0.1 Three-body phase space

In the notation of http://hitoshi.berkeley.edu/233B/phasespace.pdf,

$$\int \mathrm{d}\Phi_3 = \int \frac{\mathrm{d}s_{23}}{2\pi} \frac{\mathrm{d}\cos\theta_1}{2} \frac{\mathrm{d}\phi_1}{2\pi} \frac{\bar\beta_1(\frac{m_1^2}{s}, \frac{s_{23}}{s})}{8\pi} \frac{\mathrm{d}\cos\hat\theta_{23}}{2} \frac{\mathrm{d}\hat\phi_{23}}{2\pi} \frac{\bar\beta_{23}(\frac{m_2^2}{s_{23}}, \frac{m_3^2}{s_{23}})}{8\pi}, \tag{1}$$

where (θ_1, ϕ_1) is the solid angle for p_1 (thus p_{23}), $(\hat\theta_{23}, \hat\phi_{23})$ is that for p_2 (thus p_3) evaluated in the rest frame of p_{23}, s_{ij} are equal to m_{ij}^2 in the traditional Dalitz language, and

$$\bar\beta(x,y) := \sqrt{1-2(x+y)+(x-y)^2}; \qquad \bar\beta_1 := \bar\beta\left(\frac{m_1^2}{s}, \frac{s_{23}}{s}\right), \qquad \bar\beta_{23} := \bar\beta\left(\frac{m_2^2}{s_{23}}, \frac{m_3^2}{s_{23}}\right) \tag{2}$$

(the subscript of $\bar\beta$ is (perhaps) just redundancy for clarity).

For a spherically-symmetric integrand $|\mathcal{M}|^2$,

- we can drop $\mathrm{d}\cos\theta_1/2$ and $\mathrm{d}\phi_1/2\pi$,
- we can take $\hat\theta_{23}$ relative to $-p_1 = p_{23}$ (in $p_1+p_2+p_3$ rest frame) to drop $\mathrm{d}\hat\phi_{23}/2\pi$,

to get

$$\int \mathrm{d}\Phi_3 = \int \frac{\mathrm{d}s_{23}}{2\pi} \frac{\bar\beta_1(\frac{m_1^2}{s}, \frac{s_{23}}{s})}{8\pi} \frac{\mathrm{d}\cos\hat\theta_{23}}{2} \frac{\bar\beta_{23}(\frac{m_2^2}{s_{23}}, \frac{m_3^2}{s_{23}})}{8\pi}. \tag{3}$$

0.1.1 Energy fractions

Eq. (3) is rewritten by the energy fractions

$$x_i = \frac{E_i}{\sqrt{s}/2} \tag{4}$$

with using the relations

$$s = (E_1+E_2+E_3)^2 = (\hat E_1 + \hat E_2 + \hat E_3)^2 - |\hat{\boldsymbol{p}}_1|^2, \quad s_{23} = (\sqrt{s} - E_1)^2 - |\boldsymbol{p}_1|^2 = (\hat E_2 + \hat E_3)^2. \tag{5}$$

From the second equation, we know $s_{23} = s + m_1^2 - sx_1$ and

$$\int \mathrm{d}\Phi_3 = \int \frac{s\mathrm{d}x_1}{2\pi} \frac{\bar\beta_1(\frac{m_1^2}{s}, \frac{s_{23}}{s})}{8\pi} \frac{\mathrm{d}\cos\hat\theta_{23}}{2} \frac{\bar\beta_{23}(\frac{m_2^2}{s_{23}}, \frac{m_3^2}{s_{23}})}{8\pi}. \tag{6}$$

where

$$x_1 = \left[\frac{2m_1}{\sqrt{s}}, 1 + \frac{m_1^2 - (m_2+m_3)^2}{s}\right]. \tag{7}$$

The rest frame of p_{23} is now fixed by x_1. Setting the z-axis as the direction of $\boldsymbol{p}_1$,

$$p_1 = \frac{\sqrt{s}}{2}\begin{pmatrix} x_1 \\ 0 \\ 0 \\ s \end{pmatrix}, \qquad q_{23} = \frac{\sqrt{s}}{2}\begin{pmatrix} x_1 \\ 0 \\ 0 \\ s \end{pmatrix} \tag{8}$$

0.1.2 Dalitz plot

In PDG Review, Eq. (3) is expressed by

$$\int \mathrm{d}\Phi_3 = \int \frac{\mathrm{d}s_{12}}{2\pi} \frac{\bar\beta_3(\frac{m_3^2}{s}, \frac{s_{12}}{s})}{8\pi} \frac{\mathrm{d}\cos\hat\theta_{12}}{2} \frac{\bar\beta_{12}(\frac{m_1^2}{s_{12}}, \frac{m_2^2}{s_{12}})}{8\pi}, \tag{9}$$

where variables with hats are in the rest frame of p_{12}. Then $\hat{\theta}_{12}$ is converted to s_{23} by

$$\begin{aligned} s_{23} &= (p_2 + p_3)^2 \\ &= m_2^2 + m_3^2 + 2\left(\hat{E}_2\hat{E}_3 - \sqrt{\left(\hat{E}_2^2 - m_2^2\right)\left(\hat{E}_3^2 - m_3^2\right)}\cos\hat{\theta}_{12}\right), \end{aligned} \tag{10}$$

where note that $\hat{\theta}_{12}$ is defined by the angle between $-\boldsymbol{p}_3$ and $\boldsymbol{p}_1$, and therefore the angle between $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$. $\hat{E}_i$ are known by

$$(\hat{E}_1 + \hat{E}_2 + \hat{E}_3)^2 - |\hat{\boldsymbol{p}}_3|^2 = s, \qquad (\hat{E}_1 + \hat{E}_2)^2 = s_{12}, \tag{11}$$

as

$$E_1 = \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}}, \qquad E_2 = \frac{s_{12} - m_1^2 + m_2^2}{2\sqrt{s_{12}}}, \qquad E_3 = \frac{s - s_{12} - m_3^2}{2\sqrt{s_{12}}}. \tag{12}$$

Thus,

$$\int \mathrm{d}\Phi_3 = \int \frac{\mathrm{d}s_{12}}{2\pi} \frac{\bar{\beta}_3(\frac{m_3^2}{s}, \frac{s_{12}}{s})}{8\pi} \frac{\mathrm{d}s_{23}}{4\sqrt{(\hat{E}_2^2 - m_2^2)(\hat{E}_3^2 - m_3^2)}} \frac{\bar{\beta}_{12}(\frac{m_1^2}{s_{12}}, \frac{m_2^2}{s_{12}})}{8\pi} = \frac{1}{128\pi^3 s}\int \mathrm{d}s_{12}\mathrm{d}s_{23}. \tag{13}$$