Advanced Topics in SAT-Solving
Part III: Implementation Techniques

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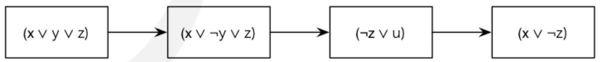
Outline

1. Basic Data Structures
2. Efficient Unit Propagation
3. Literal Selection Strategies
4. Clause Learning
5. Parallelization

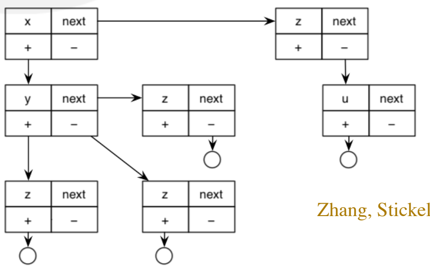
Data Structures for CNF Representation

How to represent a set of clauses?

A) As a clause list



B) As a trie data structure



Zhang, Stickel (1994)

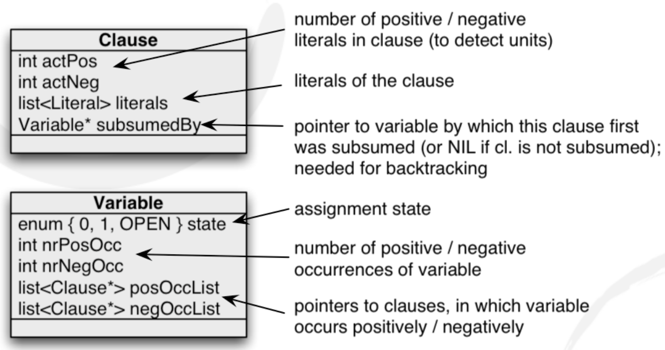
Repetition: DPLL Algorithm

```
boolean DPLL(ClauseSet S)
{
   while (S contains a unit clause {L}) {            // unit propagation
        delete from S all clauses containing L;      // u. subsumption
        delete ¬L from all clauses in S;             // u. resolution
   }
   if (∅ ∈ S) return false;
   if (S = ∅) return true;
   choose a literal L occurring in S;
   if (DP(S ∪ {{L}}) return true;
   else return DP(S ∪ {{¬L}});
}
```

Data Structures for DPLL: Requirements

- Allow for fast unit propagation
  - Detection of new units
  - Propagation of units
- Support back-tracking (restoration of clause data)
  Implementation alternatives:
  - Save copy of clause set data structure on each level
  - Remember changes (undo-stack)
  Goal: Minimize restore effort
- Compact representation of large clause sets

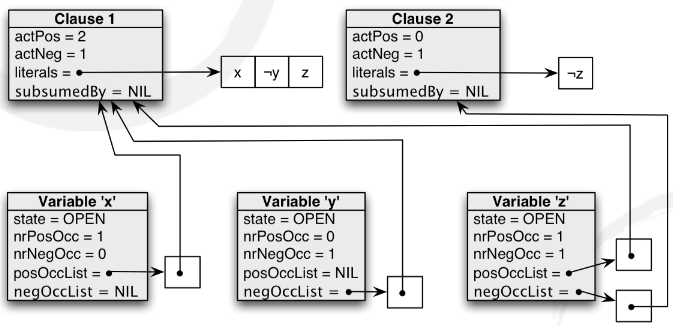
"Traditional" Approach



Crawford, Auton (1993)

Traditional Approach: Example

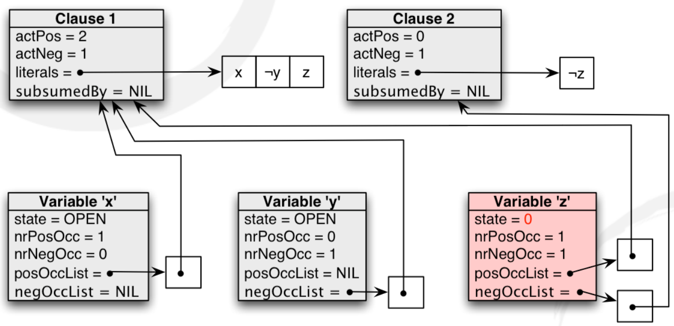
$$F = \{\{x, \neg y, z\}, \{\neg z\}\}$$



Example: Unit Propagation

$$F = \{\{x, \neg y, z\}, \{\neg z\}\}$$

Unit propagation: set z=0



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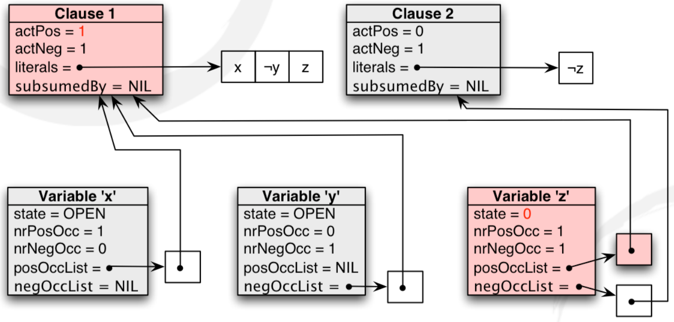

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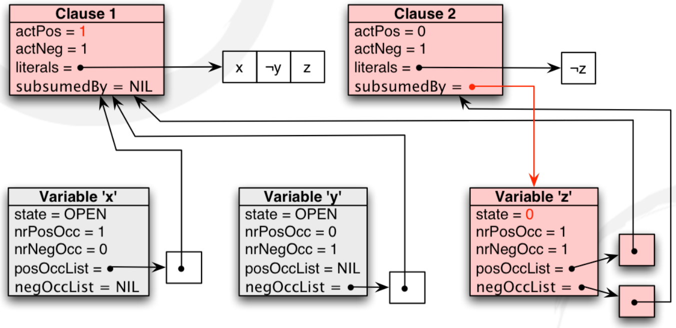


Algorithm Unit-Propagation

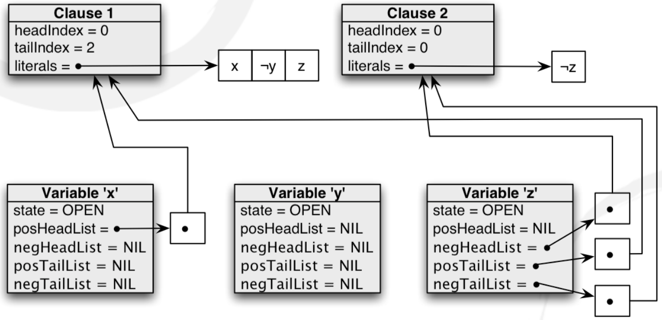
```
boolean UnitProp(Literal L)            // L: open literal; 'UnitProp' returns
{   if(L.isPositive()) {               // false on contradiction
        v = L.var(); v.state = 1;
        for(it = v.posOccList.begin(); it != v.posOccList.end(); it++) {
            clause = *it;
            if(clause.subsumedBy == NIL) clause.subsumedBy = v;
        }
        for(it = v.negOccList.begin(); it != v.negOccList.end(); it++) {
            clause = *it;  if(clause.subsumedBy == NIL) {
             clause.actPos--;          // shorten clause
             if(v.actPos + v.actNeg == 1) {   // new unit clause detected
                ok = HandleNewUnit(clause);
                if(!ok) return false;         // conflicting units?
    }   }   } }
    else {...}
    return true;
}
```

UP Alg.: Complexity and Improvements

- Setting variable x (to true) ...
  - subsumes |posOccList(x)| clauses
  - shortens |negOccList(x)| clauses
  - thus: requires a total of #occ(x) clause modifications
- Can we improve on this?
  - We only have to detect unit clauses
  - Idea (Zhang, Stickel (1996)):
    1. **Delay testing for subsumption**
       'subsumedBy' is not used any more; instead, the *test for a new unit* has to check whether clause is subsumed
    2. **Restrict unit resolution to first and last open literal in clause**
       maintain pos/negHeadList and pos/negTailList instead of pos/negOccList

Head and Tail Lists: Example

$$F = \{\{x, \neg y, z\}, \{\neg z\}\}$$



Head and Tail Lists: Unit Propagation

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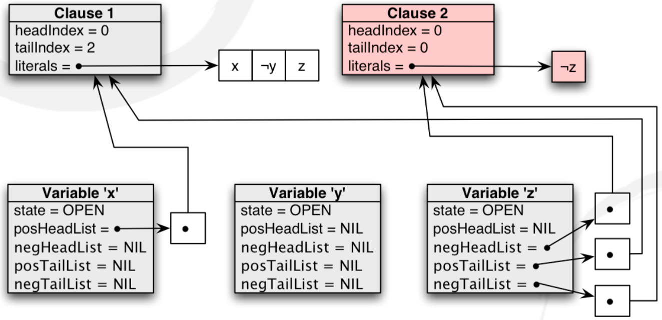
Unit propagation: set z=0



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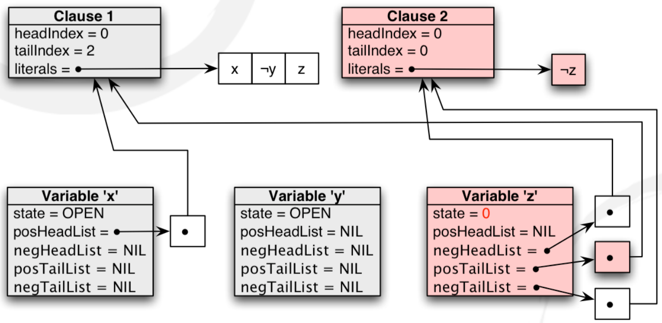

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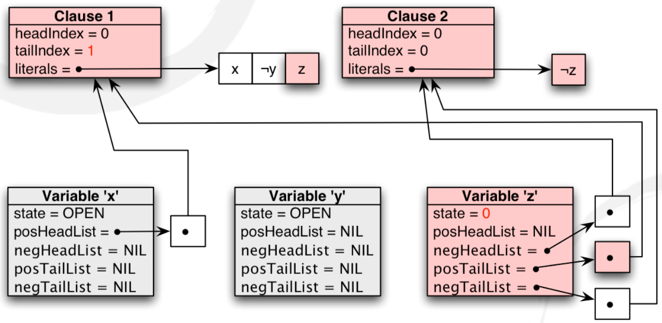

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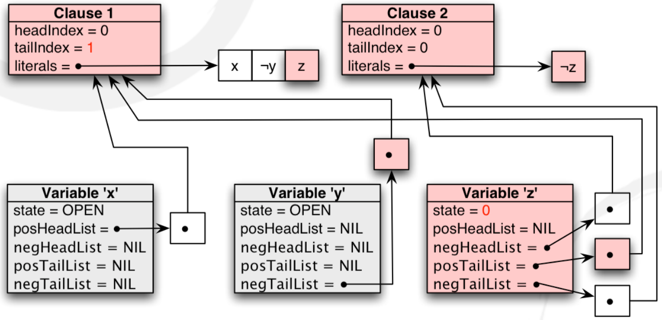
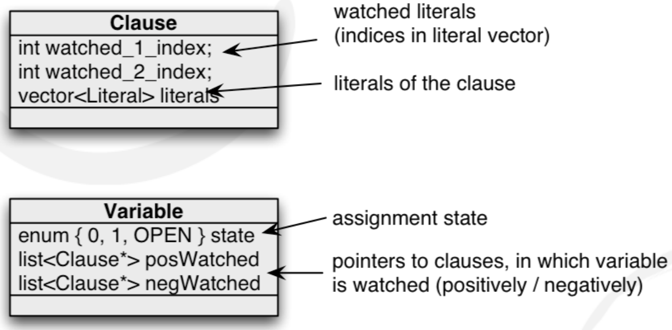


H/T Lists: Pros, Cons, Improvements

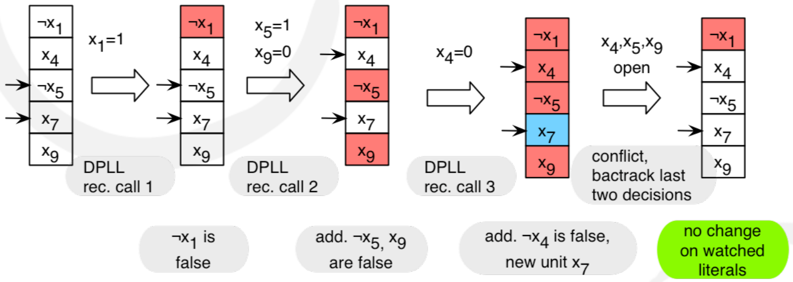
- Positive: Faster unit propagation
- Negative: Backtracking becomes more complicated (head and tail lists have to be restored)
- Further improvement: watched literals
  - Instead of head/tail literals: 2 watched literals per clause
  - Watched literals point to *arbitrary* open (different) literals
  - on backtracking: no update of data structure needed
  - First implemented in **chaff** (Moskewicz *et al.*, 2001)

Watched Literals: Data Structures



Moskewicz *et al.* (2001)

Watched Literals: Example



Literal Selection Strategies

- Which literal to select best in case-distinction step?
  - Size of search space (and thus run-time) can drastically depend on literal selection heuristics
  - Highly problem-dependent, no general "best" strategy
- Ideas for selection heuristics:
  1. Maximal simplification, e.g. maximize number of subsumed (deleted) clauses
  2. Try to reach tractable subclass of SAT, e.g. 2-SAT, Horn-SAT, only positive clauses
  3. Based on conflict analysis / clause learning (with preference for literals in recently learned clauses)

Literal Selection Strategies

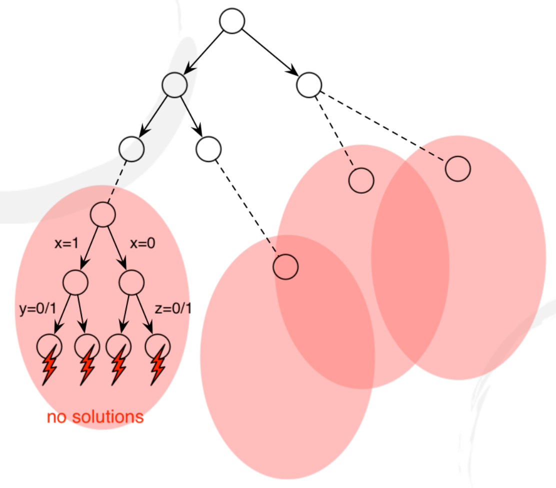
- **MOM** (maximum occurrences in minimal clauses): maximize $(occ_2(l) + occ_2(\neg l)) \cdot 2^{\alpha} + occ_2(l) \cdot occ_2(\neg l)$
  (where α is a 'large enough' number)
- **SATO**: build test set of k shortest positive clauses and choose literal that maximizes $(po(l)+1)(no(l)+1)$
  (where $po(l)$ ($no(l)$) denotes number of positive (negative) occurrences of literal l)
- **VSIDS** (variable state independent decaying sum): initial *score* is number of literal occurrences; for each learned clause, increase score by constant c for all literals in clause; periodically divide all scores by a factor f; choose literal with highest score

Conflict Analysis & Clause Learning

- Try to avoid repeated search of parts of the search tree with no solutions
- Compensate for badly selected case distinction literals
- Method: find weakest assumption under which a contradiction arises
  - Each selected branching literal counts as an 'atomic' reason
  - Find minimal necessary condition (i.e. minimal literal set) that produces the same conflict
- Also called "no-good learning" in the CSP community

Conflict Analysis: Example

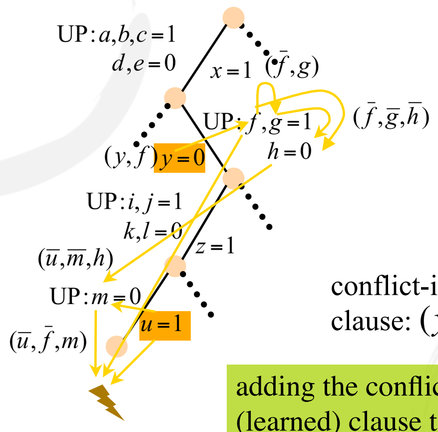
{¬x,y},
{¬x,¬y},
{x,¬z}.
{x,z}



Lemma Generation

(Marques-Silva, Sakallah, 1996)

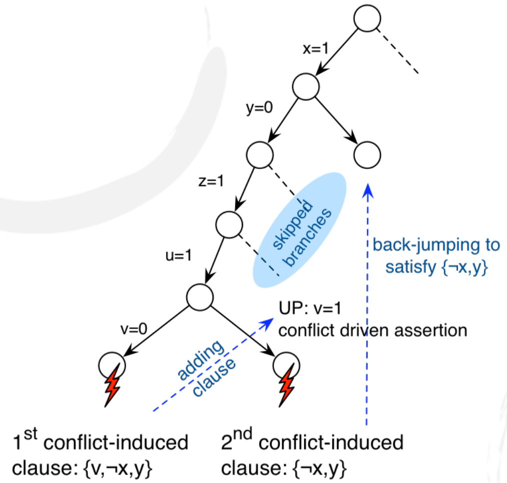
$(\bar{u},\bar{f},m)$
$(\bar{u},\bar{m},h)$
$(\bar{f},\bar{g},\bar{h})$
$(y,\bar{f})$
$(\bar{f},g)$
$\vdots$



conflict-induced clause: $(y,\bar{u})$

adding the conflict-induced (learned) clause to the instance avoids repeated search

Non-Chronological Back-Jumping



Parallelization

- Allow several processors work collaboratively on the same SAT instance
- Questions to answer:
  - How to partition search space between processors?
    - Once at the beginning or on demand during search?
    - How to deal with unreliable communication / network failure / shtudown of computers
  - Exchange learned clauses between processes?
  - Effects of combining clause learning and parallelization
- Experimental results:
  - Good speed-upds attainable on n processors (n≈8-32)
  - Parallel learning and clause exchange highly problem dependent

Dynamic Search Space Splitting

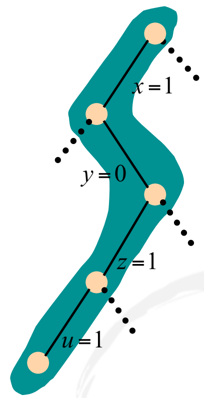
- Guiding path (H. Zhang *et al.* 1996) describes state of search, e.g.

  $((x,\mathrm{B}),(\bar{y},\mathrm{N}),(z,\mathrm{B}),(u,\mathrm{B}))$

- Partitioning of search-space at each (_,B) entry possible, e.g.

  $((x,\mathrm{N}),(\bar{y},\mathrm{N}),(z,\mathrm{B}),(u,\mathrm{B}))$

  $((\bar{x},\mathrm{N}))$



Combining Learning and Parallelization

- Acceleration by lemma generation may limit speed-ups attainable by parallelization: