Functional analysis

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Introduction

Let X be a vector space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$.

Definition. A map $f : X \to \mathbb{R}$ is called a *norm*, iff we have

1. If $x \in X \setminus 0$, then $f(x) > 0$; and $f(0) = 0$.
2. $f(kx) = |k| f(x)$ for all $k \in \mathbb{R}$, $x \in X$.
3. $f(x+y) \leq f(x) + f(y)$ for all $x, y \in X$.

Definition. A pair $(V, \|\square\|)$, where $\|\square\|$ is a norm on a vector space V, is called a *normed space*. A normed space is *Banach*, iff it is complete.

Completions

Definition. $\widehat{X}$ is called a *completion* of X, iff $\widehat{X}$ is complete and X is dense in $\widehat{X}$.

Theorem. A completion $\widehat{X}$ exists.

Proof. Call two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ equivalent, iff $\|x_n - y_n\| \to 0$, and let $\widehat{X}$ be the resulting quotient. Since the point-wise sum of two Cauchy sequences is Cauchy, in this natural way we may introduce vector space structure on $\widehat{X}$. The norm on $\widehat{X}$ is introduced as

$$\|[x_n]\| := \lim_{n \to \infty} \|x_n\|.$$

This map is defined correctly:

1. The limit on the right always exists: since $\{x_n\}$ is Cauchy, the sequence $\{\|x_n\|\}$ of reals is Cauchy, which implies it must converge.

2. If $[x_n] = [y_n]$, then $\left| \|x_n\| - \|y_n\| \right| \le \|x_n - y_n\| \xrightarrow[n\to\infty]{} 0$. Therefore, $\|[x_n]\| = \|[y_n]\|$.

X is embedded into $\widehat{X}$ by mapping $x \in X$ into the class of the constant sequence at x. It is easy to see that this map preserves norms. ■

Theorem. Let $\widehat{X}_2$ be another completion of X. Then exists a bijection $f\colon \widehat{X} \to \widehat{X}_2$ which is linear, preserves norms, and maps the embedded X into the embedded X.

These two theorems endow us with the right to never consider pathological incomplete spaces.

Proof. Map $[x_n] \in \widehat{X}$ into $\lim x_n \in \widehat{X}_2$. ■

Exercise 1. The space $C\left[a,b\right]$ with the norm $\|f\|_2 = \int |f|$ is not complete. Define

$$f_n(x) = \begin{cases} 0, & x \in [a,c], \\ n(x-c), & x \in \left[c, c+1/n\right], \\ 1, & x \in \left[c+1/n, b\right]. \end{cases}$$

Then f_n is a Cauchy sequence which does not have a limit in $C\left[a,b\right]$.

Proof. It is easy to see that the limit of $\{f_n\}$ is $[x \ge c]|_{[a,b]}$, so it does not have a continuous limit. Nevertheless, it is Cauchy, since

$$\|f_n - f_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \xrightarrow[\min\{m,n\}\to\infty]{} 0.$$

■

Equivalent norms

Definition. We say norms are *equivalent*, iff the metrics they generate are Lipschitz equivalent.

Exercise 2. Norms are equivalent iff they generate the same topology.

Exercise 3. In infinite-dimensional spaces, there are norms which are not equivalent.

Proof. For example, consider $X = C[0,1]$, L^1-norm and the sup-norm on it. It is true that L^1-norm does not surpass the sup-norm, but there is no constant for the opposite inequality: we can think of a function with an arbitrarily large sup-norm, but constant integral. ∎

Theorem. If X is finite-dimensional, then every two norms on X are equivalent.

Proof. Suppose $\dim X = n$, and $e_1, \ldots, e_n$ is a basis. Let $x = a_1 e_1 + \cdots + a_n e_n$. Let $\|\square\|$ be a norm on X. Define a new *norm* as

$$|x| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Then

$$\begin{aligned}
\|x\| &\leq \sum_{i=1}^{n} |a_i| \|e_i\| \\
&\leq M \sum_{i=1}^{n} |a_i| \\
&\leq M \left(\sum_{i=1}^{n} |a_i| \right)^{\frac{1}{2}} \\
&= M|x|.
\end{aligned}$$

The function $x \mapsto \|x\|$ is continuous in the norm $|\square|$. Let $\left|x_k - x\right| \to 0$. Then

$$\begin{aligned}
\left| \|x_k\| - \|x\| \right| &\leq \left\|x_k - x\right\| \\
&\leq M\sqrt{n}\left|x_k - x\right| \\
&\to 0.
\end{aligned}$$

Consider the set $S = \left\{x \in E \mid |x| = 1\right\}$. S is compact in $|\square|$. $\varphi|_S$ is continuous and nonzero. Then $\varphi > \delta$ for some $\delta > 0$, so

$$\left\| \frac{x}{|x|} \right\| \geq \delta \iff \|x\| \geq \delta |x|.$$

∎

Corollary. Every finite-dimensional normed vector space is complete.

Proof. Every Euclidean space is complete. ∎

Corollary. A finite-dimensional subspace of a normed space is closed.

Proof. It is complete, and every convergent sequence is Cauchy sequence. ∎

Definition. The set $M \subseteq X$ is *bounded*, iff

$$\sup_{m \in M} \|m\| < +\infty.$$

Lemma (on an almost-perpendicular)**.** Let E be a normed space, and $F < E$ its closed proper subspace. Then for every $\epsilon > 0$ exists a vector $x \in E$ such that $\|x\| = 1$ and $\operatorname{dist}(x, F) > 1 - \epsilon$.

Proof. Let $y \in E \setminus F$. Then $d = \operatorname{dist}(y, F) > 0$, since F is closed. Let $\delta > 0$. By definition of infimum, there exists $a \in F$ such that

$$d \leq \|y - a\| \leq d + \delta.$$

Put $y_2 = y - a$. Since $a \in F$, $\operatorname{dist}(y_2, F) = \operatorname{dist}(y, F) = d$. Define

$$x = \frac{y_2}{d + \delta} = \frac{y - a}{d + \delta}.$$

Then $\|x\| \leq 1$, but

$$\operatorname{dist}(x, F) = \operatorname{dist}\left(\frac{y}{d + \delta}, F\right) \geq \frac{\operatorname{dist}(y, F)}{d + \delta} = \frac{d}{d + \delta}.$$

Since the δ is arbitrary, and by increasing the norm of x we do not get closer to F, we get the desired. ∎

Theorem. Let X be a normed space. Equivalent are:

1. X is finite-dimensional.

2. Every bounded subset of X is relatively compact.

Proof of $1 \implies 2$. From corollary on page 5. ∎

Proof of $2 \implies 1$. Suppose X is not finite-dimensional. We assert that there exists a bounded sequence that does not have a convergent subsequence (so in no way the closure of a bounded subset that contains this sequence can be compact). We show this by induction: suppose $x_1, \dots, x_n$ are already built. By the almost-perpendicular lemma there exists x_{n+1} such that $\|x_{n+1}\| = 1$ and

$$\operatorname{dist}\big(x_{n+1}, \operatorname{span}\{x_1, \dots, x_n\}\big) > 1/2$$

(since X is not finite-dimensional, the span here is a proper subspace of X). Continuing to infinity, we get a sequence $\{x_n\}$. It is bounded (all its members are on the unit sphere). Nevertheless, no subsequence of it is Cauchy by construction. ∎

Linear operators

Definition. A linear operator $T : X \to Y$ is *bounded*, iff $T(B)$ is bounded, where B is the unit ball in X.

Lemma. Let X and Y be normed vector spaces, and $T : X \to Y$ a linear operator. The following are equivalent:

1. T is continuous.

2. T is continuous at 0.

3. T is bounded.

Proof of $1 \Leftrightarrow 2$. Let $x \in X$. T is continuous at x iff for every convergent $x_n \to x$ the sequence Tx_n

also converges (to Tx). Now observe that

$$\|Tx_n - Tx\| = \left\|T(x_n - x)\right\|.$$

∎

Proof of $2 \Rightarrow 3$. Consider

$$D = \left\{y \in Y \mid \left\|y\right\| \leq 1\right\}.$$

There is $\delta > 0$ such that, if $\|x\| \leq \delta$, then $Tx \in D$. Let $z \in D$. Since

$$\left\|T\delta z\right\| \leq 1,$$

we have

$$\|Tz\| \leq 1/\delta.$$

∎

Proof of $3 \Rightarrow 1$. Let $\|x\| \leq 1$. Then $\|Tx\| \leq M$, so $\|x\| < \epsilon$ implies $\|Tx\| \leq M\epsilon$. ∎

Operator norm

Definition. Let T is a continuous operator. Its *norm* is

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Exercise 4. $\|T\|$ is indeed a norm.

Lemma. An operator is bounded iff it has finite norm.

Proof. Obvious. ∎

Lemma. A linear combination of continuous operators is continuous.

Proof.

$$\left\|(kA+B)\right\| \le \left|k\right|\|A\| + \|B\|.$$

■

The space of bounded operators is complete

Definition. $\mathcal{B}(X,Y)$ is the set of bounded linear operators $X \to Y$ with the obvious structure of a vector space and the standard operator norm as the norm.

Theorem. Let Y be complete. Then $\mathcal{B}(X,Y)$ is complete.

I have seen three quite concise proofs of this theorem and understood neither. Here is a long one, my own.

Proof. Let $\{T_n\}$ be a Cauchy sequence. Fix $x \in X$. $\{T_n x\}$ is a Cauchy sequence in Y:

$$\begin{aligned} \|T_n x - T_m x\| &\le \|T_n - T_m\|\|x\| \\ &\le \epsilon\|x\|. \end{aligned}$$

By completeness of Y, there is a limit $t \xleftarrow[n\to\infty]{} T_n x$.

The map $x \mapsto t$ we have just build is a linear operator. Call it T. T_n is a Cauchy sequence, so $\|T_n\| \le M$ for some M. Then T itself is bounded:

$$\begin{aligned} \|Tx\| &\le \|T_n x\| + \epsilon \\ &\le M\|x\| + \epsilon. \end{aligned}$$

We assert that $\|T_n x - Tx\| \to 0$. Suppose otherwise:

$$\exists\,\epsilon > 0\ \forall\, n_0 \in \mathbb{N}\ \exists\, n > n_0\ \exists\, x \in B\colon \|T_n x - Tx\| > \epsilon.$$

Since $\{T_n\}$ is Cauchy,

$$\forall \delta > 0 \; \exists n_1 \in \mathbb{N} \; \forall k, l > n_1 \; \forall x \in B : \left\|T_k x - T_l x\right\| \leq \delta.$$

Fix this $\delta > 0$ and take the corresponding n_1. From the first line with quantifiers, there exist $n > n_1$ and $x \in B$ such that

$$\|T_n x - T x\| > \epsilon.$$

From the second one, for any $m > n_1$ we get

$$\begin{aligned}\|T_n x - T x\| &\leq \|T_n x - T_m x\| + \|T_m x - T x\| \\ &\leq \delta + \|T_m x - T x\|.\end{aligned}$$

Since $Tx = \lim_{m\to\infty} T_m x$, there is m_0 such that $\|T_m x - T x\| \leq \delta$ for all $m > m_0$. Take $m_1 = \max\{m_0, n_1\}$. Then, for any $m > m_1$,

$$\begin{aligned}\|T_n x - T x\| &\leq \delta + \|T_m x - T x\| \\ &\leq 2\delta.\end{aligned}$$

Now launch $n_1 \to \infty$ and, consequently, $\delta \to 0$. We get a contradiction:

$$\epsilon < \|T_n x - T x\| \leq 2\delta.$$

■

Remark. Our proof does not use the boundedness of operators in the space $\mathcal{B}(X, Y)$.

Functionals

Definition. A *functional* is a linear operator $X \to K$.

Definition. The *dual* X^* of X is the space $\mathcal{B}(X, K)$ of continuous functionals.

Corollary. X^* is complete.

Proof. Since K is complete in either case. ■

Strong convergence

Definition. A sequence of operators $T_n \to T$ converges *strongly* or *point-wise*, iff $T_n x \to Tx$ for all $x \in X$.

Lemma. If $\|T_n - T\| \to 0$, then $T_n \to T$ strongly.

Proof. Since

$$|T_n x - Tx| \le \|T_n - T\||x|.$$

■

Remark. The converse is not true.

Definition. $l^p = L^p(\mathbb{N}, \#)$ is the space of real-valued sequences which converge in the L^p norm (with respect to the cardinality measure).

Proof of the remark. Consider the operators

$$s_k(c_0, c_1, \dots) = (c_k, c_{k+1}, \dots)$$

on l^p. $\|s_k\| = 1$, since there are sequences with a single unit and other elements zero and applying s_k does not lessen the sequence norm anyway. Nevertheless, $s_k \to 0$ pointwise (strongly), since all the sequences in l^p converge. ■

More on Banach spaces

Theorem. Let Y be complete. Let $\{T_n\} \subseteq \mathcal{B}(X, Y)$ be operators with $\sup\|T_n\| < +\infty$, and let $E \subseteq X$ be dense. Suppose that, for every $e \in E$, the sequence $\{T_n e\} \subseteq Y$ converges. Then exists $T \in \mathcal{B}(X, Y)$ such that $T_n \to T$ pointwise.

Remark. If $\{T_n\}$ converges strongly, then $\sup\|T_n\| < +\infty$. This is a harder fact.

Proof. Define

$$Te := \lim_{n\to\infty} T_n e.$$

Let $x \in x$. Take $\{e_n\}$ such that $\|e_n - x\| \xrightarrow[n\to\infty]{} 0$. We assert T can be continued to a bounded operator $X \to Y$.

$\{Te_n\}$ is Cauchy.

$$\begin{aligned}\left\|Te_k - Te_k\right\| &\leq \left\|Te_j - T_n e_j\right\| + \left\|T_n e_j - T_n e_k\right\| + \left\|Te_k - T_n e_k\right\| \\ &< 2\epsilon + \left\|T_n e_j - T_n e_k\right\| \\ &\leq 2\epsilon + \|T_n\| \left\|e_j - e_k\right\| \\ &\xrightarrow[\substack{n\to\infty \\ \min\{j,k\}\to\infty}]{} 0.\end{aligned}$$

■

Theorem. Let X and Y be normed spaces, and Y complete. Let F be dense in X. Let $T : F \to Y$ be a continuous linear operator. Then exists unique $T_\sharp \in \mathcal{B}(X, Y)$ such that $\left\|T_\sharp\right\| = \|T\|$ and $T_\sharp|_F = T$.

Proof. Let $x \in X$, $f_n \in F$, $\left\|f_n - x\right\| \to 0$. Then

$$\begin{aligned}\left\|Tf_n - Tf_m\right\| &\leq \|T\| \left\|f_n - f_m\right\| \\ &\xrightarrow[\min\{m,n\}\to\infty]{0}.\end{aligned}$$

Hence $\{Tf_n\}$ is a Cauchy sequence and, as such, converges to some y. Define

$$T_\sharp x = y.$$

This definition does not depend on the choice of the sequence. ■

Three fundamental principles of functional analysis

1. Hanh-Banach theorem.

2. Closed graph and open map theorems.
3. The principle of uniform boundedness.

Hanh-Banach theorem

Definition. Let X be a vector space over K, $p: X \to \mathbb{R}$. It is said that p is a *seminorm*, iff

1. $p(x) \geq 0$.
2. $|k|p(x) = p(kx)$ for all $k \in K$.
3. $p(x+y) \leq p(x) + p(x)$.

Example. Consider $X = C(\mathbb{R})$ and $p(f) = \int_0^1 |f|$. Then p is a seminorm, but not a norm, since there is a nonzero function, which is zero on $[0,1]$.

Definition. Let X be a K-vector space. A $p: X \to \mathbb{R}$ is a *sublinear functional*, iff

1. $p(x+y) \leq p(x) + p(y)$.
2. $p(kx) = |k|p(x)$ for all $k \in K$.

Theorem (Hanh, Banach)**.** Let X be a real vector space, $p: X \to \mathbb{R}$ a sublinear functional. Let $Y \leq X$ and $f: Y \to \mathbb{R}$ a linear functional such that $f(y) \leq p(y)$ for all $y \in Y$. Then exists $F: X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \leq p(x)$ for all $x \in X$.

Lemma (Hanh-Banach in codimesion 1)**.** Ler $x_0 \in X \setminus Y$. Let $Y_\sharp = Y + \operatorname{span}\{y_0\}$. Then f can be continued to a linear functional on $f_\sharp: Y_\sharp \to \mathbb{R}$ such that $f_\sharp \leq p$ on $Y_\sharp$.

Proof. Let $y \in Y_\sharp$, $y \in Y$, $\alpha \in \mathbb{R}$.

We assert a $c \in \mathbb{R}$ can be chosen in such a way that

$$f_\sharp(y + \alpha y_0) := f(y) + \alpha c$$

satisfies

$$f_\sharp \le p \iff f(y) + \alpha c \le p\big(y + \alpha y_0\big). \tag{1}$$

If $\alpha = 0$, the inequality is satisfied.

Suppose $\alpha > 0$. Divide (1) through by α:

$$f\left(\frac{y}{\alpha}\right) + c \le p\left(\frac{y}{\alpha} + y_0\right).$$

This rewrites as

$$p(y_1 + y_0) - f(y_1) \ge c,$$

where $y_1 = y/\alpha$.

Suppose $\alpha < 0$. Dividing (1) through by $-\alpha$, we get

$$f(y_2) - p(y_2 - y_0) \le c,$$

where $y_2 = -y/\alpha$.

Now, if we show that

$$f(y_2) - p(y_2 - y_0) \le p(y_1 + y_0) - f(y_1),$$

we are done by the Cantor-Dedekind axiom. But that trivially follows from the triangle inequality for p and linearity of f. ■

Proof of the theorem of Hanh and Banach in the general case. Consider the set

$$A = \big\{(M, f_M) \mid Y \le M \le X,\ f_M \colon M \to \mathbb{R} \text{ is a linear functional, } f_M \le p\big\}.$$

We tell that $(M, f_M) \le (N, f_N)$, iff $M \le N$ and $f_N|_M = f_M$. The union of any chain is its supremum; we have a maximal element (L, F). We assert that $L = X$.

Suppose otherwise, and let $y_0 \in X \setminus L$. Define $L_\sharp = L + \operatorname{span}\{y_0\}$. By the lemma, there exists $F_\sharp \colon L_\sharp \to \mathbb{R}$ such that $F_\sharp \le p$. But then $(L_\sharp, F_\sharp) \ge (L, F)$. ■

Useful corollaries

Corollary. Let $Y \leq X$ and $f \in \mathcal{B}(Y, \mathbb{R})$. Then there exists $F\colon X \to \mathbb{R}$ such that $F|_Y = f$ and $\|F\| = \|f\|$.

Proof. Let $p(x) = \|f\|\|x\|$. Then $f(y) \leq p(y)$ for all $y \in Y$. Take F as in the HB theorem. Then

$$F(x) \leq \|f\|\|x\|.$$

Likewise, taking the $-F$ for $-f$, we get

$$\|F\| \leq \|f\|.$$

The converse inequality is evident. ■

Corollary. Let $Y \leq X$, $x_0 \in X \setminus Y$. Then exists $F \in X^*$ such that $\|F\| \leq 1$, $F|_Y = 0$, and $F\left(x_0\right) = \operatorname{dist}\left(x_0, Y\right)$.

Proof. Define $p(x) = \operatorname{dist}(x, Y)$. Since Y is closed, $d := p(x_0) > 0$. Define

$$f(y + \alpha x_0) := \alpha d.$$

Then $f \leq p$, and exists $F\colon X \to \mathbb{R}$ such that $F|_L = f|_L$ and $F \leq p$. In particular, $F|_Y = 0$ (what we need) and $F(x_0) = d$. Observe that the same applies to $-f$, and so $\|F(x)\| \leq p(x) \leq \|x\|$. Hence $\|F\| \leq 1$. ■

Corollary. Let $x_0 \in X$. Then exists $F \in X^*$ such that $\|F(x_0)\| = \|x_0\|$ and $\|F\| = 1$.

Proof. The case $Y = \{0\}$ of the previous corollary. ■

Banach limits

Definition. Let $c \leq l^\infty$ be the space of convergent subsequences. A map $L \colon l^\infty \to \mathbb{R}$ is a *Banach limit*, iff

1. It is a continuous linear functional.
2. For all $x = \{x_n\} \in c$,
$$Lx = \lim x_n.$$
3. If $x \geq 0$, then $Lx \geq 0$.
4. $L\{x_{n+1}\} = L\{x_n\}$.

Theorem. L exists.

Proof. Put, for $x = \{x_n\}$,

$$f(x) = \lim x_n.$$

$f \colon c \to \mathbb{R}$ is a linear functional. Define a majoring sublinear functional (exercise) as

$$p(x) = \limsup \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Observe that $p|_c = f$, so $f \leq p$. By HB, we have $L \colon l^\infty \to \mathbb{R}$, which, luckily, satisfies the definition of Banach limit:

1. $L|_c = f$.
2. If $x \leq 0$, then $Lx \leq p(x) \leq 0$.
3. Put $x'_i = x_{i+1} - x_i$. By linearity of L, it is sufficient to show that $L\{x'_n\} = 0$. And indeed,
$$p(x') = \lim \frac{1}{n} \sum_{i=1}^{n} (x_{n+1} - x_n) = 0.$$

∎

The complex case

Definition. Let $c \leq l^\infty(\mathbb{C})$ be the space of convergent subsequences. A map $L\colon l^\infty \to \mathbb{C}$ is a *Banach limit*, iff

1. It is a continuous linear functional.
2. For all $x = \{x_n\} \in c$,
$$Lx = \lim x_n.$$
3. If $x \in \mathbb{R}$ and $x \geq 0$, then $Lx \geq 0$.
4. $L\{x_{n+1}\} = L\{x_n\}$.
5. $\|L\| = 1$.

Proof. Let L be a real Banach limit. For $a, b \in l^\infty(\mathbb{C})$, define

$$L(a + ib) := La + iLb.$$

All of the properties now follow trivially from those of real limits, except the final one.

Simple functions are dense in l^∞, so we may prove the statement for them and be happy after using continuity of L. Let $x = \sum \alpha_k \chi_{E_k}$ for some partition $\bigsqcup_{k\in\mathbb{N}} E_k = \mathbb{N}$, and $|\alpha_\square| \leq 1$. Then $Lx = \sum \alpha_k L\chi_{E_k}$, and

$$\begin{aligned} |Lx| &\leq \sum_{k\in\mathbb{N}} L\Big(\chi_{E_k}\Big) \\ &= L\Big(\chi_{\sqcup E_k}\Big) \\ &\leq 1, \end{aligned}$$

which was asserted. ∎