

Lecture 2. System Properties

Controllability and Observability
Stability

Controller Design Problem (Infinite Horizon)

- Given a state space \mathbf{X} , a control space \mathbf{U} , an observation space \mathbf{Y} , an initial state, noise models in stochastic case
- We need to design a control law that minimizes the objective function

	Deterministic Model	Stochastic Model
Continuous time	$\dot{x} = f(x, u)$ $y = h(x, u)$ $J = \int_{t=0}^{\infty} \delta^t l_t(x, u, \dot{x}) dt$	$\dot{x} = f(x, u, w)$ $y = h(x, u, v)$ $J = \int_{t=0}^{\infty} \delta^t l_t(x, u, \dot{x}) dt$
Discrete time	$x_{k+1} = f(x_k, u_k)$ $y_k = h(x_k, u_k)$ $J = \sum_{k=0}^{\infty} \delta^k l_k(x_k, u_k, x_{k+1})$	$x_{k+1} = f(x_k, u_k, w_k)$ $y_k = h(x_k, u_k, v_k)$ $J = \sum_{k=0}^{\infty} \delta^k l_k(x_k, u_k, x_{k+1})$

Choice among Models

- In the following discussion, we may choose to use either continuous time or discrete time models, and either deterministic or stochastic models, whichever is easier to illustrate the idea.

Control-Affine Assumptions

- In the following discussion, we always assume we have a control-affine system

$$\dot{x} = f(x) + g(x)u$$

- Any nonlinear system can be written as a control-affine form through dynamic extension

Dynamic Extension

- Non-control-affine system

$$\dot{x} = f(x, u)$$

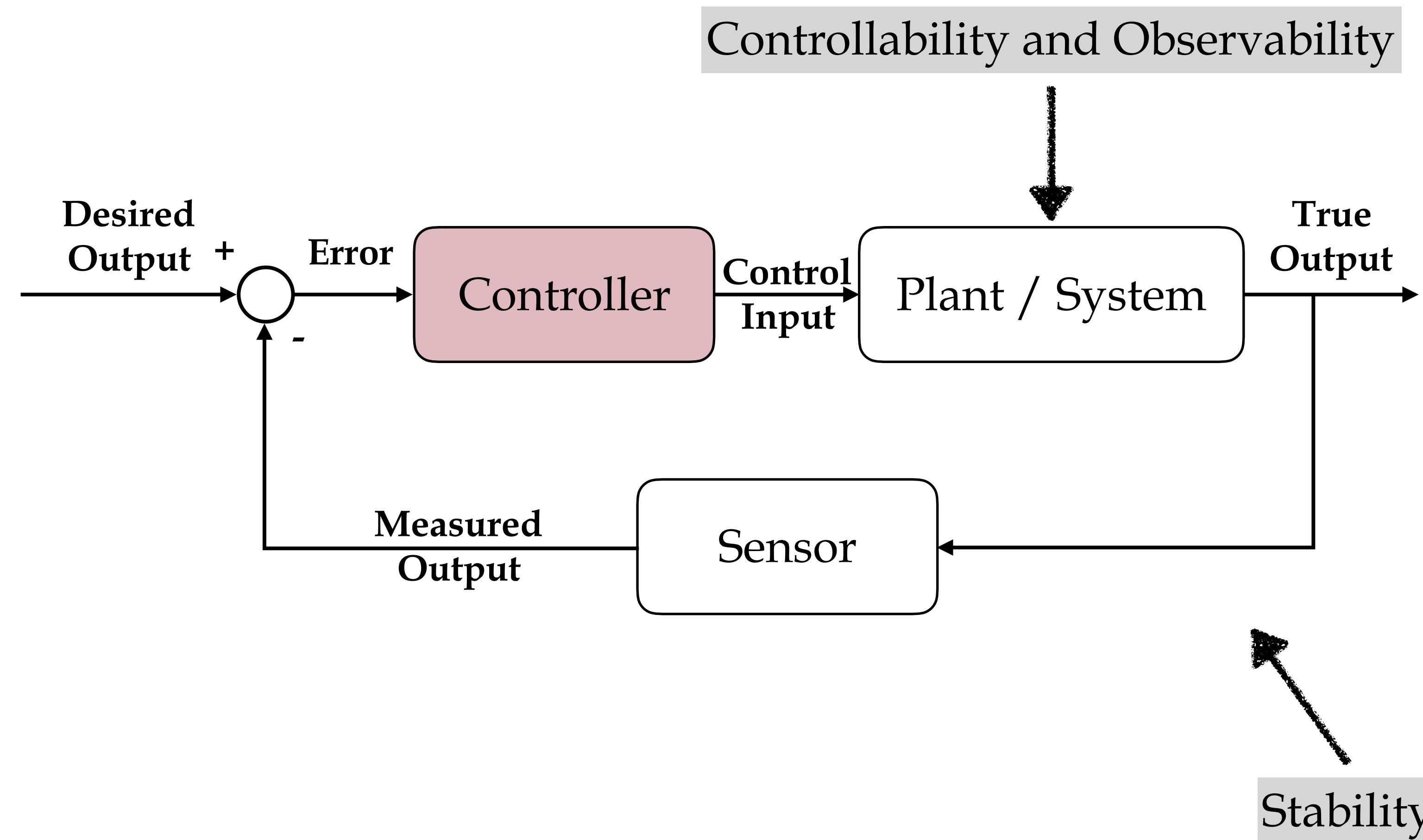
- Define extended state and extended control

$$x^e = \begin{bmatrix} x \\ u \end{bmatrix}, u^e = \dot{u}$$

- Then the extended system is control-affine

$$\dot{x}^e = \begin{bmatrix} f(x, u) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^e$$

A Closed-Loop System



System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

Controllability

$$\dot{x} = f(x) + g(x)u$$

- Controllability: the ability of the control input u to move the internal state x of a system from any **initial** state to any other **final** state in a **finite time** interval

Examples

Controllable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Uncontrollable

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Controllability (Mathematical Interpretation)

- System trajectory $x(t) = \Phi(x(0), u(0 : t))$
- Controllability is equivalent to the following feasibility test:
 - For any $x(0)$ and any x^* , there exists a finite t^* and a control sequence $u^*(0 : t^*)$ such that

$$x^* = \Phi(x(0), u^*(0 : t^*))$$

- All that matters is whether we can solve an inverse of Φ !

Controllability (Discrete Linear Case)

$$x_{k+1} = Ax_k + Bu_k$$

- System trajectory:

$$\begin{aligned} & x_0 \\ & Ax_0 + Bu_0 \\ & A^2x_0 + ABu_0 + Bu_1 \\ & \vdots \\ & A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} Bu_k \end{aligned}$$

Controllability (Discrete Linear Case)

- The feasibility test involves solving the following equation:

$$x^* = A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} B u_k$$

- Rearranging

$$x^* - A^t x_0 = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \dots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{t-1} \end{bmatrix}$$

Need to have full row rank!

Controllability (Discrete Linear Case)

- Consider the discrete-time linear system $x_{k+1} = Ax_k + Bu_k$ where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}$.
- Define the controllability matrix as

$$\mathcal{C} = [A^{n-1}B \quad A^{n-2}B \quad \dots \quad B] \quad \text{Q: Why it only goes to } A^{n-1}?$$

- Theorem: the system is controllable if and only if

$$\text{rank}(\mathcal{C}) = n$$

Uncontrollable States

- If a state is uncontrollable, then
 - If states are meaningful (physical) variables that need to be controlled, then the design of the actuators are deficient.
 - The effect of control is limited. There is also a possibility of instability.
 - Stabilizability: a system is stabilizable if all its uncontrollable states are stable.

Uncontrollable vs Underactuation

- Uncontrollable: “you can never control it”
- Underactuation: systems that cannot be commanded to follow arbitrary trajectories in configuration space.
 - Usually because the system has a lower number of actuators than degrees of freedom
- An uncontrollable system is underactuated.
- An underactuated system can be controllable.

Observability

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x, u)\end{aligned}$$

- Observability: for any possible sequence of state x and control u , the current state x_0 can be determined in **finite** time using only the outputs y .

Example

Observable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

Unobservable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x\end{aligned}$$

Observability (Mathematical Interpretation)

- System trajectory $x(t) = \Phi(x(0), u(0 : t))$, $y(t) = h(x(t), u(t))$
- Observability is equivalent to the following uniqueness test:
 - For any $u(0 : t)$ and any valid observations $y(0 : t)$, there exists a finite t^* such that there is a **unique** solution of $x(0)$ in the following equations

$$y(t) = h(\Phi(x(0), u(0 : t)), u(t)), \forall t \in [0, t^*]$$

- All that matters is whether we can solve an inverse of $h \circ \Phi$!

Observability (Discrete Linear Case)

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k \\y_k &= Cx_k + Du_k\end{aligned}$$

- System trajectory and measurements:

x_0	$Cx_0 + Du_0$
$Ax_0 + Bu_0$	$CAx_0 + CBu_0 + Du_1$
$A^2x_0 + ABu_0 + Bu_1$	$CA^2x_0 + CABu_0 + CBu_1 + Du_2$
⋮	⋮
$A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} Bu_k$	$CA^t x_0 + \sum_{k=0}^{t-1} CA^{t-1-k} Bu_k + Du_t$

Observability (Discrete Linear Case)

- The initial state x_0 can be obtained by solving the following equation

$$\begin{bmatrix} y_0 - Du_0 \\ y_1 - CBu_0 - Du_1 \\ y_2 - CABu_0 - CBu_1 - Du_2 \\ \vdots \\ y_t - \sum_{k=0}^{t-1} CA^{t-1-k} Bu_k - Du_t \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^t \end{bmatrix} x_0$$

Need to have full column rank!

Observability (Discrete Linear Case)

- Consider the discrete-time linear system

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times r}$.

- Define the observability matrix as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Q: Why it only goes to A^{n-1} ?

- Theorem: the system is observable if and only if

$$\text{rank}(\mathcal{O}) = n$$

Unobservable States

- If a state is unobservable,
 - We cannot determine its value no matter how long we observe.
 - We need to design more sensors to get more information.
 - It may be unstable.
- Detectability: A system is detectable if all the unobservable states are stable.

Unobservable vs Partially Observable

- Unobservable: “you can never know it”
- Partially observable:
 - The agent cannot directly observe the underlying state. Instead, it must maintain a probability distribution over the set of possible states, based on a set of observations and observation probabilities, and the underlying MDP.
 - Observability / unobservability is an intrinsic system property. The uncertainty does not come from stochasticity.
 - The uncertainty in partial observability comes from stochasticity.

Controllability and Observability

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned}$$

- Controllability and observability are dual problems.
- Mathematically, controllability is about existence; observability is about uniqueness.
- Controllability and observability are **independent from stochasticity**.

Controllability and Observability

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & B \end{bmatrix}$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Examples

Controllable, Observable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

Uncontrollable, Observable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x\end{aligned}$$

Controllable, Unobservable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x\end{aligned}$$

Uncontrollable, Unobservable

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x\end{aligned}$$

* The first state's dynamics are internal

Open-Ended Questions

- Any example of real-world uncontrollable / unobservable systems?
- Is a human-robot system controllable / observable from robot's perspective?
- How to modify the system to improve controllability / observability?

Why Important?

- Controllability and observability of the system affects the feasibility of the controller design problem.
- We can never design a controller to stabilize a system that is uncontrollable (with uncontrollable states that are unstable).

Nonlinear Systems*

- Controllability and observability for nonlinear systems
- Linearization
- Lie brackets

http://www.bcamath.org/documentos_public/courses/TalkBCAM20110707controllability-Coron.pdf

System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

Autonomous Systems

- An autonomous system is a system that is time-invariant

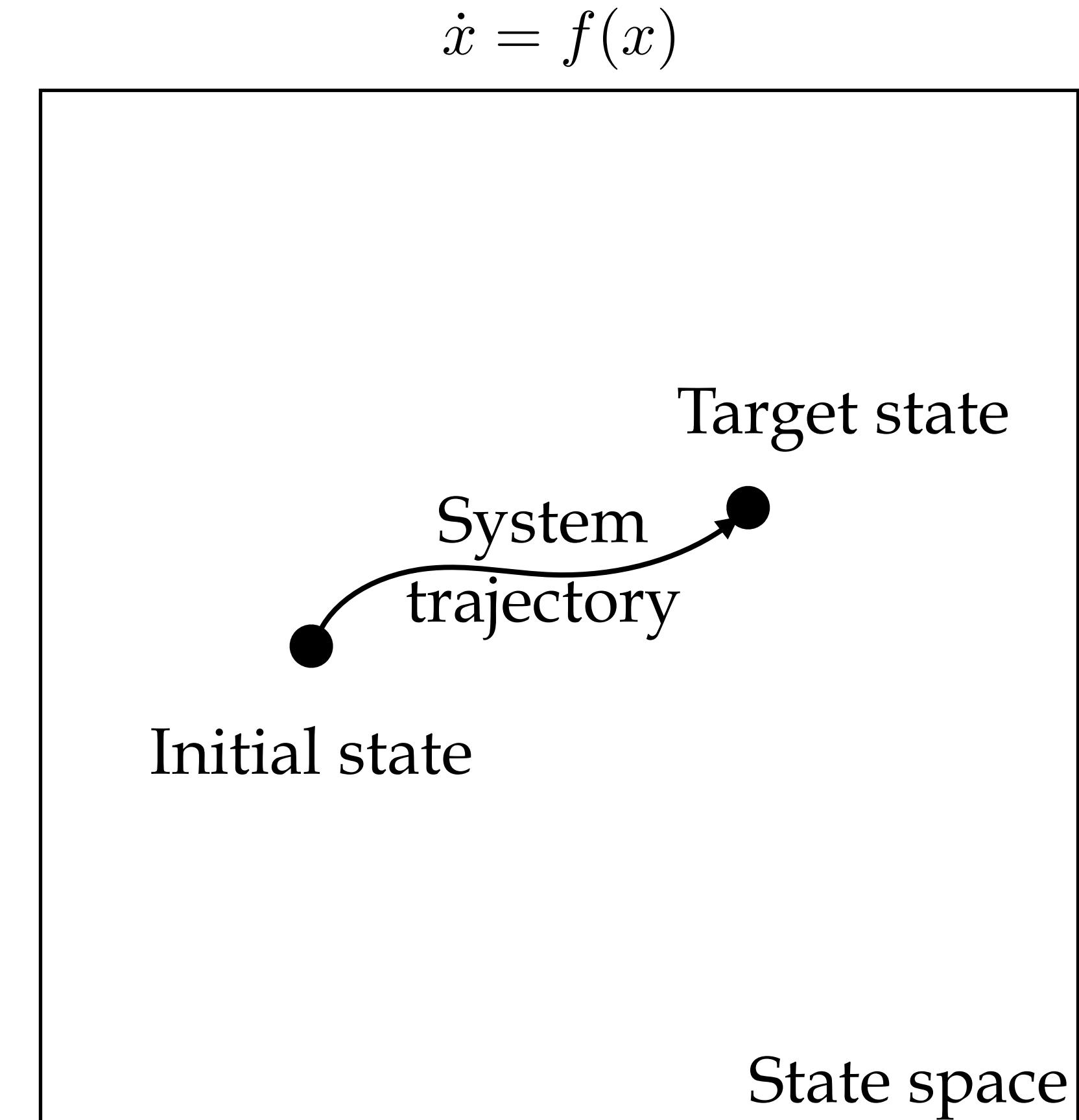
$$\dot{x} = f(x)$$

- It can be a closed-loop system

$$\dot{x} = f^o(x, u), u = g(x)$$

Analysis of Autonomous Systems

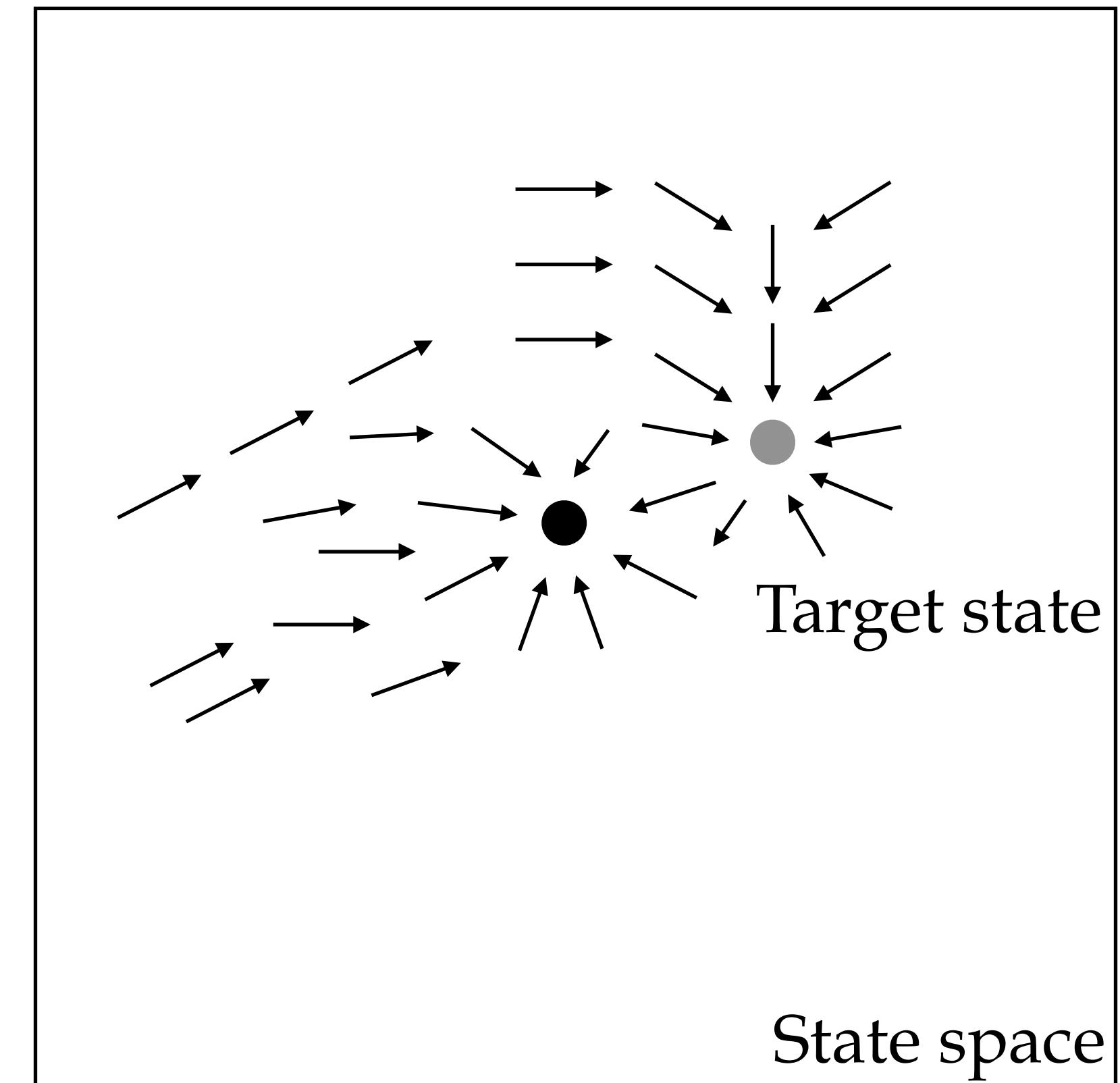
- Will the system rest in the target state?
 - Equilibrium
 - Stability



Analysis of Autonomous Systems

- Will the system rest in the target state?
 - Equilibrium
 - Stability

$\dot{x} = f(x)$ defines a vector field



Equilibrium

- Equilibrium is where $\dot{x} = f(x) = 0$
- Define $J(x) = \frac{\partial f}{\partial x}(x)$
- Finding Equilibrium point by solving the following optimization:

$$\min_x \|f(x)\|^2$$

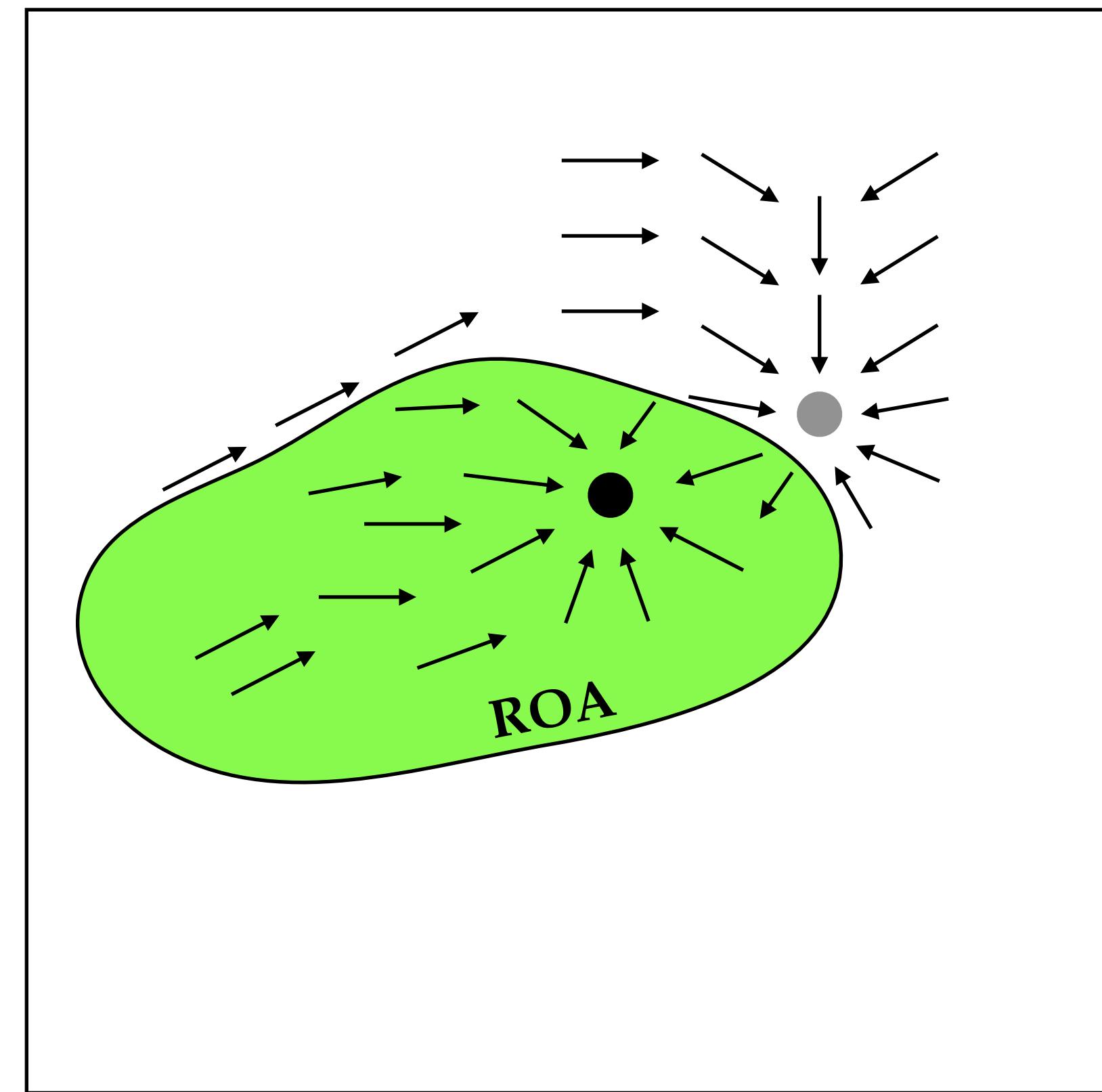
- Newton method: $x_{i+1} = x_i + \alpha J^{-1}(x_i)f(x_i)$
- Steepest descent: $x_{i+1} = x_i - \beta J^T(x_i)f(x_i)$

Equilibrium

- (Uniqueness) Global Implicit Function Theorem: the equilibrium is unique if
 - $\det [J(x)] \neq 0, \forall x$
 - $\|f(x)\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$

Region of Attraction (ROA)

- Region of attraction of an equilibrium
- The set of initial states under which the system will eventually reach the equilibrium.



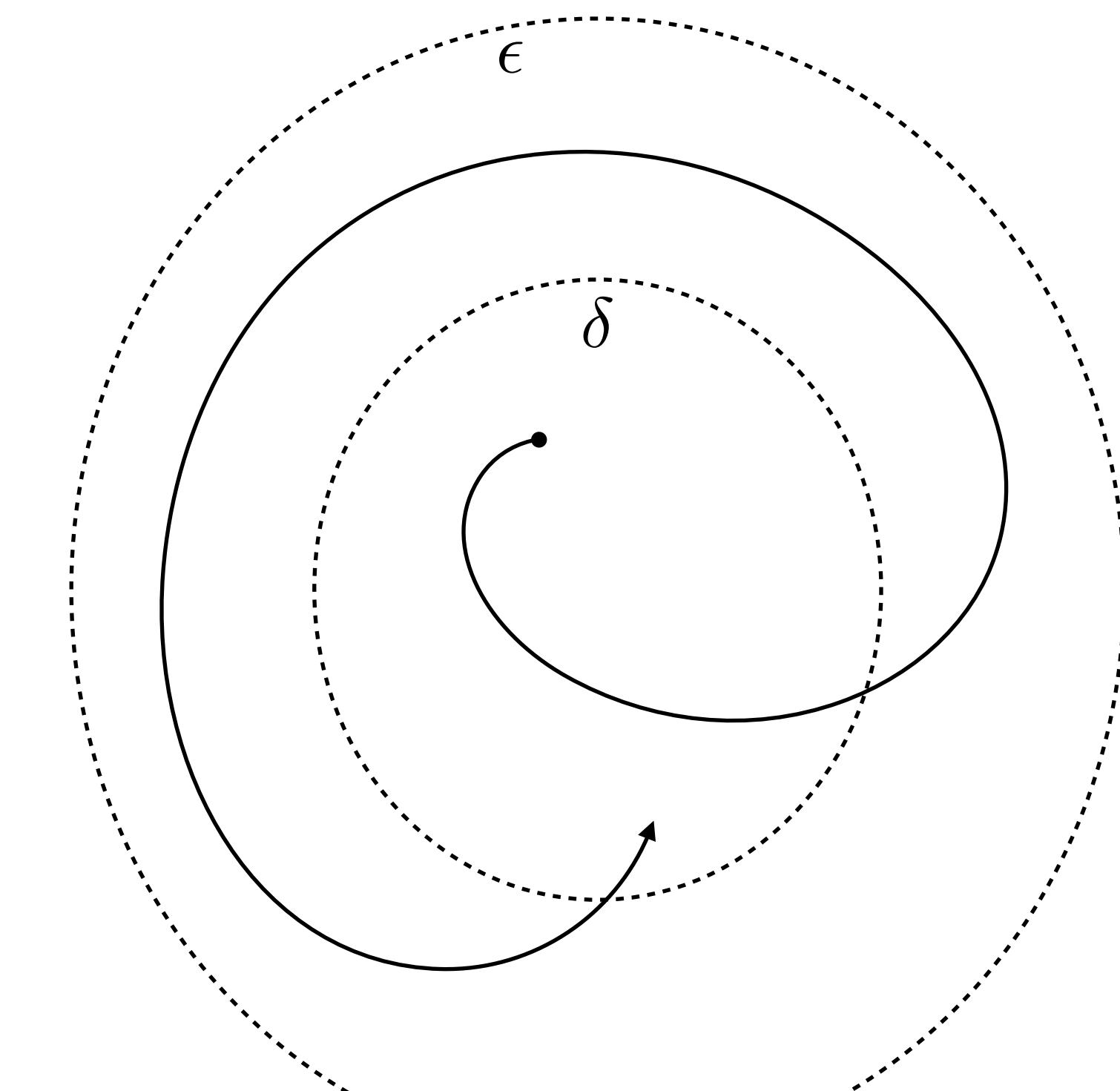
Stability

- $\epsilon - \delta$ definition
- Stability for linear systems
- Lyapunov stability

Stability

- $\epsilon - \delta$ definition:
- A system is stable at origin if and only if for any $\epsilon > 0$, there exist a $\delta > 0$ such that for all initial states at t_0 that satisfy $\|x_0\| < \delta$,

$$\|x(t)\| < \epsilon, \forall t > t_0$$



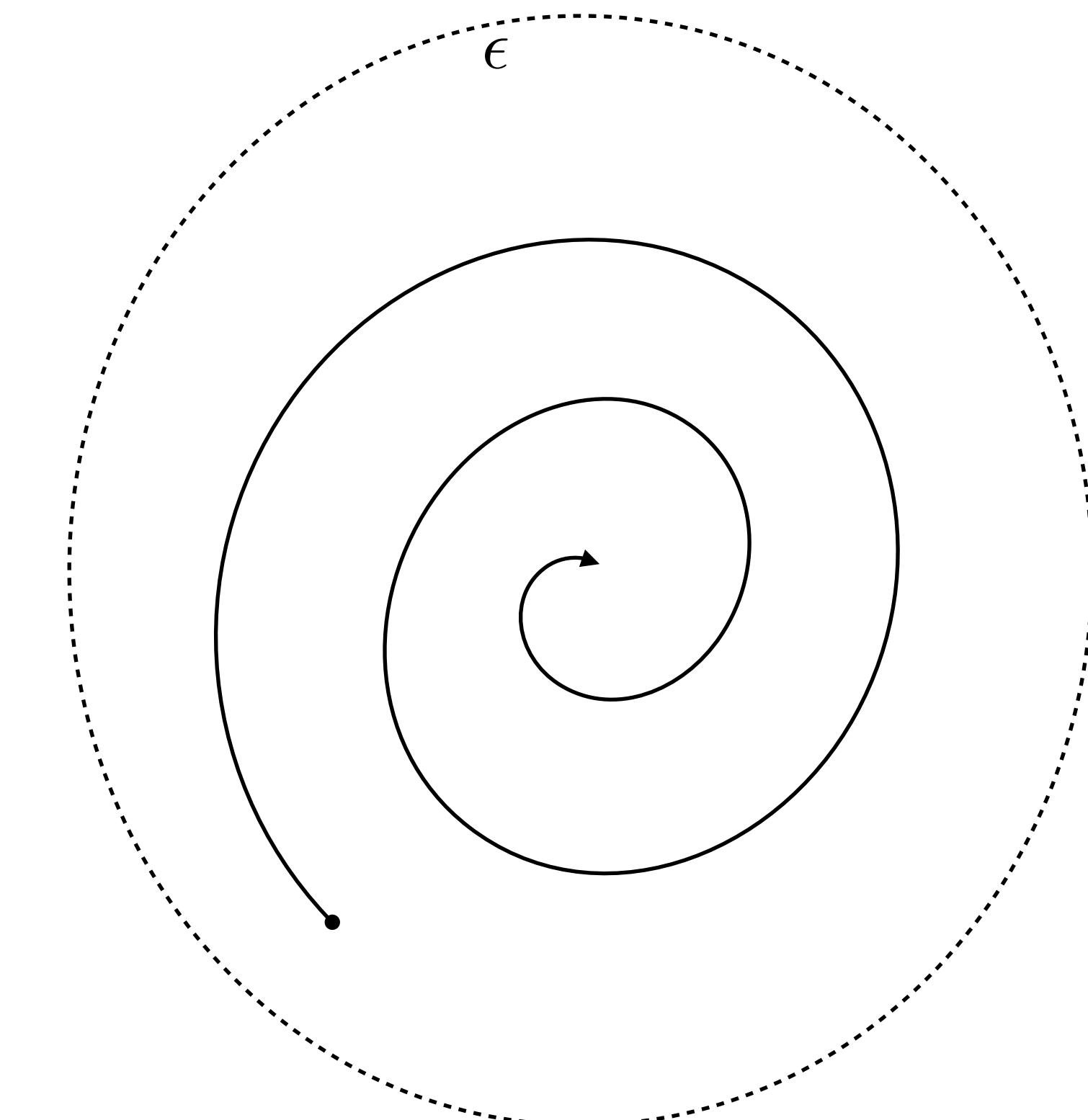
Q: Can $\delta > \epsilon$?

Asymptotic Stability

- $\epsilon - \delta$ definition:
- A system is globally asymptotically stable at origin if and only if 1) the system is stable; 2) for any initial condition x_0 and for any $\epsilon > 0$, there exist a time $t^* > t_0$ such that

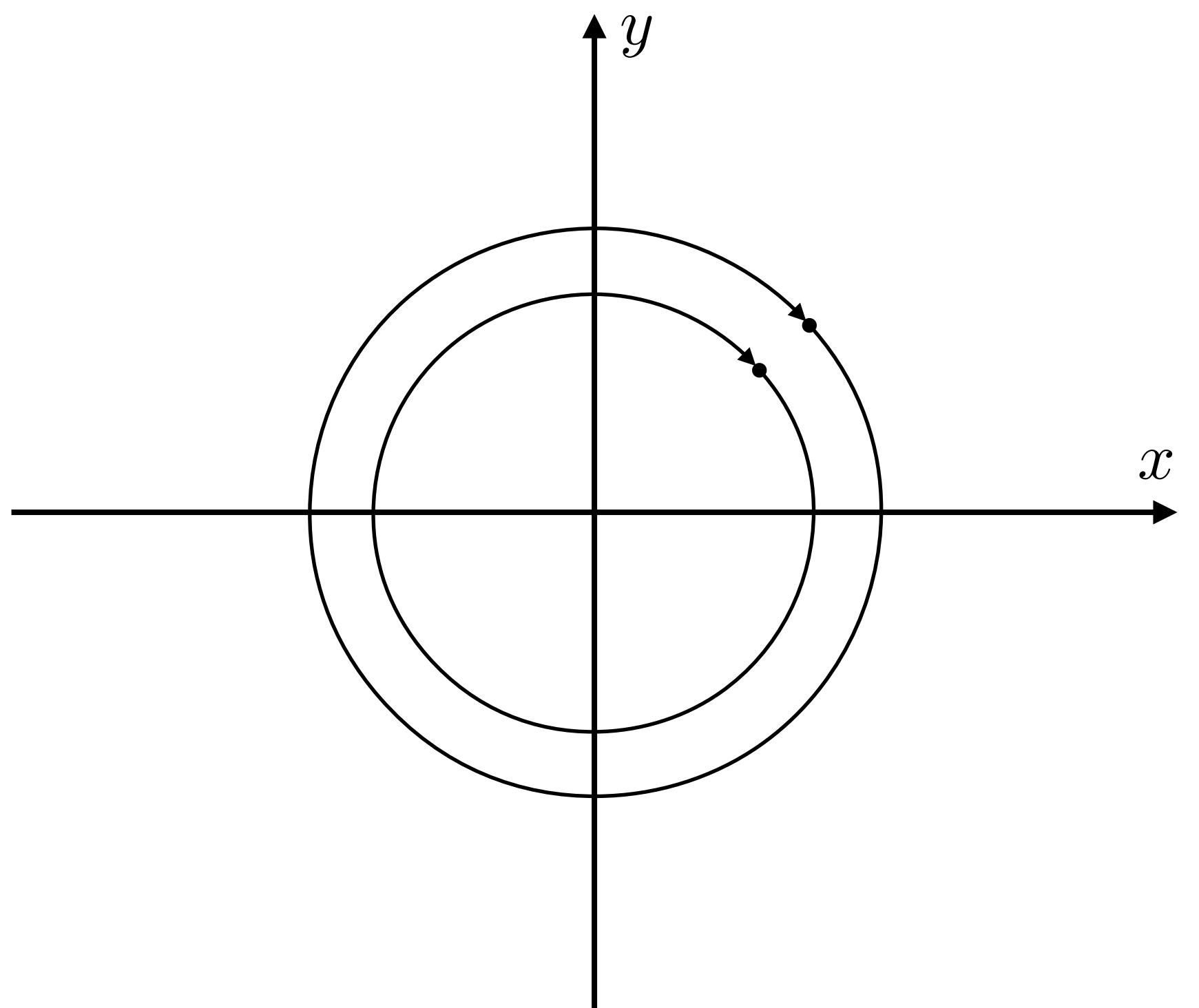
$$\|x(t)\| < \epsilon, \forall t > t^*$$

* This is a convergence condition



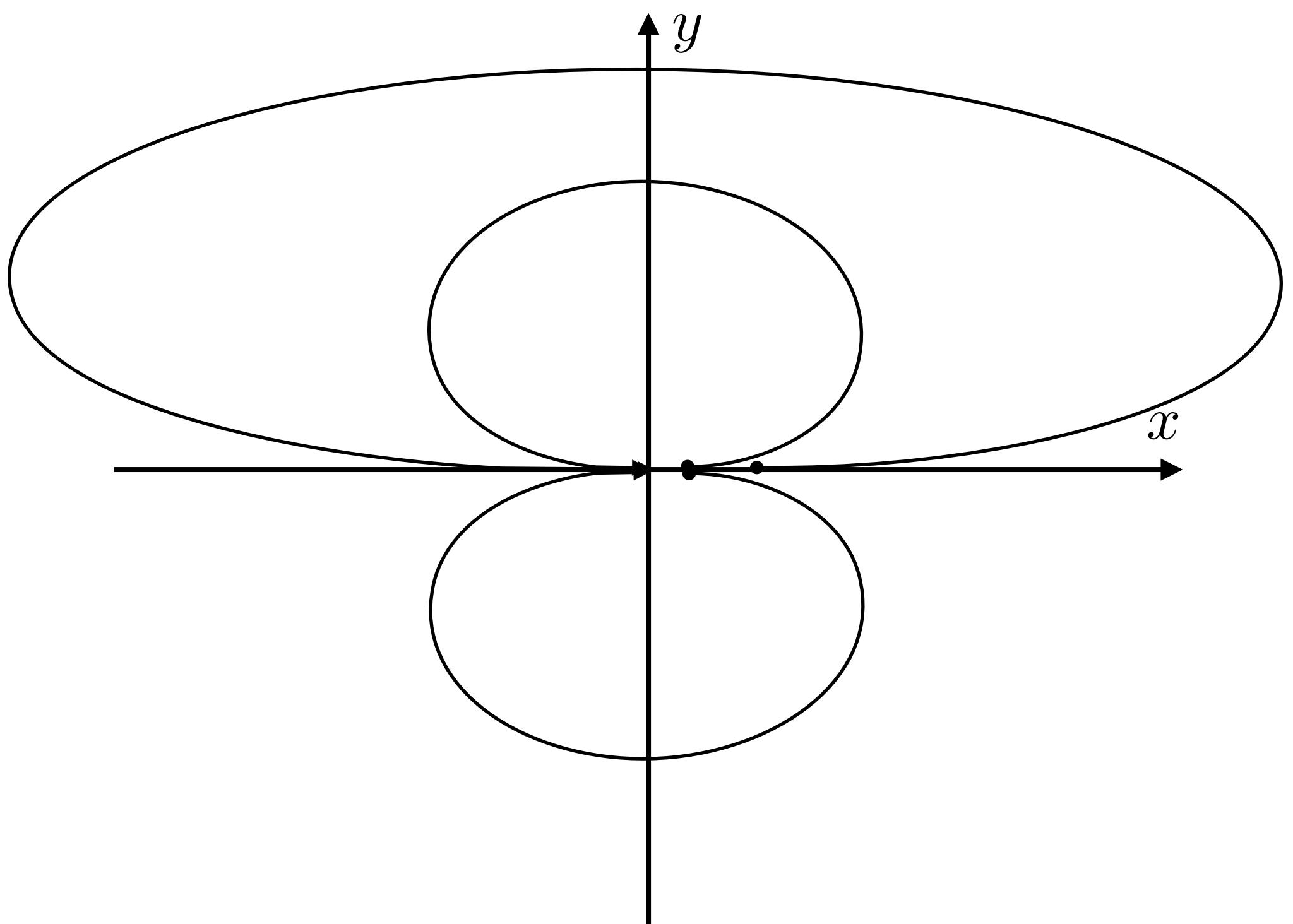
Examples

- Stable but not asymptotically stable
 - e.g., $\dot{x} = y, \dot{y} = -x$



Examples

- Converge but not stable
- e.g., $\dot{x} = x^2 - y^2, \dot{y} = 2xy$ (here we need to identify the positive infinity with the negative infinity)



Stability for Linear Systems

- Linear system $\dot{x} = Ax$
- Solution: $x(t) = x(0)e^{At}$
- Stability of the system depends on the exponential term e^{At}

Stability for Linear Systems

- Eigen-values λ_i satisfy that $Av_i = \lambda_i v_i$, where v_i is the eigen-vector associated with that eigen-value.
- Eigen-value decomposition

$$A \underbrace{[v_1 \ v_2 \ \dots \ v_n]}_V = \underbrace{[v_1 \ v_2 \ \dots \ v_n]}_V \underbrace{\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)}_{\Sigma}$$

$$A = V\Sigma V^{-1}$$

- By Similarity Transformation, the exponential term is

$$e^{At} = V \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}) V^{-1}$$

Stability for Linear Systems

- $e^{At} = V \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}) V^{-1}$
- By Euler's formula $e^{\lambda_i t} = e^{\text{Re}(\lambda_i)t} [\cos(\text{Im}(\lambda_i)t) + i \sin(\text{Im}(\lambda_i)t)]$
- Assume the system has distinct eigen-values.
- The system is stable if and only if

$$\text{Re}(\lambda_i) \leq 0, \forall i$$

- The system is asymptotically stable if and only if

$$\text{Re}(\lambda_i) < 0, \forall i$$

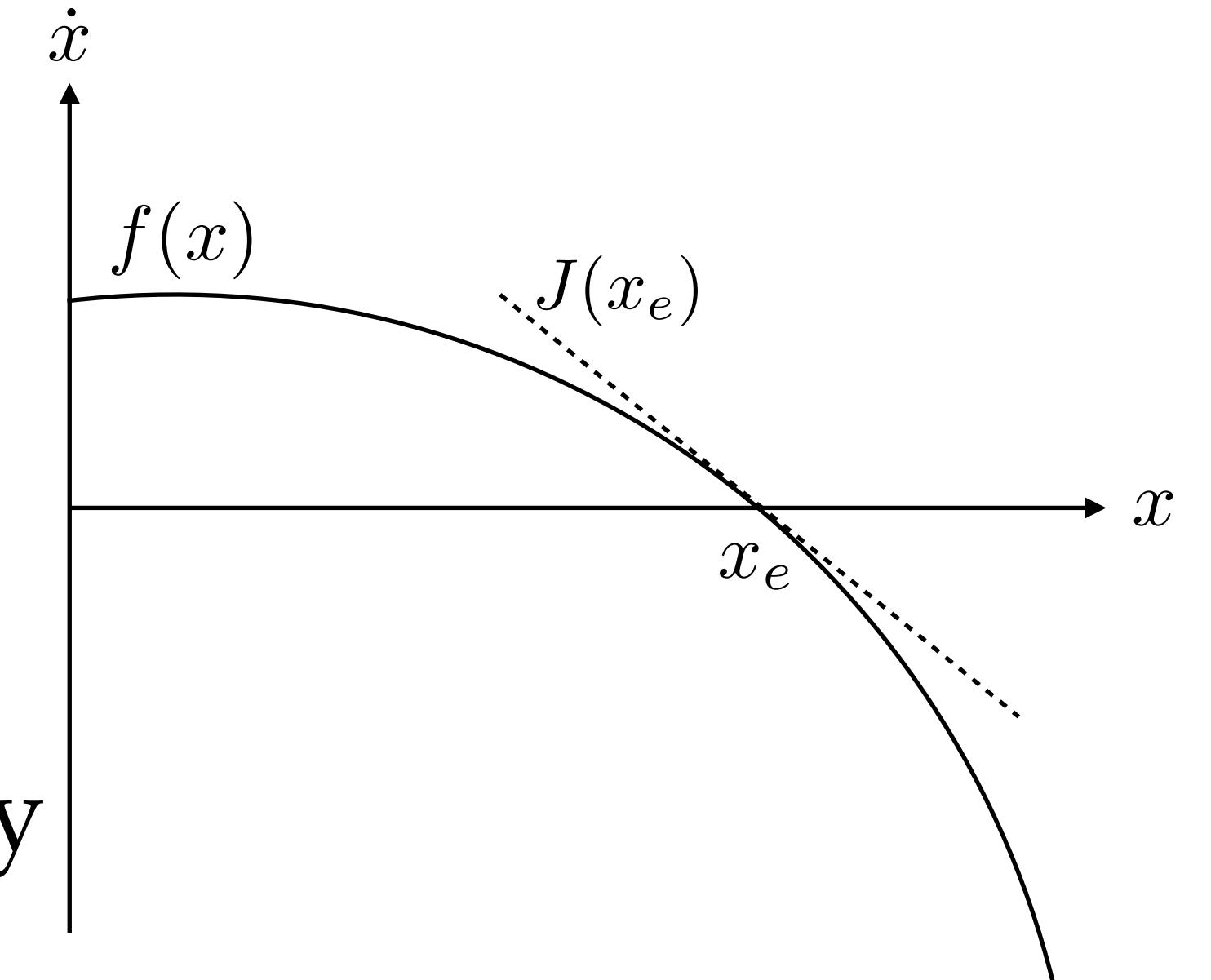
* Exponential decay

Lyapunov 1st method

- For the nonlinear system $\dot{x} = f(x)$, linearize around the equilibrium point x_e ,

$$\delta\dot{x} = J(x_e)\delta x + h(x_e, \delta x)$$

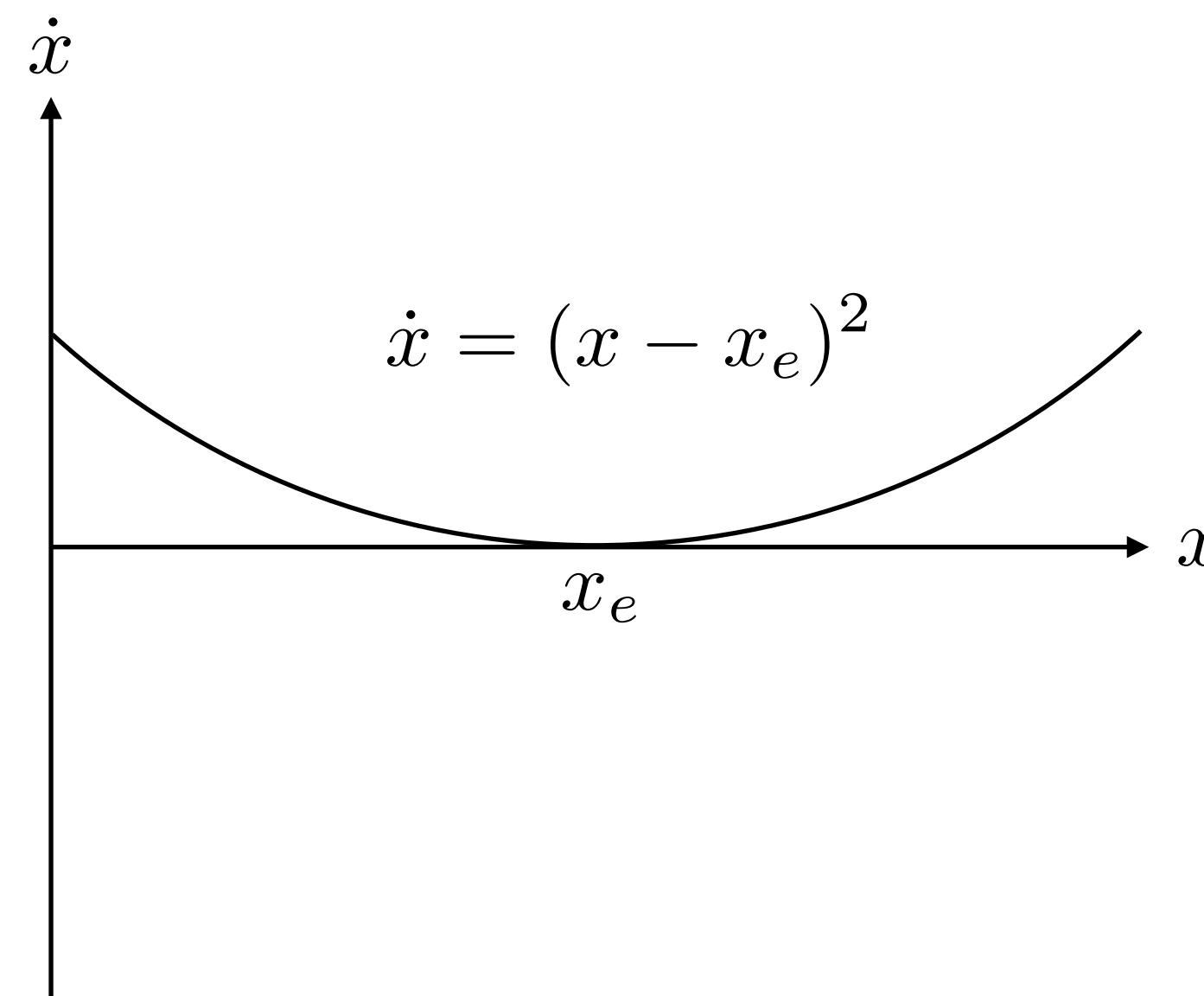
- Then linear stability analyses apply locally if
 - All eigen-values of $J(x_e)$ have non-zero real parts
 - The error term decays faster than first order decay



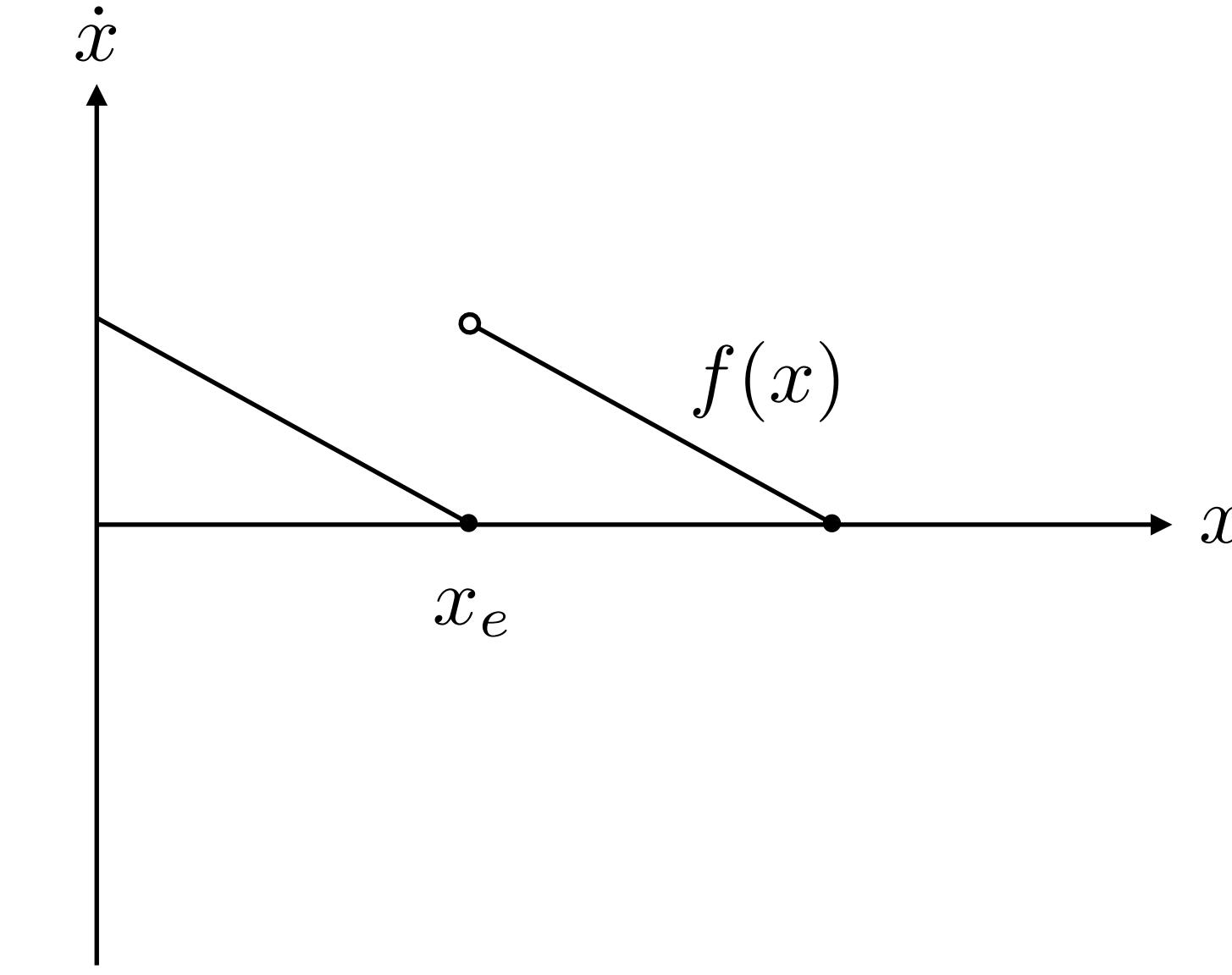
$$\lim_{\|\delta x\| \rightarrow 0} \frac{\|h(x_e, \delta x)\|}{\|\delta x\|} \rightarrow 0$$

Lyapunov 1st method

- Scenarios that cannot be analyzed by Lyapunov 1st method



Eigen-value is zero.



Error term does not decay.

Other Methods*

- Scalar energy function-based methods (to be discussed later):
 - Lyapunov 2nd method
 - Passivity

Why Important?

- Stability analysis is important to formally ensure that the objective can always be achieved.

System Properties

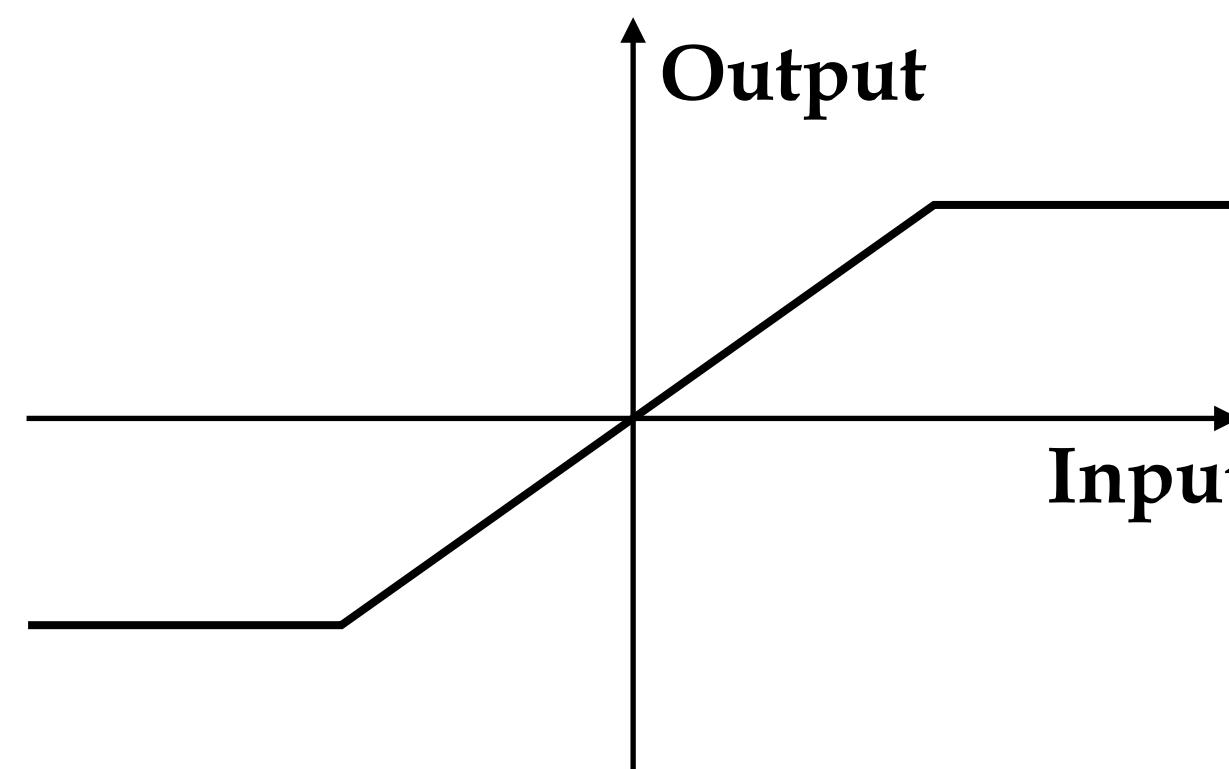
- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

Nonlinear Systems

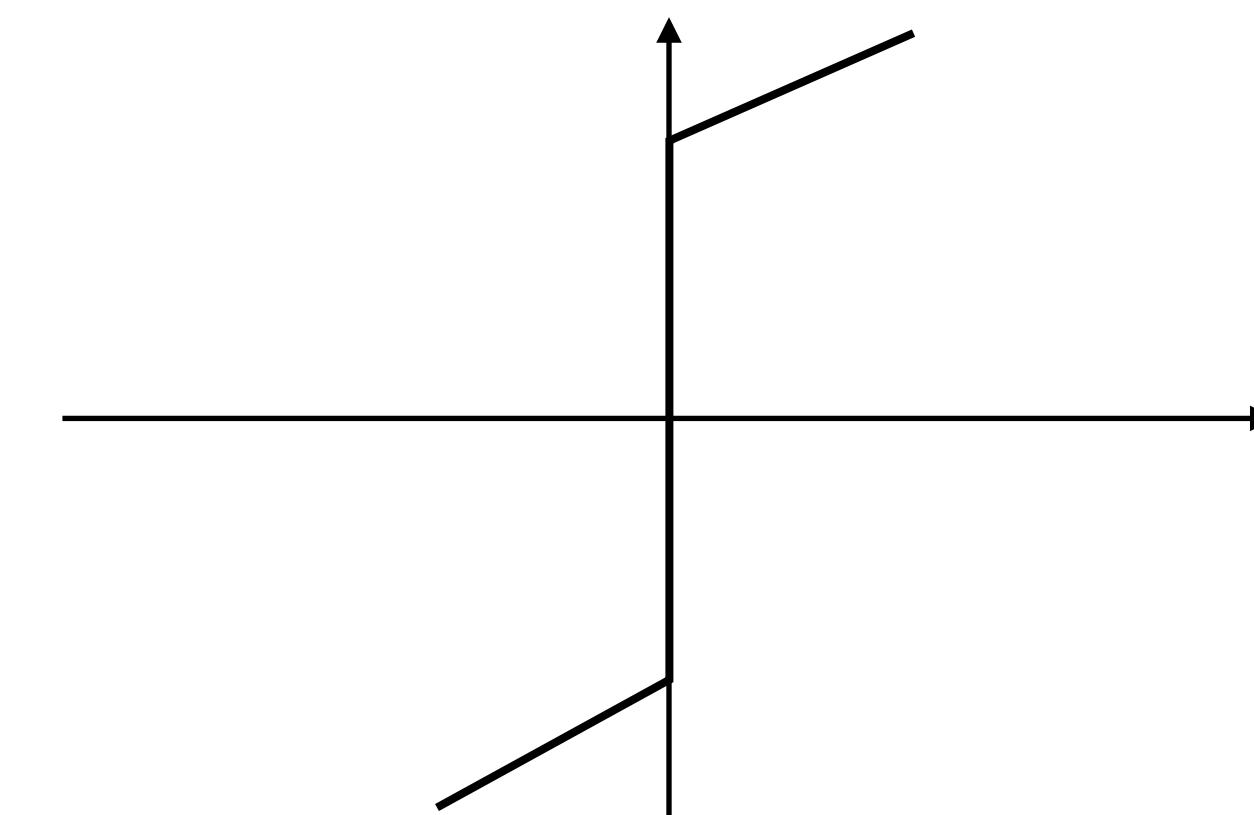
- A systems $\dot{x} = f(x)$ is nonlinear if it does not satisfy the following two properties:
 - Superposition $f(x + y) = f(x) + f(y)$
 - Homogeneity $f(kx) = kf(x)$

Types of Nonlinearity

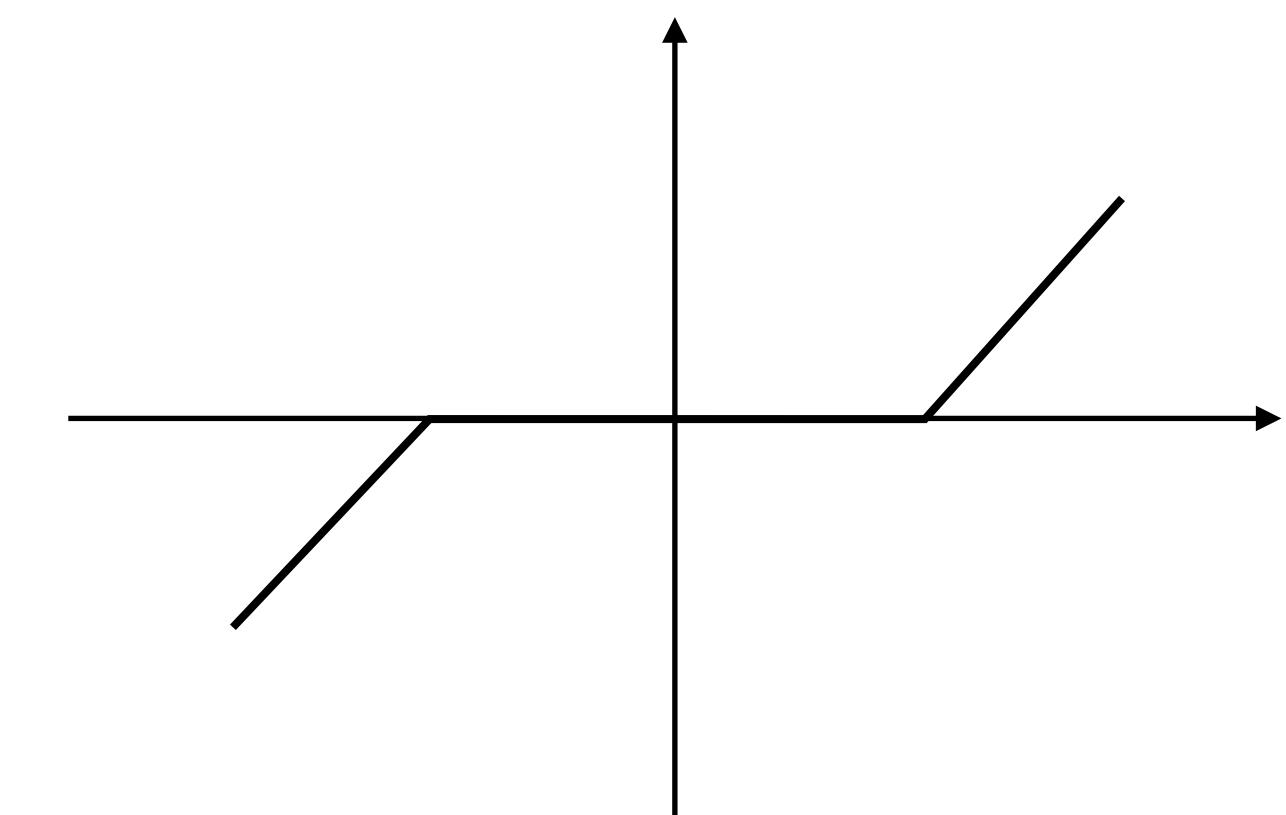
- Single-valued & time-invariant



Saturation



Friction

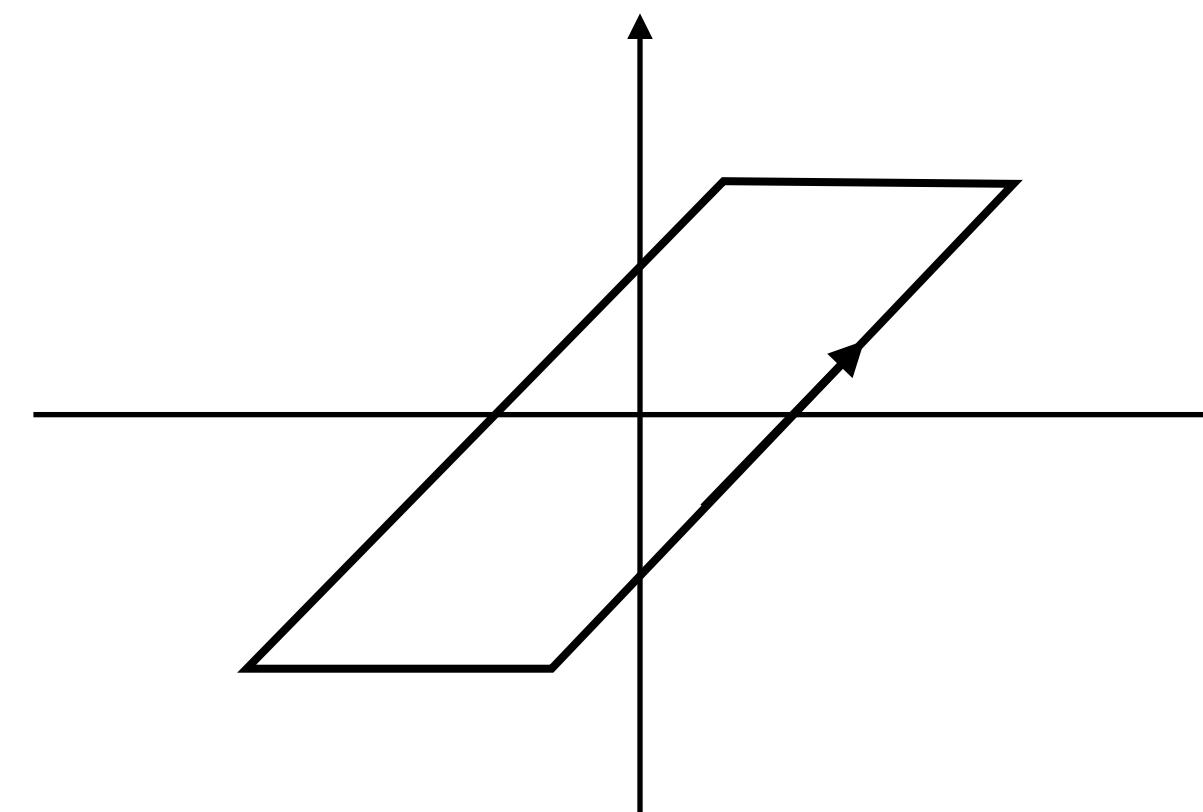


Dead zone

* Note the similarity to the neural activation functions

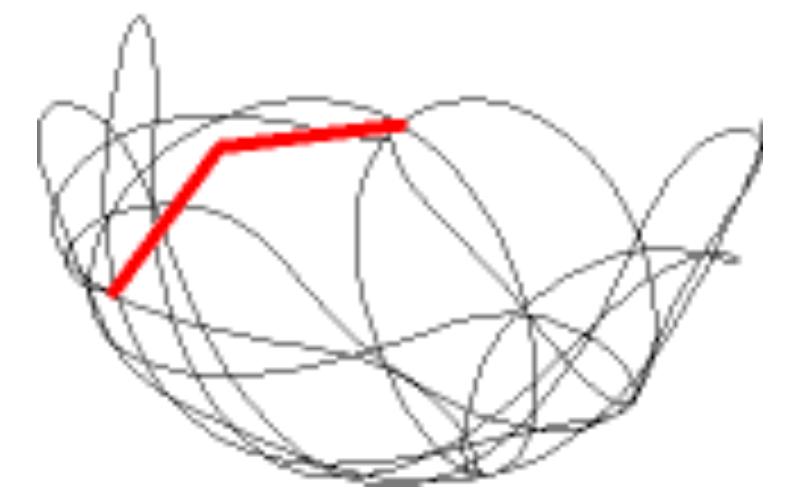
Types of Nonlinearity

- Memory
 - In general, a nonlinearity with “memory” means that we have not included all states that define the nonlinearity



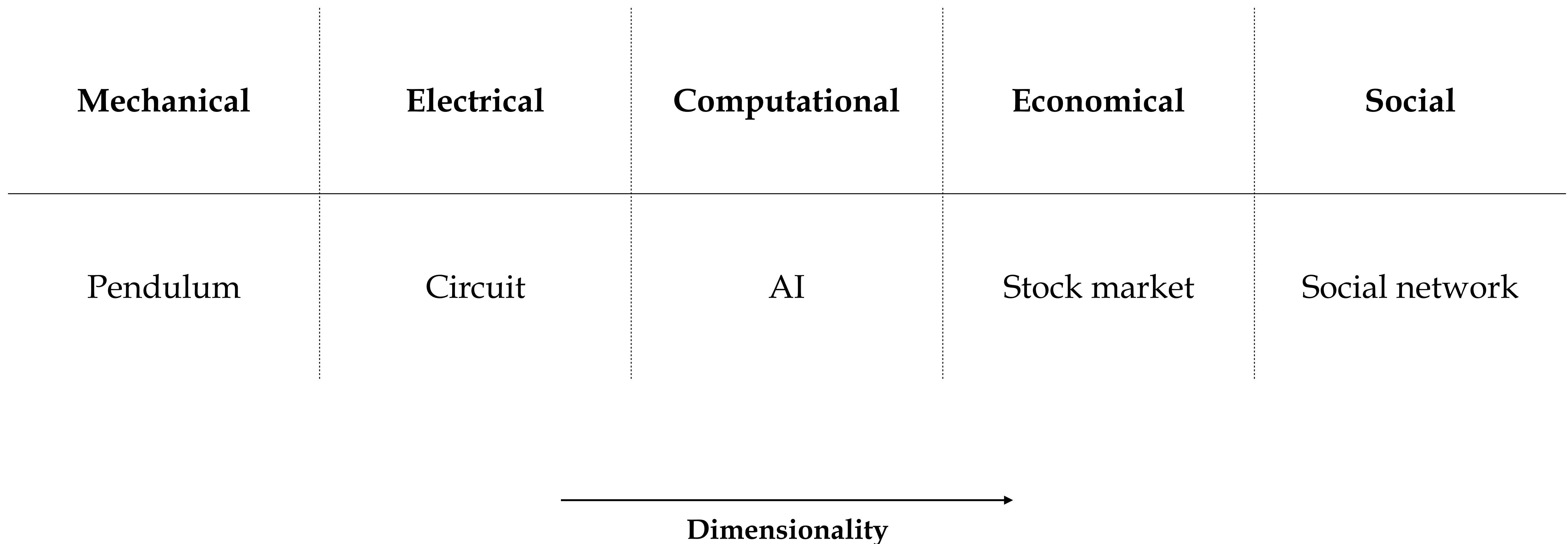
Chaos

- When the present determines the future, the **approximate** present does not **approximately** determine the future.
- A small change in initial conditions can result in huge change in the future.
- Chaos is different from uncertainty.
- Unpredictability.



A [double-rod pendulum](#) animation showing chaotic behavior. Starting the pendulum from a slightly different **initial condition** would result in a completely different **trajectory**. The double-rod pendulum is one of the simplest dynamical systems with chaotic solutions.

Nonlinear Systems



Nonlinear Systems

- Properties:
 - Multiple equilibrium points
 - Local vs. global stability
 - Response characteristics depends upon initial conditions, input type and input magnitude

System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems