# Statistical Inference for Particle and Astro Physics
# Problems on parameter estimation

The first two problems below are from an a 2012 University of London exam on Statistical Data Analsyis

**PART MARKS**

1. An experiment yields $n$ time values, $t_1,\ldots,t_n$, and a calibration value $y$, all of which are independent. The time measurements are all exponentially distributed with a mean of $\tau+\lambda$ and the calibration measurement, $y$, follows a Gaussian distribution with a mean $\lambda$ and standard deviation $\sigma$. Suppose that $\sigma$ is known and we want to estimate $\tau$ and $\lambda$.

(a) Write down the likelihood function for $\tau$ and $\lambda$, and show that the Maximum Likelihood (ML) estimators for these parameters are

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} t_i - y$$

and

$$\hat{\lambda} = y .$$

**[10]**

(b) Find the variances of $\hat{\tau}$ and $\hat{\lambda}$, and the covariance $\mathrm{cov}[\hat{\tau},\hat{\lambda}]$. Use the fact that the variance of an exponentially distributed variable is equal to the square of its mean. **[10]**

(c) Show using a sketch how a contour of constant log-likelihood can be used to determine the standard deviations of $\hat{\tau}$ and $\hat{\lambda}$.

Explain qualitatively how you would expect the variance of $\hat{\tau}$ to be different if the parameter $\lambda$ were to be known exactly. **[10]**

(d) Show that the (co)variances of $\hat{\tau}$ and $\hat{\lambda}$ obtained from the matrix of second derivatives of the log-likelihood are the same as those found in (c). Use the fact that the inverse of a $2\times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**[10]**

**PART MARKS**

2. A measurement yields two values, $n_{\mathrm{a}}$ and $n_b$, which are independent and Poisson distributed with mean values $\nu_{\mathrm{a}}$ and $\nu_{\mathrm{b}}$, respectively. From these parameters we define the total mean, $\nu$, and asymmetry parameter, $\alpha$, as

$$\nu = \nu_{\mathrm{a}} + \nu_{\mathrm{b}} \ ,$$

$$\alpha = \frac{\nu_{\mathrm{a}} - \nu_{\mathrm{b}}}{\nu_{\mathrm{a}} + \nu_{\mathrm{b}}} \ .$$

Recall that the Poisson distribution is $P(n;\nu) = \nu^n e^{-\nu} / n!$.

(a) Write down the likelihood function in terms of $\nu$ and $\alpha$ and find the Maximum Likelihood (ML) estimators for these parameters. **[10]**

(b) Estimate the variance of $\hat{\alpha}$ as a function of $\nu$ and $\alpha$ using error propagation. **[10]**

(c) Consider the Bayesian approach to inference about $\nu$ and $\alpha$, and suppose we take the prior pdf for the parameters to be

$$\pi(\nu,\alpha) \propto 1/\sqrt{\nu} \ .$$

Find as a proportionality the joint posterior pdf for $\nu$ and $\alpha$ given $n_{\mathrm{a}}$ and $n_b$.

Show that $\nu$ and $\alpha$ are independent, and that the marginal posterior pdfs follow

$$p(\nu \,|\, n_{\mathrm{a}}, n_{\mathrm{b}}) \propto \nu^{(n_{\mathrm{a}} + n_{\mathrm{b}} - 1/2)} e^{-\nu} \ ,$$

$$p(\alpha \,|\, n_{\mathrm{a}}, n_{\mathrm{b}}) \propto (1+\alpha)^{n_{\mathrm{a}}} (1-\alpha)^{n_{\mathrm{b}}} \ .$$

**[14]**

(d) Find the posterior modes for $\nu$ and $\alpha$. Comment on how these relate to the corresponding ML estimators. **[6]**