Complex analysis

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Contents

Notation

For the rest of this course, $\mathbb{N}$ contains 0.

Introduction

Definition. Unless stated otherwise, $G \subseteq \mathbb{C}$ and $H \subseteq \mathbb{R}^n$ are arbitrary domains.

Definition. A function $f\colon G \to \mathbb{C}$ is *analytic*, if for any $z_0 \in G$ there exists $r > 0$ such that $D_r(z_0) \subseteq G$, and

$$f(z) = \sum_{n\in\mathbb{N}_0} a_n(z - z_0)^n$$

for some $\{a_n\}$ and every $z \in \mathbb{D}_r(z_0)$.

Definition. $\varphi\colon G \to \mathbb{C}$ is *holomorphic* at z_0, iff exists

$$\varphi'(z_0) = \lim_{h\to 0} \frac{\varphi(z_0 + h) - \varphi(z_0)}{h}.$$

Here $h \in \mathbb{C}$. If a function f is holomorphic at every point of $E \subseteq \mathbb{C}$, we write $f \in \operatorname{Hol} E$.

Definition. A function $f \in \operatorname{Hol}\mathbb{C}$ is called *entire*.

Theorem (Cauchy-Riemann equations). $f\colon G \to \mathbb{C}$ is holomorphic at z_0 iff

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, xxx \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

at (x_0, y_0), where $u(x, y) + iv(x, y) = f(x + iy)$ and $z_0 = x_0 + iy_0$.

That is, the Jacobi matrix of $u \times v$ is in the image of the standard embedding of $\mathbb{C}$ into $M_2(\mathbb{R})$:

$$a + ib \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Has been proven in semester II.. ∎

Lemma. If $f : G \to \mathbb{C}$ is analytic, then it is holomorphic.

Proof. The series for $f(z)$ can be differentiated. ∎

Theorem (Cauchy)**.** Let $f : G \to \mathbb{C}$ be holomorphic. Then it is analytic at every $x_0 \in G$, with the radius of convergence r being equal to $r = \operatorname{dist}\left(x_0, \mathbb{C} \setminus G\right)$.

The proof will be given shortly.

Differential forms

A reminder

Definition. If we have a form

$$\omega(h) = \sum_I \omega_I \cdot h^I,$$

its integral (also a form) is defined as

$$\int \omega = \int \sum_I \omega_I \circ x \cdot D_I,$$

where D_I is the determinant of the rows I of the Jacobi matrix $\mathrm{d}x$.

Integral of a form along a path

Definition (integral along a curve)**.** Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a C^1 function. If $\varphi = f_1\,\mathrm{d}x_1 + \dots + f_n\,\mathrm{d}x_n$, where f_i are continuous complex functions on G, then

$$\int\limits_{\gamma} \varphi := \sum_{j=1}^{n} \int\limits_{t=a}^{b} f_j(\gamma(t))\gamma_j'(t).$$

Evidently, the integral over a one-dimensional submanifold does not depend on parametrisation. We will further use that to write integrals over subsets of $\mathbb{C}$, not curves.

Remark. We may only require that γ is rectifiable. In this case, the integral will be in the sense of Stieltjes:

$$\int\limits_{\gamma} \Phi = \sum_{j=1}^{n} \int\limits_{t=a}^{b} f_j(\gamma(t))\,\mathrm{d}\gamma_j.$$

We will not need this during this course.

Lemma. Integral of differential forms along a path is linear with respect to the form.

Proof. Evident. ∎

Lemma (change of variables)**.** Let $\alpha\colon [c,d] \to [a,b]$ be a C^1-homeomorphism, and $\widetilde{\gamma} = \gamma \circ \alpha$. Then

$$\int\limits_{\widetilde{\gamma}} \varphi = \pm \sum_{j=1}^{n} \int\limits_{t=c}^{d} f_j \circ \gamma \circ \alpha(s) \cdot \gamma_j' \circ \alpha(s) \cdot \alpha'(s).$$

The sign here depends on whether α is increasing or decreasing.

Proof. Follows from the change-of-variables formula for the Riemann integral. ∎

Definition. Let α, β be C^1-paths. Their *concatenation* $\alpha\beta$ is defined as

$$\gamma(t) = \begin{cases} \alpha(t), & t \in [a,b], \\ \beta \circ \varphi(t), & t \in [b,c], \end{cases}$$

where $\varphi\colon [b,c] \to [a', b']$ is a homeomorphism.

Definition. A path γ is *piecewise smooth*, iff it is a finite concatenation of smooth paths.

Definition. The integral of a form along a piecewise smooth path is the sum of integrals over its components.

Definition. If $\varphi = \sum \varphi_j \, \mathrm{d}x_j$ is a differential form, we denote

$$\|\varphi\| = \sqrt{\sum_{j=1}^{n} \varphi_j^2}.$$

Differential 1-forms are simply functions between Euclidean spaces.

Theorem (principal estimate)**.** If γ is piecewise smooth and φ is a continuous differential 1-form in a neighbourhood of $\operatorname{im}\gamma$, then

$$\left| \int_\gamma \varphi \right| \leq l(\gamma) \cdot \sup_{x \in \operatorname{im}\gamma} \|\varphi(x)\|.$$

Idea for a proof. CBS. ∎

Antiderivatives

Definition. Let ω be be a differential 1-form in G. Its *derivative* is the form

$$\mathrm{d}\omega = \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial x} \,\mathrm{d}x_j.$$

Definition. Let $G \subseteq \mathbb{R}^n$ be a differential form. $F : G \to \mathbb{C}$ is called the *antiderivative* of Φ, iff $\mathrm{d}F = \Phi$.

Definition. A differential form ω is

1. *exact*, iff it has an antiderivative;
2. *closed*, iff every point $x \in G$ has a neighbourhood where ω is exact.

Observe that this definition differs from the one given in the semester III. This one is more general: the previous one depended on smoothness.

Lemma. Suppose ω is a C^1 differential 1-form in G. Then ω is closed iff

$$\partial_i \omega_j = \partial_j \omega_i$$

for all $i, j \in \{1, \ldots, n\}$.

Proof of $\Rightarrow$. Locally, we have an antiderivative Ω, so $\partial_j \Omega = \omega_j$. Then

$$\partial_i \omega_j = \partial_i \partial_j \Omega = \partial_j \partial_i \Omega = \partial_j \omega_i.$$

■

Proof of $\Leftarrow$. We know from semester III that every differential form ω such that $\mathrm{d}\omega = 0$ is exact. But this is true of ω. ■

Lemma. Let γ be a piecewise smooth path with $\operatorname{im}\gamma \subseteq G$ and ends A, B. Then

$$\int_\gamma \mathrm{d}F = F(B) - F(A). \tag{1}$$

Proof. From the Newton-Leibniz formula. ■

Theorem. Every two points in G can be connected by piecewise linear path.

A well-known fact. ■

Definition. Let Φ be a differential form in a region $H \subseteq \mathbb{R}^n$. We call H a Φ-*balanced*[1] region, iff $\int_\gamma \Phi = 0$ for every closed curve γ with $\operatorname{im}\gamma \subseteq H$.

Theorem (reformulations of 'exact')**.** Let Φ be a differential form in $H \subseteq \mathbb{R}^n$ with continuous coefficients. Equivalent are:

1. $\int_\gamma \Phi$ depends on A and B only.
2. H is Φ-balanced.
3. Φ is exact.

Proof of $3 \Rightarrow 2$. See (1). ■

Proof of $2 \Rightarrow 1$. Concatenate paths between two points and get a closed path. But the integral (which is zero by hypothesis) splits into two. Next we use that it depends on the direction of the path. ■

[1] My own term.

Proof of $1 \Rightarrow 3$. Fix $A_0 \in G$. For any $x \in H$, let γ be a piecewise linear path $A_0 \rightsquigarrow x$. Define

$$F(x) = \int_\gamma \Phi.$$

Since the integral depends only on x, this is correctly defined function We assert that F is an antiderivative for Φ; that is, the partial derivatives of F are components of Φ. To see this, consider the path

$$A_0 \rightsquigarrow_\gamma x \rightsquigarrow_\beta x + te_j,$$

where the last part β is linear. Then

$$\begin{aligned}
\frac{F(x+te_j) - F(x)}{t} &= \frac{\int_\gamma \Phi + \int_\beta \Phi - \int_\gamma \Phi}{t} \\
&= \frac{1}{t} \int_\beta \Phi \\
&= \frac{1}{t} \sum_{k=1}^{n} \int_{\tau=0}^{t} f_k\left(x + \tau e_j\right) \beta_k'(\tau) \\
&= \frac{1}{t} \int_{\tau=0}^{t} f_j\left(x + \tau e_j\right) \\
&\xrightarrow[t\to 0]{} f_j(x).
\end{aligned}$$

■

Definition. We call a region $H \subseteq \mathbb{R}^2$ *rectangle-astroid*[2], iff there exists $x_0 \in H$ such that for every $x \in H$ the 2-dimensional rectangle with sides parallel to the axes, having x_0 and x as diametral points, lies in H together with its closure. The rectangle in this context is called *central*.

This definition is long, but it highlights what we need to use to prove the following proposition for circles.

Lemma (addition to the theorem). Let $H \subseteq \mathbb{R}^2$ be a rectangle-astroid region. Then H being Φ-balanced is also equivalent to

$$\int_Q \Phi = 0$$

for every 1-dimensional central rectangle Q.

[2]My own term.

Idea for a proof. The integral over this rectangle is equal to zero, since it is closed. Conversely, we can find an antiderivative like in the proof of the theorem. ∎

Theorem (Cauchy, Morera). Let γ be a closed continuous curve with $\operatorname{im}\gamma \subseteq G$. Then a function $f : G \to \mathbb{C}$ is holomorphic iff

$$\int_\gamma f = 0.$$

Connectivity

Lemma. Every two points of a domain $H \subseteq \mathbb{R}^n$ can be connected by a piecewise-linear path.

Proof. Fix $x_0 \in H$ and put A to be the set of all points reachable from x_0 by a piecewise-linear path over H. A is open, since H is: every point in A is the centre of a ball in H, and balls are convex. The complement $H \setminus A$ is open for the same reasons: take any point x from there, there is a ball $B \subseteq H$ around it; if there was a point of A in this B, we could connect it to x. Therefore, A is open and closed; but it is not empty either, so $A = H$ by connectedness of the domain H. ∎

Recall the theorem on equivalence of linear connectivity and connectedness for locally linearly connected spaces. The proof is almost the same.

Theorem. Every two points of a domain $H \subseteq \mathbb{R}^n$ can be connected by a C^∞ path.

We need to recall some stuff about convolutions for the proof.

Definition. A family $\{\varphi_s\}_{s>0}$ of infinitely smooth functions $\mathbb{R}^n \to \mathbb{R}$ with compact support and such that

$$\int \varphi_s = 1, \qquad \varphi_s(x) = \frac{\varphi_1(x/s)}{s^n}$$

for all $s > 0$ and $x \in \mathbb{R}^n$ is called a *standard approximative unit* or a *mollifier*.

Theorem.

1. A family $\{\varphi_\square\}$ exists.
2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then the functions $\varphi_s * f$ converge to f uniformly with $s \to 0$ and are infinitely smooth.

The *proof* was given in the third semester. Now we start with the main theorem.

Proof. Let $\gamma\colon [a,b] \to H$ be a continuous path $A \rightsquigarrow B$. Continue γ to a continuous path $\mathbb{R} \to H$ by setting

$$\gamma(t) = \begin{cases} \gamma(a), & t < a, \\ \gamma(b), & t > b, \\ \gamma(t), & t \in [a,b]. \end{cases}$$

Let $\varphi_\square$ be a standard approximative unit for functions on $\mathbb{R}$, and put for all $s > 0$

$$\widehat{\gamma}_s(x) = (\varphi_s * \gamma)(x),$$

where $*$ denotes component-wise convolution. Fix $\epsilon > 0$, $a_2 < a$, and $b_2 > b$. By the theorem on approximating with units, if s is small enough, we have $|\widehat{\gamma}_s(t) - \gamma(t)| < \epsilon$ for all $t \in [a_2, b_2]$. This ϵ can be chosen in such a way that the path $\widehat{\gamma}_s$ does not leave the domain H. Further, choose a_2, b_2 such that

$$\widehat{\gamma}_s(a_2) = \gamma(a), \qquad \widehat{\gamma}_s(b_2) = \gamma(b).$$

GAP

This ensures $\widehat{\gamma}|_{[a_2,b_2]}$ is indeed the path we need. ■

Closed forms and balanced regions

Theorem (reformulations of 'closed')**.** Let Φ be a differential form in $H \subseteq \mathbb{R}^n$ with continuous coefficients f_j. The following are equivalent:

1. Φ is closed.
2. Every $x \in H$ has a Φ-balanced neighbourhood $U \subseteq H$.

In case $n = 2$, two more reformulations are true:

3. For every $x \in H$, there exists a rectangle-astroid region $B \ni x$ such that $\int_Q \Phi = 0$ for any central rectangle Q.
4. $\int_Q \Phi = 0$ for any rectangle Q such that $\operatorname{Cl} Q \subseteq H$.

Proof of $3 \implies 4$. Split the rectangle Q into equal four, $\{Q_i\}$. Then the integral over Q is equal to the sum of integrals over $\{Q_i\}$. Cover the compact $\operatorname{Cl} Q$ with 'good' rectangle-astroid regions which lie in H completely (they exist by hypothesis). Let $\delta > 0$ be the Lebesgue number of this cover. Continue to split the rectangles into four until each of them is less than δ in diameter. Now it is clear that the integral over Q is zero itself. ■

Proof of $4 \implies 3$. G is open. ■

Change of basis

Let $z = x + iy$. Then

$$\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y \qquad \mathrm{d}\overline{z} = \mathrm{d}x + i\,\mathrm{d}y,$$

so

$$\mathrm{d}x = \frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2}, \qquad \mathrm{d}y = \frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2}.$$

Now let $\varphi = u\,\mathrm{d}x + v\,\mathrm{d}y$ be a 1-form, defined in $H \subseteq \mathbb{R}^2$. Then, if $\varphi = \mathrm{d}\Phi$ for a function $\Phi\colon H \to \mathbb{C}$, we have

$$\Phi = \frac{\partial_1\Phi - \partial_2\Phi}{2}\,\mathrm{d}z + \frac{\partial_1\Phi - \partial_2\Phi}{2}\,\mathrm{d}\overline{z}$$

by direct computation. By analogy, we define

Definition.

$$\partial_z\Phi := \frac{\partial_1\Phi - i\partial_2\Phi}{2} \qquad \partial_{\overline{z}}\Phi := \frac{\partial_1\Phi + i\partial_2\Phi}{2}.$$

Let $\Phi = p + iq$. It then can be derived that

$$\partial_{\overline{z}}\Phi = \frac{\partial_1 p - \partial_2 q}{2} + i\frac{\partial_1 q + \partial_2 p}{2}.$$

Therefore,

$$\partial_{\overline{z}}\Phi = 0 \iff \begin{cases} \partial_1 p = \partial_2 q, \\ \partial_1 q = -\partial_2 p. \end{cases}$$

These are the Cauchy-Riemann equations. Therefore,

Lemma. A function $\Phi\colon G \to \mathbb{C}$ is holomorphic iff $\mathrm{d}\Phi = \partial_z\Phi\,\mathrm{d}z$.

We can also prove the following:

Lemma. Let $\alpha\,\mathrm{d}z$ be a form in G. It is exact iff there exists a holomorphic function $A\colon G \to \mathbb{C}$ such that $A' = \alpha$.

Proof of $\Rightarrow$. Let A be the antiderivative. $\alpha\,\mathrm{d}z = \alpha\,\mathrm{d}x + i\alpha\,\mathrm{d}y$. Then $\partial_1 A = \alpha$, $\partial_2 A = i\alpha$, so

$$A' = \partial_z A = \frac{\partial_1 A - i\partial_2 A}{2} = \alpha, \qquad \partial_{\overline{z}} A = \frac{\partial_1 A + i\partial_2 A}{2} = 0.$$

■

Lemma (a variation of the principal estimate)**.** Let $\alpha\colon G \to \mathbb{C}$ be a function, $\gamma\colon [a,b] \to G$ a C^1 path. Then

$$\left|\int_\gamma \alpha \,\mathrm{d}z\right| \le l(\gamma) \cdot \sup_{z \in \operatorname{im}\gamma} |\alpha(z)|.$$

Idea for a proof. Use the principal estimate and the identity $\gamma' = \gamma_1' + i\gamma_2'$. ∎

Cauchy's theorem on closedness

Theorem (Cauchy, on closedness)**.** If $f\colon G \to \mathbb{C}$ is holomorphic, then the form $f\,\mathrm{d}z$ is closed.

That is, locally, it has antiderivatives. The proof spans the several following pages and requires a few lemmas.

The case of continuous derivative

Lemma. Suppose f' is continuous. Then the form $f\,\mathrm{d}z$ is closed.

While this might seem to give a hint of our further course of action, we won't be so blunt as to prove the continuity of f' directly.

Idea for a proof. Indeed, as we know, the closedness is then equivalent to the equality

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

This is straightforward to show using Cauchy-Riemann equations. ∎

Example. The form $\mathrm{d}z/z$ is closed and not exact. $f' = -1/z^2$ in this case is a continuous function. By the previous lemma, f is closed (in fact, its antiderivatives are logarithms). Now consider the unit circle $\mathbb{S}^1$. By parametrising with $e^{i\square}$, we can easily check that

$$\int_{\mathbb{S}^1} \frac{\mathrm{d}z}{z} = 2\pi i.$$

But this means we have found a non-balanced region of $\mathbb{C}$, so $\mathrm{d}z/z$ is not exact.

Indices

Lemma. Let $C = z_0 + r\mathbb{S}^1$ for some $r > 0$. Then

$$\int_C \frac{\mathrm{d}z}{z - z_1} = \begin{cases} 0, & |z_1 - z_0| > r, \\ 2\pi i, & |z_1 - z_0| < r. \end{cases} \tag{2}$$

Proof. Consider the case $|z_1 - z_0| > r$. Luckily, the form $\frac{\mathrm{d}z}{z - z_1}$ is closed in $H = \mathbb{C} \setminus \{z_1\}$. This H contains a rectangle around the square C. Every closed form is exact within this rectangle by a theorem from semester 3.

Suppose, for now, $|z_1 - z_0| < r$. We reduce this to the case $z_0 = z_1$, which has been considered in the example on page 15. Then

$$\begin{aligned}
\int_C \frac{\mathrm{d}z}{z - z_1} &= \int_C \frac{\mathrm{d}z}{(z - z_0) - (z_1 - z_0)} \\
&= \int_C \frac{\mathrm{d}z}{z - z_0} \cdot \frac{1}{1 - \frac{z_1 - z_0}{z - z_0}} \\
&= \int_C \frac{\mathrm{d}z}{z - z_0} \cdot \sum_{k \in \mathbb{N}} \left(\frac{z_1 - z_0}{z - z_0} \right)^k \\
&= 2\pi i + \sum_{k \in \mathbb{N} \setminus 0} \int_C \frac{\mathrm{d}z}{z - z_0} \left(\frac{z_1 - z_0}{z - z_0} \right)^k \\
&= 2\pi i.
\end{aligned}$$

■

The conclusion

Theorem (Cauchy, on closedness). If $f\colon G \to \mathbb{C}$ is holomorphic, then the form $f\,\mathrm{d}z$ is closed.

Proof. Suppose otherwise: there exists a 2-dimensional rectangle $P \subseteq G$ with $I := \int_{\partial P} f \neq 0$.

Subdivide it into four $\{Q_i\}$ such that

$$I = \sum_i \int_{\partial Q_i} f.$$

For one of them, Q_j, the modulus of the integral is at least one fourth the $|I|$. Denote $P_1 = Q_j$. Continuing this sequence, we get diminishing $\{P_j\}$ with a single point z_0 in their intersection. The function f is holomorphic at z_0, so we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z),$$

where $\varphi(z) = o(z - z_0)$ with $z \to z_0$. Select k such that $|\varphi(z)| < \epsilon |z - z_0|$ for all $z \in P_k$. Then we have

$$\begin{aligned}
\left|\frac{I}{4^k}\right| &\le \left|\int_{\partial P_k} f\right| \\
&= \left|\int_{z \in \partial P_k} \Big(\underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\text{these guys are exact in } P_k} + \varphi(z)\Big)\right| \\
&= \left|\int_{z \in \partial P_k} \varphi(z)\right| \\
&\le \epsilon \cdot (\operatorname{Diam} P_k) \cdot S_1(\partial P_k) \\
&= \epsilon \cdot \frac{(\operatorname{Diam} P) \cdot S_1(\partial P)}{4^k}.
\end{aligned}$$

But we thought that $I \neq 0$. ■

On correctible singularities

Lemma (on correctible singularities)**.** Let $a \in G$. Suppose the form $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y$ is closed in $G \setminus \{a\}$, and the coefficients f and g are continuous in G. Then ω is closed in G.

Idea for a proof. Approximate integrals with smaller rectangles. ■

The minor integral formula of Cauchy

Theorem (Cauchy, minor integral formula)**.** Let $f\colon G \to \mathbb{C}$ be holomorphic, $z_0, z_1 \in G$, $C = z_0 + r\mathbb{S}^1$, $\left|z_1 - z_0\right| < r$. Then
$$f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1}\,\mathrm{d}z.$$

'Probably the most important formula in complex analysis.' — S.K.

The 'greater' integral formula gives the same result for C not necessarily a circle.

Proof. Fix $z_0 \in G$. Consider
$$h(z) = \begin{cases} \frac{f(z)-f(z_1)}{z-z_1}, & z \neq z_1 \\ f'(z_1), & z = z_1. \end{cases}$$
h is continuous in G and holomorphic in $G \setminus \left\{z_0\right\}$. By Cauchy's theorem on closedness (page 16), the form $h\,\mathrm{d}z$ is closed in $G \setminus \{z_1\}$. By the lemma on correctible singularities (page 17), it is also closed in all of G. Let $\widehat{C}$ be a ball of slightly greater radius $r + \delta$, but still lying in G. Every closed form in $\widehat{C}$ is exact, so
$$\int_C h\,\mathrm{d}z = 0.$$
Rewriting this yields
$$f(z_1) \underbrace{\int_C \frac{\mathrm{d}z}{z - z_1}}_{=2\pi} = \int_C \frac{f(z)}{z - z_1}\,\mathrm{d}z.$$
∎

Cauchy's theorem on analyticity

Theorem (Cauchy, on analyticity)**.** Let f be holomorphic in G, $z_0 \in G$, $R = \operatorname{dist}(z_0, \partial G)$. For all $k \in \mathbb{N}$, define

$$c_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}}\, \mathrm{d}z.$$

Then

$$f(z) = \sum_{k \in \mathbb{N}} c_k (z - z_0)^k$$

for all z such that $|z - z_0| < R$. In particular, f is analytic in G.

Proof. Let C be a circle around z_0 of radius $r < R$. By Cauchy's minor integral formula from page 18, for any z inside of C we have

$$\begin{aligned}
f(z_1) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1}\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0) - (z_1 - z_0)}\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_C \frac{1}{z - z_0} \cdot \frac{f(z)}{1 - \frac{z_1 - z_0}{z - z_0}}\, \mathrm{d}z \\
&= \sum_{k \in \mathbb{N}} \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \cdot \left(\frac{z_1 - z_0}{z - z_0} \right)^k \mathrm{d}z \\
&= \sum_{k \in \mathbb{N}} (z_1 - z_0)^k \cdot \left(\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}}\, \mathrm{d}z \right),
\end{aligned}$$

which is desired. Transposing the sum with the integral is legit, since the series for geometric progression converges uniformly for all z. ∎

Remark. In the analyticity theorem,

$$c_k = \frac{f^{(k)}(z_0)}{k!}.$$

In particular,

$$f'(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta.$$

Morera's theorem

Corollary (Morera's theorem)**.** Let $f\colon G \to \mathbb{C}$ be a continuous function. The following conditions are equivalent:

1. f is analytic.
2. f is holomorphic.
3. $f\,\mathrm{d}z$ is closed.

Proof. $1 \Rightarrow 2$ follows by differentiating the series, $2 \Rightarrow 3$ is the Cauchy's theorem on closedness from page 16. We now give a proof of $3 \Rightarrow 1$. By definition, the form $f\,\mathrm{d}z$ locally has an antiderivative Φ. Then Φ is holomorphic, and $\Phi' = f$. By the analyticity theorem, Φ is holomorphic, which implies that f is, too (simply differentiate the series). ∎

The mean value theorem

Lemma (mean value theorem)**.** If $f \in \operatorname{Hol} G$ and $z \in G$, then

$$f(z) = \frac{1}{2\pi} \int_{t=0}^{2\pi} f\left(z + re^{it}\right)$$

for all $0 < r < \operatorname{dist}(z, \partial G)$.

Idea for a proof. Follows from the Cauchy's minor integral formula (page 18). ∎

Maximum modulus principle

Theorem (maximum modulus principle)**.** Let $f\colon G \to \mathbb{C}$ be holomorphic. Then $|f|$ has no strict maximum in G.

Proof. Suppose f has a non-strict maximum in G. Then

$$\begin{aligned} |f(a)| &= \frac{1}{2\pi}\left|\int_C f(z)\,\mathrm{d}z\right| \\ &\leq \frac{1}{2\pi}\int_C |f(z)|\,\mathrm{d}z \\ &\leq |f(a)|. \end{aligned}$$

Here every inequality must be an equality. Then $|f(z)| = |f(a)|$ for all z in some disk D around a.

Suppose $f'(a) \neq 0$. Then f is a local homeomorphism around a, which means it must map a ball in $\mathbb{C}$ into a ball in $\mathbb{C}$. But in any ball there are points with varying modulus.

Therefore, f is constant, what completes the proof. ∎

Liouville's theorem

Theorem (Liouville)**.** A bounded entire function is constant.

A powerful theorem.

Proof. From principal estimate and the formula for f':

$$\begin{aligned} |f'(z)| &\leq \frac{M}{2\pi}\int_C \frac{r}{(r-|z|)}\,\mathrm{d}t \\ &= \frac{Mr}{(r-|z|)^2} \\ &\xleftarrow[r\to\infty]{} 0. \end{aligned}$$

∎

Principal theorem of algebra

Corollary (principal theorem of algebra)**.** $\mathbb{C}$ is algebraically closed.

Proof. Suppose $p\colon \mathbb{C} \to \mathbb{C}$ of $\deg p > 0$ has no roots: $|p| > \delta$ for some $\delta > 0$. Then $1/p$ is an entire function, and, as such, is either unbounded or constant. It is not constant, so it must be unbounded; but $1/|p| < 1/\delta$ — a contradiction. ∎

Harmonic functions

Definition. A function $f\colon G \to \mathbb{R}$ is *harmonic*, iff there exists a holomorphic $g\colon G \to \mathbb{C}$ such that $\operatorname{Re} g = f$. If $\widetilde{f} = \operatorname{Im} g$, f and $\widetilde{f}$ are said to be *harmonic conjugates*.

Lemma (Laplace's equations)**.** If $u + iv \in \operatorname{Hol} G$, then

$$u'_{xx} + v'_{yy} = 0.$$

Proof. From Cauchy-Riemann equations. ∎

Definition. The differential operator

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \right)^2$$

is called the *Laplace operator*.

Definition. A function $f\colon H \to \mathbb{R}$ is called *harmonic*, iff

$$\Delta f = 0.$$

Theorem. If $u \in C^2(G)$ and $\Delta u = 0$, then every $a \in G$ has a neighbourhood U_a such that $u|_{U_a}$ is harmonic.

Proof. Let $\varphi = \frac{\partial u}{\partial x}$ and $\psi = -\frac{\partial u}{\partial y}$. We have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y},$$

so there exists a function v such that $\frac{\partial v}{\partial y} = \varphi$ and $\frac{\partial v}{\partial x} = \psi$. This v is harmonically conjugate to u, since the Cauchy-Riemann equations are satisfied. ∎

Integrals of closed forms

Lemma. Suppose $\varphi = \mathrm{d}\Phi$ in G. Then

$$\int_\gamma \varphi = \Phi(\gamma(b)) - \Phi(\gamma(a))$$

for every path $\gamma\colon [a,b] \to G$.

Proof. By the theorem on exact forms. ∎

Definition. A continuous function $F\colon [a,b] \to \mathbb{C}$ is called the *antiderivative along a path* $\gamma\colon [a,b] \to G$ of a 1-form φ, iff for every $t \in [a,b]$ exists an antiderivative Φ in some neighbourhood $V_t \ni \gamma(t)$ such that $F(t) = \Phi \circ \gamma(t)$ for all $t \in V_t$.

Theorem.

1. The antiderivative F exists.
2. If F_1 and F_2 are two antiderivatives, there exists $c \in \mathbb{C}$ such that $F_1 = F_2 + c$.

Proof of uniqueness. It follows from the definition that F is locally constant. ∎

Proof of existence. Split $[a,b]$ into small subintervals (small enough for the form φ to have an antiderivative on the γ-image of each). Choose constants in their intersections in such a way that they make an actual function (antiderivatives differ by a constant). ∎

Remark. This can be used as an alternative definition of the integral along a path:

$$\int_\gamma \varphi := F(b) - F(a).$$

Homotopies

Definition. A *homotopy* is a continuous map $h\colon [\alpha,\beta]\times[a,b] \to G$ (usually $\alpha = 0$, $\beta = 1$).

Definition. A *loop* is a closed path; i.e. a $\gamma\colon [a,b] \to \mathbb{C}$ that can be continued to an $(b-a)$-periodic one. A loop is *contractible*, iff it is homotopic to a constant path. G is *simply-connected*, iff every loop in G is contractible.

Example. All astroid domains are simply-connected.

Definition. Let φ be a 1-form in G, $S = [0,1]\times[a,b]$, $h\colon S \omega G$ a homotopy. A continuous $F\colon S \to G$ is an *antiderivative of φ along h*, iff for every $(t,s) \in S$ there exists a ball $B \subseteq G$ with the centre at $h_t(s)$ and an antiderivative U for φ in B.

Theorem. Such F always exists in case of a closed φ.

Proof. $h(S)$ is compact. By closedness of φ, take balls such that there exists an antiderivative. By continuity of h and Lebesgue's lemma, there exists $\delta > 0$ such that $h(e)$ lies completely in one of these balls for every e of diameter less than δ. Extract a finite subcover, and make them agree on intersections. ∎

Theorem. Let γ_t be a homotopy in G, and all paths γ_t are closed. Let φ be a closed 1-form in G, then

$$\int_{\gamma_0} \varphi = \int_{\gamma_1} \varphi.$$

Proof. Define $g(t) = \int_{\gamma_t} \varphi$. We assert this function is locally constant. Fix $t \in [a, b]$. Then

$$\int_{\gamma_t} \varphi = F\big(\gamma_t(b)\big) - F\big(\gamma_t(a)\big) \equiv \text{const}$$

for some antiderivative F along γ_t. ∎

Corollary. In a simply-connected set, every closed form is exact.

Proof. Because the integral over any closed path equals that over a constant one. ∎

Laurent series

Definition. A *Laurent series* is one of the form

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$$

for some $z, z_0, c_k \in \mathbb{C}$. It is said to *converge* at z, iff both

$$\sum_{k \geq 0} c_k (z - z_0)^k, \qquad \sum_{k < 0} c_k (z - z_0)^k$$

converge at z. The first of these is the *regular* part; the other one — *principal.*

Theorem. Let

$$A = \left\{ z \in \mathbb{C} \mid r < |z - z_0| < R \right\}$$

(*anneau*) for some $0 < r < R$. If $f \colon A \to \mathbb{C}$ is holomorphic, then f decomposes uniquely into a Laurent series at any $z_0 \in A$.

Idea for a proof. Consider integrals over the inner c and outer C components of the boundary of the ring A. For every $z \in Z$,

$$\int_c \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w = \int_C \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w.$$

From this we get an expression for $2\pi i f(z)$ as a sum of integrals of two geometric series. ∎

Singularities

Definition. Let $f\colon G \setminus \{a\} \to \mathbb{C}$ be holomorphic, $a \in G$.

- a is *removable*, iff f is bounded in a neighbourhood of a.
- a is a *pole*, iff $f(z) \to \infty$ when $z \to a$.
- a is *essential*, iff it is not removable or a pole; i.e. f has no limit at a.

Lemma. Let a be a correctible singularity of $f\colon G \setminus \{a\} \to \mathbb{C}$. Then f can be continued to an analytic function $f\colon G \to \mathbb{C}$.

Proof. We assert every coefficient of the principal part is zero. Indeed,

$$c_k = \frac{1}{2\pi i} \int_{a+rS^1} f(\zeta)(\zeta - a)^{-j-1}\, \mathrm{d}\zeta,$$

$$\begin{aligned} |c_k| &\leq \frac{A|r|^z - j - 12\pi r}{2\pi} \\ &\xrightarrow[r\to 0]{} 0. \end{aligned}$$

∎

Lemma. a is a pole iff the number of nonzero coefficients c_k at a with $k < 0$ is nonzero and finite. a is essential, iff it is infinite.

Idea for a proof. If $f \to \infty$, then a is c removable for $1/f$. ∎

Sokhotsky's theorem

Theorem (Sokhotsky)**.** Let $f \in \operatorname{Hol} G$, and a an essential singularity of f. Then, for every $\epsilon > 0$, the set $f\big(B_\epsilon(a)\big)$ is dense in $\mathbb{C}$.

Proof. Suppose otherwise: let $b \in \mathbb{C}$ be such that

$$\big|f(z) - b\big| \geq \delta$$

for all $z \in B := B_\epsilon(a)$ for some $\epsilon > 0$. The function

$$h(z) := \frac{1}{f(z) - b}$$

is analytic at a and bounded. Then h has a removable singularity at a, which allows us to think it is defined in the whole of B. Hence

$$f(z) = \frac{1}{h(z)} + b$$

has a finite number of negative powers in the Laurent decomposition, which is false. ∎

Orders of zeroes

Definition. Let $a \in G$. Every holomorphic $f : G \setminus \{a\} \to \mathbb{C}$ can be presented as

$$f(z) = (z-a)^k g(z),$$

where $k \in \mathbb{Z}$, $g \in \operatorname{Hol} G$, $g(a) \neq 0$. This k is called the *order* of a in f, and denoted

$$\operatorname{ord}_a f := k.$$

Lemma. The following are equivalent:

1. $\operatorname{ord}_a f = k$.
2. $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$, $f^{(k)}(a) \neq 0$.

Proof. Since $c_k = \frac{f^{(k)}(z_0)}{k!}$. ∎

Residues

Definition. Let $a \in G$ be an isolated singularity of f,

$$f(z) = \sum_{j \in \mathbb{Z}} c_j (z-a)^j$$

in a ball of radius $r > 0$ around a. The *residue* of f at a is the number

$$\operatorname*{Res}_a f := c_{-1} = \frac{1}{2\pi i} \int_{a+rS^1} f(z)\,\mathrm{d}z.$$

Lemma. If $-m = \operatorname{ord}_a f$ and $m > 0$, then

$$\operatorname*{Res}_a f = \frac{1}{(m-1)!} \lim_{z \to a} \Big((z-a)^m f(z) \Big)^{(m-1)}$$

Proof. Exercise. ∎

Lemma. If $f = g/h$ and $\operatorname{ord}_a h = 1$, then

$$\operatorname*{Res}_a f = \frac{g(a)}{h'(a)}.$$

Proof. Follows from the previous one. ∎

Indices

Definition. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed path. If $a \notin \operatorname{im}\gamma$, then the *index* of γ relative to a is the number

$$\operatorname*{Ind}_a \gamma := \frac{1}{2\pi i} \int\limits_\gamma \frac{\mathrm{d}z}{z-a}.$$

Full-blown representation theory.

Lemma.

1. Index does not change upon homotopies that do not contain a in their image.
2. Index is an integer.
3. Index is constant on every connected component of $\mathbb{C} \setminus \operatorname{im}\gamma$.

The residue theorem

Definition. Let $\Gamma := \{\gamma_1, \dots, \gamma_n\}$ be closed paths in $G \subseteq \mathbb{C}$. $\gamma_\square$ is a *proper path system*, iff for every $f \in \operatorname{Hol} G$ we have

$$\int\limits_\Gamma f(z)\,\mathrm{d}z := \left(\int\limits_{\gamma_1} + \dots + \int\limits_{\gamma_n} \right) f(z)\,\mathrm{d}z = 0.$$

Theorem (on residues)**.** Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a proper path system in G,

$$f \in \operatorname{Hol} G \setminus \{z_1, \dots, z_k\},$$

where $z_j \notin \gamma$ for every j and some path $\gamma\colon [0,1] \to G$. Then

$$\frac{1}{2\pi i} \int\limits_\Gamma f(z)\,\mathrm{d}z = \sum_{z \in G} \operatorname*{Res}_z(f) \operatorname*{Ind}_z(\Gamma).$$

Proof. Let $g_1, \dots, g_k$ be principal parts of Laurent series of f around $\{z_j\}$. Then

$$g_j(z) = \sum_{s=1}^{\infty} \frac{c_s}{(z - z_j)^s}$$

If $s > 1$,

$$g_s(z) = \frac{c_{-1}}{z - z_j} + h_s(z),$$

where h_s has an antiderivative in G. Integrating this, we get exactly

$$\int_{\gamma_j} f(z)\,\mathrm{d}z = \int_{\gamma_j} \underbrace{\left(f(z) - g_j(z)\right)}_{\text{has no singularity in } z_j}\,\mathrm{d}z + \operatorname*{Res}_{z_j}(f) \cdot 2\pi i \operatorname*{Ind}_{z_j}(\gamma_j).$$

Summing this over j and using properness of Γ, we get the desired. ■

Remark. This theorem still holds true, if f has an infinite number of singularities $\{z_j\}$, but with discrete induced topology.

Change of variables for non-smooth paths

Lemma. Let F be an antiderivative of $f\,\mathrm{d}z$ along γ. Then $F \circ h$ is antiderivative of the same form along $\gamma \circ h$.

Proof. Exercise. ■

Lemma. Let $G_1, G_2 \subseteq \mathbb{C}$ be domains, $g\colon G_1 \to G_2$ a holomorphic function. Let $f \in \operatorname{Hol} G_2$, and let γ be a path in G_1. Then

$$\int_{g\circ\gamma} f(z)\,\mathrm{d}z = \int_{\gamma} f\left(g(z)\right) \cdot g'(z)\,\mathrm{d}z.$$

Proof. If γ is piecewise smooth, this follows from the change of variables formula. Otherwise, let F be an antiderivative of $f\,\mathrm{d}z$ along $g \circ \gamma$. Differentiating any local antiderivative, we see that it is also an antiderivative of $\left(f \circ g\right) \cdot g'$ along γ. ■

Lemma. Index $\mathrm{Ind}_{z_0}(\gamma)$ of a continuous γ is locally constant with respect to z_0.

Proof. From the previous lemma. ■

Lemma. If z_0 is in an unbounded component of $\mathbb{C} \setminus \operatorname{im} \gamma$, then $\mathrm{Ind}_{z_0}(\gamma) = 0$.

Proof.

$$\operatorname*{Ind}_{z_0} \gamma = \frac{1}{2\pi i} \int\limits_{\gamma - z} \frac{\mathrm{d}w}{w}.$$

With $|z_0| \to \infty$, $\operatorname{im}(\gamma - z_0)$ is within a half-plane. ■

Logarithms

Definition. A continuous $\varphi \colon G \to \mathbb{C}$ is a *branch of a logarithm* in G, iff $e^{\varphi(z)} = z$ for all $z \in G$.

Lemma. Let $\varphi \colon G \to \mathbb{C}$ be a logarithm branch.

1. $0 \notin G$.
2. $\varphi'(z) = 1/z$. In particular, φ is holomorphic.

Proof.

1. Obvious.
2. From the inverse function theorem.

■

Definition. A continuous $\varphi \colon G \to \mathbb{C}$ is a *branch of a logarithm of* $F \colon G \to \mathbb{C}$ in G, iff $e^{\varphi(z)} = F(z)$ for all $z \in G$.

Theorem. If G is simply-connected, and $F \in \operatorname{Hol} G$, $F \neq 0$, then there exists a branch of the logarithm of F in G.

Proof. There exists an antiderivative H for F'/F. Differentiating $\left(e^H/F\right)'$, we get $e^H = cF$ for some $c \in \mathbb{C}$. Then $F = e^{H - \operatorname{Arg} c}$. ■

Definition. We know that

$$F(z) = e^{\Phi(z)}.$$

There is φ such that

$$\Phi(z) = \log\left|F(z)\right| + i\varphi(z),$$

where φ is some continuous $G \to \mathbb{C}$. This φ is called a *branch of the argument of* F.

Definition. A continuous $u\colon [0,1] \to \mathbb{C}$ is a *branch of a logarithm* along $\gamma[0,1] \to \mathbb{C}$, iff $u \circ \gamma(t)$ is a logarithm of $F \circ \gamma(t)$. Likewise for *argument*.

Theorem. In the conditions of the definition, there exists a logarithm branch of F, and any two such differ by $2\pi i k$ for some $k \in \mathbb{Z}$.

Proof. Uniqueness: $u_1 - u_2$ is a continuous function $[0,1] \to 2\pi i\mathbb{Z}$. Existence follows from local existence. ■

Lemma. There exists a continuous argument branch.

Proof. We can find one knowing a continuous logarithm branch. ■

Lemma. Index is an integer.

Proof. Choose a continuous logarithm branch and use uniqueness locally. ■

Jordan's curve theorem

Theorem. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a simple closed path. Then $\mathbb{C} \setminus \operatorname{im}\gamma$ consists of two connected components.

Proof. ■

The argument principle

Definition. Let γ be a simple closed path in G such that $\mathbb{C} \setminus \operatorname{im}\gamma$ splits into two connected components. The bounded of these two is called *Jordan*.

Theorem. Let f be holomorphic in a neighbourhood of a γ-Jordan $\operatorname{Cl} G$, except a finite number of poles which are not in $\operatorname{im}\gamma$. Then

$$\frac{1}{2\pi} \operatorname{Arg} \circ f \circ \gamma \Big|_0^1 = \sum_{z \in G} \operatorname{ord}_z(f).$$

Proof. On the left we have

$$\begin{aligned}\Delta &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z \\ &= \sum_{z \in G} \operatorname*{Res}_z \left(\frac{f'}{f}\right) \operatorname*{Ind}_z(\gamma) \\ &= \sum_{z \in G} \operatorname*{Res}_z \left(\frac{f'}{f}\right) \\ &= \sum_{z \in G} \operatorname{ord}_z \left(\frac{f'}{f}\right).\end{aligned}$$

The last equality is easy to check directly. ■

The theorem of Rouché

Theorem (Rouché). Let G be a γ-Jordan domain. Let f, g be holomorphic in a neighbourhood of $\operatorname{Cl} G$. Suppose that

$$\forall z \in \operatorname{im}\gamma : \left|f(z)\right| > \left|g(z)\right|.$$

Then f and $f+g$ have the same number of zeroes in G.

Idea for a proof. Use the previous theorem to compute the difference in sums of multiplicities as an integral of some f'/f. The condition on modules is needed to divide through f, and compute $\operatorname{Ind}_0 \varphi \circ \gamma$, where $\varphi(z) = 1 + g(z)/f(z)$. ∎

The Riemann sphere

Definition. $\widehat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$. A basis of nbhds of ∞ is formed by

$$\Big\{\{z \mid |z| > R\} \mid R \in \mathbb{R}\Big\}.$$

Definition. Let f be defined in a neighbourhood of ∞. f is *holomorphic at* ∞, iff $f(1/z)$ is holomorphic in a neighbourhood of 0.

Definition. Let f be holomorphic at ∞. Then

$$\operatorname*{Ind}_{\infty}(\gamma) := \operatorname*{Ind}_{0}(1/\gamma),$$
$$\operatorname*{Res}_{\infty}(f) := -\operatorname*{Res}_{z\to 0}\frac{f(1/z)}{z^2}.$$

The basis for this definition is the fact that

$$\operatorname*{Res}_{\infty}(f)\operatorname*{Ind}_{\infty}(\gamma) = \int_{1/\gamma} -\frac{f(1/z)}{z^2}\,dz = \int_{\gamma} f(z)\,dz$$

for any γ not through zero.

Another theorem on residues

Theorem. If $f \in \operatorname{Hol} \mathbb{C} \setminus \{z_1, \dots, z_k\}$, then

$$\sum_{z \in \mathbb{C}} \operatorname*{Res}_z(f) = 0.$$

Proof. Observe that, if γ is the circle around 0 of sufficiently large radius, $1/\gamma$ is passed in the opposite direction. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma f(z)\, \mathrm{d}z &= \operatorname*{Res}_\infty(f) \operatorname*{Ind}_0 \frac{1}{\gamma} \\ &= -\operatorname*{Res}_\infty(f). \end{aligned}$$

∎

Univalent functions

Definition. A function is *univalent*, iff it is holomorphic and injective.

Theorem. Let $f\colon G \to \mathbb{C}$ be univalent. Then f' is nowhere zero.

Remark. The converse is not true even for holomorphic functions.

Proof. Suppose $f'(z_0) = 0$.

$$f(z) = f(z_0) + (z - z_0)^k g(z),$$

where $k > 1$, $g(z_0) \neq 0$. In a sufficiently small ball $B \ni z_0$, g is nonzero and there is a logarithm $h\colon B \to \mathbb{C}$ of g:

$$g(z) = e^{h(z)}.$$

Then

$$f(z) = f(z_0) + \left((z - z_0)\widetilde{g}(z)\right)^k,$$

where

$$\widetilde{g}(z) = e^{\frac{h(z)}{k}}.$$

The function $\varphi(z) = (z - z_0)g(z)$ is univalent in a a neighbourhood of z_0 by a theorem from real analysis:

$$\varphi'(z) = g(z) + (z - z_0)g'(z_0) \neq 0 \text{ when } z = z_0.$$

This means that, if z_1 is sufficiently close to $f(z_0)$, φ is injective, and the equation

$$\left(\varphi(z)\right)^k = z_1 - f(z_0)$$

has k solutions in z. This contradicts univalence of f:

$$f(z) = \left(\varphi(z)\right)^k.$$

■

Lemma. If $f\colon G \to \mathbb{C}$ is univalent, then f is open.

Proof. Maps with non-degenerate differentials are open. ■

Conformal maps

Definition. We say a map $f\colon G \to \mathbb{C}$ is *conformal*, iff

$$\operatorname{Arg}(\overline{z_1}z_2) = \operatorname{Arg}\left(\overline{f'(z_0)z_1} \cdot f'(z_0)z_2\right)$$

for all $z_0, z_1, z_2 \in G$. That is, conformal maps preserve angles between tangent vectors. In fact, this is (easy to see) equivalent to univalence.

Theorem (Riemann)**.** The following conditions are equivalent:

1. G is simply-connected and $\mathbb{C} \neq G$.

2. Exists an univalent map $G \to \operatorname{Int} \mathbb{D}$.

$\Leftarrow$. From Liouville's theorem. ■

$\Rightarrow$. In April. ■

Automorphisms

Automorphisms

Definition. Any univalent $f : G \to G$ we'll call an *automorphism* of G.

Theorem. Any automorphism of $\mathbb{C}$ is an affine map

$$z \mapsto az + b.$$

The converse is trivially true.

Proof. Let $f \in \operatorname{Aut} \mathbb{C}$. If infinitely many coefficients in the Taylor expansion are nonzero, ∞ is an essential singularity for f. By Sokhotsky's theorem, for any ball $B \ni 0$ we have

$$f(B) \cap f(\mathbb{C} \setminus B) \neq \varnothing.$$

Hence f is a polynomial. But f' is nowhere zero by one of the previous theorems, which means f is affine. ■

Automorphisms of $\widehat{\mathbb{C}}$

Definition. A *Möbius transform* is a function of form

$$f : z \mapsto \frac{az+b}{cz+d},$$

where $ad \neq bc$.

As we can see, $f(-d/c) = \infty$ and $f(\infty) = a/c$.

Lemma. Möbius transforms form a group with respect to composition.

Proof. Routine check. ■

Lemma. Let G be acting on a set X. Let $H \leq G$ be acting transitively. Suppose $H \supseteq \mathrm{Stab}_G(x)$ for some $x \in X$. Then $H = G$.

Proof. Let $g \in G$. Since the action of H is transitive, there exists $h \in H$ such that $hgx = x$. Then $hg \in \mathrm{Stab}_G(x) \subseteq H$, and so $h^{-1} \cdot hg = g \in H$. ■

Theorem. The group of Möbius transforms is not smaller than $\mathrm{Aut}\,\widehat{\mathbb{C}}$.

Proof. Let M be the group of Möbius transforms, and let $S \leq \mathrm{Aut}\,\widehat{\mathbb{C}}$ be the stabiliser of ∞. It is easy to see that M acts transitively. $S \leq M$, since every $s \in S$ also leaves in place the $\mathbb{C}$, and so must be an affine function (page 37). ■

Disk automorphisms

Let $D := \mathrm{Int}\,\mathbb{D}^2$.

Definition. Let N be the set of maps of form

$$f_{a,c}\colon z \mapsto c \cdot \frac{z+a}{\bar{a}z+1},$$

where $|a| < 1$ and $|c| = 1$.

Lemma. $f_{a,c}(D) \subseteq D$.

Proof. Ultimately, it follows from the maximum modulus principle. ∎

Lemma. N is a group.

Proof.

1. For the identity map, put $a = 0$ and $c = 1$.
2. The inverse is of the same form (routine algebra).
3. The composition too (same).

∎

Lemma (Schwartz)**.** Let $f\colon D \to D$ be an analytic function, $f(0) = 0$. Then $\left|f(z)\right| \leq |z|$ for all $z \in D$. If the equality is reached for some $z \neq 0$, then exists $\alpha \in D$ such that $f(z) = \alpha z$ for all z.

In particular, $N \geq \mathrm{Stab}_{\mathrm{Aut}\,D}(0)$.

Proof. Suppose f is analytic in some ball of radius $R > 1$ with centre at 0. Let $g(z) := f(z)/z$. g is holomorphic with $|z| < R$ and $z \neq 0$. g can be continued to an analytic function in the ball of radius R around 0 (including 0). If $|z| = 1$, then, by passage to the limit,

$$\left|g(z)\right| = \left|f(z)\right|/|z| = \left|f(z)\right| \leq 1.$$

By the maximum modulus principle, the conclusion holds.

In the general case, consider $f_r(z) := f(rz)$. f_r is defined and holomorphic with $|rz| < 1$; that is, when $|z| < 1/r =: R$. For sure, $R > 1$. By the case already considered, $\left|f_r(z)\right| \le |z|$ for all $z \in D$, and so $\left|f(rz)\right| \le |z|$ for all z with $|z| < 1$.

Suppose now, the equality is reached for some $z \in D \setminus \{0\}$. Since z is internal for D, g must be constant by the maximum modulus principle. ∎

Theorem. $N = \operatorname{Aut} D$.

Proof. We show that N contains the stabiliser of zero. Let $\varphi \in \operatorname{Aut} D$, $\varphi(0) = 0$. By the Schwartz's lemma, $\left|\varphi(z)\right| \le |z|$ for all $z \in D$. By symmetry, $\left|\varphi^{-1}(z)\right| \le |z|$ for all $z \in D$. Then $|z| = \left|\varphi^{-1} \circ \varphi(z)\right| \le \left|\varphi(z)\right|$, so $\varphi(z) = |z|$ everywhere in D.

To see that N acts transitively, observe that $f(a) = 0$ and $f(0) = -ca$ for every $f \in N$. Now we can apply the lemma from page 38. ∎

Examples of conformal maps

Lemma. Let $\Pi_- = \left\{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\right\}$, $\psi \colon z \mapsto \frac{z-a}{z+\bar{a}}$, $\operatorname{Re} a < 0$. Then

$$\psi(\Pi_-) = D.$$

Proof. It is straightforward to show that $\psi(\Pi_-) \subseteq \mathbb{D}$. Likewise, $1/\psi(\Pi_+) \subseteq D$. Hence $\varphi(\Pi_-) \subseteq D$, $\varphi(\Pi_+) \subseteq \widehat{\mathbb{C}} \setminus \mathbb{D}$, $\varphi(i\mathbb{R}) \subseteq \mathbb{S}^1$. Since φ is an automorphism, this implies the desired. ∎

Lemma. All Möbius transforms map circles on the Riemann sphere into circles.

Proof. The maps $z + a, az, 1/z$ generate $\operatorname{Aut} \widehat{\mathbb{C}}$ — for them it is obvious. ∎

Lemma. A Möbius transform is uniquely determined by images of three points.

Proof.

$$\frac{\frac{z-z_1}{z-z_2}}{\frac{z_3-z_1}{z_3-z_2}} = \frac{\frac{w-w_1}{w-w_2}}{\frac{w_3-w_1}{w_3-w_2}}.$$

∎

Convergence of analytic functions

Definition. We say that a sequence $\{f_n\} \subseteq \operatorname{Hol} G$ *converges compactly* to $f\colon G \to \mathbb{C}$, iff, for every compact $K \subseteq G$,

$$f_n|_K \rightrightarrows f|_K.$$

Example. Let $G = \{z \mid |z| < 1\}$. The sequence $f_n = \frac{1}{n} \cdot \frac{1}{1-z^2}$ converges pointwise, but not uniformly; hence does not converge in the above sense.

Theorem (1st of Weierstrass)**.** If $f_n \to f$ compactly, then f is holomorphic in G.

Proof. By Stokes-Seidel theorem, f is continuous. We show that the integral over the boundary B of any rectangle in G is zero; then the theorem follows from the Cauchy's theorem on closedness, page 16. Observe that

$$\left| \int_B f(z)\,\mathrm{d}z - \int_B f_n(z)\,\mathrm{d}z \right| \leq \max_{z\in B} \left| f(z) - f_n(z) \right| \cdot \mathrm{const}$$

$$\xrightarrow[n\to\infty]{} 0.$$

But the integral over any f_n is zero (they are analytic in G). ∎

Theorem (2nd of Weierstrass)**.** If $f_n \to f$ compactly, then $f_n' \to f'$ compactly.

Proof. Let K be a closed ball of radius r with centre at some z_0, $\overline{K} \subseteq G$, $\gamma := \partial K$.

$$f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2}.$$

When $z \in \overline{K}$, the denominator is nonzero, so $f_n' \rightrightarrows f'$ on K.

If K is a general compact set, we can cover it with a finite number of balls and do the same. ∎

Normal families

Definition. Let G be a domain, F a set of functions on G. F is *normal*, iff

$$\forall \text{compact } K \subseteq G \ \exists c > 0 \ \forall f \in F : \left| f|_K \right| < c.$$

Lemma. If F is normal, then $F' := \left\{ f' \mid f \in F \right\}$ is normal.

Proof. Repeats the proof of the second theorem of Weierstrass. ∎

Theorem (Montel)**.** Equivalent are:

1. F is normal.
2. Any sequence $\left\{ f_n \right\}$ with $f_n \in F$ has a compactly convergent subsequence.

$2 \Rightarrow 1$. Suppose otherwise: on some compact the functions f_n are not uniformly bounded; i.e., there is a diverging (in this sense) sequence of functions. But they have ∎

$1 \Rightarrow 2$. For every $f \in F$, $z_1, z_2 \in G$,

$$\left| f(z_2) - f(z_1) \right| \leq \sup_{\xi \in [z_1, z_2]} \left| f'(\xi) \right| \cdot |z_2 - z_1| \leq c|z_2 - z_1|$$

for some $c > 0$, since the set $T_\delta := \left\{ z \in G \mid \operatorname{dist}(z, E) \leq \delta \right\}$ is compact, and we can bound F' on T_δ by its normality and the previous lemma. Therefore, F is uniformly equicontinuous. The Arcela–Askoli theorem states it must be uniformly bounded.

Now we apply the diagonal process. Let $E_1 \subseteq E_2 \subseteq \ldots$ be an exhausting sequence of compacts for G (that is, $E_j \subseteq \operatorname{Int} E_{j+1}$ and $\bigcup E_j = G$). Let $\left\{ f_n^m \right\}$ be a family of sequences such that $\left\{ f_n^{m+1} \right\}$

is a subsequence of $\{f_n^m\}$ that converges on E_m. Then the diagonal sequence $\{f_n^n\}_n$ converges on every of $E_\square$. ■

Analytic continuation

Let $f\colon G \to \mathbb{C}$ be analytic. How easy is it continue outwards of G?

Definition. Let $G_1, G_2 \subseteq \mathbb{C}$ be domains such that $G_1 \cap G_2$ is also connected. Let $f_i \in \operatorname{Hol} G_i$ for $i \in \{1, 2\}$. f_2 is said to be a *direct continuation* of f_1 onto G_2, iff $f_1 = f_2$ on $G_1 \cap G_2$.

Definition. G is said to be a *natural domain* for $f \in \operatorname{Hol} G$, iff f does not admit a direct continuation onto a larger domain $G' \supset G$.

Theorem (Hadamard)**.** The unit disk $D := \operatorname{Int} \mathbb{D}^2$ is the natural domain of

$$f\colon z \mapsto \sum_{n=1}^{\infty} z^{n!}.$$

Proof. This series indeed converges in D. If f can be continued to something larger, let G, this G contains $z_0 \in \mathbb{S}^1$. Let $\zeta := e^{2\pi i \alpha}$ be lying on $\mathbb{S}^1 \cap G \ni$, with $\alpha =: p/q \in \mathbb{Q}$. Exists $\lim_{r \to 1-} f(r\zeta)$. But then $f(r\zeta) \xrightarrow[r \to 1-]{} \infty$ (the addends become large from the qth one). ■

Definition. A pair (G, f), where $f \in \operatorname{Hol} G$ and G is a ball with centre at some z_0, is called an *element* of an analytic function. If the radius of r is maximal, the element is called *natural*.

Of course, G may not be the natural domain for f.

Definition. Let $(D_1, f_1), (D_2, f_2)$ be elements. The second of the them is the *continuation* of the first one along some path γ, iff exists $\varphi\colon [0,1] \to \mathbb{C}$ such that for every $t \in [a, b]$ there exists an element (D_t, f_t) of an analytic function, with D_t having the centre at $\gamma(t)$, such that $\varphi(s) = f_t\big(\gamma(s)\big)$ for all $s \in \gamma^{-1}(D_t)$.

The function φ is as smooth as the path γ.