Derivative Boot Camp Extra Session

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Preface

Welcome to your first year of university! You, who completed the true form of the Derivative Boot Camp, are now ready to take on more challenging tasks in your engineering studies. For those who still have the energy and motivation to go deeper, Sho has prepared this extra session of the Boot Camp.

This session aims to complete your knowledge and skills of derivatives to the university standard. By the end of the exercises, you will be able to calculate "simple" derivatives such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+\tan(\ln x)}{\sinh(2x^2+1)}.$$

Tasks are classified to ★ (**minimal**), *** (basic), ** (intermediate), and * (for motivated students). You don't have to solve all of them, but try to solve as many as you can. Good Luck!

[A′] (a bit tough problems)

(1) $\dfrac{d}{dx}(5x^2 - 2x + 1)^{-3}$

(2) $\dfrac{d}{dx}(x^4 + 1)^{-3/5}$

(3) $\dfrac{d}{dx}\sin^2[(x^2 + 2x)^2]$

(4) $\dfrac{d}{dx}\tan\left(x + \sqrt{2x}\right)$

(5) $\dfrac{d}{dx}\dfrac{\sin(x^2 + 1)}{\sin(x - 1)}$

(6) $\dfrac{d}{dx}\dfrac{1}{\sqrt{x^2 + 4}}$

(7) $\dfrac{d}{dx}x^{-1/2}\cos x^2$

(8) $\dfrac{d}{dx}x^2 \sin x \tan 2x$

(9) $\dfrac{d}{dx}x(x - 1)^{-3/4}$

(10) $\dfrac{d}{dx}\dfrac{2\sin x}{(x - 1)^{3/4}}$

(11) $\dfrac{d}{dx}\dfrac{x^3 \tan x}{\cos x}$

(12) $\dfrac{d}{dx}\dfrac{x \sin x}{\cos 2x}$

(13) $\dfrac{d}{dx}\dfrac{x^2 \sin^2 x}{\cos x^2}$

(14) $\dfrac{d}{dx}\dfrac{x^3 + 1}{\sqrt{x - 1}}$

(15) $\dfrac{d}{dx}\tan^2\left(x\sqrt{x}\right)$

A′.1 Workout 3: Harder Practice for motivated students

If you have not satisfied with the above problems, you can try the following ones. First, you need to understand the following procedure.

- Consider $\sin\sqrt{x^2 + 1}$ and we let $u = x^2 + 1$.

$$\frac{d}{dx}\sqrt{x^2 + 1} = \frac{d}{dx}\sqrt{u} = \frac{d\sqrt{u}}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}.$$

  Therefore, with letting $v = \sqrt{x^2 + 1}$,

$$\frac{d}{dx}\sin\sqrt{x^2 + 1} = \frac{d\sin v}{dx} = \frac{d\sin v}{dv}\frac{dv}{dx} = \cos v \cdot \frac{x}{\sqrt{x^2 + 1}} = \frac{x\cos\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}.$$

- You can calculate it at once, where we let $u(x) = x^2 + 1$ and $v = v(u) = \sqrt{u}$:

$$\frac{d}{dx}\sin\sqrt{x^2 + 1} = \frac{d\sin v}{dv}\frac{dv}{du}\frac{du}{dx} = (\cos v)\frac{1}{2\sqrt{u}}(2x) = \frac{x\cos\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}.$$

Now you are ready to calculate very complicated functions.

[A′] [*] (The final problems)

(1) $\dfrac{d}{dx}\left[\sin\left(x + \sqrt{x}\right)\right]^2$

(2) $\dfrac{d}{dx}\dfrac{1}{\sqrt{\cos(2x^2 + 1)}}$

(3) $\dfrac{d}{dx}\dfrac{3x + 1}{\sin(3x + 1)}$

(4) $\dfrac{d}{dx}\cos^2(x^2 - 1)$

(5) $\dfrac{d}{dx}(x + \tan x)^{4/3}$

(6) $\dfrac{d}{dx}\left(\dfrac{\sin x}{3x^2 + 2x}\right)^4$

(7) $\dfrac{d}{dx}\left(x - \sqrt{x^2 + 1}\right)^2$

(8) $\dfrac{d}{dx}\tan\sqrt{x + x^{-1}}$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^5(x^2 + x)$

(10) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(\sin(\cos x))$

A′.2 Exponential Functions

We want to consider the functions

$$f(x) = a^x \qquad \text{with} \quad a > 0.$$

They are called exponential functions. Let's begin with the basic part, and then try to calculate its derivative. (Recall that any derivatives can be calculated by the definition of the derivative!)

***[A′] Let $f(x) = 2^x$ and $g(x) = 0.5^x$. Calculate the following values. You can use a calculator **only for** (5) and (6).

(1) $f(0), f(1), f(2), f(10)$.

(2) $g(0), g(1), g(2)$.

(3) $f(-1), f(-3), g(-1), g(-2)$.

(4) $f(1/2), f(-1/2), g(1/2), g(-1/2)$.

(5) $f(1.585), g(1.585), f(1.585 \times 2)$.

(6) $f(0.01), f(-0.01), g(0.01), g(-0.01)$.

***[B′] Simplify the following expressions, where x and y are real numbers, $a > 0$, and $b > 0$.

(1) $4^{2.5} \times 4^{3.5}$

(2) $3^x \times 3^y$

(3) $\dfrac{7^{x+5}}{7^2}$

(4) $1/a^x$

(5) a^x/a^y

(6) $\dfrac{(ab)^3}{b^2}$

(7) $a^5 \times a^5 \times a^5$

(8) $(a^x)^2$

(9) $\dfrac{(a^x)^y}{a^y}$

**[C′] Check the following equations based on the definition od differentials.

$$\frac{\mathrm{d}}{\mathrm{d}x}2^x = 2^x \times \lim_{h\to 0}\frac{2^h - 1}{h}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}3^x = 3^x \times \lim_{h\to 0}\frac{3^h - 1}{h}.$$

**[D′] Using online resources, evaluate the above limits. You will find

$$\lim_{h\to 0}\frac{2^h - 1}{h} \simeq 0.7, \qquad \lim_{h\to 0}\frac{3^h - 1}{h} \simeq 1.1.$$

*[E′] The next goal is to find a number ξ that satisfies $\lim_{h\to 0}\dfrac{\xi^h - 1}{h} = 1$. Let us consider a function

$$\lambda(x) = \lim_{h\to 0}\frac{x^h - 1}{h};$$

then we need to find ξ satisfying $f(\xi) = 1$. Now, observing the result of the previous problem, we can guess $2 < a < 3$ because we have found $\lambda(2) \simeq 0.7$ and $\lambda(3) \simeq 1.1$.

Using online resources, evaluate $\lambda(2.7)$, $\lambda(2.71828)$, and $\lambda(2.8)$.

A′.3 Napier's Constant

The number,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = 2.7182818284590452\cdots \tag{A′.1}$$

is called Napier's constant, and is denoted by e. It is the number which satisfies $\lambda(\mathrm{e}) = 1$ in the previous page:

$$\lim_{h\to 0}\frac{\mathrm{e}^h - 1}{h} = \lim_{h\to 0}\frac{(2.718\cdots)^h - 1}{h} = 1. \tag{A′.2}$$

Logarithmic function with base e is called the natural logarithm:

$$\ln x := \log_{\mathrm{e}} x,$$

which we will review in the next page. Here, you need to recall the relation

$$a^{\log_a b} = b, \qquad \text{thus} \quad \mathrm{e}^{\ln x} = x.$$

> **Remark:** Students tend to understand this as a complicated theorem, but it isn't. Rather, this is a *definition* of $\log_a b$.
> Recall that the number x satisfying $a^x = b$ is defined as $x = \log_a b$. Then, if we put it onto the shoulder of a, it should give $a^x = a^{\log_a b} = b$.

Now we can prove this theorem:

Theorem A′.1

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}a^x = \mathrm{e}^x \ln a \qquad (a > 0). \tag{A′.3}$$

Note that this is valid even for $a = 1$. (What is $\ln 1$?)

***[F′] Prove the first part of the above theorem. You should start from the definition of derivatives and find

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \lim_{h\to 0}\frac{\mathrm{e}^h - 1}{h} \times \mathrm{e}^x = \mathrm{e}^x. \tag{A′.4}$$

***[G′] Prove the latter part of the above theorem, using $a^x = \mathrm{e}^{x\ln a}$ and the formula of derivatives for composite functions.

A′.4 Logarithm

Derivatives of logarithmic functions are also important.

Theorem A′.2

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a} \qquad (a > 0, a \neq 1). \tag{A′.5}$$

The proof of the first part, $(\ln x)' = 1/x$, is a bit tricky. We will skip the proof here. If we accept it, we can easily prove the latter part if you recall the formula

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad (a > 0,\ \ b > 0,\ \ c > 0;\quad a \neq 1,\ \ c \neq 1).$$

***[H′] Using $(\ln x)' = 1/x$, prove the latter part of the above theorem.
[Hint: Replace c in the above equation by e.]

**[I′] Calculate the following values. You will need a calculator for, but only for, (5) and (6).

(1) $\log_{10} 100, \log_{10} 0.01, \log_{10} 0.001.$

(2) $\log_2 4, \log_3 27, \log_5 \sqrt{5}, \log_9 3.$

(3) $\ln e, \ln\sqrt{e}, \ln\dfrac{1}{e}.$

(4) $\log_{0.5} 0.5, \log_{0.5} 0.25, \log_{1/3} 3.$

(5) $\ln 1, \ln 1.001, \ln 1.000001$

(6) $\ln 0.1, \ln 0.001, \ln 0.000001$

***[J′] Simplify the following expressions. Here, x and y are real numbers and $a > 0$, $a \neq 1$.

(1) $\dfrac{\log_2 8}{\log_2 4}$

(2) $\log_a 10 + \log_a 5$

(3) $\log_a 10 - \log_a 5$

(4) $\ln x - \ln y$

(5) $\log_a x^{10}$

(6) $\log_a \dfrac{1}{x^{10}}$

**[K′] Simplify the following expressions. Here, $a > 0$, $a \neq 1$, and $b > 0$, $b \neq 1$.

(1) $\dfrac{\log_a 8}{\log_a 4}$

(2) $\dfrac{\log_a 5}{\log_b 5}$

(3) $\ln 2^{20} + \ln 3^{10}$

(4) $\dfrac{\log_a 5}{\log_{1/a} 5}$

(5) $\ln 8 - 2\ln 4 + \ln\dfrac{1}{20}$

(6) $(\log_a b)(\log_b a)^2$

**[L′] Express the following expressions using natural logarithm ($\ln x$). Here, $a > 0$ and $b > 0$.

(1) $\log_2 3$

(2) $\log_2 a^b$

(3) $\log_7(3\mathrm{e}^2/49)$

A′.5 Hints for Motivated Students

These are more mathematical, more advanced, and less important. You may well skip them.

[M′] For $f(x) = a^x$, check the following facts.

(1) If $a > 0$, $f(x)$ is defined for any $x \in \mathbb{R}$. Also, $f(x)$ is always positive and continuous.

(2) If $a < 0$, $f(x)$ is defined only for $x \in \mathbb{Z}$. It is difficult to define, e.g., $(-2)^{0.49}$.

(3) If $a = 0$, $f(x)$ is defined only for $x > 0$. The value is always zero.[♯1]

> **♯1:** After you learn complex analysis, we can define a^x for negative a. Namely,
>
> $$a^x = (-1)^x|a|^x = (\mathrm{e}^{\mathrm{i}\pi})^x|a|^x = |a|^x\mathrm{e}^{\mathrm{i}\pi x},$$
>
> which is not real but complex for $x \notin \mathbb{Z}$. On the other hand, 0^x is well-defined only for $x > 0$. In fact, there is no good way to define 0^0. (We sometimes use $0^0 = 1$, but this is mathematically incorrect.)

[N′] Try to derive $(\ln x)' = 1/x$ from the definition. Notice that $x > 0$.

(1) Prove the following equation: [Hint: $\tilde{h} = h/x$]

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}\lim_{\tilde{h}\to 0}\frac{\ln(1+\tilde{h})}{\tilde{h}}$$

(2) From the equation (A′.1), prove that

$$\mathrm{e} = \lim_{h\to +0}(1+h)^{1/h}.$$

Now, we want to use $\mathrm{e} = \lim_{h\to 0}(1+h)^{1/h}$. This is slightly different from the above expression and, mathematically, an independent proof is required. However, to simplify discussion, we accept this formula.

(3) Using $\mathrm{e} = \lim_{h\to 0}(1+h)^{1/h}$, show that

$$\lim_{h\to 0}\frac{\ln(1+h)}{h} = \ln \mathrm{e}.$$

(4) Combine the above discussion to show $(\ln x)' = 1/x$.