APPENDIX

∘ Convergence

- convergence in the r-th mean (平均収束)

$$X_n \xrightarrow{L^r} \underset{\substack{\uparrow \\ \text{random variable}}}{X} \iff \lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$$

- Almost-Sure convergence ("almost everywhere", "with probability 1", "strongly"; 概収束)

$$X_n \xrightarrow{a.s.} X \iff P_r\left(\lim_{n\to\infty} X_n = X\right) = 1$$

- convergence in probability (確率収束)

$$X_n \xrightarrow{p} X \iff {}^\forall \varepsilon > 0,\ \lim_{n\to\infty} P_r(|X_n - X| > \varepsilon) = 0$$

$$\iff {}^\forall \varepsilon, \delta > 0,\ {}^\exists N_\delta \text{ s.t. } {}^\forall n > N_\delta,\ P_r(|X_n - X| > \varepsilon) < \delta$$

- Convergence in distribution (分布収束, 法則収束, 弱収束)

$$X_n \xrightarrow{d} X \iff {}^\forall x \in \mathbb{R} \text{ s.t. } F(x) \text{ is continuous},$$

$$\lim_{n\to\infty} F_n(x) = F(x)$$

where $F_n(x) = P_r(X_n \leq x)$,

$F(x) = P_r(X \leq x)$

- $L^s \underset{s>r\geq 1}{\Longrightarrow} L^r \Longrightarrow p \Longrightarrow d$
- $a.s. \Longrightarrow p \Longrightarrow d$
- $p \Longleftarrow d$ if d-convergence into a constant.

- Positive definite
  - An Hermitian matrix $M \in \mathbb{C}^{N\times N}$ is positive definite

    $\Leftrightarrow$ $z^\dagger M z > 0$ for any non-zero $z \in \mathbb{C}^N$

    $\Leftrightarrow$ all eigenvalues > 0

  - A symmetric matrix $M \in \mathbb{R}^{N\times N}$ is positive definite

    $\Leftrightarrow$ $a^T M a > 0$ for any non-zero $z \in \mathbb{R}^N$

    $\Leftrightarrow$ all eigenvalues > 0

  - For general $M \in \mathbb{C}^{N\times N}$,

    $M > 0 \overset{def}{\Longleftrightarrow} z^\dagger M z > 0 \Longrightarrow M$ is Hermitian

  - For general $M \in \mathbb{R}^{N\times N}$,

    $M > 0 \overset{def}{\Longleftrightarrow} z^\dagger M z > 0 \Rightarrow M$ is Hermitian

    $\quad\quad \Rightarrow a^T M a > 0$

    but converse does not hold.

$$\begin{cases} a^T M a > 0 \not\Leftrightarrow M > 0 \\ a^T M a > 0 \wedge M \text{ is Hermitian} \Rightarrow M > 0 \end{cases}$$

※ We can define, instead,

$M > 0 \overset{def}{\Longleftrightarrow} \mathrm{Re}(z^\dagger M z) > 0 \quad \forall z \in \mathbb{C}^N \setminus \{0\}.$

◦ Lagrange multiplier

Class C_1

For $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}^m$ and $S = \{x \subset U \mid g(x) = 0\}$,

open set $U \subset \mathbb{R}^n$

$$a \in S \text{ is an extremum}$$

$$\begin{array}{c} \\ n \end{array}\begin{array}{c} \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \\ m \end{array}$$

$\cdots n < m$ falls in this first case

$$\Leftrightarrow \operatorname{rank}[g'(a)] < m$$

or

$$\exists \lambda_0 \in \mathbb{R}_m \text{ s.t. } \Phi'(a, \lambda_0) = 0 \quad \leftarrow \Phi : U \times \mathbb{R}^m \to \mathbb{R}$$

where $\Phi(x, \underset{\mathbb{R}^m}{\lambda}) = f(x) - \lambda \cdot g(x)$.

For example,

$$f(x,y,z) \quad \text{with} \begin{cases} g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$$

$$\cdots \operatorname{rank}\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} < 2 \quad \text{OR}$$

for $\Phi(x,y,z,a,b) = f(x,y,z) - a\, g_1(x,y,z) - b(x,y,z)$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial a} = \frac{\partial \Phi}{\partial b} = 0.$$

· Gauss distribution Formulae

$$\int \Phi(\lambda a) \mathcal{N}(a|\mu, \sigma^2)\, \mathrm{d}a = \Phi\left(\frac{\mu}{(\lambda^{-2} + \sigma^2)^{1/2}}\right).$$

$$\Phi(a) = \int_{-\infty}^{a} \mathcal{N}(\theta|0,1)\, d\theta = \frac{1}{2}\left[1 + \operatorname{erf}\frac{a}{\sqrt{2}}\right]$$

$$\cdots \int \sigma(a) \mathcal{N}(a|\mu, \sigma^2) \approx \sigma\left(\frac{\mu}{(1 + \frac{\pi}{8}\sigma^2)^{1/2}}\right)$$

$$\sigma(a) \approx \Phi\left(\sqrt{\frac{\pi}{8}}\, a\right)$$

$$\int dx\, \delta(z - a \cdot x - c) \mathcal{N}(x|\mu, \Sigma) = \mathcal{N}(z | a \cdot \mu + c, a^T \Sigma a)$$