Derivative Boot Camp Extra Session

Sho Iwamoto

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Preface

Welcome to this extra session! You, who completed the true form of the Derivative Boot Camp, are now ready to take on more challenging tasks for your engineering studies. For those who still have the energy and motivation to go deeper, Sho has prepared this extra session of the Boot Camp.

This session aims to complete your knowledge and skills of derivatives to the university standard. In the previous session, **you have learned** the derivatives of

- x^a
- $\sin x$ and other trigonometric functions,
- $f(x)g(x)$ and $f(x)/g(x)$, and
- $f(g(x))$

In this extra session, **you will learn**

- exponential functions (a^x) and their derivatives
- logarithmic functions ($\log_a x$ and $\ln x$) and their derivatives, and
- hyperbolic functions ($\sinh x$ etc.) and their derivatives.

By the end, you will be able to calculate "simple" derivatives such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+\tan(\ln x)}{\sinh(2x^2+1)}.$$

Tasks are classified to ★ (minimal), *** (basic), ** (intermediate), and * (for motivated students). You don't have to solve all of them, but try to solve as many as you can. Good Luck!

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Visit https://github.com/misho104/LecturePublic for further information, updates, and to report issues.

A′.1 Highschool Review 1: Exponential Functions

We want to consider the functions

$$f(x) = b^x \qquad \text{with} \quad b > 0.$$

They are called exponential functions. Let's begin with a review of highschool mathematics.

***[A′] Let $f(x) = 2^x$ and $g(x) = 0.5^x$. Calculate the following values, but you can use a calculator **only for** (5) and (6).

(1) $f(0), f(1), f(2), f(10)$.

(2) $g(0), g(1), g(2)$.

(3) $f(-1), f(-3), g(-1), g(-2)$.

(4) $f(1/2), f(-1/2), g(1/2), g(-1/2)$.

(5) $f(1.585), g(1.585), f(1.585 \times 2)$.

(6) $f(0.01), f(-0.01), g(0.01), g(-0.01)$.

***[B′] Simplify the following expressions, where x and y are real numbers, $a > 0$, and $b > 0$.

(1) $4^{2.5} \times 4^{3.5}$

(2) $(a^2)^3$

(3) $a^2 \times a^3$

(4) $3^x \times 3^y$

(5) $\dfrac{7^{x+5}}{7^2}$

(6) $1/a^x$

(7) a^x/a^y

(8) $\dfrac{6^a}{2^a}$

(9) $\dfrac{(ab)^3}{b^2}$

(10) $a^5 \times a^5 \times a^5$

(11) $(a^x)^2$

(12) $\dfrac{(a^x)^y}{a^y}$

*[C′] It is worth noticing that the expression b^x is a bit tricky. Check and be convinced/comfortable with the following facts.

- For positive b, the exponential b^x is well-defined for any $x \in \mathbb{R}$, i.e., x can be positive, negative, fractional, or zero.
- If b is zero, $b^x = 0^x$ is well-defined only for $x > 0$. The value is always zero.
- If b is negative, b^x is well-defined only for an integer $x \in \mathbb{Z}$.

Now, choose the well-defined values from the following.

3^{-5}	$(-3)^{-2}$	$(-3)^0$	$0.2^{-0.5}$	$(-0.4)^{\pi}$	0^3	$(-0.1)^0$
$3^{0.2}$	$(-3)^2$	0.2^0	$0.2^{\sqrt{0.2}}$	$\pi^{0.5}$	$0^{\sqrt{2}}$	0^{-1}
$3^{-\sqrt{3}}$	$(-3)^{0.5}$	0.2^5	$(-0.4)^3$	$\pi^{-\pi}$	$0^{-0.5}$	0^0

> **Remark:** After you learn complex analysis, you can define b^x for $b < 0$ and any real number $x \in \mathbb{R}$, which is given by
>
> $$b^x = |b|^x(-1)^x = |b|^x(e^{i\pi})^x = |b|^x e^{i\pi x}.$$
>
> For example, $(-3)^{0.2} \simeq 1.12 + 0.81i$ and $(-\pi)^{-\sqrt{3}} \simeq 0.092 + 0.10i$. Meanwhile, there is no good way to define 0^x for $x \leq 0$. For example, the value of 0^0 is not well-defined. We sometimes use $0^0 \overset{!}{=} 1$ but it is not a global consensus.

A′.2 Highschool Review 2: Logarithmic Functions

Logarithmic functions are defined as follows. Notice the allowed range of a and x.

Definition A′.1: Logarithm and Logarithmic Functions

If a number x satisfies $b^x = A$, then y is called the **logarithm of A to the base b**. We write it by

$$x = \log_b A \qquad (A > 0, \quad b > 0,\ b \neq 1),$$
$$\therefore \quad x = \log_b A \quad \Longleftrightarrow \quad b^x = A. \tag{A′.1}$$

For $b > 0$ and $b \neq 1$, the **logarithmic function** of base b is defined by

$$f(x) = \log_b x$$

with the "domain of definition" $x > 0$.

> **Remark:** If $z = \log_b A$, then $b^z = A$. If we combine them, we get
>
> $$z = \log_b b^z \qquad \text{and} \qquad b^{\log_b A} = A.$$
>
> Students tend to understand this second equation as a complicated theorem, but it isn't. Rather, this is a *definition* of $\log_b A$.

Now you need to recall various equations related to $\log_b A$.

***[D′] Calculate the following values (without using a calculator).

(1)	$\log_{10} 100$	**(4)**	$\log_3 9$	**(7)**	$\log_5 25^{-25}$	**(10)**	$\log_{0.5} 0.5$
(2)	$\log_{10} 0.01$	**(5)**	$\log_2 16$	**(8)**	$\log_5 \sqrt{5}$	**(11)**	$\log_{0.5} 0.25$
(3)	$\log_{10} 0.001$	**(6)**	$\log_3 9^9$	**(9)**	$\log_9 3$	**(12)**	$\log_{1/3} 3$

***[E′] Simplify the following expressions. Here, $a > 0$, $a \neq 1$, and $x > 0$.

(1)	$\dfrac{\log_2 8}{\log_2 4}$	**(3)**	$\log_3 10 - \log_3 5$	**(5)**	$\log_2 x^{10}$
(2)	$\log_3 10 + \log_3 5$	**(4)**	$\log_{10} \sqrt{5} + \log_{10} \sqrt{2}$	**(6)**	$\log_2 \dfrac{1}{x^{10}}$

**[F′] Simplify the following expressions. Here, $a > 0$, $a \neq 1$, and $b > 0$, $b \neq 1$.

(1)	$2\log_{10} 8 + \log_{10} 0.25$	**(3)**	$\log_7 2^{20} + \log_7 3^{10}$	**(5)**	$(\log_a b)(\log_b a)^2$
(2)	$\dfrac{\log_a 8}{\log_a 4}$	**(4)**	$\dfrac{\log_a 5}{\log_b 5}$	**(6)**	$\dfrac{\log_a 5}{\log_{1/a} 5}$

A′.3 Toward the Derivative of Exponential Functions

Now we try to calculate their derivatives, but we don't have do do anything special. In principle, we can calculate any derivatives from the definition,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{A′.2}$$

What you need to do is to apply the substitution $f(x) = 2^x$.

**[G′] (1) Derive the following equations from Eq. (A′.2).

$$\frac{\mathrm{d}}{\mathrm{d}x} 2^x = 2^x \times \lim_{h \to 0} \frac{2^h - 1}{h}, \qquad \frac{\mathrm{d}}{\mathrm{d}x} 3^x = 3^x \times \lim_{h \to 0} \frac{3^h - 1}{h}.$$

(2) Using online resources, evaluate the above limits. The correct answer should be

$$\lim_{h \to 0} \frac{2^h - 1}{h} \simeq 0.7, \qquad \lim_{h \to 0} \frac{3^h - 1}{h} \simeq 1.1,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}x} 2^x \simeq 0.7 \times 2^x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} 3^x \simeq 1.1 \times 3^x.$$

(3) Let's define a function $\lambda(x) = \lim_{h \to 0} \frac{x^h - 1}{h}$. Then, we can write down

$$\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \lambda(2) \times 2^x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} 3^x = \lambda(3) \times 3^x.$$

Express $\frac{\mathrm{d}}{\mathrm{d}x} a^x$ for general $a > 0$ with using λ.

*[H′] We have known $\lambda(2) \simeq 0.7$ and $\lambda(3) \simeq 1.1$ in the previous problem, so there should be a number ξ that satisfies $\lambda(\xi) = 1$. Using online resources, evaluate $\lambda(2.6)$, $\lambda(2.7)$, $\lambda(2.73)$, etc., and try to find the number ξ.

A′.4 Napier's Constant and the Derivative of Exponential Functions

Definition A′.2: Napier's Constant and Natural Logarithm

The number,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = 2.7182818284590452\cdots \tag{A′.3}$$

is called Napier's constant, and is denoted by e.
The logarithmic function with base e is called the **natural logarithm**:

$$\ln x := \log_{\mathrm{e}} x \quad (= \log_{2.71828\cdots} x).$$

This number satisfies $\lambda(\mathrm{e}) = 1$ in the previous page:

$$\lim_{h\to 0}\frac{\mathrm{e}^h - 1}{h} = \lim_{h\to 0}\frac{(2.718\cdots)^h - 1}{h} = 1.$$

Namely, recalling the previous problems, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \lambda(\mathrm{e})\mathrm{e}^x = \mathrm{e}^x$, and now we can prove this theorem:

Theorem A′.3

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a \qquad (a > 0). \tag{A′.4}$$

Note that this is valid even for $a = 1$. Can you explain why? [Hint: What is ln 1?]

**[I′] Prove the following equations, where $a > 0$.

(1) $a = \mathrm{e}^{\ln a}$ **(2)** $a^x = \mathrm{e}^{x\ln a}$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$

★**[J′]** Calculate the following derivatives.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}3\mathrm{e}^x$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}5\mathrm{e}^{2x}$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}x\mathrm{e}^x$ **(7)** $\frac{\mathrm{d}}{\mathrm{d}x}x^2\mathrm{e}^{-x^2}$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}4^x$ **(4)** $\frac{\mathrm{d}}{\mathrm{d}x}4^{x+2}$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}x\mathrm{e}^{-x}$ **(8)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x}{x^2+1}$

**[K′] Calculate the following derivatives.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(\mathrm{e}^x)$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x^2-x}$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\mathrm{e}^x+1}$ **(4)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{\mathrm{e}^x-\mathrm{e}^{-x}}$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}2^{x^2-x}$

A′.5 Derivatives of Logarithmic Functions

This is the last formula for your memory: derivatives of logarithmic functions.

Theorem A′.4

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a} \qquad (a > 0, a \neq 1). \tag{A′.5}$$

The proof is a bit tricky and postponed to the later pages. You first practice the calculation.

***[L′] Calculate the following expression. Use a calculator for (2), (3), and (4).

(1) $\ln \mathrm{e}, \ln\sqrt{\mathrm{e}}, \ln\frac{1}{\mathrm{e}}$.

(2) $\ln 2, \ln 3, \ln 10$.

(3) $\ln 1, \ln 1.001, \ln 1.000001$.

(4) $\ln 0.1, \ln 0.001, \ln 0.000001$.

**[M′] Rewrite the following expressions using natural logarithm ($\ln x$) but not having other logarithms (such as $\log_2 x$). Here, $a > 0$ and $b > 0$.

(1) $\log_2 3$ **(2)** $\log_2 a^b$ **(3)** $\log_7(3\mathrm{e}^2/49)$

★[N′] Calculate the following derivatives.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}2\ln x$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}\log_{10} 99x$ **(9)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln\sin x$ **(13)** $\frac{\mathrm{d}}{\mathrm{d}x}\log_{10}\mathrm{e}^x$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^2$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2+x)$ **(10)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln\tan x$ **(14)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln 10^x$

(3) $\frac{\mathrm{d}}{\mathrm{d}x}\ln 2x$ **(7)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2+1)$ **(11)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{x}$ **(15)** $\frac{\mathrm{d}}{\mathrm{d}x}\log_2\frac{x+1}{x-1}$

(4) $\frac{\mathrm{d}}{\mathrm{d}x}\log_{10} x$ **(8)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2-1)$ **(12)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\mathrm{e}^x+1)$ **(16)** $\frac{\mathrm{d}}{\mathrm{d}x}\log_3\frac{x}{333}$

**[O′] Calculate the following derivatives, where n is a positive integer.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2+\sin x)$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^{-n}\mathrm{e}^x$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x+1)^n(x-1)$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{x^n+1}$ **(4)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{\tan x}{x}$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{(x+1)^n(x-1)}$

A′.6 Hyperbolic Functions

You've learned the derivatives of exponential and logarithmic functions, which is almost everything that you need. Here we introduce hyperbolic functions, but this is just for completeness and you only to know the names and basic properties.

Definition A′.5: Hyperbolic Functions

The hyperbolic functions are defined by

$$\sinh(x) := \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}, \qquad \tanh(x) := \frac{\sinh(x)}{\cosh(x)}. \tag{A′.6}$$

These are similar to the trigonometric functions. For example,

$$\cosh^2 x - \sinh^2 x = 1 \qquad \text{(note the sign!)}$$

***[P′] Prove the following equations. Then, based on the equations, draw graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

(1) $\sinh(-x) = -\sinh x$

(2) $\cosh(-x) = \cosh x$

(3) $\cosh^2 x - \sinh^2 x = 1$

(4) $\cosh 0 = 1$ and $\sinh 0 = 0$

(5) $\lim_{x\to\infty} \cosh x = \lim_{x\to\infty} \sinh = \infty$

(6) $\lim_{x\to\infty} \tanh x = 1$

(7) $\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \cosh x$

(8) $\frac{d}{dx}\cosh x = \sinh x$

(9) $\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$

*[Q′] Calculate the following derivatives, where n is a positive integer.

(1) $\frac{d}{dx}\frac{1}{\sinh x}$

(2) $\frac{d}{dx}\frac{e^x}{\sinh x}$

(3) $\frac{d}{dx}\sinh^n x$

(4) $\frac{d}{dx}\ln(\tanh x)$

(5) $\frac{d}{dx}\sinh x^2$

(6) $\frac{d}{dx}\sinh(\ln x)$

A′.7 Basics of Differential Equations (1)

The last part of this session, before the exercises, is a brief introduction to differential equations. A simple example is

$$f'(x) = 4x.$$

Now you need to find $f(x)$ that satisfies this equation. **One** solution is $f(x) = 2x^2$. However, you will see that there are many more solutions in the following problems.

Another simple differential equation is

$$f'(x) = f(x).$$

It is not easy to find the solution at first glance, but if you have done this Extra Session carefully, you will notice that $f(x) = \mathrm{e}^x$ satisfies this equation. Indeed, $f(x) = \mathrm{e}^x$ is **a** solution.

***[R′]** Verify each of the following statement.

(1) $f(x) = 2x^2$ satisfies the differential equation $f'(x) = 4x$.

(2) $f(x) = 2x^2 + 5$ satisfies $f'(x) = 4x$.

(3) $f(x) = 2(x^2 - 1)$ satisfies $f'(x) = 4x$.

(4) $f(x) = 2x^2 + x$ does not satisfy $f'(x) = 4x$.

(5) For any number C, $f(x) = 2x^2 + C$ satisfies $f'(x) = 4x$.

(6) If $f(x) = ax^3 + bx^2 + cx + d$ satisfies $f'(x) = 4x$, then $a = c = 0$ and $b = 2$, but d can be any number.

(7) $f(x) = \mathrm{e}^x$ satisfies the differential equation $f'(x) = f(x)$.

(8) $f(x) = -\mathrm{e}^x$ and $f(x) = 2\mathrm{e}^x$ both satisfy $f'(x) = f(x)$.

(9) $f(x) = 0$ satisfies $f'(x) = f(x)$.

(10) For any number C, $f(x) = C\mathrm{e}^x$ satisfies $f'(x) = f(x)$.

(11) If $f(x) = ax^3 + bx^2 + cx + d$ satisfies $f'(x) = f(x)$, then $a = b = c = d = 0$.

[S′] Verify each of the following statement.

(1) $f(x) = \sin x$ and $f(x) = -\sin x$ both satisfy $f''(x) = -f(x)$.

(2) $f(x) = \cos x$ and $f(x) = -\cos x$ also satisfy $f''(x) = -f(x)$.

(3) For any number A and B, $f(x) = A\cos x + B\sin x$ satisfies $f''(x) = -f(x)$.

*[T′]** Verify each of the following statement.

(1) $f(x) = \mathrm{e}^x$, $f(x) = -\mathrm{e}^x$, and $f(x) = \mathrm{e}^{-x}$ all satisfy $f''(x) = f(x)$.

(2) For any number A and B, $f(x) = A\mathrm{e}^x + B\mathrm{e}^{-x}$ satisfies $f''(x) = f(x)$.

(3) $f(x) = \sinh x$ and $f(x) = \cosh x$ also satisfy $f''(x) = f(x)$.

(4) We can express $\sinh x$ in the form of $A\mathrm{e}^x + B\mathrm{e}^{-x}$ if we choose A and B. We can do the same for $\cosh x$.

A′.8 Basics of Differential Equations (2)

If a differential equation only contains $f'(x)$ [in addition to $f(x)$ and x], it is called **first-order**. Similarly, if it contains $f''(x)$ [in addition to $f'(x)$, $f(x)$, and x], it is called **second-order**.

At this level, it is sufficient for you to know the following facts.

Theorem A′.6

Let k be a real constant. Then,

- the **general solution** of $f'(x) = kx$ is given by $f(x) = \dfrac{k}{2}x^2 + C$,
- the general solution of $f'(x) = kf(x)$ is given by $f(x) = Ce^{kx}$,

where C is any number. (Notice that they are first-order differential equations.)

Theorem A′.7

Let k be a **positive** constant. Then,

- the general solution of $f''(x) = +kf(x)$ is given by $f(x) = C_1 e^{+\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$.
- the general solution of $f''(x) = -kf(x)$ is given by $f(x) = C_1 \cos(\sqrt{k}x) + C_2 \sin(\sqrt{k}x)$,
- the general solution of $f''(x) = k$ is given by $f(x) = \dfrac{k}{2}x^2 + C_1 x + C_2$,

where C_1 and C_2 are any numbers. (Notice that they are second-order differential equations.)

In most of simple cases, the **general solution** of a first-order differential equation has one unknown constant C, and the general solution of second-order differential equation has two unknown constant C_1 and C_2. (As you see in the following problems, they may be determined by other conditions.)

**[U′] Write down the general solution for the following differential equations.

(1) $f'(x) = x$	**(4)** $f''(x) = -1$	**(7)** $f''(x) = 4f(x)$
(2) $f''(x) = 2$	**(5)** $f'(x) = 4f(x)$	**(8)** $f''(x) = -4f(x)$
(3) $f'(x) = -3x$	**(6)** $f'(x) = -4f(x)$	**(9)** $f''(x) = 0$

**[V′] For each problem, find the function that satisfies all the given equations.

(1) $f'(x) = x$, $f(0) = 0$.	**(5)** $f''(x) = 4$, $f(1) = 2$, $f'(1) = 3$.
(2) $f'(x) = -3x$, $f(0) = 10$.	**(6)** $f''(x) = 4f(x)$, $f(0) = 2$, $f'(0) = 4$.
(3) $f'(x) = -4f(x)$, $f(0) = 4$.	**(7)** $f''(x) = -4f(x)$, $f(0) = 2$, $f'(0) = 4$.
(4) $f'(x) = -4f(x)$, $f(0) = 0$.	**(8)** $f''(x) = -9f(x)$, $f(0) = 0$, $f'(0) = 0$.

A′.9 Workout 3: More Practice

Now, again, it's time to do exercise! In fact, for physics, the above discussions are not so important. It's OK if you can calculate the following, and not-OK if you can't.

*****[W′]** (basic to intermediate-level problems)

(1) $\dfrac{d}{dx}\sin 2x$	**(6)** $\dfrac{d}{dx}\cos^2 x$	**(11)** $\dfrac{d}{dx}e^x \sin x$	**(16)** $\dfrac{d}{dx}\dfrac{\ln x}{x}$
(2) $\dfrac{d}{dx}e^x$	**(7)** $\dfrac{d}{dx}\dfrac{1}{\sin^4 x}$	**(12)** $\dfrac{d}{dx}e^x \ln x$	**(17)** $\dfrac{d}{dx}\sin x\cos x$
(3) $\dfrac{d}{dx}\ln x$	**(8)** $\dfrac{d}{dx}\sqrt{x^2+2}$	**(13)** $\dfrac{d}{dx}\dfrac{\sin x}{x}$	**(18)** $\dfrac{d}{dx}e^{\sin x}$
(4) $\dfrac{d}{dx}\dfrac{1}{x}$	**(9)** $\dfrac{d}{dx}\sin(x+1)$	**(14)** $\dfrac{d}{dx}\dfrac{\sin(x^2)}{x}$	**(19)** $\dfrac{d}{dx}e^{\ln x}$
(5) $\dfrac{d}{dx}\dfrac{1}{\tan x}$	**(10)** $\dfrac{d}{dx}\ln(x-1)$	**(15)** $\dfrac{d}{dx}\dfrac{e^x}{x}$	**(20)** $\dfrac{d}{dx}3^{x^2+x}$

****[X′]** (intermediate to advanced-level problems)

(1) $\dfrac{d}{dx}\dfrac{x^3+1}{\sqrt{x-1}}$	**(8)** $\dfrac{d}{dx}e^{x^2\sinh x}$	**(15)** $\dfrac{d}{dx}\dfrac{\sin(x^2+1)}{\sin(x-1)}$
(2) $\dfrac{d}{dx}\dfrac{1}{\sqrt{\ln x}}$	**(9)** $\dfrac{d}{dx}\cosh(x^2\ln x)$	**(16)** $\dfrac{d}{dx}\dfrac{1}{\sqrt{x^2+4}}$
(3) $\dfrac{d}{dx}\sin x^3\ln(x^3)$	**(10)** $\dfrac{d}{dx}\sin(x\sinh x)$	**(17)** $\dfrac{d}{dx}x^{-1/2}\cos x^2$
(4) $\dfrac{d}{dx}(x^4+1)^{-3/5}$	**(11)** $\dfrac{d}{dx}e^{x^3}\sinh x^3$	**(18)** $\dfrac{d}{dx}\dfrac{2\sin x}{(x-1)^{3/4}}$
(5) $\dfrac{d}{dx}x(x-1)^{-3/4}$	**(12)** $\dfrac{d}{dx}(5x^2-2x+1)^{-3}$	**(19)** $\dfrac{d}{dx}\dfrac{x^3\tan x}{\cos x}$
(6) $\dfrac{d}{dx}x^{-2}e^{x^2}\sin x^2$	**(13)** $\dfrac{d}{dx}\sin^2[(x^2+2x)^2]$	**(20)** $\dfrac{d}{dx}\dfrac{x\sin x}{\cos 2x}$
(7) $\dfrac{d}{dx}e^{\sqrt{x+1}}$	**(14)** $\dfrac{d}{dx}\tan\left(x+\sqrt{2x}\right)$	**(21)** $\dfrac{d}{dx}\tan^2\left(x\sqrt{x}\right)$

A′.10 Advanced Practice

If you have not satisfied with the above problems, you can try the following ones. First, you need to understand the following procedure.

- Consider $\sin\sqrt{x^2+1}$ and we let $u = x^2+1$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+1} = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{u} = \frac{\mathrm{d}\sqrt{u}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{u}}\cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

  Therefore, with letting $v = \sqrt{x^2+1}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\sqrt{x^2+1} = \frac{\mathrm{d}\sin v}{\mathrm{d}x} = \frac{\mathrm{d}\sin v}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x} = \cos v\cdot\frac{x}{\sqrt{x^2+1}} = \frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}.$$

- You can calculate it at once, where we let $u(x) = x^2+1$ and $v = v(u) = \sqrt{u}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\sqrt{x^2+1} = \frac{\mathrm{d}\sin v}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = (\cos v)\frac{1}{2\sqrt{u}}(2x) = \frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}.$$

Now you are ready to calculate any simple functions!

*[Y′] (The final problems: Vale Tudo)

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sinh x}{x^3\ln x}$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sin\left(x+\sqrt{x}\right)\right]^2$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\sqrt{\cos(2x^2+1)}}$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{3x+1}{\sin(3x+1)}$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^2(x^2-1)$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x+\tan x)^{4/3}$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\sin x}{3x^2+2x}\right)^4$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x-\sqrt{x^2+1}\right)^2$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan\sqrt{x+x^{-1}}$

(10) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^5(x^2+x)$

(11) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^2\sin x\tanh 2x$

(12) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(\sin(\cos x))$

(13) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(\sinh(\cos x))$

(14) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2+\tan x}{\sin(2x+1)}$

(15) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2+\tan(\ln x)}{\sinh(2x^2+1)}$

A′.11 Appendix: Derivative of Logarithmic Functions

As an appendix for math-interested students, a path to the proof of the derivative of logarithmic functions is provided.

> **Remark:** In this Boot Camp, we simply accept the fact
>
> $$\mathrm{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad \text{satisfies} \qquad \lim_{h\to 0}\frac{\mathrm{e}^h-1}{h}=1$$
>
> without proof. The proof is a bit involved and will be introduced in calculus lectures.

*[Z′] Let's derive the derivative of logarithms, Eq. (A′.5). Notice that $x > 0$ throughout this problem.

(1) Write down the definition of the derivative of $\ln x$.

(2) Replace h in the equation by $h = x(\mathrm{e}^z - 1)$ and show $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln x = \dfrac{1}{x}\lim_{z\to 0}\dfrac{z}{\mathrm{e}^z-1}$.

(3) Show the two equations in Eq. (A′.5). In the derivation of the latter, you may need to use the formula $\log_a b = (\log_c b)/(\log_c a)$, which is valid for $a > 0$, $b > 0$, $c > 0$, $a \neq 1$, $c \neq 1$.