MPC Disturbance / Integral action

$$\begin{bmatrix} x_{k+1} \\ v_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_k \\ v_k \end{bmatrix} + \underbrace{\begin{bmatrix} B & 0 \\ 0 & -1 \end{bmatrix}}_{\bar{B}} \underbrace{\begin{bmatrix} u_k \\ r_k \end{bmatrix}}_{\bar{u}}$$

$$\bar{y}_k = \underbrace{\begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \text{ where } C = [1\ 0\ 0\ 0]$$

$$v_{k+1} = v_k + \bar{y}_k - \bar{r}_k, \text{ where } \bar{r}_k = \begin{bmatrix} \text{position}_k \\ 0 \end{bmatrix}$$

$$J = \sum_{k=0}^{N-1} (\bar{y}_k - \bar{r}_k)^T Q (\bar{y}_k - r_k) + \bar{u}_k^T R \underbrace{\bar{u}_k}_{= \text{decision variable}}$$

$$J(z, x_0) = [\bar{y}(0) - \bar{r}(0)]^T Q [\bar{y}(0) - \bar{r}(0)]$$

$$+ \begin{bmatrix} \bar{y}(1) - \bar{r}(1) \\ \vdots \\ \bar{y}(N) - \bar{r}(N) \end{bmatrix}^T \bar{Q} \begin{bmatrix} \bar{y}(1) - \bar{r}(1) \\ \vdots \\ \bar{y}(N) - \bar{r}(N) \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{u}_0 \\ \vdots \\ \bar{u}_{N-1} \end{bmatrix}^T \bar{R} \begin{bmatrix} \bar{u}_0 \\ \vdots \\ \bar{u}_{N-1} \end{bmatrix}$$

$$\begin{bmatrix}\bar{y}(1)\\ \vdots \\ \bar{y}(N)\end{bmatrix} - \begin{bmatrix}\bar{r}(1)\\ \vdots\\ \bar{r}(N)\end{bmatrix} = \underbrace{\begin{bmatrix} \bar{C}\bar{B} & & & 0\\ \bar{C}\bar{A}\bar{B} & \ddots & & \\ \vdots & & \ddots & \\ \bar{C}\bar{A}^{N-1}\bar{B} & \cdots & \cdots & \bar{C}\bar{B}\end{bmatrix}}_{\bar{S}}\begin{bmatrix}\bar{u}_0\\ \vdots\\ \bar{u}_{N-1}\end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix}\bar{C}\bar{A}\\ \vdots\\ \bar{C}\bar{A}^N\end{bmatrix}}_{\bar{T}}\begin{bmatrix}x(0)\\ v(0)\end{bmatrix} - \underbrace{\begin{bmatrix}\bar{r}(1)\\ \vdots \\ \bar{r}(N)\end{bmatrix}}_{\bar{R}_{ref}}$$

$$\begin{aligned}
J(z, x(0)) &= \left[\bar{S}z + \bar{T}y(0) - \bar{R}_{ref}\right]^T \bar{Q} \left[\ \text{''}\ \ \text{''}\ \right] \\
&\quad + z^T \bar{R} z + \overset{\text{no effect}}{\cancel{(\bar{y}(0) - \bar{r}(0))^T \bar{Q} \left[\ \text{''}\ \ \text{''}\ \right]}} \\
&= \left[z^T \bar{S}^T \bar{Q} \bar{S} z + z^T \bar{S}^T \bar{Q} \bar{T} y(0) - z^T \bar{S}^T \bar{Q} \bar{R}_{ref}\right] \\
&\quad + \bar{y}(0) \bar{T}^T \bar{Q} \bar{S} z - \bar{R}_{ref}^T \bar{Q} \bar{S} z \\
&= z^T \underbrace{\left[\bar{S}^T \bar{Q} \bar{S} + \bar{R}\right]}_{H} z + 2 \underbrace{\left[\bar{y}(0)^T \bar{T}^T \bar{Q} \bar{S} - \bar{R}^T \bar{Q} \bar{S}\right]}_{F} z
\end{aligned}$$

Constraints:

$$\left[\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \\ \hline -1 & & \\ & \ddots & \\ & & -1 \end{array}\right] z \leq \left[\begin{array}{c} \bar{u}_{max} \\ \vdots \\ \hline -\bar{u}_{min} \\ \vdots \end{array}\right]$$

$$\text{e.g.} \quad \bar{u}_{max} = \begin{bmatrix} 12 \\ r \end{bmatrix} \begin{array}{l} \leftarrow \text{volts} \\ \leftarrow \text{reference to input/track} \end{array}$$

* Same constraints but include "r" as input.