Functional analysis

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Contents

Introduction

Let X be a vector space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$.

Definition. A map $f: X \to \mathbb{R}$ is called a *norm*, iff we have

1. If $x \in X \setminus 0$, then $f(x) > 0$; and $f(0) = 0$.
2. $f(kx) = |k| f(x)$ for all $k \in \mathbb{R}$, $x \in X$.
3. $f(x+y) \leq f(x) + f(y)$ for all $x, y \in X$.

Definition. A pair $(V, \|\square\|)$, where $\|\square\|$ is a norm on a vector space V, is called a *normed space*. A normed space is *Banach*, iff it is complete.

Completions

Definition. $\widehat{X}$ is called a *completion* of X, iff $\widehat{X}$ is complete and X is dense in $\widehat{X}$.

Theorem. A completion $\widehat{X}$ exists.

Proof. Call two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ equivalent, iff $\|x_n - y_n\| \to 0$, and let $\widehat{X}$ be the resulting quotient. Since the point-wise sum of two Cauchy sequences is Cauchy, in this natural way we may introduce vector space structure on $\widehat{X}$. The norm on $\widehat{X}$ is introduced as

$$\left\|[x_n]\right\| := \lim_{n\to\infty} \|x_n\|.$$

This map is defined correctly:

1. The limit on the right always exists: since $\{x_n\}$ is Cauchy, the sequence $\left\{\|x_n\|\right\}$ of reals is Cauchy, which implies it must converge.
2. If $[x_n] = [y_n]$, then $\left|\|x_n\| - \|y_n\|\right| \leq \|x_n - y_n\| \xrightarrow[n\to\infty]{} 0$. Therefore, $\|[x_n]\| = \|[y_n]\|$.

X is embedded into $\widehat{X}$ by mapping $x \in X$ into the class of the constant sequence at x. It is easy to see that this map preserves norms. ∎

Theorem. Let $\widehat{X}_2$ be another completion of X. Then exists a bijection $f: \widehat{X} \to \widehat{X}_2$ which is linear, preserves norms, and maps the embedded X into the embedded X.

These two theorems endow us with the right to never consider pathological incomplete spaces.

Proof. Map $[x_n] \in \widehat{X}$ into $\lim x_n \in \widehat{X}_2$. ∎

Exercise 1. The space $C[a,b]$ with the norm $\|f\|_2 = \int |f|$ is not complete. Define

$$f_n(x) = \begin{cases} 0, & x \in [a,c], \\ n(x-c), & x \in [c, c+1/n], \\ 1, & x \in [c+1/n, b]. \end{cases}$$

Then f_n is a Cauchy sequence which does not have a limit in $C[a,b]$.

Proof. It is easy to see that the limit of $\{f_n\}$ is $[x \geq c]|_{[a,b]}$, so it does not have a continuous limit. Nevertheless, it is Cauchy, since

$$\|f_n - f_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \xrightarrow[\min\{m,n\}\to\infty]{} 0.$$

∎

Equivalent norms

Definition. We say norms are *equivalent*, iff the metrics they generate are Lipschitz equivalent.

Exercise 2. Norms are equivalent iff they generate the same topology.

Exercise 3. In infinite-dimensional spaces, there are norms which are not equivalent.

Proof. For example, consider $X = C[0,1]$, L^1-norm and the sup-norm on it. It is true that L^1-norm does not surpass the sup-norm, but there is no constant for the opposite inequality: we can think of a function with an arbitrarily large sup-norm, but constant integral. ∎

Theorem. If X is finite-dimensional, then every two norms on X are equivalent.

Proof. Suppose $\dim X = n$, and $e_1, \dots, e_n$ is a basis. Let $x = a_1 e_1 + \cdots + a_n e_n$. Let $\|\square\|$ be a norm on X. Define a new *norm* as

$$|x| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Then

$$\begin{aligned} \|x\| &\leq \sum_{i=1}^{n} |a_i| \|e_i\| \\ &\leq M \sum_{i=1}^{n} |a_i| \\ &\leq M \left(\sum_{i=1}^{n} |a_i| \right)^{\frac{1}{2}} \\ &= M|x|. \end{aligned}$$

The function $x \mapsto \|x\|$ is continuous in the norm $|\square|$. Let $|x_k - x| \longrightarrow 0$. Then

$$\begin{aligned} \big| \|x_k\| - \|x\| \big| &\leq \|x_k - x\| \\ &\leq M\sqrt{n} |x_k - x| \\ &\longrightarrow 0. \end{aligned}$$

Consider the set $S = \{x \in E \mid |x| = 1\}$. S is compact in $|\square|$. $\varphi|_S$ is continuous and nonzero. Then $\varphi > \delta$ for some $\delta > 0$, so

$$\left\| \frac{x}{|x|} \right\| \geq \delta \iff \|x\| \geq \delta |x|.$$

∎

Corollary. Every finite-dimensional normed vector space is complete.

Proof. Every Euclidean space is complete. ∎

Corollary. A finite-dimensional subspace of a normed space is closed.

Proof. It is complete, and every convergent sequence is Cauchy sequence. ∎

Definition. The set $M \subseteq X$ is *bounded*, iff

$$\sup_{m \in M} \|m\| < +\infty.$$

Lemma (on an almost-perpendicular)**.** Let E be a normed space, and $F < E$ its closed proper subspace. Then for every $\epsilon > 0$ exists a vector $x \in E$ such that $\|x\| = 1$ and $\operatorname{dist}(x, F) > 1 - \epsilon$.

Proof. Let $y \in E \setminus F$. Then $d = \operatorname{dist}(y, F) > 0$, since F is closed. Let $\delta > 0$. By definition of infimum, there exists $a \in F$ such that

$$d \leq \|y - a\| \leq d + \delta.$$

Put $y_2 = y - a$. Since $a \in F$, $\operatorname{dist}(y_2, F) = \operatorname{dist}(y, F) = d$. Define

$$x = \frac{y_2}{d + \delta} = \frac{y - a}{d + \delta}.$$

Then $\|x\| \leq 1$, but

$$\operatorname{dist}(x, F) = \operatorname{dist}\left(\frac{y}{d + \delta}, F \right) \geq \frac{\operatorname{dist}(y, F)}{d + \delta} = \frac{d}{d + \delta}.$$

Since the δ is arbitrary, and by increasing the norm of x we do not get closer to F, we get the desired. ∎

Theorem. Let X be a normed space. Equivalent are:

1. X is finite-dimensional.
2. Every bounded subset of X is relatively compact.

Proof of 1 $\implies$ 2. From corollary on page 3. ∎

Proof of 2 $\implies$ 1. Suppose X is not finite-dimensional. We assert that there exists a bounded sequence that does not have a convergent subsequence (so in no way the closure of a bounded subset that contains this sequence can be compact). We show this by induction: suppose $x_1, \dots, x_n$ are already built. By the almost-perpendicular lemma there exists x_{n+1} such that $\|x_{n+1}\| = 1$ and

$$\operatorname{dist}\left(x_{n+1}, \operatorname{Span}\{x_1, \dots, x_n\}\right) > 1/2$$

(since X is not finite-dimensional, the span here is a proper subspace of X). Continuing to infinity, we get a sequence $\{x_n\}$. It is bounded (all its members are on the unit sphere). Nevertheless, no subsequence of it is Cauchy by construction. ∎

Linear operators

Definition. A linear operator $T: X \to Y$ is *bounded*, iff $T(B)$ is bounded, where B is the unit ball in X.

Lemma. Let X and Y be normed vector spaces, and $T: X \to Y$ a linear operator. The following are equivalent:

1. T is continuous.
2. T is continuous at 0.
3. T is bounded.

Proof of 1 ⇔ 2. Let $x \in X$. T is continuous at x iff for every convergent $x_n \to x$ the sequence Tx_n also converges (to Tx). Now observe that

$$\left\|Tx_n - Tx\right\| = \left\|T(x_n - x)\right\|.$$

∎

Proof of 2 ⇒ 3. Consider

$$D = \left\{y \in Y \mid \left\|y\right\| \leq 1\right\}.$$

There is $\delta > 0$ such that, if $\|x\| \leq \delta$, then $Tx \in D$. Let $z \in D$. Since

$$\left\|T\delta z\right\| \leq 1,$$

we have

$$\|Tz\| \leq 1/\delta.$$

∎

Proof of 3 ⇒ 1. Let $\|x\| \leq 1$. Then $\|Tx\| \leq M$, so $\|x\| < \epsilon$ implies $\|Tx\| \leq M\epsilon$. ∎

Operator norm

Definition. Let T is a continuous operator. Its *norm* is

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Exercise 4. $\|T\|$ is indeed a norm.

Lemma. An operator is bounded iff it has finite norm.

Proof. Obvious. ∎

Lemma. A linear combination of continuous operators is continuous.

Proof.

$$\left\|(kA + B)\right\| \leq \left|k\right|\|A\| + \|B\|.$$

∎

The space of bounded operators is complete

Definition. $\mathcal{B}(X, Y)$ is the set of bounded linear operators $X \to Y$ with the obvious structure of a vector space and the standard operator norm as the norm.

Theorem. Let Y be complete. Then $\mathcal{B}(X, Y)$ is complete.

I have seen three quite concise proofs of this theorem and understood neither. Here is a long one, my own.

Proof. Let $\{T_n\}$ be a Cauchy sequence. Fix $x \in X$. $\{T_n x\}$ is a Cauchy sequence in Y:

$$\begin{aligned}\left\|T_n x - T_m x\right\| &\leq \left\|T_n - T_m\right\|\|x\| \\ &\leq \epsilon\|x\|.\end{aligned}$$

By completeness of Y, there is a limit $t \xleftarrow[n\to\infty]{} T_n x$.

The map $x \mapsto t$ we have just build is a linear operator. Call it T. T_n is a Cauchy sequence, so $\left\|T_n\right\| \leq M$ for some M. Then

T itself is bounded:

$$\begin{aligned}\|Tx\| &\leq \|T_n x\| + \epsilon \\ &\leq M\|x\| + \epsilon.\end{aligned}$$

We assert that $\|T_n x - Tx\| \to 0$. Suppose otherwise:

$$\exists \epsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists n > n_0\ \exists x \in B\colon \|T_n x - Tx\| > \epsilon.$$

Since $\{T_n\}$ is Cauchy,

$$\forall \delta > 0\ \exists n_1 \in \mathbb{N}\ \forall k, l > n_1\ \forall x \in B\colon \|T_k x - T_l x\| \leq \delta.$$

Fix this $\delta > 0$ and take the corresponding n_1. From the first line with quantifiers, there exist $n > n_1$ and $x \in B$ such that

$$\|T_n x - Tx\| > \epsilon.$$

From the second one, for any $m > n_1$ we get

$$\begin{aligned}\|T_n x - Tx\| &\leq \|T_n x - T_m x\| + \|T_m x - Tx\| \\ &\leq \delta + \|T_m x - Tx\|.\end{aligned}$$

Since $Tx = \lim_{m\to\infty} T_m x$, there is m_0 such that $\|T_m x - Tx\| \leq \delta$ for all $m > m_0$. Take $m_1 = \max\{m_0, n_1\}$. Then, for any $m > m_1$,

$$\begin{aligned}\|T_n x - Tx\| &\leq \delta + \|T_m x - Tx\| \\ &\leq 2\delta.\end{aligned}$$

Now launch $n_1 \to \infty$ and, consequently, $\delta \to 0$. We get a contradiction:

$$\epsilon < \|T_n x - Tx\| \leq 2\delta.$$

■

Remark. Our proof does not use the boundedness of operators in the space $\mathcal{B}(X, Y)$.

Functionals

Definition. A *functional* is a linear operator $X \to K$.

Definition. The *dual* X^* of X is the space $\mathcal{B}(X, K)$ of continuous functionals.

Corollary. X^* is complete.

Proof. Since K is complete in either case. ■

Strong convergence

Definition. A sequence of operators $T_n \to T$ converges *strongly* or *point-wise*, iff $T_n x \to Tx$ for all $x \in X$.

Lemma. If $\|T_n - T\| \to 0$, then $T_n \to T$ strongly.

Proof. Since

$$|T_n x - Tx| \leq \|T_n - T\||x|.$$

■

Remark. The converse is not true.

Definition. $l^p = L^p(\mathbb{N}, \#)$ is the space of real-valued sequences which converge in the L^p norm (with respect to the cardinality measure).

Proof of the remark. Consider the operators

$$s_k(c_0, c_1, \dots) = (c_k, c_{k+1}, \dots)$$

on l^p. $\|s_k\| = 1$, since there are sequences with a single unit and other elements zero and applying s_k does not lessen the sequence norm anyway. Nevertheless, $s_k \to 0$ pointwise (strongly), since all the sequences in l^p converge. ■

More on Banach spaces

Theorem. Let Y be complete. Let $\{T_n\} \subseteq \mathcal{B}(X, Y)$ be operators with $\sup\|T_n\| < +\infty$, and let $E \subseteq X$ be dense. Suppose that, for every $e \in E$, the sequence $\{T_n e\} \subseteq Y$ converges. Then exists $T \in \mathcal{B}(X, Y)$ such that $T_n \to T$ pointwise.

Remark. If $\{T_n\}$ converges strongly, then $\sup\|T_n\| < +\infty$. This is a harder fact.

Proof. Define

$$Te := \lim_{n\to\infty} T_n e.$$

Let $x \in x$. Take $\{e_n\}$ such that $\|e_n - x\| \xrightarrow[n\to\infty]{} 0$. We assert T can be continued to a bounded operator $X \to Y$.

$\{Te_n\}$ is Cauchy.

$$\begin{aligned}\|Te_k - Te_k\| &\le \left\|Te_j - T_n e_j\right\| + \left\|T_n e_j - T_n e_k\right\| + \|Te_k - T_n e_k\| \\ &< 2\epsilon + \left\|T_n e_j - T_n e_k\right\| \\ &\le 2\epsilon + \|T_n\|\left\|e_j - e_k\right\| \\ &\xrightarrow[\substack{n\to\infty \\ \min\{j,k\}\to\infty}]{} 0.\end{aligned}$$

∎

Theorem. Let X and Y be normed spaces, and Y complete. Let F be dense in X. Let $T: F \to Y$ be a continuous linear operator. Then exists unique $T_\sharp \in \mathcal{B}(X,Y)$ such that $\left\|T_\sharp\right\| = \|T\|$ and $T_\sharp|_F = T$.

Proof. Let $x \in X$, $f_n \in F$, $\|f_n - x\| \to 0$. Then

$$\|Tf_n - Tf_m\| \le \|T\|\|f_n - f_m\| \xrightarrow[\min\{m,n\}\to\infty]{0} .$$

Hence $\{Tf_n\}$ is a Cauchy sequence and, as such, converges to some y. Define

$$T_\sharp x = y.$$

This definition does not depend on the choice of the sequence. ∎

Three fundamental principles of functional analysis

1. Hanh-Banach theorem.
2. Closed graph and open map theorems.
3. The principle of uniform boundedness.

Hanh-Banach theorem

Definition. Let X be a vector space over K, $p: X \to \mathbb{R}$. It is said that p is a *seminorm*, iff

1. $p(x) \ge 0$.
2. $|k|p(x) = p(kx)$ for all $k \in K$.
3. $p(x+y) \le p(x) + p(x)$.

Example. Consider $X = C(\mathbb{R})$ and $p(f) = \int_0^1 |f|$. Then p is a seminorm, but not a norm, since there is a nonzero function, which is zero on $[0,1]$.

Definition. Let X be a K-vector space. A $p: X \to \mathbb{R}$ is a *sublinear functional*, iff

1. $p(x+y) \le p(x) + p(y)$.
2. $p(kx) = |k|p(x)$ for all $k \in K$.

Theorem (Hanh, Banach)**.** Let X be a real vector space, $p: X \to \mathbb{R}$ a sublinear functional. Let $Y \le X$ and $f: Y \to \mathbb{R}$ a linear functional such that $f(y) \le p(y)$ for all $y \in Y$. Then exists $F: X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \le p(x)$ for all $x \in X$.

Lemma (Hanh-Banach in codimesion 1)**.** Ler $x_0 \in X \setminus Y$. Let $Y_\sharp = Y + \mathrm{Span}\{y_0\}$. Then f can be continued to a linear functional on $f_\sharp: Y_\sharp \to \mathbb{R}$ such that $f_\sharp \le p$ on $Y_\sharp$.

Proof. Let $y \in Y_\sharp$, $y \in Y$, $\alpha \in \mathbb{R}$.

We assert a $c \in \mathbb{R}$ can be chosen in such a way that

$$f_\sharp(y + \alpha y_0) := f(y) + \alpha c$$

satisfies

$$f_\sharp \le p \iff f(y) + \alpha c \le p(y + \alpha y_0). \tag{1}$$

If $\alpha = 0$, the inequality is satisfied.

Suppose $\alpha > 0$. Divide (1) through by α:

$$f\left(\frac{y}{\alpha}\right) + c \le p\left(\frac{y}{\alpha} + y_0\right).$$

This rewrites as

$$p(y_1 + y_0) - f(y_1) \geq c,$$

where $y_1 = y/\alpha$.

Suppose $\alpha < 0$. Dividing (1) through by $-\alpha$, we get

$$f(y_2) - p(y_2 - y_0) \leq c,$$

where $y_2 = -y/\alpha$.

Now, if we show that

$$f(y_2) - p(y_2 - y_0) \leq p(y_1 + y_0) - f(y_1),$$

we are done by the Cantor-Dedekind axiom. But that trivially follows from the triangle inequality for p and linearity of f. ■

Proof of the theorem of Hanh and Banach in the general case. Consider the set

$$A = \{(M, f_M) \mid Y \leq M \leq X,\ f_M \colon M \to \mathbb{R} \text{ is a linear functional, } f_M \leq p\}$$

We tell that $(M, f_M) \leq (N, f_N)$, iff $M \leq N$ and $f_N|_M = f_M$. The union of any chain is its supremum; we have a maximal element (L, F). We assert that $L = X$.

Suppose otherwise, and let $y_0 \in X \setminus L$. Define $L_\sharp = L + \operatorname{Span}\{y_0\}$. By the lemma, there exists $F_\sharp \colon L_\sharp \to \mathbb{R}$ such that $F_\sharp \leq p$. But then $(L_\sharp, F_\sharp) \geq (L, F)$. ■

Useful corollaries

Corollary. Let $Y \leq X$ and $f \in \mathcal{B}(Y, \mathbb{R})$. Then there exists $F \colon X \to \mathbb{R}$ such that $F|_Y = f$ and $\|F\| = \|f\|$.

Proof. Let $p(x) = \|f\|\|x\|$. Then $f(y) \leq p(y)$ for all $y \in Y$. Take F as in the HB theorem. Then

$$F(x) \leq \|f\|\|x\|.$$

Likewise, taking the $-F$ for $-f$, we get

$$\|F\| \leq \|f\|.$$

The converse inequality is evident. ■

Corollary. Let $Y \leq X$, $x_0 \in X \setminus Y$. Then exists $F \in X^*$ such that $\|F\| \leq 1$, $F|_Y = 0$, and $F(x_0) = \operatorname{dist}(x_0, Y)$.

Proof. Define $p(x) = \operatorname{dist}(x, Y)$. Since Y is closed, $d := p(x_0) > 0$. Define

$$f(y + \alpha x_0) := \alpha d.$$

Then $f \leq p$, and exists $F \colon X \to \mathbb{R}$ such that $F|_L = f|_L$ and $F \leq p$. In particular, $F|_Y = 0$ (what we need) and $F(x_0) = d$. Observe that the same applies to $-f$, and so $\|F(x)\| \leq p(x) \leq \|x\|$. Hence $\|F\| \leq 1$. ■

Corollary. Let $x_0 \in X$. Then exists $F \in X^*$ such that $\|F(x_0)\| = \|x_0\|$ and $\|F\| = 1$.

Proof. The case $Y = \{0\}$ of the previous corollary. ■

Corollary. Let X be a normed space, $M \leq X$. Then

$$\operatorname{Cl} M = \bigcap \{\ker f \mid f \in X^*,\ M \subseteq \ker f\}.$$

Proof. Let N denote the intersection. Since N is closed as an intersection of closed sets, $\operatorname{Cl} M \subseteq N$. We now show that $\overline{\operatorname{Cl} M} \subseteq \overline{N}$. Let $x_0 \notin \operatorname{Cl} M$. By a corollary from page 7, there exists a functional f X* such that $\|f\| \leq 1$, $M \subseteq \ker f$ and $f(x_0) = \operatorname{dist}(x_0, M)$. f participates in the intersection on the right, but its kernel does not contain x_0. Therefore, $x_0 \notin N$, as asserted. ■

Corollary. Let X be a normed space, $M \leq X$. The following conditions are equivalent:

1. M is dense.
2. For every functional $f \in X^*$, if $M \subseteq \ker f$, then $f = 0$.

Proof. This is a reformulation of the previous theorem. ■

Banach limits

Definition. Let $c \le l^\infty$ be the space of convergent subsequences. A map $L\colon l^\infty \to \mathbb{R}$ is a *Banach limit*, iff

1. It is a continuous linear functional.
2. For all $x = \{x_n\} \in c$,
$$Lx = \lim x_n.$$
3. If $x \ge 0$, then $Lx \ge 0$.
4. $L\{x_{n+1}\} = L\{x_n\}$.

Theorem. L exists.

Proof. Put, for $x = \{x_n\}$,

$$f(x) = \lim x_n.$$

$f\colon c \to \mathbb{R}$ is a linear functional. Define a majoring sublinear functional (exercise) as

$$p(x) = \limsup \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Observe that $p|_c = f$, so $f \le p$. By HB, we have $L\colon l^\infty \to \mathbb{R}$, which, luckily, satisfies the definition of Banach limit:

1. $L|_c = f$.
2. If $x \le 0$, then $Lx \le p(x) \le 0$.
3. Put $x_i' = x_{i+1} - x_i$. By linearity of L, it is sufficient to show that $L\{x_n'\} = 0$. And indeed,
$$p(x') = \lim \frac{1}{n}\sum_{i=1}^{n}\left(x_{n+1} - x_n\right) = 0.$$

■

The complex case

Definition. Let $c \le l^\infty(\mathbb{C})$ be the space of convergent subsequences. A map $L\colon l^\infty \to \mathbb{C}$ is a *Banach limit*, iff

1. It is a continuous linear functional.
2. For all $x = \{x_n\} \in c$,
$$Lx = \lim x_n.$$
3. If $x \in \mathbb{R}$ and $x \ge 0$, then $Lx \ge 0$.
4. $L\{x_{n+1}\} = L\{x_n\}$.
5. $\|L\| = 1$.

Theorem. L exists.

Proof. Let L be a real Banach limit. For $a, b \in l^\infty(\mathbb{C})$, define

$$L(a + ib) := La + iLb.$$

All of the properties now follow trivially from those of real limits, except the final one.

Simple functions are dense in l^∞, so we may prove the statement for them and be happy after using continuity of L. Let $x = \sum \alpha_k \chi_{E_k}$ for some partition $\bigsqcup_{k\in\mathbb{N}} E_k = \mathbb{N}$, and $|\alpha_\square| \le 1$. Then $Lx = \sum \alpha_k L\chi_{E_k}$, and

$$\begin{aligned}|Lx| &\le \sum_{k\in\mathbb{N}} L\left(\chi_{E_k}\right)\\ &= L\left(\chi_{\sqcup E_k}\right)\\ &\le 1,\end{aligned}$$

which was asserted. ■

Complex Hanh-Banach

Let $K = \mathbb{C}$.

Definition. Let Z be a complex vector space. We denote by $Z^\star$ the space of bounded linear functionals $Z \to \mathbb{C}$.

Lemma. A functional $f \in Z^\star$ can be recovered from its real or imaginary part.

Proof. Let $f = u + iv$. Since f is linear over $\mathbb{C}$, we may write

$$if(x) = iu(x) - v(x) = f(ix) = u(ix) - iv(ix).$$

Since 1 and i are a basis, we infer from here that

$$-v(x) = u(ix).$$

■

Theorem (Hanh-Banach, the complex version). Let X be a complex normed space, $Y \leq X$. Let $p\colon X \to \mathbb{R}$ be a seminorm. Let $f \in Y^\star$ be a linear functional such that $|f| \leq p$. Then exists a functional $F \in X^\star$ such that $F|_Y = f$ and $|F| \leq p$.

Proof. Let $f \in Y^\star$ and $u = \operatorname{Re} f$. u is a real linear functional on Y. By the real HB, exists a functional $U \in X^*$ such that $U|_Y = u$ and $|U(x)| \leq p(x)$ for all $x \in X$. Define

$$F(x) := U(x) - iU(ix).$$

This is a complex functional on X, and $F|_Y = f$. We assert that $|F| \leq p$. Let $r \in C$ be such that

$$|F(x)| = rF(x)$$

and $|r| = 1$. Then

$$|F(x)| = F(rx) = U(rx) \leq p(rx) = |r|p(x) = p(x),$$

what was to be shown. ∎

Quotient spaces

Definition. Let X be a normed space, $M \leq X$. On X/M we can introduce a norm:

$$\|[x]\| := \operatorname{dist}(x, M).$$

In this section, we preserve this naming, using, furthermore, $[\square]$ or $q\colon X \to X/M$ for the canonical projection. (And of course, we use choice throughout.)

Lemma. This is indeed a norm.

Proof.

1. *Homogeneity.* $x = x' + m$ for some $m \in M$, $x' \in \overline{M}$. m is the closest to x point of M: for any other m_1 we have

$$\|x - m_1\| = \|x' + m - m_1\| \geq \|m - m_1\| + \|x'\|.$$

Then

$$\|[kx]\| = \|kx - m\| = |k|\|x'\| = |k|\|[x]\|.$$

2. *Triangle inequality.* Since

$$\|x + y - m\| \leq \|x - m/2\| + \|y - m/2\|.$$

3. *Nonzero on nonzero vectors.* By closedness of M.
4. *Zero on the zero vector.* As $0 \in M$.

∎

Lemma. $\|[x]\| \leq \|x\|$ for all $x \in X$.

Proof. $\operatorname{dist}(x, M) \leq \|x\|$. ∎

Lemma. If X is complete, then X/M is.

Proof. Let $\{[x_n]\}$ be a Cauchy sequence. There exists a subsequence $\{x_{n_k}\}$ such that

$$\left\|\left[x_{n_k} - x_{n_{k+1}}\right]\right\| < 2^{-k}.$$

Let $\{y_k\}$ be a sequence in M such that

$$\|x_{n_k} + y_k - x_{n_{k+1}} - y_{k+1}\| < 2^{1-k}$$

(it exists by the previous inequality and the definition of distance). Then $\{x_{n_k} - y_k\}$ is a Cauchy sequence (**exercise**). Since X is complete, it converges to some x_*. Then

$$\left[x_{n_k}\right] \to [x_0]$$

by the previous lemma. ∎

Lemma. Let $W \subseteq X/M$. $q^{-1}(W)$ is open iff W is open.

Hence the norm topology we have introduced equals the usual quotient topology.

$\Leftarrow$. Let $[x] \in W \subseteq X/M$. We want to find a neighbourhood $U \ni x$ such that $[U] \subseteq W$.

Since W is open, there exists $\epsilon > 0$ such that, if $\|[y] - [x]\|$, then $qy \in W$.

That is, if $\operatorname{dist}(y, x) < \epsilon$, then $y \in q^{-1}(U)$; as needed. ∎

$\Rightarrow$. Suppose that $q^{-1}(W)$ is open. Let $[x] \in W$. We want to find a ball around $[x]$ that would lie in W. There is one around x that lies in $q^{-1}(W)$: for all y with $\|y - x\| <$ some $\epsilon > 0$ we have $y \in q^{-1}(W)$. This implies, in particular, that

$$\|[y] - [x]\| \leq \|y - x\| < \epsilon.$$

■

Lemma. Let $U \subseteq X$. Then qU is open.

That is, q is open.

Remark. The converse is not necessarily true.

Proof. Sufficient to show that $q^{-1}(qU)$ is open. In fact,

$$q^{-1}(qU) = \bigcup_{y \in M} (U + y).$$

■

Annulator

Definition. Let $M \leq X$, as before. Its *annulator* is the set

$$M^{\perp} = \left\{ f \in X^* \mid M \subseteq \ker f \right\}.$$

Theorem. There exists a linear isometry

$$\rho\colon X^*/M^{\perp} \longrightarrow M^*,$$

defined by

$$\rho\colon [f] \longmapsto f|_M.$$

Proof. Let $[f] \in X^*/M^{\perp}$.

The ρ defined is indeed a bijective function, since for every $g \in M$ we have

$$\begin{aligned} f \sim g &\iff f - g \sim 0 \\ &\iff f - g \in M^{\perp} \\ &\iff (f - g)|_M = 0 \\ &\iff f|_M = g|_M. \end{aligned}$$

This function does map into M^*: $f|_M \in M^*$ (a restriction of a linear function onto a linear subspace is linear; a restriction of a continuous function is continuous).

We show that it is an isometry. Let $\lambda \in M^*$. By HB there exists $\Lambda\colon X \to K$ such that $\Lambda|_M = \lambda$ and $\|\Lambda\| = \|\lambda\|$. But Λ gets mapped into λ by ρ, and ρ is bijective. ■

Theorem. There exists a linear isometry

$$(X/M)^* \longleftrightarrow M^{\perp}$$

defined by

$$\rho\colon f \longmapsto f \circ Q.$$

Proof. Let $\lambda \in (X/M)^*$, and let $\Lambda := \rho(\lambda)$. Since the norm of the composition does not surpass the product of norms, we have

$$\|\Lambda\| \leq \|\lambda\|.$$

Also, note that $\Lambda_| M = 0$.

Let $\Lambda \in M^{\perp}$. Define $\lambda[x] := \Lambda x$. Then $\|\lambda\| = \|\Lambda\|$: (1) obviously, $\|\lambda\| \leq \|\Lambda\|$; (2) the other inequality has been shown. ■

Separability of duals

Theorem. Let X be normed and X^* separable. Then X is separable.

Example. The converse is false. Let $X = l^1$. Then $\left(l^1\right)^* \cong l^\infty$. But l^1 is separable, while l^∞ is not.

Proof. Let $\left\{f_j\right\}$ be a dense sequence in the unit sphere of X^*. Such exists by hypothesis.

By definition of norm in the dual, there is a sequence $\left\{x_j\right\} \in X$ with $\left\|x_j\right\| = 1$ and $\left|f_j(x_j)\right| \geq 1 - \epsilon_j$ for all j.

Let $M = \operatorname{Cl}\operatorname{Span}\left\{x_j\right\}$. By definition, this is a separable space.

Let $y \notin M$. By one of corollaries of HB there exists $f \in X^*$ such that $f|_M = 0$, $f(y) \neq 0$, $\|f\| = 1$.

Since $\left\{f_j\right\}$ is dense, there exists a convergent subsequence

g_k of $\{f_j\}$: $g_k \to f$. Assume that $|g_k(x_k)| \geq 1/2$. Then

$$\begin{aligned}|f(g_k)| &= |f(x_k) - g_k(x_k) + g_k(x_k)| \\ &\geq |g_k(x_k)| - |f(x_k) - g_k(x_k)| \\ &\geq 1/2 - o(1).\end{aligned}$$

This is in contradiction to $f|_M = 0$. ■

Open map and closed graph theorems

Theorem (on an open map)**.** Let X and Y be Banach and $A \in \mathcal{B}(X, Y)$. Then A is open.

The proof spans several lemmas.

Let $B_r \subseteq X$ be the ball of radius $r > 0$ with centre at 0.

Lemma. $0 \in \operatorname{Int}\operatorname{Cl} A(B_1)$.

Proof. Observe that

$$Y = \bigcup_{n \geq 1} \operatorname{Cl} A(B_n).$$

By Baire's theorem, one of the sets in the union has non-empty interior. But as they are all homothetic, this is true for any of them. Suppose y_0 lies in $V := \operatorname{Int}\operatorname{Cl} A(B_1)$ together with a ball B around it of radius δ.

Let $\{x_n\}$ be such that $\|x_n\| < 1$ and $Ax_n \to y_0$ (such exists, since y_0 is in the closure).

For sufficiently small y, $y_0 + y \in \operatorname{Cl} A(B_1)$.

There exists $\{z_n\}$ such that $\|z_n\| < 1$ and $Az_n \to y_0 + y$. $A(z_n - x_n) \to y$. Since $z_n, x_n \in B_1$, $z_n - x_n \in B_2$. The last two sentences say exactly that

$$y \in \operatorname{Cl} A(B_2).$$

This holds for sufficiently small y, which is equivalent to saying that

$$0 \in \operatorname{Int}\operatorname{Cl} A(B_1).$$

■

Lemma. $\operatorname{Cl} A(B_1) \subseteq A(B_2)$.

Proof. Let $y_1 \in \operatorname{Cl} A(B_1)$. $0 \in \operatorname{Int}\operatorname{Cl} A(B_{1/2})$. Since in every neighbourhood of y_1 there are points from $A(B_1)$,

$$\left(y_1 - \operatorname{Cl} A(B_{1/2})\right) \cap A(B_1) \neq \varnothing.$$

That is, exists $x_1 \in B_1$ such that

$$Ax_1 \in y_1 - \operatorname{Cl} A(B_{1/2}).$$

For some $y_2 \in \operatorname{Cl} A(B_{1/2})$,

$$Ax_1 = y_1 - y_2.$$

Likewise,

$$\left(y_2 - \operatorname{Cl} A(B_{1/4})\right) \cap A(B_{1/2}) \neq \varnothing.$$

Continuing to infinity, we get a sequence $\{x_n\}$ such that

$$\|x_n\| \leq 2/2^n,$$

and a sequence $\{y_n\}$ such that $y_n \in \operatorname{Cl} A\left(B_{2/2^n}\right)$ and

$$Ax_n = y_n - y_{n+1}.$$

Let $x := \sum x_n$ — by completeness of X, this vector indeed exists and has norm < 2. By continuity of A, $y_1 = Ax \in A(B_2)$. ■

Proof for the open graph theorem. Summing the two lemmas, we get

$$0 \in \operatorname{Int}\operatorname{Cl} A(B_1) \subseteq \operatorname{Int} A(B_1).$$

This finishes the proof, since

$$\|Ax - Ax_0\| < \epsilon \iff \|A(x - x_0)\| < \epsilon.$$

■

Inverse function theorem

Theorem (inverse function theorem)**.** Let X and Y be Banach and $A \in \mathcal{B}(X, Y)$ a bijection. Then A^{-1} is linear and continuous.

Proof. A^{-1} is linear anyway. Continuity is a direct corollary of the open map theorem. ■

Closed graph theorem

Definition. If X and Y are normed spaces, their coproduct has the following structure of a normed space:

$$\|x \oplus y\| := \|x\| + \|y\|.$$

Lemma. Let $A\colon X \to Y$ be a linear operator. Its graph is closed iff for any sequence $x_n \to 0$ such that the limit $\lim Ax_n$ exists, $Ax_n \to 0$.

Remark. The difference with continuity is that we do not expect that $\{Ax_n\}$ will automatically converge. The closed graph theorem tells it will in complete spaces.

$\Rightarrow$. As the graph is closed, there is some x so that $(x_n, Ax_n) \to (x, Ax)$.

$$\|x_n - x\| + \|A(x_n - x)\| \to 0.$$

Then

$$\|x_n - x\|, \|A(x_n - x)\| \to 0.$$

Since the sequence limit is unique, $x = 0$. Then $\|Ax_n\| \to 0$. ■

$\Leftarrow$. Let $(x_n, Ax_n) \to (x, y)$. We assert that $y = Ax$. Indeed,

$$\|x_n - x\| + \|Ax_n - y\| \to 0$$

implies that $x_n - x \to 0$. But then $A(x_n - x) \to 0$ by hypothesis, and so

$$\|Ax_n - Ax\| \to 0.$$

By uniqueness of limits, $y = Ax$. ■

Theorem (the closed graph theorem). Let X and Y be complete, $A\colon X \to Y$ linear. If the graph of A is closed, then A is continuous.

The converse is obvious.

Proof. $X \otimes Y$ is complete. Let G be the graph of A. Let $P\colon G \to X$ be defined as

$$P(x, Ax) := x.$$

P is a bijection. The inverse function theorem says that P^{-1} is continuous. Let $Q\colon G \to X$ be defined as

$$Q(x, Ax) := Ax.$$

Then Q is continuous. Then $A = QP^{-1}$ is continuous. ■

Why is the closed graph theorem important

Lemma. Let $\varphi \in L^p$. Define a linear $M\colon L^p \to L^p$ as $Mf := \varphi \cdot f$. Suppose that $\operatorname{im} M \subseteq L^p$. Then M is bounded.

Proof. We check that the graph of M is closed. Let $f_n \to 0$ in L^p and $\varphi f_n \to g$. By Hölder's inequality, the sequence φf_n also converges to 0. ■

Sobolev spaces

Definition (Sobolev spaces). The normed space $W_p^n[a,b]$ consists of $f \in C^{n-1}[a,b]$ with the norm

$$\|f\|_{W_p^n} := \left(\sum_{k=0}^{n} \left\|f^{(k)}\right\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Lemma. This is a complete space.

Theorem. Let $M\colon f \mapsto \varphi \cdot f$ for some $\varphi \in W_p^n$. Then M is bounded.

Proof. Let $f_n \to 0$ and $\varphi f_n \to g$. Then $f_n \to 0$ pointwise. Then $\varphi f_n \to 0$ pointwise, which implies $g = 0$. ■

Complementary spaces

Definition. Let X be Banach. Let $M, N \leq X$ be closed spaces. The M and N are said to be *algebraically complementary*, iff

$$M + N = X \qquad \text{and} \qquad M \cap N = \{0\}.$$

They are *topologically complementary*, iff, in addition,

$$\|x \oplus y\| \sim \|x\| + \|y\|$$

for all $x \in M$ and $y \in N$ (that is, these norms are equivalent on $M \otimes N \cong X$.

Theorem. If X is Banach and $M, N \leq X$ are algebraically complementary, then M and N are topologically complementary.

Proof. The map

$$x + y \mapsto x \otimes y$$

is an isomorphism of $M+N$ and $M \otimes N$. The norms $\|x\| + \|y\|$ and $\|x + y\|$ are equivalent. The inverse map is continuous by the inverse function theorem; hence Lipschitz. Together with the triangle inequality, this gives the equivalence of norms. ∎

Uniform boundedness principle

Theorem (uniform boundedness principle)**.** Let X be Banach and Y normed. Let $A \subseteq \mathcal{B}(X, Y)$. A is bounded iff

$$\forall x \in X \colon \sup_{a \in A} \|ax\| < \infty.$$

Proof. If A is bounded, this is trivially true. We prove the converse.

Let

$$Q_n := \{x \in X \mid \forall a \in A \colon \|ax\| \leq n\}.$$

Q_n is closed as an intersection of closed spaces (all a are continuous). By Baire's theorem, $\bigcap Q_n$ has non-empty interior. Let $x_0 \in \bigcap Q_n$ be such that

$$\|a(x_0 + x)\| \leq 1$$

whenever $\|x\| < r$ for all $a \in A$. Then

$$\|ax\| \leq 1 + \|ax_0\|$$

for all $a \in A$. This implies

$$\sup_{\|x\| < n} \|ax\| < +\infty.$$

∎

Corollary. Let X be normed, $A \subseteq X$. A is bounded iff, for every $f \in X^*$, $f(A)$ is bounded.

Proof. The dual is complete. Put $X \leftarrow X^*$ and $Y \leftarrow X$. ∎

Theorem. Let X be Banach, Y normed, and $A \subseteq \mathcal{B}(X, Y)$. A is bounded iff

$$\forall x \in X \ \forall g \in Y^* \colon \sup_{a \in A} |gax| < \infty.$$

Proof. A is bounded iff gA is bounded for every g iff

$$\forall g \in X^* \ \forall x \in X \colon \sup_{ga \in gA} |gax| < \infty$$

iff

$$\forall g \in X^* \ \forall x \in X \colon \sup_{a \in A} |gax| < \infty.$$

∎

Banach-Steinhaus theorem

Theorem. Let X, Y be Banach spaces. Let $A_n \in \mathcal{B}(X, Y)$. Suppose $A_n x$ converges in Y for all $x \in X$. Then

$$\sup_n \|A_n\| < \infty,$$

and exists $A \in \mathcal{B}(X, Y)$ such that

$$\|A_n x - Ax\| \to 0$$

for all $x \in X$.

Proof. Put $Ax := \lim(A_n x)$. If $A_n x$ converges for all x, then $A_n x$ is bounded in Y for all x. Then $\{A_n\}$ is bounded, and so

$$\|Ax\| \leq \sup_n \|A_n\| \|x\|.$$

∎

Inner products

Definition. Let H be a vector space over K. An *inner product* on H is a map $\langle \square, \square \rangle : H^2 \to K$ such that

1. $\langle \alpha x + \beta, y \rangle = \alpha \langle x, y \rangle + \beta$ for all $x, y \in H$ and $\alpha, \beta \in K$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$.
3. $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle \geq 0$.
4. $\langle x, x \rangle = 0$ implies $x = 0$.

Lemma (CBS).

$$\left| \langle x, y \rangle \right| \leq \|x\| \|y\|.$$

Proof. Let $\alpha \in K$.

$$\begin{aligned} 0 &\leq \langle x + \alpha y, x + \alpha y \rangle \\ &= \|x\|^2 + 2\operatorname{Re}\left(\alpha \langle y, x \rangle\right) + |\alpha|^2 \|y\|^2. \end{aligned}$$

$\alpha =: re^{i\varphi}$. Choose φ in such a way that $\alpha \langle y, x \rangle \in \mathbb{R}$. The resulting degree 2 polynomial in r has at most one root. ∎

Definition.

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

Lemma. This is a norm.

Proof. We check triangle inequality.

$$\begin{aligned} & \|x + y\|^2 \leq \left(\|x\| + \|y\|\right)^2 \\ \iff \quad & \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2. \end{aligned}$$

This follows from CBS. ∎

Lemma (parallelogram identity).

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Proof. Direct computation. ∎

Lemma. Let $p \neq 2$ and $p \geq 1$. Then the l^p norm is not induced by any inner product.

Proof. $2^{1/p} + 2^{1/p} \neq 2(1+1)$. ∎

Definition. We say x and y are *orthogonal* and write $x \perp y$, iff $\langle x, y \rangle = 0$.

Lemma (Pythagoras). $x_1, \ldots, x_n$ are pairwise orthogonal iff

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2.$$

Proof. Direct computation. ∎

Lemma (polarisation identity). Let $U = \{\pm 1, \pm i\}$ in case $K = \mathbb{C}$ and $U = \{\pm 1\}$ in case $K = \mathbb{R}$. Then

$$\langle x, y \rangle = \frac{1}{4} \sum_{\alpha \in U} \alpha \|x + \alpha y\|^2.$$

Proof. Direct computation. ∎

Lemma. Let $f : U \to B$ a linear operator. $\|fu\| = \|u\|$ for all $u \in U$ iff $\langle fx, fy \rangle = \langle x, y \rangle$.

Proof. From the polarisation identity. ∎

Hilbert spaces

Definition. Let H be an inner product space. H is *Hilbert*, iff it is complete with respect to the norm that is induced by the inner product.

Example. Consider $C[a, b]$ with

$$\langle f, g \rangle := \left(\int \bar{f} \cdot g \right)^{1/2}.$$

This space is not complete, so not Hilbert. But its completion is isometrically isomorphic to $L^2[a, b]$.

Projections

Theorem (existence and uniqueness of projections). Let H be a Hilbert space, C a closed convex set, $a \in H$. Then exists unique $x_0 \in C$ such that

$$\|a - x_0\| = d := \operatorname{dist}(a, C).$$

Proof of existence. We may assume that $a = 0$. We want a vector of minimal norm in C. Exists a finite sequence $x_n \in C$ such that $\|x_n\| \to d$. From parallelogram identity,

$$\left\|\frac{x_n - x_m}{2}\right\|^2 + \left\|\frac{x_n + x_m}{2}\right\|^2 = \frac{\|x_n\|^2 + \|x_m\|^2}{2}.$$

The right part tends to d^2; the right addend on the left part is at least d^2. Then $x_n \to x_0$ for some $x_0 \in C$, since H is complete. ∎

Proof of uniqueness. From the same identity we derive that the difference between two such vectors must be zero. ∎

Projections onto linear subspaces

Theorem (definitions of a projection onto a subspace). Let U be a Hilbert space, $V \leq U$. Then V is a closed convex set, and so satisfies the previous theorem. Let $x \in U$, $y_0 \in V$. The following are equivalent:

1. $\|x - y_0\| = \operatorname{dist}(X, V)$.
2. $x - y_0 \perp V$.

$2 \Rightarrow 1$. Let $y \in V$. Then

$$\begin{aligned}\|x - y\|^2 &= \left\|(x - y_0) - (y - y_0)\right\|^2 \\ &= \left\|(x - y_0)\right\|^2 - \left\|(y - y_0)\right\|^2 \\ &\geq \|x - y_0\|^2.\end{aligned}$$

∎

$1 \Rightarrow 2$.

$$\begin{aligned}&\left\|x - (y_0 + \beta y)\right\|^2 \\ &= \|x - y_0\|^2 + |\beta|^2\|y\|^2 - 2\operatorname{Re}\Big(\beta\langle y, x - y_0\rangle\Big).\end{aligned}$$

Choose β such that $\beta\langle y, x - y_0\rangle \in \mathbb{R}$ and $\beta\langle y, x - y_0\rangle < 0$. Since parabolas are convex, $y \perp x - y_0$ would lead to a contradiction. ∎

Lemma. In the conditions of the previous theorem, let $P \colon U \to V$ map every $x \in U$ into the corresponding y_0. Then P is linear.

Proof. Because

$$\alpha_1 x_1 - Px_1 + \alpha_2 x_2 - Px_2 \perp K.$$

∎

Riesz's lemma

Lemma (Riesz). Let H be a Hilbert space. The following are equivalent:

1. f is a continuous functional on H.
2. Exists $y \in H$ such that

$$fx = \langle y, x\rangle.$$

$2 \Rightarrow 1$. Clearly, f is a continuous linear functional. By CBS, $\|fx\| \leq \|x\|\|y\|$. This is sharp when $x = y$. ∎

$1 \Rightarrow 2$. Let $N := \ker f$. If $N = H$, then $f = 0$. Suppose $w \in H \setminus N$. Define $v := w - P_N w$. Obviously, $v \perp N$. Define $\Phi_v(x) = \langle v, x\rangle$. Then $\ker \Phi_v = \ker f$, and so exists $c \in K$ such that $\Phi_v = cf$. Then

$$fx = \langle v/c, x\rangle.$$

∎

Definition. A topological vector space V is *reflexive*, iff the canonical map $V \to V^{**}$ is an isomorphism.

Example. The map $\Phi_\square \colon H \to H^*$ we have built in the proof is conjugate-linear:

$$\Phi_{\alpha v} = \overline{\alpha}\Phi_v.$$

Thus any Hilbert space is reflexive.

Orthonormal bases

Definition. A subset $S \subseteq U$ of a Hilbert space U is *orthonormal*, iff $\|s\| = 1$ and $s \perp s'$ for all different $s, s' \in S$. S is a *basis*, iff it is maximal among orthonormal sets.

Lemma. An orthonormal basis B exists.

Proof. From Zorn's lemma. ∎