5 Jan 2019

RW - metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]$$

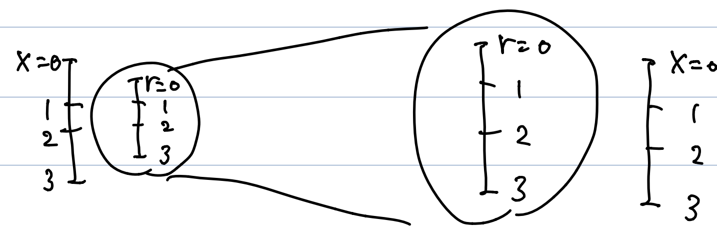
dimension-less

$a(t_0) = 1$

K must be independent of t ?

$\mathbb{r} = (r\sin\theta\cos\phi,\ r\sin\theta\cos\phi,\ r\cos\theta)$: comoving coordinate

$\mathbb{X} = a(t)\,\mathbb{r}$: proper coordinate [dimension-full !]



$a = a_0 \quad < \quad a = a_1$

To keep this RW metric, i.e. to keep the homogenity,

$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu} \quad \text{[perfect fluid]}$$

For comoving object

$\mathbb{r} = \text{const.}$ $(U_\mu = (1\,0\,0\,0))$,

$$T_{\mu\nu} = \begin{pmatrix} \rho + p & 0 \\ 0 & p\, g_{\mu\nu} \end{pmatrix}$$

⇒ Einstein eq. gives

$$\begin{cases} \left(\dfrac{\dot a}{a}\right)^2 = \dfrac{8\pi G}{3}\rho - \dfrac{k}{a^2} + \dfrac{\Lambda}{3} \\ \dfrac{\ddot a}{a} = -\dfrac{4\pi G}{3}(\rho + 3p) + \dfrac{\Lambda}{3} \end{cases}$$

OR Equivallently

$$\begin{cases} \left(\dfrac{\dot a}{a}\right)^2 = \dfrac{8\pi G}{3}\rho - \dfrac{K}{a^2} + \dfrac{\Lambda}{3} \\ \dot\rho = -3\dfrac{\dot a}{a}(\rho + p) \end{cases}$$

If $p = w\rho$, $\dot\rho = -3(1+w)\dfrac{\dot a}{a}\rho$ and $\rho \propto a^{-3(1+w)}$.

• "matter" ... non-relativistic objects $p \ll \rho$

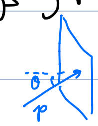
$$\dot{\rho} \simeq -3\frac{\dot{a}}{a}\rho \Rightarrow \rho \propto a^{-3} \quad [\rho \propto 1/V]$$

• "radiation" ... relativistic objects

IF the radiation is in thermal equilibrium and we can define the temperature *T* for the radiation,

$$\rho = g\int \frac{d^3p}{(2\pi)^3} E\, f(E,\mu) = \frac{4\pi g}{(2\pi)^3}\int p^2 dp\, E\, f(E,\mu)$$

$$P_z = g\int \frac{d^3p}{(2\pi)^3} p_z \cdot v_z\, f(E,\mu)$$

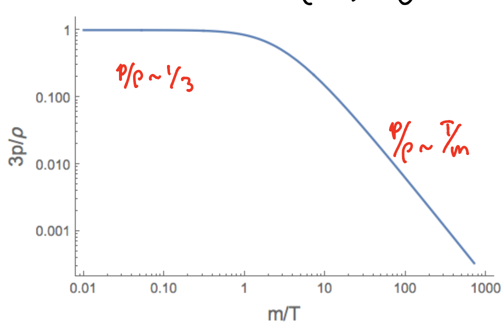


$p_z \times v_z \delta t \delta S = p\cos^2\theta\, \delta t\, \delta S$

momentum flux

$$= g\int \frac{2\pi}{(2\pi)^3} p^2 \overset{2/3}{d\cos\theta}\, dp \cdot p\cos\theta\, \frac{p}{E}\cos\theta\, f(E,\mu)$$

$$= \frac{1}{3}\cdot\frac{4\pi g}{(2\pi)^3}\int p^2 dp\, \frac{E^2-m^2}{E} f(E,\mu)$$



3p/ρ for Maxwell-Boltzmann distribution

$\Rightarrow$ for $m \ll E \sim T$,

$$p = \frac{1}{3}\rho, \quad \rho \propto a^{-4}.$$

$\frac{\dot{a}}{a} = H$ is measured, and it gives $\rho_0 = \frac{3}{8\pi G}\left(H_0^2 + \frac{\overset{\text{const.}}{K}}{\underbrace{a_0^2}_{=1}} - \frac{\overset{\text{assumed const.}}{\Lambda}}{3}\right)$

Defining $\underset{\text{(measured)}}{\rho_{c0} = \frac{3H_0^2}{8\pi G}}$, $\Omega_x = \frac{\rho_{x0}}{\rho_{c0}}$ and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$,

$$\underline{K = (\Omega_R + \Omega_m + \Omega_\Lambda - 1)H_0^2}$$

Also, $\dot{a}^2 = \left[\frac{H_0^2}{\rho_{c0}}(\rho_R + \rho_m) + \frac{\Lambda}{3}\right]a^2 - K$

$$= \left[\frac{\Omega_R}{a^2} + \frac{\Omega_m}{a} + \Omega_\Lambda a^2 - (\Omega_R + \Omega_m + \Omega_\Lambda - 1)\right]H_0^2$$

$\Downarrow$ $a \ll 1$ R.D. $a(t) \propto \sqrt{t}$

$\Downarrow$ M.D. $a \propto t^{2/3}$

$\Downarrow$ Λ.D. $a \propto \exp(\sqrt{\Omega_\Lambda} H_0 t)$

$$z = \frac{\lambda_{obs} - \lambda}{\lambda} = a^{-1} - 1$$

$$(1 + z = a^{-1})$$

BBN $T \sim \mathrm{MeV}$, $t \sim 100\,\mathrm{s}$, $\quad a_{BBN}^{-1} \sim z_{BBN} \sim 3 \times 10^8$

R-M equivalence : $\rho_m = \rho_R \Leftrightarrow 1 + z_{eq} = \Omega_m / \Omega_R \sim 3000$

e-p combination $T \sim 0.3\,\mathrm{eV}$. $t \sim 3 \times 10^5\,\mathrm{yr}$. $1 + z_{rec} \sim 1000$

structure formation $T < 30\,\mathrm{K}$. $t > 10^9\,\mathrm{yr}$. $\quad z < 10$

M-Λ equivalence : $\rho_m = \rho_\Lambda \Leftrightarrow 1 + z_\Lambda = (\Omega_\Lambda / \Omega_m)^{1/3} \sim 1.3$

$$t \equiv \int_0^{a(t)} \frac{da'}{\dot{a}'} = \frac{1}{H_0} \int_0^{(1+z)^{-1}} \left[\frac{\Omega_R}{a^2} + \frac{\Omega_m}{a} + \Omega_\Lambda a^2 - (\Omega_R + \Omega_m + \Omega_\Lambda - 1) \right]^{-1/2} da$$

For the universe w. $K = 0$ and $p \approx w\rho$,

$$\dot{a}^2 = \frac{8\pi G}{3} \rho \cdot a^2 \propto a^{-3(1+w)+2} \Rightarrow a \propto t^{\frac{2}{3(1+w)}}$$

Thermodynamics

We consider a subsystem in the Universe in which a thermal equilibrium is realized and a temperature T is defined. If the chemical potential of the system is negligible,

very strong assumption;

$$\left(\frac{\rho + p}{T} - s \right) dV = V \left(ds - \frac{1}{T} d\rho \right)$$

Considering T-dependence $\rho = \rho(T)$, $s = s(T)$ and $p = p(T)$.

$$V \left(ds - \frac{1}{T} d\rho \right) = V (s' dT - T\rho' dT)$$

Then, we can change the volume with keeping $dT = 0$, which forces

$s_{(T)} = \dfrac{\rho_{(T)} + p_{(T)}}{T}$. this and $ds = \frac{1}{T} d\rho$ yields $T p'(T) = \rho(T) + p(T)$.

note that these eq. holds only if μ is negligible.

If not, $s = \dfrac{\rho + p - \mu n}{T}$, $\quad T p'(T) = \rho + p - \mu n + T\mu' n$; $p'(T) = s + \mu'(T) n$

we can directly get this by using $T \frac{\partial f}{\partial T} = -(E - \mu + T\mu') \frac{\partial f}{\partial E}$.

Convention follows Kolb-Turner.

Boltzmann equation $\hat{L}[f] = C[f]$

$$\hat{L}_{NR} = \frac{d}{dt} + \frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}$$

$$\hat{L}_{Rel} = p^{\alpha}\frac{\partial}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\frac{\partial}{\partial p^{\alpha}}$$

Since we have assumed homogenity, $f(x^{\mu}, p^{\mu}) = f(t, p^0 = E)$

and $\frac{\partial f}{\partial p^{\theta}} = \frac{\partial f}{\partial p^{\varphi}} = \frac{\partial f}{\partial p^{r}} = 0$ [Note that $p^0 = E$ is fixed in this derivative]

$$\text{So } \hat{L}_{RW} = p^0\frac{\partial}{\partial x^0} - \Gamma^0_{\beta\gamma}p^{\beta}p^{\gamma}\frac{\partial}{\partial p^0}$$

$$= E\frac{\partial}{\partial t} - \frac{\dot{a}(t)}{a(t)}\|\mathbf{p}\|^2\frac{\partial}{\partial E} \qquad \text{where } \|\mathbf{p}\|^2 = E^2 - m^2.$$

If the particle follows some distribution :

$$n = g\int\frac{d^3\mathbf{p}}{(2\pi)^3}f(\mathbf{p}) = g\int\frac{4\pi p^2}{(2\pi)^3}f(p)\,dp = g\int\frac{4\pi E p}{(2\pi)^3}f(E)\,dE,$$

$$g\int\frac{d^3\mathbf{p}}{(2\pi)^3}\times\frac{\partial}{\partial t}f = \frac{\partial}{\partial t}n$$

$$\times\frac{\dot{a}}{a}\frac{\|\mathbf{p}\|^2}{E}\frac{\partial f}{\partial E} = \frac{\dot{a}}{a}g\int\frac{4\pi}{(2\pi)^3}p^3\frac{\partial f}{\partial E}\,dE = \frac{\dot{a}}{a}\frac{4\pi g}{(2\pi)^3}\left\{\left[p^3 f\right]_{E=m}^{E=\infty} - \int 3Epf\,dE\right\}$$

$$= -3\frac{\dot{a}}{a}n$$

$$\therefore \frac{\partial n}{\partial t} + 3\frac{\dot{a}}{a}n = g\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{E}C[f]$$

The RHS can be physically interpreted as, e.g.,

$A \leftrightarrow BX$ ← the particle we focus $\Rightarrow \langle n_A\Gamma_A\rangle - \langle n_B n_X \sigma v_{BX\to A}\rangle$.

It is common to use $\underset{0}{Y}(T) = \underset{3}{n}(T)/\underset{3}{s}(T)$ or $N(T) = n(T)/T^3$:

$$\frac{\partial Y}{\partial T} = \frac{1}{s\dot{T}}\frac{\partial n}{\partial t} - \frac{Y}{s}s' = -\left(\frac{s'(T)}{s(T)} + 3\frac{a'(T)}{a(T)}\right)Y + \frac{1}{s\frac{dT}{dt}}(\text{"RHS"})$$

$$\therefore \quad \frac{dY}{dT} + \frac{d}{dT}\log\Big[\underbrace{s(T)a^3(T)}_{\text{conserved}}\Big] = \frac{1}{s\cdot dT/dt}(\text{"RHS"})$$

OR

$$\frac{dN}{dT} + 3N\frac{d}{dT}\log\left[Ta(T)\right] = \frac{1}{T^3\cdot dT/dt}(\text{"RHS"})$$

one will specify later.

OR. with $x = \frac{m}{T}$,

$$\overset{0}{\frac{dY}{dx}} + \cancel{\frac{d}{dx}\log\left(s(x)a^3(x)\right)}^{\,0} = -\frac{m^{1}}{\underset{3}{x^2 s}\,\underset{2}{dT/dt}}\underset{4}{(\text{"RHS"})}$$

$$= \frac{x}{m^2 s\cdot\left(-\frac{1}{T^3}\frac{dT}{dt}\right)}(\text{"RHS"})$$

$$\overset{0}{\frac{dN}{dx}} + 3N\frac{d}{dx}\log\frac{a(x)}{x} = \frac{x^4}{\underset{5}{m^5}\underset{-1}{\left(-\frac{1}{T^3}\frac{dT}{dt}\right)}}\underset{4}{(\text{"RHS"})}.$$

Note: Since $s = \frac{2\pi^2}{45}g_{*s}T^3$,

$$3\frac{d}{dT}\log(Ta) = \frac{d}{dT}\log(T^3a^3) = \frac{d}{dT}\log\left(\frac{1}{g_{*s}}sa^3\right) = -\frac{d}{dT}\underbrace{\log g_{*s}}_{\substack{\text{decreasing w.r.t. time}\\ \cdots\text{"injection" to } aT.}} \leq 0$$

$$0 = \frac{d}{dt}\log sa^3 = \frac{d}{dt}\log\left(g_{*s}T^3a^3\right)$$

$$\Leftrightarrow \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} - \frac{\dot{g}_{*s}}{3g_{*s}} = -H - \frac{\dot{g}_{*s}}{3g_{*s}}$$

$$\therefore -\frac{m^{1}}{\underset{3}{x^2 s}\,\underset{2}{dT/dt}} = \frac{x}{\underset{1}{m}\underset{3}{s}\left(Hm/T^2 + m\dot{g}_{*s}/3T^2g_{*s}\right)}$$

- Annihilation

$$\langle \sigma v_{\chi\bar\chi\to\psi\bar\psi}\rangle n_\chi^{eq} n_{\bar\chi}^{eq}$$

with $\Delta g_{*s}=0$, $$\frac{dY_x}{dx} = \frac{x}{m_x s (H m_x/T^2)}\left[\langle \sigma v_{\psi\bar\psi\to\chi\bar\chi}\rangle n_\psi n_{\bar\psi} - \langle \sigma v_{\chi\bar\chi\to\psi\bar\psi}\rangle n_\chi n_{\bar\chi}\right]$$

$$k_1 R_s/\sigma_0 m_x^2 = \frac{\frac{2\pi^2}{45} g_{*s}}{\sqrt{\frac{g_*\pi^2}{90}}\sqrt{8\pi}\,\frac{m_x}{m_{pl}}} \qquad k_1 = 0.264$$

$$= \frac{s/T^3}{H m_x/T^2}\,\frac{x^4}{m_x^4}\langle\sigma v\rangle T^6\left(Y_x^{eq\,2} - Y_x^2\right) \quad \cdots ①$$

$R_s \equiv (g_{*s}/g_*^{1/2})\, m_{pl} m_x \sigma_0$

$R_x \equiv (g_x/g_*^{1/2})\, m_{pl} m_x \sigma_0$

$$= \frac{1}{s/T^3\cdot H m_x/T^2}\,\frac{x^4}{m_x^4}\langle\sigma v\rangle n_x^{eq\,2}\left(1 - \frac{Y_x}{Y_x^{eq\,2}}\right) \quad \cdots ②$$

- Cold Relic

with MB approx,

$$Y_{eq} = \left(\frac{2\pi^2}{45} g_{*s} T^3\right)^{-1}\cdot\frac{g_x}{2\pi^2} m^2 T K_2(m/T)$$

$$= \frac{\frac{1}{2\pi^2} g_x}{\frac{2\pi^2}{45} g_{*s}}\, x^2 K_2(x) = k_2 (g_x/g_{*s})\, x^{3/2} e^{-x}\left(1 + \frac{15}{8x} + \cdots\right)$$

$$k_2 = \frac{1/2\pi^2}{2\pi^2/45}\sqrt{\frac{\pi}{2}} = 0.145$$

$$Y_{eq}' = -\frac{K_1(x)}{K_2(x)} Y_{eq} = -k_2 (g_x/g_{*s})\, x^{3/2} e^{-x}\left(1 + \frac{3}{8x} + \cdots\right)$$

with $\langle\sigma v\rangle \approx \sigma_0 x^{-n}$,

$$① = \frac{k_1 R_s}{\sigma_0 m_x^2}\,\frac{x^4}{m_x^4}\langle\sigma v\rangle T^6\left(Y_x^{eq\,2} - Y_x^2\right) = k_1 R_s\, x^{-(n+2)}\left(Y_x^{eq\,2} - Y_x^2\right)$$

$\therefore\ \Delta \equiv Y - Y_{eq}$ is given by $\Delta' + Y_{eq}' = -k_1 R_s\, x^{-(n+2)}\Delta(\Delta + 2Y_{eq})$

Early time: $\Delta \sim \Delta' \sim 0$: $$\Delta \sim -\frac{Y_{eq}'\, x^{n+2}}{2 Y_{eq}\cdot k_1 R_s} = \frac{x^{n+2}}{2 k_1 R_s}\cdot\underbrace{\frac{K_1(x)}{K_2(x)}}_{\text{order one}}$$

Late time: $\Delta' = -k_1 R_s\, x^{-(n+2)}\Delta^2 \Rightarrow \Delta(t=\infty) = \dfrac{n+1}{k_1 R_s}\, x_f^{n+1}$

$k_1^{-1} \sim 3.79$

Define $\Delta(x_f) := c\, Y_{eq}(x_f)$ w. $c(c+2) = n+1$.

$$\underbrace{\Delta'(x_f)}_{\sim 0} + Y_{eq}'(x_f) = -k_1 R_s\, x_f^{-(n+2)}\, Y_{eq}^2(x_f)\, c(c+2)$$

$$\Rightarrow x_f^{n+1/2} = k R_x (n+1)\, e^{-x_f} \qquad k = k_1 k_2 = 0.038$$

$$\Rightarrow x_f = \log[(n+1) k R_x] - \left(n+\tfrac{1}{2}\right)\log x_f$$

$$\approx \log[(n+1) k R_x] - \left(n+\tfrac{1}{2}\right)\log\log[(n+1) k R_x]$$