Derivative Boot Camp (true form)

Sho Iwamoto

21-Sept-2024 22:24:00

Preface

Welcome to your first year of university! As a university student in engineering, **you must be able to calculate derivatives of "simple" functions** such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+\tan(\ln x)}{\sinh(2x^2+1)}.$$

This Boot Camp is designed to help you prepare for your first year, which is unexpectedly tough for most of you! Take your time, go through each problem carefully, and don't hesitate to ask for help!

Have you have completed the "Derivative Boot Camp (Basic)"? Congratulations! Yes, unfortunately, that was just the beginning of the Boot Camp; the problems you solved were *the minimal problems*.

This *true form* of the Boot Camp contains more problems for you. They are classified into three categories: *** (basic), ** (intermediate), and * (for motivated students). You don't have to solve all of them, but try to solve as many as you can. Good Luck!

A.1 The first step: High-school review

*[N] Let $f(x) = x^5$ and $g(x) = (x+1)^5$. Calculate $f'(0)$, $f'(1)$, $g'(0)$ and $g'(1)$. It's nice if you can find the value in a clever way, with less calculation.

A.2 Trigonometric functions

***[O] Calculate the following values.

(1) $\sin\dfrac{7\pi}{6}$ **(2)** $\tan\dfrac{8\pi}{3}$ **(3)** $\cos\dfrac{-3\pi}{2}$ **(4)** $\sin\dfrac{-5\pi}{6}$ **(5)** $\cos\dfrac{-8\pi}{3}$ **(6)** $\cos(-4\pi)$ **(7)** $\cos^2\dfrac{\pi}{6}$ **(8)** $\sin^2\dfrac{\pi}{4}$

***[P] Calculate the following values using calculators.

(1) $\sin 1$ **(2)** $\cos\pi^2$ **(3)** $\sin 0.0000123$ **(4)** $\tan 0.0000777$

A.3 Get into the University

Recall Eqs. (A.1)–(A.2) in the basic part. We saw that derivatives are **defined by** Eq. (A.1) and thus., for example,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x. \tag{A.2}$$

*[Q] Calculate the following derivatives, not using formulae but **starting from the definition**. Namely, do the same thing as (A.2) for each function.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{x}$ **(2)** $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{x^2}$ **(3)** $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$

For the last problem, the equation $\sqrt{a}-\sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$ will be useful.

***[R] We know that $f(x)$ satisfies $f''(x) = 5$. Find $f(x)$. If you can, find more than one.

**[S] We know that $f(x)$ satisfies $f'(x) = x$ and $f(1) = 3$. Find $f(x)$. (Are there more than one?)

*[T] We know that $f(x)$ satisfies $f''(x) = a$, $f'(0) = b$, and $f(0) = c$, where a, b, and c are real constants. Find $f(x)$.

A.4 The formulae you need to memorize

This section had many formulae, such as $(\sin x)' = \cos x$ or

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{f(x)} = -\frac{f'(x)}{[f(x)]^2}, \tag{A.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x), \tag{A.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{g(x)}{f(x)} = \frac{f(x)g'(x) - f'(x)g(x)}{[f(x)]^2}, \tag{A.6}$$

but we did not prove them in the basic camp. Now...

*[U] **(1)** Prove $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$ by doing the same thing as (A.2) for $f(x) = \sin x$. You may use

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{and} \qquad 1+\cos\theta = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{\sin^2\theta}{1-\cos\theta}.$$

(2) Prove Eq. (A.6), where you can use the formulae (A.4) and (A.5) without proof.

(3) Now, prove Eqs. (A.4) and (A.5). [Hint: This is not easy.]

A.5 Workout 1: Practice!

Practice makes perfect, so I can provide you with as many problems as you want!

***[V] Practice more. Here, n is a positive integer.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^2 x$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+3x)(4x^3-2x)$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{2x+1}{x^4}$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x+\sin x}{x+\cos x}$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x+\cos x)^2$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x+1}{x\cos x}$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^4+x^2+1}{x^3+x}$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\cos x}{x^3}$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-n}\sin x$

**[W] Use the formula (A.5) repeatedly to calculate the following derivatives.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x\cdot\tan x\cdot\sin x)$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+1)(x+1)(x+2)$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^3$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}x(\sin x+\cos x)^2$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sin^2 x}{\cos x}$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+1)\sin^2 x\tan x$

A.6 One more formula

(No extra problem for this part.)

A.7 The last step: Composite functions

**[X] Practice with the following problems, which are a bit tough.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(\sin x)$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(3x^2+2x)$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{5x^2+7x}$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\sqrt{5x^2+7x}}$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin\sqrt{x}$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}(3x^2+1)^{5/2}$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan(6x^2-5x)$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x^3$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan(\tan x)$

A.8 Workout 2: Practice, Practice, Practice!

***[Y] (intermediate-level problems)

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}(5x^2-2x+1)^3$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^4+1}$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(x^2+2x+2)$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan\sqrt{2x}$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{(x+1)^2}{(x-1)^2}$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x}+x\right)^2$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-1/2}\cos x$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^2\tan 2x$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x-1)^{-1/4}$

(10) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{2}{(x-1)^{1/4}}$

(11) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^3}{\cos x}$

(12) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\cos 2x}$

(13) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2}{\cos x^2}$

(14) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(x^2+1)^2$

(15) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^2(x^2+1)$

**[Z] (a bit tough problems)

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}(5x^2-2x+1)^{-3}$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^4+1)^{-3/5}$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^2[(x^2+2x)^2]$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan\left(x+\sqrt{2x}\right)$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sin(x^2+1)}{\sin(x-1)}$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\sqrt{x^2+4}}$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-1/2}\cos x^2$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^2\sin x\tan 2x$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}x(x-1)^{-3/4}$

(10) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{2\sin x}{(x-1)^{3/4}}$

(11) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^3\tan x}{\cos x}$

(12) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x\sin x}{\cos 2x}$

(13) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2\sin^2 x}{\cos x^2}$

(14) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^3+1}{\sqrt{x-1}}$

(15) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^2\left(x\sqrt{x}\right)$

Afterwords

Have you finished all the problems? Great job! You're now 100% ready for your university learning!

If you are going to take Sho's lecture, **you can email your answers to Sho**. Sho will look at it and give you feedback. This will *not* be included in the grade evaluation, but Sho will acknowledge see your hard work and you might get some recognition for your effort.

P.S. This is the end of this Boot Camp, but just the beginning of your learning. I mean, ***I am ready to provide motivated students with more advanced tasks.*** *Send me an email if you are interested.*