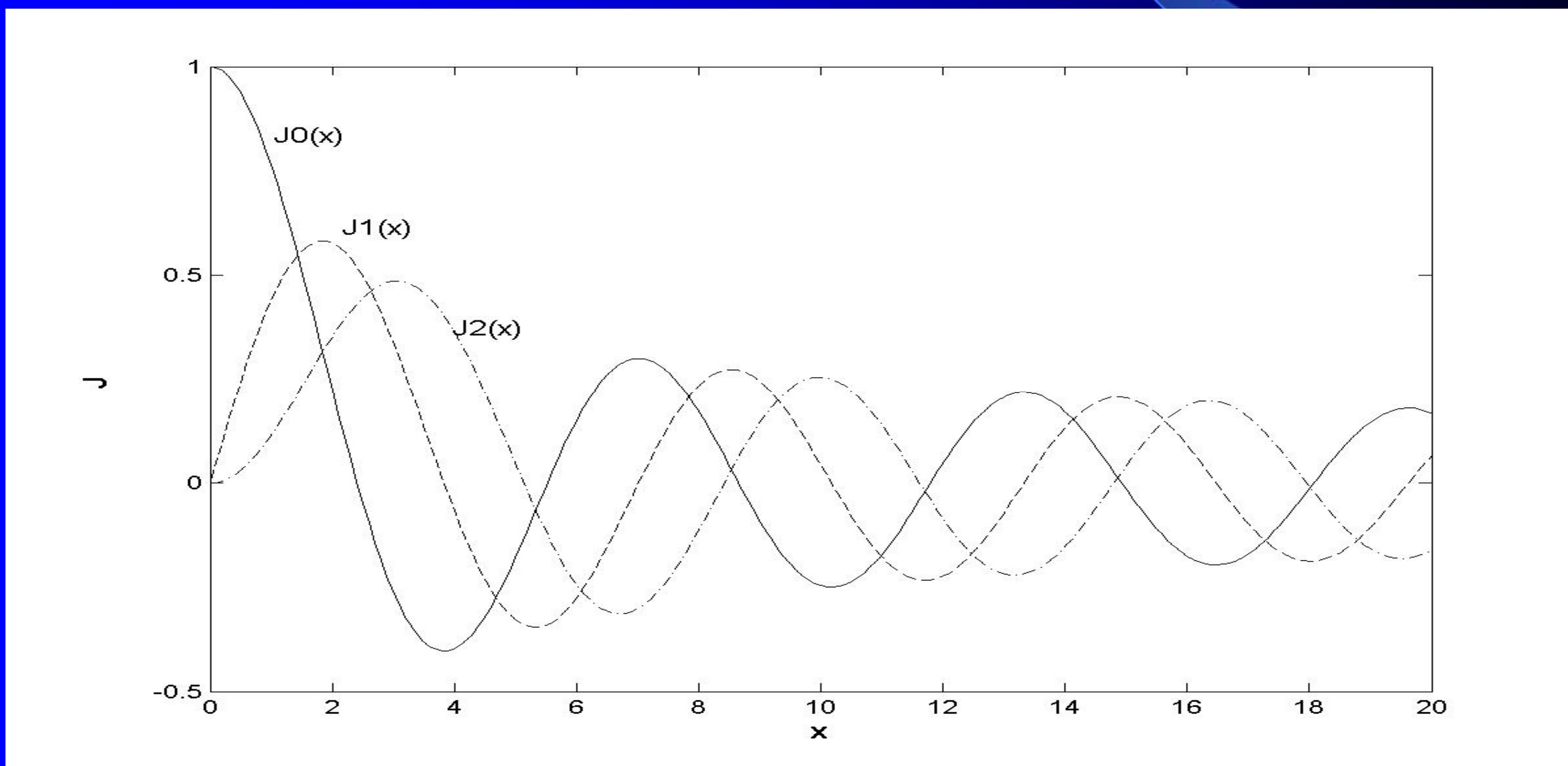
$$\Gamma(\frac{1}{2})=\sqrt{\pi}$$

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}=\sqrt{\frac{2}{\pi x}}\sin x$$

2 当 n 为整数时

$$J_0(x)=1-\left(\frac{x}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4-\frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6+\cdots$$

$$J_1(x)=\frac{x}{2}-\frac{1}{2!}\left(\frac{x}{2}\right)^3+\frac{1}{2!3!}\left(\frac{x}{2}\right)^5-\cdots$$



$$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}m!\Gamma(n+m+1)},n\geq 0$$

$$J_{-n}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{-n+2m}}{2^{-n+2m}m!\Gamma(-n+m+1)}$$

$$=(x/2)^n\sum_{m=0}^{\infty}(-1)^m\frac{(x/2)^{2(m-n)}}{m!\Gamma(m-n+1)}$$

$$=(-1)^n\sum_{k=0}^{\infty}(-1)^k\frac{(x/2)^{2k+n}}{k!\Gamma(k+n+1)}=(-1)^nJ_n(x)$$

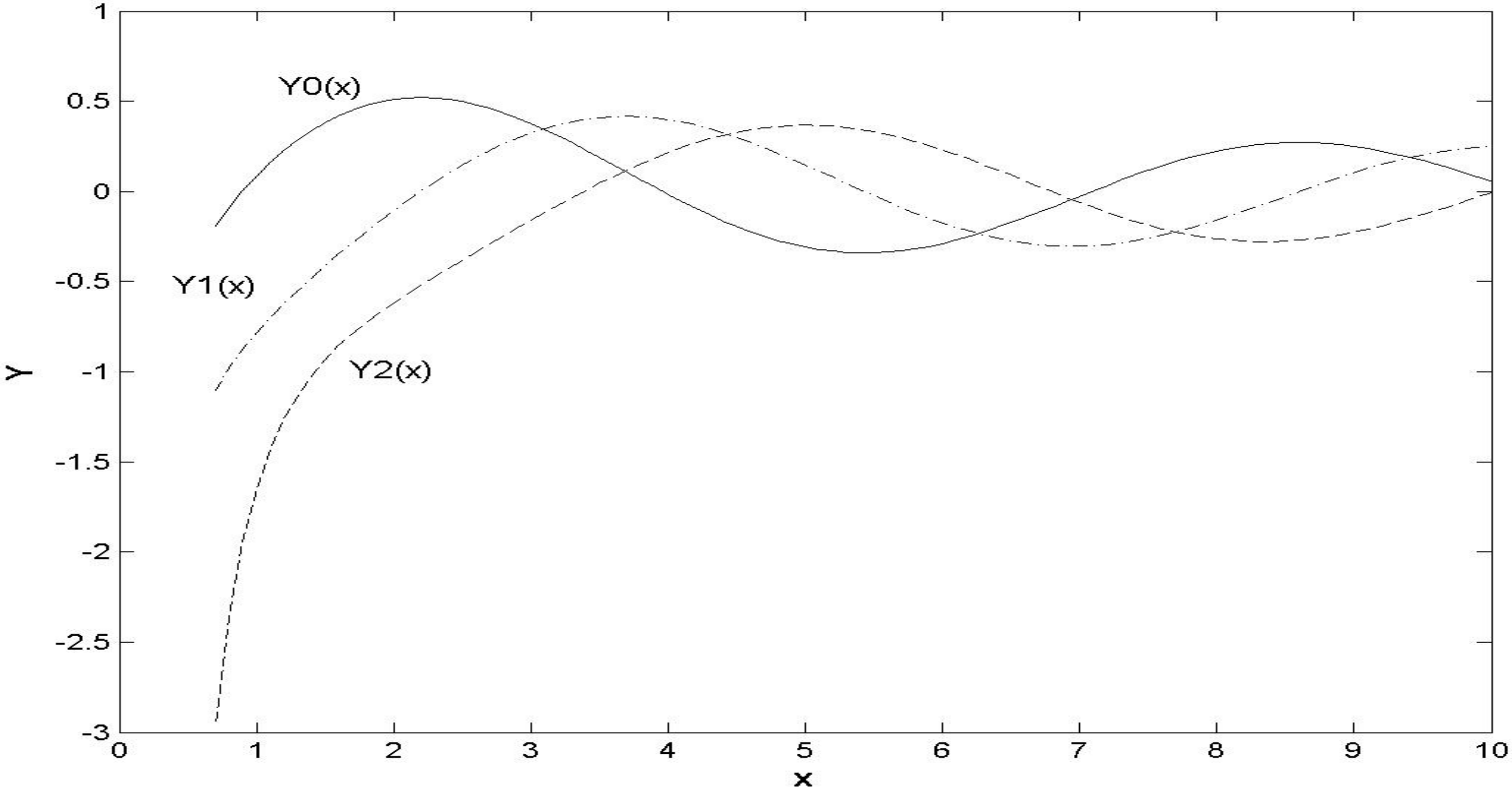
定义第二类Bessel函数为

$$Y_n(x) = \lim_{\alpha \to n} \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}$$

它既满足Bessel方程，又与 $J_n(x)$线性无关

$$Y_n(x) = \frac{2}{\pi} J_n(x)(\ln\frac{x}{2} + c) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}(\frac{x}{2})^{-n+2m}$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{n+2m}}{m!(n+m)!}(\sum_{k=0}^{n+m-1}\frac{1}{k+1} + \sum_{k=0}^{m-1}\frac{1}{k+1})$$


Y0(x)
Y1(x)
Y2(x)
Y
x
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
9
10

$$c = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n) = 0.5772\cdots$$

不论n是否为整数，**Bessel**方程的通解都可表示为

$$y = AJ_n(x) + BY_n(x)$$

§6.2 Bessel函数的母函数及递推公式

（一）Bessel函数的母函数（生成函数）

$$e^{\frac{x}{2}(z-\frac{1}{z})}=\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{2}\right)^{k}z^{k}\right]\left[\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{x}{2}\right)^{l}(-z)^{-l}\right]$$

$$e^{\frac{x}{2}(z-\frac{1}{z})}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{k!\,l!}\left(\frac{x}{2}\right)^{k+l}z^{k-l}$$

$$=\sum_{n=-\infty}^{\infty}\left[\sum_{l=0}^{\infty}\frac{(-1)^{l}}{(n+l)!\,l!}\left(\frac{x}{2}\right)^{2l+n}\right]z^{n}=\sum_{n=-\infty}^{\infty}J_{n}(x)z^{n}$$

$$e^{jx\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x) j^n e^{jn\theta}$$

$$= J_0(x) + \sum_{n=1}^{\infty} \left[J_n(x) j^n e^{jn\theta} + J_{-n}(x) j^{-n} e^{-jn\theta} \right]$$

$$= J_0(x) + 2\sum_{n=1}^{\infty} j^n J_n(x) \cos n\theta$$

$$\cos(x\cos\theta) = J_0(x) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(x)\cos 2m\theta$$

$$\sin(x\cos\theta) = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(x)\cos(2m+1)\theta$$

例1 用母函数证明：

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)$$

把母函数x换成 x+y

$$\sum_{n=-\infty}^{+\infty} J_n(x+y)z^n = \exp\left\{\frac{x+y}{2}(z-z^{-1})\right\}$$

$$=\sum_{k=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_k(x)J_m(y)z^{k+m}$$

$$=\sum_{n=-\infty}^{+\infty}\left(\sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)\right)z^n$$

$$J_n(x+y)=\sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)$$

（二）Bessel函数的积分表达式

罗朗展式的系数公式

$$C_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$$

罗朗展式的系数公式

$$J_n(x) = \frac{1}{2\pi j}\int_{.C} \frac{e^{\frac{x}{2}(z-\frac{1}{z})}}{z^{n+1}}dz$$

取C为单位圆

$$J_n(x) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{jx\sin\theta}(e^{j\theta})^{-n-1}\, je^{j\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(x\sin\theta - n\theta)} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta) d\theta, n = 0, \pm 1, \pm 2, \cdots$$

Bessel函数的递推公式

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots + (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \cdot 2!} + \frac{x^5}{2^5 \cdot 2!3!} - \frac{x^7}{2^7 \cdot 3!4!} + \cdots + (-1)^k \frac{x^{2k+1}}{2^{2k+1} k!(k+1)!} + \cdots$$

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x)$$

$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x)$$

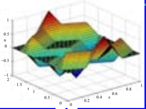
Bessel函数的递推公式

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2}{x} nJ_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

第二类Bessel函数递推公式

$$\begin{cases} \dfrac{d}{dx}[x^n Y_n(x)] = x^n Y_{n-1}(x) \\ \dfrac{d}{dx}[x^{-n} Y_n(x)] = -x^{-n} Y_{n+1}(x) \\ Y_{n-1}(x) + Y_{n+1}(x) = \dfrac{2n}{x} Y_n(x) \\ Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x) \end{cases}$$



例 利用递推公式求： $J_{3/2}(x)$

解：Bessel函数的递推公式

$$J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)-J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(-\cos x+\frac{1}{x}\sin x\right)$$

$$=-\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\cdot\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

$$=-\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\sin x}{x}\right)$$

例 计算 $\int x^3 J_0(x)dx$

解：

$$\int x^3 J_0(x)dx = \int x^2 (xJ_0(x))dx$$
$$= x^3 J_1(x) - 2\int x^2 J_1(x)dx$$
$$= x^3 J_1(x) - 2x^2 J_2(x) + C$$

$J_n(x)$ 零点

$$x_k \approx k\pi + \frac{n\pi}{2} + \frac{3\pi}{4},\ (k \in Z)$$

1．$J_n(x)$ 有无穷多个单重实零点，且这无穷多个零点在轴上关于原点是对称分布的。因而，$J_n(x)$ 必有无穷多个正的零点；

2．$J_n(x)$ 的零点与 $J_{n+1}(x)$ 的零点是彼此相间分布的，即的任意两个相邻零点之间必存在一个且仅有一个的零点；

3．以 $\mu_m^{(n)}$ 表示的正零点，则

$$\lim_{m\to+\infty}\left(\mu_{m+1}^{(n)} - \mu_m^{(n)}\right) = \pi$$

§7.3 贝塞尔函数的正交性及其应用

定理1 Bessel函数系

$$\left\{ J_n\left(\frac{\mu_m^{(n)}}{R} r\right) \right\} (m = 1,\ 2,\ \cdots)$$

具有正交性：

$$\int_0^R r J_n\left(\frac{\mu_m^{(n)}}{R} r\right) J_n\left(\frac{\mu_k^{(n)}}{R} r\right) \mathrm{d}r = \begin{cases} 0,\ (m \neq k) \\ \dfrac{1}{2} R^2 J_{n-1}^2(\mu_m^{(n)}) = \dfrac{1}{2} R^2 J_{n+1}^2(\mu_m^{(n)}),\ m = k \end{cases}$$

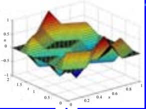
证明：

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F_1(r)}{\mathrm{d}r}\right]+\left[\left(\frac{\mu_m^{(n)}}{R}\right)^2 r-\frac{n^2}{r}\right]F_1(r)=0,\qquad \frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F_2(r)}{\mathrm{d}r}\right]+\left[a^2 r-\frac{n^2}{r}\right]F_2(r)=0$$

$$\left[\left(\frac{\mu_m^{(n)}}{R}\right)^2-a^2\right]\int_0^R rF_1(r)F_2(r)\mathrm{d}r+\int_0^R F_2(r)\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F_1(r)}{\mathrm{d}r}\right]\mathrm{d}r-\int_0^R F_1(r)\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F_2(r)}{\mathrm{d}r}\right]\mathrm{d}r=0$$

$$\left[\left(\frac{\mu_m^{(n)}}{R}\right)^2-a^2\right]\int_0^R rF_1(r)F_2(r)\mathrm{d}r+\{r[F_2(r)F_1{}'(r)-F_1(r)F_2{}'(r)]\}\Big|_0^R=0$$

$$\int_0^R rF_1(r)F_2(r)\mathrm{d}r=-\frac{R[F_2(R)F_1{}'(R)-F_1(R)F_2{}'(R)]}{\left(\frac{\mu_m^{(n)}}{R}\right)^2-a^2}$$



$$\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)\mathrm{d}r = \lim_{a\to\frac{\mu_m^{(n)}}{R}} \frac{-\mu_m^{(n)} J_n'(\mu_m^{(n)}) J_n'(aR)\cdot R}{-2a}$$

$$= \frac{R^2}{2}[J_n'(\mu_m^{(n)})]^2$$

$$\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)\mathrm{d}r = \frac{R^2}{2}J_{n-1}^2(\mu_m^{(n)}) = \frac{R^2}{2}J_{n+1}^2(\mu_m^{(n)})$$

定义1 定积分 $\int_0^R rJ_n^2\left(\frac{1}{R}\mu_m^{(n)}r\right)\mathrm{d}r$ 的平方根，称为**Bessel**函数 $J_n\left(\frac{1}{R}\mu_m^{(n)}r\right)$ 的模值。

定理2 若f(x)和f(x)的导函数在区间**[0, *R*]**至多有有限个跳跃型间断点，则f(x)在区间（**0, *R***）内在连续点处的**Bessel**展开级数收敛于该点的函数值，在间断点收敛于该点左右极限的平均值：

$$f(r)=\sum_{m=1}^{\infty}A_m J_n\left(\frac{\mu_m^{(n)}}{R}r\right) \qquad A_k=\frac{1}{\frac{R^2}{2}J_{n-1}^2(\mu_k^{(n)})}\int_0^R rf(r)J_n\left(\frac{\mu_k^{(n)}}{R}r\right)\mathrm{d}r$$

例1 设有半径R为的薄圆盘，其侧面绝热，若圆盘边界上的温度恒保持为零度，且初始温度为已知。求圆盘内的瞬时温度分布规律。

解：

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \left(x^2 + y^2 < R^2\right) \\ u(x, y, 0) = \varphi\left(x, y\right) \\ u\Big|_{x^2+y^2=R^2} = 0 \end{cases}$$

$$u(x, y, t) = V(x, y)T(t)$$

$$\frac{T'}{a^2 T} = \frac{V_{xx} + V_{yy}}{V} = -\lambda$$

$$T(t) = Ae^{-a^2\lambda t} \qquad V\Big|_{x^2+y^2=R^2} = 0$$

$$\begin{cases} \dfrac{\partial^2 V}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial V}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial \theta^2} + \lambda V = 0, (\rho < R) \\ V\Big|_{\rho = R} = 0 \end{cases}$$

$$V(\rho,\theta) = P(\rho)\Theta(\theta)$$

$$\Theta''(\theta) + \mu\Theta(\theta) = 0$$

$$\rho^2 P''(\rho) + \rho P'(\rho) + (\lambda\rho^2 - \mu)P(\rho) = 0$$

$$\begin{cases} \Theta''(\theta) + \mu\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(2\pi + \theta) \end{cases}$$

$$\mu_n = n^2$$

$$\Theta_0(\theta) = \frac{a_0}{2}(\text{为常数})$$

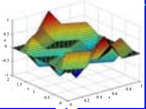
$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, (n = 1, 2, \cdots)$$

$$\begin{cases} \rho^2 P''(\rho) + \rho P'(\rho) + (\lambda\rho^2 - n^2)P(\rho) = 0 \\ P(R) = 0, \left|P(0)\right| < +\infty \end{cases}$$

$$\boldsymbol{r^2 F''(r) + rF'(r) + (r^2 - n^2)F(r) = 0}$$

$$F(r) = AJ_n(r) + BY_n(r)$$

$$P(\rho) = AJ_n(\sqrt{\lambda}\rho) + BY_n(\sqrt{\lambda}\rho)$$



$$\sqrt{\lambda}R = \mu_m^{(n)}, (m = 1, 2, \cdots)$$

$$\sqrt{\lambda} = \mu_m^{(n)} / R, (m = 1, 2, \cdots)$$

$$P_{mn}(\rho) = J_n\left(\frac{\mu_m^{(n)}}{R}\rho\right)$$

$$u = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty} c_{mn}u_{mn} = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty} c_{mn}\left(P_{mn}(\rho)\Theta_n(\theta)T_{mn}(t)\right)$$

$$u(x, y, 0) = \varphi\left(r, \theta\right) = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty} c_{mn}\left(P_{mn}(\rho)\Theta_n(\theta)T_{mn}(0)\right)$$