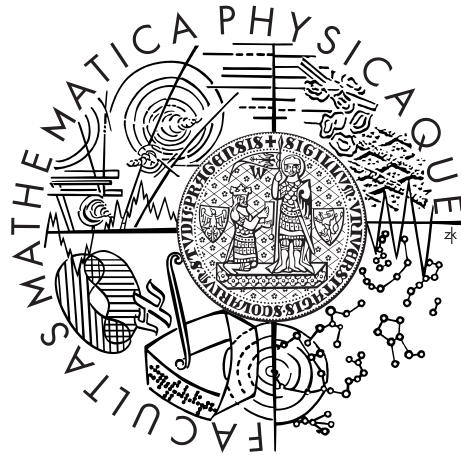


Charles University in Prague  
Faculty of Mathematics and Physics

## MASTER THESIS



Michal Szabados

## Dissections of triangles and distances of groups

Department of Algebra

Supervisor of the master thesis: prof. RNDr. Aleš Drápal, CSc., DSc.

Study programme: Mathematics

Specialization: Mathematical Structures

Prague 2013

I would like to thank Aleš Drápal for his guidance, encouragement and patience during both research and writing periods of creation of this thesis.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Coll., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 paragraph 1 of the Copyright Act.

In ..... date .....

signature of the author

*Název práce:* Delenia trojuholníkov a vzdialenosti grúp

*Autor:* Michal Szabados

*Katedra:* Katedra algebry

*Vedoucí diplomové práce:* prof. RNDr. Aleš Drápal, CSc., DSc.

*Abstrakt:*

*Klíčová slova:*

*Title:* Dissections of triangles and distances of groups

*Author:* Michal Szabados

*Department:* Department of Algebra

*Supervisor:* prof. RNDr. Aleš Drápal, CSc., DSc.

*Abstract:*

*Keywords:*

# Contents

<b>Preface</b>	<b>2</b>
<b>Introduction</b>	<b>3</b>
<b>Notation</b>	<b>5</b>
<b>1 Latin bitrades</b>	<b>6</b>
1.1 Partial latin squares . . . . .	6
1.2 Latin bitrades . . . . .	7
<b>2 Embedding latin trades into groups</b>	<b>10</b>
2.1 Sparse matrices . . . . .	10
2.2 Trade matrices . . . . .	11
2.3 Embedding latin trades into groups . . . . .	13
2.4 Definition of $\text{gdist}(n)$ . . . . .	13
<b>3 Dissections of equilateral triangles</b>	<b>15</b>
3.1 Definitions . . . . .	15
3.2 Logarithmic dissection of a rectangle . . . . .	16
3.3 Logarithmic dissection of a triangle . . . . .	17
3.4 Triangle dissections and latin bitrades . . . . .	18
3.5 Families of logarithmic dissections . . . . .	20
<b>4 Refining the bounds</b>	<b>24</b>
4.1 Padovan sequence . . . . .	24
4.2 Rosendorf . . . . .	26
4.3 Enumerations of minimal dissections . . . . .	26
<b>Conclusion</b>	<b>28</b>
<b>Appendix A: Values of <math>\hat{t}(n)</math></b>	<b>29</b>
<b>Appendix B: Minimal triangle dissections</b>	<b>31</b>
<b>Appendix C: Program in C++</b>	<b>37</b>
<b>Bibliography</b>	<b>38</b>

# Preface

@ Este neviem, ci bude predslov okrem uvodu.

# Introduction

Let us introduce two combinatorial problems:

**Problem 1.** Consider a table of addition modulo  $n$ ; it is a latin square  $n \times n$ . What is the smallest number of cells we have to change in order to get another latin square?

<span style="border: 1px solid black;">0</span>	1	2	<span style="border: 1px solid black;">3</span>	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
<span style="border: 1px solid black;">3</span>	4	<span style="border: 1px solid black;">5</span>	<span style="border: 1px solid black;">6</span>	0	1	2
4	5	<span style="border: 1px solid black;">6</span>	<span style="border: 1px solid black;">0</span>	1	2	3
<span style="border: 1px solid black;">5</span>	6	<span style="border: 1px solid black;">0</span>	1	2	3	4
6	0	1	2	3	4	5

→

<span style="border: 1px solid black;">3</span>	1	2	<span style="border: 1px solid black;">0</span>	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
<span style="border: 1px solid black;">5</span>	4	<span style="border: 1px solid black;">6</span>	<span style="border: 1px solid black;">3</span>	0	1	2
4	5	<span style="border: 1px solid black;">0</span>	<span style="border: 1px solid black;">6</span>	1	2	3
<span style="border: 1px solid black;">0</span>	6	<span style="border: 1px solid black;">5</span>	1	2	3	4
6	0	1	2	3	4	5

Figure 1: The smallest number for  $n = 7$  is nine.

**Problem 2.** Let  $\Delta_n$  be an equilateral triangle of side  $n$ . What is the smallest number of integer-sided equilateral triangles, into which  $\Delta_n$  can be dissected, such that no six of them share a common point?

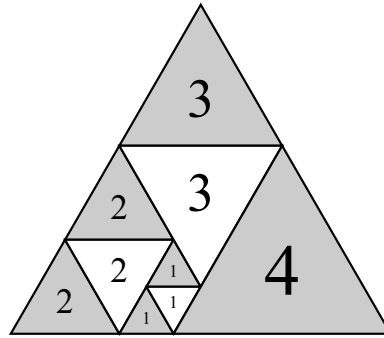


Figure 2: Dissection of a triangle of size 7 into nine triangles.

Though it is not obvious at first glance, these two problems are fundamentally related. Both triangle dissections and pairs of latin squares describe a combinatorial structure called *latin bitrade*. This structure will be of central interest throughout this work.

Let us denote by  $\text{gdist}(n)$  and  $t(n)$  the minimal numbers described in Problems 1 and 2 respectively. Our main result is a solution to the twenty-year-old conjecture of Drápal, Cavenagh and Wanless:

**Conjecture 1.** There exist positive constants  $c_1$  and  $c_2$  such that

$$c_1 \log(p) \leq \text{gdist}(p) \leq c_2 \log(p) \quad (1)$$

for sufficiently large primes  $p$ .

In other words, the conjecture states that  $\text{gdist}(n)$  is asymptotically logarithmic, the condition for  $n$  to be a prime is only a technical requirement. We also prove the same statement for  $t(n)$  in place of  $\text{gdist}(n)$ .

The lower bound in (1) was already established before. In 1989 Drápal and Kepka [9] proved the inequality for  $c_1 = e$ , and later Cavenagh [3] found an alternative proof of the same estimate. Yet another proof was given in a paper [5] by Cavenagh and Wanless, but with a slightly smaller constant.

All of these proofs are dealing with another structure which defines a latin bitrade – certain kind of 0-1 matrices. The lower bound is then determined by establishing upper bound for determinant of such a matrix. In Chapter 2 we present modified proof which leads to  $c_1 = 3 \log_3(e)$ , the best estimate known so far.

The previously known best upper bound  $\text{gdist}(p) = O(\log^2(p))$  is due to Drápal [7]. He discovered the connection between latin bitrades and dissections of equilateral triangles, and proved that  $\text{gdist}(n) \leq t(n)$ . However, he was only able to construct triangle dissections with  $O(\log^2(n))$  triangles.

In Chapter 3 we prove Conjecture 1 by constructing dissections into logarithmically many triangles. The method used is inspired by Trustrum's method [14] to dissect a square of side  $n$  into logarithmically many integer-sided squares. To be more precise, we show how to dissect an  $n \times (n+3)$  rectangle into  $5 \log_4(n) + \frac{3}{2}$  squares and how to adapt the construction to get a dissection of an equilateral triangle of side  $n$  into  $5 \log_2(n)$  triangles. We also discuss possible generalizations of our dissection method in Section 3.5.

Now, that the asymptotic behavior of  $\text{gdist}(n)$  and  $t(n)$  is known, it is natural to ask about the constants in the estimates. Putting our results together, we get

$$2.73 \approx 3 \log_3(e) \leq \frac{\text{gdist}(p)}{\log(p)} \leq \frac{t(p)}{\log(p)} \leq 5 \log_2(e) \approx 7.21. \quad (2)$$

That, however, do not seem to be the best estimates. The following is conjectured:

**Conjecture 2.** Let  $P$  be a real such that  $P^3 = P + 1$ . Then

$$\lim_{p \rightarrow \infty} \frac{\text{gdist}(p)}{\log(p)} = \lim_{p \rightarrow \infty} \frac{t(p)}{\log(p)} = 1/\log(P) \approx 3.56. \quad (3)$$

In Chapter 4 we gathered evidence which supports this claim. We expose a connection between certain triangle dissections and an integer sequence satisfying the recurrence relation  $a_{n+3} = a_{n+1} + a_n$ . We also describe a computer algorithm with which we generated the exact values of  $t(n)$  for  $n \leq 416$ . The data, together with corresponding triangulations, are listed in appendices.



# Notation

@ Logicky usporiadat.

$\mathbb{Z}, \mathbb{Z}_n$	integers, integers modulo $n$
$[a, b], [a, b)$	intervals of integers $k$ such that $a \leq k \leq b, a \leq k < b$
$[n]$	$[0, n)$
$\Delta_n$	equilateral triangle of side $n$

# 1. Latin bitrades

An  $n \times n$  table such that every row and column contains every number in  $[n]$  exactly once is a well-known combinatorial object called *latin square*. In this chapter we define *latin bitrade*, which can be thought of as an object of differences between two latin squares.

To describe a table of elements formally, we use ordered triples  $(r, c, s)$  to represent the fact that the cell in row  $r$  and column  $c$  contains the symbol  $s$ . For that we use the following notation. Let

- $R = \{r_0, \dots, r_{|R|-1}\}$  denote the set of rows,
- $C = \{c_0, \dots, c_{|C|-1}\}$  denote the set of columns and
- $S = \{s_0, \dots, s_{|S|-1}\}$  denote the set of symbols.

We consider only the case when  $R$ ,  $C$ , and  $S$  are finite. As an example, a latin square is formally a subset of  $R \times C \times S$  with  $R = C = S = [n]$ . We shall see this in more detail in a moment.

In this chapter we define only necessary notions for our purposes. For a more comprehensive introduction to latin bitrades we refer the reader to a survey by Cavenagh [4].

## 1.1 Partial latin squares

**Definition 1.1.** A *partial latin square*  $L$  is a subset of ordered triples from  $R \times C \times S$ , such that for any two given coordinates of  $(r, c, s) \in L$  the third one is determined uniquely.

A partial latin square is usually interpreted as a partially filled  $|R| \times |C|$  table. The uniqueness condition implies that the table is well defined (there is at most one symbol in every cell), and that no symbol repeats itself within a column or a row.

**Definition 1.2.** A *latin square*  $L$  is a partial latin square such that  $R = C = S$  and every cell in the table is filled. Equivalently, for every  $a_1, a_2 \in R$  there are unique  $r, c, s \in R$  such that

$$(a_1, a_2, s), (a_1, c, a_2), (r, a_1, a_2) \in L. \quad (1.1)$$

There are two important maps from partial latin squares to partial latin squares: *isotopy* and *conjugacy*.

**Definition 1.3.** Let  $A \subset R_A \times C_A \times S_A$  and  $B \subset R_B \times C_B \times S_B$  be partial latin squares. A *homotopy* is a triple of maps  $h = (h_R, h_C, h_S)$  such that

$$\begin{aligned} h_R &: R_A \rightarrow R_B, \\ h_C &: C_A \rightarrow C_B, \\ h_S &: S_A \rightarrow S_B \end{aligned}$$

and  $(r, c, s) \in A \Rightarrow (h_R(r), h_C(c), h_S(s)) \in B$ . An *isotopy* is a homotopy with homotopic inverse.

**Example 1.4.** Partial latin squares on Figure 1.4 are isotopic. The set of rows, columns and symbols is the same for both. The isotopy is given by

- $h_R$  is identity,
- $h_C$  rotates middle three columns,
- $h_S(0) = 0, h_S(1) = 2, h_S(2) = 4, h_S(3) = 1, h_S(4) = 3$ .

1	3		2			2		4	1		
4				1	3	3		2			1
		4			0				0	3	
	0	1			4		2		0	3	
$A$						$B$					

Figure 1.1: Isotopic partial latin squares.

**Definition 1.5.** Let  $A \subset R \times C \times S$  be a partial latin square and  $\sigma$  be a permutation of the 3-element set  $\{R, C, S\}$ . Then the partial latin square

$$\{(a_{\sigma(R)}, a_{\sigma(C)}, a_{\sigma(S)}) \mid (a_R, a_C, a_S) \in A\} \quad (1.2)$$

is said to be *conjugated* with  $A$ .

Note that there are six conjugacies, each one corresponding to a permutation of  $\{R, C, S\}$ .

**Definition 1.6.** Two partial latin squares are from the same *class* if one can be obtained from the other by composition of conjugacy and isotopy.

## 1.2 Latin bitrades

Now we can define latin bitrade.

**Definition 1.7.** A *latin bitrade* is a pair  $(T, T')$  of partial latin squares on  $R \times C \times S$  which are disjoint and for every  $(r, c, s) \in T$  (respectively,  $T'$ ) there exist unique  $r', c', s'$  such that

$$(r', c, s), (r, c', s), (r, c, s') \in T' \text{ (respectively, } T). \quad (1.3)$$

Let us call  $T$  and  $T'$  *latin trades*. Elements in  $T$  and  $T'$  can be paired with respect to the first two coordinates. Therefore  $|T| = |T'|$  and we shall call this number the *size* of the bitrade (or a trade).

From the tabular point of view, a latin bitrade is a pair of partial latin squares such that they occupy the same cells, but the symbols in corresponding rows and columns are permuted. Moreover, no symbol is at the same position in both of the tables.

**Example 1.8.** See Figure 1.8. The example is adapted from [4].

Note that two latin squares  $L, L'$  defined on the same set specify a latin bitrade  $(L \setminus L', L' \setminus L)$ .

	1	2	3
1	0	3	
2		0	1
3	2		0

$T$

	2	3	1
3	1	0	
1		2	0
2	0		3

$T'$

Figure 1.2: A latin bitrade on  $[4] \times [4] \times [4]$  of size 12.

**Definition 1.9** (Graph representation of a latin bitrade). A latin bitrade  $(T, T')$  is associated with a graph  $G = (V, E)$  such that

$$V = T \cup T'$$

$$E = \{(t, t') \mid t \in T, t' \in T' : t \text{ and } t' \text{ differ at exactly one coordinate}\}.$$

We call it the *graph of latin bitrade*  $(T, T')$ .

Clearly, the graph is bipartite with partitions  $T$  and  $T'$ . It is also 3-regular from the definition of latin bitrade. Moreover, it is edge 3-colorable, the edges can be colored depending on the coordinate that  $t$  and  $t'$  differ at.

With the graph representation it is easier to understand meaning of the following definitions.

**Definition 1.10.** A latin bitrade  $(T, T')$  is *connected* if there do not exist two non-empty disjoint latin bitrades  $(T_0, T'_0)$ ,  $(T_1, T'_1)$  such that  $T = T_0 \cup T_1$  and  $T' = T'_0 \cup T'_1$ .

Equivalently, a bitrade is connected if and only if its graph is connected.

**Definition 1.11.** A latin trade is called *spherical* or *planar*, if its graph is planar.

**Example 1.12.** Figure 1.3 shows a graph of a connected spherical latin bitrade  $(T, T')$  of size 6. To distinguish the elements of  $T$  and  $T'$ , the latter are typed in brackets. Solid, dashed and dotted edges join elements which differ on  $R$ -,  $C$ - and  $S$ -coordinate respectively.

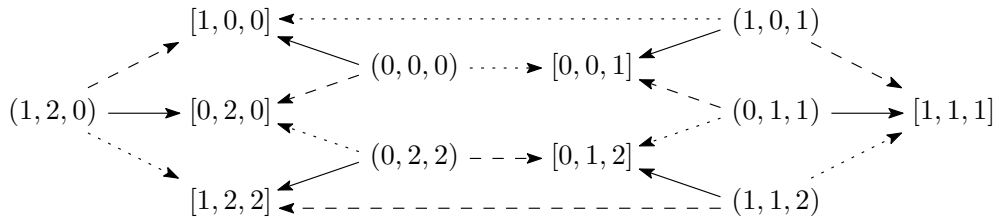


Figure 1.3: Graph of a latin bitrade of size 6 on  $[2] \times [3] \times [3]$ .

For later use, let us define maps  $\sigma_R, \sigma_C, \sigma_S : T \rightarrow T'$  such that for  $(r, c, s) \in T$

$$\sigma_R(r, c, s) = (r', c, s) \in T', \quad (1.4)$$

$$\sigma_C(r, c, s) = (r, c', s) \in T', \quad (1.5)$$

$$\sigma_S(r, c, s) = (r, c, s') \in T'. \quad (1.6)$$

The definition of the latin bitrade implies that these maps are bijections. They correspond to edges of the graph – on Figure 1.3,  $\sigma_R, \sigma_C, \sigma_S$  are represented by solid, dashed and dotted arrows respectively.

**Lemma 1.13.** *A latin bitrade  $(T, T')$  is connected if and only if for any  $t_0, t_1 \in T \cup T'$  it is possible to get  $t_1$  from  $t_0$  by consequent application of  $\sigma_R, \sigma_C, \sigma_S$ , or their inverses.*

*Proof.* Simple, see the comment above.  $\square$

**Lemma 1.14.** *Let  $\{X, Y\} \subset \{R, C, S\}$ . Then the mapping  $\sigma_Y^{-1}\sigma_X : T \rightarrow T$  is a permutation without a fixed point.*

*Proof.* The mapping is a bijection with inverse  $\sigma_X^{-1}\sigma_Y$  on a finite set, thus it is a permutation. It changes two coordinates of its argument, and therefore has no fixed points.  $\square$

We conclude this section with discussion, when it is possible to reconstruct a latin bitrade from its graph. Clearly we can do that only up to isotopy and conjugacy, as the graph representation forgets any orderings. Also, in every component of the graph we might switch roles of  $T$  and  $T'$ .

The graph of a bitrade is edge 3-colorable. By excluding edges of one color, say corresponding to  $R$ , the graph splits into (undirected) cycles, in which all elements have the same  $R$ -coordinate. If the coordinates are in different cycles different, the bitrade is called *R-separated*, analogously for  $C$  and  $S$ .

**Definition 1.15.** A latin bitrade is *separated* if it is  $R$ -,  $C$ - and  $S$ -separated.

Every latin bitrade can be transformed into a separated one – for a symbol spanning multiple cycles, it suffices to introduce a new symbol for each of them and relabel accordingly. Clearly, this new bitrade yields the same graph as the original one.

**Example 1.16.** The bitrade from Example 1.12 is separated. Figure 1.4 illustrates the cycles after deletion of edges corresponding to  $S$ .

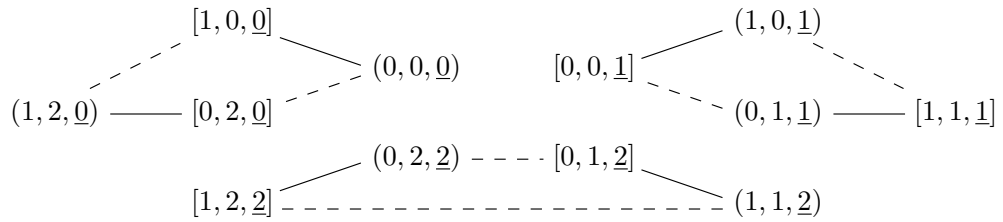


Figure 1.4: 2-color cycles in a separated latin bitrade.

**Lemma 1.17.** *Let  $G$  be a planar cubic bipartite graph. Then there exists unique face 3-coloring of  $G$ .*

*Proof.* This result was already known to Heawood. A proof of it can be found in [15].  $\square$

**Theorem 1.18.** *A connected separated spherical latin bitrade  $(T, T')$  can be reconstructed from its graph  $G$  up to isotopy, conjugacy, and switch of the roles of  $T$  and  $T'$ .*

*Proof.*  $G$  is planar cubic bipartite and therefore has unique face 3-coloring (Lemma 1.17).  $\square$

## 2. Embedding latin trades into groups

@ Only finite groups  
 @ Credit to Drápal

### 2.1 Sparse matrices

**Definition 2.1.** A matrix is *sparse* if its elements are from  $\{0, 1\}$  and there is at most one occurrence of 1 in every row.

**Lemma 2.2.** Let  $M_0, M_1$  be sparse matrices of the same number of rows such that  $M = (M_0, M_1)$  is a square block matrix. Then  $\det(M) \in \{-1, 0, 1\}$ .

*Proof.* Let  $c_0, c_1$  be the number of columns in  $M_0$  and  $M_1$  respectively. The proof is by induction on  $n := c_0 + c_1$ . The case with  $n = 1$  is trivial. Therefore assume  $n > 1$ .

There are at most two ones in every row of  $M$ .

- If there is a row with zeros only, then  $\det(M) = 0$ .
- If every row contains exactly two ones, then

$$v := (\underbrace{1, \dots, 1}_{c_0}, \underbrace{-1, \dots, -1}_{c_1})$$

is such that  $Mv^T = 0$ . Thus  $M$  is singular and  $\det(M) = 0$ .

- Otherwise there is a row  $r$  which contains only a single one in column  $c$ . Then expanding the determinant by row  $r$  yields

$$\det(M) = \pm \det(\overline{M}_r^c). \quad (2.1)$$

The matrix  $\overline{M}_r^c$  consists also from two sparse matrices, thus the result follows by induction. □

**Lemma 2.3.** Let  $M_0, M_1, M_2$  be sparse matrices of sizes  $n \times c_0, n \times c_1, n \times c_2$  such that  $M = (M_0, M_1, M_2)$  is a square block matrix. Let  $k_i$  denote the number of ones in column  $i$ . Then

$$|\det(M)| \leq \prod_{i \in [c_0]} k_i. \quad (2.2)$$

*Proof.* The proof is by induction on  $c_0$ .

1. If  $c_0 = 0$ , then  $|\det(M)| \leq \prod_{i=0}^0 k_i = 1$  holds by Lemma 2.2.
2. Otherwise expand by the first column:

$$|\det(M)| \leq \sum_{i \in [n]} M_{i,0} |\det(\overline{M}_i^1)| \leq k_0 \prod_{i \in [1, c_0)} k_i. \quad (2.3)$$

The last inequality holds since there are  $k_0$  nonzero summands and the product majorizes each subdeterminant from induction.

□

For our final result we need the following technical lemma:

**Lemma 2.4.** *Let  $n$  be a positive integer and  $k_0 + \dots + k_{m-1} = n$  for an integer  $m$ . Then*

$$\prod_{i \in [m]} k_i \leq 3^{n/3}. \quad (2.4)$$

*Proof.* For  $n = 1$  it holds trivially, let us assume  $n > 1$ . Let  $k_0 \leq \dots \leq k_{m-1}$  be lexicographically smallest such that the maximum is attained. Observe:

- $2 \cdot (k - 2) \geq k$  for  $k \geq 4$ , therefore  $k_i \leq 3$ ;
- $(1 + k) > 1 \cdot k$ , therefore  $k_i > 1$ ;
- $3 \cdot 3 > 2 \cdot 2 \cdot 2$ , therefore there are at most two twos amongst  $k_i$ .

Thus there are 3 possibilities:

$$\begin{aligned} n = 3k & \Rightarrow k_0 = \dots = k_{m-1} = 3 & \Rightarrow \prod k_i = 3^{n/3} \\ n = 3k + 2 & \Rightarrow k_0 = 2, k_1 = \dots = k_{m-1} = 3 & \Rightarrow \prod k_i = 2 \cdot 3^{(n-2)/3} \\ n = 3k + 4 & \Rightarrow k_0 = k_1 = 2, k_2 = \dots = k_{m-1} = 3 & \Rightarrow \prod k_i = 4 \cdot 3^{(n-4)/3} \end{aligned}$$

Where the products are over  $i \in [m]$ . Each of them is less than or equal to  $3^{n/3}$ , thus we are done. □

**Lemma 2.5.** *Let  $M_0, M_1, M_2$  be sparse matrices such that  $M = (M_0, M_1, M_2)$  is a square block  $n \times n$  matrix. Then*

$$|\det(M)| \leq 3^{n/3} \quad (2.5)$$

and thus  $3 \log_3(|\det(M)|) \leq n$ .

*Proof.* Let  $c_0$  be the number of columns of  $M_0$  and  $k_i$  the number of ones in the column  $i$ . Since  $M_0$  is sparse, surely  $\sum_{i \in [c_0]} k_i \leq n$ . The proof is finished by combining Lemmas 2.3 and 2.4:

$$|\det(M)| \leq \prod_{i \in [c_0]} k_i \leq 3^{n/3}. \quad (2.6)$$

□

## 2.2 Trade matrices

Recall that

- $R = \{r_0, \dots, r_{|R|-1}\}$  denotes the set of rows,
- $C = \{c_0, \dots, c_{|C|-1}\}$  denotes the set of columns and
- $S = \{s_0, \dots, s_{|S|-1}\}$  denotes the set of symbols.

Without loss of generality assume that these sets are disjoint and let  $X = R \cup C \cup S$ .

**Definition 2.6.** Let  $T$  be a latin trade. We define a matrix  $M$  of size  $|T| \times |X|$  such that if we index the rows by elements of  $T$  and columns by elements of  $X$ , then

$$t = (r, c, s) \in T \Rightarrow \begin{cases} M_{t,r} = M_{t,c} = M_{t,s} = 1, \\ M_{t,x} = 0 \end{cases} \quad \text{for } x \in X \setminus \{r, c, s\}. \quad (2.7)$$

We call it the *trade matrix* and denote by  $M_T$ .

**Lemma 2.7.** Suppose that all elements of  $X$  are used in a connected latin trade  $T$ . Then the trade matrix  $M_T$  has rank  $|X| - 2$  over  $\mathbb{Q}$ .

*Proof.* @ Prepisat dokaz! (x,y,z)

The vector

$$v = (\underbrace{a, \dots, a}_{|R|}, \underbrace{b, \dots, b}_{|C|}, \underbrace{c, \dots, c}_{|S|}) \quad (2.8)$$

is surely a solution for  $M_T v^T = 0$  if  $a + b + c = 0$ . It suffices to show that these are the only solutions.

Let  $v$  be any solution and  $f : X \rightarrow \mathbb{Q}$  is defined by

$$v = (f(r_0), \dots, f(c_0), \dots, f(s_0), \dots). \quad (2.9)$$

Let  $\bar{s} \in S$  be such that  $v_{\bar{s}}$  is maximal.  $\bar{s}$  is surely a coordinate of a triple  $(\bar{r}, \bar{c}, \bar{s}) \in T$ . Without loss of generality assume that  $(\bar{r}, \bar{c}, \bar{s}) = (0, 0, 0)$ , otherwise we can shift the coordinates. Now we want to prove that  $a = b = c = 0$ .

Since 0 is the largest element of  $f(s)$  and  $T'$  occupies the same cells as  $T$ , then for any  $(r, c, s) \in T \cup T'$

$$f(r) + f(c) \geq 0. \quad (2.10)$$

We prove that for  $(r, c, s) \in T \cup T'$  such that  $f(t) = (0, 0, 0)$  holds that  $f(\sigma_Y(r, c, s)) = (0, 0, 0)$  and  $f(\sigma_Y^{-1}(r, c, s)) = (0, 0, 0)$  for  $Y \in \{R, C, S\}$ .

Let  $(r, c, s) \in T'$  such that  $f(r, c, s) = (0, 0, 0)$ . Then  $\sigma_R^{-1}(r, c, s) = (r', c, s) \in T$  and  $0 = f(r') + f(c) + f(s) = f(r')$ . Therefore  $f(\sigma_R^{-1}(r, c, s)) = (0, 0, 0)$ . For  $\sigma_C$  and  $\sigma_S$  analogously.

Now let  $(r, c, s) \in T$  such that  $f(r, c, s) = (0, 0, 0)$ . Consider a chain of elements in  $T \cup T'$ :

$$\begin{array}{llll} (r, c, s) \in T & \xrightarrow{\sigma_C} & (r, c^1, s) \in T' & \xrightarrow{\sigma_R^{-1}} \\ (r^1, c^1, s) \in T & \xrightarrow{\sigma_C} & (r^1, c^2, s) \in T' & \xrightarrow{\sigma_R^{-1}} \\ (r^2, c^2, s) \in T & \xrightarrow{\sigma_C} & (r^2, c^3, s) \in T' & \dots \end{array}$$

The inequality gives us

$$\begin{aligned} f(r) + f(c) &= 0, f(r) + f(c^1) \geq 0, \\ f(r^1) + f(c^1) &= 0, f(r^1) + f(c^2) \geq 0, \dots \end{aligned}$$

which yields

$$\begin{aligned} 0 &= f(r) \geq f(r^1) \geq f(r^2) \geq \dots \\ 0 &= f(c) \leq f(c^1) \leq f(c^2) \leq \dots \end{aligned}$$



But it is a cycle, therefore all in the cycle are equal to zero. So  $f(\sigma_C(r, c, s)) = (0, 0, 0)$ . The result for  $\sigma_R$  can be obtained by reversing the cycle.

For  $\sigma_S$ , consider a cycle by alternating maps  $\sigma_C$  and  $\sigma_S^{-1}$ . We already know that  $f(\sigma_S^{-1}\sigma_C(r, c, s)) = \sigma_S^{-1}(0, 0, 0) = (0, 0, 0)$ . Therefore all elements in the cycle are mapped to  $(0, 0, 0)$ . By reversing the cycle we get the result.  $\square$

**Corollary 2.8.** *Let  $T$  be a latin trade which uses every symbol in  $X$ . Then  $|T| \geq |X| - 2$ .*

## 2.3 Embedding latin trades into groups

A Cayley table of a group  $G$  can be formally defined as a latin square  $L \subset G \times G \times G$  such that  $(r, c, s) \in L$  if and only if  $r \cdot c = s$ .

**Definition 2.9.** A latin trade  $T$  can be embedded in a group  $G$  if there exists an injective homotopy from  $T$  to the Cayley table of  $G$ .

Equivalently,  $T$  can be embedded in  $G$  if there are injections  $f_R, f_C, f_S : X \rightarrow G$  such that

$$(r, c, s) \in T \quad \Rightarrow \quad f_R(r) \cdot f_C(c) = f_S(s). \quad (2.11)$$

In the tabular view, a latin trade  $T$  can be embedded in a group  $G$  if we can find an isotopic copy of the partial latin square  $T$  inside of the Cayley table of  $G$  (see [Figure 1](#)).

**Lemma 2.10.** *A latin trade  $T$  can be embedded in an abelian group  $(G, +)$  if and only if there are injective maps  $f_R, f_C, f_S : X \rightarrow G$  such that*

$$(r, c, s) \in T \quad \Rightarrow \quad f_R(r) + f_C(c) + f_S(s) = 0. \quad (2.12)$$

*Proof.* The equivalence follows easily by setting  $g_R = f_R, g_C = f_C, g_S = -f_S$ .  $\square$

**Lemma 2.11.** *Let  $T$  be a trade using all symbols in  $X$  which can be embedded in  $\mathbb{Z}_p$  and  $M$  be any regular submatrix of  $M_T$ . Then  $p \mid \det(M)$ .*

*Proof.* [TODO](#)  $\square$

**Corollary 2.12.**  $p \leq 3^{|T|/3}$  and therefore  $3 \log_3(p) \leq |T|$ .

## 2.4 Definition of $\text{gdist}(n)$

**Definition 2.13.** Let  $\text{gdist}(G)$  denote the size of the smallest trade embeddable in  $G$  and let  $\text{gdist}(n)$  be the minimum across all groups of order  $n$ .

Without loss of generality now assume that  $(T, T')$  is a bitrade on  $G \times G \times G$  (i.e. that the isotopy is the identity function). In that case we can change the symbols in the cells of  $T$  with the corresponding symbols in  $T'$  to get a latin square. This can be also reversed – a latin square  $L \subset G \times G \times G$  uniquely defines a latin bitrade  $(C_G \setminus L, L \setminus C_G)$  such that  $C_G \setminus L$  is embedded in  $G$ , where  $C_G$  denotes the Cayley table of  $G$ .

Therefore  $\text{gdist}(G)$  is the smallest number of cells we have to change in a Cayley table of  $G$  to get another latin square.

Clearly if  $H$  is a subgroup of  $G$ , then  $\text{gdist}(G) \leq \text{gdist}(H)$ . Hence from Cauchy's theorem:

**Lemma 2.14.** *If  $p$  is a prime factor of  $n$ , then  $\text{gdist}(n) \leq \text{gdist}(p)$ .*

**Lemma 2.15.**  $3\log_3(p) \leq \text{gdist}(p)$ .

@ Viem urobit este odhad pre abelovske, pre ine neviem.

# 3. Dissections of equilateral triangles

The study of dissections was initiated by the paper *The dissection of rectangles into squares* by Brooks, Smith, Stone and Tutte [2]. They answered the question whether it is possible to dissect a square into some number of unequal squares (yes, it is), and developed methods to study such dissections.<sup>1</sup>

Inspired by a puzzle called *Mrs Perkins's quilt* by Dudeney [10], Conway [6] considered the case where the dissecting squares can be equally large. He proposed a question about the minimal number of integer-sided squares needed to dissect a square of side  $n$ . Because when  $n$  is divisible by  $d$  then it is possible to use a scaled up dissection of a square of side  $d$ , he considered only dissections where the dissecting squares do not have a common factor.

Conway proved that at least  $c \log(n)$  squares are needed. A year later Trustrum [14] proved that  $O(\log(n))$  is sufficient, and thus established that the answer is asymptotically logarithmic. However, the best constants in the estimates do not appear to be known.

In this chapter we prove that it is possible to dissect an equilateral triangle of side  $n$  into  $O(\log(n))$  triangles. To do so, @

@ After publishing the paper [2] on squares, Tutte generalized the theorems for dissections of equilateral triangles in [15].

## 3.1 Definitions

In the rest of this chapter, for brevity we use *triangle* instead of *equilateral triangle*, unless specified otherwise.

**Definition 3.1.** A *dissection of size  $m$  of a rectangle* is a set of  $m$  squares of integral side which cover the rectangle and overlap at most on their boundaries. Such a dissection is  $\oplus$ -free if no four squares share a common point, it is *prime* if their sides do not have a common factor.

We denote by  $r_d(n)$  the minimal size of a dissection of an  $n \times (n+d)$  rectangle.

**Definition 3.2.** A *dissection of size  $m$  of a triangle* is a set of  $m$  triangles of integral side which cover the original triangle and overlap at most on their boundaries. Such a dissection is  $\otimes$ -free if no six triangles share a common point, it is *prime* if their sides do not have a common factor.

We denote by  $t(n)$  (respectively,  $\hat{t}(n)$ ) the minimal size of a dissection (respectively, prime dissection) of a triangle of side  $n$ .

We use terms *dissection* and *tiling* interchangeably. Similarly, for squares and triangles we mean the same by *side* and *size*.

Note that in a triangle dissection only 2,3,4 or 6 triangles can share a common point. Therefore  $\otimes$ -freeness implies that actually no more than 4 triangles meet.

@ Lemma?  $t(n) \leq \hat{t}(n)$ ,  $t(p) = \hat{t}(p)$ .

---

<sup>1</sup>They showed, for example, that dissections into squares are related to electrical circuits obeying Kirchhoff's laws.

## 3.2 Logarithmic dissection of a rectangle

Let us describe an algorithm to dissect an  $n \times (n+3)$  rectangle for  $n \geq 2$ . Fix the orientation of the rectangle with the shorter side on the left. For convenience, we say that a dissection is *padded* if it has a square of side at least 2 in the upper left corner. Then the algorithm is as follows:

**Algorithm 3.3.**

- (A1) For  $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$  dissect into 4, 2, 5, 5, 3, 6, 6, 4, 7 squares respectively such that the dissection is  $\oplus$ -free and padded;
- (A2) for  $n$  of form  $4k + z$  with  $k \geq 2, z \in \{3, 4, 5, 6\}$ , depending on  $z$  dissect into 3 or 5 squares and a rectangle of size  $2k \times 2(k+3)$ . Then dissect this rectangle with two times larger tiles recursively. Figure 3.1 illustrates the method.

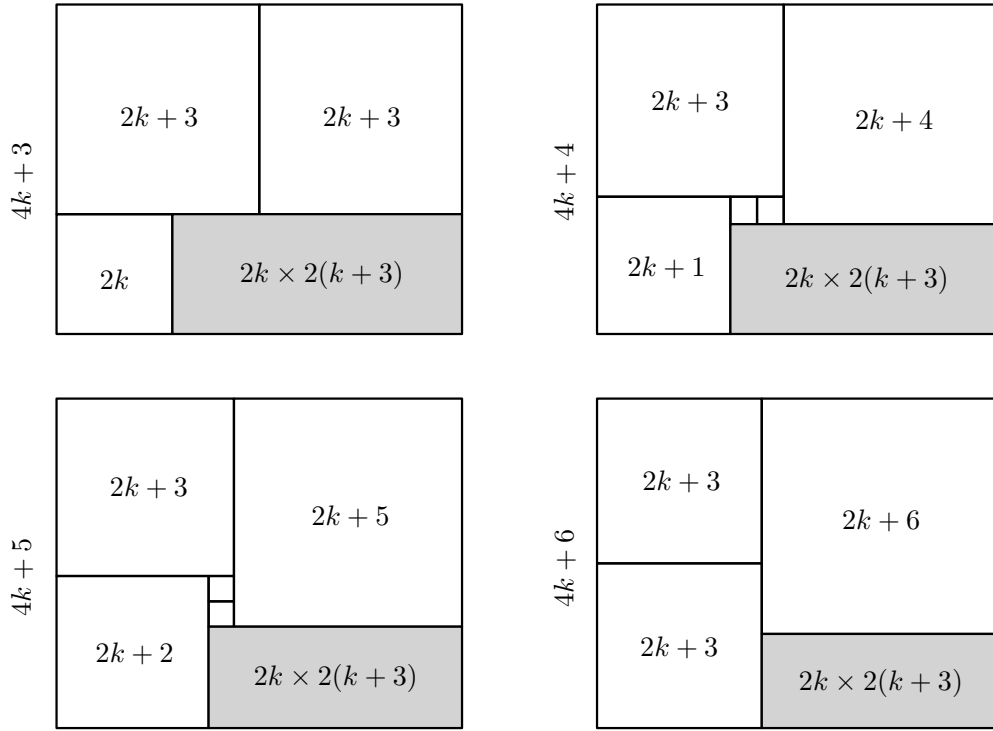


Figure 3.1: Dissecting a rectangle of size  $n \times (n+3)$

Recall that by  $r_3(n)$  we denote the smallest size of a  $\oplus$ -free dissection of an  $n \times (n+3)$  rectangle. Note that  $r_3(1) = 4$ , and let us estimate the remaining values using the algorithm:

**Lemma 3.4.** *Let  $n \geq 2$  be an integer. Then the algorithm results in a padded  $\oplus$ -free dissection of an  $n \times (n+3)$  rectangle. Furthermore  $r_3(n) \leq 5 \log_4(n) + \frac{3}{2}$ .*

*Proof.* The proof is by induction on  $n$ ; for  $n$  in (A1) the claim holds.

Let  $n = 4k + z$  where  $k \geq 2, z \in \{3, 4, 5, 6\}$ . By (A2) we clearly get a padded rectangle dissection. The inside of the recursively dissected rectangle  $2k \times 2(k+3)$  is  $\oplus$ -free by the induction hypothesis, and the outside is such by design. Therefore the only points where  $\oplus$ -freeness might be broken lie on its border.

However, the recursive dissection is padded and tiled with two times larger tiles, therefore there is a square of size at least 4 in the upper left corner which covers all possible problematic points.

Finally,

$$r_3(4k + z) \leq 5 + r_3(k) \leq 5 + 5 \log_4(k) + \frac{3}{2} \leq 5 \log_4(4k + z) + \frac{3}{2}. \quad (3.1)$$

□

### 3.3 Logarithmic dissection of a triangle

**Lemma 3.5.** *Let  $5 \leq n = 2k + 3$  be an odd integer not divisible by 3. Then  $\hat{t}(n) \leq 2r_3(k) + 2$ .*

*Proof.* Consider a triangle of side  $n$ . We can cut off triangles of sides  $k$  and  $(k+3)$  from two of its corners, which leaves us with a parallelogram of sides  $k$  and  $(k+3)$ . By a linear mapping  $f$  we can transform it into a  $k \times (k+3)$  rectangle (see Figure 3.2), which can be dissected into  $r_3(k)$  squares.

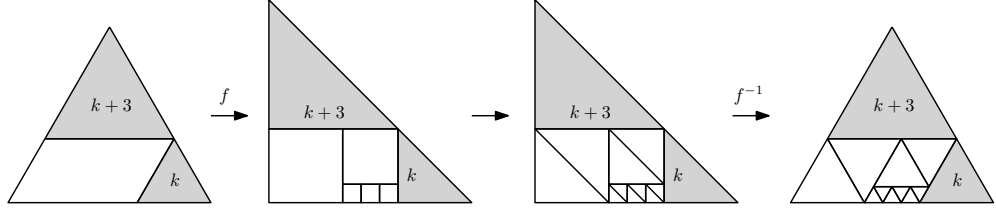


Figure 3.2: Dissecting a triangle using a dissection of a rectangle

Now, every square in the dissection can be diagonally cut into two right-angled triangles, such that after application of  $f^{-1}$  they transform into equilateral triangles. This gives us a dissection of the original triangle into  $2r_3(k) + 2$  triangles. Moreover  $\gcd(k+3, 3) = \gcd(k, 3)$  and  $3 \nmid k$ , therefore the dissection is prime.

It remains to prove  $\circledast$ -freeness. Clearly, the condition cannot be violated on the sides of the parallelogram.

Note that all the diagonal cuts have to be parallel, which means that there is at most one of them adjacent to every square corner (the rectangle dissection is  $\oplus$ -free). Thus we increase the number of shapes incident with every point at most by one and the resulting dissection is  $\circledast$ -free. □

**Corollary 3.6.** *Let  $n > 1$  be an odd integer not divisible by 3. Then  $\hat{t}(n) < 5 \log_2(n)$ .*

*Proof.* The conditions imply  $n \geq 5$ . Now by plugging  $k = \frac{n-3}{2}$  into Lemma 3.4:

$$2r_3\left(\frac{n-3}{2}\right) + 2 \leq 10 \log_4\left(\frac{n-3}{2}\right) + 5 = 5 \log_2(n-3) < 5 \log_2(n). \quad (3.2)$$

□

**Theorem 3.7.** *Let  $n \geq 2$  be an integer. Then  $\hat{t}(n) < 5 \log_2(n)$ .*

*Proof.* Set  $n = 2^p 3^q r$ , where  $p, q, r$  are nonnegative integers such that  $\gcd(r, 6) = 1$ . Then use the following algorithm to get a dissection of a triangle of side  $n$ :

- (B1) If  $p > 0$ , dissect into 4 triangles of size  $n/2$  and repeat for one of them recursively;
- (B2) If  $q > 0$ , dissect into 6 triangles and repeat for one of size  $n/3$  recursively;
- (B3) If  $r = 1$  then finish, otherwise dissect into at most  $5 \log_2(r)$  triangles as in Corollary 3.6.

Steps (B1) and (B2) are illustrated on [Figure](#). In (B3) we always use a prime dissection, therefore the resulting dissection is also prime. Clearly it is also  $\otimes$ -free.

Let us count the number of used triangles. If  $r > 1$ , then

$$\begin{aligned} \hat{t}(n) &< 3p + 5q + 5 \log_2(r) \\ &< 5p \log_2(2) + 5q \log_2(3) + 5 \log_2(r) \\ &= 5 \log_2(2^p 3^q r). \end{aligned}$$

If  $r = 1$ , then  $\hat{t}(n) \leq 3p + 5q + 1$  and

$$\begin{aligned} 3p + 5q + 1 &< 5 \log_2(2^p 3^q) && \Leftrightarrow \\ 5q + 1 &< 2p + 5q \log_2(3) && \Leftrightarrow \\ 1 &< 2p + (5 \log_2(3) - 5)q, \end{aligned}$$

which holds every time at least one of  $p, q$  is nonzero. □

### 3.4 Triangle dissections and latin bitrades

There is an interesting connection between triangle dissections and latin bitrades, first noted by Drápal [7]. Let us begin with parametrization of a triangle dissection.

Consider a plane  $\rho$  defined by  $x + y + z = 0$  in 3-dimensional Euclidean space. The planes with one fixed integer coordinate  $x = k$ ,  $y = k$ ,  $z = k$  intersect with  $\rho$ , and lines of the intersections form a triangular grid.

We identify vector  $(x_0, y_0, z_0)$  with the triangle bounded by lines  $x = x_0$ ,  $y = y_0$  and  $z = z_0$ . The number  $|x_0 + y_0 + z_0|$  is the size (or side) of the triangle. Degenerate triangles of size 0 are points in the plane  $\rho$ .

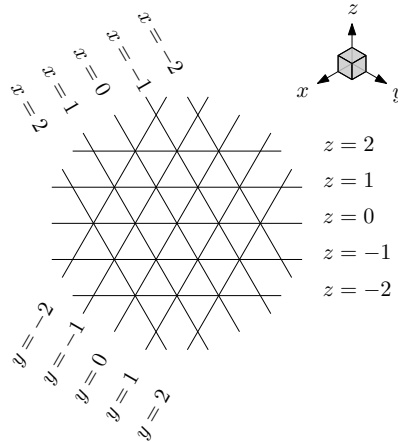


Figure 3.3: Triangular grid in plane  $x + y + z = 0$ .

Now, we can embed a triangle dissection into the grid by choosing its position and orientation.

**Definition 3.8.** For a dissection of a triangle  $\tau$  embedded in the plane  $x+y+z = 0$  define sets of vectors  $T^*$ ,  $T^\Delta$  such that

- $T^*$  is the set of vertices of the triangles in the dissection, with the vertices of  $\tau$  excluded and  $\tau$  itself included;
- $T^\Delta$  is the set of triangles in the dissection.

**Example 3.9.** See Figure 3.4.

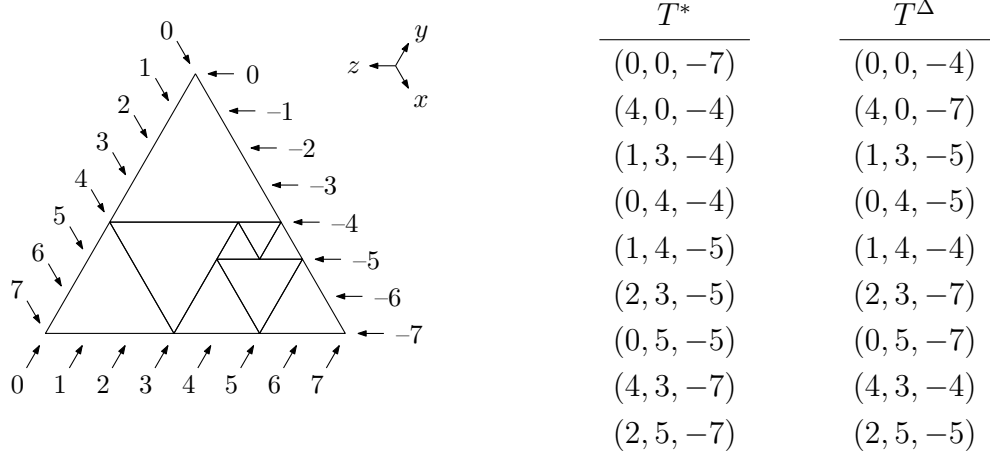


Figure 3.4: Construction of  $T^*$  and  $T^\Delta$  from a triangulation.

**Lemma 3.10.** Let  $D$  be a  $\otimes$ -free triangle dissection. Then  $(T^*, T^\Delta)$  is a latin bitrade for any embedding of  $D$  into the plane. All such latin bitrades are from the same class.

*Proof.*  $(T^*, T^\Delta)$  is a bitrade straightforwardly from  $\otimes$ -freeness. Translation of the triangle corresponds to isotopy and rotation to conjugacy, therefore the bitrades are from the same class.  $\square$

**Lemma 3.11.**  $\text{gdist}(n) \leq t(n)$ .

*Proof.* Let  $D$  be a  $\otimes$ -free dissection of a triangle  $\tau$  of size  $n$  into  $t(n)$  triangles. We claim that the homotopy

$$f : (x, y, z) \mapsto ((x \bmod n), (y \bmod n), (z \bmod n)) \quad (3.3)$$

is embedding of  $T^*$  into  $\mathbb{Z}_n$ , which would prove the statement.

Since the size of the triangle is  $n$ , it follows easily that  $f$  is injective. Since  $|x + y + z| \in \{0, n\}$  for  $(x, y, z) \in T^*$ , also  $x + y + z \equiv 0 \pmod{n}$  holds and we are done by Lemma 2.10.  $\square$

**@ The following corollary states Conjecture 1.**

**Corollary 3.12.** Let  $n \geq 2$  and  $p$  be the smallest prime factor of  $n$ . Then

$$3 \log_3(p) \leq \text{gdist}(n) < 5 \log_2(p). \quad (3.4)$$

*Proof.* The lower bound is **Theorem**. For the upper bound, combine Lemma 2.14, Lemma 3.11 and Theorem 3.7 to get

$$\text{gdist}(n) \leq \text{gdist}(p) \leq t(p) = \hat{t}(p) < 5 \log_2(n). \quad (3.5)$$

$\square$

**Corollary 3.13.** *Let  $n \geq 2$  and  $p$  be the smallest prime factor of  $n$ . Then*

$$3 \log_3(e) \leq \frac{\text{gdist}(n)}{\log(p)} < 5 \log_2(e). \quad (3.6)$$

@ This is not known:

**Conjecture 3.14.**  $\text{gdist}(p) = t(p)$

### 3.5 Families of logarithmic dissections

In previous sections we have seen how to use a logarithmic dissection into squares to get a logarithmic dissection into triangles. While the method presented gives the best results that we are aware of, in this section we show how it can be generalized as it can possibly lead to ideas, which might be able to improve the upper bound in Corollary 3.13.

Let us sketch the method first. A convex hexagon, which we call *core*, defines a dissection of a parallelogram into the core, 6 triangles and a smaller parallelogram. The sizes of the parallelograms depend on the shape of the core, and if chosen appropriately, the smaller parallelogram can be dissected recursively.

In the following, all shapes considered are aligned in a grid formed by unit equilateral triangles, i.e. all lengths are integer and all angles are multiples of  $\pi/3$ .

For this section we redefine  $t(n)$  – **we relax the  $\ast$ -freeness condition** on the dissections. We also do not concern ourselves with prime dissections.

**Definition 3.15.** A convex hexagon  $H$  in a unit triangular grid is a *core*. Let us denote its sides consecutively by  $a_0, \dots, a_5 \in \mathbb{Z}_0^+$ . We allow the hexagon to be degenerate, i.e. some of its sides can be zero. From the properties of such a hexagon, the following holds:

$$a_0 + a_1 = a_3 + a_4 =: \alpha \quad (3.7)$$

$$a_1 + a_2 = a_4 + a_5 =: \beta \quad (3.8)$$

$$a_2 + a_3 = a_5 + a_0 =: \gamma \quad (3.9)$$

Therefore the hexagon is uniquely specified by a 4-tuple  $(a_0, \alpha, \beta, \gamma)$ . We will often identify  $H = (a_0, \alpha, \beta, \gamma)$ .

Note that not every 4-tuple specifies a valid hexagon. Also note that

$$a_0 + \dots + a_5 = \alpha + \beta + \gamma \quad (3.10)$$

is the perimeter of a core.

**Definition 3.16.** A *shape*  $S$  is a union of finitely many unit triangles in the grid. Let us denote by  $t(S)$  the minimal number of triangles needed to dissect the shape  $S$ , and let  $t_d(n)$  denote  $t(P)$  for a parallelogram  $P$  of size  $n \times (n + d)$ .

We kindly ask the reader to extrapolate the formal definition of a dissection from Definition 3.2. As an example, if we denote a triangle of side  $n$  by  $\Delta_n$ , then  $t(\Delta_n) = t(n)$ .

**Lemma 3.17.** *Let  $H = (a_0, \alpha, \beta, \gamma)$  be a core and  $k$  a positive integer. Set  $n = 2k + a_0 + \alpha + \beta$  and denote by  $P_0$  and  $P_1$  parallelograms of sizes  $n \times (n + \gamma)$*



and  $k \times (k + \alpha + \beta + \gamma)$ . Then there exists a dissection of  $P_0$  into  $H$ ,  $P_1$  and six triangles. Therefore

$$t_\gamma(n) \leq 6 + t(H) + t_{\alpha+\beta+\gamma}(k) \quad (3.11)$$

*Proof.* See Figure 3.5. □

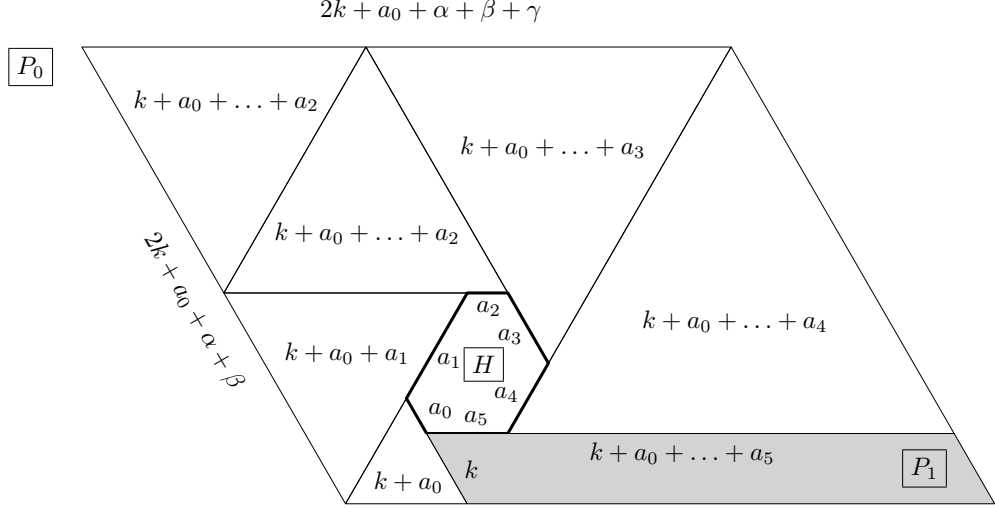


Figure 3.5: Caption

Now, let us set the variables such that we can use the tiling recursively. First, fix  $\gamma$  and  $\alpha + \beta + \gamma$ , so that  $P_0$  and  $P_1$  are always of sizes  $n \times (n + \gamma)$  and  $k \times (k + \alpha + \beta + \gamma)$ .

Next, we would like to tile  $P_1$  with tiles of sides which are multiples of an integer  $d$ . Therefore reset  $k := dk$  and set  $\alpha + \beta + \gamma = d\gamma$ . In this setting,

$P_0$  is of size  $n \times (n + \gamma)$ , and

$P_1$  is of size  $dk \times (dk + d\gamma)$

with  $n = 2dk + (d - 1)\gamma + a_0$ .

Finally, if  $n$  can be of any integer value (possibly for  $n > n_0$  for some  $n_0$ ), we can use the dissection recursively. Since  $k$  can be any integer, it suffices for  $(d - 1)\gamma + a_0$  to go through all remainders modulo  $2d$ . The term  $(d - 1)\gamma$  is a constant, therefore  $a_0$  must be such. Because  $a_0$  is nonnegative and  $a_0 \leq \gamma = a_0 + a_1$ , this gives us the final requirement  $2d - 1 \leq \gamma$ .

**Lemma 3.18.** *Let  $d, \gamma \geq 2$  be positive integers such that  $2d - 1 \leq \gamma$ . Then there exists  $n_0$  and a constant  $T$  such that*

$$t_\gamma(n) \leq 6 + T + t_\gamma(k) \quad (3.12)$$

for  $n > n_0$  and some  $k < n/(2d)$ .

*Proof.* For  $a \in [2d]$  denote

$$\begin{aligned} T_a = \min \{ & t(H) \mid H = (a_0, \alpha, \beta, \gamma) \text{ is a core,} \\ & \alpha + \beta + \gamma = d\gamma, \\ & a_0 \equiv a \pmod{2d} \} \end{aligned}$$

and define  $T = \max\{T_a \mid a \in [2d]\}$ .  $T_a$  is well-defined for every  $a$ , since it can be easily seen that there always exists a core with required parameters.

Set  $n_0 = 2d + d\gamma$  and take  $n > n_0$ . Then there is  $a \in [2d]$  such that  $n \equiv (d-1)\gamma + a \pmod{2d}$  and a core  $H = (a_0, \alpha, \beta, \gamma)$  which we have chosen such that  $t(H) = T_a$ .

Now,  $n$  can be written as  $2dk + (d-1)\gamma + a_0$  for a positive integer  $k$ . Plugging into Lemma 3.17 we get

$$t_\gamma(n) \leq 6 + t(H) + t_{d\gamma}(dk) \leq 6 + T + t_\gamma(k). \quad (3.13)$$

Clearly  $k < n/(2d)$ , which completes the proof.  $\square$

**Corollary 3.19.** *Let  $d, \gamma$  be as in Lemma 3.18. Then there exist constants  $T, C$  such that*

$$t_\gamma(n) \leq (6 + T) \log_{2d}(n) + C. \quad (3.14)$$

**Example 3.20.** Let us choose  $d = 2$  and  $\gamma = 3$ , they meet the condition  $2d - 1 \leq \gamma$ . Consider the cores on Figure 3.6, they have to have perimeter  $d\gamma = 6$ .

We chose  $a_0 \in \{0, 1, 2, 3\}$  as this is the only choice such that  $a_0 \leq \gamma$  and  $a_0$  runs through all remainders modulo  $2d = 4$ . We can set  $T = 4$  and from Corollary 3.19 we have

$$t_3(n) \leq 10 \log_4(n) + C = 5 \log_2(n) + C \quad (3.15)$$

for a constant  $C$ . The resulting tiling is in fact the tiling from Section 3.2 with every square diagonally cut in halves.

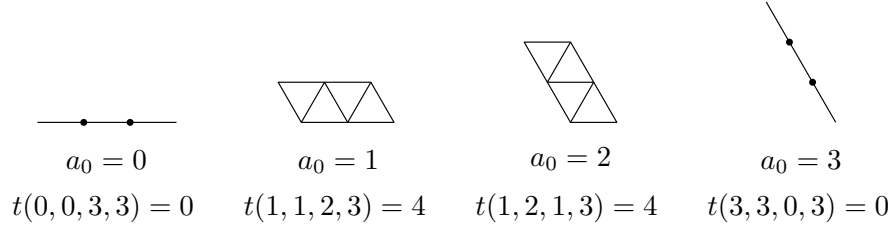


Figure 3.6: Cores for  $d = 2$ ,  $\gamma = 3$ . We denote the tiling of the corresponding core briefly by  $t(a_0, \alpha, \beta, \gamma)$ .

It would be desirable to construct a chain of better and better dissections that converge to the expected bound proposed in Chapter 4. However, the following lemmas show that using this method, this is not possible.

**Lemma 3.21.** *Let  $H = (a_0, \alpha, \beta, \gamma)$  be a core of perimeter  $d\gamma$  and  $a_0 \neq 0 \neq a_5$ . Then  $t(H) \geq d$ .*

*Proof.* Distance between the pair of parallel lines  $a_1, a_4$  is  $\frac{\sqrt{3}}{2}\gamma$ , and therefore the largest triangle that can fit in  $H$  can be of side  $\gamma$ . Therefore to cover the sides  $a_1$  and  $a_4$  we have to use at least  $(a_1 + a_4)/\gamma = (d\gamma - 2\gamma)/\gamma = d - 2$  triangles.

Since  $a_0 \neq 0 \neq a_5$ , we have to use one more triangle to cover each of these sides. These triangles have to be distinct from those lying on sides  $a_4$  and  $a_1$ , hence  $t(H) \geq d$ .  $\square$

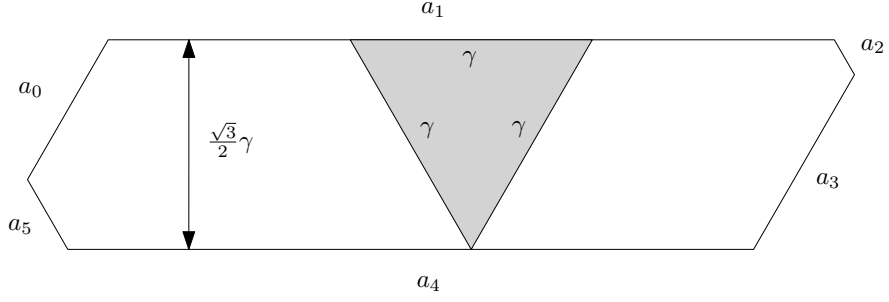


Figure 3.7: Tiling a core of perimeter  $d\gamma$ .

**Lemma 3.22.** *Let  $d, \gamma$  be as in Lemma 3.18 and  $\bar{t}_\gamma(n)$  denote the size of the dissection constructed in the same lemma. Then*

$$\bar{t}_\gamma(n) \geq (6 + d) \log_{2d}(n). \quad (3.16)$$

*Proof.* By Lemma 3.21,  $T \geq d$  in the proof of Lemma 3.18. The result follows by plugging into Corollary 3.19.  $\square$

Let us compare with the dissection into  $5 \log_2(n)$  triangles. For large  $n$ , if the method is better, then

$$\frac{\bar{t}_\gamma(n)}{\log(n)} < \frac{5 \log_2(n)}{\log(n)} \quad (3.17)$$

$$\Rightarrow \frac{6 + d}{\log(2d)} < \frac{5}{\log(2)} \quad (3.18)$$

$$\Leftrightarrow 2^{d+1} < d^5. \quad (3.19)$$

The last inequality has integer solutions only for  $d \leq 20$ . Therefore there can be only finitely many better dissections, as for fixed  $d$  the constant in the dissection depends on the value of  $T$ , which is a positive integer.

@ Not our main goal to settle this

@ Conjecture: our is the best

## 4. Refining the bounds

In previous chapters we have established that  $\text{gdist}(p)$  is asymptotically logarithmic, or, more precisely, that

$$2.73 \approx 3 \log_3(e) < \frac{\text{gdist}(p)}{\log(p)} < 5 \log_2(e) \approx 7.21 \quad (4.1)$$

for primes  $p > ?$ . The obvious question is – what are the best possible constants in these estimates?

While the question is open, in this chapter we provide evidence which suggests that the following might be true:

**Conjecture 4.1.** Let  $P$  be a real such that  $P^3 = P + 1$ . Then for primes  $p$ :

$$\lim_{p \rightarrow \infty} \frac{\text{gdist}(p)}{\log(p)} = \frac{1}{\log(P)} \approx 3.56. \quad (4.2)$$

Our argument is based on the connection of  $\text{gdist}(p)$  to dissections of triangles. Let us recall that Lemma 3.11 states that  $\text{gdist}(p) \leq t(p)$  and it is conjectured that  $\text{gdist}(p) = t(p)$  (Conjecture 3.14).

In this chapter we present computational data of Rosendorf [12] and of our own which support the following:

**Conjecture 4.2.** Let  $P$  be a real such that  $P^3 = P + 1$ . Then

$$\lim_{n \rightarrow \infty} \frac{\hat{t}(n)}{\log(n)} = \frac{1}{\log(P)}. \quad (4.3)$$

Note that Conjectures 4.2 and 3.14 imply Conjecture 4.1.

In some sense, as we will see, it is easier to approach Conjecture 4.2. One of the reasons is that we do not have to restrict ourselves only to primes, but can regard all numbers equally.

### 4.1 Padovan sequence

**Definition 4.3.** *Padovan sequence* is a linear recurring sequence  $(a_k)_{k \geq 1}$  defined by

$$a_1 = a_2 = a_3 = 1, \quad a_{k+3} = a_{k+1} + a_k \quad \text{for } n \geq 1. \quad (4.4)$$

The first few terms are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, ...

For more information about the sequence see e.g. [11].

Let  $P, \lambda_1, \lambda_2$  be roots of the polynomial  $x^3 - x - 1$ , where  $P$  is the only real root. Then we can write explicitly

$$a_k = c_0 P^k + c_1 \lambda_1^k + c_2 \lambda_2^k \quad (4.5)$$

for some complex constants  $c_0, c_1, c_2$ . Computing the values, we get  $|\lambda_1|, |\lambda_2| < 1$  and  $c_0 \approx 0.545$  is a real. Therefore  $a_k \sim c_0 P^k$ , or  $\log_P(a_k) \sim k$ .

The number  $P \approx 1.325$  is called *plastic constant*. As a side note, along with its mathematical properties, it has also its application in architecture [13].

**Definition 4.4.** Let  $n$  be a positive integer. By  $\text{spb}(n)$  we denote an integer such that

$$a_{\text{spb}(n)-1} < n \leq a_{\text{spb}(n)}. \quad (4.6)$$

Note that  $a_{\text{spb}(n)}$  is the nearest term in Padovan sequence which is larger or equals to  $n$ . Also  $\text{spb}(n) \sim \log_P(n)$ .

Consider a trapezoid consisting of three unit triangles. In each step, we can attach a triangle to the longest side of the shape to get a pentagon. This way we get a spiral-like tiling. By adding two more triangles to the pentagon we obtain a  $\oplus$ -free dissection of a triangle. (Figures 4.1, 4.2.)

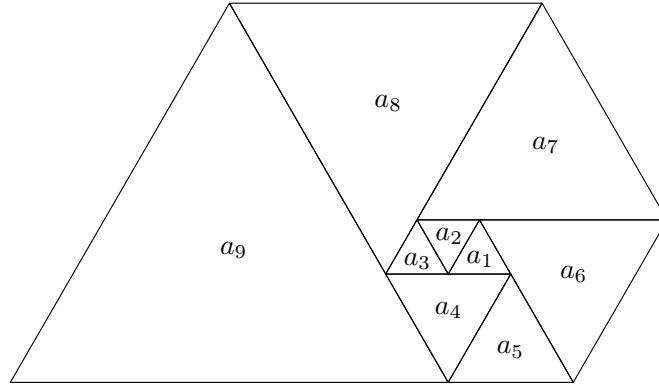


Figure 4.1: Spiral tiling.

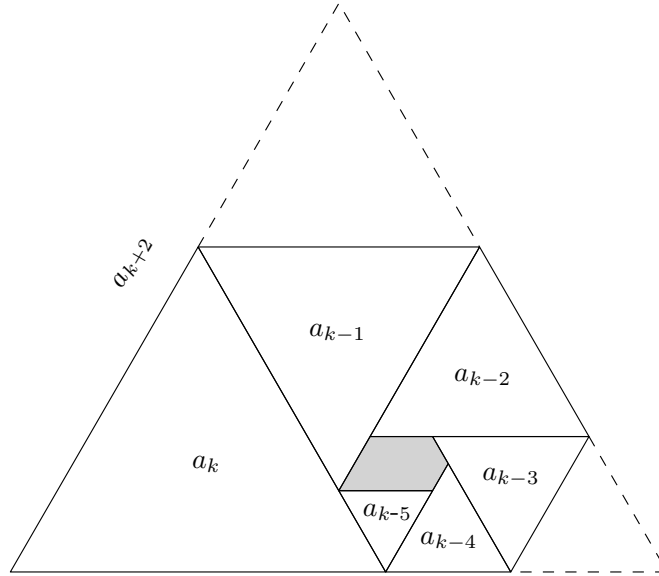


Figure 4.2: Completion of a pentagon into a triangle.

It is easy to derive that the sizes of the triangles are exactly the terms of Padovan sequence. Therefore we can construct a dissection of a triangle of side  $a_{k+2}$  into  $k + 2$  triangles. Since in such a dissection there is always a triangle of side 1, we obtain

$$\hat{t}(a_k) \leq k = \text{spb } a_k \sim \log_P(a_k). \quad (4.7)$$

## 4.2 Rosendorf

Based on Drápal's suggestion, Rosendorf studied in his master's thesis [12] a modification of the spiral tiling, which can be applied to triangles of any size. Consider the following algorithm:

**Algorithm 4.5.**

- In the beginning, from two corners of the original triangle cut off two triangles to get a pentagon or a parallelogram;
- Then, until the remaining shape is a triangle, cut off a triangle from the current shape to get either a pentagon, a parallelogram, a trapezoid or a triangle.

The algorithm is nondeterministic – if the current shape is not a pentagon, we can choose the placement and the size of the triangle to be cut off. Rosendorf proved that these dissections are exactly those which are  $\otimes$ -free and do not contain a subset of triangles forming a proper convex hexagon. Following his notation, let us denote such a dissection as (M6), standing for "missing hexagon".

Rosendorf enumerated all minimal (M6) dissections of triangles of side less than 10252. The data shows that

$$\hat{t}(n) \leq \text{spb}(n) + 2 \quad \text{for } n < 10252. \quad (4.8)$$

On the other hand, he also proved that at least  $\text{spb}(p)$  triangles are needed in an (M6) dissection of a triangle of prime side  $p$ . That, however, is not true when we allow all  $\otimes$ -free dissections.

## 4.3 Enumerations of minimal dissections

To support Conjecture 4.2, we generated all triangle dissections up to size 23 and therefore established values of  $t(n)$  and  $\hat{t}(n)$  for  $n \leq 416$ . For comparison, in [8] Drápal and Hämäläinen were able to generate dissections up to size 20, which corresponds to  $n \leq 160$ . They were, however, primarily interested in the number of triangulations of given size and perfect dissections (no two triangles of the same orientation have the same size), not in the values of  $t(n)$  and  $\hat{t}(n)$ .

We use essentially the same algorithm as in [8].<sup>1</sup> As we have shown, from a triangle dissection it is possible to construct a latin bitrade, and thus also a graph of the bitrade. All such graphs can be effectively generated by plantri software [1]. Our algorithm computes all triangle dissections which correspond to a given graph.

@ Potrbujem najprv odseparovat??

**Lemma 4.6.** *Let  $D$  be a dissection of  $\Delta_n$  and  $(T^*, T^\Delta)$  the corresponding bitrade. Then*

$$@M_{T^*} x^T = (n, 0, \dots, 0)^T \quad (4.9)$$

*has the only solution  $(x_1, \dots, x_i, \dots)$ .*

---

<sup>1</sup>Although it is not absolutely clear from their paper, our implementation has probably better time complexity.

*Proof.* @ Spherical ... dimension ... full rank.

@ The solution is solution □

**Lemma 4.7.** *Let  $D$  be a prime dissection of  $\Delta_n$  and  $(T^*, T^\Delta)$  the corresponding bitrade. Then*

$$@M_{T^*}x^T = (1, 0, \dots, 0)^T \quad (4.10)$$

*has the only solution in which  $n = \text{lcm}(\text{blah})$ .*

*Proof.* @ proof □

**Corollary 4.8.** *The vector  $x$  is  $k$ -th column of  $M_T^{-1}$ .*

The algorithm to generate all triangle dissections is as follows:

**Algorithm 4.9.**

1. Use plantri to generate planar Eulerian triangulation.
2. Construct corresponding separated bitrade  $(T, T')$ .
3. Find  $M_T^{-1}$ . Every **row** describes a homotopy of  $T$  into  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ .
4. Check that the homotopy is embedding. Then it must correspond to a triangle dissection.
5. Repeat for  $T'$ .

The generated values of  $\hat{t}(n)$  are listed in **Table**. Because we generated all dissections up to size 23, we also know that  $\hat{t}(n) = 24$  for those  $n$ , for which we know at least one dissection into 24 triangles. The values suggestively keep close to  $\text{spb}(n)$ . We conjecture the following:

**Conjecture 4.10.**  $\text{spb}(n) - 1 \leq \hat{t}(n) \leq \text{spb}(n)$ .

Conjencture 4.2 then follows by  $\text{spb}(n) \sim \log_P(n) = \log(n)/\log(P)$ .

# Conclusion



# Appendix A: Values of $\hat{t}(n)$

Table A.1 contains sizes of minimal  $\ast$ -free prime dissections of  $\Delta_n$  for  $n \leq 465$ , compared with corresponding spiral bounds. Our algorithm generated only data with  $\hat{t}(n) \leq 23$ . The last three rows in the table rely on data by Rosendorf [12]. He found dissections into 24 and 25 triangles for  $n$  in those rows.

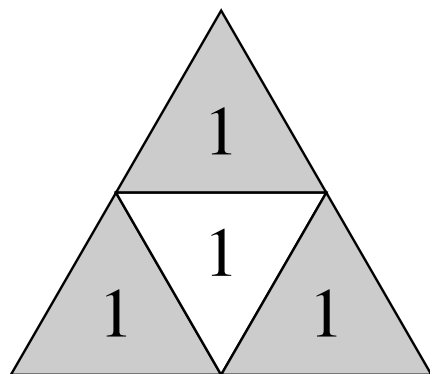
The table lists all  $n$  with  $\text{spb}(n) \leq 24$ .

$n$	$\hat{t}(n)$	$\text{spb}(n)$
2	4	4
3	6	6
4	7	7
5	8	8
6-7	9	9
8-9	10	10
10-12	11	11
13-16	12	12
17-21	13	13
22-28	14	14
29-37	15	15
39	15	16
38, 40-49	16	16
50-65	17	17
66-67	17	18
68-86	18	18
87, 90-91, 93	18	19
88-89, 92, 94-114	19	19
115-117, 120, 122, 130	19	20
118-119, 121, 123-129, 131-151	20	20
152-160, 162, 165	20	21
161, 163-164, 166-200	21	21
201-220, 225-226, 235	21	22
221-224, 227-234, 236-265	22	22
266-295, 300-301, 304-306, 315, 319	22	23
296-299, 302-303, 307-314, 316-318, 320-351	23	23
352-382, 384-388, 390-395, 397, 400-401, 404, 408-412, 414, 433	23	24
383, 389, 396, 398-399, 402-403, 405-407, 413, 415-416,	24	24
418-429, 431-432, 435, 437, 441, 444-445, 447, 449, 465	24	24
417, 430, 434, 436, 438-440, 442-443, 446, 448, 450-464	24-25	24

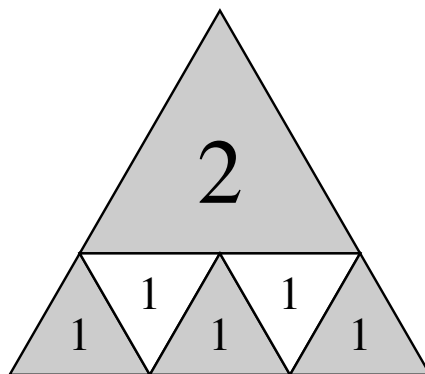
Table A.1: Values of  $\hat{t}(n)$  and  $\text{spb}(n)$  for  $n \leq 465$ .

# Appendix B: Minimal triangle dissections

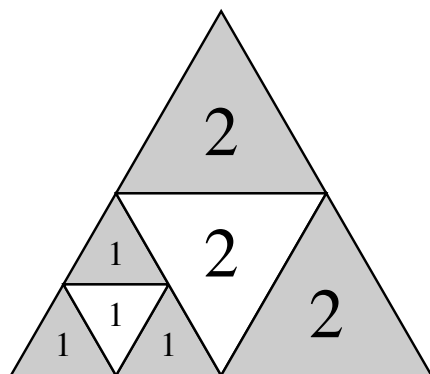
Figure B.1 shows examples of minimal  $\otimes$ -free prime dissections for triangles of sizes up to 37. The full list is attached as Appendix C.



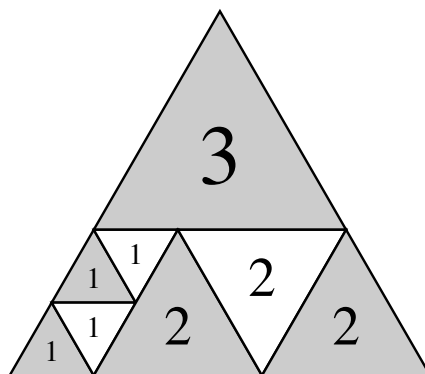
size = 2, triangles = 4



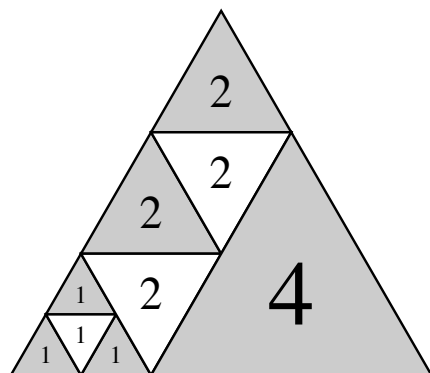
size = 3, triangles = 6



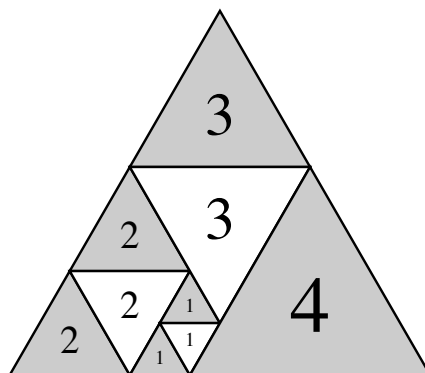
size = 4, triangles = 7



size = 5, triangles = 8

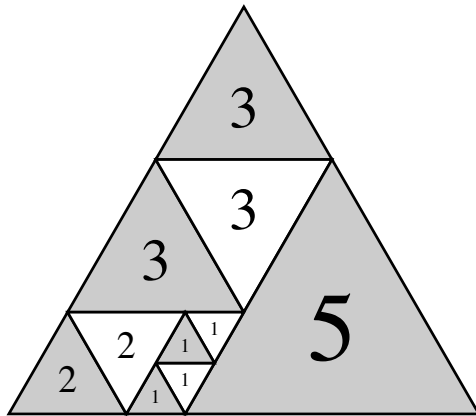


size = 6, triangles = 9

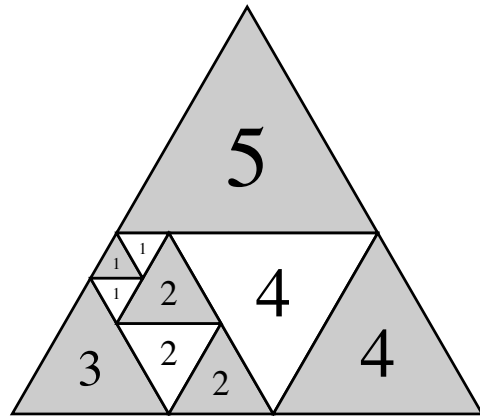


size = 7, triangles = 9

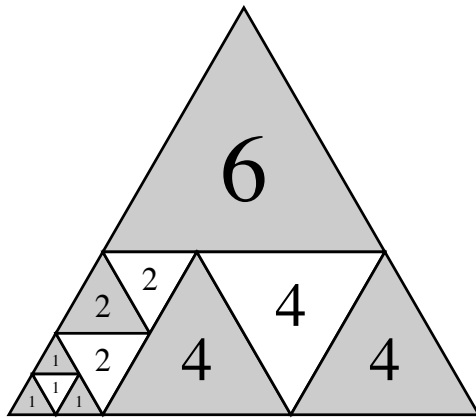
Figure B.1: Minimal triangle dissections.



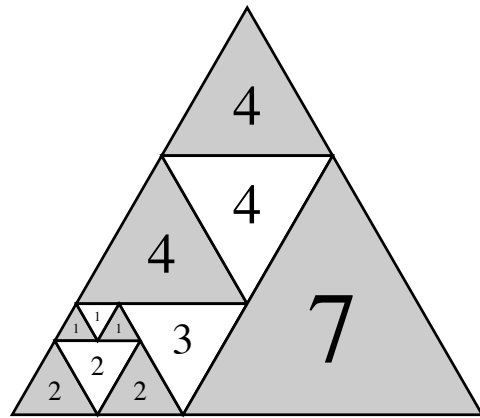
size = 8, triangles = 10



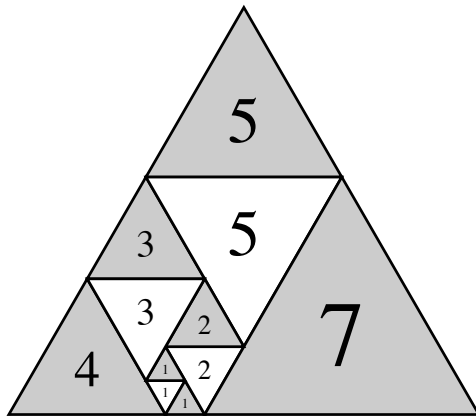
size = 9, triangles = 10



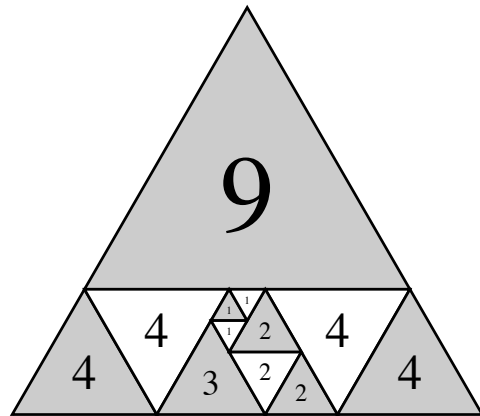
size = 10, triangles = 11



size = 11, triangles = 11

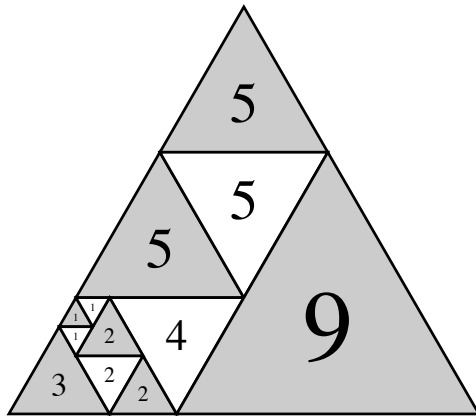


size = 12, triangles = 11

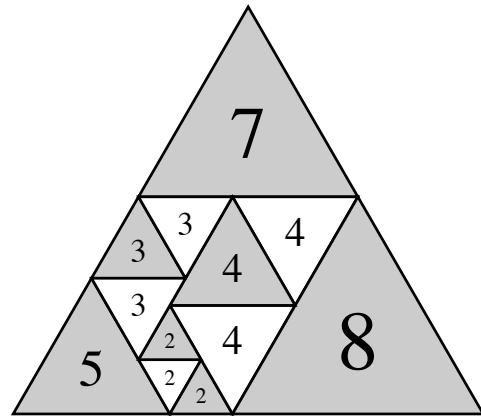


size = 13, triangles = 12

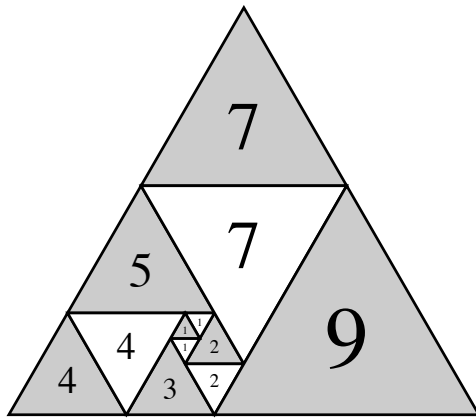
Figure B.1: Minimal triangle dissections.



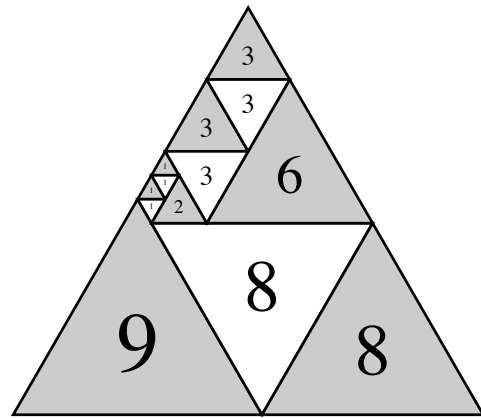
size = 14, triangles = 12



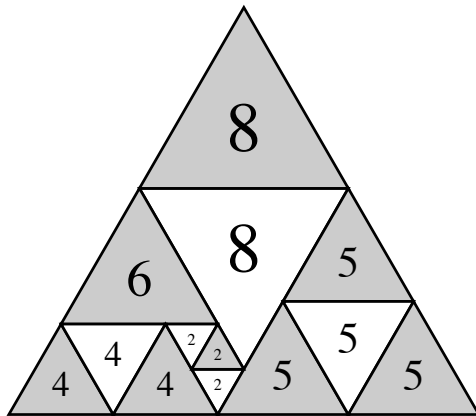
size = 15, triangles = 12



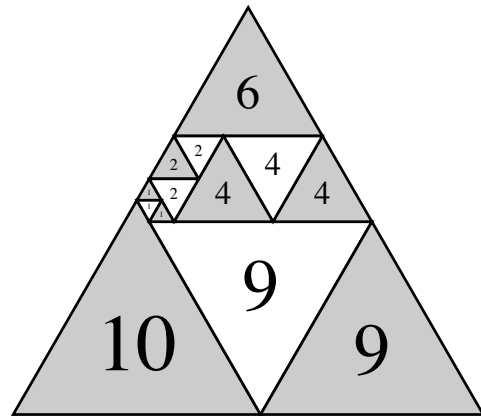
size = 16, triangles = 12



size = 17, triangles = 13

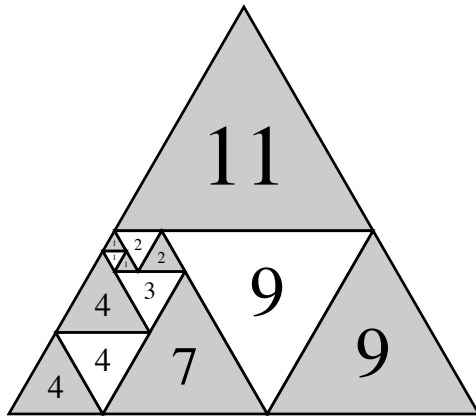


size = 18, triangles = 13

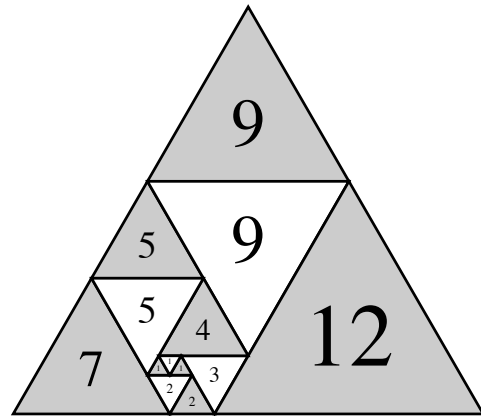


size = 19, triangles = 13

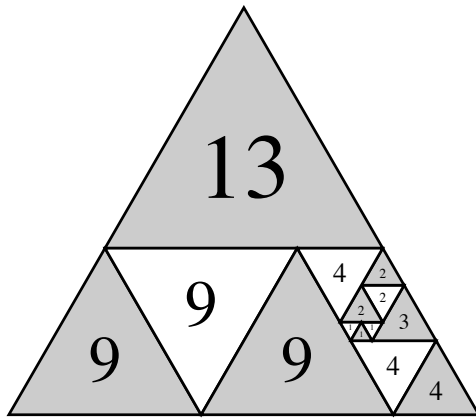
Figure B.1: Minimal triangle dissections.



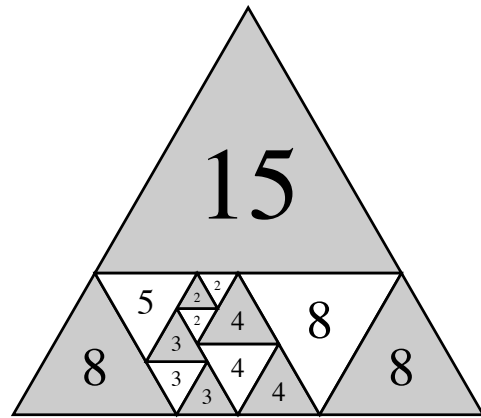
size = 20, triangles = 13



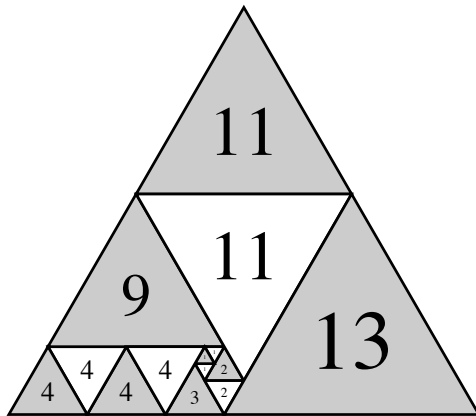
size = 21, triangles = 13



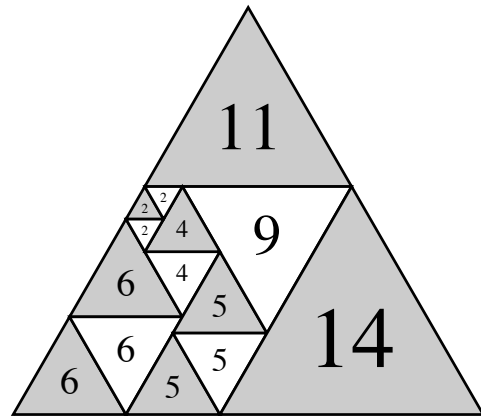
size = 22, triangles = 14



size = 23, triangles = 14

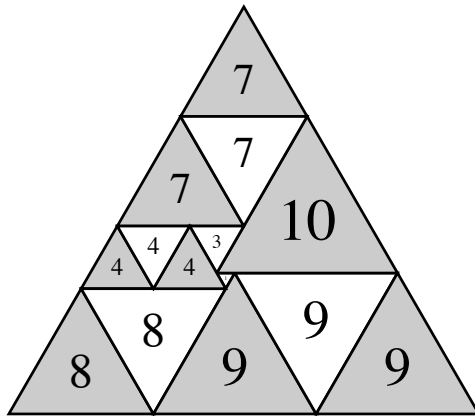


size = 24, triangles = 14

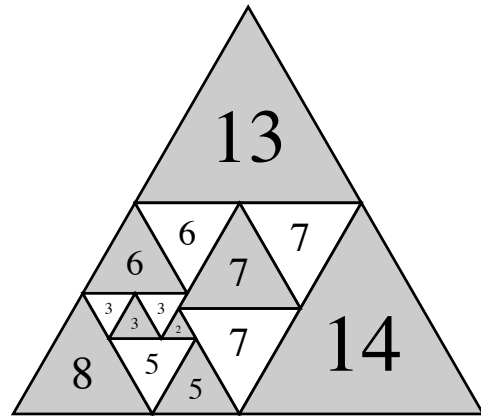


size = 25, triangles = 14

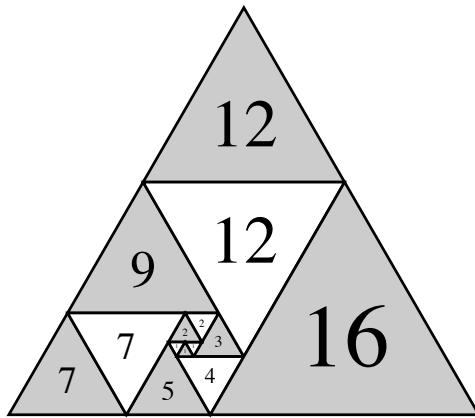
Figure B.1: Minimal triangle dissections.



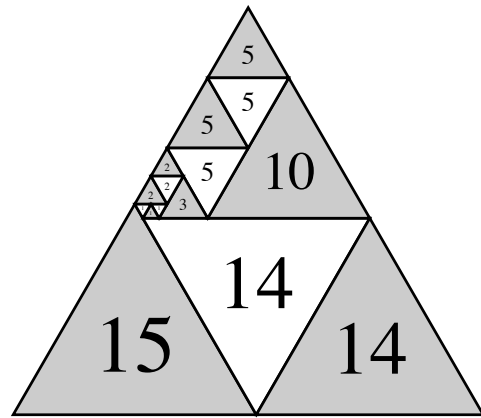
size = 26, triangles = 14



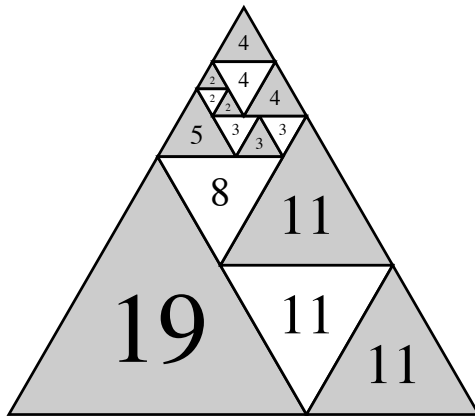
size = 27, triangles = 14



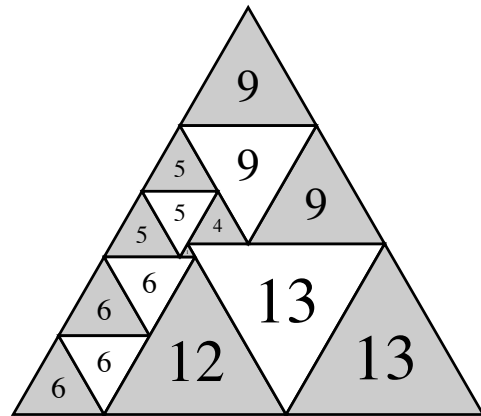
size = 28, triangles = 14



size = 29, triangles = 15

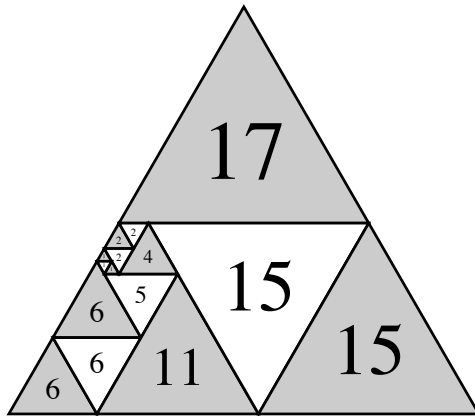


size = 30, triangles = 15

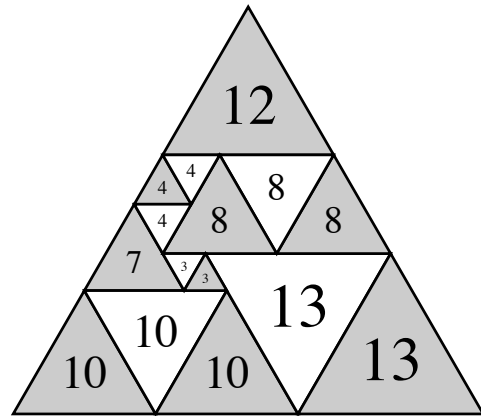


size = 31, triangles = 15

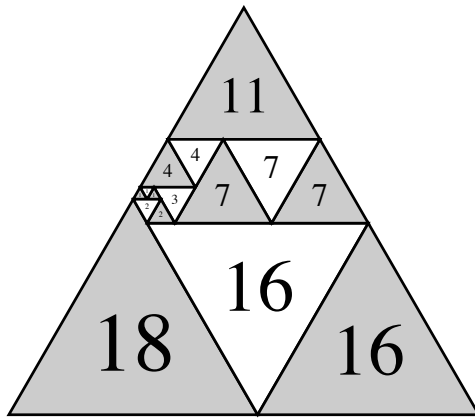
Figure B.1: Minimal triangle dissections.



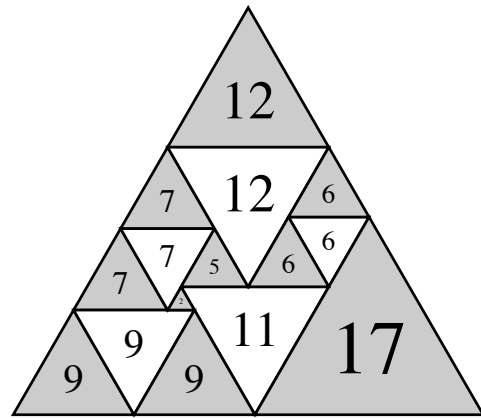
size = 32, triangles = 15



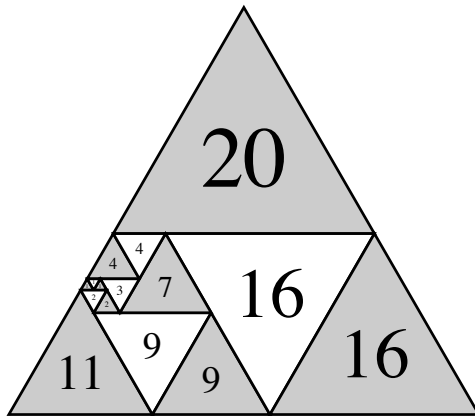
size = 33, triangles = 15



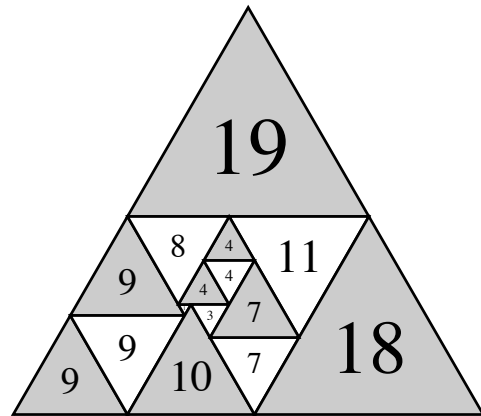
size = 34, triangles = 15



size = 35, triangles = 15



size = 36, triangles = 15



size = 37, triangles = 15

Figure B.1: Minimal triangle dissections.



# Appendix C: Program in C++

The attached CD contains a C++ program which was used to generate data presented in this thesis.

Benchmark?

Description?

# Bibliography

- [1] G. Brinkmann and B.D. McKay. Fast generation of some classes of planar graphs. *Electronic Notes in Discrete Mathematics*, 3(0):28 – 31, 1999. 6th Twente Workshop on Graphs and Combinatorial Optimization.
- [2] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. *Duke Math. J.*, 7:312–340, 1940.
- [3] Nicholas J. Cavenagh. The size of the smallest latin trade in a back circulant latin square, 2003.
- [4] Nicholas J. Cavenagh. The theory and application of latin bitrades: A survey. *Mathematica Slovaca*, 58:691–718, 2008.
- [5] Nicholas J. Cavenagh and Ian M. Wanless. Latin trades in groups defined on planar triangulations. *Journal of Algebraic Combinatorics*, 30:323–347, 2009.
- [6] J. H. Conway. Mrs perkins’s quilt. *Mathematical Proceedings of the Cambridge Philosophical Society*, 60:363–368, 6 1964.
- [7] Aleš Drápal. On a planar construction of quasigroups. *Czechoslovak Math. J.*, 41(116):538–548, 1991.
- [8] Ales Drápal and Carlo Hämmäläinen. An enumeration of equilateral triangle dissections. *Discrete Applied Mathematics*, 158(14):1479–1495, 2010.
- [9] Aleš Drápal and Tomáš Kepka. On a distance of groups and latin squares. *Commentationes Mathematicae Universitatis Carolinae*, 30(4):621–626, 1989.
- [10] H.E. Dudeney. *Amusements in Mathematics*. Dover New York, 1917.
- [11] OEIS Foundation Inc. Padovan sequence. <http://oeis.org/A000931>, 2011.
- [12] D. Rosendorf. Rozklady trojúhelníků a cyklické grupy. Master’s thesis, Charles University in Prague, 2004.
- [13] I. Stewart. Mathematical recreations. *Scientific American*, June 1996.
- [14] G. B. Trustrum. Mrs perkins’s quilt. *Mathematical Proceedings of the Cambridge Philosophical Society*, 61:7–11, 1965.
- [15] W. T. Tutte. The dissection of equilateral triangles into equilateral triangles. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44:463–482, 9 1948.