Complex analysis

Mike Antonenko

Based on lectures by Sergey Kislyakov

Typeset on March 2, 2021

Contents

Notation

For the rest of this course, $\mathbb{N}$ contains 0.

Introduction

Definition. Unless stated otherwise, $G \subseteq \mathbb{C}$ and $H \subseteq \mathbb{R}^n$ are arbitrary domains.

Definition. A function $f : G \to \mathbb{C}$ is *analytic*, if for any $z_0 \in G$ there exists $r > 0$ such that $D_r(z_0) \subseteq G$, and

$$f(z) = \sum_{n \in \mathbb{N}_0} a_n (z - z_0)^n$$

for some $\{a_n\}$ and every $z \in {}_r(z_0)$.

Definition. $\varphi : G \to \mathbb{C}$ is *holomorphic* at z_0, iff exists

$$\varphi'(z_0) = \lim_{h \to 0} \frac{\varphi(z_0 + h) - \varphi(z_0)}{h}.$$

Here $h \in \mathbb{C}$. If a function f is holomorphic at every point of $E \subseteq \mathbb{C}$, we write $f \in \operatorname{Hol} E$.

Definition. A function $f \in \operatorname{Hol} \mathbb{C}$ is called *entire*.

Theorem (Cauchy-Riemann equations). $f : G \to \mathbb{C}$ is holomorphic at z_0 iff

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, xxx \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

at (x_0, y_0), where $u(x, y) + iv(x, y) = f(x + iy)$ and $z_0 = x_0 + iy_0$.

That is, the Jacobi matrix of $u \times v$ is in the image of the standard embedding of $\mathbb{C}$ into $\mathrm{M}_2(\mathbb{R})$:

$$a + ib \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Has been proven in semester II.. ■

Lemma. If $f: G \to \mathbb{C}$ is analytic, then it is holomorphic.

Proof. The series for $f(z)$ can be differentiated. ■

Theorem (Cauchy)**.** Let $f: G \to \mathbb{C}$ be holomorphic. Then it is analytic at every $x_0 \in G$, with the radius of convergence r being equal to $r = \mathrm{dist}\left(x_0, \mathbb{C} \setminus G\right)$.

The proof will be given shortly.

Differential forms

A reminder

Definition. If we have a form

$$\omega(h) = \sum_I \omega_I \cdot h^I,$$

its integral (also a form) is defined as

$$\int \omega = \int \sum_I \omega_I \circ x \cdot D_I,$$

where D_I is the determinant of the rows I of the Jacobi matrix $\mathrm{d}x$.

Integral of a form along a path

Definition (integral along a curve)**.** Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a C^1 function. If $\varphi = f_1\,\mathrm{d}x_1 + \cdots + f_n\,\mathrm{d}x_n$, where f_i are continuous complex functions on G, then

$$\int_\gamma \varphi := \sum_{j=1}^{n} \int_{t=a}^{b} f_j(\gamma(t))\gamma_j'(t).$$

Evidently, the integral over a one-dimensional submanifold does not depend on parametrisation. We will further use that to write integrals over subsets of $\mathbb{C}$, not curves.

Remark. We may only require that γ is rectifiable. In this case, the integral will be in the sense of Stieltjes:

$$\int_\gamma \Phi = \sum_{j=1}^{n} \int_{t=a}^{b} f_j(\gamma(t))\,\mathrm{d}\gamma_j.$$

We will not need this during this course.

Lemma. Integral of differential forms along a path is linear with respect to the form.

Proof. Evident. ∎

Lemma (change of variables)**.** Let $\alpha\colon [c,d] \to [a,b]$ be a C^1-homeomorphism, and $\widetilde{\gamma} = \gamma \circ \alpha$. Then

$$\int_{\widetilde{\gamma}} \varphi = \pm \sum_{j=1}^{n} \int_{t=c}^{d} f_j \circ \gamma \circ \alpha(s) \cdot \gamma_j' \circ \alpha(s) \cdot \alpha'(s).$$

The sign here depends on whether α is increasing or decreasing.

Proof. Follows from the change-of-variables formula for the Riemann integral. ∎

Definition. Let α, β be C^1-paths. Their *concatenation* $\alpha\beta$ is defined as

$$\gamma(t) = \begin{cases} \alpha(t), & t \in [a,b], \\ \beta \circ \varphi(t), & t \in [b,c], \end{cases}$$

where $\varphi\colon [b,c] \to [a', b']$ is a homeomorphism.

Definition. A path γ is *piecewise smooth*, iff it is a finite concatenation of smooth paths.

Definition. The integral of a form along a piecewise smooth path is the sum of integrals over its components.

Definition. If $\varphi = \sum \varphi_j \,\mathrm{d}x_j$ is a differential form, we denote

$$\|\varphi\| = \sqrt{\sum_{j=1}^{n} \varphi_j^2}.$$

Differential 1-forms are simply functions between Euclidean spaces.

Theorem (principal estimate)**.** If γ is piecewise smooth and φ is a continuous differential 1-form in a neighbourhood of $\operatorname{im}\gamma$, then

$$\left| \int_\gamma \varphi \right| \leq l(\gamma) \cdot \sup_{x \in \operatorname{im}\gamma} \|\varphi(x)\|.$$

Proof idea. CBS. ∎

Antiderivatives

Definition. Let ω be be a differential 1-form in G. Its *derivative* is the form

$$d\omega = \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial x} \, dx_j.$$

Definition. Let $G \subseteq \mathbb{R}^n$ be a differential form. $F \colon G \to \mathbb{C}$ is called the *antiderivative* of Φ, iff $dF = \Phi$.

Definition. A differential form ω is

1. *exact*, iff it has an antiderivative;
2. *closed*, iff every point $x \in G$ has a neighbourhood where ω is exact.

Observe that this definition differs from the one given in the semester III. This one is more general: the previous one depended on smoothness.

Lemma. Suppose ω is a C^1 differential 1-form in G. Then ω is closed iff

$$\partial_i \omega_j = \partial_j \omega_i$$

for all $i, j \in \{1, \ldots, n\}$.

Proof of $\Rightarrow$. Locally, we have an antiderivative Ω, so $\partial_j \Omega = \omega_j$. Then

$$\partial_i \omega_j = \partial_i \partial_j \Omega = \partial_j \partial_i \Omega = \partial_j \omega_i.$$

■

Proof of $\Leftarrow$. We know from semester III that every differential form ω such that $d\omega = 0$ is exact. But this is true of ω. ■

Lemma. Let γ be a piecewise smooth path with $\operatorname{im}\gamma \subseteq G$ and ends A, B. Then

$$\int_\gamma \mathrm{d}F = F(B) - F(A). \tag{1}$$

Proof. From the Newton-Leibniz formula. ■

Theorem. Every two points in G can be connected by piecewise linear path.

A well-known fact. ■

Definition. Let Φ be a differential form in a region $H \subseteq \mathbb{R}^n$. We call H a Φ-*balanced*[1] region, iff $\int_\gamma \Phi = 0$ for every closed curve γ with $\operatorname{im}\gamma \subseteq H$.

Theorem (reformulations of 'exact'). Let Φ be a differential form in $H \subseteq \mathbb{R}^n$ with continuous coefficients. Equivalent are:

1. $\int_\gamma \Phi$ depends on A and B only.
2. H is Φ-balanced.
3. Φ is exact.

Proof of $3 \Rightarrow 2$. See (1). ■

Proof of $2 \Rightarrow 1$. Concatenate paths between two points and get a closed path. But the integral (which is zero by hypothesis) splits into two. Next we use that it depends on the direction of the path. ■

[1] My own term.

Proof of 1 $\Rightarrow$ 3. Fix $A_0 \in G$. For any $x \in H$, let γ be a piecewise linear path $A_0 \rightsquigarrow x$. Define

$$F(x) = \int_{\gamma} \Phi.$$

Since the integral depends only on x, this is correctly defined function We assert that F is an antiderivative for Φ; that is, the partial derivatives of F are components of Φ. To see this, consider the path

$$A_0 \rightsquigarrow_{\gamma} x \rightsquigarrow_{\beta} x + te_j,$$

where the last part β is linear. Then

$$\begin{aligned}
\frac{F(x+te_j) - F(x)}{t} &= \frac{\int_\gamma \Phi + \int_\beta \Phi - \int_\gamma \Phi}{t} \\
&= \frac{1}{t} \int_{\beta} \Phi \\
&= \frac{1}{t} \sum_{k=1}^{n} \int_{\tau=0}^{t} f_k\left(x + \tau e_j\right) \beta_k'(\tau) \\
&= \frac{1}{t} \int_{\tau=0}^{t} f_j\left(x + \tau e_j\right) \\
&\xrightarrow[t \to 0]{} f_j(x).
\end{aligned}$$

■

Definition. We call a region $H \subseteq \mathbb{R}^2$ *rectangle-astroid*[2], iff there exists $x_0 \in H$ such that for every $x \in H$ the 2-dimensional rectangle with sides parallel to the axes, having x_0 and x as diametral points, lies in H together with its closure. The rectangle in this context is called *central*.

This definition is long, but it highlights what we need to use to prove the following proposition for circles.

Lemma (addition to the theorem). Let $H \subseteq \mathbb{R}^2$ be a rectangle-astroid region. Then H being Φ-balanced is also equivalent to

$$\int_{Q} \Phi = 0$$

for every 1-dimensional central rectangle Q.

[2]My own term.

Proof idea. The integral over this rectangle is equal to zero, since it is closed. Conversely, we can find an antiderivative like in the proof of the theorem. ■

Theorem (Cauchy, Morera)**.** Let γ be a closed continuous curve with $\operatorname{im}\gamma \subseteq G$. Then a function $f: G \to \mathbb{C}$ is holomorphic iff

$$\int_\gamma f = 0.$$

Connectivity

Lemma. Every two points of a domain $H \subseteq \mathbb{R}^n$ can be connected by a piecewise-linear path.

Proof. Fix $x_0 \in H$ and put A to be the set of all points reachable from x_0 by a piecewise-linear path over H. A is open, since H is: every point in A is the centre of a ball in H, and balls are convex. The complement $H \setminus A$ is open for the same reasons: take any point x from there, there is a ball $B \subseteq H$ around it; if there was a point of A in this B, we could connect it to x. Therefore, A is open and closed; but it is not empty either, so $A = H$ by connectedness of the domain H. ■

Recall the theorem on equivalence of linear connectivity and connectedness for locally linearly connected spaces. The proof is almost the same.

Theorem. Every two points of a domain $H \subseteq \mathbb{R}^n$ can be connected by a C^∞ path.

We need to recall some stuff about convolutions for the proof.

Definition. A family $\{\varphi_s\}_{s>0}$ of infinitely smooth functions $\mathbb{R}^n \to \mathbb{R}$ with compact support and such that

$$\int \varphi_s = 1, \qquad \varphi_s(x) = \frac{\varphi_1(x/s)}{s^n}$$

for all $s > 0$ and $x \in \mathbb{R}^n$ is called a *standard approximative unit* or a *mollifier*.

Theorem.

1. A family $\{\varphi_\square\}$ exists.
2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then the functions $\varphi_s * f$ converge to f uniformly with $s \to 0$ and are infinitely smooth.

The *proof* was given in the third semester. Now we start with the main theorem.

Proof. Let $\gamma\colon [a,b] \to H$ be a continuous path $A \rightsquigarrow B$. Continue γ to a continuous path $\mathbb{R} \to H$ by setting

$$\gamma(t) = \begin{cases} \gamma(a), & t < a, \\ \gamma(b), & t > b, \\ \gamma(t), & t \in [a,b]. \end{cases}$$

Let $\varphi_\square$ be a standard approximative unit for functions on $\mathbb{R}$, and put for all $s > 0$

$$\widehat{\gamma}_s(x) = (\varphi_s * \gamma)(x),$$

where $*$ denotes component-wise convolution. Fix $\epsilon > 0$, $a_2 < a$, and $b_2 > b$. By the theorem on approximating with units, if s is small enough, we have $\left|\widehat{\gamma}_s(t) - \gamma(t)\right| < \epsilon$ for all $t \in [a_2, b_2]$. This ϵ can be chosen in such a way that the path $\widehat{\gamma}_s$ does not leave the domain H. Further, choose a_2, b_2 such that

$$\widehat{\gamma}_s(a_2) = \gamma(a), \qquad \widehat{\gamma}_s(b_2) = \gamma(b).$$

GAP

This ensures $\widehat{\gamma}|_{[a_2,b_2]}$ is indeed the path we need. ■

Closed forms and balanced regions

Theorem (reformulations of 'closed')**.** Let Φ be a differential form in $H \subseteq \mathbb{R}^n$ with continuous coefficients f_j. The following are equivalent:

1. Φ is closed.
2. Every $x \in H$ has a Φ-balanced neighbourhood $U \subseteq H$.

In case $n = 2$, two more reformulations are true:

3. For every $x \in H$, there exists a rectangle-astroid region $B \ni x$ such that $\int_Q \Phi = 0$ for any central rectangle Q.
4. $\int_Q \Phi = 0$ for any rectangle Q such that $\operatorname{cl} Q \subseteq H$.

Proof of 3 $\Rightarrow$ 4. Split the rectangle Q into equal four, $\{Q_i\}$. Then the integral over Q is equal to the sum of integrals over $\{Q_i\}$. Cover the compact $\operatorname{cl} Q$ with 'good' rectangle-astroid regions which lie in H completely (they exist by hypothesis). Let $\delta > 0$ be the Lebesgue number of this cover. Continue to split the rectangles into four until each of them is less than δ in diameter. Now it is clear that the integral over Q is zero itself. ■

Proof of 4 $\Rightarrow$ 3. G is open. ■

Change of basis

Let $z = x + iy$. Then

$$\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y \qquad\qquad \mathrm{d}\overline{z} = \mathrm{d}x + i\,\mathrm{d}y,$$

so

$$\mathrm{d}x = \frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2}, \qquad \mathrm{d}y = \frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2}.$$

Now let $\varphi = u\,\mathrm{d}x + v\,\mathrm{d}y$ be a 1-form, defined in $H \subseteq \mathbb{R}^2$. Then, if $\varphi = \mathrm{d}\Phi$ for a function $\Phi\colon H \longrightarrow \mathbb{C}$, we have

$$\Phi = \frac{\partial_1\Phi - \partial_2\Phi}{2}\,\mathrm{d}z + \frac{\partial_1\Phi - \partial_2\Phi}{2}\,\mathrm{d}\overline{z}$$

by direct computation. By analogy, we define

Definition.

$$\partial_z\Phi := \frac{\partial_1\Phi - i\partial_2\Phi}{2} \qquad \partial_{\overline{z}}\Phi := \frac{\partial_1\Phi + i\partial_2\Phi}{2}.$$

Let $\Phi = p + iq$. It then can be derived that

$$\partial_{\overline{z}}\Phi = \frac{\partial_1 p - \partial_2 q}{2} + i\frac{\partial_1 q + \partial_2 p}{2}.$$

Therefore,

$$\partial_{\overline{z}}\Phi = 0 \iff \begin{cases} \partial_1 p = \partial_2 q, \\ \partial_1 q = -\partial_2 p. \end{cases}$$

These are the Cauchy-Riemann equations. Therefore,

Lemma. A function $\Phi\colon G \longrightarrow \mathbb{C}$ is holomorphic iff $\mathrm{d}\Phi = \partial_z\Phi\,\mathrm{d}z$.

We can also prove the following:

Lemma. Let $\alpha\,\mathrm{d}z$ be a form in G. It is exact iff there exists a holomorphic function $A\colon G \longrightarrow \mathbb{C}$ such that $A' = \alpha$.

Proof of $\Rightarrow$. Let A be the antiderivative. $\alpha\,\mathrm{d}z = \alpha\,\mathrm{d}x + i\alpha\,\mathrm{d}y$. Then $\partial_1 A = \alpha$, $\partial_2 A = i\alpha$, so

$$A' = \partial_z A = \frac{\partial_1 A - i\partial_2 A}{2} = \alpha, \qquad \partial_{\overline{z}}A = \frac{\partial_1 A + i\partial_2 A}{2} = 0.$$

■

Lemma (a variation of the principal estimate)**.** Let $\alpha\colon G \to \mathbb{C}$ be a function, $\gamma\colon [a,b] \to G$ a C^1 path. Then

$$\left|\int_\gamma \alpha \, \mathrm{d}z\right| \leq l(\gamma) \cdot \sup_{z \in \operatorname{im}\gamma} \left|\alpha(z)\right|.$$

Proof idea. Use the principal estimate and the identity $\gamma' = \gamma_1' + i\gamma_2'$. ■

Cauchy's theorem on closedness

Theorem (Cauchy, on closedness)**.** If $f\colon G \to \mathbb{C}$ is holomorphic, then the form $f \, \mathrm{d}z$ is closed.

That is, locally, it has antiderivatives. The proof spans the several following pages and requires a few lemmas.

The case of continuous derivative

Lemma. Suppose f' is continuous. Then the form $f \, \mathrm{d}z$ is closed.

While this might seem to give a hint of our further course of action, we won't be so blunt as to prove the continuity of f' directly.

Proof idea. Indeed, as we know, the closedness is then equivalent to the equality

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

This is straightforward to show using Cauchy-Riemann equations. ■

Example. The form $\mathrm{d}z/z$ is closed and not exact. $f' = -1/z^2$ in this case is a continuous function. By the previous lemma, f is closed (in fact, its antiderivatives are logarithms). Now consider the unit circle $\mathbb{S}^1$. By parametrising with $e^{i\square}$, we can easily check that

$$\int_{\mathbb{S}^1} \frac{\mathrm{d}z}{z} = 2\pi i.$$

But this means we have found a non-balanced region of $\mathbb{C}$, so $\mathrm{d}z/z$ is not exact.

Indices

Lemma. Let $C = z_0 + r\mathbb{S}^1$ for some $r > 0$. Then

$$\int_C \frac{\mathrm{d}z}{z - z_1} = \begin{cases} 0, & |z_1 - z_0| > r, \\ 2\pi i, & |z_1 - z_0| < r. \end{cases} \tag{2}$$

Proof. Consider the case $|z_1 - z_0| > r$. Luckily, the form $\frac{\mathrm{d}z}{z - z_1}$ is closed in $H = \mathbb{C} \setminus \{z_1\}$. This H contains a rectangle around the square C. Every closed form is exact within this rectangle by a theorem from semester 3.

Suppose, for now, $|z_1 - z_0| < r$. We reduce this to the case $z_0 = z_1$, which has been considered in the example on page 14. Then

$$\begin{aligned}
\int_C \frac{\mathrm{d}z}{z - z_1} &= \int_C \frac{\mathrm{d}z}{(z - z_0) - (z_1 - z_0)} \\
&= \int_C \frac{\mathrm{d}z}{z - z_0} \cdot \frac{1}{1 - \frac{z_1 - z_0}{z - z_0}} \\
&= \int_C \frac{\mathrm{d}z}{z - z_0} \cdot \sum_{k \in \mathbb{N}} \left(\frac{z_1 - z_0}{z - z_0}\right)^k \\
&= 2\pi i + \sum_{k \in \mathbb{N} \setminus 0} \int_C \frac{\mathrm{d}z}{z - z_0} \left(\frac{z_1 - z_0}{z - z_0}\right)^k \\
&= 2\pi i.
\end{aligned}$$

■

The conclusion

Theorem (Cauchy, on closedness). If $f : G \to \mathbb{C}$ is holomorphic, then the form $f\,\mathrm{d}z$ is closed.

Proof. Suppose otherwise: there exists a 2-dimensional rectangle $P \subseteq G$ with $I := \int_{\partial P} f \neq 0$.

Subdivide it into four $\{Q_i\}$ such that

$$I = \sum_i \int_{\partial Q_i} f.$$

For one of them, Q_j, the modulus of the integral is at least one fourth the $|I|$. Denote $P_1 = Q_j$. Continuing this sequence, we get diminishing $\{P_j\}$ with a single point z_0 in their intersection. The function f is holomorphic at z_0, so we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z),$$

where $\varphi(z) = o(z - z_0)$ with $z \to z_0$. Select k such that $|\varphi(z)| < \epsilon |z - z_0|$ for all $z \in P_k$. Then we have

$$\begin{aligned}
\left|\frac{I}{4^k}\right| &\leq \left|\int_{\partial P_k} f\right| \\
&= \left|\int_{z \in \partial P_k} \big(\underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\text{these guys are exact in } P_k} + \varphi(z)\big)\right| \\
&= \left|\int_{z \in \partial P_k} \varphi(z)\right| \\
&\leq \epsilon \cdot (\operatorname{diam} P_k) \cdot S_1(\partial P_k) \\
&= \epsilon \cdot \frac{(\operatorname{diam} P) \cdot S_1(\partial P)}{4^k}.
\end{aligned}$$

But we thought that $I \neq 0$. ■

On correctible singularities

Lemma (on correctible singularities)**.** Let $a \in G$. Suppose the form $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y$ is closed in $G \setminus \{a\}$, and the coefficients f and g are continuous in G. Then ω is closed in G.

Proof idea. Approximate integrals with smaller rectangles. ■

The minor integral formula of Cauchy

Theorem (Cauchy, minor integral formula)**.** Let $f\colon G \to \mathbb{C}$ be holomorphic, $z_0, z_1 \in G$, $C = z_0 + r\mathbb{S}^1$, $\left|z_1 - z_0\right| < r$. Then

$$f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1}\,\mathrm{d}z.$$

'Probably the most important formula in complex analysis.' — S.K.

The 'greater' integral formula gives the same result for C not necessarily a circle.

Proof. Fix $z_0 \in G$. Consider

$$h(z) = \begin{cases} \frac{f(z)-f(z_1)}{z-z_1}, & z \neq z_1 \\ f'(z_1), & z = z_1. \end{cases}$$

h is continuous in G and holomorphic in $G \setminus \left\{z_0\right\}$. By Cauchy's theorem on closedness (page 15), the form $h\,\mathrm{d}z$ is closed in $G \setminus \{z_1\}$. By the lemma on correctible singularities (page 16), it is also closed in all of G. Let $\widehat{C}$ be a ball of slightly greater radius $r + \delta$, but still lying in G. Every closed form in $\widehat{C}$ is exact, so

$$\int_C h\,\mathrm{d}z = 0.$$

Rewriting this yields

$$f(z_1) \underbrace{\int_C \frac{\mathrm{d}z}{z - z_1}}_{=2\pi} = \int_C \frac{f(z)}{z - z_1}\,\mathrm{d}z.$$

∎

Cauchy's theorem on analyticity

Theorem (Cauchy, on analyticity)**.** Let f be holomorphic in G, $z_0 \in G$, $R = \operatorname{dist}(z_0, \partial G)$. For all $k \in \mathbb{N}$, define

$$c_k = \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{(z - z_0)^{k+1}}\,\mathrm{d}z.$$

Then

$$f(z) = \sum_{k \in \mathbb{N}} c_k (z - z_0)^k$$

for all z such that $|z - z_0| < R$. In particular, f is analytic in G.

Proof. Let C be a circle around z_0 of radius $r < R$. By Cauchy's minor integral formula from page 17, for any z inside of C we have

$$\begin{aligned}
f(z_1) &= \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - z_1}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{(z - z_0) - (z_1 - z_0)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int\limits_C \frac{1}{z - z_0} \cdot \frac{f(z)}{1 - \frac{z_1 - z_0}{z - z_0}}\,\mathrm{d}z \\
&= \sum_{k \in \mathbb{N}} \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - z_0} \cdot \left(\frac{z_1 - z_0}{z - z_0}\right)^k \mathrm{d}z \\
&= \sum_{k \in \mathbb{N}} (z_1 - z_0)^k \cdot \left(\frac{1}{2\pi i} \int\limits_C \frac{f(z)}{(z - z_0)^{k+1}}\,\mathrm{d}z\right),
\end{aligned}$$

which is desired. Transposing the sum with the integral is legit, since the series for geometric progression converges uniformly for all z. ∎

Remark. In the analyticity theorem,

$$c_k = \frac{f^{(k)}(z_0)}{k!}.$$

In particular,

$$f'(z) = \frac{1}{2\pi i} \int\limits_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta.$$

Morera's theorem

Corollary (Morera's theorem)**.** Let $f : G \to \mathbb{C}$ be a continuous function. The following conditions are equivalent:

1. f is analytic.
2. f is holomorphic.
3. $f\, \mathrm{d}z$ is closed.

Proof. $1 \Rightarrow 2$ follows by differentiating the series, $2 \Rightarrow 3$ is the Cauchy's theorem on closedness from page 15. We now give a proof of $3 \Rightarrow 1$. By definition, the form $f\, \mathrm{d}z$ locally has an antiderivative Φ. Then Φ is holomorphic, and $\Phi' = f$. By the analyticity theorem, Φ is holomorphic, which implies that f is, too (simply differentiate the series). ■

The mean value theorem

Lemma (mean value theorem)**.** If $C \subseteq G$ is a circle with the centre at $z_0 \in G$, then

$$f(z) = \frac{1}{2\pi} \int_C f(z)\, \mathrm{d}z.$$

Proof idea. Follows from the Cauchy's minor integral formula (page 17). ■

Maximal modulus principle

Theorem (maximal modulus principle)**.** Let $f : G \to \mathbb{C}$ be holomorphic. Then $|f|$ has no strict maximum in G.

Proof. Suppose f has a non-strict maximum in G. Then

$$\begin{aligned}\left|f(a)\right| &= \frac{1}{2\pi}\left|\int\limits_C f(z)\,\mathrm{d}z\right| \\ &\le \frac{1}{2\pi}\int\limits_C \left|f(z)\right|\mathrm{d}z \\ &\le \left|f(a)\right|.\end{aligned}$$

Here every inequality must be an equality. Then $\left|f(z)\right| = \left|f(a)\right|$ for all z in some disk D around a.

Suppose $f'(a) \neq 0$. Then f is a local homeomorphism around a, which means it must map a ball in $\mathbb{C}$ into a ball in $\mathbb{C}$. But in any ball there are points with varying modulus.

Therefore, f is constant, what completes the proof. ∎

Liouville's theorem

Theorem (Liouville)**.** A bounded entire function is constant.

A powerful theorem.

Proof. From principal estimate and the formula for f':

$$\begin{aligned}\left|f'(z)\right| &\le \frac{M}{2\pi}\int\limits_C \frac{r}{(r-|z|)}\,\mathrm{d}t \\ &= \frac{Mr}{(r-|z|)^2} \\ &\xleftarrow[r\to\infty]{} 0.\end{aligned}$$

∎

Principal theorem of algebra

Corollary (principal theorem of algebra)**.** $\mathbb{C}$ is algebraically closed.

Proof. Suppose $p\colon \mathbb{C} \to \mathbb{C}$ of $\deg p > 0$ has no roots: $|p| > \delta$ for some $\delta > 0$. Then $1/p$ is an entire function, and, as such, is either unbounded or constant. It is not constant, so it must be unbounded; but $1/|p| < 1/\delta$ — a contradiction. ■

Harmonic functions

Definition. A function $f\colon G \to \mathbb{R}$ is *harmonic*, iff there exists a holomorphic $g\colon G \to \mathbb{C}$ such that $\operatorname{Re} g = f$. If $\widetilde{f} = \operatorname{Im} g$, f and $\widetilde{f}$ are said to be *harmonic conjugates*.

Lemma (Laplace's equations)**.** If $u + iv \in \operatorname{Hol} G$, then

$$u'_{xx} + v'_{yy} = 0.$$

Proof. From Cauchy-Riemann equations. ■

Definition. The differential operator

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^2$$

is called the *Laplace operator*.

Definition. A function $f\colon H \to \mathbb{R}$ is called *harmonic*, iff

$$\Delta f = 0.$$

Theorem. If $u \in C^2(G)$ and $\Delta u = 0$, then every $a \in G$ has a neighbourhood U_a such that $u|_{U_a}$ is harmonic.

Proof. Let $\varphi = \frac{\partial u}{\partial x}$ and $\psi = -\frac{\partial u}{\partial y}$. We have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y},$$

so there exists a function v such that $\frac{\partial v}{\partial y} = \varphi$ and $\frac{\partial v}{\partial x} = \psi$. This v is harmonically conjugate to u, since the Cauchy-Riemann equations are satisfied. ■

Integrals of closed forms

Lemma. Suppose $\varphi = \mathrm{d}\Phi$ in G. Then

$$\int_\gamma \varphi = \Phi(\gamma(b)) - \Phi(\gamma(a))$$

for every path $\gamma\colon [a,b] \to G$.

Proof. By the theorem on exact forms. ■

Definition. A continuous function $F\colon [a,b] \to \mathbb{C}$ is called the *antiderivative along a path* $\gamma\colon [a,b] \to G$ of a 1-form φ, iff for every $t \in [a,b]$ exists an antiderivative Φ in some neighbourhood $V_t \ni \gamma(t)$ such that $F(t) = \Phi \circ \gamma(t)$ for all $t \in V_t$.

Theorem.

1. The antiderivative F exists.
2. If F_1 and F_2 are two antiderivatives, there exists $c \in \mathbb{C}$ such that $F_1 = F_2 + c$.

Proof of uniqueness. It follows from the definition that F is locally constant. ■

Proof of existence. Split $[a,b]$ into small subintervals (small enough for the form φ to have an antiderivative on the γ-image of each). Choose constants in their intersections in such a way that they make an actual function (antiderivatives differ by a constant). ■