HEP-PH Cheat Sheet

Sho Iwamoto / Misho

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Important notes

- Minkowski metric: $\eta = (+,-,-,-)$ unless otherwise noted.
- Levi-Civita symbol: $\epsilon^{12} = \epsilon_{12} = \epsilon^{123} = \epsilon_{123} = \epsilon^{1234} = \epsilon_{1234} = \cdots = 1$.
- Levi-Civita Lorentz tensor: $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.
- Pauli matrices: $\sigma^i \coloneqq \{\sigma_x, \sigma_y, \sigma_z\}$, hence $\sigma_i = -\sigma^i$ for $i = 1, 2, 3$, unless otherwise noted.
- Symbols with this color follows a locally-defined "different" convention.
- Elementary charge: $|e| \simeq 0.303$, always in absolute-value symbols. Note $\epsilon_0 = 1/(\mu_0 c^2) = 1$.

1 Notation and Convention

Convention

$$\text{Pauli matrices:}\quad \sigma_0 := \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \sigma_x := \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y := \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\ \sigma_z := \begin{pmatrix}1&0\\0&-1\end{pmatrix};\quad \sigma^\mu := (1,\boldsymbol{\sigma}),\ \bar\sigma^\mu := (1,-\boldsymbol{\sigma});$$

$$\sigma_\pm := \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\Big/\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \sigma_x = \sigma_+ + \sigma_-,\quad \sigma_y = -\mathrm{i}(\sigma_+ - \sigma_-). \tag{1.1}$$

$$\text{Fourier transf.:}\quad \tilde f(k) := \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}kx} f(x);\quad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}kx}\tilde f(k). \tag{1.2}$$

$$\text{Minkowski metric:}\quad \eta_{\mu\nu} := \eta^{\mu\nu} = \mathrm{diag}(+,-,-,-),\quad \varepsilon^{0123} := 1,\quad \varepsilon_{0123} = -1. \tag{1.3}$$

$$\text{coordinates:}\quad x^\mu := (t,x,y,z),\quad \partial_\mu = \left(\frac{\partial}{\partial t},\boldsymbol{\nabla}\right),\quad p^\mu = (E,p_x,p_y,p_z). \tag{1.4}$$

$$\text{gamma matrices:}\quad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu},\quad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3;\quad \{\gamma^\mu,\gamma_5\} = 0,\quad \gamma^5\gamma^5 = 1. \tag{1.5}$$

$$\text{chiral notation:}\quad \overline{\psi} := \psi^\dagger\gamma^0;\quad \gamma^\mu := \begin{pmatrix}0&\sigma^\mu\\\bar\sigma^\mu&0\end{pmatrix},\quad \gamma_5 = \begin{pmatrix}-1&0\\0&1\end{pmatrix};\quad P_\mathrm{L} = \frac{1-\gamma_5}{2},\quad P_\mathrm{R} = \frac{1+\gamma_5}{2}. \tag{1.6}$$

Electromagnetism

$$A^\mu = (\phi,\boldsymbol{A}),^{\sharp1}\quad \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}},\quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}. \tag{1.7}$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu,\quad \{F_{01},F_{02},F_{03}\} = \boldsymbol{E},\quad \{F_{23},F_{31},F_{12}\} = -\boldsymbol{B};\quad F_{\mu\nu}F^{\mu\nu} = 2(\|\boldsymbol{B}\|^2 - \|\boldsymbol{E}\|^2). \tag{1.8}$$

$$\text{Maxwell equations:}\quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \iff \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0,\quad \boldsymbol{\nabla}\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0,$$

$$\partial_\mu F^{\mu\nu} = j^\nu := (\rho,\boldsymbol{j}) \iff \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho,\quad \boldsymbol{\nabla}\times\boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{j}. \tag{1.9}$$

♯1: The definition of A^μ is determined by that of x^μ (up to an overall sign). We cannot lower the index.

2 Kinematics

Decay rate and cross section ($\mathcal{M}$ has a mass dimension of $4 - N_\mathrm{i} - N_\mathrm{f}$.)

$$\text{decay rate (rest frame; } \sqrt{s} = M_0\text{):} \quad \mathrm{d}\Gamma = \frac{\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}}{2M_0}\left|\mathcal{M}\left(M_0 \to \{p_1, p_2, \cdots, p_{N_\mathrm{f}}\}\right)\right|^2. \tag{2.1}$$

$$\text{cross section (Lorentz invariant):} \quad \mathrm{d}\sigma = \frac{\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}}{4E_A E_B\, v_{\mathrm{Møl}}}\left|\mathcal{M}\left(k_A, k_B \to \{p_1, p_2, \cdots, p_{N_\mathrm{f}}\}\right)\right|^2, \tag{2.2}$$

Lorentz-invariant phase space: $\mathrm{d}\Pi \coloneqq \dfrac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\dfrac{1}{2E_{\boldsymbol{p}}}$, $\quad \overline{\mathrm{d}\Pi^n} \coloneqq \mathrm{d}\Pi_1 \cdots \mathrm{d}\Pi_n\, (2\pi)^4\, \delta^{(4)}\left(P_0 - \sum p_n\right)$,

Møller parameter: $4E_A E_B v_{\mathrm{Møl}} = 2s\,\lambda^{1/2}(1, m_A^2/s, m_B^2/s)$.

Mandelstam variables For $(k_A, k_B) \to (p_1, p_2)$ collision,

$$\begin{aligned}
&s = (k_A + k_B)^2 = (p_1 + p_2)^2, && t = (p_1 - k_A)^2 = (p_2 - k_B)^2, && u = (p_1 - k_B)^2 = (p_2 - k_A)^2;\\
&k_A \cdot k_B = (s - m_A^2 - m_B^2)/2, && k_A \cdot p_1 = (m_1^2 + m_A^2 - t)/2, && s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2,\\
&p_1 \cdot p_2 = (s - m_1^2 - m_2^2)/2, && k_A \cdot p_2 = (m_2^2 + m_A^2 - u)/2; &&\\
&(k_A - k_B)^2 = 2(m_A^2 + m_B^2) - s, && (p_1 - p_2)^2 = 2(m_1^2 + m_2^2) - s. &&
\end{aligned}$$

Two-body final state in the rest frame With final momenta $(E_{1,2}, \pm\boldsymbol{p})$ to angle $\Omega = (\theta, \phi)$,

$$\|\boldsymbol{p}\| = \frac{\sqrt{s}}{2}\,\lambda^{1/2}\left(1, \frac{m_1^2}{s}, \frac{m_2^2}{s}\right), \quad E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \quad E_2 = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}}, \quad p_1 \cdot p_2 = \frac{s - (m_1^2 + m_2^2)}{2}.$$

$$\overline{\mathrm{d}\Pi^2}\Big|_{\mathrm{CM}} = \frac{\|\boldsymbol{p}\|}{4\pi\sqrt{s}}\frac{\mathrm{d}\Omega}{4\pi} = \frac{\|\boldsymbol{p}\|}{8\pi\sqrt{s}}\mathrm{d}\cos\theta = \frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{16\pi}\mathrm{d}\cos\theta \qquad \left(\sqrt{s} = M_0 \text{ or } E_{\mathrm{CM}}\right).$$

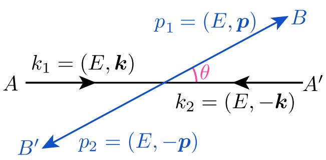
Decay rates and 2-to-2 cross sections are

$$\mathrm{d}\Gamma \overset{\mathrm{CM}}{=} \frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{32\pi M_0}\mathrm{d}\cos\theta\,|\mathcal{M}|^2, \qquad \mathrm{d}\sigma \overset{\mathrm{CM}}{=} \frac{1}{32\pi s}\frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{\lambda^{1/2}(1, m_A^2/s, m_B^2/s)}\mathrm{d}\cos\theta\,|\mathcal{M}|^2 \tag{2.3}$$

For "same mass" collisions $(m_A, m_A) \to (m_1, m_1)$,

$$t = m_A^2 + m_1^2 - s/2 + 2kp\cos\theta, \qquad k = \sqrt{s/4 - m_A^2},$$

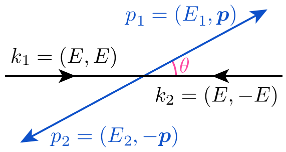
$$u = m_A^2 + m_1^2 - s/2 - 2kp\cos\theta, \qquad p = \sqrt{s/4 - m_1^2}.$$



For "initially massless" collisions $(0, 0) \to (m_1, m_2)$,

$$t = (m_1^2 + m_2^2 - s)/2 + p\sqrt{s}\cos\theta, \qquad p = (\sqrt{s}/2)\,\lambda^{1/2}(1, m_1^2/s, m_2^2/s),$$

$$u = (m_1^2 + m_2^2 - s)/2 - p\sqrt{s}\cos\theta.$$



Three-body final state Mandelstam-like variables can be defined, for $P \to (p_1, p_2, p_3)$, as

$$s_{ij} = (p_i + p_j)^2; \quad t_{0i} = (P - p_i)^2 = s_{jk}; \quad s_{12} + s_{23} + s_{31} = P^2 + p_1^2 + p_2^2 + p_3^2.$$

For spherically-symmetric processes, the phase-space integral is reduced to, at the center-of-mass frame,

$$\int \overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s}, \boldsymbol{0}) = \frac{1}{128\pi^3}\frac{1}{s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}\mathrm{d}s_{23}\int \mathrm{d}s_{13}; \tag{2.4}$$

$$\begin{aligned}
(s_{13})^{\max}_{\min} &= \frac{(s + m_3^2 - m_1^2 - m_2^2)^2}{4s_{23}} - \frac{1}{4s_{23}}\left[\lambda^{1/2}(s, m_1^2, s_{23}) \mp \lambda^{1/2}(s_{23}, m_2^2, m_3^2)\right]^2\\
&= (E_1^* + E_3^*)^2 - \left(\sqrt{E_1^{*2} - m_1^2} \mp \sqrt{E_3^{*2} - m_3^2}\right)^2,
\end{aligned} \tag{2.5}$$

where $E_1^* = \frac{s - s_{23} - m_1^2}{2\sqrt{s_{23}}}$, and $E_3^* = \frac{s_{23} - m_2^2 + m_3^2}{2\sqrt{s_{23}}}$.

2.1 Fundamentals

Lorentz-invariant phase space:

$$\int \mathrm{d}\Pi = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\sqrt{m^2+\|\boldsymbol{p}\|^2}} = \int \frac{\mathrm{d}p_0\mathrm{d}^3\boldsymbol{p}}{(2\pi)^4}(2\pi)\,\delta\left(p_0^2-\|\boldsymbol{p}\|^2-m^2\right)\Theta(p_0)$$

Källén function:

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx = (x-y-z)^2-4yz;$$

$$\lambda(1;\alpha_1^2,\alpha_2^2) = (1-(\alpha_1+\alpha_2)^2)(1-(\alpha_1-\alpha_2)^2) = (1+\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2)(1+\alpha_1-\alpha_2)(1-\alpha_1+\alpha_2).$$

$$\lambda^{1/2}(s;m_1^2,m_2^2) = s\,\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right); \qquad \lambda^{1/2}\left(1;\frac{m^2}{s},\frac{m^2}{s}\right) = \sqrt{1-\frac{4m^2}{s}},$$

$$\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = \sqrt{1-\frac{2(m_1^2+m_2^2)}{s}+\frac{(m_1^2-m_2^2)^2}{s^2}}, \qquad \lambda^{1/2}\left(1;\frac{m_1^2}{s},0\right) = \frac{s-m_1^2}{s}.$$

Two-body phase space If $f(p_1^\mu,p_2^\mu)$ is Lorentz invariant, $f \equiv f(p_1^2,p_2^2,p_1^\mu p_{2\mu}) \equiv f(p_1,p_2,\cos\theta_{12})$. Meanwhile,

$$\int \mathrm{d}\Pi_1\mathrm{d}\Pi_2 = \int \frac{\mathrm{d}^3\boldsymbol{p_1}}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p_2}}{(2\pi)^3}\frac{1}{2E_12E_2} = \int \frac{(4\pi)\mathrm{d}p_1\,p_1^2}{(2\pi)^3}\frac{(2\pi)\mathrm{d}p_2\,p_2^2\,\mathrm{d}\cos\theta_{12}}{(2\pi)^3}\frac{1}{2E_12E_2} = \int \frac{\mathrm{d}E_+\mathrm{d}E_-\mathrm{d}s}{128\pi^4}, \tag{2.6}$$

with the replacement of the variables

$$E_\pm = E_1 \pm E_2, \qquad s = (p_1+p_2)^2 = m_1^2+m_2^2+2E_1E_2-2\|\boldsymbol{p}_1\|\|\boldsymbol{p}_2\|\cos\theta_{12};$$

$$\left|\frac{\mathrm{d}(E_+,E_-,s)}{\mathrm{d}(p_1,p_2,\cos\theta_{12})}\right| = \frac{4p_1^2p_2^2}{E_1E_2}, \qquad \left|\frac{\mathrm{d}(E_1,E_2,s)}{\mathrm{d}(p_1,p_2,\cos\theta_{12})}\right| = \frac{2p_1^2p_2^2}{E_1E_2}.$$

Therefore,

$$\int \mathrm{d}\Pi_1\mathrm{d}\Pi_2 = \frac{1}{128\pi^4}\int_{(m_1+m_2)^2}^{\infty}\mathrm{d}s\int_{\sqrt{s}}^{\infty}\mathrm{d}E_+\int_{\min}^{\max}\mathrm{d}E_-, \tag{2.7}$$

where the boundary of E_- is given by

$$\cos\theta_{12} = \frac{E_+^2-E_-^2+2\left(m_1^2+m_2^2-s\right)}{\sqrt{(E_++E_-)^2-4m_1^2}\sqrt{(E_+-E_-)^2-4m_2^2}} \in [-1,1]$$

$$\therefore \quad \left|E_- - \frac{m_1^2-m_2^2}{s}E_+\right| \le \sqrt{E_+^2-s}\cdot\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = 2p\sqrt{\frac{E_+^2-s}{s}}.$$

Two-body phase space with momentum conservation In a frame with total four-momentum being $(E_0,\boldsymbol{P}_0)$,

$$\overline{\mathrm{d}\Pi^2} = \frac{\mathrm{d}^3\boldsymbol{p}_1}{16\pi^2}\frac{\delta(E_0-\sqrt{m_1^2+p_1^2}-\sqrt{m_2^2+\|\boldsymbol{P}_0-\boldsymbol{p}_1\|^2})}{E_1E_2} = \frac{1}{8\pi}\frac{\mathrm{d}E_1}{P_0}\quad\left(=\frac{1}{8\pi}\frac{p_1^2\mathrm{d}\cos\theta_1}{E_0p_1-P_0E_1\cos\theta_1}\right), \tag{2.8}$$

where $\cos\theta_1$ and p_1 are related by

$$2P_0p_1\cos\theta_1 = 2E_0\sqrt{p_1^2+m_1^2}-\mathcal{M}; \qquad \mathcal{M} := E_0^2-P_0^2+m_1^2-m_2^2. \tag{2.9}$$

If $\boldsymbol{P}_0=\boldsymbol{0}$, Eq. (2.9) fixes p_1 and any θ_1 is allowed, which is the CM result ($E_0=\sqrt{s}$). Otherwise, Eq. (2.9) associates θ_1 with zero, one, or two values of p_1:

$$\cos\theta_1 = \frac{2E_0E_1-\mathcal{M}}{2P_0p_1}, \qquad p_1 = \frac{\mathcal{M}P_0\cos\theta_1 \pm \mathcal{R}E_0}{2(E_0^2-P_0^2\cos^2\theta_1)}, \qquad E_1 = \frac{\mathcal{M}E_0 \pm \mathcal{R}P_0\cos\theta_1}{2(E_0^2-P_0^2\cos^2\theta_1)} \tag{2.10}$$

with $\mathcal{R} = \sqrt{\mathcal{M}^2-4m_1^2(E_0^2-P_0^2\cos^2\theta_1)}$.

2.2 Decay rate and Cross section

As $\langle\text{out}|\text{in}\rangle = (2\pi)^4\,\delta^{(4)}(p_\mathrm{i}-p_\mathrm{f})\mathrm{i}\mathcal{M}$ (for in $\neq$ out) and $\langle\boldsymbol{p}|\boldsymbol{p}\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\,\delta^{(3)}(\boldsymbol{0}) = 2E_{\boldsymbol{p}}V$ for one-particle state,

$$\frac{N_\text{ev}}{\prod_\text{in}N_\text{particle}} = \int \mathrm{d}\Pi^\text{out}\frac{|\langle\text{out}|\text{in}\rangle|^2}{\langle\text{in}|\text{in}\rangle} = \int \mathrm{d}\Pi^\text{out}\frac{(2\pi)^8|\mathcal{M}|^2}{\prod_\text{in}(2E)V}\frac{VT}{(2\pi^4)}\delta^{(4)}(p_\mathrm{i}-p_\mathrm{f}) = VT\int\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}\frac{|\mathcal{M}|^2}{\prod_\text{in}(2E)V}. \tag{2.11}$$

Therefore, decay rate (at the rest frame) is given by

$$\mathrm{d}\Gamma := \frac{1}{T}\frac{\mathrm{d}N_\text{ev}}{N_\text{particle}} = \frac{1}{T}VT\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}\frac{|\mathcal{M}|^2}{(2E)V} = \frac{1}{2M_0}\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}|\mathcal{M}|^2. \tag{2.12}$$

We also define Lorentz-invariant cross section σ by $N_\text{ev} =: (n_Av_\text{Møl}T\sigma)N_B = (n_Av_\text{Møl}T\sigma)(n_BV)$ with number density n, or

$$\mathrm{d}\sigma := \frac{\mathrm{d}N_\text{ev}}{n_Av_\text{Møl}TN_B} = \frac{V}{v_\text{Møl}T}VT\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}\frac{|\mathcal{M}|^2}{2E_A2E_BV^2} = \frac{1}{2E_A2E_B\,v_\text{Møl}}\overline{\mathrm{d}\Pi^{N_\mathrm{f}}}|\mathcal{M}|^2. \tag{2.13}$$

where the Møller parameter $v_{\text{Møl}}$ is equal to $v_{\text{rel}}^{\text{NR}} = \|\boldsymbol{v}_A - \boldsymbol{v}_B\|$ if $\boldsymbol{v}_A \parallel \boldsymbol{v}_B$ (cf. Ref. [1]). Generally,

$$v_{\text{Møl}} := \frac{\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}}{E_AE_B} = \frac{\sqrt{\lambda(s, m_A^2, m_B^2)}}{2E_AE_B} = \frac{p_A\cdot p_B}{E_AE_B}v_{\text{rel}} = (1-\boldsymbol{v}_A\cdot\boldsymbol{v}_B)v_{\text{rel}}, \tag{2.14}$$

where v_{rel} is the actual relative velocity

$$v_{\text{rel}} = \sqrt{1-\frac{(1-v_A^2)(1-v_B^2)}{1-(\boldsymbol{v}_A\cdot\boldsymbol{v}_B)^2}} = \frac{\sqrt{\|\boldsymbol{v}_A-\boldsymbol{v}_B\|^2 - \|\boldsymbol{v}_A\times\boldsymbol{v}_B\|^2}}{1-\boldsymbol{v}_A\cdot\boldsymbol{v}_B} = \frac{\lambda^{1/2}(s, m_A^2, m_B^2)}{s-(m_A^2+m_B^2)} \neq v_{\text{rel}}^{\text{NR}}. \tag{2.15}$$

(Note that $p_A\cdot p_B/E_AE_B = 1$ if $\boldsymbol{p}_A = 0$ or $\boldsymbol{p}_B = 0$. Also, Each of v_{rel}, VT, and $E_AE_Bv_{\text{Møl}}$ is Lorentz invariant.)

2.3 Three body phase space

The phase-space reduction utilizes the identity [2]

$$1 = \int\frac{\mathrm{d}^4p_{ij}}{(2\pi)^4}(2\pi)^4\,\delta^{(4)}(p_{ij}-p_i-p_j)\Theta(p_{ij}^0) \tag{2.16}$$

$$= \int\frac{\mathrm{d}^4p_{ij}}{(2\pi)^4}\left[\int\frac{\mathrm{d}s}{2\pi}(2\pi)\,\delta(s-p_{ij}^2)\right](2\pi)^4\,\delta^{(4)}(p_{ij}-p_i-p_j)\Theta(p_{ij}^0) \tag{2.17}$$

$$= \int\frac{\mathrm{d}^3\boldsymbol{p}_{ij}}{(2\pi)^3}\frac{\mathrm{d}s}{2\pi}\frac{1}{2p_{ij}^0}(2\pi)^4\,\delta^{(4)}(p_{ij}-p_i-p_j)\Big|_{p_{ij}^0=\sqrt{s+\|\boldsymbol{p}_{ij}\|^2}}. \tag{2.18}$$

For three-body phase space,

$$\begin{aligned}\overline{\mathrm{d}\Pi^3} &= \int\mathrm{d}\Pi_1\frac{\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^8}(2\pi)\,\delta(p_2^2-m_2^2)(2\pi)\,\delta(p_3^2-m_3^2)\Theta(p_2^0)\Theta(p_3^0)(2\pi)^4\,\delta^{(4)}\,(P-p_1-p_2-p_3)\\ &\quad\times\frac{\mathrm{d}^3\boldsymbol{p}_{23}}{(2\pi)^3}\frac{\mathrm{d}s_{23}}{2\pi}\frac{1}{2p_{23}^0}(2\pi)^4\,\delta^{(4)}(p_{23}-p_2-p_3)\Big|_{p_{23}^0=\sqrt{s_{23}+\|\boldsymbol{p}_{23}\|^2}}\end{aligned} \tag{2.19}$$

$$\begin{aligned}&= \int\frac{\mathrm{d}s_{23}}{2\pi}\int\mathrm{d}\Pi_1\frac{\mathrm{d}^3\boldsymbol{p}_{23}}{(2\pi)^3}\frac{1}{2p_{23}^0}(2\pi)^4\,\delta^{(4)}(P-p_1-p_{23})\Big|_{p_{23}^0=\sqrt{s_{23}+\|\boldsymbol{p}_{23}\|^2}}\\ &\quad\times\frac{\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^8}(2\pi)\,\delta(p_2^2-m_2^2)(2\pi)\,\delta(p_3^2-m_3^2)\Theta(p_2^0)\Theta(p_3^0)(2\pi)^4\,\delta^{(4)}(p_{23}-p_2-p_3).\end{aligned} \tag{2.20}$$

$$= \int\frac{\mathrm{d}s_{23}}{2\pi}\int\mathrm{d}\Pi_1\frac{\mathrm{d}^3\boldsymbol{p}_{23}}{(2\pi)^3}\frac{1}{2p_{23}^0}(2\pi)^4\,\delta^{(4)}(P-p_1-p_{23})\Big|_{p_{23}^0=\sqrt{s_{23}+\|\boldsymbol{p}_{23}\|^2}}\times\overline{\mathrm{d}\Pi^2}(p_{23}^0,\boldsymbol{p}_{23}) \tag{2.21}$$

and $\overline{\mathrm{d}\Pi^2}(p_{23}^0,\boldsymbol{p}_{23})$ is given by Eq. (2.8); explicitly,

$$\overline{\mathrm{d}\Pi^2}(p_{23}^0,\boldsymbol{p}_{23}) = \frac{\mathrm{d}\cos\theta_2}{8\pi}\frac{p_2^2}{p_{23}^0p_2-\|\boldsymbol{p}_{23}\|\sqrt{p_2^2+m_2^2}\cos\theta_2}; \tag{2.22}$$

$$p_2 = \frac{(s_{23}+m_2^2-m_3^2)\|\boldsymbol{p}_{23}\|\cos\theta_2 + p_{23}^0\sqrt{\lambda(s_{23},m_2^2,m_3^2)-4m_2^2\|\boldsymbol{p}_{23}\|^2\sin^2\theta_2}}{2(s_{23}+\|\boldsymbol{p}_{23}\|^2\sin^2\theta_2)}, \tag{2.23}$$

where θ_2 is the angle between $\boldsymbol{p}_{23}$ and $\boldsymbol{p}_2$ (in the lab frame).

If the matrix element to integrate is spherically symmetric, so as $\overline{\mathrm{d}\Pi^2}(p_{23}^0,\boldsymbol{p}_{23})|\mathcal{M}|^2$, i.e., it is independent of the angle of $\boldsymbol{p}_{23}$. Then one can simply evaluate $\int\mathrm{d}^3\boldsymbol{p}_{23}$, which leads to, in the center-of-mass frame,

$$\overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s},\boldsymbol{0}) = \frac{\mathrm{d}s_{23}\mathrm{d}\cos\theta_2}{64\pi^3}\frac{p_1}{\sqrt{s}}\frac{p_2^2}{p_2\sqrt{s_{23}+p_1^2}-p_1\sqrt{p_2^2+m_2^2}\cos\theta_2}\Big|_{p_1^2=\lambda(s,m_1^2,s_{23})/4s} = \frac{s}{128\pi^3}\mathrm{d}x_1\mathrm{d}x_2, \tag{2.24}$$

where we defined $x_i := 2E_i/\sqrt{s}$. Noting that $s_{23} = s+m_1^2-2E_1\sqrt{s} = s(1-x_1)+m_1^2$ etc.,

$$\overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s},\boldsymbol{0}) = \frac{1}{128\pi^3}\frac{1}{s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}\mathrm{d}s_{23}\int\mathrm{d}s_{13}; \tag{2.25}$$

$$(s_{13})_{\min}^{\max} = \frac{(s+m_3^2-m_1^2-m_2^2)^2}{4s_{23}} - \frac{1}{4s_{23}}\left[\lambda^{1/2}(s,m_1^2,s_{23})\mp\lambda^{1/2}(s_{23},m_2^2,m_3^2)\right]^2. \tag{2.26}$$

This is equal to the PDG-Eq. (47.23)[PDG2018].

2.4 Two-body decay of boosted particles

A particle with $(P, \Theta, \Phi; M)$ decaying to two particles; at the CM frame the momenta of the decay products are characterized by $\boldsymbol{q} = (q, \theta, \phi)$ with $q = (M_0/2)\,\lambda^{1/2}(1, m_1^2/M_0^2, m_2^2/M_0^2)$. Their lab-frame momenta are given by

$$P = \begin{pmatrix} E_0 \\ P_0 \mathrm{s}_\Theta \mathrm{c}_\Phi \\ P_0 \mathrm{s}_\Theta \mathrm{s}_\Phi \\ P_0 \mathrm{c}_\Theta \end{pmatrix}, \qquad p_1 = \begin{pmatrix} (E_0 \mathcal{E}_1 + P_0 q \mathrm{c}_\theta)/M_0 \\ q \mathrm{c}_\Theta \mathrm{c}_\Phi \mathrm{s}_\theta \mathrm{c}_\phi - q \mathrm{s}_\Phi \mathrm{s}_\theta \mathrm{s}_\phi + r_1 \mathrm{s}_\Theta \mathrm{c}_\Phi \\ q \mathrm{c}_\Theta \mathrm{s}_\Phi \mathrm{s}_\theta \mathrm{c}_\phi + q \mathrm{c}_\Phi \mathrm{s}_\theta \mathrm{s}_\phi + r_1 \mathrm{s}_\Theta \mathrm{s}_\Phi \\ -q \mathrm{s}_\Theta \mathrm{s}_\theta \mathrm{c}_\phi + r_1 \mathrm{c}_\Theta \end{pmatrix}, \qquad p_2 = \begin{pmatrix} (E_0 \mathcal{E}_2 - P_0 q \mathrm{c}_\theta)/M_0 \\ -q \mathrm{c}_\Theta \mathrm{c}_\Phi \mathrm{s}_\theta \mathrm{c}_\phi + q \mathrm{s}_\Phi \mathrm{s}_\theta \mathrm{s}_\phi + r_2 \mathrm{s}_\Theta \mathrm{c}_\Phi \\ -q \mathrm{c}_\Theta \mathrm{s}_\Phi \mathrm{s}_\theta \mathrm{c}_\phi - q \mathrm{c}_\Phi \mathrm{s}_\theta \mathrm{s}_\phi + r_2 \mathrm{s}_\Theta \mathrm{s}_\Phi \\ q \mathrm{s}_\Theta \mathrm{s}_\theta \mathrm{c}_\phi + r_2 \mathrm{c}_\Theta \end{pmatrix} \tag{2.27}$$

with $r_1 = (P_0 \mathcal{E}_1 + E_0 q \mathrm{c}_\theta)/M_0$, $r_2 = (P_0 \mathcal{E}_2 - E_0 q \mathrm{c}_\theta)$, and $\mathcal{E}_i = \sqrt{m_i^2 + q^2}$.

3 Gauge theory

SU(N) Fundamental rep. $N \sim (\tau^a)_{ij}$ (Hermitian), $\overline{N} \sim (-\tau^{a*})_{ij}$, and adjoint rep. **adj.** $\sim (f^a)^{bc}$.[*1]

$$\mathrm{Tr}(\tau_a\tau_b) = \frac{1}{2}\delta_{ab}, \qquad [\tau_a, \tau_b] = \mathrm{i} f_{abc}\tau_c, \qquad [\tau_a, [\tau_b, \tau_c]] = [[\tau_a, \tau_b], \tau_c] + [\tau_b, [\tau_a, \tau_c]],$$
$$f^{abc} = -2\mathrm{i}\,\mathrm{Tr}([\tau^a, \tau^b]\tau^c) \,:\, \text{real, anti-symmetric}, \qquad f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0.$$

$$N_i \mapsto [\exp(\mathrm{i}g\theta^a\tau^a)]_{ij}N_j \simeq N_i + \mathrm{i}g\theta^a\tau^a_{ij}N_j \tag{3.1}$$
$$\overline{N}_i \mapsto \overline{N}_j[\exp(-\mathrm{i}g\theta^a\tau^a)]_{ji} = [\exp(-\mathrm{i}g\theta^a\tau^{a*})]_{ij}\overline{N}_j \qquad (\text{i.e., } \overline{N}_i = N_i^*)$$
$$\simeq \overline{N}_j - \mathrm{i}g\theta^a\overline{N}_j\tau^a_{ji} \simeq \overline{N}_j - \mathrm{i}g\theta^a\tau^{a*}_{ij}\overline{N}_j \tag{3.2}$$

SU(2) Fundamental representation $\mathbf{2} \sim T^a \equiv \sigma^a/2$ and adjoint representation $\mathbf{3} \sim \epsilon^{abc}$.

$$\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}, \qquad [T_a, T_b] = \mathrm{i}\epsilon_{abc}T_c, \qquad \overline{\mathbf{2}} = \mathbf{2}^* = \epsilon\mathbf{2} \quad (\because T^* = \epsilon T\epsilon = -\epsilon T\epsilon^{-1}),$$

where the last identity comes as follows:

$$\epsilon_{ij}\mathbf{2}_j \mapsto \epsilon_{ij}\left([\exp(\mathrm{i}g\theta^aT^a)]_{jk}\mathbf{2}_k\right) = [\epsilon\exp(\mathrm{i}g\theta^aT^a)\epsilon^{-1}]_{ij}\epsilon\mathbf{2}_j = [\exp(-\mathrm{i}g\theta^aT^{a*})]_{ij}(\epsilon_{jk}\mathbf{2}_k). \tag{3.3}$$

SU(3) Fundamental rep. $\mathbf{3} \sim \tau^a \equiv \lambda^a/2$, $\overline{\mathbf{3}} \sim (-\tau^{a*})$, and adjoint rep. $8 \sim (f^a)^{bc}$.

$$\mathbf{3}:\ \phi_a \to [\exp(\mathrm{i}g\theta^\alpha\tau^\alpha)]_{ab}\phi_b, \qquad \overline{\mathbf{3}}:\ \phi_a \to [\exp(-\mathrm{i}g\theta^\alpha\tau^{\alpha*})]_{ab}\phi_b,$$
$$\phi_a^* \to [\exp(-\mathrm{i}g\theta^\alpha\tau^{\alpha*})]_{ab}\phi_b^*, \qquad \phi_a^* \to [\exp(\mathrm{i}g\theta^\alpha\tau^\alpha)]_{ab}\phi_b^*. \tag{3.4}$$

[*1] Upper and lower gauge indices are equivalent, while Lorentz indices and Weyl-spinor indices are different for super- and subscripts because they are raised/lowered by, e.g., metric tensors.

3.1 Gell-Mann matrices

Gell-Mann matrices and a Mathematica code to generate them are:

$$\lambda_{1\text{–}8} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.5}$$

```
GellMann[0] := DiagonalMatrix[{1,1,1}]/Sqrt[3/2]
GellMann[8] := DiagonalMatrix[{1,1,-2}]/Sqrt[3]
GellMann[a:1|2|3|4|5|6|7] := Module[
   {p=Switch[a,1|2|3,{1,2,0},4|5,{1,0,2},6|7,{0,1,2}]},
   Table[If[i*j==0, 0, PauliMatrix[{1,2,3,1,2,1,2}[[a]]][[i,j]]], {i,p}, {j,p}]]
```

4 Spinors

$$\text{Gamma matrices：}\quad \{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu},\ \gamma_5=\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3;\quad \{\gamma^\mu,\gamma_5\}=0,\ \gamma^5\gamma^5=1. \tag{4.1}$$

$$\text{conjugates：}\quad \overline{\psi}=\psi^\dagger\beta,\qquad \psi^{\mathrm{c}}=C(\overline{\psi})^{\mathrm{T}} \tag{4.2}$$

chiral notation

$$\overline{\psi}=\psi^\dagger\gamma^0;\ \gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix},\ \gamma_5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix};\ P_{\mathrm{L}}=\frac{1-\gamma_5}{2},\ P_{\mathrm{R}}=\frac{1+\gamma_5}{2}. \tag{4.3}$$

$$(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0,\ (\gamma^\mu)^*=\gamma^2\gamma^\mu\gamma^2,\ (\gamma^\mu)^{\mathrm{T}}=\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0, \tag{4.4}$$

(cf. Dirac notation)

$$\hat{\gamma}^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \hat{\gamma}^i=\begin{pmatrix}0 & \sigma^i\\ \bar{\sigma}^i & 0\end{pmatrix},\ \hat{\gamma}_5=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \hat{P}_{\mathrm{L}}=\frac{1-\hat{\gamma}_5}{2},\ \hat{P}_{\mathrm{R}}=\frac{1+\hat{\gamma}_5}{2}. \tag{4.5}$$

$$(\hat{\gamma}^\mu)^\dagger=\hat{\gamma}^0\hat{\gamma}^\mu\hat{\gamma}^0,\ (\hat{\gamma}^\mu)^*=\hat{\gamma}^2\hat{\gamma}^\mu\hat{\gamma}^2,\ (\hat{\gamma}^\mu)^{\mathrm{T}}=\hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^\mu\hat{\gamma}^2\hat{\gamma}^0, \tag{4.6}$$

$$(\overline{\psi_1}\psi_2)^*=(\psi_2)^\dagger(\overline{\psi}_1)^\dagger=\overline{\psi_2}\psi_1. \tag{4.7}$$

4.1 Verbose derivation

We here derive the fermion convention under the most generic with signs $h_i = \pm 1$, following Refs. [3].

Lorentz group and Lorentz tensors The Lorentz transformation $\Lambda^\mu{}_\nu$ is defined as a linear transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ that conserves $x^2 = \eta_{\mu\nu}x^\mu x^\nu$, where x^μ is a spacetime point and η is the Minkowski metric:

$$\eta^{\mu\nu} = \eta_{\mu\nu} \stackrel{\text{def}}{=} h_\eta \times \operatorname{diag}(+1,-1,-1,-1), \qquad \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu; \qquad \eta_{\rho\sigma} \stackrel{\text{def}}{=} \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \text{ (defining equation).} \tag{4.8}$$

Its nice to denote its inverse, which satisfies $x_\nu \mapsto x_\mu(\Lambda^{-1})^\mu{}_\nu$, by $\Lambda_\nu{}^\mu$:

$$(\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\alpha}\Lambda^\beta{}_\alpha\eta_{\beta\nu} \equiv \Lambda_\nu{}^\mu; \qquad x_\mu \mapsto \Lambda_\mu{}^\nu x_\nu, \qquad \delta^\alpha_\beta = \Lambda_\mu{}^\alpha\Lambda^\mu{}_\beta = \Lambda^\alpha{}_\mu\Lambda_\beta{}^\mu. \tag{4.9}$$

They form a group $L \cong \mathrm{O}(1,3)$ (Lorentz group), which has four disconnected parts:

$$\begin{aligned} L_0 &= \{\Lambda \mid \det\Lambda = +1, \Lambda^0{}_0 \ge 1\} \cong \mathrm{SO}^+(1,3), & L_\mathrm{P} &= \{\Lambda \mid \det\Lambda = -1, \Lambda^0{}_0 \ge 1\},\\ L_\mathrm{T} &= \{\Lambda \mid \det\Lambda = +1, \Lambda^0{}_0 \le -1\}, & L_\mathrm{PT} &= \{\Lambda \mid \det\Lambda = -1, \Lambda^0{}_0 \le -1\}. \end{aligned} \tag{4.10}$$

Tensors $T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ and pseudo-tensors $\tilde T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ are objects that satisfy

$$T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda_{\nu_1}{}^{\beta_1}\cdots T^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}, \qquad \tilde T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} \mapsto (\det\Lambda)\Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda_{\nu_1}{}^{\beta_1}\cdots \tilde T^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}. \tag{4.11}$$

There are two constants that qualify to be (pseudo-)tensors: $\eta^{\mu\nu}$ (and $\eta_{\mu\nu}$, δ^μ_ν) and the anti-symmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$:

$$\eta^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta\eta^{\alpha\beta} = \eta^{\mu\nu}, \qquad \varepsilon^{\mu\nu\rho\sigma} \mapsto \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\varepsilon^{\mu\nu\rho\sigma}; \qquad \varepsilon^{0123} \stackrel{\text{def}}{=} 1, \quad \varepsilon_{0123} = -1. \tag{4.12}$$

Infinitesimal transformation and Lorentz algebra Equation (4.8) gives the representation for a proper orthochronous Lorentz transformation $\Lambda \in L_0$:

$$\Lambda = 1 + \lambda + \mathcal{O}(\lambda^2); \quad \lambda = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ -\omega_x & 0 & +\theta_z & -\theta_y \\ -\omega_y & -\theta_z & 0 & +\theta_x \\ -\omega_z & +\theta_y & -\theta_x & 0 \end{pmatrix} \stackrel{\text{def}}{=} \mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{K} + \mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J} \stackrel{\text{def}}{=} \frac{-\mathrm{i}}{2}d^{\mu\nu}M_{\mu\nu} \tag{4.13}$$

where θ_i is the angle of a *passive* rotation around i-axis and ω_i describes the *passive* boost along i-axis with velocity $\beta = \tanh\omega$.

The last definition of Eq. (4.13) is used to consider Lorentz algebra $\{M\}$. As $[M_{\rho\sigma}]^\mu{}_\nu$ should be a tensor, the parameter d should also be a tensor and thus $d^{\mu\nu} = k\cdot\eta^{\mu\rho}\lambda^\nu{}_\rho$ (k is a constant), i.e.,

$$M_{\mu\nu} = -M_{\nu\mu}, \qquad d^{\mu\nu} = -d^{\nu\mu}, \qquad \{d^{01},d^{02},d^{03}\} = kh_\eta\times\boldsymbol{\omega}, \qquad \{d^{32},d^{13},d^{21}\} = kh_\eta\times\boldsymbol{\theta}. \tag{4.14}$$

Then the element of the Lorentz algebra is given by $(M_{\rho\sigma})^\mu{}_\nu = (\mathrm{i}/k)(\delta^\mu_\rho\eta_{\nu\sigma} - \delta^\mu_\sigma\eta_{\nu\rho})$, which gives the Lorentz algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = (-\mathrm{i}/k)\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right). \tag{4.15}$$

We will take $k = +1$ to match the notation of Ref. [3].

Rotation and boost The boost and rotation operators, $\boldsymbol{K}$ and $\boldsymbol{J}$, are now given in an abstract form by

$$\boldsymbol{J} = h_\eta(M_{23}, M_{31}, M_{12}), \qquad \boldsymbol{K} = -h_\eta(M_{01}, M_{02}, M_{03}), \tag{4.16}$$

so their commutation relation is read from Eq. (4.15):

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k, \tag{4.17}$$

which leads to

$$\boldsymbol{A} \stackrel{\text{def}}{=} \frac{\boldsymbol{J}+\mathrm{i}\boldsymbol{K}}{2}, \qquad \boldsymbol{B} \stackrel{\text{def}}{=} \frac{\boldsymbol{J}-\mathrm{i}\boldsymbol{K}}{2}; \qquad [A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0. \tag{4.18}$$

This means $\mathfrak{so}(1,3)$ is somewhat similar to $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, or in fact, as discussed in **??**, $\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C}\oplus\mathfrak{su}(2)_\mathbb{C}$.

More isomorphism Let us see a bit more of mathematical structure, following the discussion in **??** (cf. Refs. [4, 5]). The isomorphic groups $\mathrm{Spin}(1,3)^+ \cong \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Sp}(2,\mathbb{C})$ is a double cover of L_0; in particular, $L_0 \cong \mathrm{PSL}(2;\mathbb{C}) = \mathrm{SL}(2;\mathbb{C})/\mathbb{Z}_2$.

Meanwhile, the Lorentz algebra $\mathfrak{so}(1,3)$ is isomorphic to $\mathfrak{sl}(2;\mathbb{C})$ viewed as a real Lie albegra [6, §7.8], and its complexification $\mathfrak{so}(1,3)_\mathbb{C}$ is isomorphic to $\mathfrak{su}(2)_\mathbb{C}\oplus\mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$.

Representation of Clifford algebra As summarized in **??**, the spin group $\mathrm{Spin}(1,3)^+$ generated by Clifford algebra $\mathfrak{C}_{1,3}$ is a double cover of L_0, and thus we can consider a representation of L_0 based on $\mathfrak{C}_{1,3}$.

To construct an irreducible representation of $\mathfrak{C}_{1,3}$, we utilize the fact that we can form two sets of creation-annihilation operators

$$a^\pm = \sqrt{h_\eta}\frac{e^0 \pm e^3}{2}, \qquad b^\pm = \sqrt{h_\eta}\frac{\pm e^2 - \mathrm{i}e^1}{2}; \qquad \{a^+, a^-\} = 1,\ \{b^+, b^-\} = 1,\ \{(\text{others})\} = 0. \tag{4.19}$$

These ladder operator allows us to construct four states starting from $|00\rangle$, which is a non-zero state with $a^-|00\rangle = b^-|00\rangle = 0$, and to construct an irreducible representation of $\mathfrak{C}_{1,3}$ (and, in fact, it is unique for even dimension):

$$|10\rangle = a^+|00\rangle, \quad |01\rangle = b^+|00\rangle, \quad |11\rangle = a^+b^+|00\rangle \quad \rightsquigarrow \quad a^+ = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad b^+ = \begin{pmatrix} 0&-1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \tag{4.20}$$

and $a^- = (a^+)^\dagger$, $b^- = (b^+)^\dagger$. We then obtain a representation γ, which is called "standard representation."[*2] They are not Hermitian, but as we will see, this non-Hermiticity is solved by amending the inner product by a matrix β: $(\psi, \gamma^\mu\psi) := \psi^\dagger\beta\gamma^\mu\psi$.

Although ψ forms an irreducible representation γ^μ of $\mathfrak{C}_{1,3}$, the resulting representation $S_{\mu\nu}$ (see the next paragraph) is a reducible representation of $\mathrm{Spin}(1,3)^+$. This is confirmed by

$$\gamma_5\gamma_5 = 1, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad [\gamma_5, S_{\mu\nu}] = 0; \qquad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{4.21}$$

and $P^{\mathrm{L}}_{\mathrm{R}} = (1 \mp \gamma_5)/2$ works as the projection operators. In addition, four state are eigenstates of $J_3 = h_\eta S_{12}$ because

$$[J_3, b^+b^-] = 0,\ J_3 = b^+b^- - \frac{1}{2}, \tag{4.22}$$

which also guarantees that spinors have spin 1/2. In summary,

$$|00\rangle = |-_\mathrm{L}\rangle, \quad |10\rangle =: |-_\mathrm{R}\rangle, \quad |01\rangle =: |+_\mathrm{R}\rangle, \quad |11\rangle =: |+_\mathrm{L}\rangle; \tag{4.23}$$

$$J_3\,|\pm_\mathrm{H}\rangle = \pm\tfrac{1}{2}\,|\pm_\mathrm{H}\rangle, \quad P_\mathrm{L}\,|\pm_\mathrm{L}\rangle = |\pm_\mathrm{L}\rangle,\ P_\mathrm{R}\,|\pm_\mathrm{R}\rangle = |\pm_\mathrm{R}\rangle; \quad P_\mathrm{L}\,|\pm_\mathrm{R}\rangle = P_\mathrm{R}\,|\pm_\mathrm{L}\rangle = 0. \tag{4.24}$$

For example, in chiral notation with $(+,-,-,-)$, the Lorentz generators $S_{\mu\nu}$ are block diagonal and $|\pm_\mathrm{L}\rangle$ ($|\pm_\mathrm{R}\rangle$) has non-zero component only in the upper (lower) two component:

$$|-_\mathrm{L}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ |-_\mathrm{R}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ |+_\mathrm{R}\rangle = \begin{pmatrix}0\\0\\\mathrm{i}\\0\end{pmatrix},\ |+_\mathrm{L}\rangle = \begin{pmatrix}\mathrm{i}\\0\\0\\0\end{pmatrix}. \tag{4.25}$$

Four-spinors and Lorentz transformation The above "theoretical" discussion can be seen more explicitly, starting from spinors and a matrix representation γ^μ given by

$$\overline{\psi} = \psi^\dagger\beta, \quad \beta\beta = 1, \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}; \qquad \psi \mapsto T\psi, \quad \overline{\psi} \mapsto \overline{\psi}\beta T^\dagger\beta; \quad T \in \mathrm{Spin}(1,3)^+. \tag{4.26}$$

For $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\psi$ to be respectively scalar and vector, T should satisfy

$$T^{-1}\gamma^\mu T = \Lambda^\mu{}_\nu\gamma^\nu, \quad \beta T^\dagger\beta T = 1, \tag{4.27}$$

or in infinitesimal form $T = 1 + (-\mathrm{i}/2)d^{\mu\nu}S_{\mu\nu}$,

$$(S_{\mu\nu})^\dagger = \beta S_{\mu\nu}\beta, \quad [S_{\mu\nu}, \gamma^\alpha] = -(M_{\mu\nu})^\alpha{}_\beta\gamma^\beta; \qquad \therefore S_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu]; \quad [\gamma_\mu^\dagger, \gamma_\nu^\dagger] = \beta[\gamma_\mu, \gamma_\nu]\beta; \tag{4.28}$$

the first condition leads to a representation of the Lorentz group

$$\Lambda \overset{\text{rep}}{=} \exp\left(\frac{-\mathrm{i}}{2}d^{\mu\nu}S_{\mu\nu}\right); \qquad [S_{\mu\nu}, S_{\rho\sigma}] = -\mathrm{i}\left(\eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\sigma}S_{\mu\rho}\right) \tag{4.29}$$

as seen in Eq. (4.15), while the second condition determines what β should be, as given in, e.g., Eq. (4.3) and Eq. (4.5).

Charge conjugation and Majorana spinor The charge conjugation of ψ is something like ψ^* but should obey the same representation as ψ does, i.e., $C'\psi^*$ with C' being a unitary matrix such that $B\psi^* \to TB\psi^*$. Or it can be seen that the, the previous procedure with ψ^* may generate another irreducible representation and it should be related to γ^μ by an unitary matrix. Anyway, we define the charge conjugation by

$$\psi^\mathrm{c} = C(\overline{\psi})^\mathrm{T} = C\beta^\mathrm{T}\psi^*; \quad CC^\dagger = 1. \quad \therefore C^*\beta[\gamma_\mu^\dagger, \gamma_\nu^\dagger]\beta C^\mathrm{T} = [\gamma_\mu^\mathrm{T}, \gamma_\nu^\mathrm{T}]. \tag{4.30}$$

Combining with Eq. (4.28),

$$\beta\beta = 1, \quad CC^\dagger = 1, \quad \beta\gamma^\mu\beta = h_\beta(\gamma^\mu)^\dagger, \quad C\gamma_\mu^* C^\dagger = h_C\gamma_\mu^\dagger. \tag{4.31}$$

In even dimensions, the expressions have many choices as seen in the signs h_C and h_β; moreover, the sign h_{cc} defined in $(\psi^\mathrm{c})^\mathrm{c} = h_{\mathrm{cc}}\psi$ depends on the definition. It is thus useful to use a specific notation for further discussion.

For example, the construction of a Majorana spinor ψ_M, which satisfies $(\psi_\mathrm{M})^\mathrm{c} = \psi_\mathrm{M}$, is simply done as $\psi_\mathrm{M} \propto \psi + \psi^\mathrm{c}$ if $\eta_{\mathrm{cc}} = 1$, but needs some phases if $\eta_{\mathrm{cc}} = -1$.

Weyl spinor

4.2 Convention ♣TODO:WIP!♣

First we prepare a vector x^μ and a symmetric matrix $\eta^{\mu\nu}$, which we call "contravariant vector" x^μ and the metric $\eta^{\mu\nu}$. Then we perform a Lorentz transformation on x^μ to obtain $(x')^\mu$, with which we can define a matrix $\Lambda(\boldsymbol{v}a, \theta)^\mu_\nu$ through $\Lambda^\mu{}_\nu x^\nu = (x')^\mu$.

We then consider Λs for infinitesimal transformations and define $\boldsymbol{S}$, $\boldsymbol{J}$, and $\boldsymbol{K}$ by

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu - \mathrm{i}(\theta\cdot\boldsymbol{J}^\mu{}_\nu + \boldsymbol{\beta}\cdot\boldsymbol{K}^\mu{}_\nu) \simeq \delta^\mu_\nu - \frac{\mathrm{i}}{2}\left[\Lambda^{\alpha\beta}S_{\alpha\beta}\right]^\mu{}_\nu \tag{4.32}$$

Imposing "Lorentz condition" (♣TODO:what?♣), we get the expression for $S = \mathrm{i}(\delta\cdots)$ and $[J^i, J^j] = \cdots$; further, we get $\Lambda^\mu_\nu = \exp(-\mathrm{i}\theta\cdot\boldsymbol{J} - \mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{K})$, $\theta = (\theta_{23}, \theta_{31}, \theta_{12})$, $\boldsymbol{\xi} = \hat{\boldsymbol{v}}\tanh^{-1}\|\boldsymbol{v}\| = (\theta^{10}, \theta^{20}, \theta^{30})$; $J = (S_{23}, S_{31}, S_{12})$, $K = (S^{01}, S^{02}, S^{03})$....?

[*2] The chiral notation (4.3) and Dirac notation (4.5) are equivalent to this standard representation, i.e., related by unitary matrices.

Lorentz transformation with a rotation θ around an axis $\hat{\boldsymbol{\theta}}$ and a boost $\boldsymbol{v}$ are given by

$$\Lambda = \exp\left[-\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right]; \qquad \boldsymbol{\theta} := \theta\hat{\boldsymbol{\theta}}, \quad \boldsymbol{\beta} := \hat{\boldsymbol{v}}\tanh^{-1}\|\boldsymbol{v}\|, \tag{4.33}$$

♣TODO:check!♣

Lorentz transformation (infinitesimal): $\Lambda = \begin{pmatrix} 0 & & \boldsymbol{\beta}^{\mathrm{T}} & \\ & 0 & -\theta_z & \theta_y \\ \boldsymbol{\beta} & \theta_z & 0 & -\theta_x \\ & -\theta_y & \theta_x & 0 \end{pmatrix}$

$[J_{\mu\nu}]^{\alpha}{}_{\beta} = \mathrm{i}(\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta})$

Lorentz tensor $M^{\mu_1\mu_2\cdots\mu_n} \propto \bar{\sigma}^{\mu_1\dot{\beta}_1\alpha_1}\cdots M_{\alpha_1\cdots\dot{\beta}_1\cdots}$

Especially $V^{\mu} =: \frac{1}{2}\bar{\sigma}^{\mu\dot{\beta}\alpha}V_{\alpha\dot{\beta}}$, $V_{\alpha\dot{\beta}} = V^{\mu}\sigma_{\mu\alpha\dot{\beta}}$; hermite $V_{\alpha\dot{\beta}} \Leftrightarrow real V^{\mu}$.

$$(V^{\mathrm{T}})_{\alpha\dot{\beta}} = V_{\dot{\beta}\alpha}, \text{♣TODO:(correct? possibly wrong dot-positions?)♣} \tag{4.34}$$

$$(V^{*})_{\dot{\alpha}\beta} := (V_{\alpha\dot{\beta}})^{*}, \tag{4.35}$$

$$(V^{\dagger})_{\alpha\dot{\beta}} := (V_{\beta\dot{\alpha}})^{*} = (V^{*})_{\dot{\beta}\alpha} \tag{4.36}$$

♣TODO:anyway not very sure about the reasoning; though my old note says like this...♣

In general, metric is symmetric.

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \eta_{\nu\rho}\Lambda^{\rho}{}_{\sigma}(\eta^{-1})^{\sigma\mu} =: \Lambda_{\nu}{}^{\mu} \tag{4.37}$$

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