Phase 2: Two-Qubit Interaction and Trotterization

In Phase 2 we move from a single qubit to a simple two-qubit system so we can start seeing how interactions change the dynamics. The Hamiltonian we use is a basic Ising model with a coupling term and local X terms. This is the smallest example where different parts of the Hamiltonian do not commute, which means we can no longer write the time evolution as one clean exponential like we did in Phase 1. Instead, we have to approximate the evolution by breaking it into smaller pieces and applying them in sequence (Trotterization). The goal here is to understand what this approximation is doing, how interactions produce correlated behavior between qubits, and how the accuracy changes when we vary the number of Trotter steps or introduce noise from real hardware.

Two-Qubit Hilbert Space

In Phase 2 we extend from a single qubit to a system of two interacting qubits. The joint state now lives in a four-dimensional Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2,$$

with the computational basis

$$\{\, |00\rangle, |01\rangle, |10\rangle, |11\rangle \,\}.$$

We use the same Pauli matrices as in Phase 1:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For a two-qubit system, operators acting on a specific qubit are defined using tensor products:

$$\sigma_{z_1} = \sigma_z \otimes I, \qquad \sigma_{z_2} = I \otimes \sigma_z,$$

and similarly for σ_{x_1} and σ_{x_2}.

Two-Qubit Ising Hamiltonian

In this phase we study a simple two-qubit Ising-type Hamiltonian:

$$H = J\,\sigma_{z_1}\sigma_{z_2} + \frac{\Omega}{2}\big(\sigma_{x_1} + \sigma_{x_2}\big)$$

We separate this into two parts:

$$H = H_{\mathrm{ZZ}} + H_{\mathrm{local}},$$

where

$$H_{\mathrm{ZZ}} = J\,\sigma_{z_1}\sigma_{z_2}, \qquad H_{\mathrm{local}} = \frac{\Omega}{2}\big(\sigma_{x_1} + \sigma_{x_2}\big)$$

Coupling term $H_{\mathbf{ZZ}}$

The product $\sigma_{z_1}\sigma_{z_2}$ means

$$\sigma_{z_1}\sigma_{z_2} = (\sigma_z \otimes I)(I \otimes \sigma_z) = \sigma_z \otimes \sigma_z$$

In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ this operator has the matrix

$$\sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

States where the two qubits are aligned ($|00\rangle$ or $|11\rangle$) have one energy, while states where they are anti-aligned ($|01\rangle$ or $|10\rangle$) have the opposite energy. The coupling strength J controls how strongly the two qubits "care" about matching or not.

Local X terms $H_{\mathbf{local}}$

$$H_{\text{local}} = \frac{\Omega}{2}\big(\sigma_{x_1} + \sigma_{x_2}\big),$$

acts independently on each qubit and plays a similar role to the driving term in Phase 1. These terms try to flip each qubit between $|0\rangle$ and $|1\rangle$. The parameter Ω sets the overall strength of this local drive.

Time Evolution and the Need for Trotterization

The time evolution operator for a time-independent Hamiltonian is

$$U(t) = e^{-iHt} = e^{-i(H_{\text{ZZ}}+H_{\text{local}})t}$$

In general the two parts do not commute:

$$[H_{\text{ZZ}}, H_{\text{local}}] \neq 0$$

Because of this, we cannot simply write the evolution as an exact product $e^{-iHt} = e^{-iH_{\text{ZZ}}t}e^{-iH_{\text{local}}t}$.

Instead, we approximate the evolution using the Trotter–Suzuki method. We split the total time t into n small time steps of size

$$\Delta t = \frac{t}{n},$$

and alternate the simpler pieces of the Hamiltonian.

First-order Trotterization.

The first-order formula:

$$e^{-i(H_{\text{ZZ}}+H_{\text{local}})t} \approx \big(e^{-iH_{\text{local}}\Delta t}e^{-iH_{\text{ZZ}}\Delta t}\big)^n$$

As n increases, the Trotter error becomes smaller, but the corresponding quantum circuit becomes deeper (more gates).

Second-order Trotterization.

A common improvement is the second-order Trotter formula:

$$e^{-i(H_{\mathrm{ZZ}}+H_{\mathrm{local}})t} \approx \left(e^{-iH_{\mathrm{local}}\Delta t/2}e^{-iH_{\mathrm{ZZ}}\Delta t}e^{-iH_{\mathrm{local}}\Delta t/2}\right)^n$$

This reduces the approximation error for the same number of time steps, at the cost of a slightly more complicated circuit for each step.

In the notebook for Phase 2, we will:

- Construct the matrix form of H on the two-qubit space.
- Use exact matrix exponentials e^{-iHt} as a reference.
- Build first- and second-order Trotter circuits implementing H_{ZZ} and H_{local}.
- Compare the Trotterized evolution with the exact evolution.