Spin formalism 9405376

Particle : (J^2, J_z, S^2, S_z)
$\equiv (L^2, L_z, S^2, S_z)$
$\equiv (J^2, J_z, S^2, \Lambda)$

where $\Lambda := \mathbb{J}\cdot\hat{p} = (\mathbb{L}+\mathbb{S})\cdot\hat{\mathbb{p}} = (\mathbb{r}\times\mathbb{p}+\mathbb{S})\cdot\hat{\mathbb{p}} = \mathbb{S}\cdot\hat{\mathbb{p}}$

"helicity"

- invariant under $\begin{cases}\text{rotations}\\ \text{boosts along the momentum axis}\end{cases}$ (as long as the direction is not reversed)

$$U[R(\phi,\theta,\gamma)] := e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\gamma J_z}$$

$\|\mathbb{J}\|^2 = j(j+1)$
$J_z = m$

$$D^{(j)}_{mm'}(R)\,\delta_{jj'} := \langle jm | U[R] | j'm' \rangle$$
$$= e^{-i(m\phi+m'\gamma)} \underbrace{\langle jm | e^{-i\theta J_y} | j'm' \rangle}_{=:\, d^{j}_{mm'}(\theta)\,\delta_{jj'}}$$

$\langle \hat{n} | \ell m \rangle$
$= Y_\ell^m(\theta,\phi)$
$= Y_\ell^m(\hat{n})$
(Sakurai §3.5)

$$\begin{cases} e^{-i\pi J_y} |jm\rangle = (-1)^{j-m} |j\ -m\rangle \\ Y_{\ell m}(\theta,\phi) = \sqrt{\dfrac{2\ell+1}{4\pi}}\, D^{\ell}_{m0}(\phi,\theta,\gamma)^* \\ \int d\Omega\, D^{(j)}_{m\lambda}(R)\, D^{(j')}_{m'\lambda}(R)^* = \dfrac{4\pi}{2j+1}\,\delta_{jj'}\,\delta_{mm'} \end{cases} \quad \cdots ☆1$$

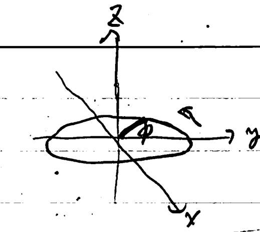
$$|\mathbb{p}, \lambda\rangle = U[R(\phi,\theta,-\phi)]\,|p\hat{z}, \lambda\rangle \quad \cdots ☆2$$

w. $J_z |p\hat{z}, \lambda\rangle$
$= S_z |p\hat{z}, \lambda\rangle = \lambda |p\hat{z}, \lambda\rangle$

because

$$U[R(\phi,\theta,-\phi)] = e^{-i\phi J_z} e^{-i\theta J_y} e^{i\phi J_z}$$
$$= e^{-i\theta(J_y\cos\phi - J_x\sin\phi)}$$

is equivalent to a single rotation of θ around $(-\sin\phi, \cos\phi, 0)$



$$|\mathbb{P},\lambda\rangle = \sum\sum |p,jm\rangle\langle p,jm|\,U[R(\phi,\theta,-\phi)]\,|p,j'm'\rangle\langle p,j'm'|p\hat{z},\lambda\rangle$$

$$= \sum\sum |p,jm\rangle D^{(j)}_{mm'}(R)\underbrace{\langle p,jm'|p\hat{z},\lambda\rangle}$$

because $\Lambda = \mathbb{S}\cdot\hat{\mathbb{p}} = S_z$, $\underline{\lambda = m'}$.

$$= \sum_{j=|\lambda|}^{\infty}\sum_{m=-j}^{j} |p,jm\rangle D^{(j)}_{m\lambda}(R)\langle p,j\lambda|p\hat{z},\lambda\rangle$$

integer / half-integer

not independent

Impose : $\langle p,jm\lambda|p',j'm'\lambda'\rangle = \frac{\delta(p-p')}{p^2}\delta_{jj'}\delta_{mm'}\delta_{\lambda\lambda'}$

$$\int d\Omega\, D^{(j)}_{m\lambda}(R)^*|\mathbb{P},\lambda\rangle = \frac{4\pi}{2j+1}\; |p,jm\rangle\underbrace{\langle p,j\lambda|p\hat{z},\lambda\rangle}_{\text{constant}}$$

$$\therefore\ |p,jm\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}\int d\Omega\, D^{(j)}_{m\lambda}(R)^*|\mathbb{P},\lambda\rangle \qquad f(p,\lambda) = \sqrt{\frac{2j+1}{4\pi}}$$

use ☆1 to this

In scattering processes, at CM-frame $\mathbb{P} := \mathbb{P}_1 = -\mathbb{P}_2$

$$\Lambda_{tot} := \Lambda_1 + \Lambda_2 = (\mathbb{J}_1 - \mathbb{J}_2)\cdot\hat{\mathbb{P}}$$

$$\mathbb{J}\cdot\hat{\mathbb{P}} := (\mathbb{J}_1 + \mathbb{J}_2)\cdot\hat{\mathbb{P}} = \Lambda_1 - \Lambda_2.$$

Second particle ?

→ o construct $|p\hat{z},\lambda\rangle$ and rotate to $-p\hat{z}$
o construct $|0,-\lambda\rangle$ and boost to $-p\hat{z}$

} different by phase

"Jacob-Wick second particle convention"

$$|-p\hat{z},\lambda\rangle = (-1)^{s-\lambda} e^{-i\pi J_y} |p\hat{z},\lambda\rangle$$

so that $\langle p\hat{z},-\lambda|-p\hat{z},\lambda\rangle \to 1$ for $p\to 0$.

$$|\mathbf{p};\lambda_1,\lambda_2\rangle := U[R(\phi,\theta,-\phi)]\Big(|p\hat{z},\lambda_1\rangle\otimes|-p\hat{z},\lambda_2\rangle\Big)$$

$$\mathbf{J}\cdot\hat{\mathbf{p}}\,|\mathbf{p};\lambda_1,\lambda_2\rangle = (\lambda_1-\lambda_2)|\mathbf{p};\lambda_1,\lambda_2\rangle$$

$$|p,jm\lambda_1\lambda_2\rangle = \sqrt{\frac{2j+1}{4\pi}}\int d\Omega\, D^{(j)}_{m\lambda}(R)^*\,|\mathbf{p};\lambda_1\lambda_2\rangle$$

$$O(|a\rangle\otimes|b\rangle) = (O|a\rangle)\otimes|b\rangle + |a\rangle\otimes(O|b\rangle) = (O_a+O_b)(|a\rangle\otimes|b\rangle)$$

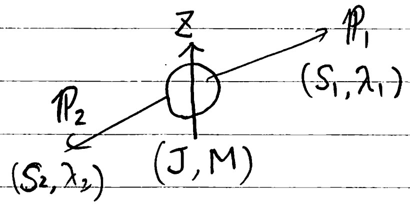
No. 4

Sho Iwamoto

Date

$M = L_z + S_z$

(I) Decay



$J_z |\text{Initial}\rangle = M |\text{initial}\rangle$

$$d\Gamma = \frac{1}{2M}\int d\Pi_2 \, |M|^2 = \frac{1}{2M}\cdot\frac{2\|\mathbb{p}_f\|}{8\pi M}\int\frac{d\Omega}{4\pi}|M|^2$$

$$\therefore \frac{d\Gamma}{d\Omega} = \frac{p_f}{32\pi^2 M^2}\left|\mathcal{M}^{JM}_{\lambda_1\lambda_2}(\theta,\phi)\right|^2$$

$$\mathcal{M}^{JM}_{\lambda_1\lambda_2}(\theta.\phi) = \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M\lambda}(\phi\,\theta\,-\phi)^* \underbrace{\mathcal{M}^J_{\lambda_1\lambda_2}}_{\text{Independent of } \phi,\theta}$$

(II) Scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_A 2E_B v_{rel}}\,\frac{1}{32\pi^2}\,\frac{2p_f}{E_{CM}}|M|^2$$

$$= \frac{2p_f}{4 s\, p_i}\cdot\frac{1}{32\pi^2}|M|^2 = \frac{1}{64\pi^2 s}\,\frac{p_f}{p_i}\left|\mathcal{M}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,\theta,\phi)\right|^2$$

$$\mathcal{M}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s.\theta.\phi) = \sum_{J=\max(\lambda_i,\lambda_f)}^{\infty}(2J+1)\, d^J_{\lambda_i\lambda_f}(\theta)\, e^{i(\lambda_i-\lambda_f)\phi}\,\mathcal{M}^J_{\lambda_c.\lambda_d.\lambda_a\lambda_b}(s)$$

$\lambda_i = \lambda_a - \lambda_b$

$\lambda_f = \lambda_c - \lambda_d$

For spinless process,

$$\mathcal{M}(s.\theta.\phi) = \sum_{J=0}^{\infty}(2J+1)\,P_J(\cos\theta)\,\mathcal{M}^J$$

$$s = (E_A+E_B)^2 = m_A^2 + m_B^2 + 2p_i^2 + 2E_A E_B$$

$$v_{rel} = \left|\frac{\mathbb{p}_A}{E_A} - \frac{\mathbb{p}_B}{E_B}\right| = \frac{p_i}{E_A E_B}(E_A+E_B)$$

- Parity conservation

$$\Rightarrow \begin{cases} \mathcal{M}^{J}_{\lambda_1\lambda_2} = \eta_0\eta_1\eta_2(-1)^{S_1+S_2-J}\mathcal{M}^{J}_{-\lambda_1,-\lambda_2} \\ \mathcal{M}^{J}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \dfrac{\eta_c\eta_d}{\eta_a\eta_b}(-1)^{S_c+S_d-S_a-S_b}\mathcal{M}^{J}_{-\lambda_c,-\lambda_d;-\lambda_a-\lambda_b}(s) \end{cases}$$

η: intrinsic parity

- Time Reversal (CP)

$$\Rightarrow \mathcal{M}^{J}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s) = \mathcal{M}^{J}_{\lambda_a\lambda_b,\lambda_c\lambda_d}(s)$$

- Identical particle

$$a=b \Rightarrow \mathcal{M}^{J}_{\lambda_c\lambda_d,\lambda_a\lambda_b} = (-1)^J\mathcal{M}^{J}_{\lambda_c\lambda_d,\lambda_b\lambda_a}$$

$$c=d \Rightarrow \qquad = (-1)^J\mathcal{M}^{J}_{\lambda_d\lambda_c,\lambda_a\lambda_b}$$

Sho Iwamoto

VII Spin 1/2 wave functions

Dirac representation

$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\gamma^i = \begin{pmatrix} & +\sigma \\ -\sigma & \end{pmatrix}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$C = i\gamma^0\gamma^2 = \begin{pmatrix} & i\sigma_2 \\ i\sigma_2 & \end{pmatrix}$

$$u(p) = \sqrt{2m}\begin{pmatrix} \cosh\frac{\xi}{2} & \chi \\ \sinh\frac{\xi}{2} & (\sigma\cdot\hat{p})\chi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E+m} & \chi \\ \frac{1}{\sqrt{E+m}}\, p\cdot\sigma & \chi \end{pmatrix}$$

rapidity ξ : $\begin{cases} \cosh\xi/2 = \sqrt{\frac{E+m}{2m}} \\ \sinh\xi/2 = \sqrt{\frac{E-m}{2m}} \end{cases}$

$E = m\cosh\xi$

$p = m\sinh\xi$

$$v(p) = C\bar{u}^T(p)$$

$$= i\gamma^0\gamma^2\gamma^0 u^*$$

$$= \begin{pmatrix} \frac{1}{\sqrt{E+m}}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} p\cdot\sigma^*\chi^* \\ \sqrt{E+m}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\chi^* \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{E+m}}\, p\cdot\sigma\,(i\sigma^2)\chi^* \\ \sqrt{E+m}\,(i\sigma^2)\chi^* \end{pmatrix}$$

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$S^{ij} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$\overset{\text{Dirac}}{=} \frac{-i}{4}[\sigma^i, \sigma^j]\otimes\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$

helisity eigenstate $\frac{1}{2}\sigma\cdot\hat{p}\,\chi_\lambda = \lambda\chi_\lambda$

for $\hat{p} = (\theta, \phi)$, $\chi_{1/2} = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$ $\chi_{-1/2} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\theta/2 \end{pmatrix}$

$$\chi_\lambda(\hat{p}) = e^{-i\theta n\cdot\sigma/2}\chi_\lambda(\hat{z})$$

w. $n = (-\sin\phi \;\; \cos\phi \;\; 0)$

$$\boxed{\chi_{-\lambda}(\hat{p}) = -2\lambda\cdot i\sigma_2\,\chi_\lambda^*(\hat{p})}$$

Explicitly
$$\begin{cases} u(p,\lambda) = \sqrt{2m}\begin{pmatrix} \cosh\xi/2 & \chi_\lambda(\hat{p}) \\ 2\lambda\sinh\xi/2 & \chi_\lambda(\hat{p}) \end{pmatrix} \\ v(p,\lambda) = \sqrt{2m}\begin{pmatrix} \sinh\xi/2 & \chi_{-\lambda}(\hat{p}) \\ -2\lambda\cosh\xi/2 & \chi_{-\lambda}(\hat{p}) \end{pmatrix} \end{cases}$$

two-particle state (compatible w. Jacob-Wick convention)

$$\chi_\lambda(-\hat{p}) = e^{-i\phi J_z} e^{-i\theta J_y} e^{i\phi J_z} \cdot (-1)^{s-\lambda} e^{-i\pi J_y} \chi_\lambda(p\hat{z})$$

Calculating explicitly,

$$\chi_{\pm\frac{1}{2}}(-\hat{p}) = \chi_{\mp\frac{1}{2}}(\hat{p})$$

※2 in P.1

Dirac rep. only ?

$$\begin{cases} v(p,\lambda) = -2\lambda\gamma_5 u(p,-\lambda) \\ \cancel{u(-p,-\lambda) = \xi_\lambda \gamma^0 u(p,\lambda)} \\ v(-p,-\lambda) = \xi_{-\lambda}\gamma^0 v(p,\lambda) \end{cases} \quad \text{w. } \cancel{\xi_\lambda = 2\lambda e^{-2i\lambda\phi}}$$

helisity-defining direction

Independent of representation,

rest : $S^\mu = \begin{pmatrix} 0 \\ 2\lambda\hat{p} \end{pmatrix}$

$\beta = 1$: $S^\mu = 2\lambda p^\mu/m$

spin 4-vector $S^\mu = \dfrac{2\lambda}{m}\begin{pmatrix} \|p\| \\ E\hat{p} \end{pmatrix}$ $\quad \begin{cases} S\cdot p = 0 \\ S^2 = -1 \end{cases}$

$$\begin{cases} \not{p} u = m u & \gamma_5 \not{S} u = u \\ \not{p} v = -m v & \gamma_5 \not{S} v = v \end{cases}$$

$$\rightarrow u(p,\lambda)\bar{u}(p,\lambda) = \frac{1}{2}(1+\gamma_5\not{S})(\not{p}+m) \qquad (m \neq 0)$$
$$v(p,\lambda)\bar{v}(p,\lambda) = \frac{1}{2}(1+\gamma_5\not{S})(\not{p}-m)$$

$$u(p,\lambda)\bar{u}(p,\lambda) = \frac{1}{2}(1+2\lambda\gamma_5)\not{p} \qquad (m=0)$$
$$v(p,\lambda)\bar{v}(p,\lambda) = \frac{1}{2}(1-2\lambda\gamma_5)\not{p}$$