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CALCULATIONS OF RELIC DENSITIES IN THE EARLY UNIVERSE

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The cosmological relic densities of stable particles are calculated to high precision The results depend on assumptions made about the quark–hadron phase transition We discuss in detail the calculation of the thermally averaged annihilation cross section We consider the specific cases of photinos, higgsinos, and Dirac and Majorana neutrinos

1. Introduction

In recent years, many theories have been proposed that predict the existence of new stable particles that could account for the observed dark matter in the universe. These theories have free parameters which, because of the high energies and weak couplings involved, cannot currently be probed by terrestrial accellerators. This has led physicists to consider ways in which astrophysical data can be used to put constraints on the masses and couplings of dark-matter candidates.

One of the most important constraints is the requirement that the relic mass density of dark matter be compatible with present observations. In the early universe, these particles would have been present in large numbers in thermal equilibrium, and as the universe cooled they could reduce their density only through pair annihilation [1, 2, 3]. As their density decreases, however, it becomes more and more difficult for particles to find others with which to annihilate, and the comoving density can become constant, or "freeze out". This results in a relic density many orders of magnitude higher than that naively expected from thermodynamic arguments.

The purpose of this paper is to calculate the relic mass densities of stable particles as accurately as possible. Since we are dealing with particles which are not in thermal equilibrium, this involves tracing the evolution of the mass density with a precise rate equation. In sect. 2 we discuss this rate equation in detail. In sect. 3 we

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study the role played by the expansion of the universe and calculate the effective degrees of freedom in the universe as a function of temperature. In sect. 4 we discuss the thermally averaged annihilation cross section and introduce a new approximation which allows us to significantly improve on the accuracy of previous calculations. In sects. 5 and 6, we use our techniques to calculate the relic mass densities of photinos, higgsinos, and heavy neutrinos (both Dirac and Majorana). In an appendix we discuss the (often confusing) factors of two related to correctly averaging over initial spins and particle–antiparticle states.

2. The rate equation

The rate equation for a hypothetical stable particle χ is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3\frac{\dot{R}}{R}n - \langle\sigma v_{\mathrm{rel}}\rangle\left(n^2 - n_0^2\right), \tag{1}$$

where n is the actual number density of χ particles, summed over spins and particle–antiparticle versions (see the appendix); n_0 is the number density the χ particles would have in thermal equilibrium at temperature T; R is the cosmic scale factor; σ is the annihilation cross section, properly averaged over initial spins and particle–antiparticle states (see the appendix); v_{rel} is the relative velocity of the annihilating particles; and $\langle\ \rangle$ implies a thermal average which we will explain in sect. 4.

It is convenient to replace time in the above equation by the photon temperature. For this purpose we define effective degrees of freedom $h(T)$ and $g(T)$ by

$$h(T) \equiv \frac{3}{2}\left(\frac{15}{\pi^2}\right)\frac{s(T)}{T^3}, \tag{2a}$$

$$g(T) \equiv 2\left(\frac{15}{\pi^2}\right)\frac{\rho(T)}{T^4}, \tag{2b}$$

where $s(T)$ and $\rho(T)$ are, respectively, the total entropy and energy densities of the universe, and T is the temperature of the photon gas. With these definitions photons contribute $+2$ to both $g(T)$ and $h(T)$; fully relativistic electrons and positrons together contribute $+\frac{7}{2}$; nonrelativistic particles (in chemical equilibrium) do not contribute. The total entropy S is proportional to hT^3R^3, and entropy conservation therefore implies

$$\frac{\dot{R}}{R} = -\frac{\dot{T}}{T} - \frac{1}{3}\frac{h'}{h}\dot{T}, \tag{3}$$

where $h' \equiv \mathrm{d}h/\mathrm{d}T$.

If we assume a standard Robertson–Walker cosmology, then we can express the time rate of change of the scale factor as

$$\frac{\dot{R}}{R} = \left(\tfrac{8}{3}\pi G\rho\right)^{1/2}. \tag{4}$$

This, along with eqs. (2b) and (3), allows us to put eq (1) in the convenient form

$$\frac{\mathrm{d}q}{\mathrm{d}x} = M\left(\tfrac{4}{45}\pi^3 Gg\right)^{-1/2}\left(h + \tfrac{1}{3}Mxh'\right)\langle\sigma v_{\mathrm{rel}}\rangle\left(q^2 - q_0^2\right), \tag{5}$$

where M is the mass of a χ particle (assumed to be at least a few GeV), $x \equiv T/M$, $q \equiv n/T^3h$, and $q_0 \equiv n_0/T^3h$. It is important to remember that T is the temperature of the photon gas, which may differ from that of the χ particles (more on this below).

Eq. (5), which is our starting point for numerical analysis, differs somewhat from the one used in most previous calculations. The difference is that we have retained terms related to the change of h with temperature; in particular the second term on the right-hand side of eq. (3) has usually not been included (an exception is in ref. [4]). This has its greatest effect at $T \simeq 100$–400 MeV when the quark–hadron transition occurs, and at $T \simeq 0.5$ MeV when electrons and positrons annihilate away. The effect of the h' term at $T \simeq 0.5$ MeV can be estimated analytically, since by this late time the right-hand side of eq. (5) is negligible and q is essentially constant and equal to its value at $T = 0$. Thus $n(T) \simeq q(0)h(T)T^3$. When electrons and positrons annihilate, $h(T)$ decreases by a factor of approximately $\frac{4}{11}$. Therefore, so does $n(T)$. This reduction has been done "by hand" in previous calculations, whereas our numerical integration of eq. (5) does it automatically (and more accurately).

h' は

100~400 ~~to~~ MeV の [illegible]

Other qualitative features of the solution $q(x)$ can be deduced from eq. (5). For $x \gg 1$, the χ particles are relativistic, and we expect them to be in chemical equilibrium with the photons so that $q = q_0$. In this regime $n_0 \sim T^3$ so $q_0 \sim$ constant. When $x \sim 1$, q_0 begins to fall like $\mathrm{e}^{-M/T}$, and the χ particles begin to annihilate. As their density decreases, however, it becomes harder for the χ's to find others to annihilate with, until at some temperature T_{f}, usually called the "freeze out" temperature, $q(x)$ ceases to track $q_0(x)$ and remains roughly constant as $T \to 0$. At this point the χ particles are no longer in *chemical* equilibrium with the photons. However, the χ particles can still be in *mechanical* equilibrium with the photon gas through scatterings with particles of other species, such as electrons. Because of this, in all cases of interest the photon gas and the χ gas stay at the same temperature down to temperatures of order a few MeV. By this time, $\langle\sigma v_{\mathrm{rel}}\rangle$ is essentially constant and $n_0 \ll n$. Therefore, the subsequent discrepancy between the photon

few MeV の 根拠 は?

and χ temperatures does not affect the numerical solution of eq. (5), and we can ignore it.

To obtain the numerical solution of eq. (5), we integrate from $x \gg 1$ (where $q = q_0$) down to $x = 0$, thus finding $q(0)$. The present mass density of the χ particles is then simply given by

$$\rho_\chi = Mq(0)h(0)T_0^3 , \tag{6}$$

where $T_0 = 2.7$ K is the present photon temperature, and $h(0)$ is computed in sect. 3. It is convenient to define, as usual, $\Omega_\chi = \rho_\chi/\rho_{\rm crit}$, where $\rho_{\rm crit} = 3H^2/8\pi G$ is the critical density, $H = (100\ {\rm km\ s^{-1}\ Mpc^{-1}})h$ is the present value of the Hubble parameter $\dot{R}/R$, and $0.5 \lesssim h \lesssim 1$ encodes the uncertainty in its value*. We then find

$$\Omega_\chi h^2 = (1.555 \times 10^8)(M/{\rm GeV})h(0)q(0) . \tag{7}$$

3. Effective degrees of freedom

In order to solve our rate equation, we must be able to compute the functions $g(T)$ and $h(T)$. Here we follow, with some modification, the treatment in ref. [5].

Suppose we have a particle species i present at temperature T whose interaction energy is small compared with its free particle energy. In general, the equilibrium density of particles of type i with momenta in a range d^3p centered on $\boldsymbol{p}$ is $f(E_i)\,d^3p$, where

$$f(E_i) = \frac{\kappa_i}{(2\pi)^3}\left[\exp(E_i/T) \pm 1\right]^{-1} . \tag{8}$$

Here κ_i is the number of spin and particle–antiparticle states, $E_i = (m_i^2 + p^2)^{1/2}$, and the plus (minus) sign is for fermions (bosons). In terms of integrals over $f(E_i)$, the contribution of species i to the energy and entropy densities is

$$\rho_i = \int d^3p f(E_i)E_i , \qquad s_i = \int d^3p f(E_i)\frac{3m_i^2 + 4p^2}{3E_iT} . \tag{9),(10}$$

We then have

$$\rho = \sum_i \rho_i = \frac{\frac{1}{2}g(T)\pi^2T^4}{15} , \qquad s = \sum_i s_i = \frac{\frac{2}{3}h(T)\pi^2T^3}{15} . \tag{11),(12}$$

* The parameter h should not be confused with the entropy function $h(T)$

This treatment assumes that all the particle species are in thermal equilibrium with the photons. This breaks down ~~if the interaction rate of a particular species~~ becomes of order the expansion rate of the universe. In this case that species is said to have decoupled, and its entropy and the entropy of the other particles (which continue to interact) are separately conserved. In particular, if a massive species annihilates and reheats the interacting particles, the decoupled species will not share in the released energy and so will differ in temperature from the interacting particles. The temperature of the decoupled particles is calculated from conservation of entropy, and is then used to find their contribution to $g(T)$ and $h(T)$.

Suppose a species i decouples at temperature $T = T_d$. (Of course, the decoupling temperature is actually a range of temperatures, but the following treatment proves more than accurate enough for our needs.) We know that $hT^3R^3 =$ constant, and we can rescale R so that $hT^3R^3 \equiv 1$. Below T_d, species-i particles no longer interact with the other particles, so their entropy is conserved separately. Thus for $T < T_d$ we can write

$$h_i(T)T_i^3R(T)^3 = \gamma, \qquad h_o(T)T^3R(T)^3 = 1 - \gamma, \tag{13a, b}$$

where h_i is the contribution of species i to h, T_i is the temperature of species i, $h_o = \Sigma_{j \neq i} h_j$ (the subscript o means "other"), and the constant γ is specified by

$$\gamma = h_i(T_d)T_d^3R(T_d)^3, \qquad 1 - \gamma = h_o(T_d)T_d^3R(T_d)^3 \tag{14a, b}$$

Taking the ratio of eqs. (14a) and (14b) we find that

$$\gamma = \frac{1}{1 + h_o(T_d)/h_i(T_d)} . \quad ? \tag{15}$$

Using $h(T) = 1/T^3R(T)^3$ along with eqs. (13b) and (15), we deduce that for $T < T_d$

$$\begin{aligned} h(T) &= \frac{1}{1-\gamma} h_o(T) \\ &= \left[1 + \frac{h_i(T_d)}{h_o(T_d)}\right] h_o(T) . \end{aligned} \tag{16}$$

Furthermore from eqs. (13a), (13b), and (15) we find

$$\frac{T_i^3}{T^3} = \frac{h_i(T_d)h_o(T)}{h_o(T_d)h_i(T)} . \tag{17}$$

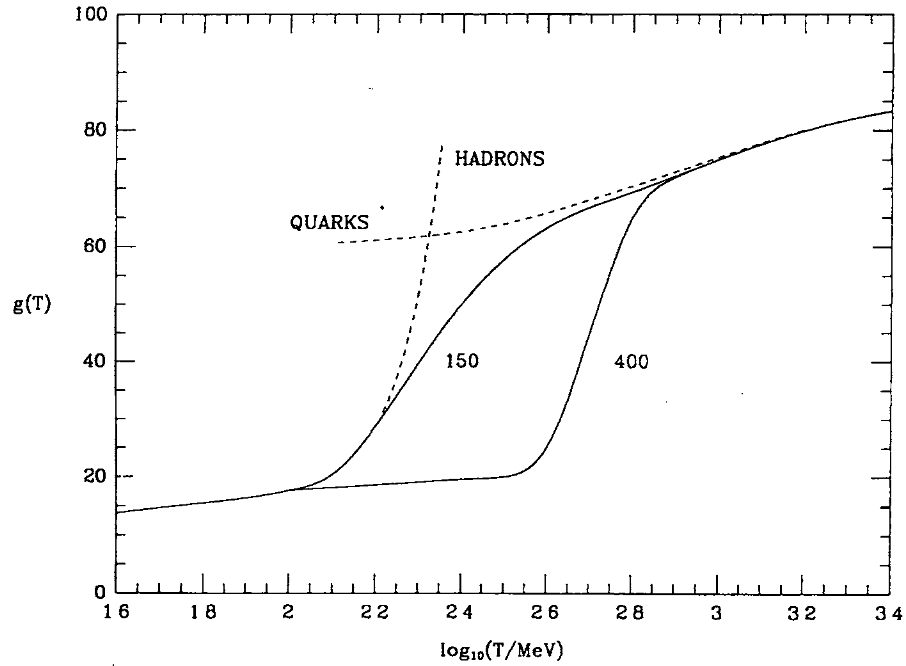

Fig 1 The scaled energy density $g(T)$ for our two assumptions about the quark–hadron transition The curve labeled "150" has this transition near 150 MeV, and the curve labeled "400" has it near 400 MeV Curves for free quarks (with current algebra masses) and free hadrons are shown with dashed lines

We need eq. (17) in order to compute $g(T)$

$$g(T) = g_0(T) + g_i(T_i)\frac{T_i^4}{T^4}\,. \tag{18}$$

We are concerned with the decoupling of light neutrinos which are fully relativistic at $T = T_d$. In this case both $g_i(T)$ and $h_i(T)$ are constants; each is equal to $\frac{7}{4}$ times the number of light-neutrino types.

If another species decouples at a lower temperature, the procedure above must be repeated, using eqs. (16) and (18) as the starting expressions for $h(T)$ and $g(T)$. To actually compute $g(T)$ and $h(T)$ at a given temperature T, we perform the integrals in eqs. (9) and (10) for each relevant species, and then do the sums in eqs. (11) and (12). We must decide whether we should be considering quarks or hadrons as particles to be included in the sum. We first consider temperatures well below the quark–hadron transition, $T \ll T_c \sim$ 100–400 MeV. Here, we include the contributions from all the baryons and mesons*, as well as the leptons and the photons. At

* We used the tables in ref [5] Particles with masses above ~ 3 GeV contribute negligibly to $g(T)$ and $h(T)$ for $T \lesssim 400$ MeV

temperatures of order a few MeV, neutrinos decouple and must be treated according to the procedure outlined above. We assumed [6] that electron neutrinos decouple at $T_d = 2.0$ MeV, and muon and tau neutrinos at $T_d = 3.5$ MeV. The temperatures differ since electron neutrinos interact with electrons via W exchange as well as Z exchange. We also took into account the fact that electrons are not fully relativistic at these temperatures. We ultimately find $h(0) = 3.915$ and $g(0) = 3.368$ for three generations of light neutrinos, and $h(0) = 4.553$ and $g(0) = 3.824$ for four generations of light neutrinos.

For temperatures approaching the quark–hadron transition, the particle density is so high that we no longer have an ideal gas of hadrons. As the interparticle distances become small, the hadrons begin to overlap and the interaction energy becomes important [7]. At even earlier times, the temperature is high enough so that the quarks become asymptotically free, and it is a good approximation to treat the matter as an ideal gas of quarks, gluons, leptons, W and Z bosons, and photons. At present we have no reliable model from which to obtain the values of g and h in the region between the free hadron phase and the free quark phase. It is believed that the quark–hadron phase transition takes place between 100 MeV and 400 MeV, with the lower end favored by recent results in lattice gauge theory [8]. We consider two illustrative cases, one with a quark–hadron transition near 150 MeV, and one

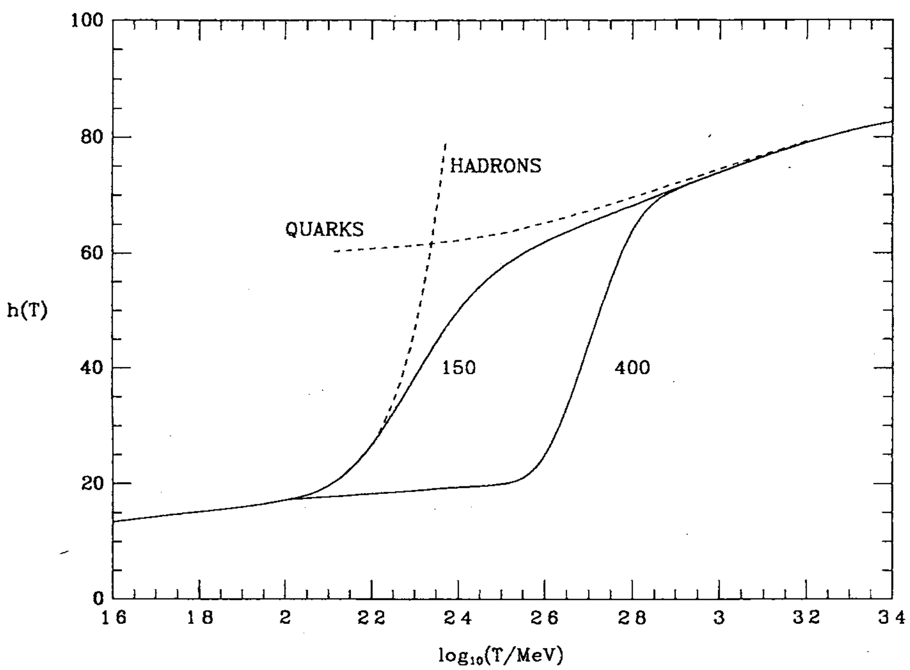

Fig 2 Same as fig (1), except for the scaled entropy density $h(T)$

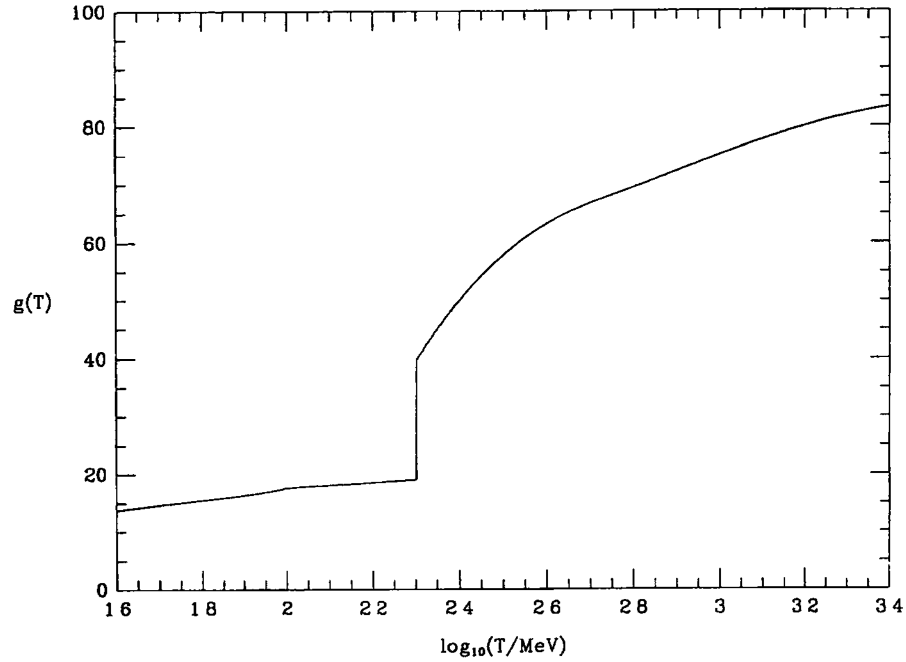

Fig 3 The scaled energy density $g(T)$ assuming a first order quark–hadron transition at $T_c = 200$ MeV

with this transition near 400 MeV. These two possibilities for $g(T)$ and $h(T)$ are shown in figs. 1 and 2, along with the free hadron and free quark curves. It is reasonably likely that the true curves lie between these two extremes, so that by calculating quantities using both choices we can infer the error introduced by our lack of knowledge of this regime*.

One interesting possibility (supported by the lattice results [8]) is that the quark–hadron phase transition was first order. This would imply that the temperature of the universe was kept constant during the transition by the energy released. If this were the case, then $\mathrm{d}h/\mathrm{d}T$ would be infinite at the transition temperature, and one might worry that this could change the results radically from the cases that we describe above. This turns out not to happen. We performed a variety of calculations with the g curve in fig. 3 (and a corresponding h curve) and found that the results still lay between those obtained using the h and g curves of figs. 1 and 2.

* We should also note that the contribution of the χ particles themselves to $g(T)$ and $h(T)$ is not included in the figures since it varies with the assumed value of M Furthermore, as discussed in sect 2 we really only need to know $g(T)$ and $h(T)$ for $T \leq T_f$ where the χ contribution is completely negligible

4. Thermal averaging

Let $|\mathscr{T}|^2$ be the absolute square of the reduced matrix element for the annihilation of two χ particles, summed over final spins and suitably averaged over initial spins and particle–antiparticle states (see the appendix for details). For fixed incoming momenta p_1 and p_2 we define as usual

$$\sigma v_{\text{rel}} = \frac{1}{4E_1E_2}\int \mathrm{d}\,\text{LIPS}\,|\mathscr{T}|^2\,, \tag{19}$$

where d LIPS (Lorentz Invariant Phase Space) is given by

$$\mathrm{d}\,\text{LIPS} = (2\pi)^4\delta^4\Big(p_1 + p_2 - \sum_j p_j\Big)\prod_i \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2p_i^0}\,. \tag{20}$$

The sum and product are each over the outgoing particles. Here we have assumed that Pauli blocking or enhancing factors for the phase space of the final state particles can be ignored. We will justify this in a moment.

Clearly $\int \mathrm{d}\,\text{LIPS}\,|\mathscr{T}|^2$ is Lorentz invariant, and therefore depends on the momenta only through the Mandelstam variable $s = -(p_1+p_2)^2$. It is convenient to define

$$w(s) \equiv \tfrac{1}{4}\int \mathrm{d}\,\text{LIPS}\,|\mathscr{T}|^2 = E_1E_2\sigma v_{\text{rel}}\,. \tag{21}$$

We now suppose that the initial χ particles have an energy distribution $f(E)$, where $E = (\boldsymbol{p}^2 + M^2)^{1/2}$. We then define the thermal average as [9]

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$$\langle \sigma v_{\text{rel}}\rangle \equiv \frac{1}{n_0^2}\int \mathrm{d}^3p_1\,\mathrm{d}^3p_2\, f(E_1)f(E_2)\frac{1}{E_1E_2}w(s)\,, \tag{22}$$

→ Tによって決まる?

where $n_0 = \int \mathrm{d}^3p f(E)$ has already made its appearance in eq. (5). Note, however, that $\langle\sigma v_{\text{rel}}\rangle$ does not depend on the overall normalization of $f(E)$.

$\langle \sigma v_{rel}\rangle$ は MB approx では M によらない.

We know from our qualitative discussion that the χ-particles' density will begin to differ from its equilibrium value only after T falls below the freezing temperature T_f. Typically $T_f \sim \frac{1}{20}M$, so that for our purposes we need to know $\langle\sigma v_{\text{rel}}\rangle$ only at temperatures where it is a very good approximation to take $f(E)$ to be a Boltzmann distribution regardless of the statistics of the particle. This is a great simplification, and justifies our ignoring any Pauli blocking/enhancing factors in eq. (19).

$T_f < \frac{1}{20}M$ しか 考えないので MB でok

For a general cross section, the integrals in eq. (22) cannot be done analytically. Further, since this is a multiple integral, numerical solutions are expensive and difficult to obtain. The solution is to expand $\langle\sigma v_{\text{rel}}\rangle$ in powers of $x \equiv T/M$. One can then do all the integrals analytically and obtain a relatively simple expression for

$\langle\sigma v_{\rm rel}\rangle$. Since, as discussed above, we are only interested in regimes where $x \lesssim \frac{1}{20}$, we need only take a few terms in the expansion to get very good accuracy. In practice we keep terms through order x^3

To perform the expansion in x, we first write

$$s = -(p_1+p_2)^2 = 2\left(M^2 + E_1E_2 - p_1p_2\cos\theta\right), \tag{23}$$

where θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. This is the only dependence on angle in the integrand. Using the Boltzmann limit of eq. (8), $f(E) = \kappa(2\pi)^{-3}e^{-E/T}$ where κ is the number of spin states of the χ particle (see the appendix), we can write

$$\begin{aligned}\langle\sigma v_{\rm rel}\rangle &= \frac{\kappa^2}{(2\pi)^6 n_0^2}\int d^3p_1\, d^3p_2\, e^{-E_1/T}e^{-E_2/T}\frac{1}{E_1E_2}w(s),\\ &= \frac{\kappa^2}{8\pi^4 n_0^2}\int_0^\infty dp_1\, dp_2\,\frac{p_1^2p_2^2}{E_1E_2}e^{-E_1/T}e^{-E_2/T}\int_{-1}^{+1} d\cos\theta\, w(s),\\ &= \frac{\kappa^2}{8\pi^4 n_0^2}\int_M^\infty dE_1\, dE_2\, p_1p_2\, e^{-E_1/T}e^{-E_2/T}\int_{-1}^{+1} d\cos\theta\, w(s),\end{aligned} \tag{24}$$

where we have used $p\,dp = E\,dE$. We now make the following change of variables

$$E_a = M(1+xy_a), \qquad p_a = M(2x)^{1/2}\left(y_a + \tfrac{1}{2}xy_a^2\right)^{1/2}, \tag{25}$$

where $a = 1,2$ indexes the two incoming χ particles. We then find

$$\begin{aligned}\langle\sigma v_{\rm rel}\rangle = \frac{\kappa^2M^4x^3e^{-2/x}}{4\pi^4n_0^2}&\int_0^\infty dy_1\, dy_2\left(y_1+\tfrac{1}{2}xy_1^2\right)^{1/2}\left(y_2+\tfrac{1}{2}xy_2^2\right)^{1/2}\\ &\times e^{-y_1}e^{-y_2}\int_{-1}^{+1} d\cos\theta\, w(s).\end{aligned} \tag{26}$$

We can now also rewrite eq. (23) as

$$\frac{s}{4M^2} = 1 + \tfrac{1}{2}x(y_1+y_2) + \tfrac{1}{2}x^2y_1y_2 - x\left(y_1+\tfrac{1}{2}xy_1^2\right)^{1/2}\left(y_2+\tfrac{1}{2}xy_2^2\right)^{1/2}\cos\theta, \tag{27}$$

and expand the integrand of eq. (26) in powers of x. This involves making a Taylor expansion of $w(s)$ about $s/4M^2 = 1$. All the integrals can then be done in closed form. We also need to compute n_0, both because it appears in the definition of $\langle\sigma v_{\rm rel}\rangle$ and because it is present explicitly in our basic equation (5). Making the derivation should be followed.

change of variable (25), we get

$$
\begin{aligned}
n_0 &= \int \mathrm{d}^3 p f(E) \\
&= \left[\kappa(2\pi x)^{-3/2}\mathrm{e}^{-1/x}T^3\right] 2\pi^{-1/2}\int_0^\infty \mathrm{d}y(1+xy)\left(y+\tfrac{1}{2}xy^2\right)^{1/2}\mathrm{e}^{-y} \\
&= \left[\kappa(2\pi x)^{-3/2}\mathrm{e}^{-1/x}T^3\right]\left[1+\tfrac{15}{8}x+\tfrac{105}{128}x^2-\tfrac{315}{1024}x^3+\mathrm{O}(x^4)\right], \qquad (28)
\end{aligned}
$$

$$
\frac{1}{n_0^2} = \frac{8\pi^3\mathrm{e}^{2/x}}{\kappa^2 x^3 M^6}\left[1-\tfrac{15}{4}x+\tfrac{285}{32}x^2-\tfrac{2115}{128}x^3+\mathrm{O}(x^4)\right]. \qquad (29)
$$

We ultimately arrive at the general formula

すごいる—

$$
\boxed{\begin{aligned}
\langle\sigma v_{\mathrm{rel}}\rangle = \frac{1}{M^2}\Big[& w-\tfrac{3}{2}(2w-w')x+\tfrac{3}{8}(16w-8w'+5w'')x^2 \\
& -\tfrac{5}{16}(30w-15w'+3w''-7w''')x^3+\mathrm{O}(x^4)\Big]_{s/4M^2=1}, \qquad (30)
\end{aligned}}
$$

where primes denote derivatives with respect to $s/4M^2$ (rather than s itself), and w and its derivatives are all to be evaluated at $s/4M^2=1$.

Eq. (30) is one of our central results. In previous calculations, $\langle\sigma v_{\mathrm{rel}}\rangle$ has usually been truncated to w/M^2. The potential importance of the term linear in x was pointed out in ref [10]. However, the coefficient of x has never before been correctly computed, and there have been no previous attempts to compute the quadratic and cubic terms.

In all cases of interest, the dominant annihilation is into particle–antiparticle pairs. Furthermore, the interaction is typically mediated by a heavy particle (such as a W, Z, scalar quark, etc.), and so can often be treated as a fundamental four-point interaction. In this case the phase-space integrals in eq. (21) can be easily done, and $w(s)$ takes the form

$$
w(s) = \sum_f \left(1-\frac{4m_f^2}{s}\right)^{1/2}\left[C_{0f}+C_{1f}\frac{s}{4M^2}+C_{2f}\left(\frac{s}{4M^2}\right)^2\right], \qquad (31)
$$

where m_f is the mass of the outgoing particle, C_{0f}, C_{1f}, and C_{2f} are dimensionless constants (which in general depend on the type of the outgoing particle–antiparticle pair), and the sum is over the allowed types of outgoing particle–antiparticle pairs.

Plugging this $w(s)$ into eq. (30) yields

$$\langle \sigma v_{\text{rel}} \rangle = \sum_f \frac{1}{M^2}\left(1 - \frac{m_f^2}{M^2}\right)^{1/2} \left[a_f + b_f x + c_f x^2 + d_f x^3 + \mathrm{O}(x^4)\right], \qquad (32)$$

where

$$a_f = C_{0f} + C_{1f} + C_{2f}, \qquad (33a)$$

$$b_f = -\tfrac{3}{2}(2C_{0f} + C_{1f}) + \tfrac{3}{4}\beta_f(C_{0f} + C_{1f} + C_{2f}), \qquad (33b)$$

$$c_f = \tfrac{3}{4}(8C_{0f} + 4C_{1f} + 5C_{2f}) - \tfrac{3}{8}\beta_f(9C_{0f} + 4C_{1f} - C_{2f}) - \tfrac{15}{32}\beta_f^2(C_{0f} + C_{1f} + C_{2f}), \qquad (33c)$$

$$d_f = -\tfrac{15}{16}(10C_{0f} + 5C_{1f} + 2C_{2f}) + \tfrac{15}{32}\beta_f(21C_{0f} + 5C_{1f} + 3C_{2f}) + \tfrac{15}{64}\beta_f^2(15C_{0f} + 8C_{1f} + C_{2f}) + \tfrac{105}{128}\beta_f^3(C_{0f} + C_{1f} + C_{2f}), \qquad (33d)$$

and $\beta_f \equiv m_f^2/(M^2 - m_f^2)$. The sum is again over the different types of particle–antiparticle pairs into which the χ's annihilate.

MM → $m_f m_f$
[illegible]

It should be noted that we have expanded $(1 - 4m_f^2/s)^{1/2}$ about $s(x=0) = 4M^2$. Clearly this is not a valid expansion if $M \simeq m_f$. In cases of interest, m_f is just the mass of an ordinary quark or lepton into which χ annihilates. Since M is at least a few GeV, we need only worry about annihilations into the tau lepton and the charm, bottom, and top quarks. For M near the mass of one of these particles, the corresponding annihilations tend to contribute very little to the total cross section. To check this we performed a calculation for $M = m_b$ using a valid expansion for the bottom quark contribution. The result was negligible compared with the terms from annihilation into lighter quarks and leptons. There is a narrow range of values of M just above m_b (and also m_c and m_τ) for which our expansion is not applicable; we estimate that this range is no more than ~ 0.2 GeV.

It is sometimes necessary to account for the finite mass of the exchanged particle to compute an accurate cross section. If the exchange is always in the s channel (as in the case of heavy neutrino annihilation through a virtual Z) eq. (31) is modified by the inclusion of a factor of $(1 - s/M_Z^2)^{-2}$. This, of course, then changes eqs. (32) and (33). The resulting corrected expressions are rather lengthy and uninformative, and we will not reproduce them here; we did, however, use them for our numerical work. If the exchange is in the t or u channels, eq. (31) itself must be replaced with a much more cumbersome expression. Also, if one considers the case of $M > M_W$ (which we do not), then one must also include annihilation into W^+W^- pairs [11].

5. Photinos, higgsinos, and Majorana neutrinos

In this paper we consider two generic cases for the χ particle: Majorana fermions and Dirac fermions. Majorana fermions include supersymmetric particles such as photinos and higgsinos. More precisely, the lightest (and therefore stable) particle in minimal supersymmetric extensions of the standard model is usually a linear combination of the fermionic partners of neutral gauge and Higgs bosons [12]. There are large regions of parameter space where it is possible to treat this lightest particle as a pure photino or higgsino, and ignore mixing effects. These cases of pure photino and higgsino are those we will analyze in detail in this paper (More general cases could be handled by computing the appropriate values of C_{0f}, C_{1f}, and C_{2f}.) In this section we will also consider the case of a Majorana neutrino as the χ particle, although there is no particular reason for a Majorana neutrino to be stable. In sect. 6 we will consider a Dirac neutrino; the same caveat applies.

The effective lagrangian for a Majorana fermion χ to interact with an ordinary quark or lepton f can be written as

$$\mathscr{L}_{\text{eff}} = \sum_f \bar{\chi}\gamma^{\mu}\gamma_5\chi \bar{f}\gamma_{\mu}(V_f + A_f\gamma_5)f , \tag{34}$$

where χ is a Majorana spinor field and f is a Dirac spinor field. Using the usual techniques* we evaluate the reduced matrix element and extract C_{0f}, C_{1f}, and C_{2f}. We find

$$C_{0f} = \frac{2}{3\pi} M^4 \left[7\xi_f A_f^2 - 2\xi_f V_f^2 \right] , \tag{35a}$$

$$C_{1f} = \frac{4}{3\pi} M^4 \left[-(2 + 2\xi_f) A_f^2 - (2 - \xi_f) V_f^2 \right] , \tag{35b}$$

$$C_{2f} = \frac{8}{3\pi} M^4 \left[A_f^2 + V_f^2 \right] , \tag{35c}$$

where $\xi_f \equiv m_f^2/M^2$ and m_f is the mass of f. These expressions must be inserted in eq. (33), and the resulting $\langle \sigma v_{\text{rel}} \rangle$ summed over quarks and leptons f with mass less than the mass of the χ particle, M.

For pure photinos, we have

$$V_f = 0 , \qquad A_f = 2\pi\alpha Q_f^2 / M_{\text{sf}}^2 , \tag{36}$$

where α is the fine structure constant, Q_f is the electric charge of f in units of the

* We can pretend that χ is a Dirac field provided that the resulting $|\mathscr{T}|^2$ is multiplied by a factor of four See ref [13] for details

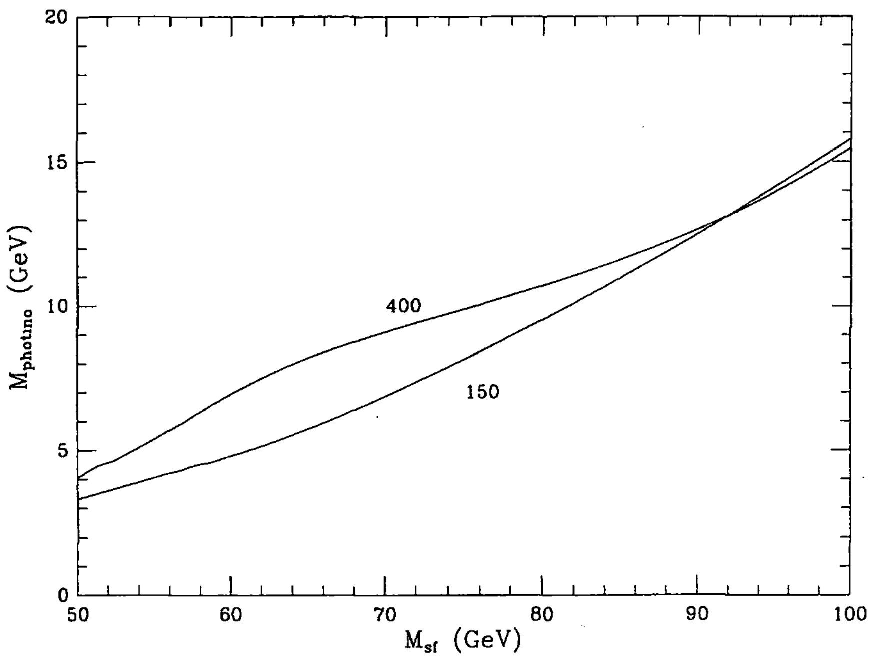

Fig 4 The photino mass needed to have $\Omega_\chi h^2 = 0.25$ versus the (common) scalar quark and lepton mass for our two assumptions about the quark–hadron transition

proton charge, and M_{sf} is the mass of the scalar partner of the quark or lepton f*. In our numerical work we did not correct eq. (31) for finite scalar masses. (The exchange is in the t channel, which leads to very messy expressions.) Furthermore, the corrections, which are of order M^2/M_{sf}^2, are no larger than those needed to correct for the mixing of the photino with other neutral gauge/Higgs fermions. Because we have not included these model-dependent effects, our results should be regarded as illustrative rather than definitive.

In fig. 4 and table 1 we have displayed the photino mass and the (common) scalar mass necessary to have $\Omega_\chi h^2 = 0.25$. We show the results for this value since $\Omega = 1$ is favored by inflation, while $h \simeq 0.5$ is favored by numerical simulations of galaxy formation when $\Omega = 1$. We show the results for the two cases of the quark–hadron transition depicted in figs. 1 and 2.

For both pure higgsinos and Majorana neutrinos, annihilation occurs through Z exchange in the s channel. In this case we have

$$V_f = \sqrt{\tfrac{1}{2}\lambda}\, G\left(T_f^{3L} - 2Q_f \sin^2\theta_W\right), \qquad A_f = \sqrt{\tfrac{1}{2}\lambda}\, G\left(-T_f^{3L}\right), \tag{37}$$

* Actually, there are separate scalar partners for the right- and left-handed parts of each quark and lepton For simplicity we assume that all scalars have a common mass M_{sf}, and that there is negligible mixing between them

where G is Fermi's constant, T_f^{3L} is the weak isospin of the left-handed part of the quark or lepton f, Q_f is the electric charge of f (in units of the proton charge), θ_W is the weak mixing angle, and λ is a numerical factor between zero and one. For higgsinos, $\lambda = \cos^2(2\alpha)$, where $\tan\alpha = V_2/V_1$ is the ratio of the two Higgs expectation values that appear in all supersymmetric extensions of the standard model. For higgsinos, there may also be contributions due to scalar exchange. We ignore these contributions (although they could alter our results if the scalar quark and lepton masses are of order M_Z or smaller) since the enlargement of the parameter space makes the presentation of the results cumbersome (see ref. [12]). For Majorana neutrinos, λ is some combination of Cabibbo-like mixing angles. We corrected eqs. (32) and (33) for the finite value of M_Z.

We show our results in two ways. First, in fig. 5 and table 2, we fix $\lambda = 1$, and display $\Omega_\chi h^2$ against the mass of the Majorana neutrino (or higgsino). Then, in fig. 6, we fix $\Omega_\chi h^2 = 0.25$, and display the needed value of λ against the mass.

6. Dirac neutrinos

In this section we consider a heavy, stable Dirac neutrino as the dark-matter particle. If such a particle is stable, it is likely that it carries a conserved quantum number and does not mix with ordinary neutrinos. The effective lagrangian for a Dirac fermion χ to interact with an ordinary quark or lepton f can be written as

$$\mathscr{L}_{\text{eff}} = \sum_f \bar{\chi}\gamma^\mu(1-\gamma_5)\chi \bar{f}\gamma_\mu(V_f + A_f\gamma_5)f, \tag{38}$$

where χ and f are both Dirac spinor fields. The constants V_f and A_f are again given by eq. (37), with $\lambda = 1$ if there is no mixing of the heavy neutrino χ with ordinary neutrinos. Using the usual techniques* we find

$$C_{0f} = \frac{1}{12\pi}M^4\left[5\xi_f A_f^2 - \xi_f V_f^2\right], \tag{39a}$$

$$C_{1f} = \frac{1}{6\pi}M^4\left[-(1+4\xi_f)A_f^2 - (1-2\xi_f)V_f^2\right], \tag{39b}$$

$$C_{2f} = \frac{2}{3\pi}M^4\left[A_f^2 + V_f^2\right], \tag{39c}$$

where $\xi_f \equiv m_f^2/M^2$. These expressions must be inserted in eq. (33) and then corrected for the finite mass of the Z.

* Note that, when χ is a Dirac fermion, our σ is *one half* of the usual spin-averaged, particle–antiparticle annihilation cross section (see the appendix)

TABLE 1

The (common) scalar quark and lepton mass needed to have $\Omega_\chi h^2 = 0\ 25$ for a given photino mass, for our two assumptions about the quark–hadron transition

$M_{\tilde\gamma}$ (GeV)	$T_c = 150$ MeV M_{sf} (GeV)	$T_c = 400$ MeV M_{sf} (GeV)
3 0	47 8	46 2
4 0	54 7	49 9
5 0	61 2	53 7
6 0	66 3	57 0
7 0	70 5	60 2
8 0	74 4	64 1
9 0	78 1	69 4
10 0	81 6	75 6
15 0	97 6	98 5
20 0	112	112

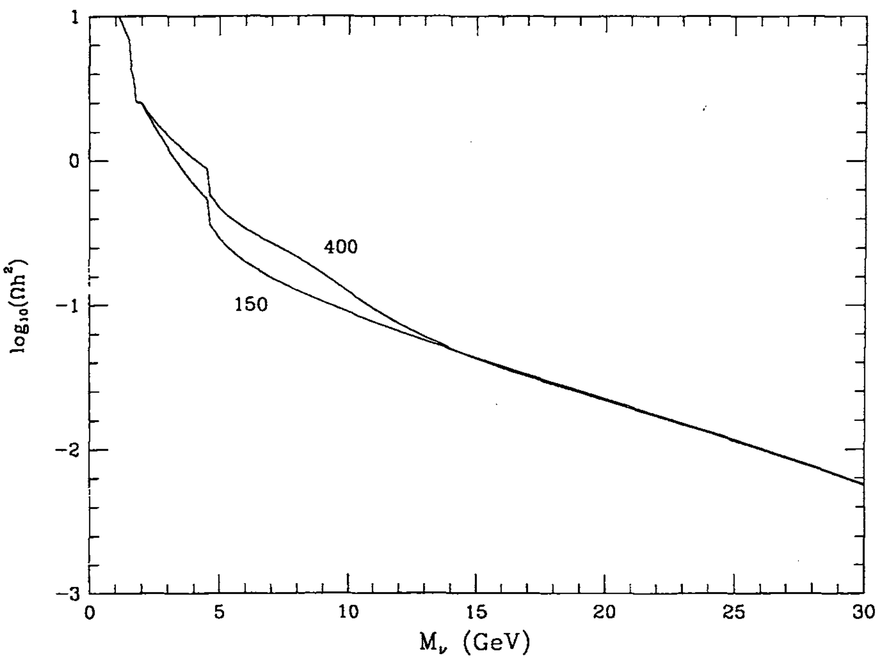

Fig 5 The value of $\Omega_\chi h^2$ for a Majorana neutrino (or higgsino) with $\lambda = 1$ versus its mass for our two assumptions about the quark–hadron transition

TABLE 2
The value of $\Omega_\chi h^2$ for a Majorana neutrino (or higgsino with $\lambda = 1$) for a given value of its mass, for our two assumptions about the quark–hadron transition

M_{ν_M} (GeV)	$T_c \simeq 150$ MeV $\Omega_\chi h^2$	$T_c \simeq 400$ MeV $\Omega_\chi h^2$
3 0	1 22	1 50
4 0	0 685	1 03
5 0	0 291	0 472
6 0	0 200	0 341
7 0	0 155	0 271
8 0	0 126	0 216
9 0	0 106	0 165
10 0	0 0893	0 124
15 0	0 0431	0 0425
20 0	0 0224	0 0217
25 0	0 0117	0 0114
30 0	0 00569	0 00560

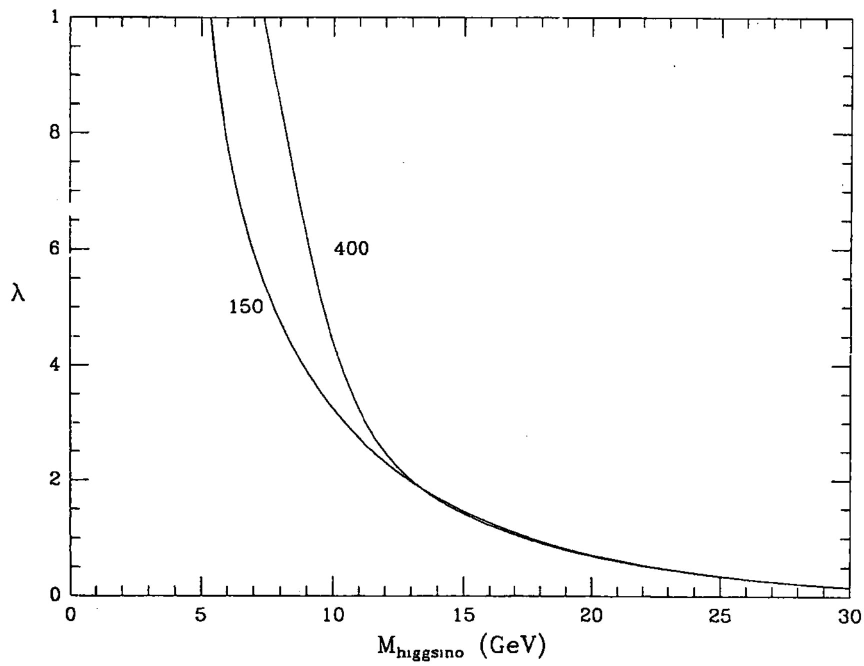

Fig 6 The value of λ versus the higgsino mass for our two assumptions about the quark–hadron transition

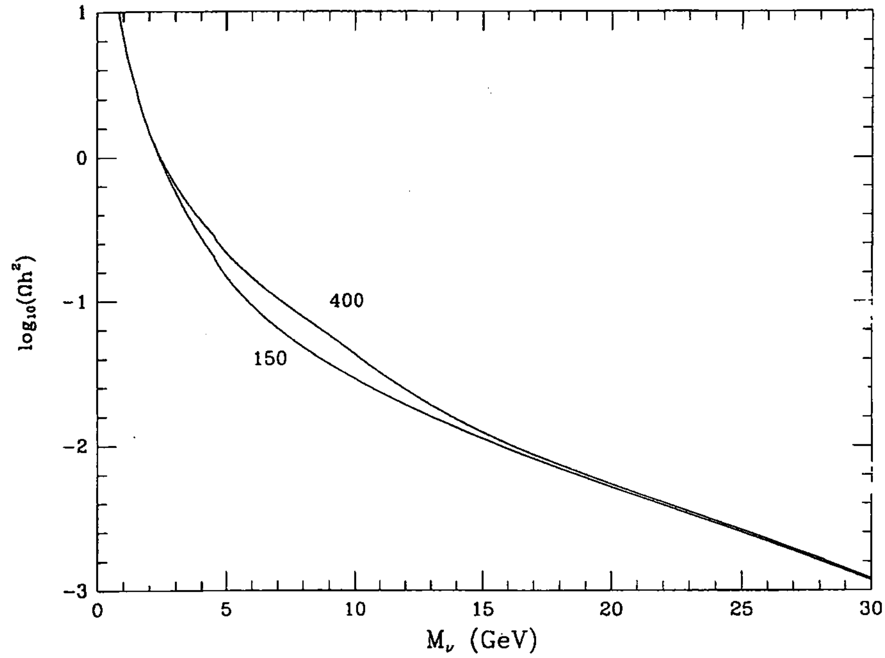

Fıg 7 The value of $\Omega_\chi h^2$ for a Dırac neutrıno versus ıts mass for our two assumptıons about the quark–hadron transıtıon

TABLE 3

The value of $\Omega_\chi h^2$ for a Dırac neutrıno for a gıven value of ıts mass, for our two assumptıons about the quark–hadron transıtıon

M_{ν_D} (GeV)	$T_c = 150$ MeV $\Omega_\chi h^2$	$T_c = 400$ MeV $\Omega_\chi h^2$
2 0	1 48	1 49
3 0	0 573	0 645
4 0	0 275	0 362
5 0	0 148	0 216
6 0	0 0933	0 145
7 0	0 0649	0 105
8 0	0 0479	0 0778
9 0	0 0368	0 0581
10 0	0 0290	0 0431
15 0	0 0112	0 0124
20 0	0 00521	0 00545
25 0	0 00254	0 00262

In fig 7 and table 3, we have displayed the value of $\Omega_\chi h^2$ against the Dirac neutrino mass for the two cases of the quark–hadron transition we have discussed.

7. Conclusions

We have presented equations whose numerical solution gives accurate estimates of the mass density of hypothetical stable particles remaining after the Big Bang. We believe that all effects at the level of one percent or larger have been included. The greatest uncertainty is from our imprecise knowledge of the equation of state of quarks and hadrons at temperatures of a few hundred million electron volts

We presented our numerical results for pure photinos, pure higgsinos, Majorana neutrinos, and Dirac neutrinos, using our two different assumptions about the quark–hadron transition. A direct consequence of these calculations is a lower limit on the mass of each of the particles we considered. For photinos, we find that $M > 4.80$ GeV (6.93 GeV) for a quark–hadron transition near 150 MeV (400 MeV) assuming that $M_{sf} > 60$ GeV [14] and $\Omega_\chi h^2 < 0.25$. This is comparable to previous results [10, 15, 12]. Clearly the largest uncertainty comes from the details of the quark–hadron transition. For Majorana neutrinos (with $\lambda = 1$), $M > 5.34$ GeV (7.37 GeV), again comparable to previous results [16, 17]. For higgsinos, there is no model-independent lower limit on the mass, since it depends on the value of λ and on the scalar quark and lepton masses (which we took to be very large). Finally, for Dirac neutrinos, $M > 4.15$ GeV (4.70 GeV), which is somewhat higher than previous estimates [1–3, 17–19], due either to differences in the cross section itself or the method of thermal averaging. Our calculations also impact on any computations of astrophysical or laboratory signatures of dark-matter candidates Our general technique should be useful when and if more concrete information on the nature of dark matter becomes available.

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Appendix

SPIN AND PARTICLE–ANTIPARTICLE AVERAGING

Suppose the χ particle comes in κ spin types, including antiparticles, so that Dirac fermions would have $\kappa = 4$ and Majorana fermions would have $\kappa = 2$. Then if n_i is the number density of the ith spin type, the contribution of annihilations to $\mathrm{d}n_i/\mathrm{d}t$ can be written as

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -\sum_{j=1}^{\kappa} \langle \sigma_{ij} v_{\rm rel} \rangle n_i n_j , \tag{40}$$

where σ_{ij} is the cross section for type i to annihilate with type j, $v_{\rm rel}$ is the relative velocity, and $\langle\ \rangle$ implies the thermal average of sect. 4.

It should be stressed that this formula is correct without any additional factors of two. For $i \neq j$, we can think of type j impinging on type i, and it should be clear that eq. (40) is correct as it stands. For $i = j$, there is an extra factor of two since two type-i particles are destroyed in each annihilation, but this is balanced by an extra factor of one half introduced to avoid double counting the number of interactions.

Summing eq. (40) over i we get

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\sum_{i,j=1}^{\kappa} \langle \sigma_{ij} v_{\rm rel} \rangle n_i n_j , \tag{41}$$

where $n \equiv \sum_{i=1}^{\kappa} n_i$ is the total number density of the χ particles The equipartition theorem tells us that $n_i = n/\kappa$ for each spin type i, so we get

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= -\frac{1}{\kappa^2}\sum_{i,j=1}^{\kappa} \langle \sigma_{ij} v_{\rm rel} \rangle n^2 \\ &= -\langle \sigma v_{\rm rel} \rangle n^2 ,\end{aligned} \tag{42}$$

where we have defined the spin and particle–antiparticle averaged cross section

$$\sigma \equiv \frac{1}{\kappa^2}\sum_{i,j=1}^{\kappa} \sigma_{ij} . \tag{43}$$

It is this cross section which appears throughout the text.

We demonstrate for Dirac and Majorana fermions. A Dirac fermion has $\kappa = 4$: two spin states for each of the fermion f and the antifermion $\bar{f}$. If f carries a conserved quantum number, then $\sigma_{ff} = \sigma_{\bar{f}\bar{f}} = 0$. Furthermore, $\sigma_{f\bar{f}} = \sigma_{\bar{f}f}$, and $\sigma_{\bar{f}f}$ represents four different cross sections, corresponding to the two possible spin states for each of f and $\bar{f}$. Thus we get

$$\begin{aligned}\sigma &= \tfrac{1}{16}\sum_{\rm spins} (\sigma_{\bar{f}f} + \sigma_{f\bar{f}} + \sigma_{ff} + \sigma_{\bar{f}\bar{f}}) \\ &= \tfrac{1}{8}\sum_{\rm spins} \sigma_{\bar{f}f} \\ &= \tfrac{1}{2}\bar{\sigma}_{\bar{f}f} ,\end{aligned} \tag{44}$$

where $\bar{\sigma}_{\bar{f}f}$ is the usual spin-averaged cross section.

Most papers about Dirac fermions as dark matter are written in terms of the density of just the fermions, $n_f = \frac{1}{2}n$. Thus

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 2\frac{\mathrm{d}n_f}{\mathrm{d}t} = -\tfrac{1}{2}\langle\bar{\sigma}_{\bar{f}f}v_{\mathrm{rel}}\rangle(2n_f)^2, \tag{45}$$

so that

$$\frac{\mathrm{d}n_f}{\mathrm{d}t} = -\langle\bar{\sigma}_{\bar{f}f}v_{\mathrm{rel}}\rangle n_f^2, \tag{46}$$

which is the usual form of the rate equation for Dirac fermions in the literature. (After solving for n_f, the answer must be multiplied by two to get the relic density of the fermions *and* antifermions.)

For Majorana fermions there are no antiparticles, so $\kappa = 2$. In this case we have

$$\begin{aligned}\frac{1}{\kappa^2}\sum_{i,j=1}^{\kappa}\sigma_{ij} &= \tfrac{1}{4}\sum_{\text{spins}}\sigma_{ff}\\ &= \bar{\sigma}_{ff}\end{aligned} \tag{47}$$

where $\bar{\sigma}_{ff}$ is the usual spin-averaged cross section. This again reproduces the equation used in earlier work However, one should be aware of extra factors of two involved in calculating cross sections with Majorana spinors. These are thoroughly explained in ref. [13].

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Three exceptions in the calculation of relic abundances

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The calculation of relic abundances of elementary particles by following their annihilation and freeze-out in the early Universe has become an important and standard tool in discussing particle dark-matter candidates. We find three situations, all occurring in the literature, in which the standard methods of calculating relic abundances fail. The first situation occurs when another particle lies near in mass to the relic particle and shares a quantum number with it. An example is a light squark with neutralino dark matter. The additional particle must be included in the reaction network, since its annihilation can control the relic abundance. The second situation occurs when the relic particle lies near a mass threshold. Previously, annihilation into particles heavier than the relic particle was considered kinematically forbidden, but we show that if the mass difference is $\sim 5-15$ %, these "forbidden" channels can dominate the cross section and determine the relic abundance. The third situation occurs when the annihilation takes place near a pole in the cross section. Proper treatment of the thermal averaging and the annihilation after freeze-out shows that the dip in relic abundance caused by a pole is not nearly as sharp or deep as previously thought.

I. INTRODUCTION

The calculation of the present-day density of elementary particles which were in thermal equilibrium in the early Universe has become quite commonplace.[1] Of particular interest is the so-called Lee-Weinberg[2,3] calculation in which annihilation after a particle species has become nonrelativistic determines the present-day abundance of that species. Standard approximate solutions to the Boltzmann equation exist for this calculation and have been tested numerically. In this paper we wish to point out three cases where naive application of the standard methods fails to give correct results and a modified treatment is required. All three cases exist in the literature, and in all three cases erroneous conclusions have been drawn. For each case we present appropriate approximate solutions to the Boltzmann equation(s) and describe the values of the parameters for which they apply.

The first case occurs when the relic particle is the lightest of a set of similar particles whose masses are nearly degenerate. In this case the relic abundance of the lightest particle is determined not only by its annihilation cross section, but also by the annihilation of the heavier particles, which will later decay into the lightest. We call this the case of "coannihilation." As an example, consider a supersymmetric theory in which the scalar quarks or scalar electrons are only slightly more massive than the lightest supersymmetric particle (LSP), usually taken to be a neutralino. Previous calculations of the relic abundance which consider only the LSP annihilation can be in error by more than two orders of magnitude.

The second case concerns annihilation into particles which are more massive than the relic particle. Previous treatments regarded this as kinematically forbidden, but we show that if the heavier particles are only 5–15% more massive, these channels can dominate the annihilation cross section and determine the relic abundance. We call this the "forbidden" channel annihilation case. Examples include annihilation into $b\bar{b}$, $t\bar{t}$, W^+W^-, or Higgs bosons, when the annihilating particle is lighter than the final-state particle.

The third case occurs when the annihilation takes place near a pole in the cross section. This happens, for example, in Z^0-exchange annihilation when the mass of the relic particle is near $m_Z/2$. Previous treatments have incorrectly handled the thermal averages and the integration of the Boltzmann equation in these situations. The dip in relic abundance caused by a pole is broader and not nearly as deep as previous treatments imply.[4]

For all three cases we present simple formulas which allow for a more correct treatment. We also present examples for each case and describe the precise conditions under which the modified treatment is necessary. In Sec. II we review the standard method for performing the Lee-Weinberg calculation and describe the approximations within which we will work. In Sec. III we discuss the coannihilation case, in Sec. IV we discuss the forbidden channel case, and in Sec. V we discuss annihilation near a pole.

II. STANDARD CALCULATION OF RELIC ABUNDANCE

Here we summarize the standard technique for calculating the relic abundance of a particle species χ in the Lee-Weinberg scenario. First, a note about the philoso-

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phy of this presentation. We are considering cases where the "standard" technique fails by a factor of 2 or more, and so we wish to highlight the modifications necessary to avoid these gross errors. Thus we are not overly concerned with 10–20% corrections, and our presentation will not emphasize some recent improvements[3] which make the calculations more precise, but more cumbersome.

The relic abundance is found by solving the Boltzmann equation for the evolution of the χ number density n:

$$\frac{dn}{dt}=-3Hn-\langle\sigma v\rangle[n^2-(n^{\mathrm{eq}})^2]\ , \tag{1}$$

where H is the Hubble parameter, n^{eq} is the χ equilibrium number density, v is the "relative velocity," and $\langle\sigma v\rangle$ is the thermal average of the annihilation cross section ($\chi+\chi\to$all). For the equilibrium number density, we will use the nonrelativistic approximation $n^{\mathrm{eq}}\approx g(mT/2\pi)^{3/2}\exp(-m/T)$, where g is the number of spin, color, etc., degrees of freedom of χ, T is the temperature, and m is the mass of the relic. For particles which may potentially play the role of cold dark matter, the relevant temperatures are of order $m/25$ and the nonrelativistic equilibrium abundance is well justified. Typically,[5] the cross section is Taylor expanded in v^2 before the thermal average is taken, $\sigma=a+bv^2$. The thermal average then shows a linear temperature dependence $\langle\sigma v\rangle=a+6bT/m$.

Equation (1) can be solved numerically, but a convenient and accurate approximation exists. At early times n is approximated by n^{eq}, but as the temperature drops below the mass m, n^{eq} drops exponentially, and eventually a point, denoted "freeze-out," is reached where the reaction rate is not fast enough to maintain equilibrium. From this point on, the n^{eq} term in Eq. (1) can be ignored and the resulting equation is easily integrated. Thus the solution of Eq. (1) is given by solving in two regimes and matching those solutions at the freeze-out point.

For ease of presentation we will follow the method detailed in Ref. 1. The freeze-out point is given in terms of the scaled inverse temperature $x=m/T$:

$$x_f=\ln\frac{0.038gm_{\mathrm{Pl}}m\langle\sigma v\rangle}{g_*^{1/2}x_f^{1/2}}\ , \tag{2}$$

where $m_{\mathrm{Pl}}=1.22\times10^{19}$ GeV and g_* is the total number of effectively relativistic degrees of freedom at the time of freeze-out. Equation (2) is usually solved iteratively. In Ref. 1 this is done analytically by substituting for x_f on the right side of Eq. (2). For the cases considered in this paper, the thermal averaged cross section changes rapidly with temperature and we will take a numerical approach to solving Eq. (2).

At freeze-out the abundance of relic particles is usually taken to be the equilibrium abundance; however, after freeze-out there is further significant annihilation of relic particles which reduces the abundance to its final and present value. The efficiency of this post-freeze-out annihilation is expressed through the integral J:

$$J(x_f)=\int_{x_f}^{\infty}\frac{\langle\sigma v\rangle}{x^2}dx\ . \tag{3}$$

The present-day mass density of χ particles is then given by

$$\Omega h^2\approx\frac{1.07\times10^9\ \mathrm{GeV}^{-1}}{Jg_*^{1/2}m_{\mathrm{Pl}}}\ , \tag{4}$$

where Ω is the present-day mass density divided by the critical density for closure and h is the present day Hubble parameter[6] in units of 100 km s^{-1} Mpc^{-1}. For the standard scenarios where $\langle\sigma v\rangle$ is expressed in terms of its Taylor expansion, the present-day abundance of χ particles is approximately

$$\Omega h^2=\frac{1.07\times10^9x_f}{g_*^{1/2}m_{\mathrm{Pl}}(\mathrm{GeV})(a+3b/x_f)}\ . \tag{5}$$

We now list the known weaknesses with the calculation as summarized above. First, as already mentioned, the derivation assumes the nonrelativistic approximation in calculating equilibrium abundances. It also assumes a nonrelativistic approach in calculating thermal average cross sections, and as pointed out in Ref. 3, this may lead to 10% errors. Next, the derivation assumes that g_* remains constant throughout the period of annihilation, which is not generally correct. The approximation is worst when freeze-out occurs during the quark-hadron transition, but even in this case the errors[3,7] are not large. Except where noted, we will ignore these small omissions for the remainder of this paper.

Potentially more problematic, for our discussion, is the determination of x_f or the matching point for the two regimes where the Boltzmann equation may be easily solved. The value of x_f is derived by assuming that at early times $n\approx n^{\mathrm{eq}}$. It is then simple to solve for the difference $\delta=n-n^{\mathrm{eq}}$. The freeze-out point is given by when $\delta=cn^{\mathrm{eq}}$, where the constant c is determined empirically by comparing to some numerical integrations of the Boltzmann equation. For annihilation cross sections that are relatively temperature insensitive, the choice of c is not critical. A convenient choice[8] for thermal-averaged cross sections with power-law temperature dependence, $\langle\sigma v\rangle\sim T^n$, is $c(c+2)=n+1$. The value of x_f given in Eq. (2) uses $n=0$, $c=\sqrt{2}-1$.

For the cases discussed later in this paper, the question of matching and determining of x_f may have to be readdressed since the cross sections we consider have much stronger temperature dependences than simple power laws. Thus, whereas a small error in x_f usually makes little difference to the annihilation integral J, in our case it may. With this in mind, for most of our paper we present results based on the use of Eq. (2) ($c=\sqrt{2}-1$); however, we will show some explicit numerical integrations of the Boltzmann equation to show that this choice does not lead to large errors.

The plan for the remainder of this paper is now straightforward. (a) Put the problem in the form of Eq. (1). (b) Solve for x_f using Eq. (2) and an appropriate $\langle\sigma v\rangle$. (c) Evaluate the annihilation integral J and plug

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Simple way of calculating the cosmological relic density

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A simple procedure is presented which leads to a dramatic simplification in the calculation of the relic density of stable particles in the Universe.

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Any stable species χ contributes to the total mass-energy density in the Universe. If its number density cannot be reduced efficiently enough before it decouples from the thermal equilibrium, its relic abundance $\Omega_\chi h_0^2$ can be sizable and it can affect the evolution and the age of the Universe. A conservative estimate that the Universe is at least 10×10^9 years old requires $\Omega_\chi h_0^2<\Omega_{\rm tot}h_0^2<1$ [1]. Furthermore, there is growing evidence for dark matter at both galactic and larger length scales [1] which would most likely require the existence of some type of exotic neutral particles. Since such particles are often present in many theories beyond the standard model, it is important to develop a simple practical procedure for calculating their relic abundance with enough precision.

Two groups [2, 3] have developed equivalent frameworks for properly calculating the relic density of χ's, including corrections due to thermally averaging the momentum of each incoming particle separately. (In Ref. [1] a common heat bath is assumed for both incoming particles – see the *Note added.*) The method of Ref. [2] is in practice applicable away from poles and new final-state thresholds, which is most often the case. In Ref. [3] also the vicinity of poles and thresholds has been carefully studied. Essentially, one needs to calculate the thermally averaged product of the $\chi\bar{\chi}$ annihilation cross section and their relative velocity $\langle\sigma v_{\rm rel}\rangle$. In practice, one expands $\langle\sigma v_{\rm rel}\rangle=a+bx+O(x^2)$ in powers of $x\equiv T/m_\chi\sim1/20$ in order to avoid difficult numerical integrations and approximates $\langle\sigma v_{\rm rel}\rangle$ by a and b. Both techniques give equivalent results [3, 4] in the overlapping region (away from poles and thresholds).

Unfortunately, in practice the actual calculation of even the first two terms of the expansion is typically very complicated and tedious. In this paper I report on a dramatic technical simplification in practical applications of the method of Ref. [2].

Consider an annihilation of particles χ, $\bar{\chi}$ into a two-body final state. Furthermore, in many cases of interest, the final-state particles have an equal mass m_F. (A general case of unequal final-state masses will be presented elsewhere [5].) Let the momenta of the two initial states χ, $\bar{\chi}$ and the two final states F, $\bar{F}$ be p_1, p_2 and k_1, k_2, respectively. Srednicki *et al.* [2] introduce the function $w(s)$ defined as

$$w(s)\equiv\frac{1}{4}\int dX_{\rm LIPS}|\overline{\mathcal{M}}|^2=E_1E_2\,\sigma v_{\rm rel},\tag{1}$$

where $dX_{\rm LIPS}$, the Lorentz-invariant phase space, in this case takes the form

$$dX_{\rm LIPS}=(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2},\tag{2}$$

where $E_{1,2}=\mathbf{k}_{1,2}^2+m_F^2$, and $|\overline{\mathcal{M}}|^2$ is the square of the reduced matrix element for the annihilation process $\bar{\chi}\chi\to\bar{F}F$ summed over the spins of the final-state particles and averaged over the spins of the initial particles [2].

Denote

$$f\equiv|\overline{\mathcal{M}}|^2.\tag{3}$$

In general, $f=f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_1,\mathbf{k}_2)$. The integral Eq. (1) can be conveniently evaluated in the center-of-mass frame (c.m.) in which $\mathbf{p}_2=-\mathbf{p}_1$ and $\mathbf{k}_2=-\mathbf{k}_1$. After a few elementary steps one obtains

$$w(s)=\frac{1}{2^6\pi}\sqrt{1-\frac{4m_F^2}{s}}\times\int_{-1}^{1}d\cos\theta\, f\left(\mathbf{k}_2=-\mathbf{k}_1,|\mathbf{k}_1|=\sqrt{s/4-m_F^2}\right).\tag{4}$$

Using the kinematic relation between the Mandelstam variables

$$t=(p_1-k_1)^2=m_\chi^2+m_F^2-\frac{s}{2}\left[1-\sqrt{1-\frac{4m_\chi^2}{s}}\sqrt{1-\frac{4m_F^2}{s}}\cos\theta\right],\tag{5}$$

one can express Eq. (4) as

$$w(s)=\frac{1}{2^5\pi}\frac{1}{s\sqrt{1-4m_\chi^2/s}}\int_{t_-(s)}^{t_+(s)}dt\,f(s,t),\tag{6}$$

where $t_\pm\equiv t(\cos\theta=\pm1)=t_0\pm\Delta t$,

$$t_0(s)=m_\chi^2+m_F^2-\frac{s}{2},\tag{7}$$

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and

$$\Delta t(s) = \frac{s}{2}\sqrt{1-\frac{4m_\chi^2}{s}}\sqrt{1-\frac{4m_F^2}{s}}. \tag{8}$$

Notice that one can always express f as a function of the Mandelstam variables s and t only [2]. In particular, u can be eliminated by using the relation $s+t+u = 2m_\chi^2+2m_F^2$. Furthermore, in calculating the relic density it is convenient to introduce [2]

$$z \equiv \frac{s}{4m_\chi^2}, \tag{9}$$

in terms of which Eq. (6) can be simply rewritten as

$$w(z) = \frac{1}{2^7\pi m_\chi^2}\frac{1}{\sqrt{z}\sqrt{z-1}}\int_{t_-(z)}^{t_+(z)} dt\, f(z,t). \tag{10}$$

In order to calculate the relic abundance of χ's one needs to solve the Boltzmann (rate) equation. The actual quantity that appears in the rate equation is the thermally averaged product $\langle\sigma v_{\rm rel}\rangle$, which is usually approximated by $a+bx$, as mentioned above. One of the main results of Srednicki *et al.* [2] was to show that

$$a = \frac{1}{m_\chi^2}w(z=1), \tag{11}$$

$$b = \frac{1}{m_\chi^2}\left[\frac{3}{2}\frac{dw(z=1)}{dz} - 3w(z=1)\right]. \tag{12}$$

While these formulas look deceptively simple, the actual calculations can be, and in practice *are*, very cumbersome and often virtually unmanageable. For example, in a relatively simple case of calculating the interference term between two t-channel amplitudes (with the masses of the exchanged particles denoted by μ_1 and μ_2) one needs to evaluate several integrals of the type $\int_{t_-}^{t_+} dt\, t^n(t-\mu_1^2)^{-1}(t-\mu_2^2)^{-1}$, and in general in calculating $w(z)$ one has to compute a whole multitude of more complicated integrals. Once such a (lengthy) expression for $w(z)$ is found, one needs to next take a derivative dw/dz which in general leads to even more cumbersome formulas. Additional highly nontrivial computational complications arise when the masses of the final-state particles are not equal.

Below I show that a great deal of these difficulties can be avoided. In fact, one can avoid performing any integrals *completely*. First, one can always conveniently express [2] a and b in terms of "reduced" variables a^0 and b^0:

$$a = \sqrt{1-m_F^2/m_\chi^2}\, a^0, \tag{13}$$

$$b = \sqrt{1-m_F^2/m_\chi^2}\left\{\left[-3+\frac{3}{4}\frac{m_F^2/m_\chi^2}{(1-m_F^2/m_\chi^2)}\right]a^0 + b^0\right\}. \tag{14}$$

I will show below that a^0 and b^0 can be written in a simple and elegant form as

$$a^0 = \frac{1}{2^5\pi m_\chi^2}f(z=1), \tag{15}$$

$$b^0 = \frac{1}{2^5\pi m_\chi^2}\left[-3m_\chi^2\frac{\partial f(z=1)}{\partial t} + m_\chi^2(m_\chi^2-m_F^2)\frac{\partial^2 f(z=1)}{\partial t^2} + \frac{3}{2}\frac{\partial f(z=1)}{\partial z}\right], \tag{16}$$

and $f(z=1)$ should be understood as $f(z=1, t_0(z=1))$, etc. Equations (15) and (16) are the main result of this paper. They can also be readily generalized to the case of unequal final-state masses [5]. Higher order terms of the expansion can also be easily derived. It is clear that the whole procedure of calculating a^0 and b^0 has now been reduced to merely writing down $|\overline{\mathcal{M}}|^2$, substituting all the variables in $|\overline{\mathcal{M}}|^2$ in terms of z and t, and next taking a few relatively simple derivatives. In fact, one can easily do all these steps entirely with the help of any advanced algebraic program. The truly difficult part of computing a^0 and b^0—performing complicated integrations in deriving $w(z)$—has been completely eliminated.

In order to prove Eqs. (15) and (16), notice that for any regular function $g(z,t(z))$ and its integrand $G(z) = \int dt\, g(z,t(z))$ one can show that

$$\lim_{z\to1}\left[\frac{1}{\sqrt{z-1}}G(z)|_{t_-(z)}^{t_+(z)}\right] = \lim_{z\to1}\left[\frac{1}{\sqrt{z-1}}\int_{t_-(z)}^{t_+(z)} dt\, g(z,t(z))\right]$$

$$g(z=1,t_0(z=1)) = 4m_\chi^2\sqrt{1-m_F^2/m_\chi^2}\, g(z=1,t_0(z=1)), \tag{17}$$

where I have used $\lim_{z\to1}t_\pm = t_0(z=1)$, $\lim_{z\to1}\Delta t = 2m_\chi^2\sqrt{1-m_F^2/m_\chi^2}\sqrt{z-1}$, and

$$\lim_{z\to1}G(z,t_\pm) = \lim_{z\to1}\Big[G(z,t_0) \pm g(z,t_0)\Delta t + \frac{1}{2}\frac{\partial g(z,t_0)}{\partial t}(\Delta t)^2 \pm \frac{1}{6}\frac{\partial^2 g(z,t_0)}{\partial t^2}(\Delta t)^3 + O((\Delta t)^4)\Big]. \tag{18}$$

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Cosmological heavy-neutrino problem

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A new method of deriving the Lee-Weinberg equation is presented. This method makes the approximations clear and suggest how they can be improved. A nearly exact formula for the number of heavy neutrinos left over after the annihilation process is given.

I. INTRODUCTION

An important problem that arises in determining the present particle content of the Universe is the calculation of the number of particles of each species that have survived the creation and annihilation processes that took place in the early Universe.

In 1977 Lee and Weinberg (LW) wrote a seminal paper[1] on the heavy-lepton content of the Universe. The starting point of their discussion, their Eq. (2), is an equation governing the annihilation of heavy neutral leptons $L,\bar{L}$ into light leptons ($\nu\bar{\nu},e\bar{e},\ldots$) in an expanding space-time:

$$\frac{dn}{dt}=-3\frac{\dot{R}}{R}n-\langle\sigma v\rangle(n^2-n_0{}^2)\ . \tag{1}$$

The notation in this formula, which was written down without derivation, is as follows: n is the number of heavy leptons per unit volume with n_0 the corresponding number in thermal equilibrium, R is the universal expansion factor, and $\langle\sigma v\rangle$ is a thermally averaged annihilation rate which LW do not define precisely. The heart of the LW paper is the numerical and approximate analytical solution to this formula. This analysis has the important conclusion that if a heavy lepton has a mass greater than a few MeV, then unless its mass $M\geq 2$ GeV, the present heavy-lepton number density is so high that the energy density exceeds the observed bound derived from the expansion rate of the Universe. While Lee and Weinberg do not imply that their formula (1) is *ab initio* exact, subsequently it has entered the literature[2] as the "exact" Boltzmann equation for the process. In this paper we shall show precisely how formula (1) arises from a justified approximation to the correct Boltzmann equation. We shall also present a simple and accurate approximate solution to the formula. We have chosen the LW process as a paradigm. At the end of the paper we shall describe briefly other processes to which the same methods are applicable.

II. BOLTZMANN EQUATION

To simplify our exposition, we shall explicitly work in a spatially flat universe described by the Robertson-Walker interval

$$ds^2=R(t)^2d\mathbf{x}^2-dt^2\ , \tag{2}$$

but it should be clear that our results also apply when space is curved in a homogeneous, isotropic manner. We describe the momentum of particles by the momentum vector $\mathbf{p}$ measured in a local Lorentz frame. Thus, for example, the energy and momentum of the heavy leptons are related by

$$E(p)=(\mathbf{p}^2+M^2)^{1/2}\ . \tag{3}$$

In the absence of collisions, the motion along a geodesic is described by

$$\frac{d\mathbf{p}}{d\lambda}=-\frac{\dot{R}}{R}E\mathbf{p}\ , \tag{4}$$

where the overdot denotes a derivative with respect to the comoving time t, and where λ is an affine parameter which is conveniently chosen to give

$$E=\frac{dt}{d\lambda}\ . \tag{5}$$

The Boltzmann equation describes the time evolution of the phase-space density $f(p,t)$ of the heavy leptons. (We assume, of course, that this density is homogeneous and isotropic so that it depends only upon $p=|\mathbf{p}|$ and t.) In the absence of collisions, the phase-space density is constant along a particle's world line,

$$f(p(\lambda+d\lambda),t(\lambda+d\lambda))=f(p(\lambda),t(\lambda))\ . \tag{6}$$

In view of Eqs. (4), (5), and (6) it is easy to see that, taking account of collisions, we have the well-known covariant Boltzmann equation

$$\left[E\frac{\partial}{\partial t}-\frac{\dot{R}}{R}E\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}}\right]f(p,t)=C(p,t)\ . \tag{7}$$

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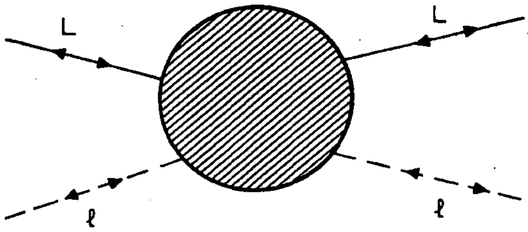

FIG. 1. Elastic collisions with light leptons (l) maintain the heavy-lepton (L) phase-space distribution in mechanical thermal equilibrium at temperature $T(t)$.

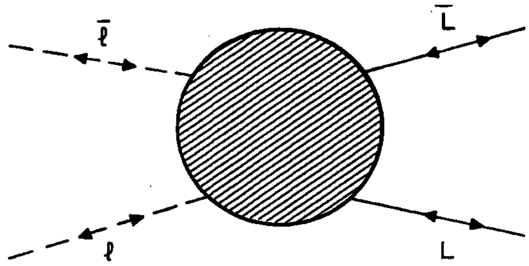

FIG. 2. Inelastic collisions maintain the chemical composition of the heavy leptons—their overall number density—at the thermal equilibrium value until the temperature $T(t)$ falls well below the heavy-lepton mass M.

The collision term $C(p,t)$ has two essentially different pieces: an elastic part $C_E(p,t)$ corresponding to the processes depicted generically in Fig. 1 and an inelastic part $C_I(p,t)$ corresponding to the processes depicted generically in Fig. 2,

$$C(p,t)=C_E(p,t)+C_I(p,t)\ . \tag{8}$$

The elastic part is a sum of terms each of the form

$$\begin{aligned} C_E(p,t)=\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E(p')}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3q'}{(2\pi)^3}\frac{1}{2q'^0}&(2\pi)^4\delta^{(4)}(p+q-p'-q')\,|\,T_E\,|^2\\ &\times\{[1-f(p,t)][1-g(q,t)]f(p',t)g(q',t)\\ &-[1-f(p',t)][1-g(q',t)]f(p,t)g(q,t)\}\ . \end{aligned}\tag{9}$$

Here the $g(q,t)$ is the phase-space density of a light lepton which is kept in thermal equilibrium at temperature $T(t)$ by other collision processes which are very rapid in comparison to the expansion rate $\dot{R}/R$. In the epoch which concerns us, the temperature $T(t)$ is much larger than any light-lepton mass so that we can take $q^0=|\,\mathbf{q}\,|$ and

$$g(q,t)=\frac{1}{\exp[q^0/T(t)]+1}\ . \tag{10}$$

The effect of Fermi statistics is accounted for in Eq. (9) by the Pauli blocking factor $[1-f][1-g]$. Since the density of light leptons is large, the elastic collision integrals (9) give rise to a relaxation time τ_E that is much shorter than the expansion time $R/\dot{R}$. Hence the elastic collisions keep the heavy-lepton phase-space density close to (kinetic) thermal equilibrium. To a good approximation, discussed in Appendix A, this implies that $f(p,t)$ is given by the "equilibrium" distribution

$$f(p,t)\simeq f_0(p,t)=\frac{1}{\exp\left[\alpha(t)+\dfrac{E(p)}{T(t)}\right]+1}\ . \tag{11}$$

When $f_0(p,t)$ is substituted into Eq. (9), $C_E(p,t)$ vanishes for any $\alpha(t)$, where $\alpha(t)$ is a time-dependent effective chemical potential, which must be present in $f(p,t)$ to allow the chemical composition, which is not determined by the elastic collisions, to vary with time.

The distribution $\bar{f}(p,t)$ of the antiparticles $\bar{L}$ has exactly the same form (11) but with $\alpha(t)$ replaced with $\bar{\alpha}(t)$. We assume, with LW, that in the beginning $\alpha(t)=\bar{\alpha}(t)=0$. Since the evolution equations for $f(p,t)$ and $\bar{f}(p,t)$ are identical, at later times we have $\alpha(t)=\bar{\alpha}(t)$.

Upon substituting f_0 of Eq. (11) for f in the inelastic collision integral, we obtain a sum of terms of the form

$$\begin{aligned} C_I(p,t)=\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E(\bar{p})}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{2\bar{q}^0}&(2\pi)^4\delta^{(4)}(p+\bar{p}-q-\bar{q})\,|\,T_I\,|^2\\ &\times\{[1-f_0(p,t)][1-f_0(\bar{p},t)]g(q,t)g(\bar{q},t)\\ &-[1-g(q,t)][1-g(\bar{q},t)]f_0(p,t)f_0(\bar{p},t)\}\ . \end{aligned}\tag{12}$$

The time development of the number density and hence of the "chemical potential" $\alpha(t)$ is obtained by dividing the

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TASI LECTURES ON DARK MATTER*

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Observational evidence and theoretical motivation for dark matter are presented and connections to the CMB and BBN are made. Problems for baryonic and neutrino dark matter are summarized. Emphasis is placed on the prospects for supersymmetric dark matter.

1. Lecture 1

The nature and identity of the dark matter of the Universe is one of the most challenging problems facing modern cosmology. The problem is a long-standing one, going back to early observations of mass-to-light ratios by Zwicky[1]. Given the distribution (by number) of galaxies with total luminosity L, $\phi(L)$, one can compute the mean luminosity density of galaxies

$$\mathcal{L} = \int L\phi(L)dL \tag{1}$$

which is determined to be[2]

$$\mathcal{L} \simeq 2 \pm 0.2 \times 10^8 h_o L_\odot Mpc^{-3} \tag{2}$$

where $L_\odot = 3.8 \times 10^{33}$ erg s^{-1} is the solar luminosity. In the absence of a cosmological constant, one can define a critical energy density, $\rho_c = 3H^2/8\pi G_N = 1.88 \times 10^{-29} {h_o}^2$ g cm^{-3}, such that $\rho = \rho_c$ for three-space curvature $k = 0$, where the present value of the Hubble parameter has been defined by $H_o = 100 h_o$ km Mpc^{-1} s^{-1}. We can now define a critical mass-to-light ratio is given by

$$(M/L)_c = \rho_c/\mathcal{L} \simeq 1390 h_o (M_\odot/L_\odot) \tag{3}$$

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which can be used to determine the cosmological density parameter

$$\Omega_m = \frac{\rho}{\rho_c} = (M/L)/(M/L)_c \tag{4}$$

Mass-to-light ratios are, however, strongly dependent on the distance scale on which they are determined[3]. In the solar neighborhood $M/L \simeq 2 \pm 1$ (in solar units), yielding values of Ω_m of only $\sim .001$. In the bright central parts of galaxies, $M/L \simeq (10-20)h_o$ so that $\Omega_m \sim 0.01$. On larger scales, that of binaries and small groups of galaxies, $M/L \sim (60-180)h_o$ and $\Omega_m \simeq 0.1$. On even larger scales, that of clusters, M/L may be as large as $(200-500)h_o$ giving $\Omega_m \simeq 0.3$. This progression in M/L seems to have halted, as even on the largest scales observed today, mass-to-light ratios imply values of $\Omega_m \lesssim 0.3-0.4$. Thus when one considers the scale of galaxies (and their halos) and larger, the presence of dark matter (and as we shall see, non-baryonic dark matter) is required.

Direct observational evidence for dark matter is found from a variety of sources. On the scale of galactic halos, the observed flatness of the rotation curves of spiral galaxies is a clear indicator for dark matter. There is also evidence for dark matter in elliptical galaxies, as well as clusters of galaxies coming from the X-ray observations of these objects. Also, direct evidence has been obtained through the study of gravitational lenses. On the theoretical side, we predict the presence of dark matter (or dark energy) because 1) it is a strong prediction of most inflation models (and there is at present no good alternative to inflation) and 2) our current understanding of galaxy formation requires substantial amounts of dark matter to account for the growth of density fluctuations. One can also make a strong case for the existence of non-baryonic dark matter in particular. The recurrent problem with baryonic dark matter is that not only is it very difficult to hide baryons, but given the amount of dark matter required on large scales, there is a direct conflict with primordial nucleosynthesis if all of the dark matter is baryonic. In this first lecture, I will review the observational and theoretical evidence supporting the existence of dark matter.

1.1. *Observational Evidence*

Assuming that galaxies are in virial equilibrium, one expects that one can relate the mass at a given distance r, from the center of a galaxy to its rotational velocity by

$$M(r) \propto v^2 r/G_N \tag{5}$$

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COSMIC ABUNDANCES OF STABLE PARTICLES: IMPROVED ANALYSIS

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An exact relativistic single-integral formula for the thermal average of the annihilation cross section times velocity, the key quantity in the determination of the cosmic relic abundance of a species, is obtained. Since it does not require expansion of the cross section at low relative velocity, it can also be used when the cross section varies rapidly with energy, e.g. near the formation of a resonance or the opening of a new annihilation channel. We discuss approximate formulas in these cases, and we find that dips in the relic density near resonances are significantly broader and shallower than previously thought and that spurious reductions near thresholds disappear.

1. Introduction

We will examine relativistic effects on the thermal average of the annihilation cross section times the "relative velocity" $\langle \sigma v \rangle$, the key quantity in the determination of the cosmic relic abundance of a species.

For non-relativistic gases, the thermally averaged annihilation cross section $\langle \sigma v \rangle$ has sometimes been approximated with the value of σv at $s = \langle s \rangle$, the thermally averaged center-of-mass squared energy, and the expansion $\langle s \rangle = 4m^2 + 6mT$ has been taken. Such an approximation is good only when σv is almost linear in s, i.e. unfortunately almost never. A better approximation for non-relativistic gases is to expand $\langle \sigma v \rangle$ in powers of $x^{-1} \equiv T/m$. A common way of doing this (cf. e.g. ref. [1]) is to write $s = 4m^2 + m^2 v^2$, expand σv in powers of v^2 and

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take the thermal average to obtain

$$\begin{aligned}\langle\sigma v\rangle &= \langle a + bv^2 + cv^4 + \dots\rangle \\ &= a + \tfrac{3}{2}bx^{-1} + \tfrac{15}{8}cx^{-2} + \dots . \end{aligned} \tag{1.1}$$

Most species are not completely non-relativistic at decoupling: when x is of order 20–25 (a typical value at freeze-out for weakly interacting particles) the mean rms velocity of the particles is of order $c/4$, and relativistic corrections of order 5–10% are expected. The relativistic expansion in powers of x^{-1} was first found in ref. [2].

All these formulations are based on the expansion of the cross section σ in powers of the "relative velocity" v. They become inappropriate either when the cross section is poorly approximated by its expansion, as near the formation of a resonance, or when its expansion diverges, as at the opening of a new annihilation channel.

We will propose a single-integral formula for $\langle\sigma v\rangle$ (eq. (3.8) below) valid for all temperatures $T \lesssim 3m$, which does not require expansion of the cross section and which can be evaluated without the numerical difficulties associated with multiple integrals.

We will begin in sect. 2 by clarifying what the "relative velocity" v appearing in $\langle\sigma v\rangle$ is in the relativistic context. For this purpose, we will briefly recall the derivation of the Boltzmann equation for the evolution of the spatial density of a species in the early universe. We will then derive the general single-integral formula for $\langle\sigma v\rangle$ in sect. 3 and we will compare the expansions in powers of the temperature of the relativistic and non-relativistic thermal averages. The relativistic corrections are important only when a precision of a few per cent is required. If so, the Boltzmann equation should be solved at least with the same precision. This can be done either numerically or by an analytic approximation. In sects. 4 and 5 we will present a sufficiently precise relativistic generalization of the usual non-relativistic approximation to the solution of the Boltzmann equation. In particular, sect. 4 is devoted to the effective degrees of freedom in the early universe. We will then discuss the non-relativistic treatment of the thermal average in the presence of a resonance (sect. 6) and near the threshold of a new annihilation channel (sect. 7). Finally, we will present a specific example of resonances and thresholds: a light higgsino in the "minimally non-minimal" supersymmetric model, with mass in the region of the Υ resonances and the $b\bar{b}$ threshold.

2. The Boltzmann equation

The evolution of the phase-space density $f(\boldsymbol{p}, \boldsymbol{x}, t)$ of a particle species is described by the Boltzmann equation, which can be written as [1,3–5]

$$L[f] = C[f], \tag{2.1}$$