Vector Boot Camp

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Preface

Welcome to your first year of university! As a university student in engineering, **you must be able to calculate derivatives of "simple" functions** such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+\tan x}{\sin(2x+1)}.$$

This Boot Camp is designed to help you prepare for your first year, which is unexpectedly tough for most of you! Take your time, go through each problem carefully, and don't hesitate to ask for help!

To motivate you, we will have a **mini test** at the beginning of the second lecture; the problems will be from this Boot Camp. I hope this preparation will make your university life easier, more enjoyable, and more satisfactory. Good luck!

Background While Sho heard that vectors are covered in 高中, it seems that most of you have not practiced enough and do not have experience enough to proceed to Chapter 7–8. However, we do not have time to do it in the lecture hours (it will consume 1.5 week). So, Sho has to ask you to do this exercise **before Nov. 1**. This is **critical** for you, so a large point (4.0 points) are given as a reward.

To motivate you, we will have a **mini test** at the beginning of the second lecture; the problems will be from this Boot Camp. I hope this preparation will make your university life easier, more enjoyable, and more satisfactory. Good luck!

Remarks

Sho never provides you with solutions. **You students** need to make the solution. To this end,

- Use online resources such as Wolfram Alpha[*1].
- Share your answers to other colleagues, using LINE or Google Docs[*2]. Compare your answers with theirs.
- Ask questions to colleagues, to the TA, or to Sho. You can utilize Sho's office hours[*3].

[*1] https://www.wolframalpha.com/
[*2] https://docs.google.com/
[*3] https://www2.nsysu.edu.tw/iwamoto/

B.1 Definition of Vectors

Definition B.1: Vectors (for physics)

- A vector is a physical quantity that has **both magnitude and direction**.
- We describe them by $\vec{a}$, $\vec{A}$, $\vec{p}_{\mathrm{F}}$, etc.[♯1], and their magnitudes by $|\vec{a}|$, $|\vec{A}|$, $|\vec{p}_{\mathrm{F}}|$, etc.
- We often describe them by arrows. The arrows' direction should match the vector's direction. The arrows' length should be *proportional to* the vector's magnitude.

♯1: Professionals use boldface, e.g., $\boldsymbol{a}$, $\boldsymbol{A}$, $\boldsymbol{p}_{\mathrm{F}}$, etc., but in this lecture we only use the beginners' style.

Definition B.2: Vector quantity and Scalar quantity

- Physical quantities with direction are called **vector quantity**.
  - They can be described by vectors.[♯2]
- Physical quantities without direction are called **scalar quantity**.
  - They can be described by numbers.
- Consider a vector quantity $\vec{A}$. Its magnitude $|\vec{A}|$ is a scalar quantity.

♯2: If the space considered is one-dimensional, we can describe the direction by $+$ or $-$ and a vector quantity can be described by a number. Here, however, **you** must specify which direction is positive.

For example, **mass** m and **temperature** T are scalar quantities. **Velocity** $\vec{v}$ is a vector quantity and its magnitude $|\vec{v}|$ (it has a special name "**speed**") is a scalar quantity. Acceleration $\vec{a}$ is a vector quantity.

★**[A]** Choose vector quantities. Choose scalar quantities.

temperature	Sho's height	time difference	resistor
air pressure	Sho's weight	time	resistance
wind speed	Sho's mass	duration	conductor
velocity	city name	distance	conductance
electric charge	position	kilometer	conductivity

B.2 How to describe directions

> **Remark:** Using natural English is nice, but it is more important, and thus we should pay greater attention, to ensure that the expression is clear and does not cause any misunderstanding.

Basic *We* need to learn how to describe directions of vectors **in English**. The easiest one is

- leftward = toward the left = to the left
- rightward = toward the right = to the right
- upward = toward the top = to the top
- downward = toward the bottom = to the bottom

If you need to describe the third dimension which is perpendicular to the textbook's page or the sheet (or the blackboard), you can use

- into the (sheet | page | blackboard) = toward the (sheet | page | blackboard)
- out of the (sheet | page | blackboard) = away from the (sheet | page | blackboard)

For 2d case, **once you specify the "north" direction**, you can use the following expressions:

- northward, southward, eastward, westward (= (to | toward) the north, etc.)
- northwestward, southwestward, etc. (= (to | toward) the northwest, etc.)

With axes Usually, we define x-axis and y-axis (and z-axis, if 3d). **If such axes are defined** (or once you have defined them), we can use

- in the positive x-direction = in the $+x$ direction
- in the positive y-direction = in the $+y$ direction
- in the negative x-direction = in the $-x$ direction
- in the negative y-direction = in the $-y$ direction

In particular, for 2d cases, we can use the angle from the positive x-axis (counterclockwise):

- $30°$ from the $+x$ axis (= $30°$ counterclockwise from the positive x-axis)
- $90°$ from the $+x$ axis (= in the $+y$ direction)
- $170°$ from the $+x$ axis (= $10°$ clockwise from the negative x-axis)
- $-15°$ from the $+x$ axis (= $15°$ clockwise from the positive x-axis)
- angle θ from the $+x$ axis, where $\tan\theta = 0.1$ and $0 < \theta < \pi/2$.

Sometimes the word "the horizontal" is useful:

- $10°$ above the horizontal / $30°$ below the horizontal

> **Remark:** It is difficult to describe general 3d directions by words. For this purpose, we usually use mathematical expressions (i.e., vectors).

Rotations To describe the direction of rotation, we use

- clockwise / counterclockwise

If you have an arrow and discuss rotations about the arrow, you can use the following expressions:

- following the right-hand rule (= counterclockwise when viewed from the arrow's direction)
- following the left-hand rule (= clockwise when viewed from the arrow's direction)

We will often use these expressions in General Physics 2.

B.3 Vectors: Magnitude and Direction

In this Boot Camp, we forget about physics[♯3]. Focusing on mathematics, we will discuss vectors in its mathematical aspects. We will begin with a few more expressions useful for vectors.

If the angle between $\vec{a}$ and $\vec{b}$ is 90°, we say

- $\vec{a}$ is **perpendicular to** $\vec{b}$.
- $\vec{a}$ is orthogonal to $\vec{b}$.
- $\vec{a}$ is normal to $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are perpendicular to each other.
- $\vec{a}$ and $\vec{b}$ are orthogonal to each other.
- $\vec{a}$ and $\vec{b}$ are normal to each other.

(all of them are correct but the first one is the most common). If the angle is 0°, we say[♯4]

- $\vec{a}$ is **in the same direction as** $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are in the same direction.

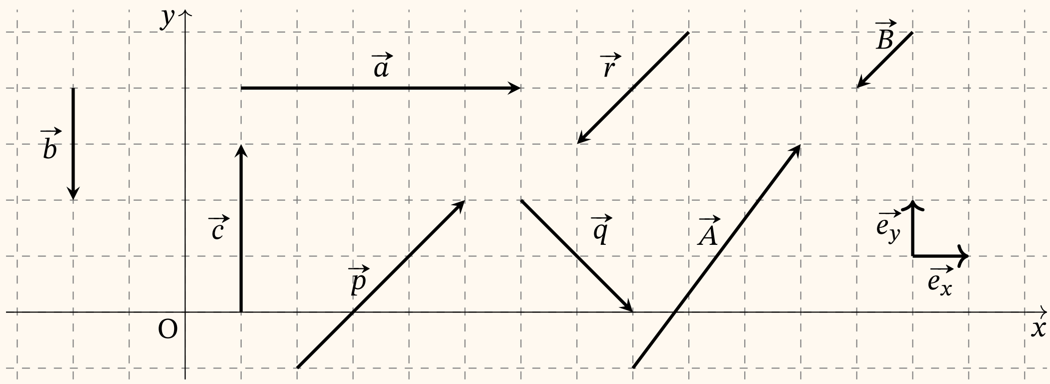
Finally, if the angle is 180°,

- $\vec{a}$ is **anti-parallel to** $\vec{b}$.
- $\vec{a}$ is opposite to $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are anti-parallel.
- $\vec{a}$ and $\vec{b}$ are in opposite directions.

♯3: We forget units and significant figures, which you will learn in the beginning of General Physics 1 lectures. In other words, in the lectures, **you must not forget units and significant figures** of vector quantities.

♯4: Avoid the word "parallel" because some people think it includes both 0° and 180° cases. Clarity is important in science.

★[B] Vectors are drawn on the grid, which has a spacing of 1. Answer the following questions.



(1) Describe the direction (in English words) and magnitude of each vector. Try to use multiple expressions.

(2) Describe relationship between the directions of the following vector pairs: ($\vec{a}$ and $\vec{b}$), ($\vec{b}$ and $\vec{c}$), ($\vec{p}$ and $\vec{q}$), ($\vec{p}$ and $\vec{r}$), ($\vec{e}_x$ and $\vec{e}_y$), and ($\vec{a}$ and $\vec{e}_x$). For example, "$\vec{a}$ is perpendicular to $\vec{b}$".

B.4 Vector Basics

These are a few basic things that you have learned in highschool:

- **(Zero vector)** There is a special vector $\vec{0}$. Its length is 0 (zero) and it has no direction.
- **(Scalar multiplication)** If $\vec{v}$ is a vector, $2\vec{v}$ and $-3\vec{v}$ are both vectors.
- **(Addition)** If $\vec{p}$ and $\vec{q}$ are vectors, $\vec{p}+\vec{q}$ is a vector.

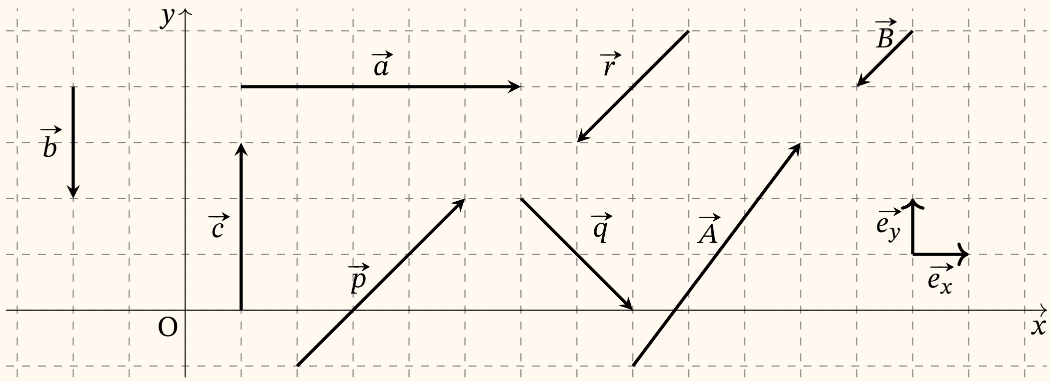
By extending the last two items, we can obtain the following statement.

- **(Linear combination)** If a and b are scalars (i.e., real numbers) and $\vec{p}$ and $\vec{q}$ are vectors,

  $$a\vec{p}+b\vec{q}$$

  is a vector. Here, a and b can be positive, zero, or negative.

★**[C]** Vectors are drawn on the grid, which has a spacing of 1.



(1) Describe $\vec{a}$ by using $\vec{e}_x$ (and a number). Describe $\vec{b}$ and $\vec{c}$ by using $\vec{e}_y$.

(2) Describe $\vec{B}$ by using $\vec{r}$. Describe $\vec{p}$ by using $\vec{r}$.

(3) Describe $\vec{A}$ and $\vec{B}$ by using $\vec{e}_x$ and $\vec{e}_y$.

(4) Describe $\vec{e}_x$ and $\vec{e}_y$ by using $\vec{A}$ and $\vec{B}$.
[Hint: See your answer of the previous problem.]

(5) Draw a vector $\vec{\beta}$ that is in the same direction as $\vec{b}$ and $|\vec{\beta}|=1$.

★**[D]** Consider a vector $\vec{s}$ whose length is 3, i.e., $|\vec{s}|=3$.

(1) Calculate the magnitude of $2\vec{s}$, $-3\vec{s}$, and $0\vec{s}$.

(2) Calculate the magnitude of $\dfrac{\vec{s}}{|\vec{s}|}$.

(3) Describe relationship between the directions of the following vector pairs:
($\vec{s}$ and $4\vec{s}$), ($0.1\vec{s}$ and $-4\vec{s}$), and ($\vec{s}$ and $\dfrac{\vec{s}}{|\vec{s}|}$).

(4) Let $\vec{S}=-3\vec{s}$ and k be a real number. Find the direction and magnitude of $\vec{s}+k\vec{S}$.

Here is one more important concept:

- **(Unit vector)** If a vector has a magnitude of one, it is called a **unit vector**.

If $\vec{a}$ is not the zero vector,

- the unit vector in the same direction as $\vec{a}$ is given by $\dfrac{\vec{a}}{|\vec{a}|}$; it is denoted by $\hat{\vec{a}}$ (hat + arrow);
- the unit vector that is anti-parallel to $\vec{a}$ is given by $-\dfrac{\vec{a}}{|\vec{a}|}$ $(= -\hat{\vec{a}})$.

> **Remark:** In the textbook, vectors are written by $\vec{\mathbf{a}}$ and unit vectors are by $\hat{\mathbf{a}}$. (Some of you will be confused by this notation in General Physics 2.) Although the notation is a bit messy, Sho will always use $\vec{a}$ and $\hat{\vec{a}}$.

***[E] Consider $\vec{s}$ and $\vec{t}$, which satisfy $|\vec{s}| = 3$ and $|\vec{t}| = 2$. Here, $k > 0$.

(1) Find the unit vector whose direction is the same as $\vec{s}$.

(2) Find the unit vector whose direction is the same as $-2\vec{t}$.

(3) Find the unit vector whose direction is the same as $4\vec{s} + 3\vec{t}$.

(4) Find the unit vector anti-parallel to $\vec{s}$.

(5) Find the vector which is anti-parallel to $\vec{s}$ and has a magnitude of 12.

(6) Find the vector which is in the same direction as $\vec{t}$ and has a magnitude of k.

We will later come back to more exercise related to unit vectors.

B.5 Inner product

We use this definition, which may be different from what you learned in highschool.

Definition B.3: Inner product

For vectors $\vec{a}$ and $\vec{b}$, the **inner product** of $\vec{a}$ and $\vec{b}$ is defined by

$$\vec{a}\cdot\vec{b} \stackrel{\text{def}}{=} |\vec{a}||\vec{b}|\cos\theta, \qquad \text{where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}.$$

***[F] Prove the following equations **from the above definition**.

(1) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ for any vectors $\vec{a}$ and $\vec{b}$.

(2) $(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b})$ for any vectors $\vec{a}$ and $\vec{b}$ and any number k.

(3) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ for any vector $\vec{a}$.

(4) $\vec{a}\cdot\vec{b} = 0$ if $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, or $\vec{a}$ is perpendicular to $\vec{b}$.

(5) $-|\vec{a}||\vec{b}| \le \vec{a}\cdot\vec{b} \le |\vec{a}||\vec{b}|$ for any vectors $\vec{a}$ and $\vec{b}$.

*[G] Prove $(\vec{a}+\vec{b})\cdot\vec{c} = (\vec{a}\cdot\vec{c}) + (\vec{b}\cdot\vec{c})$ from the above definition. [Hint: Very difficult.]

The next two problems are the most important ones in this Boot Camp.

★[H] Two vectors $\vec{x}$ and $\vec{y}$ satisfy $|\vec{x}| = |\vec{y}| = 1$ and $\vec{x}\cdot\vec{y} = 1/2$.

(1) Calculate the angle between $\vec{x}$ and $\vec{y}$.

(2) Expand $|a\vec{x}+b\vec{y}|^2$ and calculate its value. Calculate $|a\vec{x}+b\vec{y}|$.

Now, assume $\vec{s} = 3\vec{x} + 3\vec{y}$ and $\vec{t} = 2\vec{x} - \vec{y}$. Also, let $\hat{\vec{s}}$ be the unit vector whose direction is the same as $\vec{s}$.

(3) Calculate $|\vec{s}|$, $|\vec{t}|$, and $\vec{s}\cdot\vec{t}$. Find the angle between $\vec{s}$ and $\vec{t}$.

(4) Describe $\hat{\vec{s}}$. [Hint: The answer will contain $\vec{x}$ and $\vec{y}$.]

(5) Assume $\vec{u} = \vec{x} + k\vec{y}$ is perpendicular to $\vec{s}$. Find the value of k.

★[I] Two vectors $\vec{e_x}$ and $\vec{e_y}$ satisfy $|\vec{e_x}| = |\vec{e_y}| = 1$ and $\vec{e_x}\cdot\vec{e_y} = 0$.

(1) Calculate the angle between $\vec{e_x}$ and $\vec{e_y}$.

(2) Expand $|a\vec{e_x}+b\vec{e_y}|^2$ and calculate its value. Calculate $|a\vec{e_x}+b\vec{e_y}|$.

Now, assume $\vec{p} = a\vec{e_x} + b\vec{e_y}$, which is not $\vec{0}$, and $\vec{q} = 3\vec{e_x} + 4\vec{e_y}$.

(3) Calculate $|\vec{p}|$, $\vec{p}\cdot\vec{e_x}$, and $\vec{p}\cdot\vec{e_y}$.

(4) Calculate $|\vec{q}|$ and $\vec{p}\cdot\vec{q}$.

(5) Find the unit vector whose direction is the same as $\vec{p}$.

[J] Three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ satisfy $|\vec{A}| = 2$, $|\vec{B}| = 3$, $\vec{A} \cdot \vec{B} = 3\sqrt{2}$, and $\vec{A} \cdot \vec{C} = -1$.

★(1) Find the angle between $\vec{A}$ and $\vec{B}$.

★(2) Calculate $|\vec{A} + \vec{B}|^2$, $|\vec{A} - \vec{B}|^2$, and $|2\vec{A} + 4\vec{B}|^2$.

★(3) Calculate $(\vec{A} - 2\vec{B}) \cdot (2\vec{A} + \vec{B} + \vec{C}) + 2\vec{B} \cdot \vec{C}$.

★(4) Assume $|\vec{A} + k\vec{B}| = \sqrt{10}$. Find the value of k.

*****(5)** Assume $\vec{B} + c\vec{C}$ is perpendicular to $\vec{A}$. Find the value of c.

****(6)** What do we know about the values of $|\vec{C}|$ and $|\vec{A} - \vec{C}|$?