# Lecture 2. System Properties

Controllability and Observability
Stability

# Controller Design Problem (Infinite Horizon)

- Given a state space **X**, a control space **U**, an observation space **Y**, an initial state, noise models in stochastic case
- We need to design a control law that minimizes the objective function

| | **Deterministic Model** | **Stochastic Model** |
|---|---|---|
| **Continuous time** | $\dot{x} = f(x,u)$<br>$y = h(x,u)$<br>$J = \int_{t=0}^{\infty} \delta^t l_t(x,u,\dot{x})dt$ | $\dot{x} = f(x,u,w)$<br>$y = h(x,u,v)$<br>$J = \int_{t=0}^{\infty} \delta^t l_t(x,u,\dot{x})dt$ |
| **Discrete time** | $x_{k+1} = f(x_k,u_k)$<br>$y_k = h(x_k,u_k)$<br>$J = \sum_{k=0}^{\infty} \delta^k l_k(x_k,u_k,x_{k+1})$ | $x_{k+1} = f(x_k,u_k,w_k)$<br>$y_k = h(x_k,u_k,v_k)$<br>$J = \sum_{k=0}^{\infty} \delta^k l_k(x_k,u_k,x_{k+1})$ |

# Choice among Models

- In the following discussion, we may choose to use either continuous time or discrete time models, and either deterministic or stochastic models, whichever is easier to illustrate the idea.

# Control-Affine Assumptions

- In the following discussion, we always assume we have a control-affine system

$$\dot{x} = f(x) + g(x)u$$

- Any nonlinear system can be written as a control-affine form through dynamic extension

# Dynamic Extension

- Non-control-affine system

$$\dot{x} = f(x, u)$$

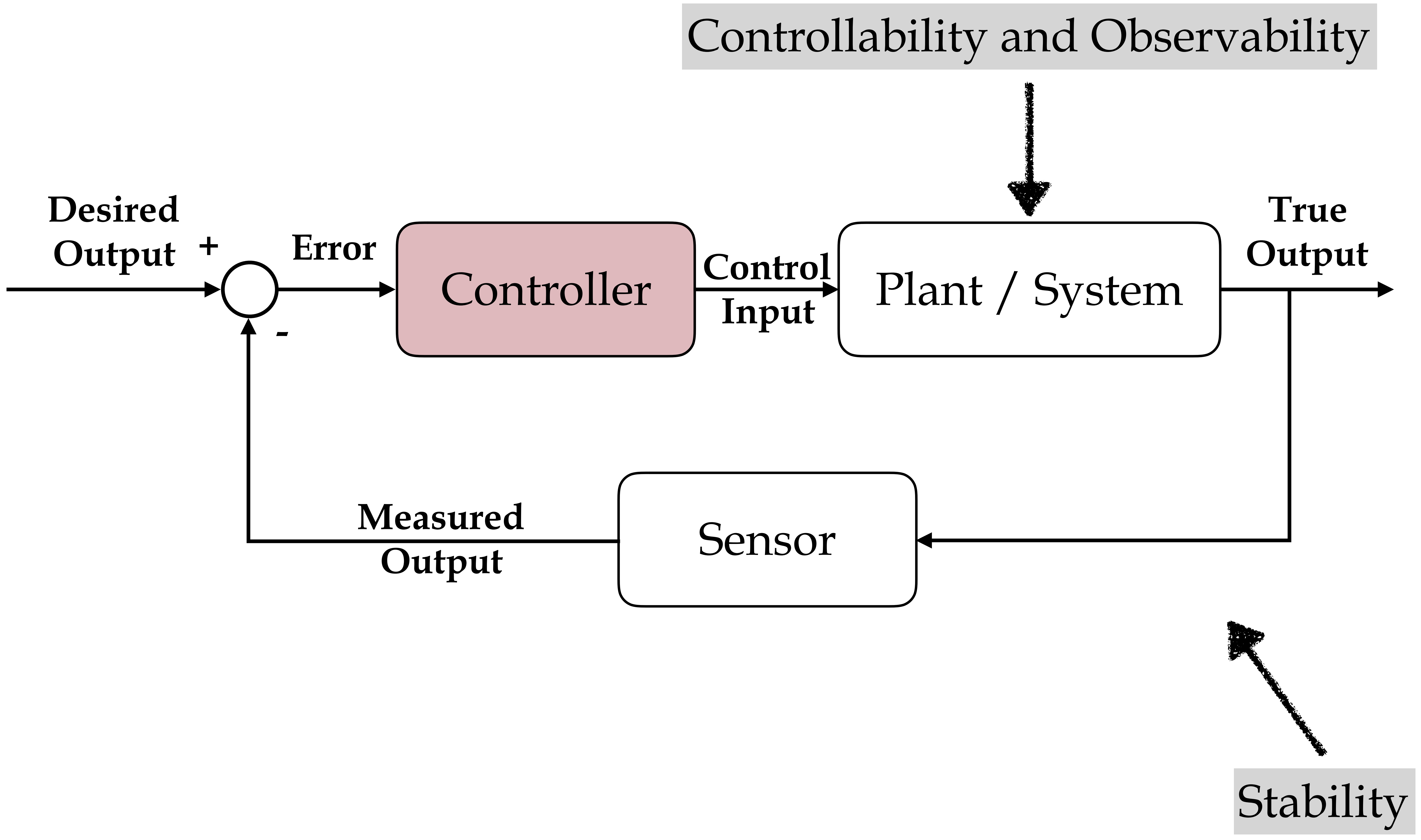
- Define extended state and extended control

$$x^e = \left[ \begin{array}{c} x \\ u \end{array} \right], u^e = \dot{u}$$

- Then the extended system is control-affine

$$\dot{x}^e = \left[ \begin{array}{c} f(x, u) \\ 0 \end{array} \right] + \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] u^e$$

# A Closed-Loop System



# System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

# Controllability

$$\dot{x} = f(x) + g(x)u$$

- Controllability: the ability of the control input u to move the internal state x of a system from any **initial** state to any other **final** state in a **finite time** interval

# Examples

Controllable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Uncontrollable

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

# Controllability (Mathematical Interpretation)

- System trajectory $x(t) = \Phi(x(0), u(0:t))$
- Controllability is equivalent to the following feasibility test:
  - For any $x(0)$ and any $x^*$, there exists a finite $t^*$ and a control sequence $u^*(0:t^*)$ such that

$$x^* = \Phi(x(0), u^*(0:t^*))$$

- All that matters is whether we can solve an inverse of $\Phi$!

# Controllability (Discrete Linear Case)

$$x_{k+1} = Ax_k + Bu_k$$

- System trajectory:

$$\begin{aligned}
&x_0\\
&Ax_0 + Bu_0\\
&A^2x_0 + ABu_0 + Bu_1\\
&\vdots\\
&A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} Bu_k
\end{aligned}$$

# Controllability (Discrete Linear Case)

- The feasibility test involves solving the following equation:

$$x^* = A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} B u_k$$

- Rearranging

$$x^* - A^t x_0 = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & B \end{bmatrix}}_{\textbf{Need to have full row rank!}} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{t-1} \end{bmatrix}$$

# Controllability (Discrete Linear Case)

- Consider the discrete-time linear system $x_{k+1} = Ax_k + Bu_k$ where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}$.
- Define the controllability matrix as

$$\mathcal{C} = \left[\begin{array}{cccc} A^{n-1}B & A^{n-2}B & \cdots & B \end{array}\right]$$

**Q: Why it only goes to $A^{n-1}$?**

- Theorem: the system is controllable if and only if

$$rank(\mathcal{C}) = n$$

# Uncontrollable States

- If a state is uncontrollable, then
  - If states are meaningful (physical) variables that need to be controlled, then the design of the actuators are deficient.
  - The effect of control is limited. There is also a possibility of instability.
- Stabilizability: a system is stabilizable if all its uncontrollable states are stable.

# Uncontrollable vs Underactuation

- Uncontrollable: "you can never control it"
- Underactuation: systems that cannot be commanded to follow arbitrary trajectories in configuration space.
  - Usually because the system has a lower number of actuators than degrees of freedom
- An uncontrollable system is underactuated.
- An underactuated system can be controllable.

# Observability

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x, u)$$

- Observability: for any possible sequence of state x and control u, the current state $x_0$ can be determined in **finite** time using only the outputs y.

# Example

Observable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Unobservable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

# Observability (Mathematical Interpretation)

- System trajectory $x(t) = \Phi(x(0), u(0:t))$, $y(t) = h(x(t), u(t))$
- Observability is equivalent to the following uniqueness test:
  - For any $u(0:t)$ and any valid observations $y(0:t)$, there exists a finite $t^*$ such that there is a **unique** solution of $x(0)$ in the following equations

$$y(t) = h(\Phi(x(0), u(0:t)), u(t)), \forall t \in [0, t^*]$$

- All that matters is whether we can solve an inverse of $h \circ \Phi$!

# Observability (Discrete Linear Case)

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

- System trajectory and measurements:

$$\begin{aligned} &x_0 & &Cx_0 + Du_0 \\ &Ax_0 + Bu_0 & &CAx_0 + CBu_0 + Du_1 \\ &A^2x_0 + ABu_0 + Bu_1 & \Rightarrow \quad &CA^2x_0 + CABu_0 + CBu_1 + Du_2 \\ &\vdots & &\vdots \\ &A^t x_0 + \sum_{k=0}^{t-1} A^{t-1-k} Bu_k & &CA^t x_0 + \sum_{k=0}^{t-1} CA^{t-1-k} Bu_k + Du_t \end{aligned}$$

# Observability (Discrete Linear Case)

- The initial state $x_0$ can be obtained by solving the following equation

$$\begin{bmatrix} y_0 - Du_0 \\ y_1 - CBu_0 - Du_1 \\ y_2 - CABu_0 - CBu_1 - Du_2 \\ \vdots \\ y_t - \sum_{k=0}^{t-1} CA^{t-1-k}Bu_k - Du_t \end{bmatrix} = \underline{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^t \end{bmatrix}} x_0$$

**Need to have full column rank!**

# Observability (Discrete Linear Case)

- Consider the discrete-time linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times r}$.

- Define the observability matrix as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

**Q: Why it only goes to $A^{n-1}$?**

- Theorem: the system is observable if and only if

$$rank(\mathcal{O}) = n$$

# Unobservable States

- If a state is unobservable,
  - We cannot determine its value no matter how long we observe.
  - We need to design more sensors to get more information.
  - It may be unstable.
- Detectability: A system is detectable if all the unobservable states are stable.

# Unobservable vs Partially Observable

- Unobservable: “you can never know it”
- Partially observable:
  - The agent cannot directly observe the underlying state. Instead, it must maintain a probability distribution over the set of possible states, based on a set of observations and observation probabilities, and the underlying MDP.
- Observability / unobservability is an intrinsic system property. The uncertainty does not come from stochasity.
- The uncertianty in partial observability comes from stochasity.

# Controllability and Observability

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

- Controllability and observability are dual problems.
- Mathematically, controllability is about existence; observability is about uniqueness.
- Controllability and observability are **independent from stochasticity**.

# Controllability and Observability

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix}$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

# Examples

Controllable, Observable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Uncontrollable, Observable

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

Controllable, Unobservable

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Uncontrollable, Unobservable

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

*** The first state's dynamics are internal**

# Open-Ended Questions

- Any example of real-world uncontrollable / unobservable systems?
- Is a human-robot system controllable / observable from robot's perspective?
- How to modify the system to improve controllability / observability?

# Why Important?

- Controllability and observability of the system affects the feasibility of the controller design problem.
- We can never design a controller to stablize a system that is unstablizable (with uncontrollable states that are unstable).

# Nonlinear Systems*

- Controllability and observability for nonlinear systems
- Linearization
- Lie brackets

http://www.bcamath.org/documentos_public/courses/TalkBCAM20110707controllability-Coron.pdf

# System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

# Autonomous Systems

- An autonomous system is a system that is time-invariant

$$\dot{x} = f(x)$$

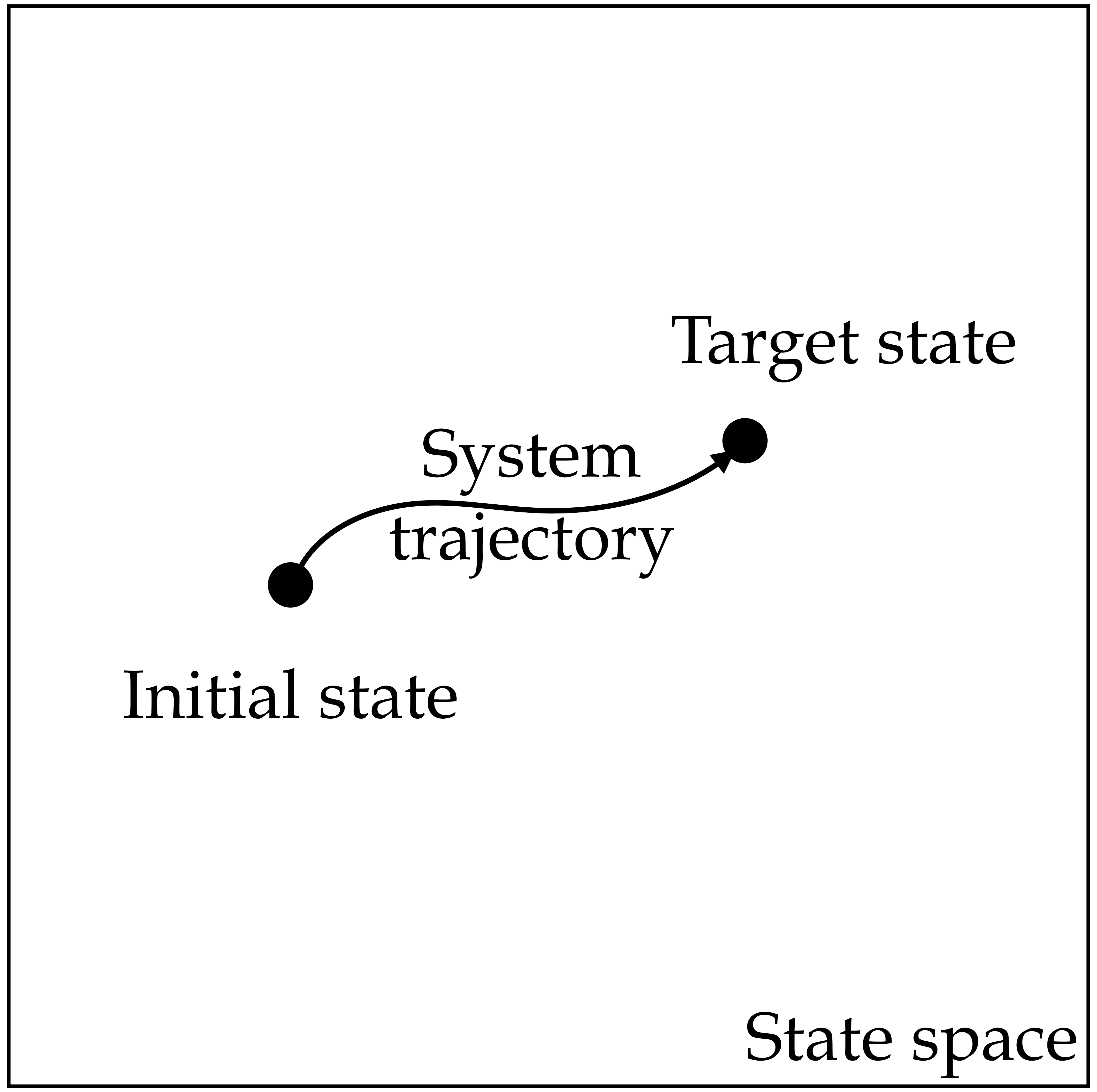
- It can be a closed-loop system

$$\dot{x} = f^o(x, u), u = g(x)$$

# Analysis of Autonomous Systems

- Will the system rest in the target state?
  - Equilibrium
  - Stability

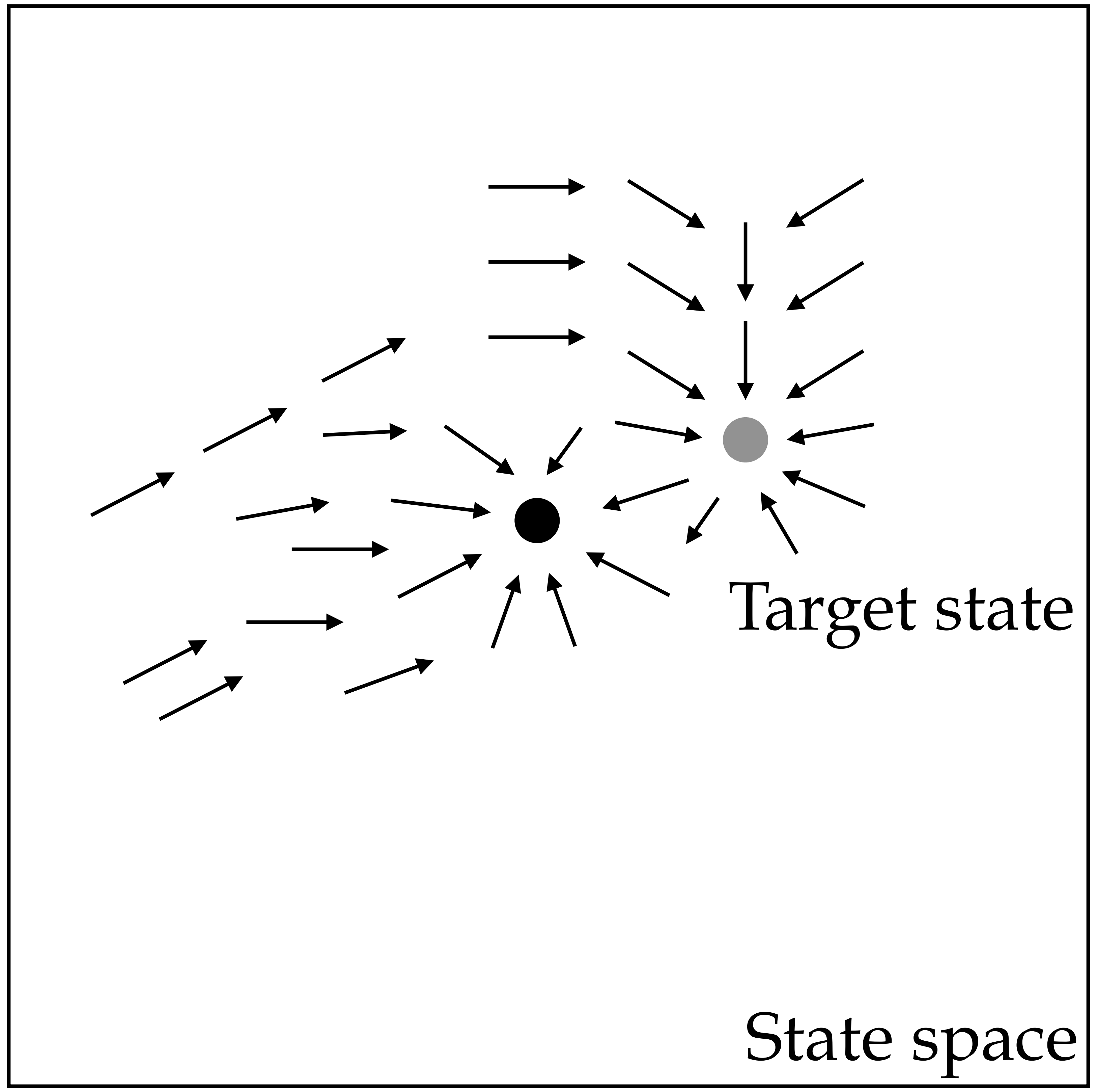
$\dot{x} = f(x)$



# Analysis of Autonomous Systems

- Will the system rest in the target state?
    - Equilibrium
    - Stability

$\dot{x} = f(x)$ defines a vector field



# Equilibrium

- Equilibrium is where $\dot{x} = f(x) = 0$

- Define $J(x) = \frac{\partial f}{\partial x}(x)$

- Finding Equilibrium point by solving the following optimization:

$$\min_x \|f(x)\|^2$$

  - Newton method: $x_{i+1} = x_i + \alpha J^{-1}(x_i) f(x_i)$

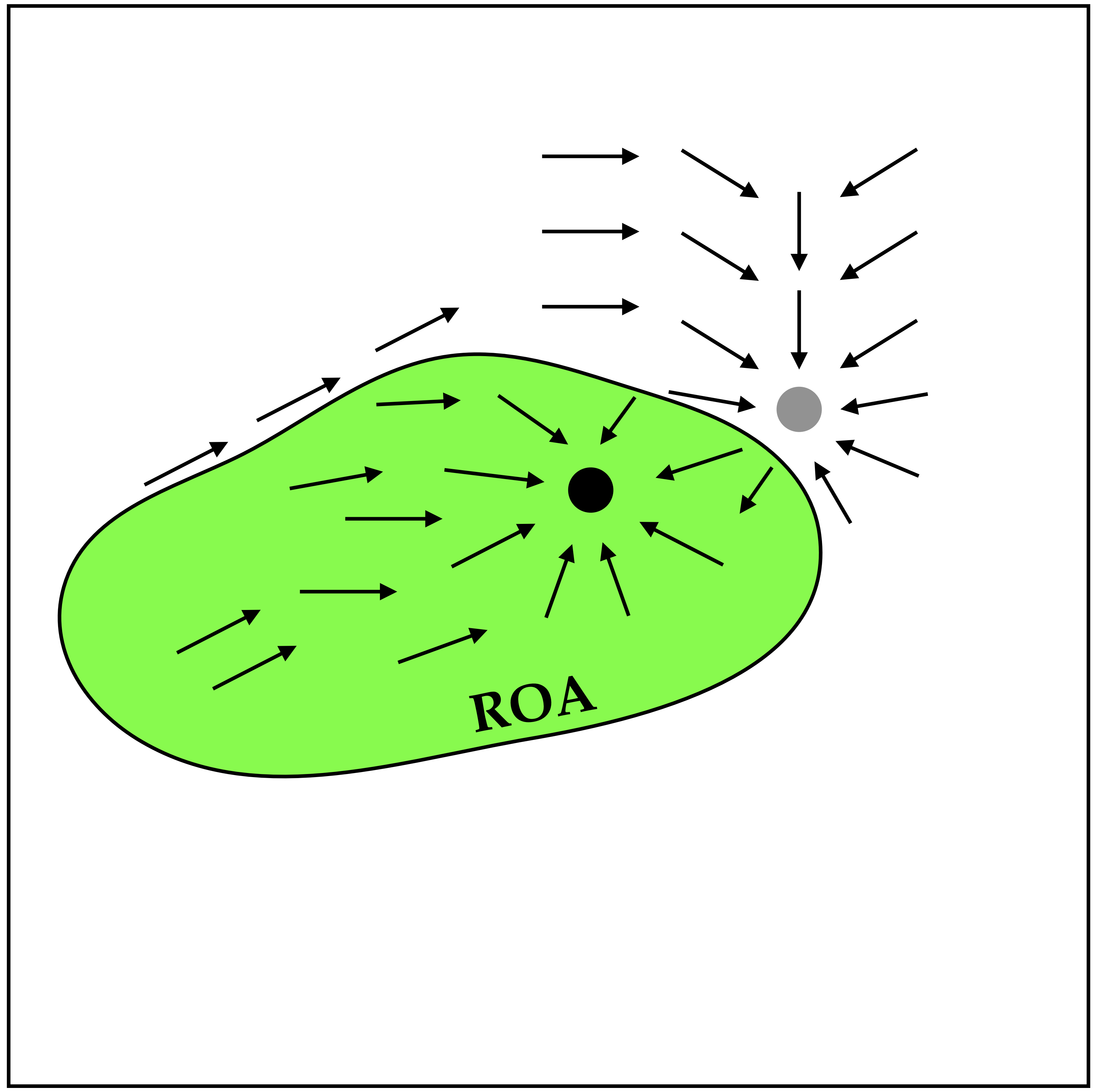
  - Steepest descent: $x_{i+1} = x_i - \beta J^T(x_i) f(x_i)$

# Equilibrium

- (Uniqueness) Global Implicit Function Theorem: the equilibrium is unique if
  - $\det [J(x)] \neq 0, \forall x$
  - $\|f(x)\| \to \infty$ as $\|x\| \to \infty$

# Region of Attraction (ROA)

- Region of attraction of an equilibrium
  - The set of initial states under which the system will eventually reach the equilibrium.



# Stability

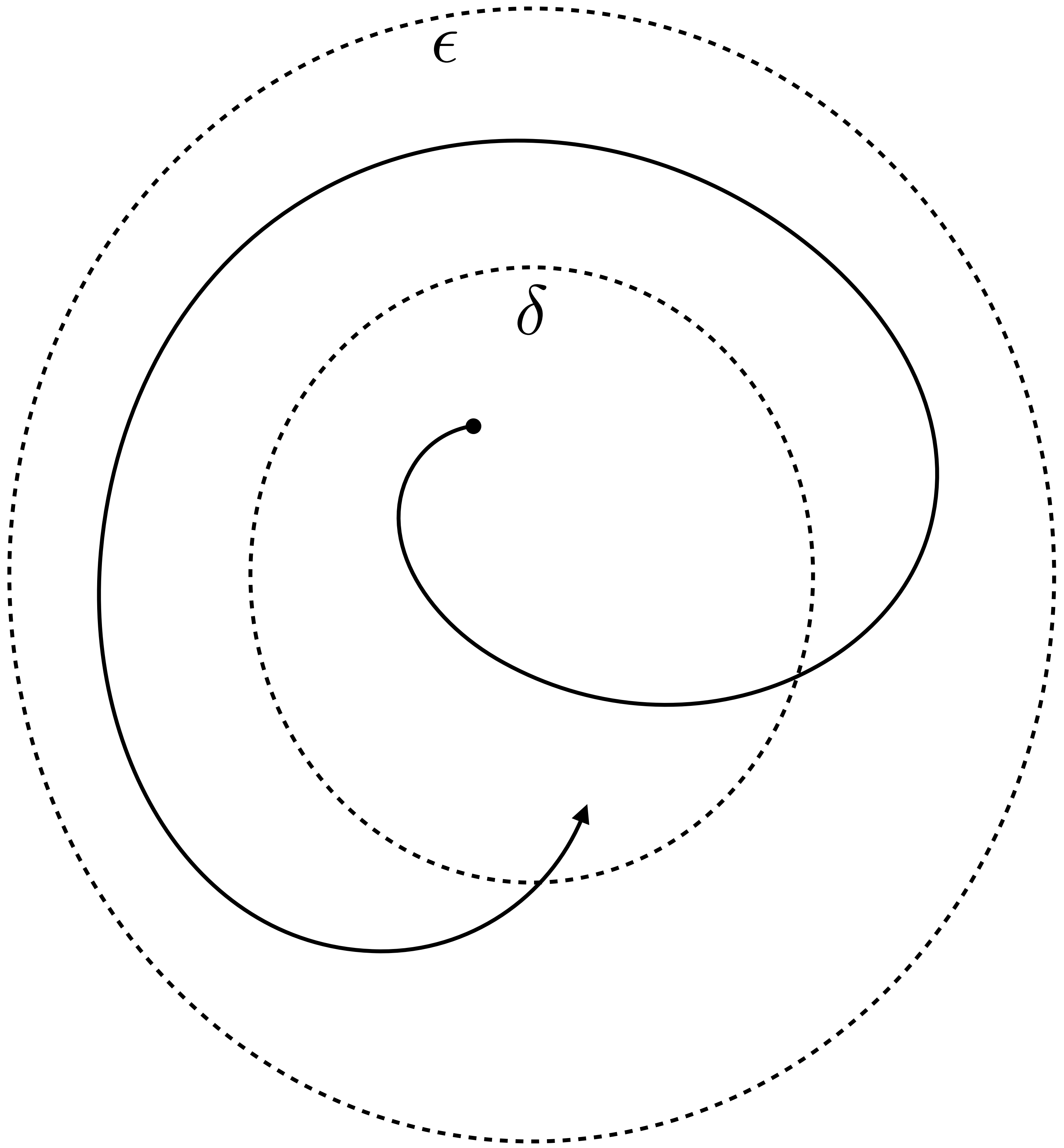
- $\epsilon - \delta$ definition
- Stability for linear systems
- Lyapunov stability

# Stability

- $\epsilon - \delta$ definition:

- A system is stable at origin if and only if for any $\epsilon > 0$, there exist a $\delta > 0$ such that for all initial states at $t_0$ that satisfy $\|x_0\| < \delta$,

$$\|x(t)\| < \epsilon, \forall t > t_0$$



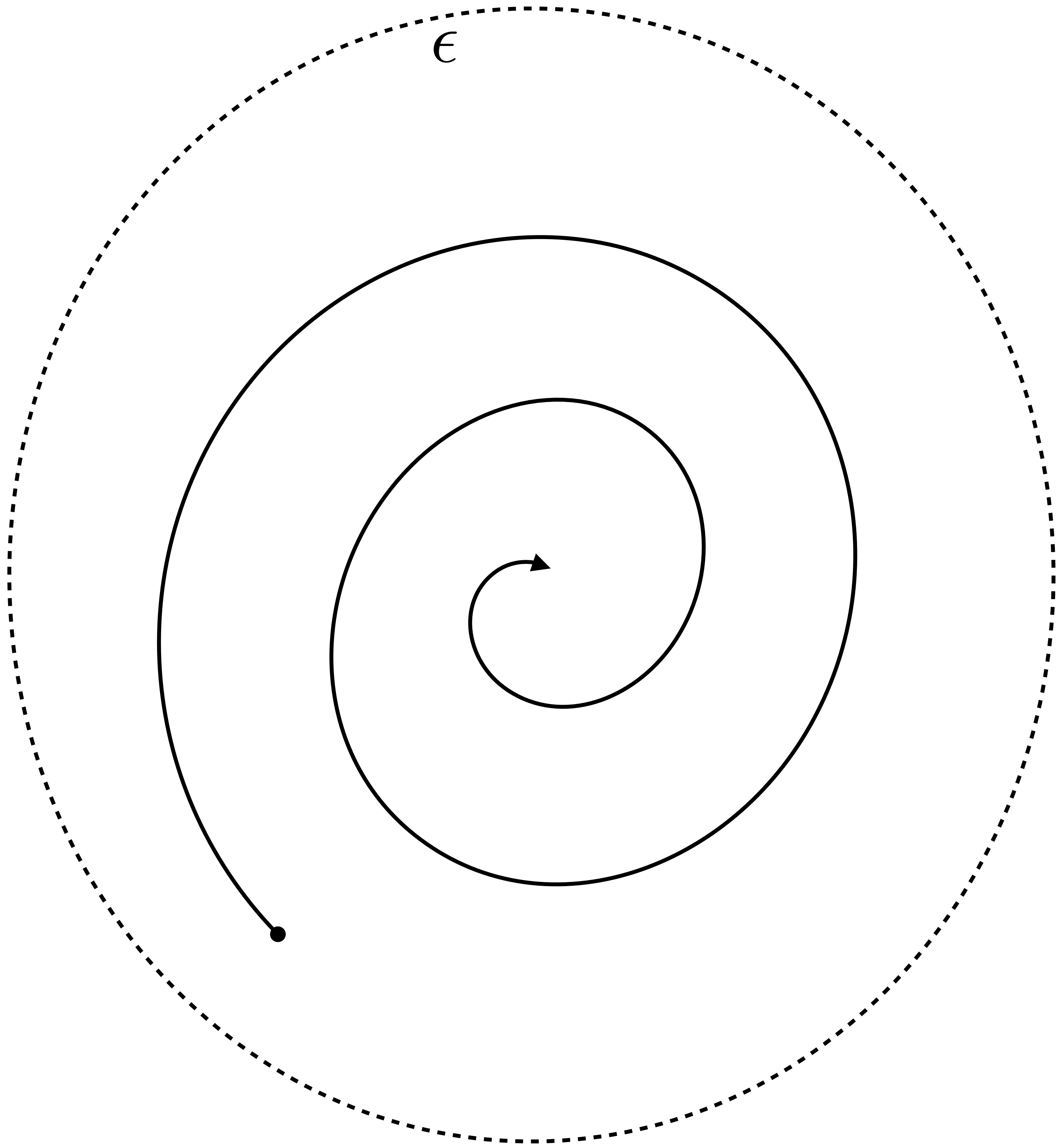

**Q: Can $\delta > \epsilon$?**

# Asymptotic Stability

- $\epsilon - \delta$ definition:
- A system is globally asymptotically stable at origin if and only if 1) the system is stable; 2) for any initial condition $x_0$ and for any $\epsilon > 0$, there exist a time $t^* > t_0$ such that

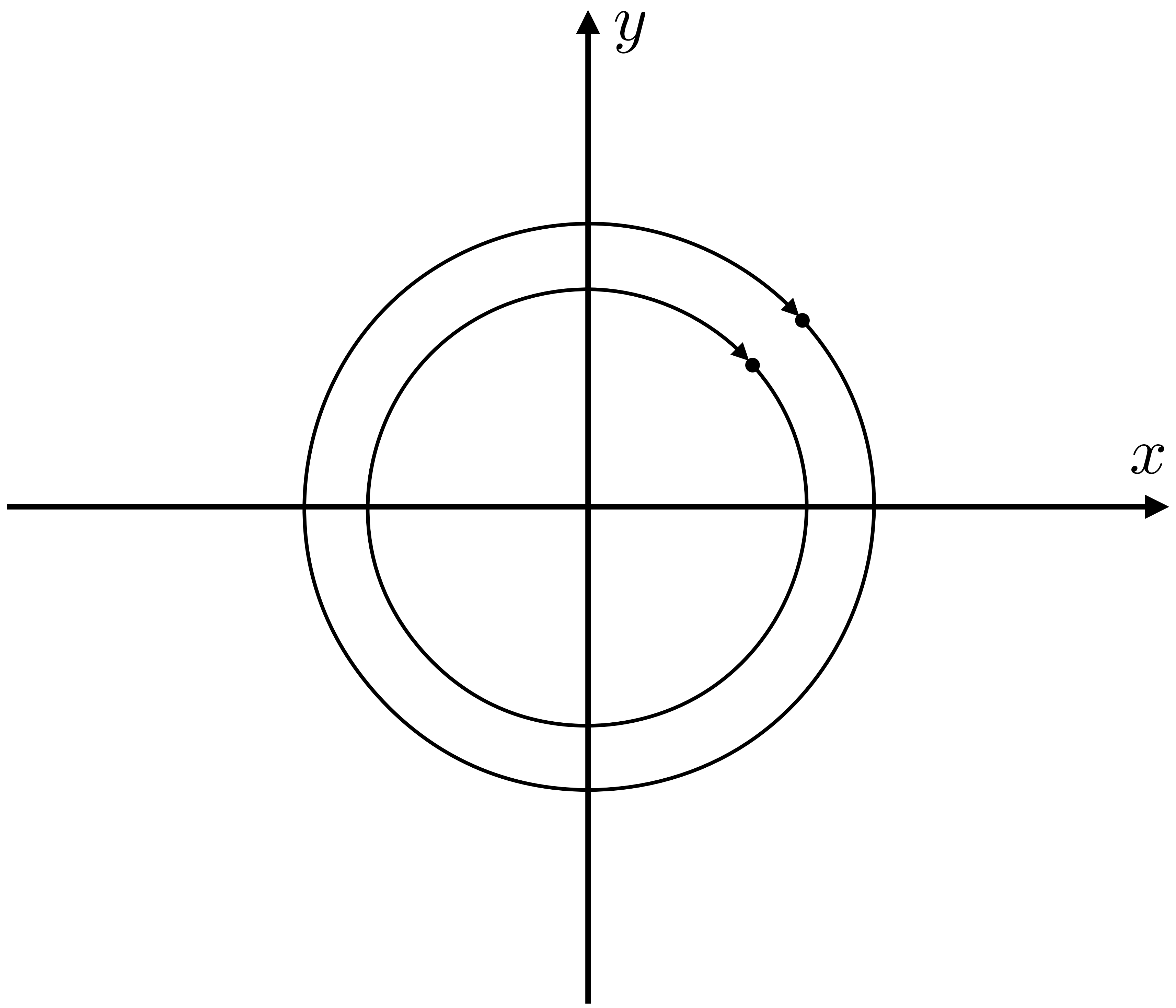
$$\|x(t)\| < \epsilon, \forall t > t^*$$

*** This is a convergence condition**



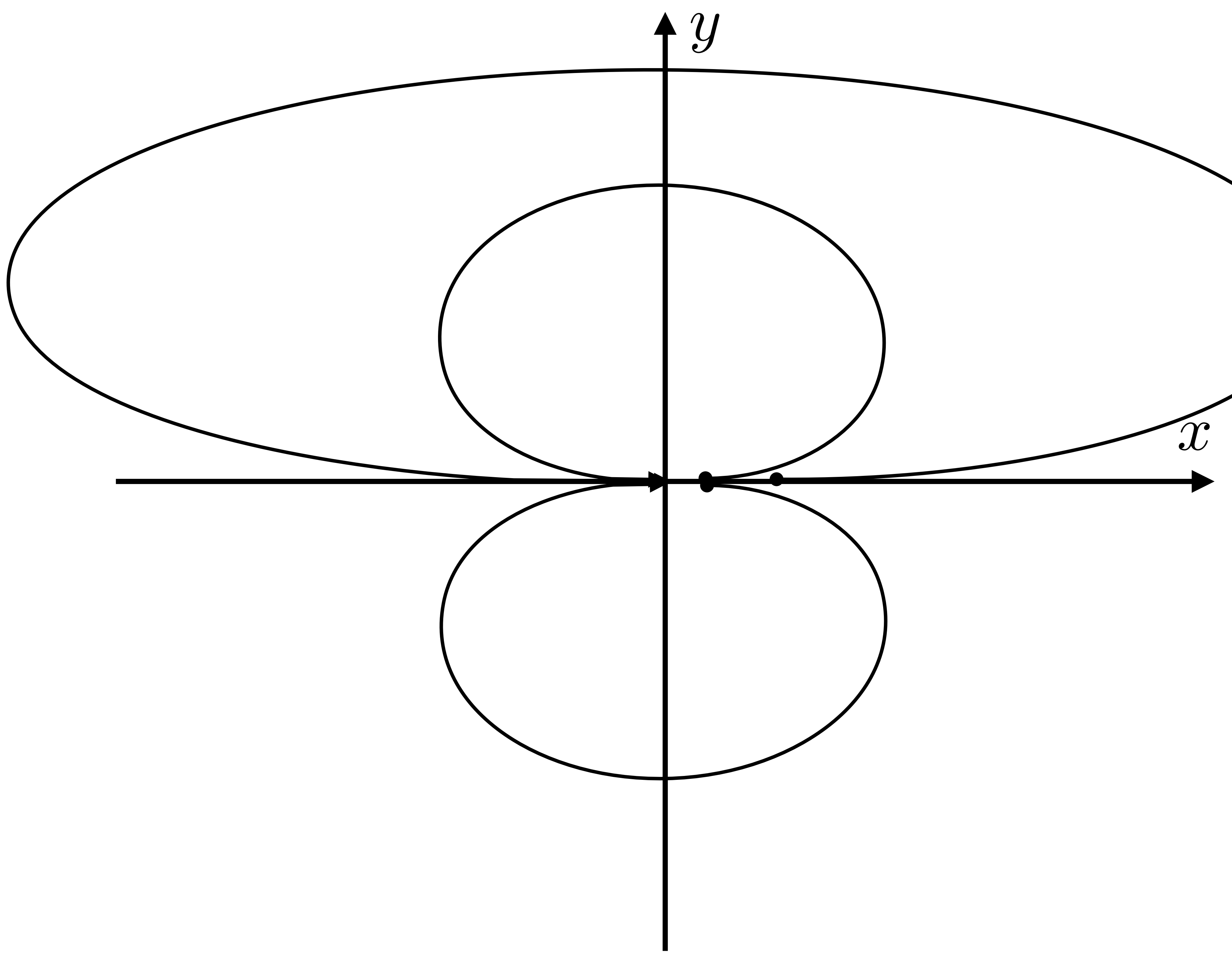
# Examples

- Stable but not asymptotically stable
  - e.g., $\dot{x} = y, \dot{y} = -x$



# Examples

- Converge but not stable
  - e.g., $\dot{x} = x^2 - y^2, \dot{y} = 2xy$ (here we need to identify the positive infinity with the negative infinity)



# Stability for Linear Systems

- Linear system $\dot{x} = Ax$
- Solution: $x(t) = x(0)e^{At}$
- Stability of the system depends on the exponential term $e^{At}$

# Stability for Linear Systems

- Eigen-values $\lambda_i$ satisfy that $Av_i = \lambda_i v_i$, where $v_i$ is the eigen-vector associated with that eigen-value.

- Eigen-value decomposition

$$A \underbrace{[v_1 \; v_2 \; \dots \; v_n]}_{V} = \underbrace{[v_1 \; v_2 \; \dots \; v_n]}_{V} \underbrace{\operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)}_{\Sigma}$$

$$A = V \Sigma V^{-1}$$

- By Similarity Transformation, the exponential term is

$$e^{At} = V \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}) V^{-1}$$

# Stability for Linear Systems

- $e^{At} = V\text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})V^{-1}$
- By Euler's formula $e^{\lambda_i t} = e^{\text{Re}(\lambda_i)t}[\cos(\text{Im}(\lambda_i)t) + i\sin(\text{Im}(\lambda_i)t)]$
- Assume the system has distinct eigen-values.
- The system is stable if and only if

$$\text{Re}(\lambda_i) \leq 0, \forall i$$

- The system is asymptotically stable if and only if

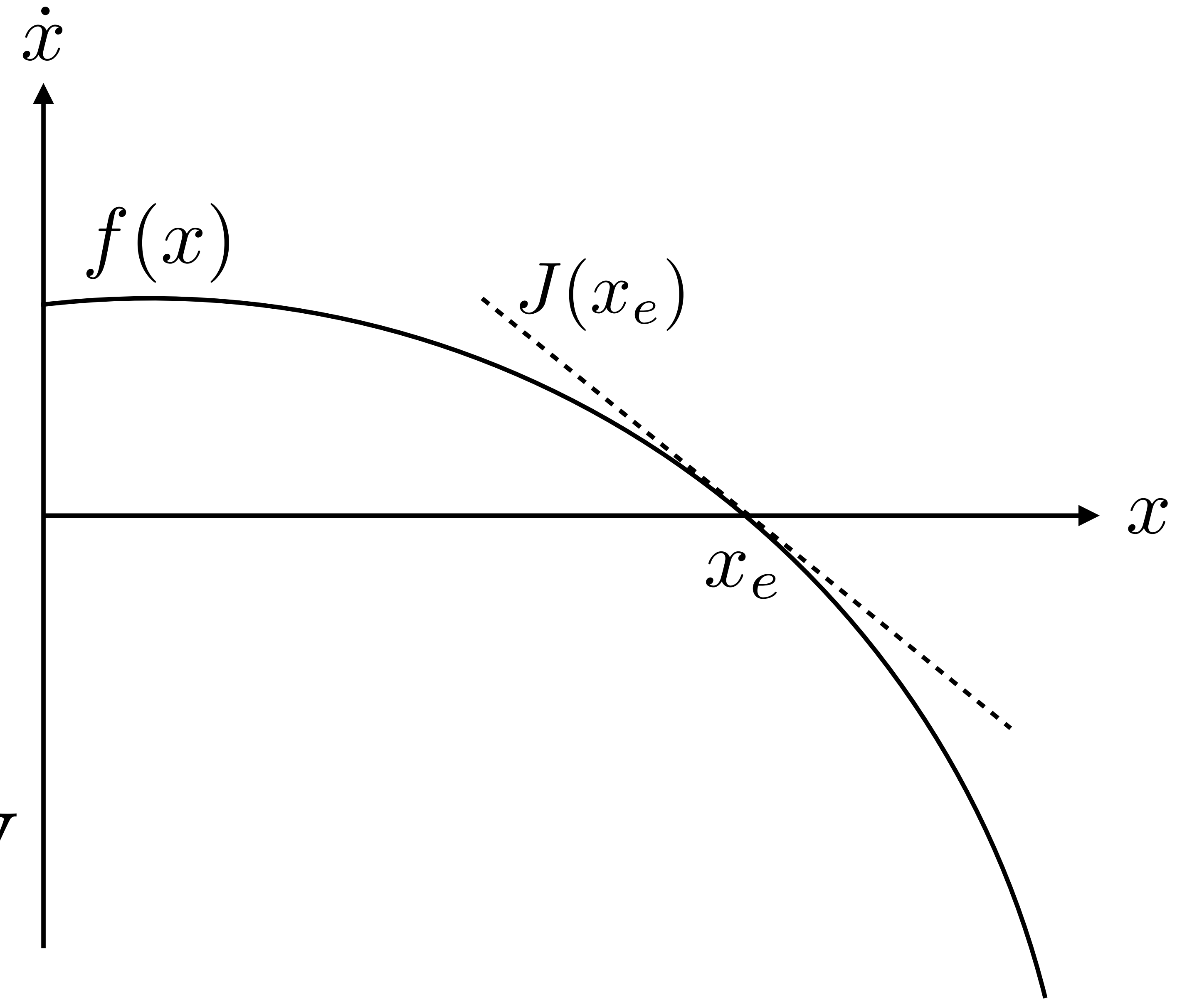
$$\text{Re}(\lambda_i) < 0, \forall i$$

*** Exponential decay**

# Lyapunov 1st method

- For the nonlinear system $\dot{x} = f(x)$, linearize around the equilibrium point $x_e$,

$$\delta\dot{x} = J(x_e)\delta x + h(x_e, \delta x)$$

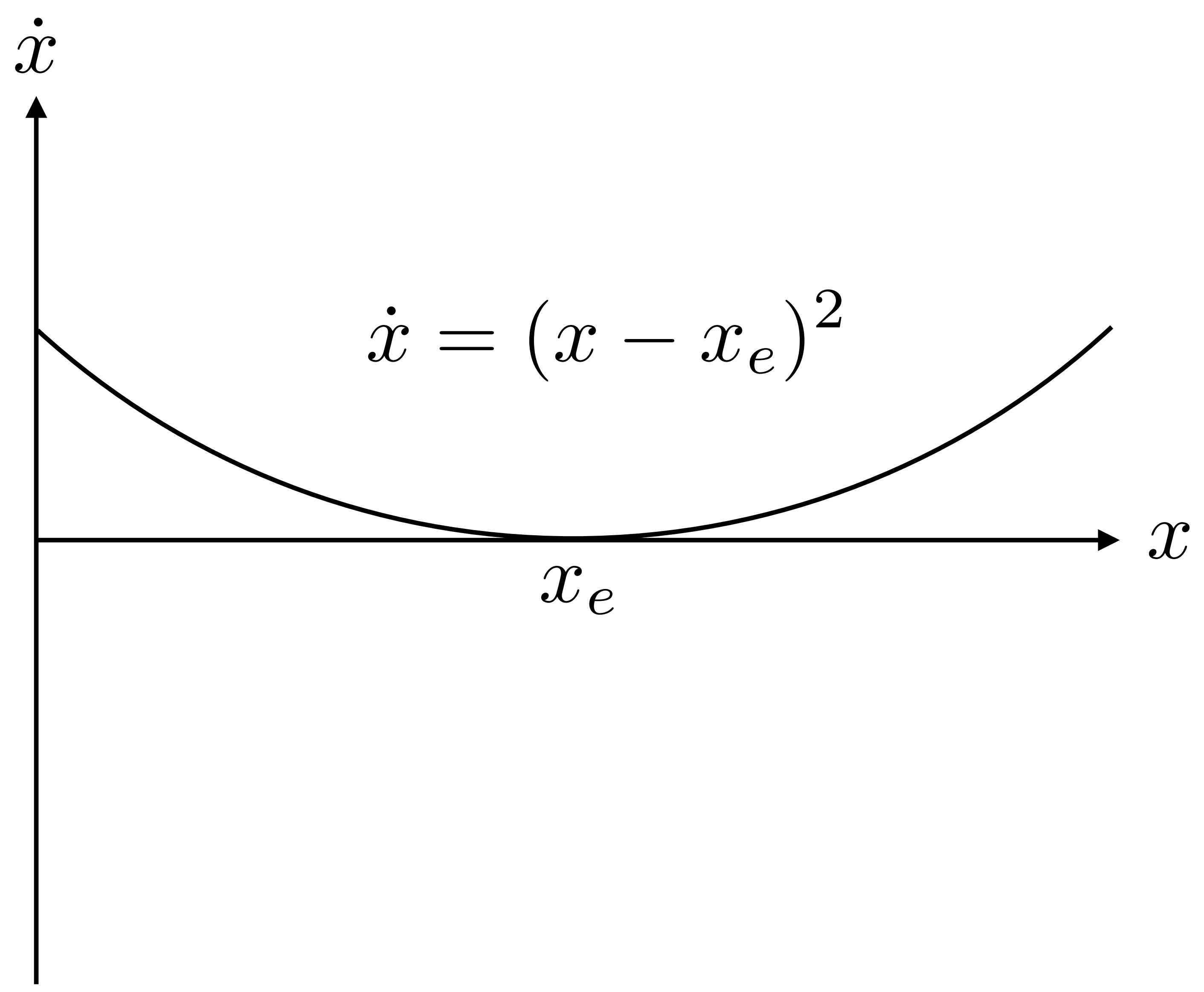



- Then linear stability analyses apply locally if
  - All eigen-values of $J(x_e)$ have non-zero real parts
  - The error term decays faster than first order decay

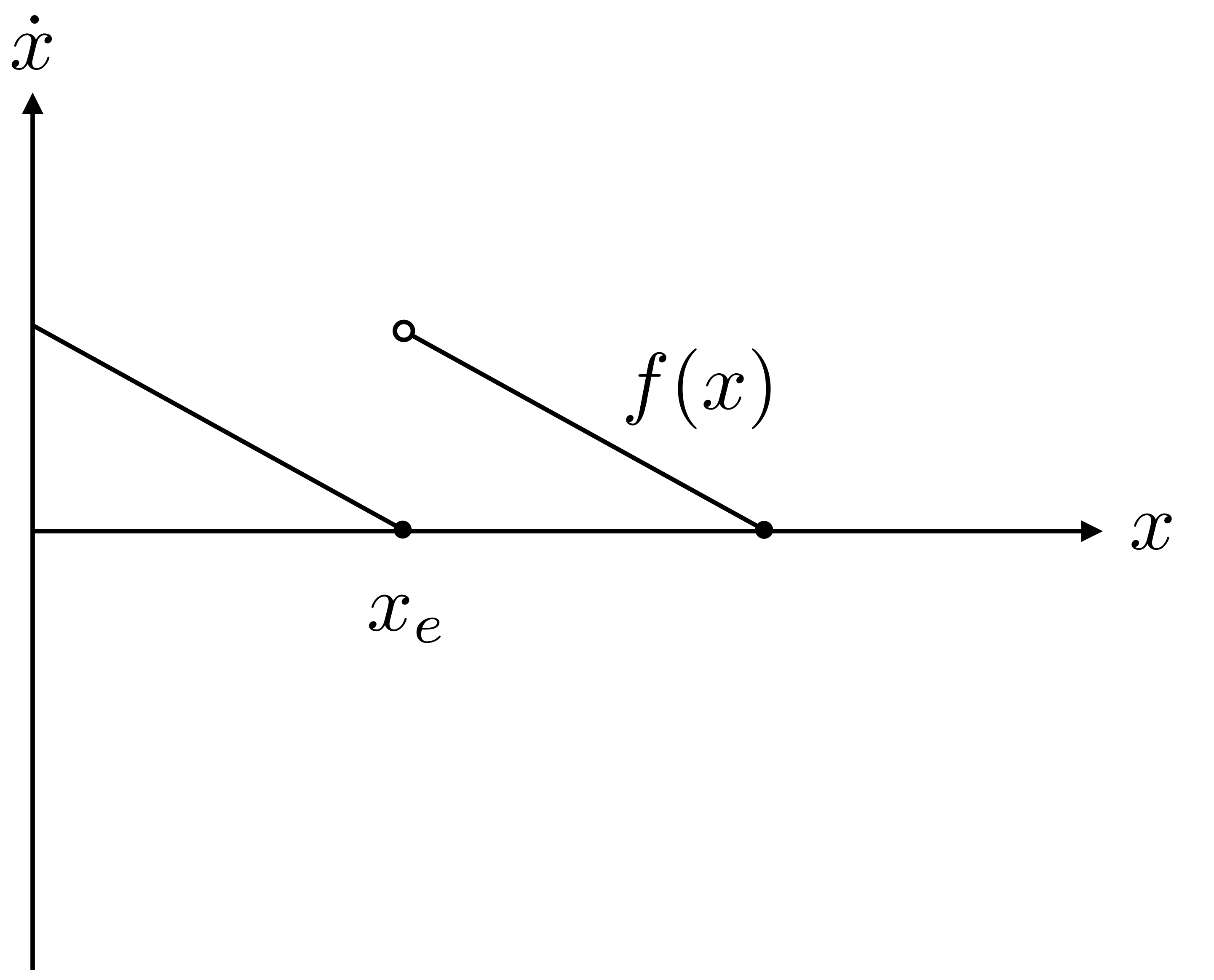
$$\lim_{\|\delta x\| \to 0} \frac{\|h(x_e, \delta x)\|}{\|\delta x\|} \to 0$$

# Lyapunov 1st method

- Scenarios that cannot be analyzed by Lyapunov 1st method




**Eigen-value is zero.**




**Error term does not decay.**

# Other Methods*

- Scalar energy function-based methods (to be discussed later):
  - Lyapunov 2nd method
  - Passivity

# Why Important?

- Stability analysis is important to formally ensure that the objective can always be achieved.

# System Properties

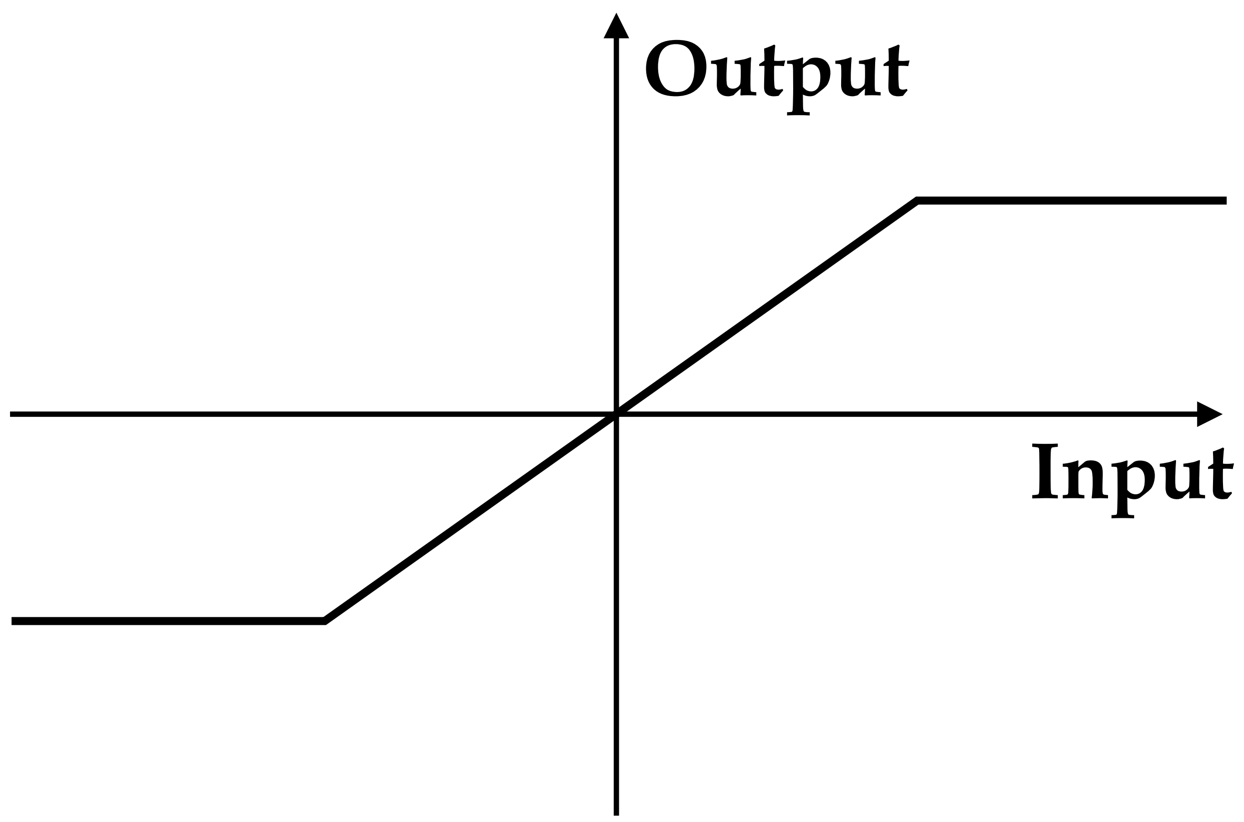
- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems

# Nonlinear Systems

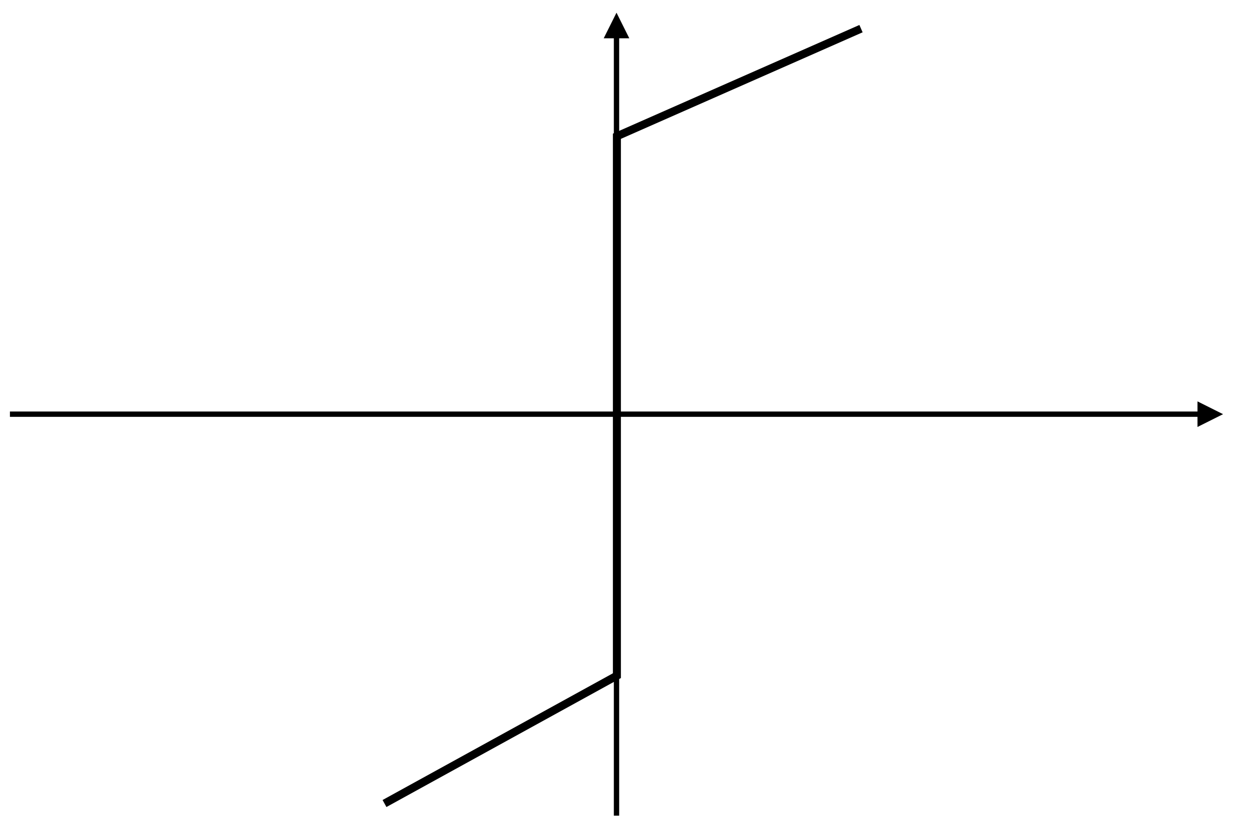
- A systems $\dot{x} = f(x)$ is nonlinear if it does not satisfy the following two properties:
  - Superposition $f(x+y) = f(x) + f(y)$
  - Homogeneity $f(kx) = kf(x)$

# Types of Nonlinearity

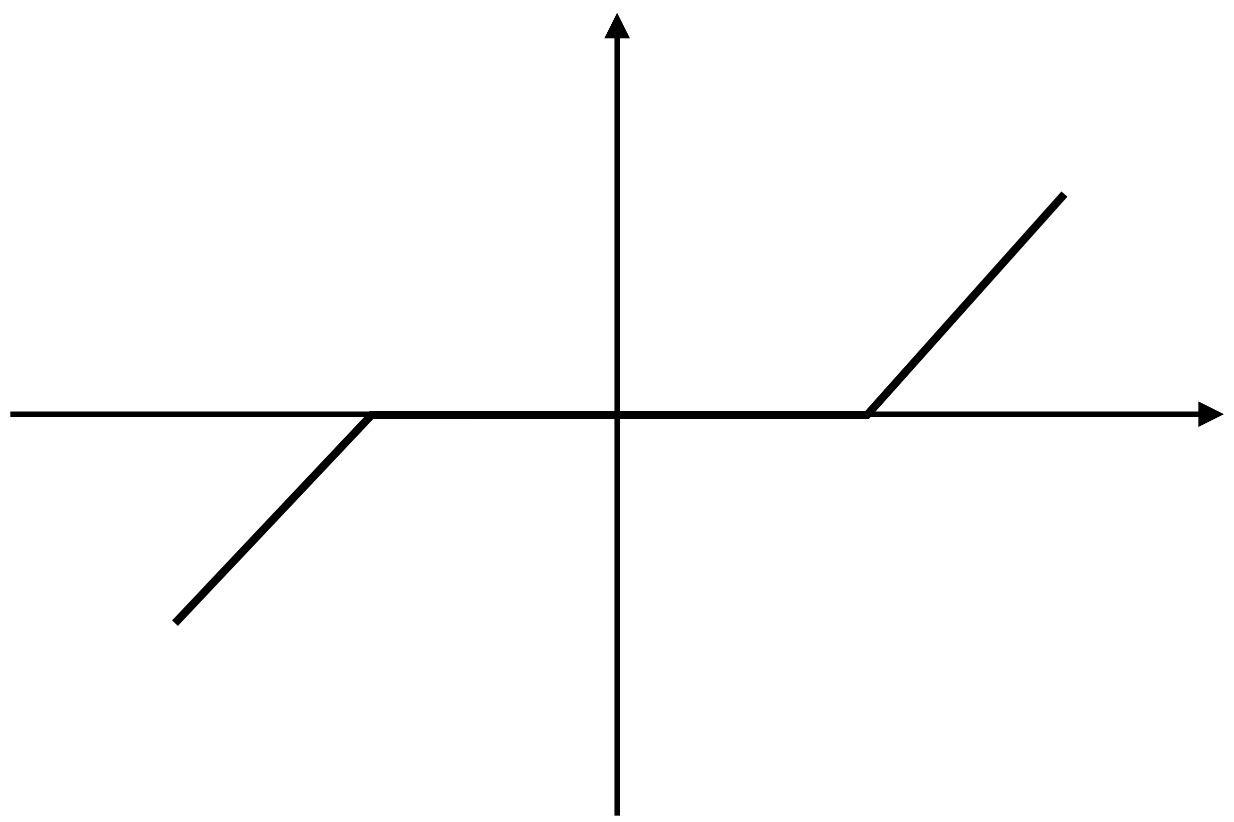
- Single-valued & time-invariant




Saturation



Friction

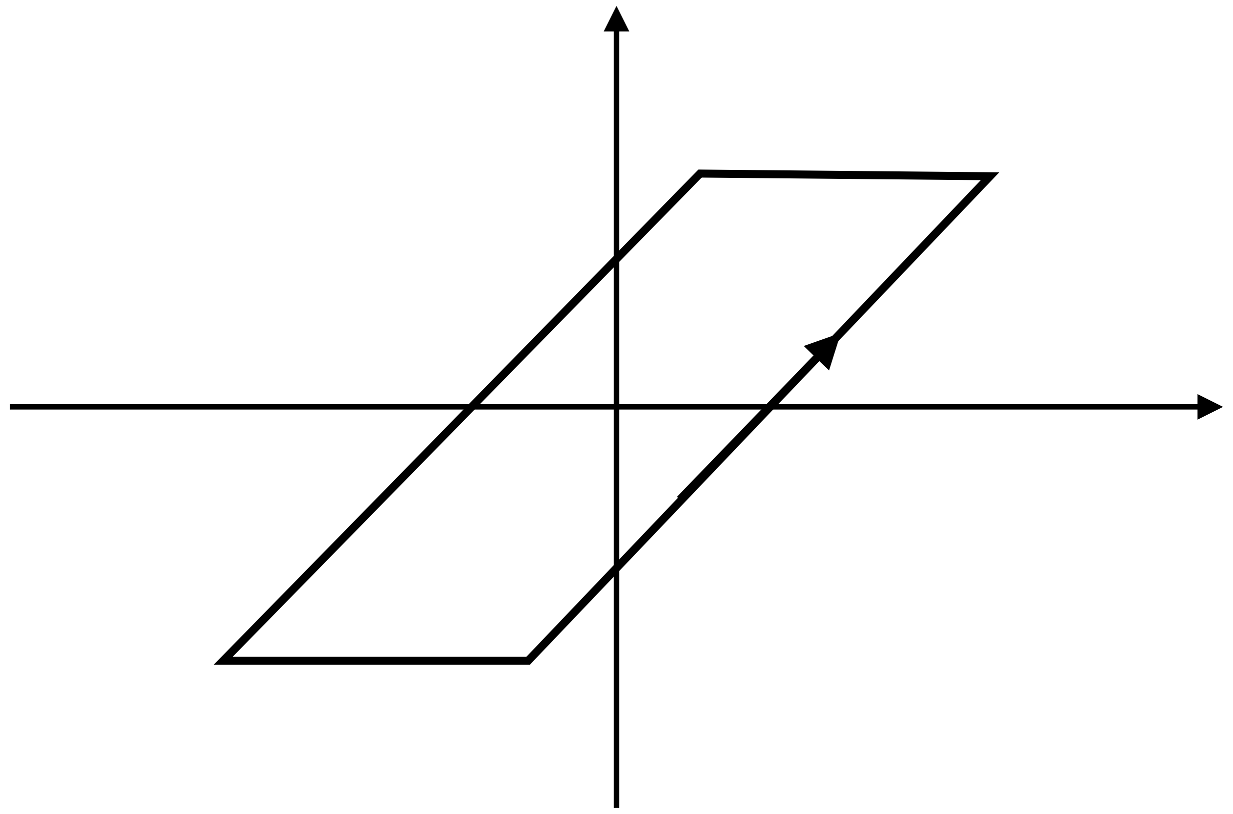


Dead zone

* Note the similarity to the neural activation functions

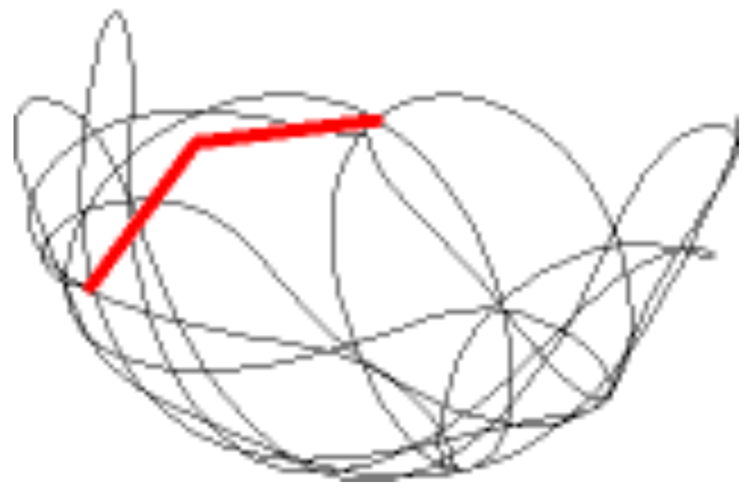
# Types of Nonlinearity

- Memory
  - In general, a nonlinearity with "memory" means that we have not included all states that define the nonlinearity



# Chaos

- When the present determines the future, the **approximate** present does not **approximately** determine the future.
- A small change in initial conditions can result in huge change in the future.
- Chaos is different from uncertainty.
- Unpredictability.



**A double-rod pendulum animation showing chaotic behavior. Starting the pendulum from a slightly different initial condition would result in a completely different trajectory. The double-rod pendulum is one of the simplest dynamical systems with chaotic solutions.**

# Nonlinear Systems

| **Mechanical** | **Electrical** | **Computational** | **Economical** | **Social** |
|---|---|---|---|---|
| Pendulum | Circuit | AI | Stock market | Social network |

$\longrightarrow$

**Dimensionality**

# Nonlinear Systems

- Properties:
    - Multiple equilibrium points
    - Local vs. global stability
    - Response characteristics depends upon initial conditions, input type and input magnitude

# System Properties

- Controllability and Observability
- Stability
- General Properties of Nonlinear Systems