Flag Based Fault Tolerance for 3 Qubit Repetition Code

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I. Introduction

Quantum computation holds the potential to achieve significant computational speedups for certain clas sically intractable problems. However, harnessing this potential on current quantum devices is chal lenging due to the presence of hardware noise. To mitigate errors that arise during quantum computa tions, the concept of quantum fault-tolerant compu tation has emerged as a viable solution [1]. Despite its promise, conventional Fault-Tolerant Quantum Computation (FTQC) schemes impose a consid erable resource burden, necessitating a substantial number of ancillary qubits to store redundancy information for stabilizer checks [2, 3, 4].

Recent advancements have sought to address this challenge by introducing the innovative idea of incorporating flags into the original FTQC schemes [4, 5, 6, 7]. This report focuses on the imple mentation of flag-based fault tolerance, specifically tailored for the *3-qubit repetition code*. Remark ably, this approach requires only two additional qubits while enabling the measurement of stabiliz ers through a flag-based scheme.

Our methodology involves deliberately introduc ing randomized errors using circuits and developing an error-correcting scheme. To ensure the robust ness of our approach, we initiate the testing phase by simulating the model, subsequently transitioning to the implementation on noisy quantum computing hardware. This thorough testing and implementa tion process aims to provide valuable insights into the effectiveness of flag-based fault tolerance for the 3-qubit code in real-world quantum computing scenarios.

II. Bit-flip channel

The Bit-Flip channel serves as a quantum analogue to a classical error channel, introducing errors in the form of bit flips during quantum operations. In the classical realm, this is akin to a scenario where bits randomly transition from 0 to 1 and vice versa. Transposing this notion to the quantum context, where quantum bits or qubits are manipulated, the Bit-Flip map becomes a crucial mathematical repre sentation of potential errors in quantum operations. This map articulates how quantum information may undergo corruption, wherein the quantum state is subject to ”flipping.” Mathematically, this quantum bit flip is expressed through the map:

$$\rho \rightarrow \rho' = p \cdot \rho + (1 - p) \cdot X\rho X,$$

where p represents the probability of error, ρ denotes the initial quantum state, and X symbolizes the quantum operation corresponding to a bit flip as the Pauli operator X acts as a bit flip because $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$, and $(1 - p)$ is the probability of a bit flip per timestep.

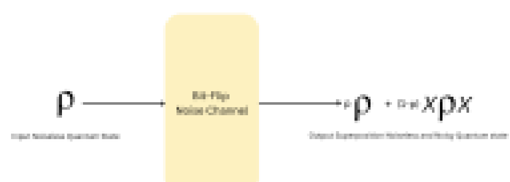

Figure 1. The Bit Flip Noise Channel

In the next section, we will explore the method to adopt for protecting the qubit against bit flip errors.

III. 3 Qubit Repetition Code

In our strategy for encoding against noise in the quantum channel, we leverage the ability to clone orthogonal quantum states. Specifically, we employ a classically-inspired encoding scheme represented by:

$$|0\rangle \mapsto |000\rangle = |\bar{0}\rangle,$$

$$|1\rangle \mapsto |111\rangle = |\bar{1}\rangle.$$

To protect the qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$, we encode it as a 3-qubit codeword

$$|\psi'\rangle = a|000\rangle + b|111\rangle.$$

The resulting encoded qubit state $|\psi\rangle$ becomes entangled across all three registers. This entangle ment proves essential for safeguarding our informa tion against channel noise.

All three qubits are susceptible to identical bit-flip noise, characterized by the error set $E = \{I, X_1, X_2, X_3, X_1X_2, X_2X_3, X_1X_3, X_1X_2X_3\}$, where X_i denotes the Pauli X operator acting on qubit i: $X_1 \equiv XII$, $X_2 \equiv IXI$, $X_3 \equiv IIX$. This error model captures the potential bit-flip errors in the quantum information storage and transmission process. In the classical context,

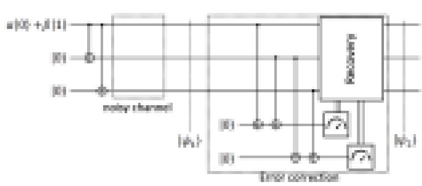

Figure 2. The Bit Flip Error Correcting Circuit

single bit-flip errors are typically corrected by measuring the three bits of the codeword and then taking a majority vote. However, when considering the action of errors from the set $\{I, X_1, X_2, X_3\}$ on the state $|\psi\rangle$, it results in four mutually orthogonal states. Consequently, there exists a quantum measurement capable of discerning which of these four subspaces the state occupies without necessitating a

projection onto a basis state. This is crucial as projecting onto a basis state would destroy the superposition inherent in the quantum state. Once we identify the subspace in which the state resides, it becomes possible to transform the state back to $|\psi\rangle$ by applying one of the unitary operators $\{I, X_1, X_2, X_3\}$. Notably, applying these operators does not require knowledge of the specific state being stored. This insight forms the cornerstone of quantum error correction: in a well-designed Quantum Error Correction Code (QECC), there exists a measurement that reveals the error without disclosing any information about the encoded state.

Bit-flip	$\|\psi_1\rangle$	M_1	M_2	Recovery	$\|\psi_2\rangle$
-	$\alpha\|000\rangle + \beta\|111\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
1	$\alpha\|100\rangle + \beta\|011\rangle$	1	0	$X \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
2	$\alpha\|010\rangle + \beta\|101\rangle$	1	1	$I \otimes X \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
3	$\alpha\|001\rangle + \beta\|110\rangle$	0	1	$I \otimes I \otimes X$	$\alpha\|000\rangle + \beta\|111\rangle$

Figure 3. Error Syndrome and Recovery Scheme

A. What are Stabilizers?

A stabilizer is an operator that commutes with all the elements of the code space, leaving the code space invariant. Mathematically, for a QECC with a code space C, a set of stabilizers $S = \{S_1, S_2, \ldots, S_k\}$ is defined such that for any state $|\psi\rangle$ in the code space: $S_i|\psi\rangle = |\psi\rangle$, for all $i = 1, 2, \ldots, k$. And the *stabilizer generators* are a set of commuting operators that generate the entire stabilizer group for the given quantum code.

Stabilizer codes are equipped with the property that errors in the quantum state induce changes that can be detected by measuring the eigenvalues of the stabilizers.

The outcomes of these measurements, known as the error syndromes, provide information about the errors that occurred.

B. Stabilizer Generators for 3 Qubit Repetition Code

In the context of this specific quantum code, determining the subspace in which the state resides is equivalent to measuring two commuting observ ables, namely $Z_1Z_2 \equiv ZZI$ and $Z_2Z_3 \equiv IZZ$. Measuring Z_1Z_2 or Z_2Z_3 is similar to measuring the parity of qubits 1 and 2 or 2 and 3. Notably, these observables represent joint measurements on two qubits, allowing for measurement without ex plicitly determining the values of Z on the individ ual qubits.

These observables, Z_1Z_2 and Z_2Z_3, are referred to as stabilizer generators of this code and the stabi lizer group of this code is *{III, ZZI, IZZ, ZIZ}*. The outcomes of these two observables serve as the error syndrome, uniquely identifying the error that occurred.

IV. Fault Tolerant Quantum Computation

FTQC encompasses a set of principles that enable the implementation of repeated error correction throughout an extended quantum computation, all while ensuring that the error-correction process itself does not introduce more errors than it rectifies. [8, 9]

A. Flag Based Fault Tolerance

One fundamental aspect of fault-tolerant quantum computing is the ability to perform syndrome mea surements in a fault-tolerant manner. However, im plementing circuits with a large number of ancillas on near-term quantum devices becomes impractical due to the growing noise associated with increased circuit depth [10, 11, 12]. To tackle this challenge, several algorithms have been proposed to reduce the number of ancilla qubits , however the issue remains of exponential order [7]. Following the procedures outlined in [7] , it becomes feasible to achieve fault-tolerant syndrome measurements for certain families of codes using only 2 additional qubits.

V. Flag Based Fault Tolerance for 3 Qubit Repetiton Code

We construct the circuit for the bit-flip code and incorporate two ancillas to measure any er rors that may occur, obtaining the error syndrome. Subsequently, additional ancillas are introduced to ensure the accuracy of the syndrome measurement, serving as our flag qubits. (a) If the flag qubit is measured as $|-\rangle$, then employ the unflagged circuits to extract the two syndromes for rectifying the error propagated from the ancilla. Conclude by applying the corresponding correction.

(b) Alternatively, if the syndrome is −1, indi cating that the syndrome qubit is measured as $|1\rangle$,

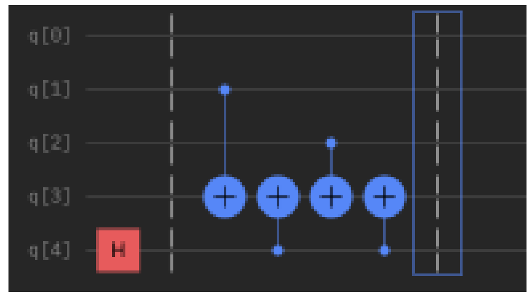

Figure 4. Syndrome Detection using Flag

utilize the unflagged circuits to extract all four syndromes. Complete the process by applying the corresponding correction.

VI. Results and Conclusion

In conclusion, our successful demonstration of flag-based fault tolerance for the 3-qubit repetition code revealed a significant decrease in average fidelity to 2.34% when implemented on IBM's quantum processor. This decline underscores the impact of decoherence, highlighting the influence of noise within these quantum systems.

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