ES-A-Austin Community College-1

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Executive Summary

Problem: To construct a population model of two neighboring patrilineal clans with an associated pool of females per clan.

The object of this report is to map a mathematical model that can be used to create accurate predictions of the population distribution of males and females resulting from conflict between neighbouring patrilineal clans. The model created is intended to access the sustainability of such a social tradition.

Our model presents a modified Lotka- Volterra model that is able to:

1. Predict equilibrium points in population dynamics of two interacting tribes
2. Assume a non-uniform pool of females

$$y'_1 = y_1\left(r_1 x_1 - c_{11} y_1 - c_{12} y_2\right)$$
$$y'_2 = y_2\left(r_2 x_2 - c_{22} y_2 - c_{21} y_1\right)$$
$$x'_1 = a_1 x_1\left(1 - \frac{x_1}{K_1}\right)$$
$$x'_2 = a_2 x_2\left(1 - \frac{x_2}{K_2}\right)$$

Here y_1' and y_2' represent the male's population dynamics, and x_1' and x_2' represent the female's population dynamics. Notice that the populations of females grow logistically and are autonomous. We assume that Tribe 1's males will not interact with Tribe 2's females, and vice versa. Therefore our model could be extended into a $2n$ equation system to model n interacting clans in the following manner:

$$y'_1 = y_1\left(r_1 x_1 - \sum_{i=1}^{n} c_{1i} \cdot y_i\right)$$
$$\vdots$$
$$y'_n = y_n\left(r_n x_n - \sum_{i=1}^{n} c_{ni} \cdot y_i\right)$$
$$x'_n = a_n x_n\left(1 - \frac{x_n}{K_n}\right)$$

Continuing the exploration on our model of two male populations and two female populations, we are interested in whether competitive patrilineal clans are a sustainable cultural arrangement or not. We make the following assumptions in our model:

1. The parameters $\{r_i\}_{i=1}^{n}$, $\left\{c_{ij}\right\}_{1\le i,j\le n}$, a_n, K_n are all positive real numbers.
2. Competition is more intense than self-restraint: $c_{ij} > c_{kk}$ for all $i,j,k \in [n]$ with $i \neq j$.

3. The initial conditions $(y_1(0), y_2(0), ..., y_n(0), x_1(0), x_2(0), ..., x_n(0))$ lies in the positive orthant $(\Re^{+}_{>0})^{n+1}$ and also satisfying $x_n(0) < K_n$.

Analyzing our system for fixed points we came up with the following set of equilibria:

$$\varepsilon = \left\{ (0,0,0,0),\ (0,0,K_1,0),\ (0,0,0,K_2),\ (0,0,K_1,K_2),\ \left(\frac{r_1 K_1}{c_{11}},0,K_1,0\right),\left(\frac{r_1 K_1}{c_{11}},0,K_1,K_2\right),\right.$$

$$\left.\left(0,\frac{r_2 K_2}{c_{22}},0,K_2\right),\left(0,\frac{r_2 K_2}{c_{22}},K_1,K_2\right),\left(\frac{r_1 c_{22} K_1 - r_2 c_{12} K_2}{c_{11} c_{22} - c_{12} c_{21}},\ \frac{r_2 c_{11} K_2 - r_1 c_{21} K_1}{c_{11} c_{22} - c_{12} c_{21}},\ K_1,\ K_2\right)\right\}$$

Through the Jacobian method we were able to analyze the stability of the fixed points and will present our finding in table format:

Fixed Points	**Signs of eigenvalues**	**Stability of Fixed point**
$(0,\ 0,\ 0,\ 0)$	One zero eigenvalue of multiplicity 4	Unstable
$(0,\ 0,\ K_1,\ 0)$	One zero multiplicity 2, one positive, one negative	Unstable saddle point
$(0,\ 0,\ 0,\ K_2)$	One zero multiplicity 2, one positive, one negative	Unstable saddle point
$(0,\ 0,\ K_1,\ K_2)$	Two positive, Two negative	Unstable Saddle point
$(\frac{r_1}{c_{11}} K_1, 0, K_1, 0)$	One zero, three negative	Locally Stable
$(\frac{r_1}{c_{11}} K_1, 0, K_1, K_2)$	Three negative, one positive	Locally Stable
$(0, \frac{r_2}{c_{22}} K_2, 0, K_2)$	One positive, three negative	Locally Stable
$(0, \frac{r_2}{c_{22}} K_2, K_1, K_2)$	Three negative, one positive	Locally Stable
$(\frac{r_1 c_{22} K_1 - r_2 c_{12} K_2}{c_{11} c_{22} - c_{12} c_{21}}, \frac{r_2 c_{11} K_2 - r_1 c_{21} K_1}{c_{11} c_{22} - c_{12} c_{21}}, K_1, K_2)$	Two positive, Two negative	Unstable saddle point

Based on our analytics of fixed points we deduct that when patrilineal clans compete against each other one group will go extinct. A decline of patrilineal clans is inevitable as it is impossible to maintain a population equilibrium in the conditions of intense conflict between neighbouring clans.

References:

[1] Collins, Nathan. 2018. Wars and clan structure may explain a strange biological event 7,000 years ago, Stanford researchers find. Stanford University News Service. 30 May. https://news.stanford.edu/pressreleases/2018/05/30/war-clan-structubiological-event/. Accessed 3 September 2018.

[2] Zeng, Tian Chen, Alan J. Aw, and Marcus W. Feldman. 2018. Cultural hitchhiking and competition between patrilineal kin groups explain the post-Neolithic Y-chromosome bottleneck. Nature Communications. Volume 9, Article number: 2077. https://www.nature.com/articles/s41467-018-04375-6. Accessed 3 September 2018. Freely downloadable. This article is licensed under a Creative Commons Attribution 4.0 International License.

$$\lambda = \frac{-c_{21}\cdot r_1\cdot c_{22}\cdot K_1 - c_{21}\cdot c_{12}\cdot r_2\cdot K_2 + 2\cdot c_{11}\cdot c_{22}\cdot r_2\cdot K_2 + c_{12}\cdot c_{21}\left(\left(c_{11}\cdot r_2\cdot K_2 - r_1\cdot K_1\cdot c_{21}\right)^2\right)}{c_{21}\cdot c_{12} - c_{11}\cdot c_{22}}$$

$$\frac{r_1}{r_2}\frac{K_1}{K_2} < \frac{\left(c_{21}\cdot c_{22} - 2\cdot c_{11}\cdot c_{22} - c_{12}\cdot c_{21}\cdot c_{11}{}^2\cdot r_1\cdot K_1\right)}{\left(r_1\cdot K_1\cdot c_{21}{}^2 - 2\cdot c_{22}\cdot c_{21} - 2\cdot r_2\cdot K_2\cdot c_{11}\cdot c_{21}{}^2\cdot c_{12}\right)}$$