Derivative Boot Camp (Basic)

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Preface

Welcome to your first year of university! As a university student in engineering, **you must be able to calculate derivatives of "simple" functions** such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+\tan x}{\sin(2x+1)}.$$

This Boot Camp is designed to help you prepare for your first year, which is unexpectedly tough for most of you! Take your time, go through each problem carefully, and don't hesitate to ask for help!

To motivate you, we will have a **mini test** at the beginning of the second lecture; the problems will be from this Boot Camp. I hope this preparation will make your university life easier, more enjoyable, and more satisfactory. Good luck!

Remarks

Sho never provides you with solutions. **You students** need to make the solution. To this end,

- Use online resources such as Wolfram Alpha[♯1].
- Share your answers to other colleagues, using LINE or Google Docs[♯2]. Compare your answers with theirs.
- Ask questions to colleagues, to the TA, or to Sho. You can utilize Sho's office hours[♯3].

> **♯1:** https://www.wolframalpha.com/
>
> **♯2:** https://docs.google.com/
>
> **♯3:** https://www2.nsysu.edu.tw/iwamoto/

A.1 The first step: High-school review

First we review high-school mathematics, but with taking care of a typical pitfall. Namely, some students are confused by the notation of derivatives.

Consider a function $f(x)$. The derivative of $f(x)$ is written by $f'(x)$, which is (usually) a different function from $f(x)$. For example, if $f(x) = x^2$, then $f'(x) = 2x$. We may express it in several ways:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x);$$

these expressions are all equivalent. Don't be confused!

Sho recommends you to identify $\frac{\mathrm{d}f}{\mathrm{d}x} \overset{!}{=} f'$. Then, you will easily see that

- $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x)$ is a function,
- but we often omit the "(x)" part and write f', $\frac{\mathrm{d}f}{\mathrm{d}x}$, or $\frac{\mathrm{d}}{\mathrm{d}x}f$.
- *If a is a constant*, $f'(a) = \frac{\mathrm{d}f}{\mathrm{d}x}(a) = \frac{\mathrm{d}}{\mathrm{d}x}f(a)$ means the value of $f'(x)$ at $x = a$.

★**[A]** Calculate the following derivatives, where a is a real constant and $f(x) = x^4$.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}x^2$	**(4)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^8 + x^3\right)$	**(7)** $\frac{\mathrm{d}}{\mathrm{d}x}f(1)$
(2) $\frac{\mathrm{d}}{\mathrm{d}x}(-4x^3)$	**(5)** $\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^3$	**(8)** $f''(1)$
(3) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^5$	**(6)** $\frac{\mathrm{d}f}{\mathrm{d}x}$	**(9)** $f''(a)$

A.2 Trigonometric functions

For arguments of **trigonometric functions** $\sin x$, $\cos x$, etc., we usually use **radians** instead of degrees. The degree 180° is equal to π radian, so

$$180\,\mathrm{deg} = 180^\circ = \pi\,\mathrm{rad}, \qquad \text{i.e.,} \qquad 1\,\mathrm{rad} = \frac{180^\circ}{\pi} \approx 57.30^\circ.$$

Furthermore, the unit "rad" is often omitted. Namely, e.g.,

$$\cos\frac{\pi}{3} = \cos\left(\frac{\pi}{3}\,\mathrm{rad}\right) = \cos 60^\circ = \frac{1}{2}$$

and you need to get accustomed to this convention.

In particular, notice that **the small circle ° means "degrees"**. If you write the small circle, it is equivalent to "degrees". Meanwhile, if you do not write the circle °, we interpret it in radians. So,

$$\sin 30^\circ = \frac{1}{2}, \quad \sin 30 \simeq -0.988, \qquad \left(\text{or} \quad \sin(30\,\mathrm{deg}) = \frac{1}{2}, \quad \sin(30\,\mathrm{rad}) \simeq -0.988\right).$$

In general,

$$\sin x^\circ = \sin\left(\frac{\pi x}{180}\,\mathrm{rad}\right) = \sin\frac{\pi x}{180}. \tag{A.1}$$

Remark: Almost all students are confused by the notation, $(\sin x^2) \neq (\sin x)^2$.

$$\text{Namely,} \quad \sin^2 x \;=\; (\sin x)^2 \;\neq\; \sin x^2 \;=\; \sin(x^2).$$

Remark: We should use this "$\sin^k x$" notation only for $k > 0$. Sho suggests you to use specifically for positive-*integer* k, such as $\cos^2 x$ or $\tan^2 4\theta$. For negative or fractional exponents, it is preferable to use expressions like $(\sin x)^{3/2}$ or $(\sin x)^{-2}$:

$$(\sin x)^{-2} = \frac{1}{\sin^2 x}, \qquad (\tan x)^{-1/2} = \frac{1}{(\tan x)^{1/2}} = \frac{1}{\sqrt{\tan x}} = (\cot x)^{1/2}, \quad \text{etc.}$$

Avoid ambiguous expressions.[♯4♯5]

♯4: Do not write $\sin(x)^2$. It will only confuse readers.

♯5: You will soon learn "inverse trigonometric functions" such as $\arcsin x$ and $\arctan\theta$. They are sometimes written as $\sin^{-1} x$ or $\tan^{-1}\theta$.

★[C] Calculate the following values.

(1) $\sin\frac{\pi}{6}$	**(3)** $\sin\frac{\pi}{3}$	**(5)** $\tan\frac{\pi}{6}$	**(7)** $\tan 0$
(2) $\sin\frac{\pi}{4}$	**(4)** $\cos\frac{2\pi}{3}$	**(6)** $\tan\frac{\pi}{3}$	**(8)** $\cos 2\pi$

★[E] Find the following values, using calculators or online resources.

(1) $\sin 1$	**(2)** $\sin 1^\circ$	**(3)** $\sin^2 3$	**(4)** $\sin 3^2$

A.3 Get into the University

Recall that derivatives are **defined by** $\dfrac{df}{dx}(x_0) = \lim\limits_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$, or equivalently,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{A.2}$$

Any derivatives can be calculated based on this definition. For example, with $f(x) = x^2$,

$$\frac{d}{dx} x^2 = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(2x+h) = 2x. \tag{A.3}$$

★[G] We can calculate other derivatives by the same method as in (A.3). Fill the blanks.

(1) $$\frac{d}{dx} x^3 = \lim_{h\to 0} \frac{\boxed{}}{h} = \lim_{h\to 0} \frac{\boxed{}}{\boxed{}} = \lim_{h\to 0}\left(\boxed{}\right) = 3x^2.$$

(2) $$\frac{d}{dx} 4x = \lim_{h\to 0} \frac{\boxed{}}{\boxed{}} = \lim_{h\to 0} \frac{\boxed{}}{\boxed{}} = \lim_{h\to 0}\left(\boxed{}\right) = 4.$$

★[I] We know that a function $f(x)$ satisfies an equation $f'(x) = 4x$. Can you find what $f(x)$ is? Is it unique? Or can we find more than one possibilities?

A.4 The formulae you need to memorize

Now it's time for more complicated functions. First, you need to memorize the following formulae. I mean, *you need to do exercise until you've memorized them and can use them without any hesitation*.

Theorem A.1

Derivatives of trigonometric functions are given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{1}{\cos^2 x}. \tag{A.4}$$

Theorem A.2

Derivatives of the reciprocal, product, and quotient of a function(s) are given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{f(x)} = -\frac{f'(x)}{[f(x)]^2}, \tag{A.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x), \tag{A.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{g(x)}{f(x)} = \frac{f(x)g'(x) - f'(x)g(x)}{[f(x)]^2}, \tag{A.7}$$

where $f(x)$ and $g(x)$ are any differentiable functions.

★[K] Use online resources to check that these are correct (i.e., Sho didn't make any typo).

★[L] Calculate the following derivatives.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}3\sin x$ **(2)** $\frac{\mathrm{d}}{\mathrm{d}x}(2\cos x - \sin x)$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}(1 + x + \tan x)$

★[M] Calculate the following derivatives, using Eqs. (A.5)–(A.7).

(1) $\frac{\mathrm{d}}{\mathrm{d}x}(x+1)(x^2+2)$ **(4)** $\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x$ **(7)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+1}{x+1}$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}x\sin x$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x+1}$ **(8)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}$

(3) $\frac{\mathrm{d}}{\mathrm{d}x}(3x^2+2x+1)^2$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\tan x}$ **(9)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x}$

A.5 Workout 1: Practice!

Practice makes perfect!

★[N] Practice for the formulae (A.5) and (A.6).

(1) $\dfrac{d}{dx}\dfrac{1}{\sin x}$ (4) $\dfrac{d}{dx}(x-1)(\cos x + x)$ (7) $\dfrac{d}{dx}(x^2+1)^2$

(2) $\dfrac{d}{dx}\dfrac{1}{3x^2+1}$ (5) $\dfrac{d}{dx}\sin x \cos x$ (8) $\dfrac{d}{dx}x^5 \cos x$

(3) $\dfrac{d}{dx}\dfrac{1}{x^2+2x+1}$ (6) $\dfrac{d}{dx}\cos x \tan x$ (9) $\dfrac{d}{dx}(x^2 \cdot x^3)$

★[O] Practice for the formulae (A.5) and (A.7). Here, n is a positive integer.

(1) $\dfrac{d}{dx}\dfrac{x+1}{x-1}$ (4) $\dfrac{d}{dx}\dfrac{\sin x}{x^2}$ (7) $\dfrac{d}{dx}\dfrac{1}{x^n}$

(2) $\dfrac{d}{dx}\dfrac{x+1}{x^2+1}$ (5) $\dfrac{d}{dx}\dfrac{1-\cos x}{1+\cos x}$ (8) $\dfrac{d}{dx}x^{-3}\cos x$

(3) $\dfrac{d}{dx}\dfrac{x^2-1}{x-1}$ (6) $\dfrac{d}{dx}\dfrac{x^3+1}{x^2+1}$ (9) $\dfrac{d}{dx}\dfrac{x^3-1}{x-1}$

[Advanced note: Did you notice a better way to calculate (3) and (9)?]

A.6 One more formula

The formula you know well, $(x^n)' = nx^{n-1}$, can be generalized to any real number a.

Theorem A.3

For **any real number** a,
$$\frac{d}{dx}x^a = ax^{a-1}. \tag{A.8}$$

Notice you can use this formula for $a = 1/2$, $a = -1$, $a = -3/2$, or even $a = 0$.

★[R] Calculate the following derivatives based on the theorem above.

(1) $\dfrac{d}{dx}x^{64}$ (4) $\dfrac{d}{dx}\dfrac{1}{x}$ (7) $\dfrac{d}{dx}\dfrac{1}{x^{1/3}}$ (10) $\dfrac{d}{dx}x^0$

(2) $\dfrac{d}{dx}x^{-10}$ (5) $\dfrac{d}{dx}\sqrt{x}$ (8) $\dfrac{d}{dx}\dfrac{1}{\sqrt{x}}$ (11) $\dfrac{d}{dx}x^{0.1}$

(3) $\dfrac{d}{dx}x^{-2}$ (6) $\dfrac{d}{dx}x^{1/3}$ (9) $\dfrac{d}{dx}\dfrac{1}{x\sqrt{x}}$ (12) $\dfrac{d}{dx}x^{4\pi}$

A.7 The last step: Composite functions

Now, we consider **composite functions**, which are functions of functions. For example, consider $f(x) = \sin(x^2)$. Its derivative can be calculated with the next theorem.

Theorem A.4

Consider $f(u)$, which is a function of u. Assume $u = u(x)$ is a function of x. Then, we can consider $f(u)$ as a function of x, i.e., $f(x) = f(u(x))$, and

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{A.9}$$

This theorem is complicated but you need to get accustomed to. For example,

- For $f(x) = \sin(x^2)$, we set $u = x^2$ and $f(u) = \sin u$. Then, $\frac{\mathrm{d}f}{\mathrm{d}u} = \cos u$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, which lead to the conclusion $\frac{\mathrm{d}f}{\mathrm{d}x} = (\cos u)\cdot 2x = 2x\cos x^2$.
- Consider $g(x) = (x^2 + 2x + 1)^4$. We use the theorem with $f(u) = u^4$ and $u = x^2 + 2x + 1$. Then, $\frac{\mathrm{d}f}{\mathrm{d}u} = 4u^3$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2$, which lead to $\frac{\mathrm{d}f}{\mathrm{d}x} = 4(x^2 + 2x + 1)^3 \cdot (2x + 2)$.
  [Notice that this is equal to $8(x + 1)^7$.]
- This theorem helps us a lot. For example, the derivative of the function $(x^2 + 1)^3$ can be easily calculated, with $u = x^2 + 1$, as $3u^2 \cdot 2x = 6x(x^2 + 1)^2$.

You can calculate more complicated functions. One example is $\cos(\sin x)$. If we let $u = \sin x$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(\sin x) = \frac{\mathrm{d}\cos u}{\mathrm{d}x} = \frac{\mathrm{d}\cos u}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin u\cdot\cos x = -\sin(\sin x)\cdot\cos x.$$

★**[S]** Practice.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^3$ **(5)** $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x+1}$ **(9)** $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)^{-2}$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+2x+1)^4$ **(6)** $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\tan x}$ **(10)** $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)^{-1/2}$

(3) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x^4$ **(7)** $\frac{\mathrm{d}}{\mathrm{d}x}\sin x^{-2}$ **(11)** $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{\tan x}}$

(4) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^4 x$ **(8)** $\frac{\mathrm{d}}{\mathrm{d}x}\cos\sqrt{x}$

You will also be asked to combine with the formulae you've learned so far. For example,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{\cos(x^2+1)} &= \frac{(x)'\cos(x^1+1) - x\left[\cos(x^2+1)\right]'}{[\cos(x^2+1)]^2} = \frac{\cos(x^2+1) - x(-\sin u)(u)'}{\cos^2(x^2+1)}\\ &= \frac{\cos(x^2+1) + x\cdot 2x\cdot\sin(x^2+1)}{\cos^2(x^2+1)} = \frac{1+2x^2\tan(x^2+1)}{\cos(x^2+1)}\end{aligned}$$

where we let $u = x^2 + 1$.

A.8 Workout 2: Practice, Practice, Practice!

Once you have learned, you need to get accustomed to the calculation. To this end, you need to practice more. Some of the following may be a bit complicated, but you can solve them by combining the previous formulae. If you are lost, try using online resources, ask your colleagues, or ask Sho.

★[U] Practice more.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}(5x^2 - 2x + 1)$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^4 + \sqrt{x})$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x + \cos x)$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x + \sqrt{2x})$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2 + 1}{x - 1}$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x} + x^2\right)$

(7) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\cos x$

(8) $\dfrac{\mathrm{d}}{\mathrm{d}x}x^2 \sin x$

(9) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos x^2$

(10) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^2 x$

(11) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x - 1)^{-1/2}$

(12) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\cos x}$

(13) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\cos x}$

(14) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\cos x^2}{x^2 + 1}$

(15) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\sqrt{\cos x}}$

Afterwords

Have you finished all the problems? Great job! You're now ready for the mini test and well-prepared for your upcoming university life!

If you are going to take Sho's lecture, **you can send your answers/questions to Sho** via an email (`mailto:iwamoto@g-mail.nsysu.edu.tw`). Sho will look at it and give you feedback. It will *not* be included in the grade evaluation, but Sho will acknowledge your hard work and you might get some recognition for your effort.

By the way... Did you notice the problem numbering had some gaps...? Yes, this is not an end, but just the beginning of your university learning. I mean, ***just the beginning of the Derivative Boot Camp*** *and many problems are hidden in this document. You will find more challenging problems and further extra topics.*