We consider a situation where the government wants to invest in a public good financed by an income tax. The cost of the project is 18k and the question is whether the tax raised by a nonlinear income tax schedule will be enough to cover the cost. We estimate the parameters of the (Pareto) income distribution and then see whether the expected tax income exceeds the cost.

Due to the nonlinearity of the tax scheme, it is hard to propagate the parameter uncertainty into the uncertainty of the tax revenue using a frequentist approach. With a Bayesian analysis this is straightforward to do: we feed the posterior distribution of the parameters into the tax function.

Another advantage of Bayesian analysis is that one can do a scenario analysis, say the cost of the project can be low, average or high with certain probabilities.

We first generate a sample from the theoretical income distribution. With this sample of 50 individuals, we will estimate the parameters of our model.

```
import numpy as np
import pymc as pm
from pymc import do, observe
N=50
individuals = np.arange(N)

with pm.Model(coords={"individuals":individuals}) as model_income:
    alpha = pm.HalfNormal("alpha",1)
    m = pm.Normal("m",30000,5000)
    income = pm.Pareto('income', alpha=alpha, m=m,dims="individuals")


true_values = {
    "alpha": 3.0,
    "m": 30000
}

income_simulate = do(model_income, true_values)

with income_simulate:
    simulate = pm.sample_prior_predictive(samples=1)


income_data = simulate.prior.income.values
```

```
# income_data
```

```
Sampling: [income]
```

Given the `income_data` that we have, the following code block generates the posterior distribution of the parameters α, m.

```
model_inference = observe(model_income,\
                {"income":income_data[0,0]})

with model_inference:
    idata = pm.sample(progressbar=False)
```

```
Initializing NUTS using jitter+adapt_diag...
Multiprocess sampling (4 chains in 4 jobs)
NUTS: [alpha, m]
Sampling 4 chains for 1_000 tune and 1_000 draw iterations (4_000 + 4_000 draws total)
There were 3065 divergences after tuning. Increase `target_accept` or reparameterize.
The rhat statistic is larger than 1.01 for some parameters. This indicates problems dur
The effective sample size per chain is smaller than 100 for some parameters.  A higher
```

We can view the estimates and the values for `r_hat` (which are not great but not so relevant for our application).

```
import arviz as az
headers = ['mean', 'sd', 'hdi_3%', 'hdi_97%',\
           'ess_bulk', 'r_hat']
variables = ["m","alpha"]
df_summary = az.summary(idata,var_names=variables)[headers]
df_summary
```

	mean	sd	hdi_3%	hdi_97%	ess_bulk	r_hat
m	29838.667	233.688	29405.822	30072.144	298.0	1.01
alpha	2.731	0.375	2.065	3.498	262.0	1.03

Next we generate our posterior predictive distribution of income.

```
idata_posterior_predictive = pm.sample_posterior_predictive(idata,model=model_inference
posterior_predictive_incomes = idata_posterior_predictive.posterior_predictive.income.v
```

```
Sampling: [income]
```

The following code block defines the nonlinear tax function. And we calculate the posterior distribution for the tax revenue.

```python
import pytensor
import pytensor.tensor as pt


def piecewise_linear_tax_scalar(income, thresholds, rates):
    """
    Calculates the tax for a given income based on a piecewise linear tax function.
    This function works for inputs of any dimension due to broadcasting.
    """
    t1, t2 = thresholds
    r1, r2, r3 = rates

    # Tax for the first bracket
    tax = np.minimum(income, t1) * r1

    # Tax for the second bracket
    tax += np.maximum(0, np.minimum(income, t2) - t1) * r2

    # Tax for the third bracket
    tax += np.maximum(0, income - t2) * r3

    return tax

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```python
    tax += pt.maximum(0, pt.minimum(income, t2) - t1) * r2

    # Tax for the third bracket
    tax += pt.maximum(0, income - t2) * r3

    return tax


# pt.dtensor3 creates a placeholder for a 3D tensor with double-precision floats.
income_tensor_3d = pt.dtensor3('income_3d')

# Define the realistic parameters for the Netherlands (2025 system)
thresholds = (38441, 76817)
rates = (0.3582, 0.3748, 0.4950)

tax_due_3d = piecewise_linear_tax(income_tensor_3d, thresholds, rates)
calculate_tax_3d = pytensor.function(inputs=[income_tensor_3d], outputs=tax_due_3d)

# Use the compiled function to calculate the taxes
tax_income_per_head = calculate_tax_3d(posterior_predictive_incomes)

tax_income_per_head.shape
```

4	1000	50

In a frequentist approach one can calculate the expected tax revenue. This expectation may (or may not) differ significantly from the target of 18k needed to finance the public good. Whether or not it does, what do we learn from this?

```python
tax_income_per_head.mean(axis=2).mean(), tax_income_per_head.mean(axis=2).std()
```

17824.886616071733	3251.493091037539

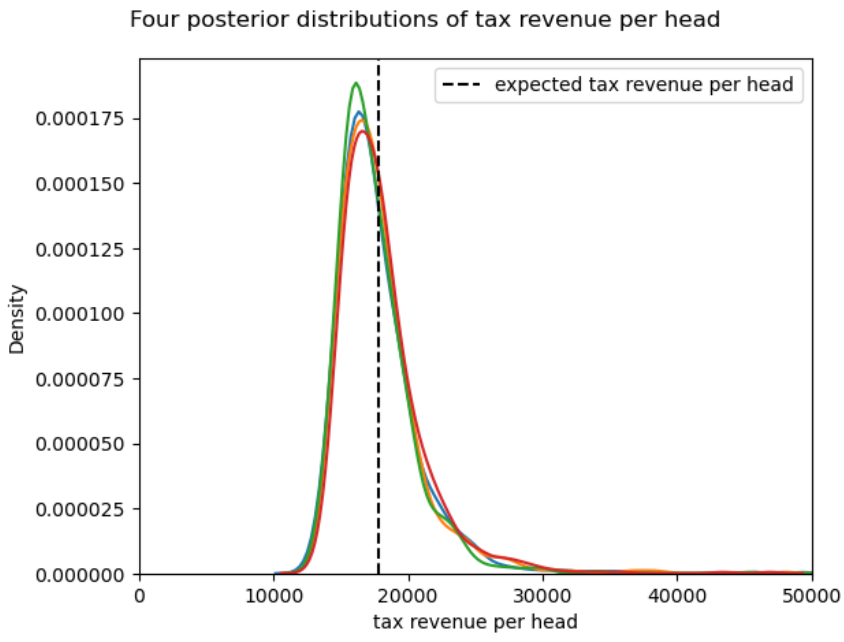
The Bayesian approach allows us to quantify the uncertainty. The following figure shows 4 different distributions of the tax revenue. This figure shows that whether or not the mean tax revenue is above or below 18k is hardly relevant.

```
import matplotlib.pyplot as plt
import seaborn as sns

# Assuming your tensor is called 'data_tensor'
data = tax_income_per_head.mean(axis=2)  # Convert to numpy array if using PyTorch

for i in range(4):
    sns.kdeplot(data[i, :])

plt.axvline(x=tax_income_per_head.mean(), color='black', linestyle='--',label="expectec
# plt.axvline(x=piecewise_linear_tax_scalar(posterior_predictive_incomes.mean(),thresho
plt.legend()
plt.xlim(0,50000)
plt.xlabel("tax revenue per head")
plt.suptitle('Four posterior distributions of tax revenue per head')
plt.tight_layout();
```



With the Bayesian approach we can answer the question: how likely is it that tax revenue falls below the 18k threshold:

```
threshold = 18000
```

```python
print(data.mean())
print(np.mean(data < threshold))
```

```
17824.886616071733
0.635
```

Also do a scenario analysis with different values for the threshold?