Practical Option Valuation with Negative Underlying Prices

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Abstract

The detection of radon levels is important for reducing home and workplace exposure to ionizing radiation. For the detector under consideration, this detection is done by counting the number of alpha decays in any given time period, and a single measurement cannot distinguish between the different isotopes of radon that may be present. In this paper, we present a method to approximate the relative amounts of radon-222 and radon-220 in a particular sampling sequence by performing a linear regression to the theoretical expected count rate from each of these isotopes.

Problem Statement

Residential radon (Rn) progeny exposure is "the leading cause of lung cancer in non-smokers, and the second leading cause of lung cancer in smokers" [1]. Uranium (U) and thorium (Th) in the soil eventually decay into radon, which can then seep into basements and low-lying areas of the house. The two main radon isotopes are Rn-222, which is part of the U-238 decay chain, and Rn-220, also called thoron, which is part of the Th-232 decay chain. There is currently much interest in the Rn-220 contribution to radon progeny exposure, which has so far been largely ignored. Though Rn-220 has a relatively short half life and usually decays before it reaches the living areas in a house, its radioactive progeny can still pose a problem.

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Radon Sniffer [2], which is used by radon mitigators and building scientists to find radon entry points. The sniffer works by pumping air through a filter that removes all radon progeny, after which it is passed into a detector that counts alpha particle emissions from the decaying particles. The detector only counts the total number of alpha decays in a given period, so it cannot distinguish between Rn-222, Rn-220, or their progeny without further processing. Currently, these sniffers assume the only radon species present is Rn-222.

The problem we were presented with was to determine a sampling scheme and algorithm that can reliably determine the approximate amounts of Rn-220 and Rn222 in any particular radon-containing sample of air, given the existing capabilities of the sniffer.

Method

Primer on Radioactive Decay.

The nuclei of some atoms can randomly and spontaneously decay. Such atoms, and the substances they comprise, are said to be unstable, or radioactive. Nuclear decay is accompanied by the emission of a particle

There is a variety of particles that can be emitted on decay, but here we are only concerned with two: alpha (α) particles and beta (β) particles. The type of particle emitted in the decay is called the mode of that decay. Alpha or beta decay change the nuclear species (i.e. the number of protons and neutrons) of the given nucleus. Another common mode of decay is gamma decay, but this does not alter the nuclear species, and for this and other reasons it does not play a part in the problem at hand. The nuclear species of an atom is also known as that atom's nuclide.

When an atom decays, the result may also be unstable. A sequence of such decays form what is known as a decay chain, where an unstable substance A decays into another unstable substance B, which itself decays into substance C, etc. These decay chains continue until a stable nuclide is reached. An important quantity describing any radioactive substance is its half-life $t_{1/2}$, which is the amount of time over which a given particle of the substance has a 50% probability of having decayed. Equivalently, it is the time by which 50% of the substance is expected to have decayed.

The last point to make is that the probability of decay for each atom in any interval of time depends only on the length of that interval (i.e. the decay process is memoryless). It does not depend on how long that atom has existed overall, nor does it depend on the atoms around it. As such, mixing different radioactive substances does not change their individual behaviour. From the memoryless property of the process, it follows that the time it takes for a given atom to decay follows an exponential distribution, while the number of decays in a given period from a large amount of a pure radioactive substance approximately follows a Poisson distribution.

Modelling Expected Values

Modelling Expected Values. Suppose at time 0 you have a collection of $N(0)$ radioactive atoms, all of a single nuclide. On average, the number $N(t)$ of atoms you expect to find remaining at time t is given by

$$N(t) = N(0)e^{-\lambda t}$$

Where λ, called the *decay constant* of the nuclide, is related to the half-life $t_{1/2}$ by $\lambda = \ln(2)/t_{1/2}$. The decay rate of this collection of atoms at time t is $\lambda N(t)$.

Before we proceed, a word about units. Radioactive quantities are frequently described not by mass or number of particles but in terms of activity, which in the equation above is the quantity $\lambda N(t)$. Activity carries units of decays per unit time. The most common units of activity in the applications being considered are becquerels (Bq) and picocuries (pCi). By definition, 1Bq is 1 decay per second, and 1pCi is equivalent to 0.037Bq. In the context of radon mitigation, quantities are usually described in activity per volume, and the units are becquerels per cubic metre ($\mathrm{Bq/m^3}$) and picocuries per litre ($\mathrm{pCi\,L^{-1}}$). One finds that 1pCi/L is equal to $37\mathrm{Bq/m^3}$.

As mentioned before, the progeny of a radionuclide can also be radioactive, resulting in so-called decay chains. The word "chain" may be somewhat misleading, as some radionuclides can decay in more than one way. For example, Bi-212 decays by alpha emission into Tl-208 with a probability of 33.7%, and decays by beta emission into Po-212 with a probability of 66.3%. The decay series which are relevant to this project are those of U-238 and Th-232, the relevant portions of which are illustrated in [Figure 1 and 2].

Consider a (linearly ordered) decay chain $X_0 \to X_1 \to \ldots \to X$ in which the decay constant of X_j is λ_j for $j = 0, 1, \ldots, l-1$, and X is stable. We assume throughout that $\lambda_0, \lambda_1, \ldots, \lambda_{l-1}$ are distinct and positive. Starting at time 0 with a collection $N_0(0)$ atoms of X_0, $N_1(0)$ atoms of X_1, etc., let $N_0(t), N_1(t), \ldots, N_{l-1}(t)$ denote the expected number of atoms of the respective type in the given decay chain remaining at time t. When $0 < j < l$, the number $N_j(t)$ can change over time either by the decay of an atom of type X_j into one of type X_{j+1} (decreasing $N_j(t)$ in the process) or by an atom of X_{j-1} decaying into an atom of X_j (thereby increasing $N_j(t)$). The time evolution of the ensemble is thus described by the following system of ordinary differential equations:

$$\frac{dN_0}{dt} = -\lambda_0 N_0, \qquad \frac{dN_j}{dt} = -\lambda_j N_j + \lambda_{j-1} N_{j-1} \qquad (0 < j < l)$$

References

[1] "Radon," *Lung Cancer Canada*. Available: https://www.lungcancercanada.ca/Lung-Cancer/Radon.aspx

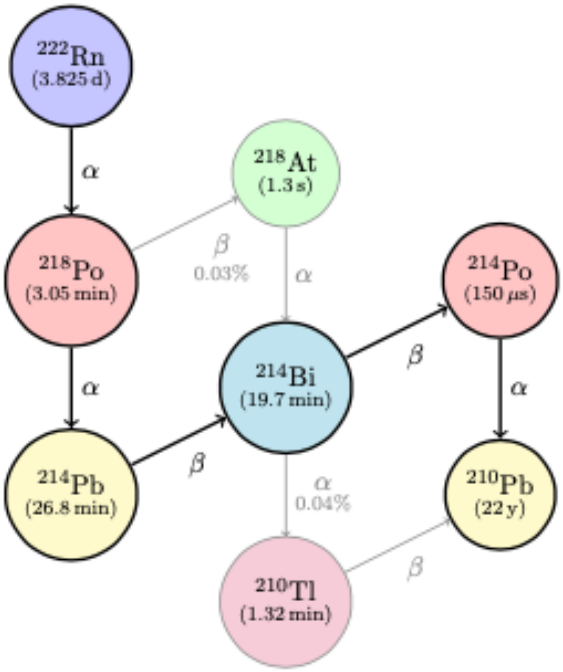

Figure 1: Rn-222 Decay chain. Based on [2, Ch. 15]. Effective linear chains are indicated in bold.

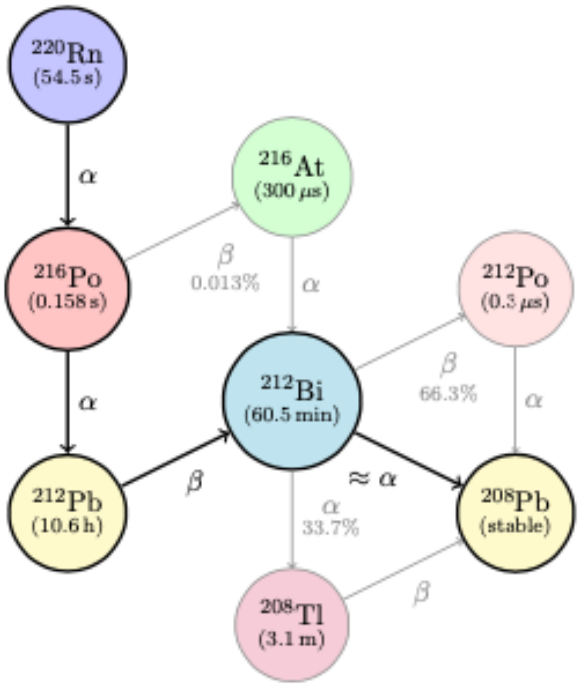

Figure 2: Rn-220 Decay chain. Based on [2, Ch. 15]. Effective linear chains are indicated in bold.

[2] R. B. Leighton, *Principles of modern physics*, vol. 1st ed. McGraw-Hill, 1959.