REASONING ABOUT KNOWLEDGE AND CHANGE

DYNAMIC EPISTEMIC LOGIC AND THE MUDDY CHILDREN PUZZLE

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Dynamic Epistemic Logic

Introduction

Founded by Georg Henrik von Wright[1], with major contributions by Jaakko Hintikka[2], and based upon Saul Kripke's modal logic[3], dynamic epistemic logic (*DEL*) is "the logic of knowledge"[4]. Its major aim is to express the information considered to be true (or believed as such) by a group of agents in a given universe. Before we can proceed we must give a number of definitions:

World: Contains a set of known formulas, sentences or relations. Information contained within the same world cannot be contradictory (p and $\neg p$ cannot be known in the same world).

Model: Determines the "universe" within which the given worlds exist. Within this introduction to epistemic logic we will never consider more than one different model.

Accessible: There are a number of relations which can be established in-between worlds (we will look at a few below), but such relations will only hold in-between two accessible worlds, and "two worlds *i* and *j* [are accessible] if all propositions that are true in *i* are possible in *j*)[5].

Agent: Agents are entities that can travel through relations in-between worlds. "A particular world *j* is […] accessible to an agent *a* in world *i*, iff the set of all propositions *p* that *a* knows in *i* are compatible with all true propositions in *j*"[6]. This means that agents won't be able to access world *j*s containing statements in contradiction with what they know in their original world *i*s.

Having defined agents, worlds, and relationships between the two, we will now look at the possible different kinds of access relationships that can be established by agents between worlds:

Knowledge: Agent *c* in world *i*, has knowledge of *p* in world *j*, if in all worlds $i_1..i_n$ *c* has knowledge of, *p* holds. This very definition was given by Hintikka (1962) as:
"K_c(A): in all possible worlds compatible with what c knows, it is the case that A"[7]
Where "K" expresses the Knowledge Relationship, "c" is the Agent, and "A" is the logical expression that is meant to hold.

Belief: Agent *c* in world *i*, believes *p* in world *j*, if in all worlds $i_1..i_n$ *c* has knowledge of, *p* is possible. This very definition was given by Hintikka (1962) as:
"B_c(A): in all possible worlds compatible with what c believes, it is the case that A"[8]
Where "B" expresses the Belief Relationship, "c" is the Agent, and "A" is the logical expression that is meant to hold.

We could therefore define "belief as the attitude of assent towards the truth of [a] particular proposition, and knowledge as [a] justified true belief"[9].
Having introduced the syntax, we can now say that the following are perfectly acceptable statements in Epistemic Logic:

K_c(p) : Agent *c* knows *p*

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[1] http://www.britannica.com/biography/G-H-von-Wright, [Accessed March 2016]
[2] http://thebiography.us/en/hintikka-kaarlo-jaakko, [Accessed March 2016]
[3] http://www.britannica.com/biography/Saul-Kripke, [Accessed March 2016]
[4] See reference A.
[5] See reference F at minute 1:15.
[6] See reference F at minute 2:00.
[7] See reference B.
[8] See reference B.
[9] See reference G at minute 2:00.

$B_c(p)$: Agent *c* believes *p*
$K_c(p \rightarrow q)$: Agent *c* knows that *p* implies *q*
$K_c(K_a(p))$: Agent *c* knows that agent *a* knows *p*
$K_c(K_c(p))$: Agent *c* knows that agent *c* knows *p*

Models

Having seen a part of the syntax for Epistemic Logic, we can now to proceed to analyse how Knowledge and Belief relationships apply in actual models. We will introduce the following last bit of graphical notation[10 11 12], with regards to *Figure 0*.

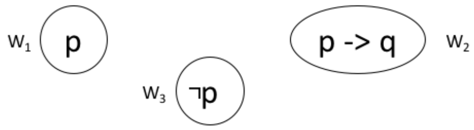

Figure 0

$M, w_1 \vDash p$: Within model *M*, p is true in the situation represented by world w_1
$M, w_2 \vDash p \rightarrow q$: Within model *M*, $p \rightarrow q$ is true in the situation represented by world w_2
$M, w_3 \vDash \neg p$: Within model *M*, $\neg p$ is true in the situation represented by world w_3

Given the above notation, we can introduce Accessibility Relationships into our EL Models:

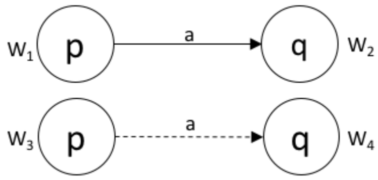

Figure 1a

$M, w_1 \vDash K_a(q)$: Within model *M*, agent *a* knows q in the situation represented by world w_1
$M, w_3 \vDash B_a(q)$: Within model *M*, agent *a* believes q in the situation represented by world w_3

We can see how the above statements agree with the above definitions, but lets take a closer look at the following figure:

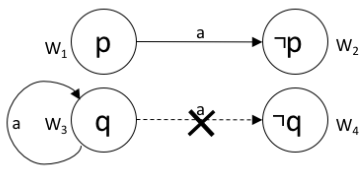

Figure 1b

Within such image the following holds:
$M, w_1 \vDash K_a(\neg p)$,

[10] See reference C. The notation used by Eric Pacuit will be used to express (and thus simplify) our Coin Toss & Muddy Children problem.
[11] See reference D. Even though Agents are not explicitally expressed in this notation, it helps in comprehending and expressing Epistemic Logic Models and Sentences.
[12] See reference E, pages 4-6. Our graphical notation for models was taken from Wesley H. Holliday's report.

As M, $w_1 \vDash K_a(p)$ is NOT TRUE (i.e. There is NO Knowledge Relationship (KR) from w_1 to w_1 with regards to agent *a*.

If we take a closer look at w_3 and w_4, though, we see that the following should hold:
$M, w_3 \vDash K_a(q)$
$M, w_3 \vDash K_a(\neg q)$
This is a contradiction and it means that *M* is NOT well-modelled. According to the definitions given above, such a case should never occur.

Even though it is true that no agent *a* can know a contradiction (as in p and ¬p), *Figure 2a* shows two valid examples:

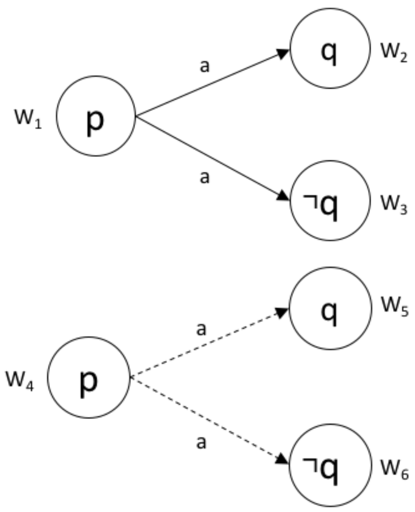

Figure 2a

$M, w_1 \vDash K_a(\neg p \vee p)$
$M, w_4 \vDash B_a(\neg p \vee p)$
Both of the above are not contradictions, as agent a knows *p* OR ¬*p*. Could he ever know both? No. Once agent *a* knows *p*, he won't be able to have knowledge of ¬*p*, and such KR is dropped from the model *M*.

If, instead we consider *Figure 2b*:

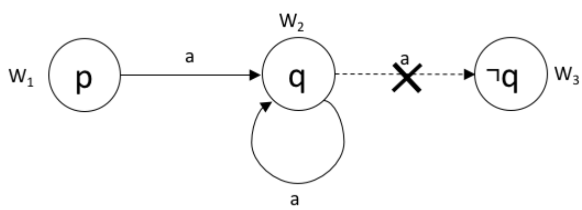

Figure 2b

The model would say:
$M, w_1 \vDash K_a(q)$
$M, w_2 \vDash K_a(q \wedge \neg q)$
Which is a contradiction. (*M* is not well-modelled).

Having discussed and analysed what are the characteristics of a well-modelled model *M*, we can now look at a more complex situation expressed by *Figure 3*:

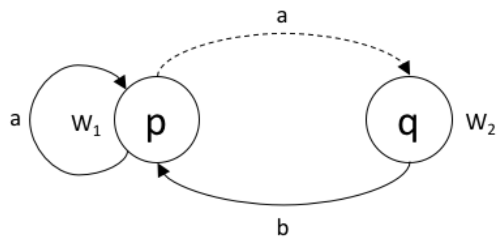

Figure 3

In such a model, all of the following would hold:
$M, w_1 \vDash K_a(p)$
$M, w_1 \vDash B_a(q)$
$M, w_1 \vDash B_a(K_a(p))$
$M, w_1 \vDash K_a(K_a(K_a(K_a(p))))$
$M, w_2 \vDash K_a(B_a(q))$

We now have an example of the power of Epistemic Logic. It can express the cumulative information that is known by groups of individuals at any given time.

Consider now the following scenario: two agents Ann and Bob (*a* and *b* for short), are each given a coin to toss. After tossing their coin they will each know whether their coin is displaying heads or tails, while they'll ignore the result obtained by the other player.[13] Such a situation can be described by the model in *Figure 4*:

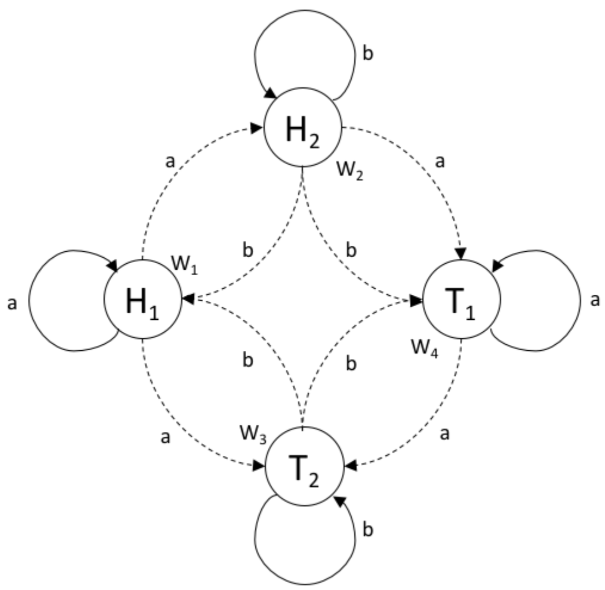

Figure 4

[13] This example has been thoroughly explained by Eric Pacuit in his report (See reference C, pages 3-4) and can be found in different forms at the following: reference A (the example is given as red and white cards), reference E (the example is given with a spymaster).

Every agent will only ever know the state of his/her coin, without having knowledge of the other's one. It is though true that they can believe the opponent's coin to be in a particular state, and without any external help, at any moment in time agent a will always believe that agent *b*'s coin will be in either state H_2 or T_2.

It is to be noted that $H_1 = \neg T_1$, therefore if an agent knows T_1 (i.e. that coin 1 has resulted with a tails), he won't be able to know at the same time H_1 (i.e. that coin 1 has resulted with heads), as it would be a contradiction.
Firstly let's explicitly state that the following hold:
$M, w_1 \vDash K_a(H_1)$: Within model *M*, agent *a* knows H_1 in the situation represented by w_1
$M, w_4 \vDash K_a(T_1)$: Within model *M*, agent *a* knows T_1 in the situation represented by w_4
$M, w_2 \vDash K_b(H_2)$: Within model *M*, agent *b* knows H_2 in the situation represented by w_2
$M, w_3 \vDash K_b(T_2)$: Within model *M*, agent *b* knows T_2 in the situation represented by w_3
And so far the system holds. It is though worth noticing that *Figure 4* NEVER expresses the following:
$M, w_1 \vDash K_a(H_1 \wedge T_1)$
As it would be a contradiction.

But there is a lot more to it. Let's say that we'd like to express the following statement: "Agent *a* believes that agent *b* knows that he's either got Heads or Tails on coin two". We would all agree that this is a perfectly logical sentence, and, as such, it can in fact be translated as the following:
$M, w_1, w_4 \vDash B_a(K_b(H_2) \vee K_b(T_2))$
Such statements hold for both w_1 and w_4.

All that is expressed by *Figure 4* can be shown with the following notation[14] which takes into account all possible combinations of the states the above figure:

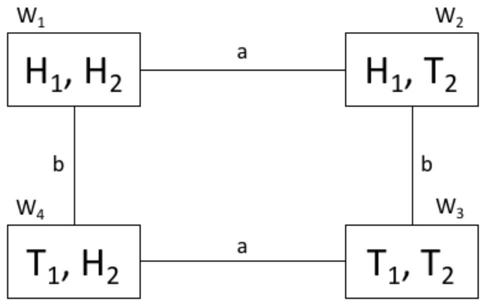

Figure 5

What we can see is a listing of all the possible combinations of the states of our model, with accessibility relations between different worlds. It is worth noticing that each agent is related to one and only one state, as in the state of coin 1 will always be related to agent *a*. But what does the notation expressed in *Figure 5* imply?

1) An agent *c* has knowledge of *p* in worlds w_1 and w_2 if it is the case that *p* in both w_1 and w_2
2) An agent *c* in world w_1 believes *p* in world w_2 if it is NOT the case that *p* in world w_1

According to the above definitions we could thus say that the following hold:
$M, w_1 \vDash K_a(H_1)$
$M, w_1 \vDash B_b(T_1)$
Or in more complicated cases:
$M, w_1 \vDash K_a(H_1) \wedge K_b(H_2)$: Holds according to definition 1
$M, w_1 \vDash B_a(K_b(H_2) \vee K_b(T_2))$: Holds according to definition 2

[14] As expressed by Eric Pacuit: see reference C, page 4.

As we are talking about coins which can only ever be attributed two distinct states (either Heads or Tails), we could re-write *Figure 5* with binary values only, making it look like the following:

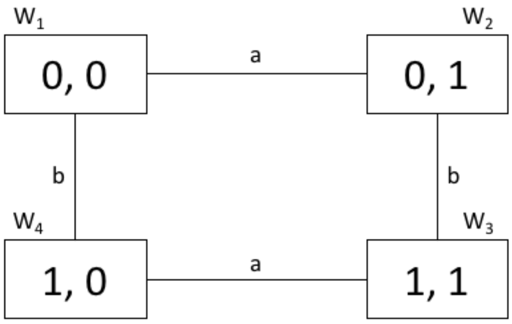

Figure 6

All that could be expressed by *Figure 5* can also be expressed by the simplified binary *Figure 6*.

Dynamic Epistemic Logic

Up to know we've looked at epistemic models which analysed static situations. We could express formulas in them, and analyse their truthfulness or falsity, but we were limited to what the model itself could express. Differently from propositional logic, dynamic logic can update the semantics of a problem during our analysis of the problem itself. This can be achieved by various means, one of them are announcements:

Announcement: "A public announcement is a communicative event where all agents receive the same information and it is common knowledge among them that this is so. […] An announcement is modelled by removing the states where the announcement is false, i.e. by going to a submodel. "[15]

According to the given definition of announcement, let's consider the model in *Figure 4*. Lets now imagine that agent *a* publicly announces that he/she has obtained Heads on their coin (i.e. coin 1). We must chance the model accordingly, and such changes are present in *Figure 7*:

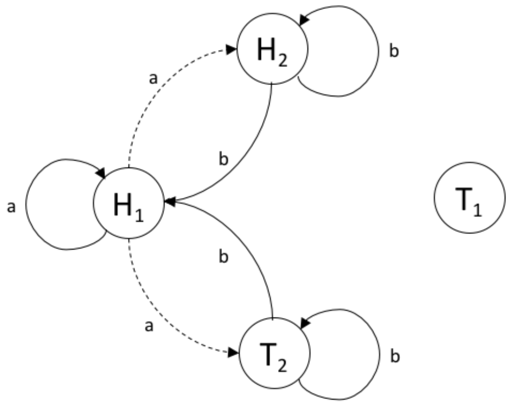

Figure 7

We can clearly see how w_4 is no longer reachable from any other world, as it contains a situation which has been confirmed to be false. The model has also changed in such a way that the following are true:

[15] See reference A, Page 4.

M, $w_2 \vDash K_b(H_1)$
M, $w_3 \vDash K_b(H_1)$

The changes operated on *Figure 4* could also be translated into the combinatorial-model in the following way:

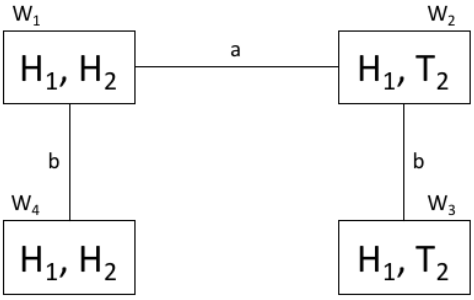

Figure 8

We can now see that within all worlds w_1, w_2, w_3, w_4 agents have knowledge of H_1. In fact also in this case:
M, $w_1 \vDash K_a(H_1)$
M, $w_2 \vDash K_a(H_1)$
M, $w_3 \vDash K_b(H_1)$
M, $w_4 \vDash K_b(H_1)$

There just seems to be one obvious flaw with the model: what is the point of distinguishing w_1 from w_4 and w_2 from w_3? There in fact isn't, and we could assimilate the two together and for the following *Figure 9*.

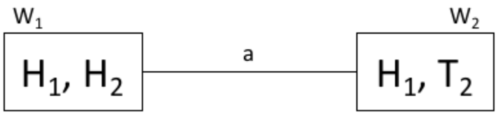

Figure 9

This appears to be the most concise notation to express the model in *Figure 4* after the announcement that agent *a* has obtained Heads on coin 1.

Therefore it is important to consider how announcements can dynamically model epistemic models determining changes in accessibilities between worlds.

The Muddy Children Problem

The muddy children problem, which can be understood in depth here, can be modelled through Dynamic Epistemic Logic, with a few differences.

1) Every child can be in either of two states *M* (muddy) or *C* (clean)
2) Every child will have the knowledge of the states of all the other children in the model, without knowing anything about themselves. They will believe to be either muddy or clean, but they won't have any knowledge of that.
3) The father won't be modelled into the model *M*, he is instead considered to have omniscient knowledge about the system (as he, in fact, can see, and thus knows, the state of every child).
4) All public announcements made by the father will be considered to be true by all the children.

Given the above rules, we could express the Muddy Children Problem, in the case of only 2 children, with the following model:

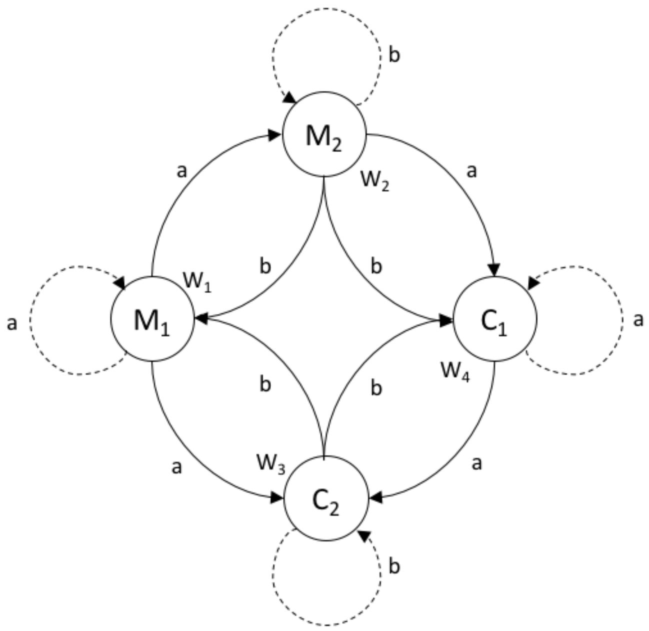

Figure 11a

Figure 11a will be able to express the following:
M, $w_1 \vDash K_a(M_2)$
M, $w_1 \vDash B_a(C_1) \vee B_a(M_1)$
M, $w_3 \vDash K_a(K_b(K_a(B_b(C_2) \vee B_b(M_2))))$

To simplify Figure 11a, we could introduce *Figure 11b*

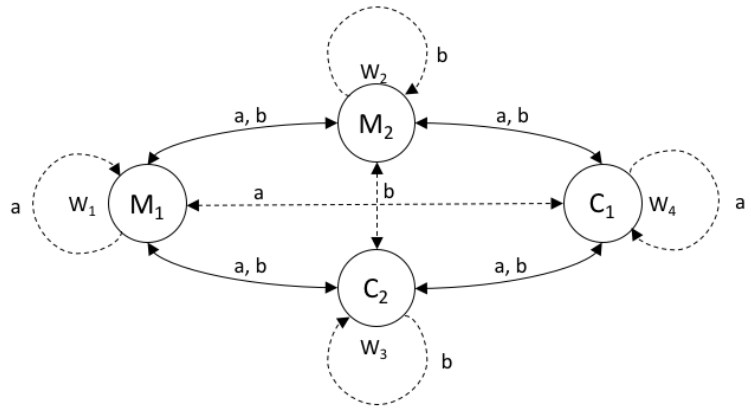

Figure 11b

Where the following are true:
M, $w_1 \vDash K_a(M_2)$
M, $w_2 \vDash K_b(M_1)$
But these, even though it mightn't look like it, are NOT true:
M, $w_2 \vDash K_a(M_1)$
M, $w_1 \vDash K_b(M_2)$

Similarly to the coin problem, we can group the states expressed within the single-state-worlds, into binary-worlds, expressing all the possible combinatorial states of our model:

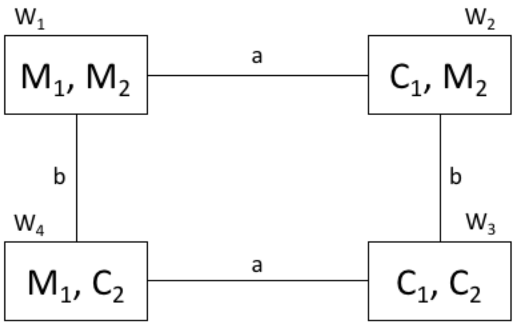

Figure 12

All the statements valid above will also hold in *Figure 12*. The main difference with the coin problem, is that with *Figure 12* we imply that agents have no knowledge of their particular state (as if they didn't know what the result of their coin toss was), but they are at any given time perfectly aware and knowledgeable of all the states of all other agents in the model.

Having considered the epistemic logic modelling for a two-muddy children problem, lets now expand it into containing three of the kids. We want to be able to express the following:

M, w⊨If $(B_a(M_1) \vee B_a(C_1))$ then $((K_a(M_2) \vee K_a(C_2)) \wedge .. \wedge (K_a(M_n) \vee K_a(C_n)))$

Which, translated into English, would mean: If an agent *a* has no knowledge of himself (i.e. only believes he could be in either state), then he must know the states of all other children.

Such a Relationship holds for the following diagram (we used the simplified notation introduced for *Figure 11b*):

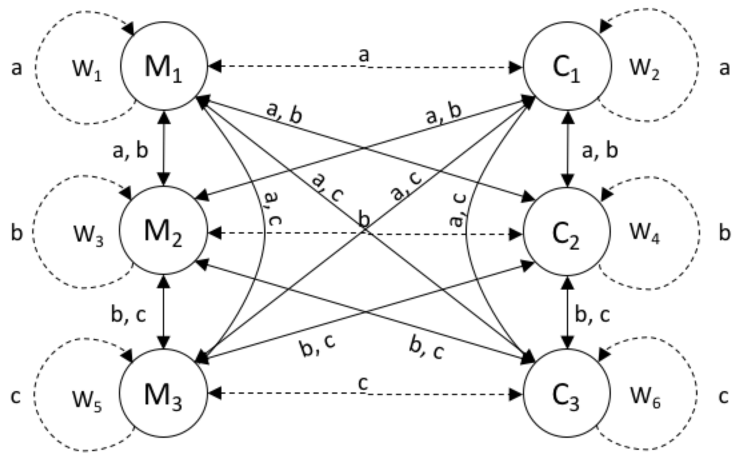

Figure 13

All of the following hold within the model *M*:

M, w_1⊨$(K_a(M_2) \vee K_a(C_2)) \wedge (K_a(M_3) \vee K_a(C_3)) \wedge (B_a(M_1) \vee B_a(C_1))$

M, w_3⊨$(K_b(M_1) \vee K_b(C_1)) \wedge (K_b(M_3) \vee K_b(C_3)) \wedge (B_b(M_2) \vee B_b(C_2))$

M, w_5⊨$(K_c(M_1) \vee K_c(C_1)) \wedge (K_c(M_2) \vee K_c(C_2)) \wedge (B_c(M_3) \vee B_c(C_3))$

We can thus express the knowledge of all agents within the model, and their respective beliefs. We could also reformat the above in the previously used combinatorial model:

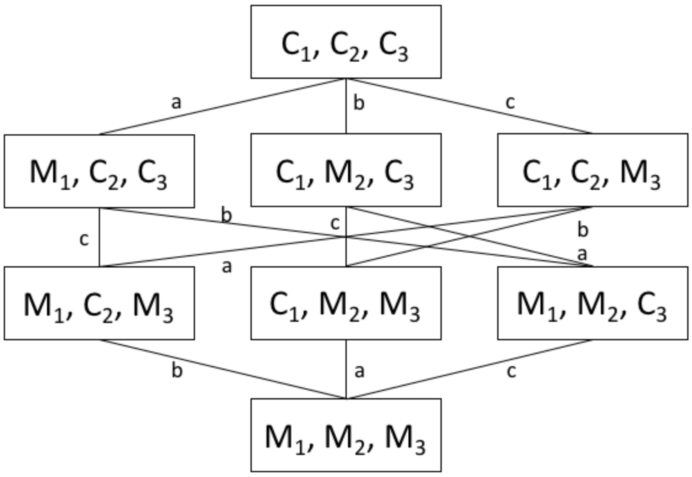

Figure 14

And, possibly, in binary notation:

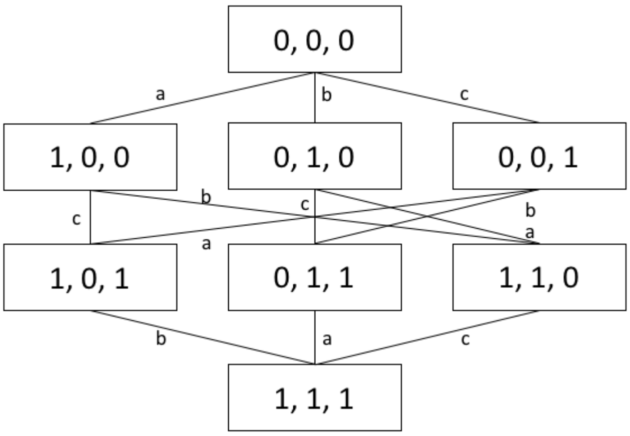

Figure 15

Given such a model, we can now fully understand how announcements will be able to cut down, and simplify, the structure, up to when we reach a world which contains the exact same pattern of muddy children as the situation we are looking at.

The Muddy Children Puzzle

Introduction

A group of children has been playing outside and they are called back into the house by their father[16]. The children gather round him. As one may imagine, some of them have become dirty from playing. In particular: they may have mud on their face. Children can only see whether other children are muddy, but are unable to notice if there is any mud on their own face. All this is commonly known,

[16] The following problem is analysed in depth by Reference H (chapter 3) and Reference I

and the children are, obviously, perfect logicians. Their father now says: ‘‘At least one of you is muddy.’’ And then: ‘‘Those who know whether they are muddy must step forward.’’ If nobody steps forward, the father keeps repeating the request. At some stage all muddy children will step forward.

Let’s break the problem down:

1. *n* children meet their father after playing in the mud. The father notices that *k* of the children are muddy.
2. Each child sees everybody but himself.
3. The father says: “*At least one of you has mud on his forehead*.”
4. The father then says: “*Do any of you know that you are muddy? If you do, step forward.*”
5. No one steps forward.
6. The father repeats the question, and again no one moves.
7. After exactly *k* repetitions, all the muddy children step forward simultaneously.

And let’s now analyse it logically.

Base Case: One Child

Let’s start from the simplest case: we have only one child.
This child is muddy, so suppose k = 1.
When the father says at least one is muddy, he concludes that it’s him.

This is trivial, he knows he is muddy right away.

The following, expressed with the Dynamic Epistemic Logic notation (explained at this link), would give us *Figure 16*:

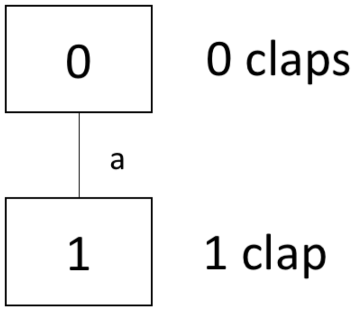

Figure 16

If no claps occur, then the starting situation (or world) w_1 must represent reality. If one clap occurs, then the child in question must be muddy, as expressed by the second world w_2.

Analysis with Two Children

As before, Alice and Bob played outside. They both got mud on their face. And also as before father informs them that at least one of them is muddy. Then he says, ‘‘I will clap my hands. If you know whether you are muddy, please step forward.’’ He claps his hands, but neither Alice nor Bob steps forward. He then repeats what he said before, ‘‘I will clap my hands. If you know whether you are muddy, please step forward.’’ This time, when he claps his hands, Alice and Bob both step forward.

From the above we can suppose k = 2 (both muddy).
And the situation could be described by the following *DEL* model:

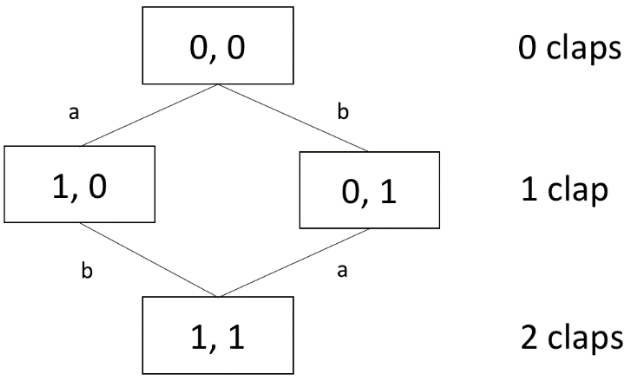

Figure 17a

After the first announcement, each muddy child sees the other, so they suppose the other is the only muddy child. So can therefore delete our base case w_1 (with no muddy children) from *Figure 17a* and look at the simplified *Figure 17b*.

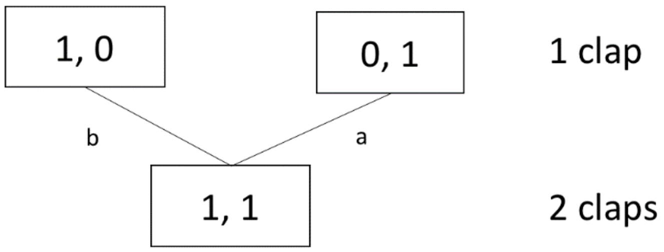

Figure 17b

Now, if the situation described by w_2 or w_3 were correct it would mean that one of the two children would step forward (as they wouldn't see any other muddy child, but at the same time they'd know there is one: them). But this is not the case: each child notices that the other one didn't step forward. They thus each realise that they must be muddy as well. So both will step forward with the next clap.

Analysis with Three Children

Alice, Bob, and Chris are coming home from playing outside and all three have mud on their face. Father tells them that at least one of them is muddy and then says, ''In a moment I will clap my hands. If you know whether you are muddy, please step forward.'' He claps his hands. Nothing happens. He repeats this twice. The third time he claps his hands, all three children step forward.

To solve this puzzle, with start with considering all possible situations.
Each child can be either clean or muddy, therefore we obtain the 8 possibilities listed below in *Figure 17*. (This is also *Figure 14* from the Dynamic Epistemic Logic page).

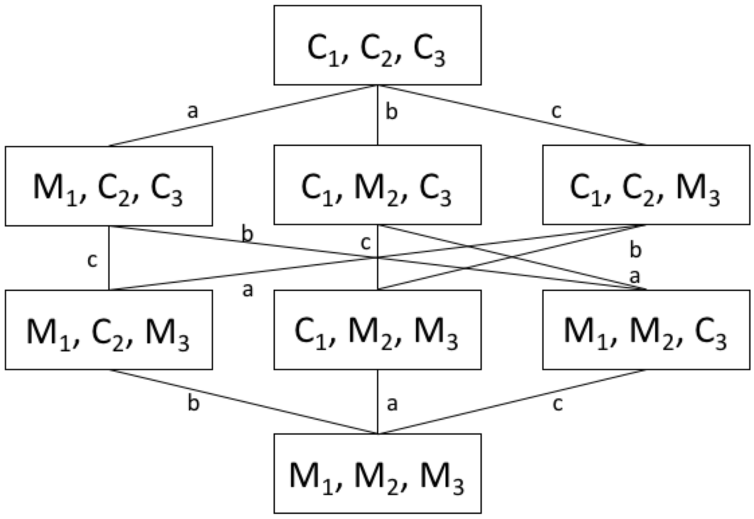

Figure 18

We can also rewrite the graph using 0s and 1s instead on C and M, in order to use a clearer binary notation.
So 0 will stand for "clean" and 1 will stand for muddy.

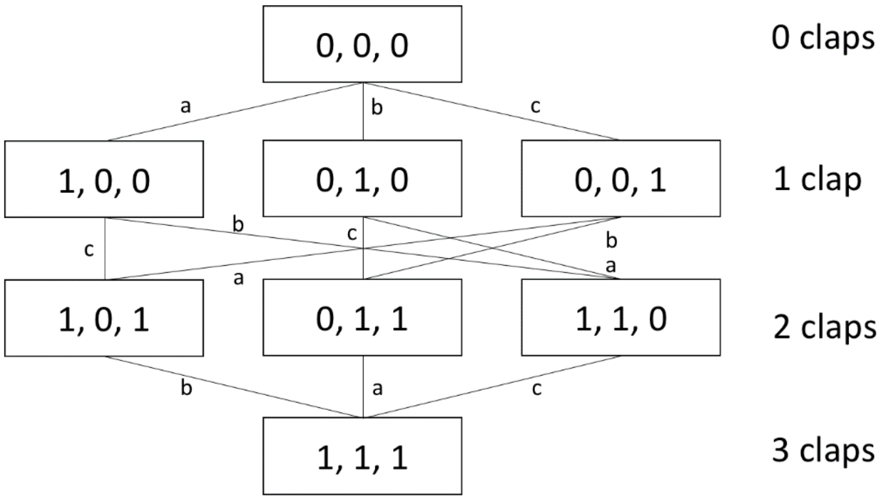

Figure 19a

Some of the situations have been linked by lines and labels (with the notation explained at the following link).
If for example two worlds are linked with an '*a'* label, then it means that agent *a* (aka Alice), will not be able to distinguish between the two worlds. Such a concept can be described in *DEL* as:

$$\exists\ w_1 R(a) w_2 \leftrightarrow \forall(p) \in (w_1 \wedge w_2)\ ((K_a(p)_{w1} \leftrightarrow K_a(p)_{w2}) \wedge (B_a(p)_{w1} \leftrightarrow B_a(p)_{w2}))$$

There is a relationship, with regards to agent *a*, between two worlds w_1 and w_2 iff for every *p* in world w_1 and w_2, *a* knows *p* knows in w_1 iff a knows *p* in w_2, and *a* believes *p* in w_1 iff *a* believes *p* in w_2.

So, if we consider the situation at 0 claps, Alice, Bob and Chris are all uncertain of whether they are muddy or not as they cannot discriminate between these two distinct beliefs.

Now, how does the father's announcement change the graph?
The first announcement "At least one of you is muddy" rules out the [0, 0, 0] case, so the now the graph will be:

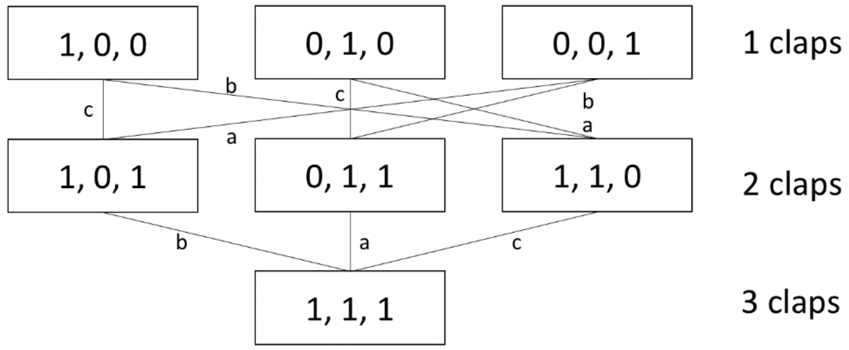

Figure 19b

Now, if the situation is [1, 0, 0] Alice will know that she is muddy.
If the situation is [0, 1, 0] Bob will know that he is muddy.
If the situation is [0, 0, 1] Chris will know that she is muddy.
(We can deduce this just following the labels from the first graph).
But nobody steps forward, as all children can clearly see more than 1 other muddy child in their line of sight.

The father will know clap his hands again. This event rules out these three cases ([1, 0, 0] – [0, 1, 0] – [0, 0, 1]), as someone should have stepped forward, but nobody did. This lack of action is informative for the children.
We can now revise again the graph. And obtain the following:

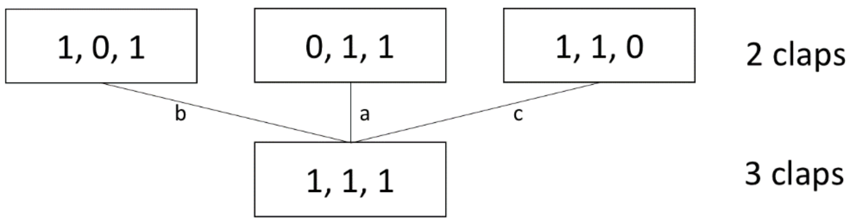

Figure 19c

Now consider the situations [1,0, 1] – [0, 1, 1] – [1, 1, 0].
In these situations, there are no outgoing lines for two of the three children.
In other words, there is now remaining a unique possibility.

Let's take case [0, 1, 1] for instance.
Bob and Caroline now know they are muddy.
But why?
In [0, 1, 1], Bob *would* see that Chris is muddy and that Alice is clean. Initially, it remained possible that only Chris was muddy. But we rule that case out because nobody steps forward and the father claps its hands again.
Therefore, Bob concludes that Chris sees a muddy child. That must be himself.
Chris can draw the conclusion in the same way.

But what if the father claps again its hands because nobody stepped forward?
We must rule out the cases [1, 0, 1] – [0, 1, 1] – [1, 1, 0], and we are left with [1, 1, 1], as expressed by *Figure 19d*.

Figure 19d

Having now reached the state that is compatible with reality, all of the children that are effectively muddy will step forward (in this case, all of them).

General Case

So the question now is, what is the general rule to describe the muddy children problem?
Such a query could be broken down to the following two sub-points:

A. When will a muddy child know for certain that he is muddy?
B. After how many claps will all the muddy children step forward?

Let's consider the following figure which describes a general model for the muddy children problem:

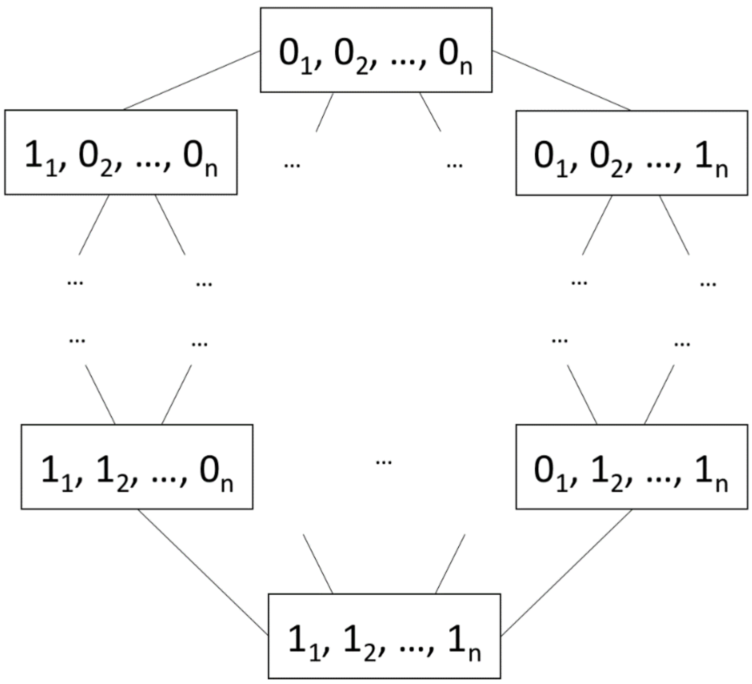

Figure 20

In order to fully comprehend the problem we will analyse a simpler (non-general) situation described by the case in which we need to consider four children.

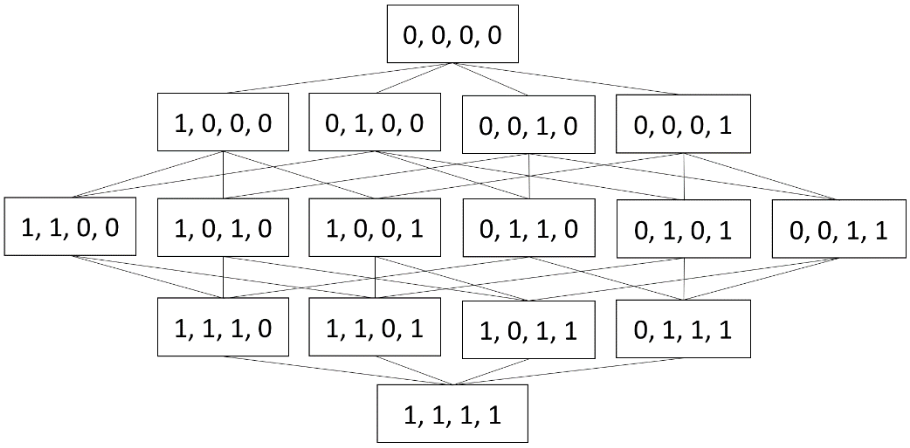

Figure 21

In order to achieve readability we have omitted which agent can travel between which worlds (it can de deduced from the above graphs). Now, lets look at when the problem would end case by case:

1) With only 1 muddy child, the problem will end after 1 clap by the father. In any situation all children will at most see 1 other muddy child.
2) With 2 muddy children, the problem will end after 2 claps by the father. In any situation all children will at most see 2 other muddy children.
3) With 3 muddy children, the problem will end after 3 claps by the father. In any situation all children will at most see 3 other muddy children.
4) With 4 muddy children, the problem will end after 4 claps by the father. In any situation all children will at most see 4 other muddy children.

So, when exactly do the muddy children step forward? Let's look at the following diagram:

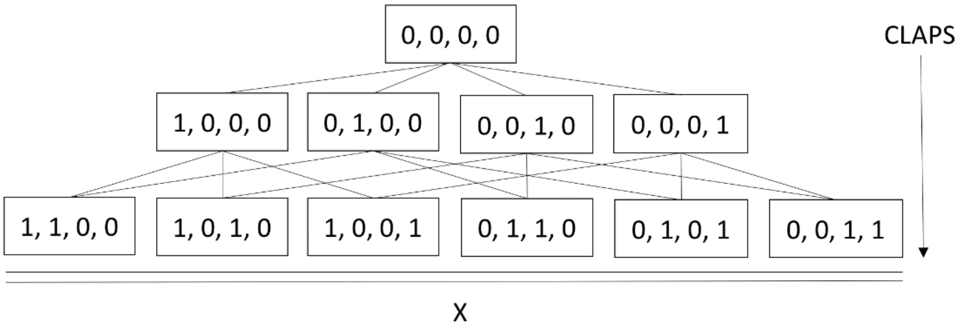

Figure 22

We know that the problem will terminate when the muddy children step forward. That happens when they deduce that no other possible branching for their problem is possible. There therefore must be an "X" value which cuts the model after a given amounts of claps. How can we try and determine "X"? Well, all non-muddy children will in any situation see n other muddy siblings, given that n is the actual number of muddy children. Instead, if a child is muddy, in any situation he will see $n-1$ other muddy children.

So if after m number of claps a child sees $m-1$ other muddy children, he knows that he must step forward. And such a thought occurs to all other muddy children simultaneously. "X" therefore represents the actual number of muddy children plus one.

So we now know that:

A) A muddy child will know for certain that he is muddy if he can see *n* other muddy children, but *n+1* claps have occurred.
B) If there are *m* muddy children present around the father, they will all step forward together after *m* number of claps.

And this is the general solution to the muddy children problem.

One Blind Child

Of the many possible extensions to the muddy children problem, we decided to look at the following one:
"Consider another family, consisting of two children: Anne and Bill. Bill is blind. So, he cannot see his own face, but he can also not see Anne's face. They play outside and both get mud on their face. After they have come home, father says, ''At least one of you is muddy.'' Anne asks her father, ''Am I muddy?'' Even before father answers the question, Bill leaves to clean his face. Why?"[17]

Let's try to model the problem with Dynamic Epistemic Logic:

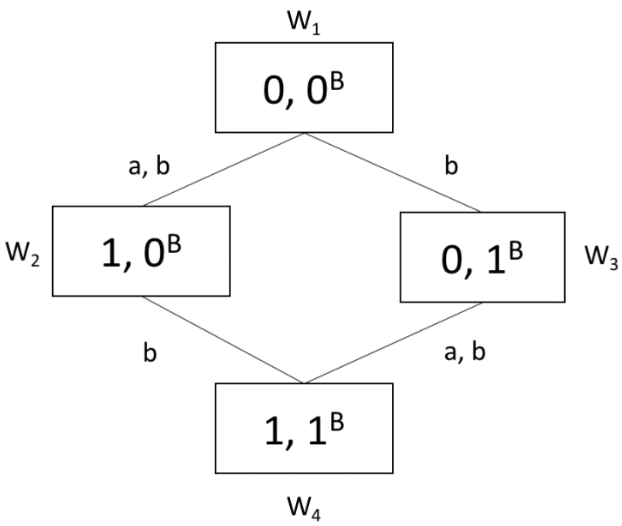

Figure 23

[17] See Reference H at page 24.

In *Figure 23*, we introduced the notation 0^B to say that the given child (in the particular case, clean) is blind; 1^B would mean that a blind child is muddy.
What we can see from the model is that starting from w_1, agent *b* is unable to distinguish not only w_1 from w_3, but also w_1 from w_2! Differently from agent *a*, which in fact hasn't knowledge of his own state, but knows whether or not *b* is muddy, agent *b* is totally unaware of the states of any of the agents in the model.

But there is one extra aspect to the problem: all non-blind children can ask their father whether they are muddy or not (without receiving an answer). So, when would such a query be asked?

Let's look at the case where no claps have occurred (*Figure 23*), the only two worlds that agent *a* (i.e. not blind) cannot distinguish upon, are worlds w_1 and w_2. So, if no claps occur agent *a* has no reason to wonder if himself, or agent *b* are muddy.

But let's consider that the father claps once.

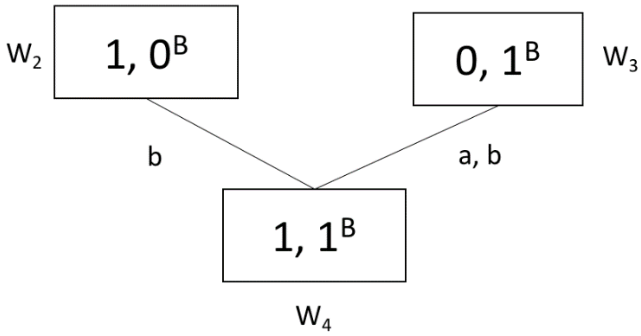

Figure 24

We have now obtained *Figure 24*. There are now two possibilities:
1) Agent *b* is muddy
2) Agent *b* is not muddy

If we consider case 2, what would happen? Well, we know that there is at least 1 muddy child, due to the father clapping, so therefore we must be in world w_2. As we consider children to be good logicians, we will immediately see that agent *a* will step forward, as he has realised that he is the only muddy child. The blind child, knowing that others have stepped forward, will know that he is clean.

If we instead consider case 2, we might be in two distinct worlds: w_3 or w_4. We must also explicitally point out (even though it can be inferred from *Figure 24*) that agent *a* has no way to determine whether or not he is muddy, and so is unable to distinguish the two very different situations expressed by w_3 and w_4. It is thus at this point that agent *a* will inevitably ask the question: "father am I muddy or not". As he is in pure doubt.

At the same time, though, we see that in both w_3 and w_4, agent *b* is always muddy. Therefore, by the fact that agent *a* is in doubt, he deduces that we must be talking about case 1 (as no questions would have been asked to the father if we had been talking about case 2). It is at this point that the blind child will have certainty that he is muddy and step forward.

Having taken the blind child out of the circle, we will now to proceed to analyse the simpler for of *Figure 24* without agent *b*.

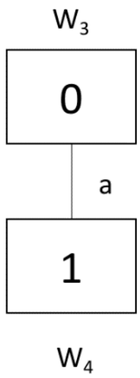

Figure 25

As we are talking about a muddy children example with no blind children, we can refer back to what stated above for the general muddy children case.

But what would happen if the total amount of children was greater than two? We could start by analysing the case in which we had three:

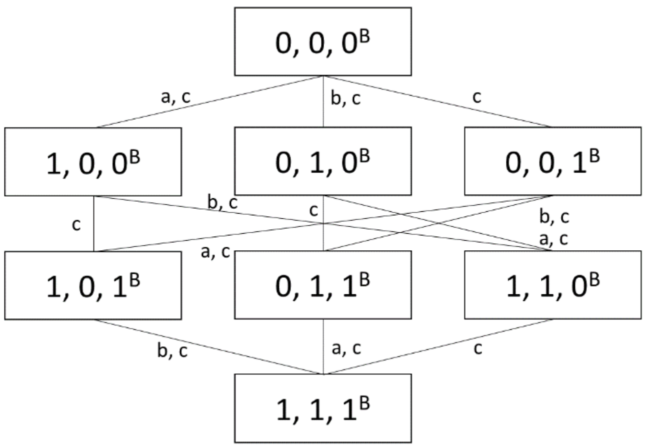

Figure 26

Well, there are mainly two possibilities:

1) All of the non-blind children are also non-muddy
2) There is at least one muddy and non-blind child

If we consider case 1, then immediately, after the first clap (in case the blind child is muddy), all of the non-blind children will ask their father whether they are muddy or not. This is due to two factors:

a) All non-blind children can only see one other muddy child (i.e. the blind one)
b) They will ask themselves whether they are muddy or not all at the same time as they are perfect logicians.

At this point the blind child will know that he is muddy (see above) and we will focus on a sub-problem with only the non-blind children.

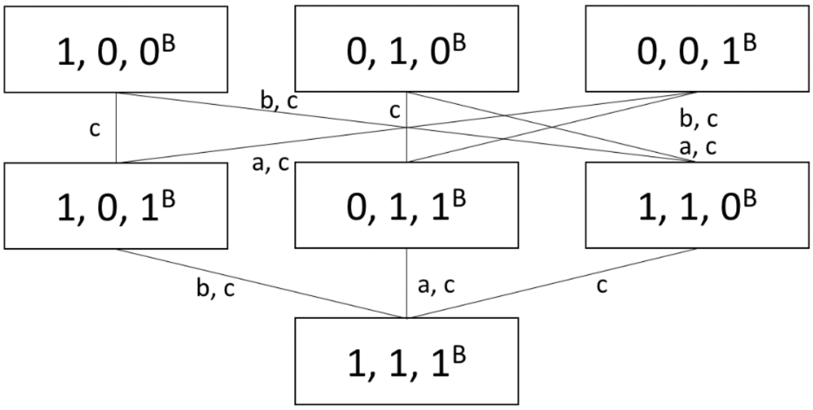

Figure 27

If, instead, we consider case 2, we would have at least 1 (muddy, even though he doesn't know it) child which at a certain point would see *m* other muddy children after *m* claps of the father. He then will ask the father whether or not he is muddy. This tells the blind child that he is indeed muddy, as it is a follow-up to the case with only two children (of which one blind). After the blind child has stepped forward, the problem proceeds as normal.

If the blind child is not muddy, then all other children will consider it as a normal situation and will, thus, never ask any question to the father. The blind child is considered as a traditional non-muddy member of the puzzle.

Multiple Blind Children

The case in which we have multiple blind children is very troublesome. We must make an assumption: every child (blind or not) knows the total number of children present at the puzzle.

Given this, we have come to the conclusion that the problem could be solved only in the following cases:

1) All the blind children are muddy
2) All the blind children are clean

If, in fact, we are in any of the above cases we could consider the set of all blind children as just one entity that will respond to the questions raised by other children to their father, in the same ways as described above (i.e. they would all step forward, knowing that they are all muddy).

If, instead, we are not in one of the two cases above, well …

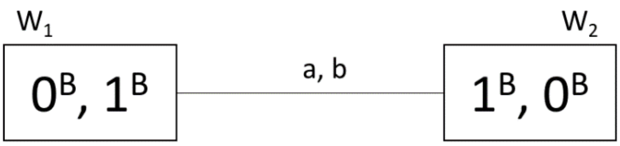

Figure 28

In this case the problem will never solve itself. Neither agent *a*, nor agent *b*, know who of the two is muddy, and who isn't. The father won't be able to clap anymore (as he would otherwise be lying), and they won't be able to receive any external form of help. This case turns the puzzle into an inconclusive problem.

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