Statistical Statistical Inference for Particle and Astro Physics
Solutions to problem on exponential fit

**1(a) [4 marks]** The exponentially distributed time measurements, $t_1, \ldots, t_n$, and the Gaussian distributed calibration measurement $y$ are all independent, so the likelihood is simply the product of the corresponding pdfs:

$$L(\tau, \lambda) = \prod_{i=1}^{n} \frac{1}{\tau + \lambda} e^{-t_i/(\tau+\lambda)} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\lambda)^2/2\sigma^2} .$$

The log-likelihood is therefore

$$\ln L(\tau, \lambda) = -n \ln(\tau + \lambda) - \frac{1}{\tau + \lambda} \sum_{i=1}^{n} t_i - \frac{(y-\lambda)^2}{2\sigma^2} + C ,$$

where $C$ represents terms that do not depend on the parameters and therefore can be dropped. Differentiating $\ln L$ with respect to the parameters gives

$$\begin{aligned}
\frac{\partial \ln L}{\partial \tau} &= -\frac{n}{\tau + \lambda} + \frac{\sum_{i=1}^{n} t_i}{(\tau + \lambda)^2} \\
\frac{\partial \ln L}{\partial \lambda} &= -\frac{n}{\tau + \lambda} + \frac{\sum_{i=1}^{n} t_i}{(\tau + \lambda)^2} + \frac{y - \lambda}{\sigma^2} .
\end{aligned}$$

Setting the derivatives to zero and solving for $\tau$ and $\lambda$ gives the ML estimators,

$$\begin{aligned}
\hat{\tau} &= \frac{1}{n} \sum_{i=1}^{n} t_i - y \\
\hat{\lambda} &= y .
\end{aligned}$$

**1(b) [4 marks]** The variances of $\hat{\lambda}$ and $\hat{\tau}$ and their covariance are

$$\begin{aligned}
V[\hat{\lambda}] &= V[y] = \sigma^2 , \\
V[\hat{\tau}] &= V\left[\frac{1}{n} \sum_{i=1}^{n} t_i - y\right] = \frac{1}{n^2} \sum_{i=1}^{n} V[t_i] + V[y] = \frac{(\tau + \lambda)^2}{n} + \sigma^2 \\
\mathrm{cov}[\hat{\tau}, \hat{\lambda}] &= \mathrm{cov}\left[\frac{1}{n} \sum_{i=1}^{n} t_i - y, y\right] = -V[y] = -\sigma^2 ,
\end{aligned}$$

For the covariance we used the fact that $t_i$ and $y$ are independent and thus have zero covariance.

**1(c) [4 marks]** The standard deviations of $\hat{\tau}$ and $\hat{\lambda}$ can be determined from the contour of $\ln L(\tau, \lambda) = \ln L_{\max} - 1/2$, as shown in Fig. 1. The standard can be approximated by the distance from the maximum of $\ln L$ to the tangent line to the contour (in either direction).

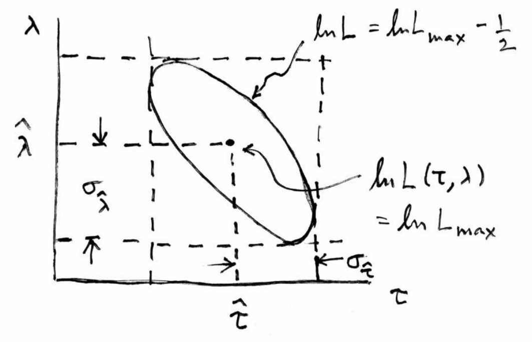


Figure 1: Illustration of the method to find $\sigma_{\hat{\tau}}$ and $\sigma_{\hat{\lambda}}$ from the contour of $\ln L(\tau, \lambda) = \ln L_{\max} - 1/2$ (see text).

If $\lambda$ were to be known exactly, then the standard deviation of $\hat{\tau}$ would be less. This can be seen from Fig. 1, for example, since the distance one need to move $\tau$ away from the maximum of $\ln L$ to get to $\ln L_{\max} - 1/2$ would be less if $\lambda$ were to be fixed at $\hat{\lambda}$.

**1(d) [4 marks]** The second derivatives of $lnL$ are

$$
\begin{aligned}
\frac{\partial^2 \ln L}{\partial \tau^2} &= \frac{n}{(\tau+\lambda)^2} - \frac{2\sum_{i=1}^n t_i}{(\tau+\lambda)^3} \, , \\
\frac{\partial^2 \ln L}{\partial \lambda^2} &= \frac{n}{(\tau+\lambda)^2} - \frac{2\sum_{i=1}^n t_i}{(\tau+\lambda)^3} - \frac{1}{\sigma^2} \, , \\
\frac{\partial^2 \ln L}{\partial \tau \partial \lambda} &= \frac{n}{(\tau+\lambda)^2} - \frac{2\sum_{i=1}^n t_i}{(\tau+\lambda)^3} \, .
\end{aligned}
$$

Using $E[t_i] = \tau + \lambda$ we find the expectation values of the second derivatives,

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln L}{\partial \tau^2}\right] &= \frac{n}{(\tau+\lambda)^2} - \frac{2n(\tau+\lambda)}{(\tau+\lambda)^3} = -\frac{n}{(\tau+\lambda)^2} \, , \\
E\left[\frac{\partial^2 \ln L}{\partial \lambda^2}\right] &= -\frac{n}{(\tau+\lambda)^2} - \frac{1}{\sigma^2} \, , \\
E\left[\frac{\partial^2 \ln L}{\partial \tau \partial \lambda}\right] &= -\frac{n}{(\tau+\lambda)^2} \, .
\end{aligned}
$$

The inverse covariance matrix of the estimators is given by

$$
V_{ij}^{-1} = -E\left[\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right]
$$

where here we can take, e.g., $\theta_1 = \tau$ and $\theta_2 = \lambda$. We are given the formula for the inverse of the corresponding $2 \times 2$ matrix, and by substituting in the ingredients we find

$$
V = \begin{pmatrix} \frac{(\tau+\lambda)^2}{n} + \sigma^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2 \end{pmatrix}
$$

which are the same as what was found in (c).