Functional analysis

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Introduction

Let X be a vector space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$.

Definition. A map $f : X \to \mathbb{R}$ is called a *norm*, iff we have

1. If $x \in X \setminus 0$, then $f(x) > 0$; and $f(0) = 0$.
2. $f(kx) = |k| f(x)$ for all $k \in \mathbb{R}$, $x \in X$.
3. $f(x+y) \le f(x) + f(y)$ for all $x, y \in X$.

Definition. A pair $(V, \|\square\|)$, where $\|\square\|$ is a norm on a vector space V, is called a *normed space*. A normed space is *Banach*, iff it is complete.

Completions

Definition. $\widehat{X}$ is called a *completion* of X, iff $\widehat{X}$ is complete and X is dense in $\widehat{X}$.

Theorem. A completion $\widehat{X}$ exists.

Proof. Call two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ equivalent, iff $\|x_n - y_n\| \to 0$, and let $\widehat{X}$ be the resulting quotient. Since the point-wise sum of two Cauchy sequences is Cauchy, in this natural way we may introduce vector space structure on $\widehat{X}$. The norm on $\widehat{X}$ is introduced as

$$\|[x_n]\| := \lim_{n \to \infty} \|x_n\|.$$

This map is defined correctly:

1. The limit on the right always exists: since $\{x_n\}$ is Cauchy, the sequence $\{\|x_n\|\}$ of reals is Cauchy, which implies it must converge.

2. If $[x_n] = [y_n]$, then $\left| \|x_n\| - \|y_n\| \right| \leq \|x_n - y_n\| \xrightarrow[n\to\infty]{} 0$. Therefore, $\|[x_n]\| = \|[y_n]\|$.

X is embedded into $\widehat{X}$ by mapping $x \in X$ into the class of the constant sequence at x. It is easy to see that this map preserves norms. ■

Theorem. Let $\widehat{X}_2$ be another completion of X. Then exists a bijection $f : \widehat{X} \to \widehat{X}_2$ which is linear, preserves norms, and maps the embedded X into the embedded X.

These two theorems endow us with the right to never consider pathological incomplete spaces.

Proof. Map $[x_n] \in \widehat{X}$ into $\lim x_n \in \widehat{X}_2$. ■

Exercise 1. The space $C[a,b]$ with the norm $\|f\|_2 = \int |f|$ is not complete. Define

$$f_n(x) = \begin{cases} 0, & x \in [a,c], \\ n(x-c), & x \in [c, c+1/n], \\ 1, & x \in [c+1/n, b]. \end{cases}$$

Then f_n is a Cauchy sequence which does not have a limit in $C[a,b]$.

Proof. It is easy to see that the limit of $\{f_n\}$ is $[x \geq c]|_{[a,b]}$, so it does not have a continuous limit. Nevertheless, it is Cauchy, since

$$\|f_n - f_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \xrightarrow[\min\{m,n\}\to\infty]{} 0.$$

■

Equivalent norms

Definition. We say norms are *equivalent*, iff the metrics they generate are Lipschitz equivalent.

Exercise 2. Norms are equivalent iff they generate the same topology.

Exercise 3. In infinite-dimensional spaces, there are norms which are not equivalent.

Proof. For example, consider $X = C[0,1]$, L^1-norm and the sup-norm on it. It is true that L^1-norm does not surpass the sup-norm, but there is no constant for the opposite inequality: we can think of a function with an arbitrarily large sup-norm, but constant integral. ∎

Theorem. If X is finite-dimensional, then every two norms on X are equivalent.

Proof. Suppose $\dim X = n$, and $e_1, \ldots, e_n$ is a basis. Let $x = a_1e_1 + \cdots + a_ne_n$. Let $\|\square\|$ be a norm on X. Define a new *norm* as

$$|x| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Then

$$\begin{aligned}\|x\| &\le \sum_{i=1}^{n} |a_i|\|e_i\| \\ &\le M\sum_{i=1}^{n}|a_i| \\ &\le M\left(\sum_{i=1}^{n}|a_i|\right)^{\frac{1}{2}} \\ &= M|x|.\end{aligned}$$

The function $x \mapsto \|x\|$ is continuous in the norm $|\square|$. Let $\left|x_k - x\right| \to 0$. Then

$$\begin{aligned}\left|\|x_k\| - \|x\|\right| &\le \left\|x_k - x\right\| \\ &\le M\sqrt{n}\left|x_k - x\right| \\ &\to 0.\end{aligned}$$

Consider the set $S = \left\{x \in E \mid |x| = 1\right\}$. S is compact in $|\square|$. $\varphi|_S$ is continuous and nonzero. Then $\varphi > \delta$ for some $\delta > 0$, so

$$\left\|\frac{x}{|x|}\right\| \ge \delta \iff \|x\| \ge \delta|x|.$$

■

Corollary. Every finite-dimensional normed vector space is complete.

Proof. Every Euclidean space is complete. ■

Corollary. A finite-dimensional subspace of a normed space is closed.

Proof. It is complete, and every convergent sequence is Cauchy sequence. ■

Definition. The set $M \subseteq X$ is *bounded*, iff

$$\sup_{m \in M} \|m\| < +\infty.$$

Lemma (on an almost-perpendicular)**.** Let E be a normed space, and $F < E$ its closed proper subspace. Then for every $\epsilon > 0$ exists a vector $x \in E$ such that $\|x\| = 1$ and $\operatorname{dist}(x, F) > 1 - \epsilon$.

Proof. Let $y \in E \setminus F$. Then $d = \operatorname{dist}(y, F) > 0$, since F is closed. Let $\delta > 0$. By definition of infimum, there exists $a \in F$ such that

$$d \leq \left\|y - a\right\| \leq d + \delta.$$

Put $y_2 = y - a$. Since $a \in F$, $\operatorname{dist}(y_2, F) = \operatorname{dist}(y, F) = d$. Define

$$x = \frac{y_2}{d + \delta} = \frac{y - a}{d + \delta}.$$

Then $\|x\| \leq 1$, but

$$\operatorname{dist}(x, F) = \operatorname{dist}\left(\frac{y}{d + \delta}, F\right) \geq \frac{\operatorname{dist}(y, F)}{d + \delta} = \frac{d}{d + \delta}.$$

Since the δ is arbitrary, and by increasing the norm of x we do not get closer to F, we get the desired. ■

Theorem. Let X be a normed space. Equivalent are:

1. X is finite-dimensional.
2. Every bounded subset of X is relatively compact.

Proof of $1 \implies 2$. From corollary on page 5. ■

Proof of $2 \implies 1$. Suppose X is not finite-dimensional. We assert that there exists a bounded sequence that does not have a convergent subsequence (so in no way the closure of a bounded subset that contains this sequence can be compact). We show this by induction: suppose $x_1, \ldots, x_n$ are already built. By the almost-perpendicular lemma there exists x_{n+1} such that $\|x_{n+1}\| = 1$ and

$$\operatorname{dist}\big(x_{n+1}, \operatorname{span}\{x_1, \ldots, x_n\}\big) > 1/2$$

(since X is not finite-dimensional, the span here is a proper subspace of X). Continuing to infinity, we get a sequence $\{x_n\}$. It is bounded (all its members are on the unit sphere). Nevertheless, no subsequence of it is Cauchy by construction. ■

Linear operators

Definition. A linear operator $T : X \to Y$ is *bounded*, iff $T(B)$ is bounded, where B is the unit ball in X.

Lemma. Let X and Y be normed vector spaces, and $T : X \to Y$ a linear operator. The following are equivalent:

1. T is continuous.
2. T is continuous at 0.
3. T is bounded.

Proof of $1 \Leftrightarrow 2$. Let $x \in X$. T is continuous at x iff for every convergent $x_n \to x$ the sequence Tx_n

also converges (to Tx). Now observe that

$$\|Tx_n - Tx\| = \left\|T(x_n - x)\right\|.$$

■

Proof of 2 ⇒ 3. Consider

$$D = \left\{y \in Y \mid \left\|y\right\| \le 1\right\}.$$

There is $\delta > 0$ such that, if $\|x\| \le \delta$, then $Tx \in D$. Let $z \in D$. Since

$$\left\|T\delta z\right\| \le 1,$$

we have

$$\|Tz\| \le 1/\delta.$$

■

Proof of 3 ⇒ 1. Let $\|x\| \le 1$. Then $\|Tx\| \le M$, so $\|x\| < \epsilon$ implies $\|Tx\| \le M\epsilon$. ■

Operator norm

Definition. Let T is a continuous operator. Its *norm* is

$$\|T\| = \sup_{x \ne 0} \frac{\|Tx\|}{\|x\|}.$$

Exercise 4. $\|T\|$ is indeed a norm.

Lemma. An operator is bounded iff it has finite norm.

Proof. Obvious. ■

Lemma. A linear combination of continuous operators is continuous.

Proof.

$$\left\|(kA + B)\right\| \le \left|k\right| \|A\| + \|B\|.$$

∎

The space of bounded operators is complete

Definition. $\mathcal{B}(X, Y)$ is the set of bounded linear operators $X \to Y$ with the obvious structure of a vector space and the standard operator norm as the norm.

Theorem. Let Y be complete. Then $\mathcal{B}(X, Y)$ is complete.

I have seen three quite concise proofs of this theorem and understood neither. Here is a long one, my own.

Proof. Let $\{T_n\}$ be a Cauchy sequence. Fix $x \in X$. $\{T_n x\}$ is a Cauchy sequence in Y:

$$\begin{aligned} \|T_n x - T_m x\| &\le \|T_n - T_m\| \|x\| \\ &\le \epsilon \|x\|. \end{aligned}$$

By completeness of Y, there is a limit $t \xleftarrow[n \to \infty]{} T_n x$.

The map $x \mapsto t$ we have just build is a linear operator. Call it T. T_n is a Cauchy sequence, so $\|T_n\| \le M$ for some M. Then T itself is bounded:

$$\begin{aligned} \|Tx\| &\le \|T_n x\| + \epsilon \\ &\le M\|x\| + \epsilon. \end{aligned}$$

We assert that $\|T_n x - Tx\| \to 0$. Suppose otherwise:

$$\exists \epsilon > 0 \; \forall n_0 \in \mathbb{N} \; \exists n > n_0 \; \exists x \in B \colon \|T_n x - Tx\| > \epsilon.$$

Since $\{T_n\}$ is Cauchy,

$$\forall \delta > 0 \; \exists n_1 \in \mathbb{N} \; \forall k, l > n_1 \; \forall x \in B : \left\|T_k x - T_l x\right\| \leq \delta.$$

Fix this $\delta > 0$ and take the corresponding n_1. From the first line with quantifiers, there exist $n > n_1$ and $x \in B$ such that

$$\|T_n x - T x\| > \epsilon.$$

From the second one, for any $m > n_1$ we get

$$\begin{aligned} \|T_n x - T x\| &\leq \|T_n x - T_m x\| + \|T_m x - T x\| \\ &\leq \delta + \|T_m x - T x\|. \end{aligned}$$

Since $Tx = \lim_{m\to\infty} T_m x$, there is m_0 such that $\|T_m x - T x\| \leq \delta$ for all $m > m_0$. Take $m_1 = \max\{m_0, n_1\}$. Then, for any $m > m_1$,

$$\begin{aligned} \|T_n x - T x\| &\leq \delta + \|T_m x - T x\| \\ &\leq 2\delta. \end{aligned}$$

Now launch $n_1 \to \infty$ and, consequently, $\delta \to 0$. We get a contradiction:

$$\epsilon < \|T_n x - T x\| \leq 2\delta.$$

■

Remark. Our proof does not use the boundedness of operators in the space $\mathcal{B}(X, Y)$.

Functionals

Definition. A *functional* is a linear operator $X \to K$.

Definition. The *dual* X^* of X is the space $\mathcal{B}(X, K)$ of continuous functionals.

Corollary. X^* is complete.

Proof. Since K is complete in either case. ■

Strong convergence

Definition. A sequence of operators $T_n \to T$ converges *strongly* or *point-wise*, iff $T_n x \to T x$ for all $x \in X$.

Lemma. If $\|T_n - T\| \to 0$, then $T_n \to T$ strongly.

Proof. Since

$$|T_n x - T x| \leq \|T_n - T\| |x|.$$

■

Remark. The converse is not true.

Definition. $l^p = L^p(\mathbb{N}, \#)$ is the space of real-valued sequences which converge in the L^p norm (with respect to the cardinality measure).

Proof of the remark. Consider the operators

$$s_k(c_0, c_1, \dots) = (c_k, c_{k+1}, \dots)$$

on l^p. $\left\|s_k\right\| = 1$, since there are sequences with a single unit and other elements zero and applying s_k does not lessen the sequence norm anyway. Nevertheless, $s_k \to 0$ pointwise (strongly), since all the sequences in l^p converge. ■