Examining Interaction Amongst Patrilineal Clans Using Differential Equations

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Modified Lotka-Volterra Model

$$y_1' = y_1\left(r_1 x_1 - \sum_{i=1}^{n} c_{1i} \cdot y_i\right)$$

$$\vdots$$

$$y_n' = y_n\left(r_n x_n - \sum_{i=1}^{n} c_{ni} \cdot y_i\right)$$

$$x_n' = a_n x_n\left(1 - \frac{x_n}{K_n}\right)$$

Case where n=2

$$y'_1 = y_1\left(r_1 x_1 - c_{11} y_1 - c_{12} y_2\right)$$

$$y'_2 = y_2\left(r_2 x_2 - c_{22} y_2 - c_{21} y_1\right)$$

$$x'_1 = a_1 x_1\left(1 - \frac{x_1}{K_1}\right)$$

$$x'_2 = a_2 x_2\left(1 - \frac{x_2}{K_2}\right)$$

$r_1,\ r_2 = \text{growth rate of male population}$

$c_{11},\ c_{22} = \text{self} - \text{restraining coefficient}$

$c_{12},\ c_{21} = \text{competition coefficients}$

$a_1,\ a_2 = \text{growth rate of female population}$

$K_1,\ K_2 = \text{carrying capacity for female population}$

Assumptions:

1. *The parameters* $\{r_i\}_{i=1}^{n}$, $\{c_{ij}\}_{1 \le ij \le n}$, a_n, K_n *are all positive.*

2. *Competition is more intense than self − restraint:* $c_{ii} > c_{kk}$ *for all* $i, j, k \in [n]$ *with* $i \neq j$

3. *Initial conditions* $\left(y_1(0),\, y_2(0),\, \ldots, y_n(0),\, x_1(0), x_2(0),\, \ldots, x_n(0)\right)$ *lie in the positive orthant* $\left(\Re^{+}_{>0}\right)^{2n}$ *and also satisfy* $x_n(0) < K_n$

Jacobian Matrix and Equilibrium Points

$$J\left(y_1, y_2, x_1, x_2\right) = \begin{bmatrix} -2y_1c_{11} - c_{12}r_2 + r_1x_1 & -c_{12}y_2 & r_1y_1 & 0 \\ -c_{21}y_2 & -c_{21}y_1 - 2c_{22}y_2 + r_2x_x & 0 & r_2y_2 \\ 0 & 0 & a_1 - \dfrac{2a_1x_1}{k_1} & 0 \\ 0 & 0 & 0 & a_2 - \dfrac{2a_2x_2}{k_2} \end{bmatrix}$$

$$\varepsilon = \left\{ (0,0,0,0),\ (0,0,K_1,0),\ (0,0,0,K_2),\ (0,0,K_1,K_2),\ \left(\frac{r_1K_1}{c_{11}},0,K_1,0\right),\left(\frac{r_1K_1}{c_{11}},0,K_1,K_2\right),\right.$$
$$\left.\left(0,\frac{r_2K_2}{c_{22}},0,K_2\right),\left(0,\frac{r_2K_2}{c_{22}},K_1,K_2\right),\left(\frac{r_1c_{22}K_1 - r_2c_{12}K_2}{c_{11}c_{22} - c_{12}c_{21}},\ \frac{r_2c_{11}K_2 - r_1c_{21}K_1}{c_{11}c_{22} - c_{12}c_{21}},\ K_1,\ K_2\right)\right\}$$

What do these points mean? Are they sustainable?

No Survival

$$f(y^*_1, y^*_2, x^*_1, x^*_2) = \{(0, 0, 0, 0)\}$$

- Trivial solution, all populations go to extinction
- One zero eigenvalue with multiplicity 4
- Unstable saddle point

Female only Survival

$$f(y^*_1, y^*_2, x^*_1, x^*_2) = \{(0,0,k_1,0), (0,0,0,k_2), (0,0,k_1,k_2)\}$$

- Only one female population survives
  - 1 zero eigenvalue with multiplicity 2
  - 1 negative eigenvalue
  - 1 positive eigenvalue
- semi-stable

- Both female populations survive
  - Two positive eigenvalues
  - Two negative eigenvalues
- Semi stable

One tribe and corresponding pool of females

$$f\left(y^*_1, y^*_2, x^*_1, x^*_2\right) = \left\{\left(\frac{r_1}{c_{11}}k_1, 0, k_1, 0\right), \left(0, \frac{r_2}{c_{22}}k_2, 0, k_2\right)\right\}$$

- One zero eigenvalue
- Three negative eigenvalues
- Stable
  - Also stable in 3d analog

One Tribe and Both Female Populations Survive

$$f\left(y^*_1, y^*_2, x^*_1, x^*_2\right) = \left\{\left(\frac{r_1}{c_{11}}k_1, 0, k_1, k_2\right), \left(0, \frac{r_2}{c_{22}}k_2, k_1, k_2\right)\right\}$$

- Both have three negative eigenvalues
- For the case where tribe survives
  - If $\frac{r_1 \cdot K_1}{r_2 \cdot K_2} > c_{11}$ then all negative eigenvalues
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- For the case where tribe 2 survives
  - If $\frac{r_1 \cdot K_1}{r_2 \cdot K_2} > \frac{c_{11}}{c_{22}}$ then all negative eigenvalues

All Population Groups Survive

$$f\left(y^*_1, y^*_2, x^*_1, x^*_2\right) = \left\{ \frac{r_1 c_{22} k_1 - r_2 c_{12} k_2}{c_{11} c_{22} - c_{12} c_{21}}, \frac{r_2 c_{22} k_2 - r_1 c_{21} k_1}{c_{11} c_{22} - c_{12} c_{21}}, k_1, k_2 \right\}$$

- Two negative eigenvalues
- 1 positive
- 1 undetermined

$$\lambda = \frac{-c^{\circ}_{21} \cdot r_1 \cdot c_{22} \cdot K_1 - c_{21} \cdot c_{12} \cdot r_2 \cdot K_2 + 2 \cdot c_{11} \cdot c_{22} \cdot r_2 \cdot K_2 + c_{12} \cdot c_{21}\left(\left(c_{11} \cdot r_2 \cdot K_2 - r_1 \cdot K_1 \cdot c_{21}\right)^2\right)}{c_{21} \cdot c_{12} - c_{11} \cdot c_{22}}$$

Negative if $\frac{r_1}{r_2} \frac{K_1}{K_2} < \frac{\left(c_{21} \cdot c_{22} - 2 \cdot c_{11} \cdot c_{22} - c_{12} \cdot c_{21} \cdot c_{11}^2 \cdot r_1 \cdot K_1\right)}{\left(r_1 \cdot K_1 \cdot c_{21}^2 - 2 \cdot c_{22} \cdot c_{21} - 2 \cdot r_2 \cdot K_2 \cdot c_{11} \cdot c_{21}^2 \cdot c_{12}\right)}$

Mobility Modeling

This would affect the competition coefficients. c(ij) would go up if a tribe had access to quicker ways of traveling to interact with other tribes. This will likely only affect the equilibrium point where all populations survive.

Potential Further Analysis

- Find associated eigenvectors
- Linearization analysis
- Lyapunov Stability Analysis
- Generalize coefficient relationship for n clans

Questions?