Mathematics for AI and Data Science : Vectors and Vector Operation

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Learning Outcome

- To learn to represent vectors

- To know how to perform basic operations like addition, scalar multiplication, projection using vectors

- To understand how to perform dot product and cross product of vectors

Outline

- Introduction and Motivation of Vectors
- Vector Operations
  - Addition
  - Scalar multiplication
  - Linear Combination
- Component Representation
  - Vector Length
  - Vector Algebra
- Unit Vector
- Dot product of Vectors
- Projection
  - Scalar
  - Vector
- Cross-Product of Vectors

Motivation

Uses of Vectors in Data Science

- ❖ Vectors are important for many different areas of **machine learning** and pattern processing

- ❖ In **machine learning**, feature **vectors** are **used** to represent numeric or symbolic characteristics, called features, of an object in a mathematical, easily analyzable way.

Uses of Vectors in Data Science

- There are many quantities which require only 1 measurement to describe them. e.g Length of a string, or area of any shape or temperature of any surface (Such quantities are called ***scalars)***

- However, most of the quantities or datasets require at **least 2 measurements** to describe them. Along with the magnitude, they have a "direction" associated e.g velocity or force. Hence, they need vector representation.

Example

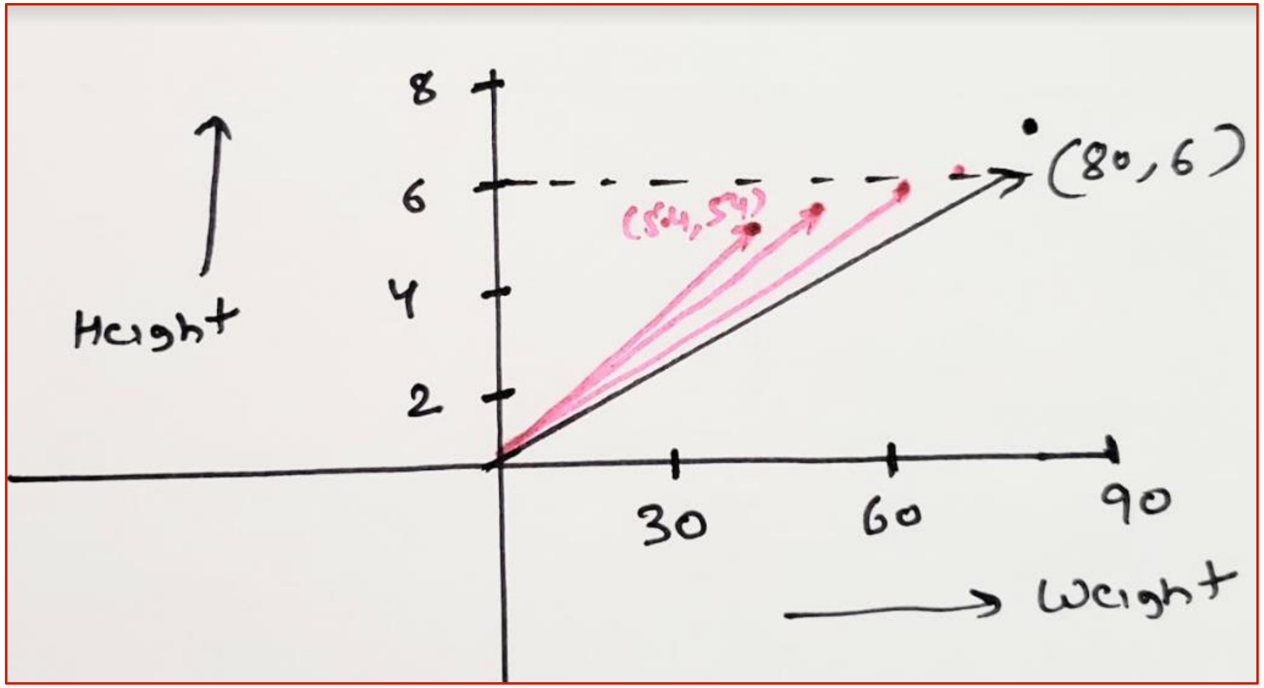
- Let's say you are collecting some data about a group of students in a class. You are measuring the height and weight of each student and the data collected for 5 students is as follows:

Heigt (in feets)	Weight (in Kgs)
6	80
5.4	54
5	50
5.7	65
5.8	72

- Each individual measurement here is a scalar quantity. So height or weight viewed stand-alone are scalars.

- However, when you look at the observation about each student as a whole i.e height and weight together for every student, you can think of it as a **vector.**

Example



- Hence, we can represent the data all the 5 students using vectors.
- Similarly, machine learning algorithms **use vectors representation to process high dimensional data**

Introduction to Vectors

Vectors

- The term vector is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both **magnitude and direction.**

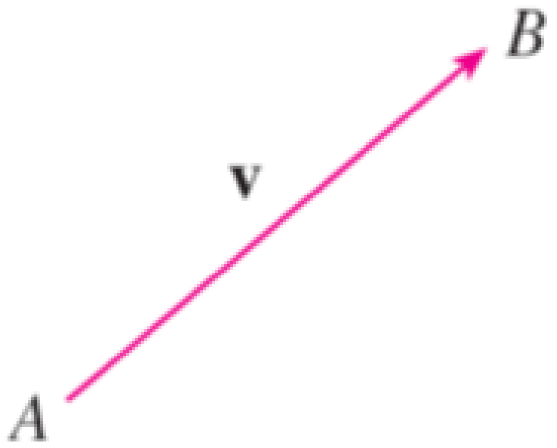
- A vector is often represented by an **arrow or a directed line segment**. The length of the arrow represents the **magnitude** of the vector and the **arrow** **points in the direction of the vector.**

- We denote a vector by printing a letter in boldface (v) or by putting an arrow above the letter $(\vec{v})$

Vectors

❖For instance, suppose a particle moves along a line segment from point A to point B.

❖The corresponding displacement vector v, shown in Figure 1, has initial point A (the tail) and terminal point B (the tip) and we indicate this by writing

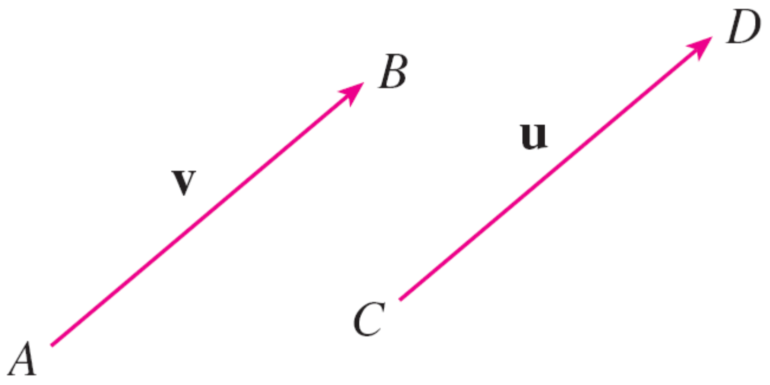


$$\mathbf{v} = \overrightarrow{AB}$$

Equivalent Vectors

❖Notice that the vector u has the same length and the same direction as v even though it is in a different position.

❖We say that u and v are **equivalent (or equal)** and we write u = v.



$\mathbf{u} = \overrightarrow{CD}$

$\mathbf{v} = \overrightarrow{AB}$

Figure 1

Equivalent vectors

Zero Vectors

❖ The zero vector, denoted by 0, has length 0. It is the only vector with no specific direction.

Combining/ Adding Vectors

Combining Vectors

- Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$.

- Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$ as in Figure 2.

- The combined effect of these displacements is that the particle has moved from A to C.

- The resulting displacement vector $\overrightarrow{AC}$ is called the **sum of** $\overrightarrow{AB}$ **and** $\overrightarrow{BC}$ **, we write**

$$\boxed{\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}}$$

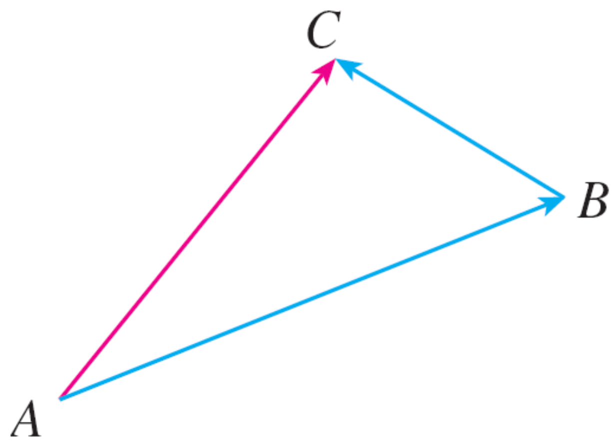


Figure 2

Combining Vectors

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of **u** to the terminal point of **v**.

❖The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law.**

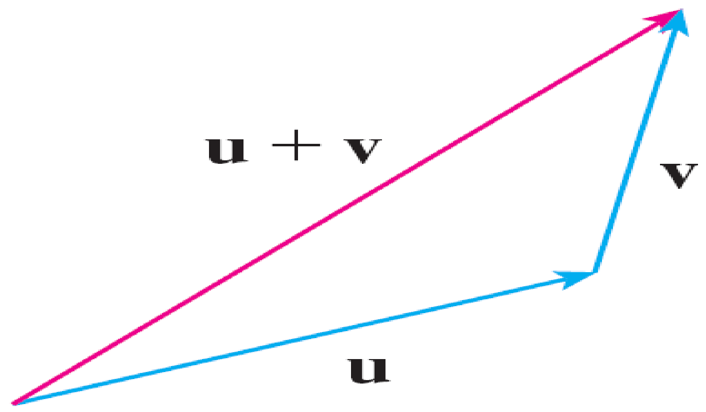


Figure 3

The Triangle Law

Combining Vectors

❖In Figure 4 we start with the same vectors u and v as in Figure 3 and draw another copy of v with the same initial point as u.

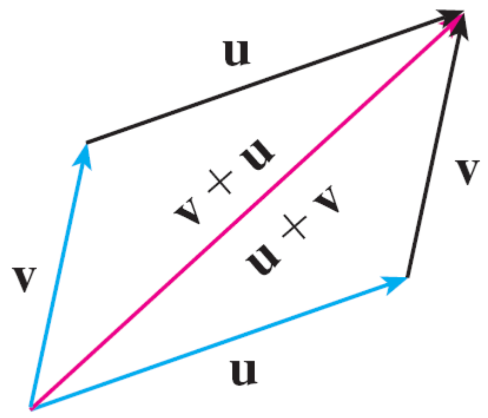


Figure 4

The Parallelogram Law

❖Completing the **parallelogram**, we see that u + v = v + u.

❖**Parallelogram Law :** If we place u and v so they start at the same point, then u + v lies along the **diagonal** of the parallelogram with u and v as sides. (This is called the Parallelogram Law.)

Scalar Multiplication

Scalar Multiplication

❖It is possible to multiply a vector by a **real number c**. (In this context we call the real number c a scalar to distinguish it from a vector.)

❖For instance, we want 2v to be the same vector as v + v, which has the **same direction** as v but is **twice** as long. In general, we multiply a vector by a scalar as follows.

Definition of Scalar Multiplication If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Scalar Multiplication

❖This definition is illustrated in Figure 7.

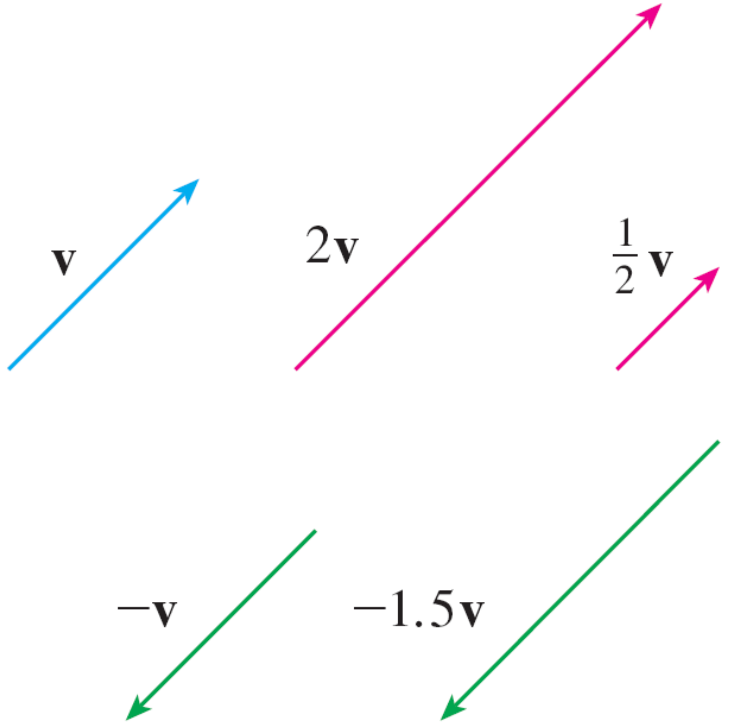


Figure 7

Scalar multiples of **v**

Scalar Multiplication

❖ Notice that two nonzero vectors are parallel if they are scalar multiples of one another.

❖ In particular, the vector –v = (–1)v has the s**ame length as v but points in the opposite direction.** We call it the negative of v.

❖ By the difference u – v of two vectors we mean

❖ $$u - v = u + (-v)$$

Subtraction of Vectors

❖So we can construct u – v by first drawing the negative of v, –v, and then adding it to u by the Parallelogram Law as in Figure 8(a).

❖Alternatively, since v + (u – v) = u, the vector u – v, when added to v, gives u. So we could construct u – v as in Figure 8(b) by means of the Triangle Law.

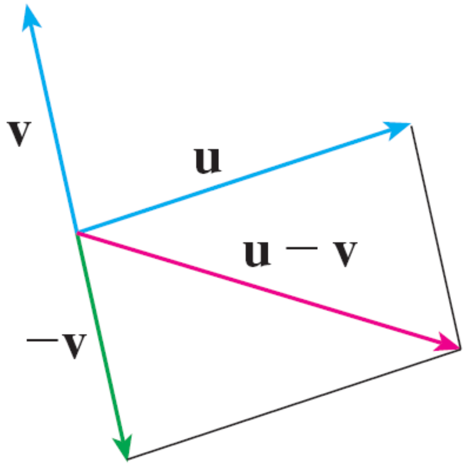


Figure 8(a)

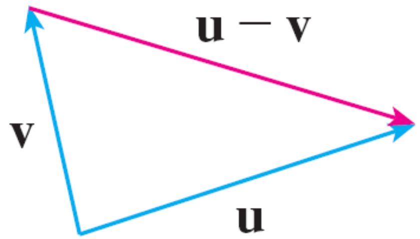


Figure 8(b)

Drawing **u – v**

Vector Representation

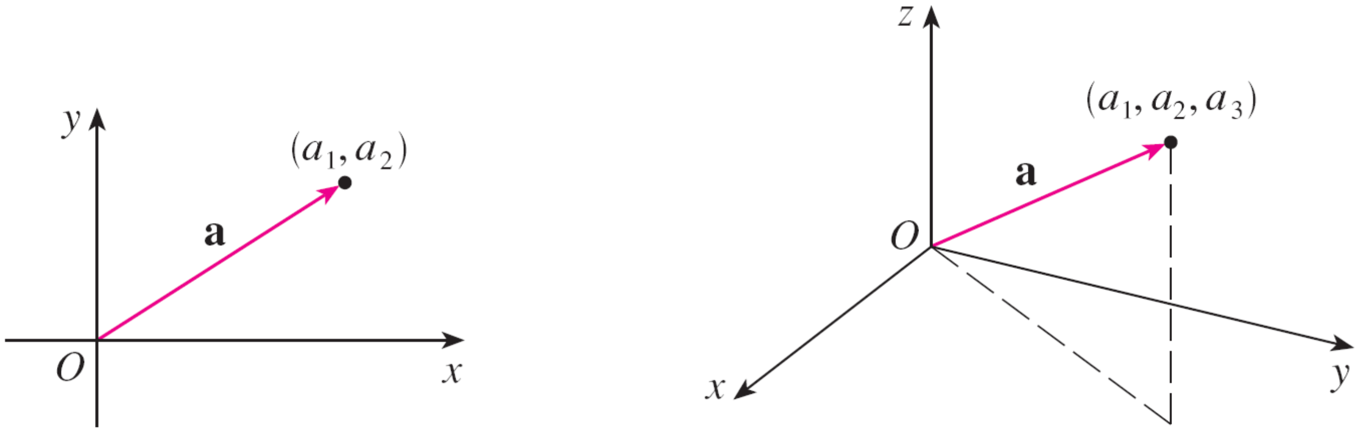
Vector Representation: Components

❖For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

❖If we place **the initial point of a vector a at the origin of a rectangular coordinate** system, then the terminal point of a has **coordinates** of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).

❖These coordinates are called the **components** of a and we write

a = ⟨a1, a2⟩   or   a = ⟨a1, a2, a3⟩

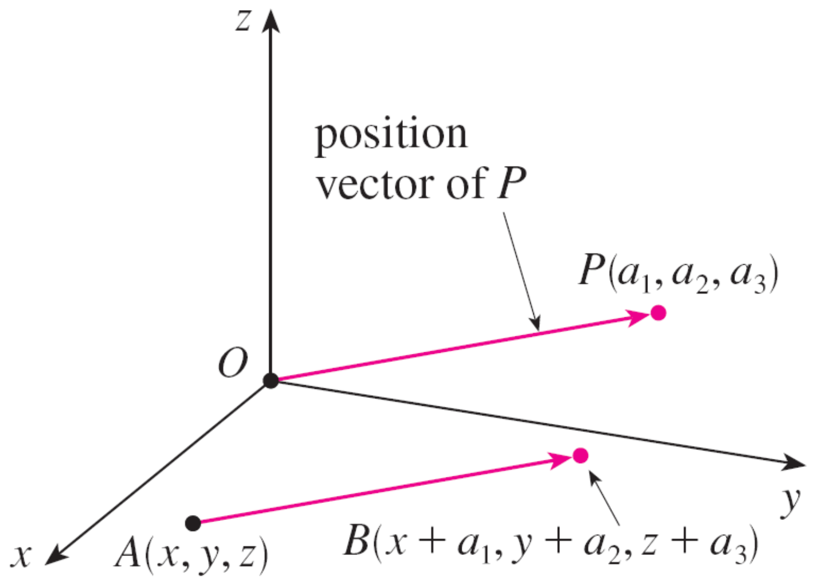


$\mathbf{a} = \langle a_1, a_2 \rangle$ $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Figure 11

Position Vector

❖In three dimensions, the vector a = $\overrightarrow{OP}$ = ⟨a1, a2, a3⟩ is the **position vector** of the point P(a1, a2, a3). (See Figure 13.)



Representations of **a** = $\langle a_1, a_2, a_3 \rangle$

Figure 13

Length or Magnitude of Vector

❖The **magnitude or length** of the vector v is denoted by the symbol | v | or || v ||.

❖By using the distance formula to compute the length of a segment OP(a vector through origin), we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector Addition (Algebra)

- **How do we add vectors algebraically?**
- Figure 14 shows that if a = ⟨a1, a2⟩ and b = ⟨b1, b2⟩,
- The sum is a + b = ⟨a1 + b1, a2 + b2⟩,
- (for the case where the components are **positive**).
- In other words, to **add algebraic vectors we add their components**. Similarly, to **subtract vectors we subtract components**.

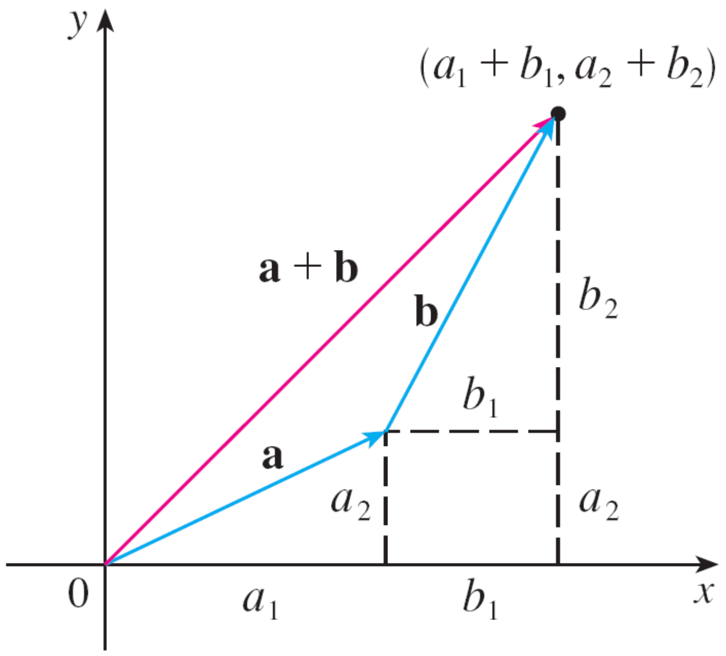


Figure 14

Scalar Multiplication (Algebra)

❖ To multiply a vector by a scalar we **multiply each component by that** scalar.

❖ From the similar triangles in Figure 15 we see that the components of ca are ca1 and ca2.

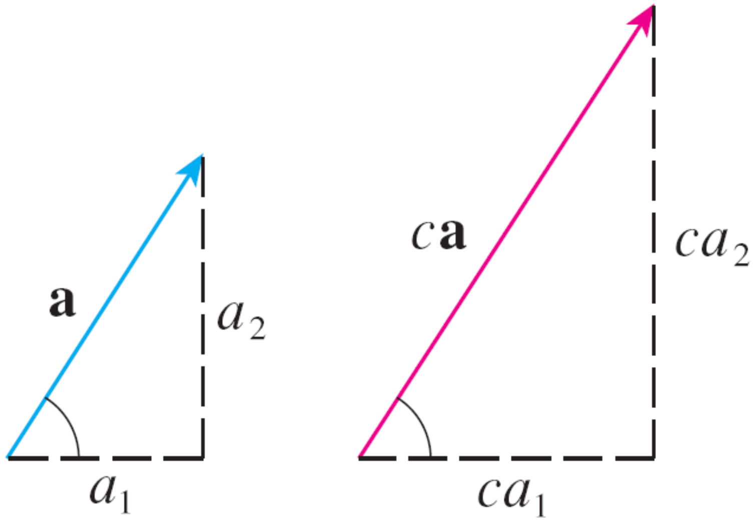


Figure 15

Linear Combination

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.

- Important Questions
  - What is the picture of all combination of c***u***?
  - What is the picture of all combination of c***u*** +d***v?***

Linear Combination

- Important Questions
  - What is the picture of all combination of c***u***?: Fill a line through (0,0)
  - What is the picture of all combination of c***u*** +d***v*** **: Fill a plane through (0,0)**

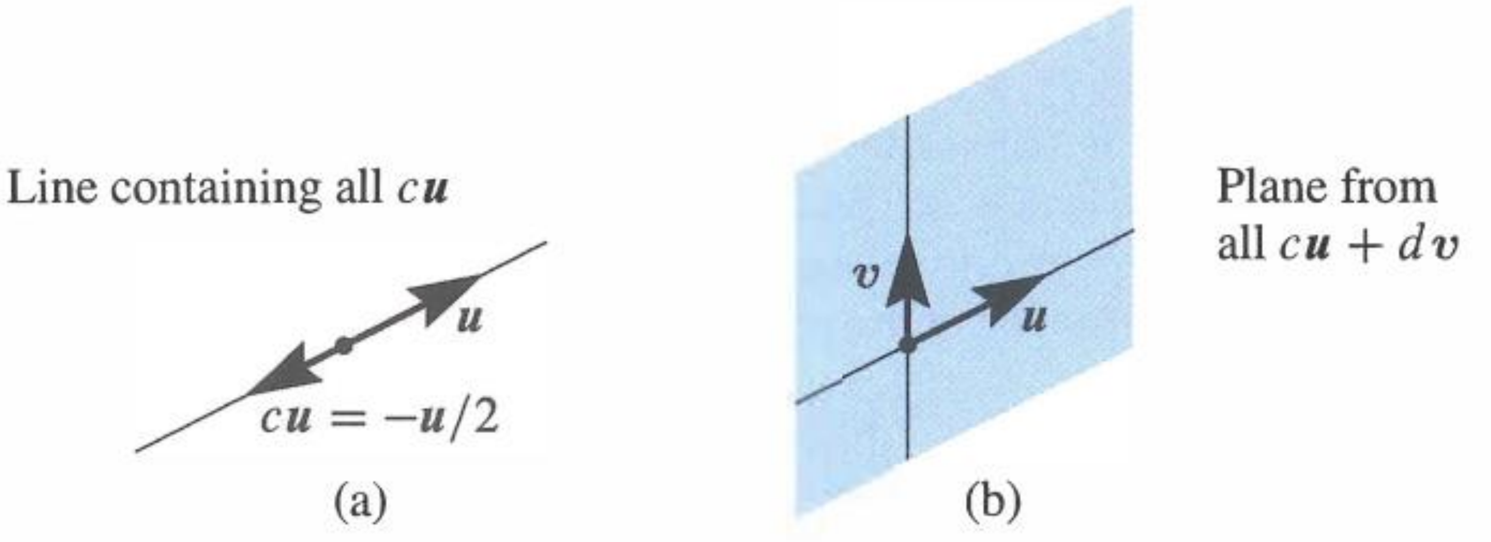


Figure 1.3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

Introduction to linear algebra,Gilbert Strang

Properties of Vector

❖ Addition and scalar multiplication are defined in terms of components just as for the cases n = 2 and n = 3.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

The standard basis vectors

❖Three vectors in V3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

❖These vectors i, j, and k are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define i = ⟨1, 0⟩ and j = ⟨0, 1⟩. (See Figure 17.)

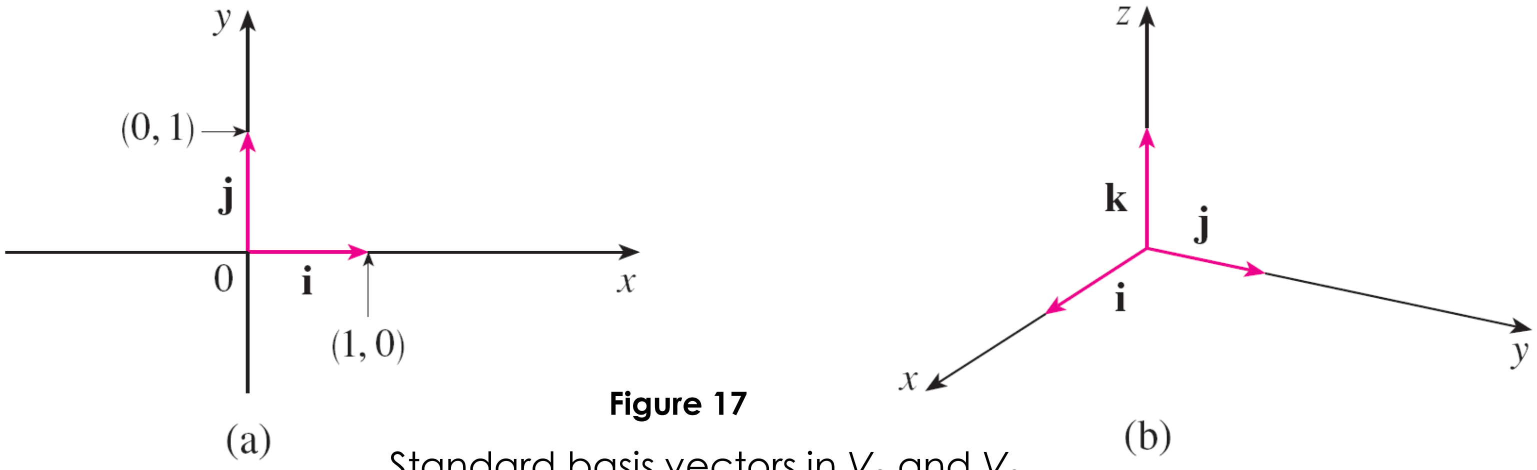


Figure 17

Standard basis vectors in V_2 and V_3

The standard basis vectors

- If $a = \langle a_1, a_2, a_3 \rangle$, then we can write
- $a = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$
- $= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$
- $a = a_1 i + a_2 j + a_3 k$
- Thus any vector in V3 can be expressed in terms of i, j, and k. For instance,
- $\langle 1, -2, 6 \rangle = i - 2j + 6k$
- Similarly, in two dimensions, we can write
- $a = \langle a_1, a_2 \rangle = a_1 i + a_2 j$

Unit Vector

- **A unit vector is a vector whose length is 1**.

- For instance, i, j, and k are all unit vectors. In general, if a ≠ 0, then the unit vector that has the same direction as a is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

- In order to verify this, we let c = 1/| a |. Then u = ca and c is a positive scalar, so **u has the same direction as a.**

- **Example:** $\mathbf{q} = < -2, 1 >,$

- Magnitude: $|q| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ **Unit Vector:** $\boxed{u = \frac{< -2, 1 >}{\sqrt{5}} = < -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} >}$

Dot Product

The Dot Product

❖So far we have added two vectors and multiplied a vector by a scalar. The question arises: **Is it possible to multiply two vectors so that their product** is a useful quantity? One such product is the dot product, whose definition follows.

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

❖Thus, to find the dot product of a and b, we multiply corresponding components and add.

The Dot Product

- The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the scalar product (or inner product).
- Although Definition 1 is given for three-dimensional vectors, the dot product of n-dimensional vectors is defined in a similar fashion:
- Example n=2:
- $\langle a1, a2\rangle \cdot \langle b1, b2\rangle = a1 * b1 + a2 * b2$

The Dot Product

Examples:

- $\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1) = 2$
- $\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\frac{1}{2}\rangle = (-1)(6) + 7(2) + 4(-\frac{1}{2}) = 6$
- $(\mathrm{i} + 2\mathrm{j} - 3\mathrm{k}) \cdot (2\mathrm{j} - \mathrm{k}) = 1(0) + 2(2) + (-3)(-1) = 7$

Properties of the Dot Product

❖The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The Dot Product

- These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

- 3. a • (b + c) = ⟨a1, a2, a3⟩ • ⟨b1 + c1, b2 + c2, b3 + c3⟩
- = a1(b1 + c1) + a2(b2 + c2) + a3(b3 + c3)
- = a1b1 + a1c1 + a2b2 + a2c2 + a3b3 + a3c3
- = (a1b1 + a2b2 + a3b3) + (a1c1 + a2c2 + a3c3)
- = a • b + a • c

Geometric Interpretation of the Dot Product

❖The dot product a • b can be given a geometric interpretation **in terms of the angle θ** between a and b, which is defined to be the angle between the representations of a and b that start at the origin, where $0 \le \theta \le \pi$.

In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1.

❖Notethat if a and b are **parallel** vectors, then $\theta = 0$ or $\theta = \pi$.

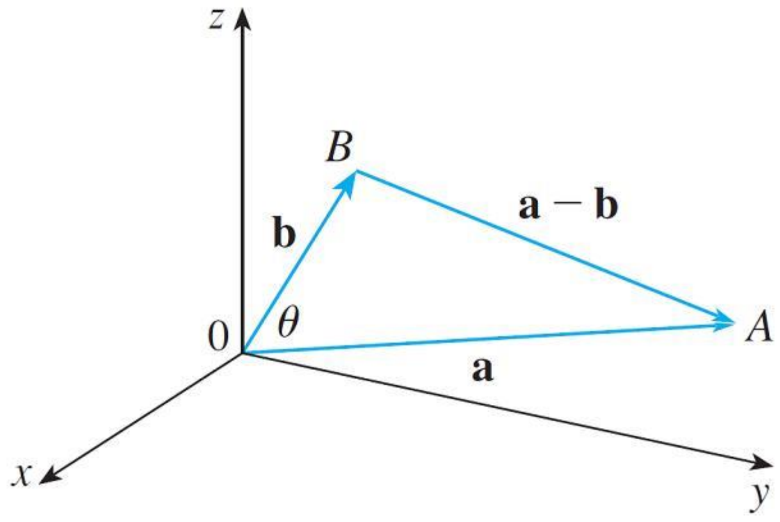


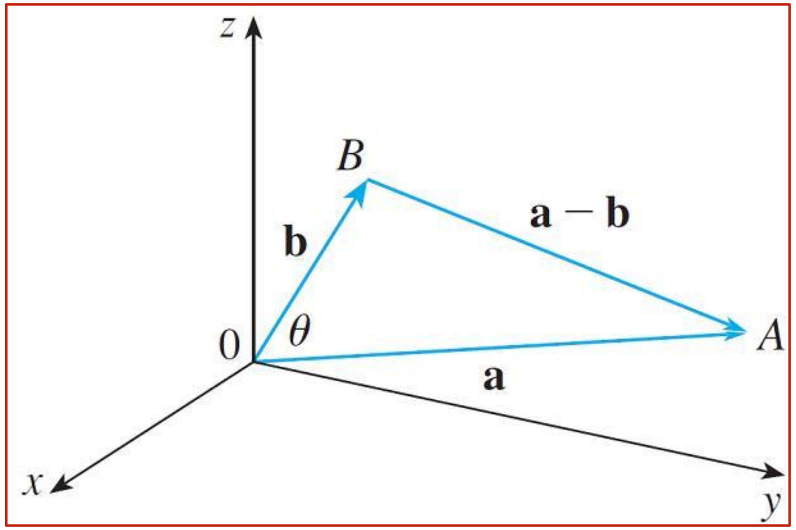
Figure 1

Geometric Interpretation of the Dot Product

❖ The formula in the following theorem is used by physicists as the definition of the dot product.

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$



Geometric Interpretation of the Dot Product

- **Example:** If the vectors a and b have lengths 4 and 6, and the angle between them is $\pi/3$, **find a • b?**

- Solution: Using Theorem 3, we have

- $$a \cdot b = |a|\,|b| \cos(\pi/3)$$
-
- $$= 4 \cdot 6 \cdot \frac{1}{2}$$
-
- $$= 12$$

Angle between two Vectors

❖The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

❖Note:

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Example 3

❖Find the **angle** between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ **and** $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Example 3

❖Find the **angle** between the vectors **a = ⟨2, 2, –1⟩ and b = ⟨5, –3, 2⟩.**

❖Solution:

❖Since, $|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$ and $|\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$

❖a • b = 2(5) + 2(–3) + (–1)(2) = 2

❖ $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{2}{3\sqrt{38}}$

❖

❖So the angle between a and b is $\theta = \cos^{-1}\left(\dfrac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$

The Dot Product of Orthogonal or Perpendicular Vectors

❖Two nonzero vectors a and b are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

❖$a \cdot b = |a||b| \cos(\pi/2) = 0$

❖and conversely if $a \cdot b = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$.

❖The zero vector 0 is considered to be perpendicular to all vectors.

❖Therefore we have the **following method for determining whether two vectors are orthogonal.**

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The Dot Product of Orthogonal or Perpendicular Vectors

- **Example :** Show that $2i + 2j – k$ is perpendicular to $5i – 4j + 2k$?

- **Solution**:

- $(2i + 2j – k) \cdot (5i – 4j + 2k) = 2(5) + 2(–4) + (–1)(2) = 0$

- these vectors are perpendicular.

The Dot Product as a Measure of Direction of Two Vectors

- Because $\cos\theta > 0$ if $0 \le \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \le \pi$, we see that $a \cdot b$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$.
- We can think of $a \cdot b$ as measuring the extent to which a and b point in the **same direction.**
- $a \cdot b$ is positive : If a and b point in the same general direction,
- $a \cdot b$ is 0: If a and b they are perpendicular
- $a \cdot b$ is negative: If a and b point generally opposite directions (see Figure 2).

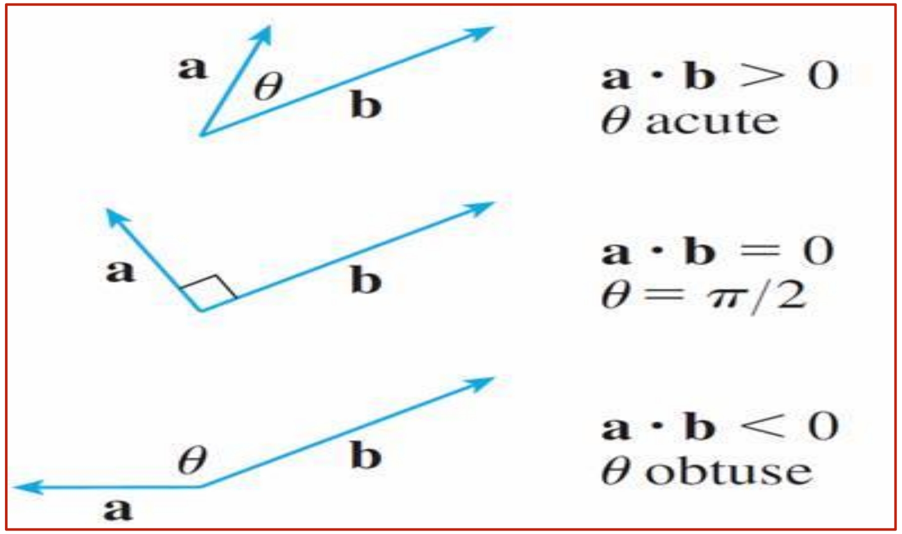


Figure 2

Direction Angles and Direction Cosines

Direction Angles and Direction Cosines

❖The direction angles of a nonzero vector a are the angles α, β, and γ (in the interval [0, π]) that a makes with the **positive** x-, y-, and z-axes. (See Figure 3.)

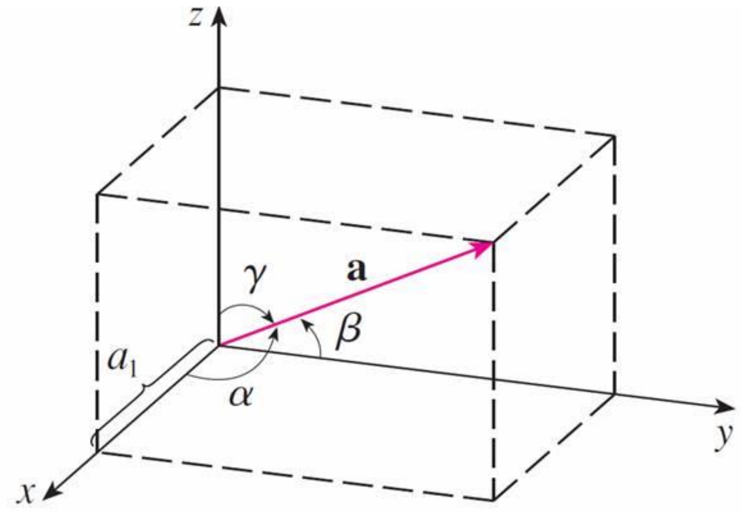


Figure 3

Direction Angles and Direction Cosines

❖The cosines of these direction angles, cos α, cos β, and cos γ, are called the direction cosines of the vector a. Using Corollary 6 with b replaced by i, we obtain

$$\boxed{8} \qquad \cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

❖(This can also be seen directly from Figure 3.) Similarly, we also have

$$\boxed{9} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

Direction Angles and Direction Cosines

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

- Using the above equations we can write
- $a = \langle a1, a2, a3\rangle = \langle |a| \cos\alpha, |a| \cos\beta, |a| \cos\gamma\rangle$
- $= |a| \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

Direction Angles and Direction Cosines

- **Example:**
- Find the direction angles of the vector a = ⟨1, 2, 3⟩.
- Solution: $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$,
- Since Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

- and so

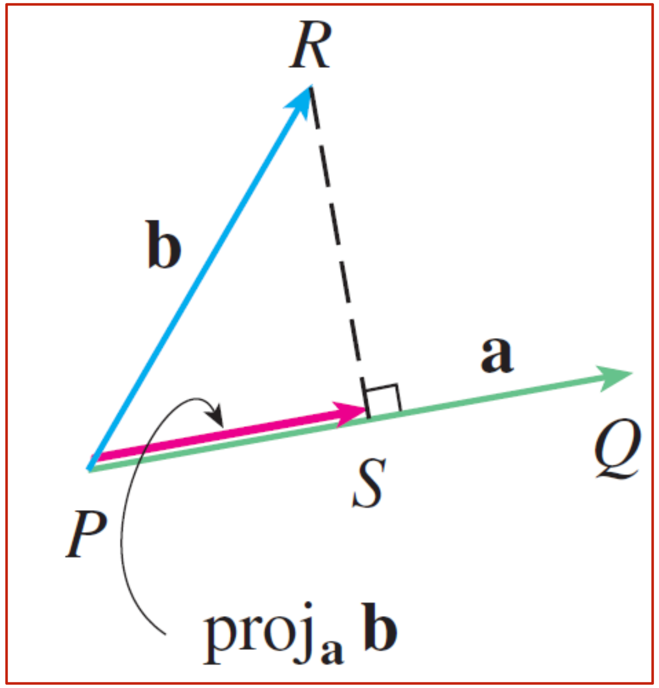
$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

Projections

Projections

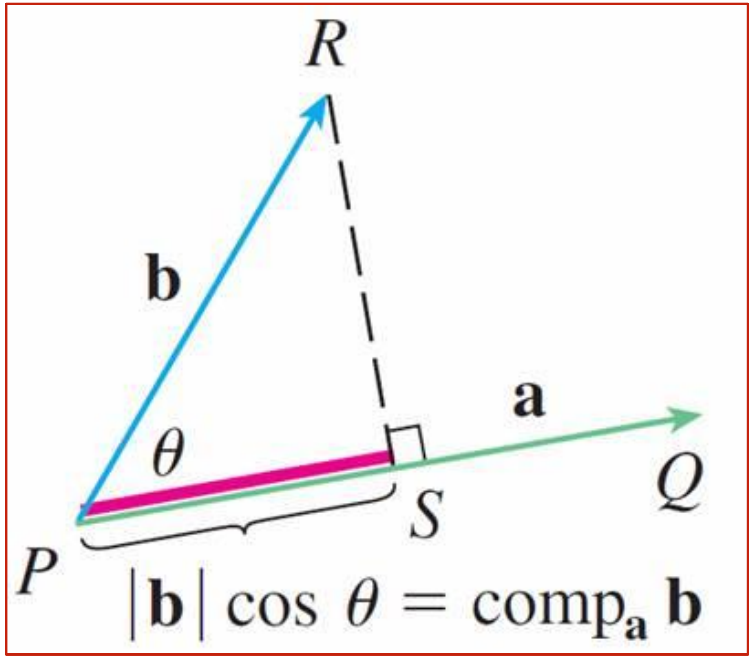
❖Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors a and b with the same initial point P.

❖ Projections are of two types: **Vector and Scalar**



Vector projections

Figure 4

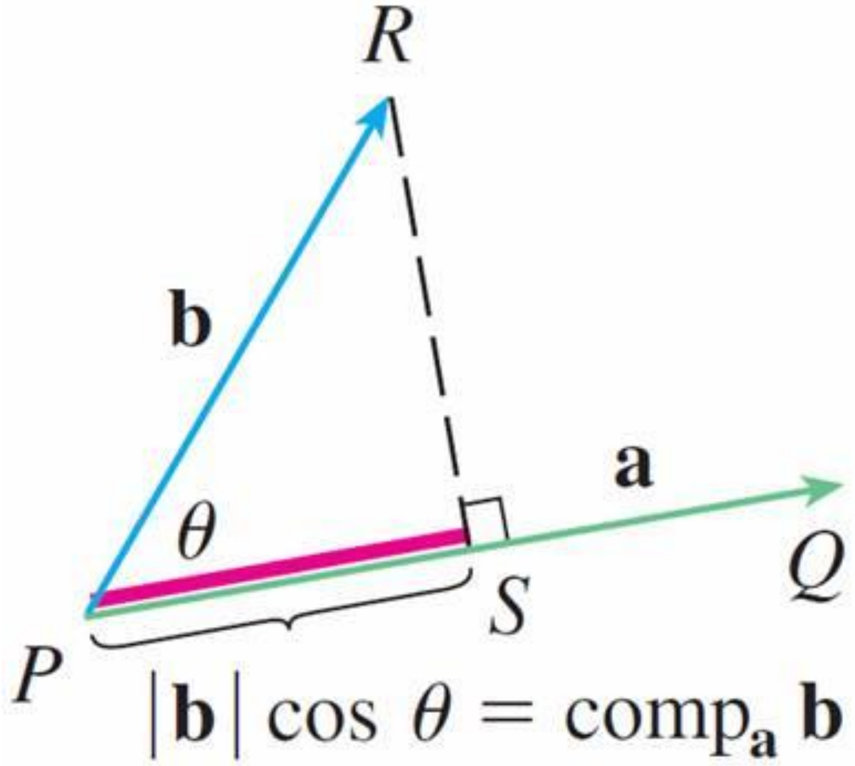


Scalar projections

Scalar Projections

❖**Scalar Projection**: The scalar **projection of b onto a** is defined to be **the component of b along a**, which is the number **| b | cos** θ, where **θ** is the angle between a and b. (See Figure 5.)

❖Scalar projection is denoted by **comp$_a$b.**



Scalar projection

Figure 5

Relation between Dot Product and Scalar Projection

❖The equation $a \cdot b = |a||b| \cos\theta = |a|(|b| \cos\theta)$

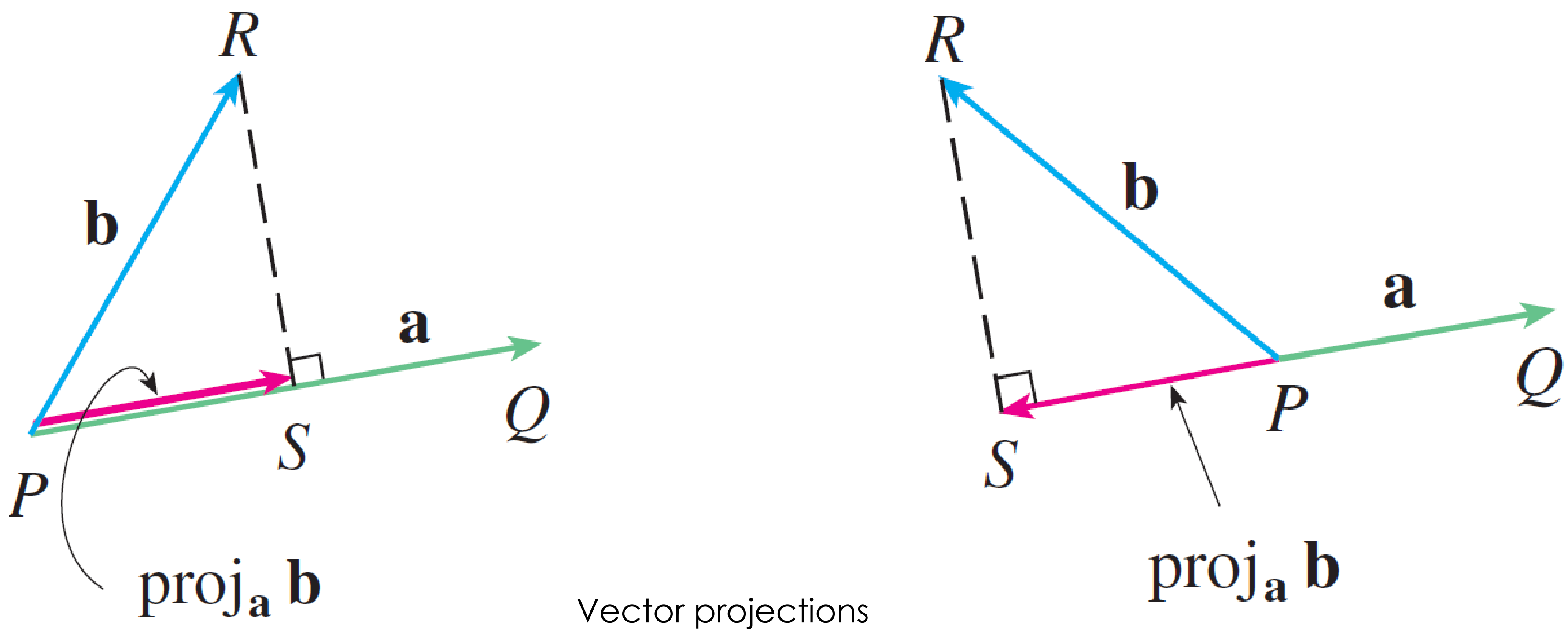
❖shows that the **dot product** of a and b can be interpreted as the **length of a times the scalar projection of b onto a.** Since

$$\boxed{|\mathbf{b}| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}}$$

❖The **$\text{comp}_a b$** can be computed by taking the dot product of b with the unit vector in the direction of a.

Vector Projections

❖**Vector Projection**: If S is the foot of the perpendicular from R to the line containing PQ, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of b onto a and is denoted by **proj_a b**. (**You can think of it as a shadow of b**).



Vector projections

Figure 4

Vector Projections

❖We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}}\, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad \text{proj}_{\mathbf{a}}\, \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\, \mathbf{a}$

❖Notice that the vector projection is the **scalar projection times the unit vector in the direction of a.**

Projection

❖**Example:** Find the scalar projection and vector projection of b = ⟨1, 1, 2⟩ onto a = ⟨−2, 3, 1⟩.

❖Solution:

❖The **scalar projection** of b onto a is: $\boxed{\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}}$

$$|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14},$$

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

Example 6 – Solution

cont'd

❖**Vector projection:** is this scalar projection times the unit vector in the direction of a:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \frac{3}{14}\,\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

The Cross Product

The Cross Product

❖The cross product (also known as vector product) $a \times b$, is a vector c that is **perpendicular** to both a and b.

$\boxed{8}$ **Theorem** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The Cross Product

- Given two nonzero vectors $a = \langle a1, a2, a3\rangle$ and $b = \langle b1, b2, b3\rangle$, we want to find c that is perpendicular to both a and b,

- If $c = \langle c1, c2, c3\rangle$ is such a vector, then **a • c = 0 and b • c = 0** and so

- $$a1c1 + a2c2 + a3c3 = 0$$

- $$b1c1 + b2c2 + b3c3 = 0$$

The Cross Product

4 **Definition** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

- Notice that the cross product a × b of two vectors a and b, unlike the dot product, is a vector. For this reason it is also called **the vector product**.

- Note that a × b is defined only when a and b are **three-dimensional vectors**.

The Cross Product using Determinant

❖A **determinant of order 2 :**

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

❖For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

The Cross Product using Determinant

Cont'd

❖**A determinant of order 3 :**

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

❖The number a_i in the first row is multiplied by the second-order determinant (obtained from the left side by deleting the row and column in which a_i appears.)

❖Notice also the minus sign in the second term.

The Cross Product

❖**Example:**

❖If a = ⟨1, 3, 4⟩ and b = ⟨2, 7, −5⟩, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

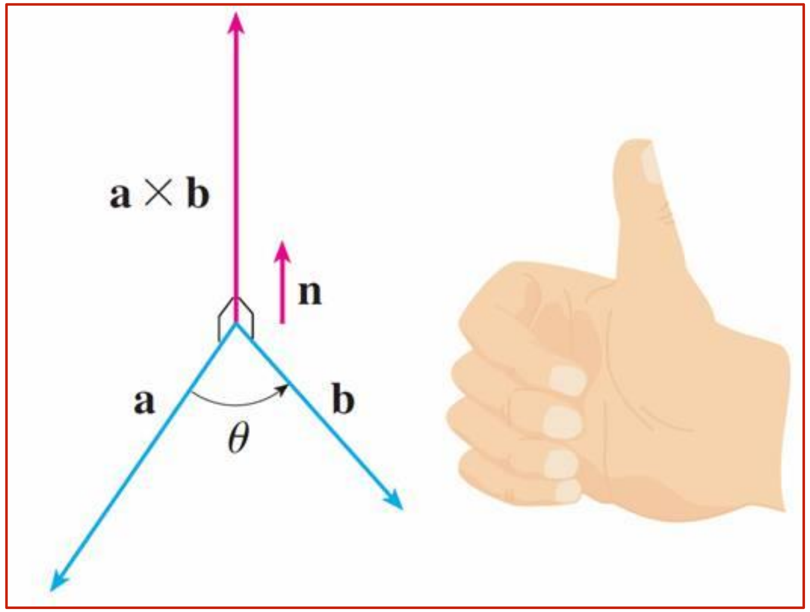
$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

❖ $= (-15 - 28)i - (-5 - 8)j + (7 - 6)k$

❖ $= -43i + 13j + k$

Right-hand Rule: The Cross Product

- If a and b are represented by directed line segments with the same initial point (as in Figure 1), then the cross product **a × b points in a direction** perpendicular to the plane through a and b.



The right-hand rule gives the direction of **a** × **b**.

Figure 1

Right-hand Rule: The Cross Product

- **The right-hand rule:** If the fingers of your **right hand curl** in the direction of a rotation (through an angle less than 180°) from to a to b, then your **thumb points in the direction of a × b.**

- Now that we know the direction of the vector a × b, the remaining thing we need to complete its geometric description is its **length | a × b |**. This is given by the following theorem.

$\boxed{9}$ **Theorem** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Properties of The Cross Product

❖The following theorem summarizes the properties of vector products.

11 Theorem If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Summary

- Introduction and Motivation of Vectors
- Vector Operations
  - Addition
  - Scalar multiplication
  - Linear Combination
- Component Representation
  - Vector Length
  - Vector Algebra
- Unit Vector
- Dot product of Vectors
- Projection
  - Scalar
  - Vector
- Cross-Product of Vectors

References

❖Some portion of these slides are adopted from the following places:

- ❖Motivation example:
  - ❖ https://towardsdatascience.com/why-data-is-represented-as-a-vector-in-data-science-problems-a195e0b17e99
- ❖https://college.cengage.com/mathematics/larson/calculus_analytic/8e/chapters/chapter12.html

Recommended Reading

- Book Chapter/Section:
  - Chapter 1 (sec 1.1, sec 1.2), Introduction to Linear Algebra, Gilbert Strange, 5^{th} Edition
  - Chapter 12 (section 12.1 to 12.4) of Early Transcendentals by James Stewart, 7th Ed