

# Complex analysis

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## Notation

For the rest of this course,  $\mathbb{N}$  contains 0.

## Introduction

**Definition.** Unless stated otherwise,  $G \subseteq \mathbb{C}$  and  $H \subseteq \mathbb{R}^n$  are arbitrary domains.

**Definition.** A function  $f: G \rightarrow \mathbb{C}$  is *analytic*, if for any  $z_0 \in G$  there exists  $r > 0$  such that  $D_r(z_0) \subseteq G$ , and

$$f(z) = \sum_{n \in \mathbb{N}_0} a_n (z - z_0)^n$$

for some  $\{a_n\}$  and every  $z \in D_r(z_0)$ .

**Definition.**  $\varphi: G \rightarrow \mathbb{C}$  is *holomorphic* at  $z_0$ , iff exists

$$\varphi'(z_0) = \lim_{h \rightarrow 0} \frac{\varphi(z_0 + h) - \varphi(z_0)}{h}.$$

Here  $h \in \mathbb{C}$ . If a function  $f$  is holomorphic at every point of  $E \subseteq \mathbb{C}$ , we write  $f \in \text{Hol } E$ .

**Definition.** A function  $f \in \text{Hol } \mathbb{C}$  is called *entire*.

**Theorem** (Cauchy-Riemann equations).  $f: G \rightarrow \mathbb{C}$  is holomorphic at  $z_0$  iff

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

at  $(x_0, y_0)$ , where  $u(x, y) + iv(x, y) = f(x + iy)$  and  $z_0 = x_0 + iy_0$ .

That is, the Jacobi matrix of  $u \times v$  is in the image of the standard embedding of  $\mathbb{C}$  into  $M_2(\mathbb{R})$ :

$$a + ib \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Has been proven in semester II.. ■

**Lemma.** If  $f: G \rightarrow \mathbb{C}$  is analytic, then it is holomorphic.

*Proof.* The series for  $f(z)$  can be differentiated. ■

**Theorem** (Cauchy). Let  $f: G \rightarrow \mathbb{C}$  be holomorphic. Then it is analytic at every  $x_0 \in G$ , with the radius of convergence  $r$  being equal to  $r = \text{dist}(x_0, \mathbb{C} \setminus G)$ .

The proof will be given shortly.

## Differential forms

### A reminder

**Definition.** If we have a form

$$\omega(h) = \sum_I \omega_I \cdot h^I,$$

its integral (also a form) is defined as

$$\int \omega = \int \sum_I \omega_I \circ x \cdot D_I,$$

where  $D_I$  is the determinant of the rows  $I$  of the Jacobi matrix  $dx$ .

## Integral of a form along a path

**Definition** (integral along a curve). Let  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  be a  $C^1$  function. If  $\varphi = f_1 dx_1 + \dots + f_n dx_n$ , where  $f_i$  are continuous complex functions on  $G$ , then

$$\int_{\gamma} \varphi := \sum_{j=1}^n \int_{t=a}^b f_j(\gamma(t)) \gamma'_j(t) dt.$$

Evidently, the integral over a one-dimensional submanifold does not depend on parametrisation. We will further use that to write integrals over subsets of  $\mathbb{C}$ , not curves.

**Remark.** We may only require that  $\gamma$  is rectifiable. In this case, the integral will be in the sense of Stieltjes:

$$\int_{\gamma} \Phi = \sum_{j=1}^n \int_{t=a}^b f_j(\gamma(t)) d\gamma_j.$$

We will not need this during this course.

**Lemma.** Integral of differential forms along a path is linear with respect to the form.

*Proof.* Evident. ■

**Lemma** (change of variables). Let  $\alpha: [c, d] \rightarrow [a, b]$  be a  $C^1$ -homeomorphism, and  $\tilde{\gamma} = \gamma \circ \alpha$ . Then

$$\int_{\tilde{\gamma}} \varphi = \pm \sum_{j=1}^n \int_{t=c}^d f_j \circ \gamma \circ \alpha(s) \cdot \gamma'_j \circ \alpha(s) \cdot \alpha'(s) ds.$$

The sign here depends on whether  $\alpha$  is increasing or decreasing.

*Proof.* Follows from the change-of-variables formula for the Riemann integral. ■

**Definition.** Let  $\alpha, \beta$  be  $C^1$ -paths. Their *concatenation*  $\alpha\beta$  is defined as

$$\gamma(t) = \begin{cases} \alpha(t), & t \in [a, b], \\ \beta \circ \varphi(t), & t \in [b, c], \end{cases}$$

where  $\varphi: [b, c] \rightarrow [a', b']$  is a homeomorphism.

**Definition.** A path  $\gamma$  is *piecewise smooth*, iff it is a finite concatenation of smooth paths.

**Definition.** The integral of a form along a piecewise smooth path is the sum of integrals over its components.

**Definition.** If  $\varphi = \sum \varphi_j dx_j$  is a differential form, we denote

$$\|\varphi\| = \sqrt{\sum_{j=1}^n \varphi_j^2}.$$

Differential 1-forms are simply functions between Euclidean spaces.

**Theorem** (principal estimate). If  $\gamma$  is piecewise smooth and  $\varphi$  is a continuous differential 1-form in a neighbourhood of  $\text{im } \gamma$ , then

$$\left| \int_{\gamma} \varphi \right| \leq l(\gamma) \cdot \sup_{x \in \text{im } \gamma} \|\varphi(x)\|.$$

*Proof idea.* CBS. ■

## Antiderivatives

**Definition.** Let  $\omega$  be a differential 1-form in  $G$ . Its *derivative* is the form

$$d\omega = \sum_{j=1}^n \frac{\partial \omega_j}{\partial x} dx_j.$$

**Definition.** Let  $G \subseteq \mathbb{R}^n$  be a domain.  $F: G \rightarrow \mathbb{C}$  is called the *antiderivative* of  $\Phi$ , iff  $dF = \Phi$ .

**Definition.** A differential form  $\omega$  is

1. *exact*, iff it has an antiderivative;
2. *closed*, iff every point  $x \in G$  has a neighbourhood where  $\omega$  is exact.

Observe that this definition differs from the one given in the semester III. This one is more general: the previous one depended on smoothness.

**Lemma.** Suppose  $\omega$  is a  $C^1$  differential 1-form in  $G$ . Then  $\omega$  is closed iff

$$\partial_i \omega_j = \partial_j \omega_i$$

for all  $i, j \in \{1, \dots, n\}$ .

*Proof of  $\Rightarrow$ .* Locally, we have an antiderivative  $\Omega$ , so  $\partial_j \Omega = \omega_j$ . Then

$$\partial_i \omega_j = \partial_i \partial_j \Omega = \partial_j \partial_i \Omega = \partial_j \omega_i.$$

■

*Proof of  $\Leftarrow$ .* We know from semester III that every differential form  $\omega$  such that  $d\omega = 0$  is exact. But this is true of  $\omega$ .

■

**Lemma.** Let  $\gamma$  be a piecewise smooth path with  $\text{im } \gamma \subseteq G$  and ends  $A, B$ . Then

$$\int_{\gamma} dF = F(B) - F(A). \quad (1)$$

*Proof.* From the Newton-Leibniz formula. ■

**Theorem.** Every two points in  $G$  can be connected by piecewise linear path.

*A well-known fact.* ■

**Definition.** Let  $\Phi$  be a differential form in a region  $H \subseteq \mathbb{R}^n$ . We call  $H$  a  $\Phi$ -balanced<sup>1</sup> region, iff  $\int_{\gamma} \Phi = 0$  for every closed curve  $\gamma$  with  $\text{im } \gamma \subseteq H$ .

**Theorem** (reformulations of ‘exact’). Let  $\Phi$  be a differential form in  $H \subseteq \mathbb{R}^n$  with continuous coefficients. Equivalent are:

1.  $\int_{\gamma} \Phi$  depends on  $A$  and  $B$  only.
2.  $H$  is  $\Phi$ -balanced.
3.  $\Phi$  is exact.

*Proof of 3  $\Rightarrow$  2.* See (1). ■

*Proof of 2  $\Rightarrow$  1.* Concatenate paths between two points and get a closed path. But the integral (which is zero by hypothesis) splits into two. Next we use that it depends on the direction of the path. ■

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<sup>1</sup>My own term.



*Proof of 1  $\Rightarrow$  3.* Fix  $A_0 \in G$ . For any  $x \in H$ , let  $\gamma$  be a piecewise linear path  $A_0 \rightsquigarrow x$ . Define

$$F(x) = \int_{\gamma} \Phi.$$

Since the integral depends only on  $x$ , this is correctly defined function We assert that  $F$  is an antiderivative for  $\Phi$ ; that is, the partial derivatives of  $F$  are components of  $\Phi$ . To see this, consider the path

$$A_0 \rightsquigarrow_{\gamma} x \rightsquigarrow_{\beta} x + te_j,$$

where the last part  $\beta$  is linear. Then

$$\begin{aligned} \frac{F(x + te_j) - F(x)}{t} &= \frac{\int_{\gamma} \Phi + \int_{\beta} \Phi - \int_{\gamma} \Phi}{t} \\ &= \frac{1}{t} \int_{\beta} \Phi \\ &= \frac{1}{t} \sum_{k=1}^n \int_{\tau=0}^t f_k(x + \tau e_j) \beta'_k(\tau) \\ &= \frac{1}{t} \int_{\tau=0}^t f_j(x + \tau e_j) \\ &\xrightarrow{t \rightarrow 0} f_j(x). \end{aligned}$$

■

**Definition.** We call a region  $H \subseteq \mathbb{R}^2$  *rectangle-astroid*<sup>2</sup>, iff there exists  $x_0 \in H$  such that for every  $x \in H$  the 2-dimensional rectangle with sides parallel to the axes, having  $x_0$  and  $x$  as diametral points, lies in  $H$  together with its closure. The rectangle in this context is called *central*.

This definition is long, but it highlights what we need to use to prove the following proposition for circles.

**Lemma** (addition to the theorem). Let  $H \subseteq \mathbb{R}^2$  be a rectangle-astroid region. Then  $H$  being  $\Phi$ -balanced is also equivalent to

$$\int_Q \Phi = 0$$

for every 1-dimensional central rectangle  $Q$ .

---

<sup>2</sup>My own term.

*Proof idea.* The integral over this rectangle is equal to zero, since it is closed. Conversely, we can find an antiderivative like in the proof of the theorem. ■

**Theorem** (Cauchy, Morera). Let  $\gamma$  be a closed continuous curve with  $\text{im } \gamma \subseteq G$ . Then a function  $f: G \rightarrow \mathbb{C}$  is holomorphic iff

$$\int_{\gamma} f = 0.$$

## Connectivity

**Lemma.** Every two points of a domain  $H \subseteq \mathbb{R}^n$  can be connected by a piecewise-linear path.

*Proof.* Fix  $x_0 \in H$  and put  $A$  to be the set of all points reachable from  $x_0$  by a piecewise-linear path over  $H$ .  $A$  is open, since  $H$  is: every point in  $A$  is the centre of a ball in  $H$ , and balls are convex. The complement  $H \setminus A$  is open for the same reasons: take any point  $x$  from there, there is a ball  $B \subseteq H$  around it; if there was a point of  $A$  in this  $B$ , we could connect it to  $x$ . Therefore,  $A$  is open and closed; but it is not empty either, so  $A = H$  by connectedness of the domain  $H$ . ■

Recall the theorem on equivalence of linear connectivity and connectedness for locally linearly connected spaces. The proof is almost the same.

**Theorem.** Every two points of a domain  $H \subseteq \mathbb{R}^n$  can be connected by a  $C^\infty$  path.

We need to recall some stuff about convolutions for the proof.

**Definition.** A family  $\{\varphi_s\}_{s>0}$  of infinitely smooth functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  with compact support and such that

$$\int \varphi_s = 1, \quad \varphi_s(x) = \frac{\varphi_1(x/s)}{s^n}$$

for all  $s > 0$  and  $x \in \mathbb{R}^n$  is called a *standard approximative unit* or a *mollifier*.

**Theorem.**

1. A family  $\{\varphi_s\}$  exists.
2. Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function. Then the functions  $\varphi_s * f$  converge to  $f$  uniformly with  $s \rightarrow 0$  and are infinitely smooth.

The *proof* was given in the third semester. Now we start with the main theorem.

*Proof.* Let  $\gamma: [a, b] \rightarrow H$  be a continuous path  $A \rightsquigarrow B$ . Continue  $\gamma$  to a continuous path  $\mathbb{R} \rightarrow H$  by setting

$$\gamma(t) = \begin{cases} \gamma(a), & t < a, \\ \gamma(b), & t > b, \\ \gamma(t), & t \in [a, b]. \end{cases}$$

Let  $\varphi_\square$  be a standard approximative unit for functions on  $\mathbb{R}$ , and put for all  $s > 0$

$$\widehat{\gamma}_s(x) = (\varphi_s * \gamma)(x),$$

where  $*$  denotes component-wise convolution. Fix  $\epsilon > 0$ ,  $a_2 < a$ , and  $b_2 > b$ . By the theorem on approximating with units, if  $s$  is small enough, we have  $|\widehat{\gamma}_s(t) - \gamma(t)| < \epsilon$  for all  $t \in [a_2, b_2]$ . This  $\epsilon$  can be chosen in such a way that the path  $\widehat{\gamma}_s$  does not leave the domain  $H$ . Further, choose  $a_2, b_2$  such that

$$\widehat{\gamma}_s(a_2) = \gamma(a), \quad \widehat{\gamma}_s(b_2) = \gamma(b).$$

**G A P**

This ensures  $\widehat{\gamma}|_{[a_2, b_2]}$  is indeed the path we need. ■

## Closed forms and balanced regions

**Theorem** (reformulations of ‘closed’). Let  $\Phi$  be a differential form in  $H \subseteq \mathbb{R}^n$  with continuous coefficients  $f_j$ . The following are equivalent:

1.  $\Phi$  is closed.
2. Every  $x \in H$  has a  $\Phi$ -balanced neighbourhood  $U \subseteq H$ .

In case  $n = 2$ , two more reformulations are true:

3. For every  $x \in H$ , there exists a rectangle-astroid region  $B \ni x$  such that  $\int_Q \Phi = 0$  for any central rectangle  $Q$ .
4.  $\int_Q \Phi = 0$  for any rectangle  $Q$  such that  $\text{cl } Q \subseteq H$ .

*Proof of  $3 \Rightarrow 4$ .* Split the rectangle  $Q$  into equal four,  $\{Q_i\}$ . Then the integral over  $Q$  is equal to the sum of integrals over  $\{Q_i\}$ . Cover the compact  $\text{cl } Q$  with ‘good’ rectangle-astroid regions which lie in  $H$  completely (they exist by hypothesis). Let  $\delta > 0$  be the Lebesgue number of this cover. Continue to split the rectangles into four until each of them is less than  $\delta$  in diameter. Now it is clear that the integral over  $Q$  is zero itself. ■

*Proof of  $4 \Rightarrow 3$ .*  $G$  is open. ■

## Change of basis

Let  $z = x + iy$ . Then

$$dz = dx + i dy$$

$$d\bar{z} = dx - i dy,$$

so

$$dx = \frac{dz + d\bar{z}}{2}, \quad dy = \frac{dz - d\bar{z}}{2}.$$

Now let  $\varphi = u dx + v dy$  be a 1-form, defined in  $H \subseteq \mathbb{R}^2$ . Then, if  $\varphi = d\Phi$  for a function  $\Phi: H \rightarrow \mathbb{C}$ , we have

$$\Phi = \frac{\partial_1 \Phi - \partial_2 \Phi}{2} dz + \frac{\partial_1 \Phi + \partial_2 \Phi}{2} d\bar{z}$$

by direct computation. By analogy, we define

**Definition.**

$$\partial_z \Phi := \frac{\partial_1 \Phi - i \partial_2 \Phi}{2} \quad \partial_{\bar{z}} \Phi := \frac{\partial_1 \Phi + i \partial_2 \Phi}{2}.$$

Let  $\Phi = p + iq$ . It then can be derived that

$$\partial_{\bar{z}} \Phi = \frac{\partial_1 p - \partial_2 q}{2} + i \frac{\partial_1 q + \partial_2 p}{2}.$$

Therefore,

$$\partial_{\bar{z}} \Phi = 0 \iff \begin{cases} \partial_1 p = \partial_2 q, \\ \partial_1 q = -\partial_2 p. \end{cases}$$

These are the Cauchy-Riemann equations. Therefore,

**Lemma.** A function  $\Phi: G \rightarrow \mathbb{C}$  is holomorphic iff  $d\Phi = \partial_z \Phi dz$ .

We can also prove the following:

**Lemma.** Let  $\alpha dz$  be a form in  $G$ . It is exact iff there exists a holomorphic function  $A: G \rightarrow \mathbb{C}$  such that  $A' = \alpha$ .

*Proof of  $\Rightarrow$ .* Let  $A$  be the antiderivative.  $\alpha dz = \alpha dx + i\alpha dy$ . Then  $\partial_1 A = \alpha$ ,  $\partial_2 A = i\alpha$ , so

$$A' = \partial_z A = \frac{\partial_1 A - i \partial_2 A}{2} = \alpha, \quad \partial_{\bar{z}} A = \frac{\partial_1 A + i \partial_2 A}{2} = 0.$$

■

**Lemma** (a variation of the principal estimate). Let  $\alpha: G \rightarrow \mathbb{C}$  be a function,  $\gamma: [a, b] \rightarrow G$  a  $C^1$  path. Then

$$\left| \int_{\gamma} \alpha \, dz \right| \leq l(\gamma) \cdot \sup_{z \in \text{im } \gamma} |\alpha(z)|.$$

*Proof idea.* Use the principal estimate and the identity  $\gamma' = \gamma'_1 + i\gamma'_2$ . ■

## Cauchy's theorem on closedness

**Theorem** (Cauchy, on closedness). If  $f: G \rightarrow \mathbb{C}$  is holomorphic, then the form  $f \, dz$  is closed.

That is, locally, it has antiderivatives. The proof spans the several following pages and requires a few lemmas.

### The case of continuous derivative

**Lemma.** Suppose  $f'$  is continuous. Then the form  $f \, dz$  is closed.

While this might seem to give a hint of our further course of action, we won't be so blunt as to prove the continuity of  $f'$  directly.

*Proof idea.* Indeed, as we know, the closedness is then equivalent to the equality

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

This is straightforward to show using Cauchy-Riemann equations. ■

**Example.** The form  $dz/z$  is closed and not exact.  $f' = -1/z^2$  in this case is a continuous function. By the previous lemma,  $f$  is closed (in fact, its antiderivatives are logarithms). Now consider the unit circle  $\mathbb{S}^1$ . By parametrising with  $e^{i\varphi}$ , we can easily check that

$$\int_{\mathbb{S}^1} \frac{dz}{z} = 2\pi i.$$

But this means we have found a non-balanced region of  $\mathbb{C}$ , so  $dz/z$  is not exact.

## Indices

**Lemma.** Let  $C = z_0 + r\mathbb{S}^1$  for some  $r > 0$ . Then

$$\int_C \frac{dz}{z - z_1} = \begin{cases} 0, & |z_1 - z_0| > r, \\ 2\pi i, & |z_1 - z_0| < r. \end{cases} \quad (2)$$

*Proof.* Consider the case  $|z_1 - z_0| > r$ . Luckily, the form  $\frac{dz}{z - z_1}$  is closed in  $H = \mathbb{C} \setminus \{z_1\}$ . This  $H$  contains a rectangle around the square  $C$ . Every closed form is exact within this rectangle by a theorem from semester 3.

Suppose, for now,  $|z_1 - z_0| < r$ . We reduce this to the case  $z_0 = z_1$ , which has been considered in the example on page 14. Then

$$\begin{aligned} \int_C \frac{dz}{z - z_1} &= \int_C \frac{dz}{(z - z_0) - (z_1 - z_0)} \\ &= \int_C \frac{dz}{z - z_0} \cdot \frac{1}{1 - \frac{z_1 - z_0}{z - z_0}} \\ &= \int_C \frac{dz}{z - z_0} \cdot \sum_{k \in \mathbb{N}} \left( \frac{z_1 - z_0}{z - z_0} \right)^k \\ &= 2\pi i + \sum_{k \in \mathbb{N} \setminus 0} \int_C \frac{dz}{z - z_0} \left( \frac{z_1 - z_0}{z - z_0} \right)^k \\ &= 2\pi i. \end{aligned}$$

■

## The conclusion

**Theorem** (Cauchy, on closedness). If  $f: G \rightarrow \mathbb{C}$  is holomorphic, then the form  $f dz$  is closed.

*Proof.* Suppose otherwise: there exists a 2-dimensional rectangle  $P \subseteq G$  with  $I := \int_{\partial P} f \neq 0$ .

Subdivide it into four  $\{Q_i\}$  such that

$$I = \sum_i \int_{\partial Q_i} f.$$

For one of them,  $Q_j$ , the modulus of the integral is at least one fourth the  $|I|$ . Denote  $P_1 = Q_j$ . Continuing this sequence, we get diminishing  $\{P_j\}$  with a single point  $z_0$  in their intersection. The function  $f$  is holomorphic at  $z_0$ , so we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z),$$

where  $\varphi(z) = o(z - z_0)$  with  $z \rightarrow z_0$ . Select  $k$  such that  $|\varphi(z)| < \epsilon|z - z_0|$  for all  $z \in P_k$ . Then we have

$$\begin{aligned} \left| \frac{I}{4^k} \right| &\leq \left| \int_{\partial P_k} f \right| \\ &= \left| \int_{z \in \partial P_k} \underbrace{\left( f(z_0) + f'(z_0)(z - z_0) \right)}_{\text{these guys are exact in } P_k} + \varphi(z) \right| \\ &= \left| \int_{z \in \partial P_k} \varphi(z) \right| \\ &\leq \epsilon \cdot (\text{diam } P_k) \cdot S_1(\partial P_k) \\ &= \epsilon \cdot \frac{(\text{diam } P) \cdot S_1(\partial P)}{4^k}. \end{aligned}$$

But we thought that  $I \neq 0$ . ■

## On correctible singularities

**Lemma** (on correctible singularities). Let  $a \in G$ . Suppose the form  $\omega = f dx + g dy$  is closed in  $G \setminus \{a\}$ , and the coefficients  $f$  and  $g$  are continuous in  $G$ . Then  $\omega$  is closed in  $G$ .

*Proof idea.* Approximate integrals with smaller rectangles. ■



## The minor integral formula of Cauchy

**Theorem** (Cauchy, minor integral formula). Let  $f: G \rightarrow \mathbb{C}$  be holomorphic,  $z_0, z_1 \in G$ ,  $C = z_0 + r\mathbb{S}^1$ ,  $|z_1 - z_0| < r$ . Then

$$f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1} dz.$$

‘Probably the most important formula in complex analysis.’ — S.K.

The ‘greater’ integral formula gives the same result for  $C$  not necessarily a circle.

*Proof.* Fix  $z_0 \in G$ . Consider

$$h(z) = \begin{cases} \frac{f(z) - f(z_1)}{z - z_1}, & z \neq z_1 \\ f'(z_1), & z = z_1. \end{cases}$$

$h$  is continuous in  $G$  and holomorphic in  $G \setminus \{z_0\}$ . By Cauchy’s theorem on closedness (page 15), the form  $h dz$  is closed in  $G \setminus \{z_1\}$ . By the lemma on correctible singularities (page 16), it is also closed in all of  $G$ . Let  $\widehat{C}$  be a ball of slightly greater radius  $r + \delta$ , but still lying in  $G$ . Every closed form in  $\widehat{C}$  is exact, so

$$\int_C h dz = 0.$$

Rewriting this yields

$$f(z_1) \underbrace{\int_C \frac{dz}{z - z_1}}_{=2\pi} = \int_C \frac{f(z)}{z - z_1} dz.$$

■

## Cauchy’s theorem on analyticity

**Theorem** (Cauchy, on analyticity). Let  $f$  be holomorphic in  $G$ ,  $z_0 \in G$ ,  $R = \text{dist}(z_0, \partial G)$ . For all  $k \in \mathbb{N}$ , define

$$c_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz.$$

Then

$$f(z) = \sum_{k \in \mathbb{N}} c_k (z - z_0)^k$$

for all  $z$  such that  $|z - z_0| < R$ . In particular,  $f$  is analytic in  $G$ .

*Proof.* Let  $C$  be a circle around  $z_0$  of radius  $r < R$ . By Cauchy's minor integral formula from page 17, for any  $z$  inside of  $C$  we have

$$\begin{aligned} f(z_1) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1} dz \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0) - (z_1 - z_0)} dz \\ &= \frac{1}{2\pi i} \int_C \frac{1}{z - z_0} \cdot \frac{f(z)}{1 - \frac{z_1 - z_0}{z - z_0}} dz \\ &= \sum_{k \in \mathbb{N}} \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \cdot \left( \frac{z_1 - z_0}{z - z_0} \right)^k dz \\ &= \sum_{k \in \mathbb{N}} (z_1 - z_0)^k \cdot \left( \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz \right), \end{aligned}$$

which is desired. Transposing the sum with the integral is legit, since the series for geometric progression converges uniformly for all  $z$ . ■

**Remark.** In the analyticity theorem,

$$c_k = \frac{f^{(k)}(z_0)}{k!}.$$

## Morera's theorem

**Corollary** (Morera's theorem). Let  $f: G \rightarrow \mathbb{C}$  be a continuous function. The following conditions are equivalent:

1.  $f$  is analytic.
2.  $f$  is holomorphic.
3.  $f \, dz$  is closed.

*Proof.*  $1 \Rightarrow 2$  follows by differentiating the series,  $2 \Rightarrow 3$  is the Cauchy's theorem on closedness from page 15. We now give a proof of  $3 \Rightarrow 1$ . By definition, the form  $f \, dz$  locally has an antiderivative  $\Phi$ . Then  $\Phi$  is holomorphic, and  $\Phi' = f$ . By the analyticity theorem,  $\Phi$  is holomorphic, which implies that  $f$  is, too (simply differentiate the series). ■

## The mean value theorem

**Lemma** (mean value theorem). If  $C \subseteq G$  is a circle with the centre at  $z_0 \in G$ , then

$$f(z) = \frac{1}{2\pi} \int_C f(z) \, dz.$$

*Proof idea.* Follows from the Cauchy's minor integral formula (page 17). ■