Charles University in Prague
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MASTER THESIS

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Dissections of triangles and distances of groups

Department of Algebra

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Contents

Preface

@ Este neviem, ci bude predslov okrem uvodu.

Introduction

Let us introduce two combinatorial problems:

Problem 1. Consider a table of addition modulo n; it is a latin square $n \times n$. What is the smallest number of cells we have to change in order to get another latin square?

$$
\begin{array}{|ccccccc|}
\hline
\boxed{0} & 1 & 2 & \boxed{3} & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 \\
\boxed{3} & 4 & \boxed{5} & \boxed{6} & 0 & 1 & 2 \\
4 & 5 & \boxed{6} & \boxed{0} & 1 & 2 & 3 \\
\boxed{5} & 6 & \boxed{0} & 1 & 2 & 3 & 4 \\
6 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|ccccccc|}
\hline
\boxed{3} & 1 & 2 & \boxed{0} & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 \\
\boxed{5} & 4 & \boxed{6} & \boxed{3} & 0 & 1 & 2 \\
4 & 5 & \boxed{0} & \boxed{6} & 1 & 2 & 3 \\
\boxed{0} & 6 & \boxed{5} & 1 & 2 & 3 & 4 \\
6 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{array}
$$

Figure 1: The smallest number for $n = 7$ is nine.

Problem 2. Let Δ_n be an equilateral triangle of side n. What is the smallest number of integer-sided equilateral triangles, into which Δ_n can be dissected, such that no six of them share a common point?

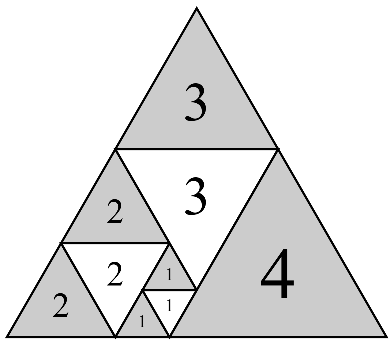

Figure 2: Dissection of a triangle of size 7 into nine triangles.

Though it is not obvious at first glance, these two problems are fundamentally related. Both triangle dissections and pairs of latin squares describe a combinatorial structure called *latin bitrade*. This structure will be of central interest throughout this work.

Let us denote by $\mathrm{gdist}(n)$ and $t(n)$ the minimal numbers described in Problems 1 and 2 respectively. Our main result is a solution to the twenty-year-old conjecture of Drápal, Cavenagh and Wanless:

Conjecture 1. There exist positive constants c_1 and c_2 such that

$$c_1 \log(p) \leq \mathrm{gdist}(p) \leq c_2 \log(p) \tag{1}$$

for sufficiently large primes p.

In other words, the conjecture states that gdist(n) is asymptotically logarithmic, the condition for n to be a prime is only a technical requirement. We also prove the same statement for $t(n)$ in place of gdist(n).

The lower bound in (1) was already established before. In 1989 Drápal and Kepka [11] proved the inequality for $c_1 = e$, and later Cavenagh [4] found an alternative proof of the same estimate. Yet another proof was given in a paper [6] by Cavenagh and Wanless, but with a slightly smaller constant.

All of these proofs are dealing with another structure which defines a latin bitrade – certain kind of 0-1 matrices. The lower bound is then determined by establishing upper bound for determinant of such a matrix. In Chapter 2 we present modified proof which leads to $c_1 = 3\log_3(e)$, the best estimate known so far.

The previously known best upper bound $\mathrm{gdist}(p) = O(\log^2(p))$ is due to Drápal [8]. He discovered the connection between latin bitrades and dissections of equilateral triangles, and proved that $\mathrm{gdist}(n) \le t(n)$. However, he was only able to construct triangle dissections with $O(\log^2(n))$ triangles.

In Chapter 3 we prove Conjecture 1 by constructing dissections into logarithmically many triangles. The method used is inspired by Trustrum's method [16] to dissect a square of side n into logarithmically many integer-sided squares. To be more precise, we show how to dissect an $n \times (n+3)$ rectangle into $5\log_4(n) + \frac{3}{2}$ squares and how to adapt the construction to get a dissection of an equilateral triangle of side n into $5\log_2(n)$ triangles. We also discuss possible generalizations of our dissection method in Section 3.5.

Now, that the asymptotic behavior of gdist(n) and $t(n)$ is known, it is natural to ask about the constants in the estimates. Putting our results together, we get

$$2.73 \approx 3\log_3(e) \le \frac{\mathrm{gdist}(p)}{\log(p)} \le \frac{t(p)}{\log(p)} \le 5\log_2(e) \approx 7.21. \tag{2}$$

That, however, do not seem to be the best estimates. The following is conjectured:

Conjecture 2. Let P be a real such that $P^3 = P + 1$. Then

$$\lim_{p\to\infty} \frac{\mathrm{gdist}(p)}{\log(p)} = \lim_{p\to\infty} \frac{t(p)}{\log(p)} = 1/\log(P) \approx 3.56. \tag{3}$$

In Chapter 4 we gathered evidence which supports this claim. We expose a connection between certain triangle dissections and an integer sequence satisfying the recurrence relation $a_{n+3} = a_{n+1} + a_n$. We also describe a computer algorithm with which we generated the exact values of $t(n)$ for $n \le 416$. The data, together with corresponding triangulations, are listed in appendices.

Notation

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We define empty sum to equal zero and empty product to equal one.

$\mathbb{Z}, \mathbb{Z}_n$	integers, integers modulo n
$[a, b], [a, b)$	intervals of integers k such that $a \leq k \leq b$, $a \leq k < b$
$[n]$	$[1, n]$
Δ_n	equilateral triangle of side n
$\overline{M}_r^c$	matrix M with row r and column c excluded
$f(n) \sim g(n)$	$\lim_{n\to\infty}(f(n)/g(n)) = 1$

1. Latin bitrades

An $n \times n$ table such that every row and column contains every number in $[n]$ exactly once is a well-known combinatorial object called *latin square*. In this chapter we define *latin bitrade*, which can be thought of as an object of differences between two latin squares.

To describe a table of elements formally, we use ordered triples (r, c, s) to represent the fact that the cell in row r and column c contains the symbol s. For that we use the following notation. Let

- $R = \{r_1, \ldots, r_{|R|}\}$ denote the set of rows,
- $C = \{c_1, \ldots, c_{|C|}\}$ denote the set of columns and
- $S = \{s_1, \ldots, s_{|S|}\}$ denote the set of symbols.

We consider only the case when R, C, and S are finite. As an example, a latin square is formally a subset of $R \times C \times S$ with $R = C = S = [n]$. We shall see this in more detail in a moment.

In this chapter we define only necessary notions for our purposes. For a more comprehensive introduction to latin bitrades we refer the reader to a survey by Cavenagh [5].

1.1 Partial latin squares

Definition 1.1. *A partial latin square* L is a subset of ordered triples from $R \times C \times S$, such that if $(r, c, s), (r', c', s') \in L$ agree on two coordinates, then $(r, c, s) = (r', c', s')$. We say that L is *on* $R \times C \times S$.

A partial latin square is usually interpreted as a partially filled $|R| \times |C|$ table. The condition implies that the table is well defined (there is at most one symbol in every cell), and that no symbol repeats itself within a column or a row.

Definition 1.2. *A latin square* L is a partial latin square such that $R = C = S$ and every cell in the table is filled. Equivalently, for every $a, a' \in R$ there are unique $r, c, s \in R$ such that

$$(r, a, a'), (a, c, a'), (a, a', s) \in L. \tag{1.1}$$

There are two important maps from partial latin squares to partial latin squares: *isotopy* and *conjugacy*.

Definition 1.3. Let $A \subset R_A \times C_A \times S_A$ and $B \subset R_B \times C_B \times S_B$ be partial latin squares. *A homotopy* h is defined by triple of maps

$$h_R : R_A \to R_B, \quad h_C : C_A \to C_B, \quad h_S : S_A \to S_B$$

such that

$$\begin{array}{rccc} h: & A & \to & B \\ & (r, c, s) & \mapsto & \big(h_R(r), h_C(c), h_S(s)\big). \end{array}$$

We write $h = (h_R, h_C, h_S)$. A homotopy is *trivial* if there is only one point in its image. *An isotopy* is a homotopy with homotopic inverse.

Example 1.4. Partial latin squares on Figure 1.1 are isotopic. The set of rows, columns and symbols is the same for both. The isotopy is given by

- h_R is identity,
- h_C rotates middle three columns,
- $h_S(1) = 2,\ h_S(2) = 4,\ h_S(3) = 1,\ h_S(4) = 3,\ h_S(5) = 5.$

1	3		2	
4			1	3
	4		5	
	5	1		4

A

2		4	1	
3		2		1
		5	3	
	2		5	3

B

Figure 1.1: Isotopic partial latin squares.

Definition 1.5. Let $A \subset R \times C \times S$ be a partial latin square and σ be a permutation of the 3-element set $\{R, C, S\}$. Then the partial latin square

$$\{(a_{\sigma(R)}, a_{\sigma(C)}, a_{\sigma(S)}) \mid (a_R, a_C, a_S) \in A)\} \tag{1.2}$$

is said to be *conjugated* with A.

Note that there are six conjugacies, each one corresponding to a permutation of $\{R, C, S\}$.

Definition 1.6. Two partial latin squares are from the same *class* if one can be obtained from the other by composition of conjugacy and isotopy.

1.2 Latin bitrades

Now we can define latin bitrade.

Definition 1.7. *A latin bitrade* is a pair (T, T') of partial latin squares on $R \times C \times S$ which are disjoint and for every $(r, c, s) \in T$ (respectively, T') there exist unique r', c', s' such that

$$(r', c, s), (r, c', s), (r, c, s') \in T' \text{ (respectively, } T). \tag{1.3}$$

Let us call T and T' *latin trades.* Elements of T and T' can be paired with respect to the first two coordinates. Therefore $|T| = |T'|$ and we shall call this number the *size* of the bitrade (or a trade).

From the tabular point of view, a latin bitrade is a pair of partial latin squares such that they occupy the same cells, but the symbols in corresponding rows and columns are permuted. Moreover, no symbol is at the same position in both of the tables.

Example 1.8. See Figure 1.2. The example is adapted from [5].

Note that two latin squares L, L' defined on the same set specify a latin bitrade $(L \setminus L', L' \setminus L)$.

	1	2	3
1	0	3	
2		0	1
3	2		0

T

	2	3	1
3	1	0	
1		2	0
2	0		3

T'

Figure 1.2: A latin bitrade on $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ of size 12.

Definition 1.9 (Graph representation of a latin bitrade)**.** A latin bitrade (T, T') is associated it with a graph $G = (V, E)$ such that

$V = T \cup T'$

$E = \{(t, t') \mid t \in T, t' \in T' : \ t \text{ and } t' \text{ differ at exactly one coordinate}\}.$

We call it the *graph of latin bitrade* (T, T').

Clearly, the graph is bipartite with partitions T and T'. It is also 3-regular from the definition of latin bitrade. Moreover, it is edge 3-colorable, the edges can be colored depending on the coordinate that t and t' differ at.

With the graph representation it is easier to understand the purpose of definitions in the rest of this section.

Definition 1.10. A latin bitrade (T, T') is *connected* if there do not exist two non-empty disjoint latin bitrades (T_1, T_1'), (T_2, T_2') such that $T = T_1 \cup T_2$ and $T' = T_1' \cup T_2'$.

Equivalently, a bitrade is connected if and only if its graph is connected.

Definition 1.11. A latin trade is called *spherical* or *planar*, if its graph is planar.

Example 1.12. Figure 1.3 shows a graph of a connected spherical latin bitrade (T, T') of size 6. To distinguish the elements of T and T', the latter are typed in brackets. Solid, dashed and dotted edges join elements which differ on R-, C- and S-coordinate respectively.

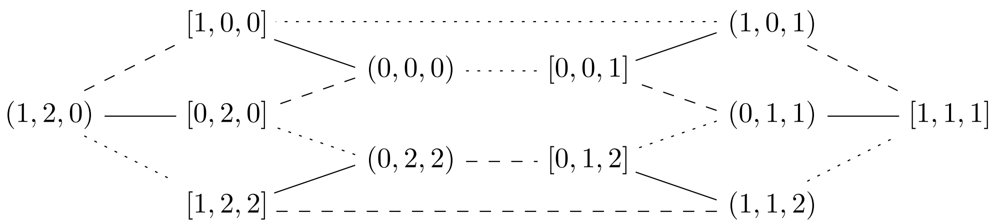


Figure 1.3: Graph of a latin bitrade of size 6 on $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

For later use, let us define maps $\sigma_R, \sigma_C, \sigma_S : T \cup T' \to T \cup T'$ by

$$\sigma_R(r, c, s) = (r', c, s) \text{ with } r \neq r', \tag{1.4}$$

$$\sigma_C(r, c, s) = (r, c', s) \text{ with } c \neq c', \tag{1.5}$$

$$\sigma_S(r, c, s) = (r, c, s') \text{ with } s \neq s'. \tag{1.6}$$

The definition of the latin bitrade implies that these maps are involutions. They correspond to the edges of the graph – on Figure 1.3, $\sigma_R, \sigma_C, \sigma_S$ are represented by solid, dashed and dotted edges respectively.

Lemma 1.13. *A latin bitrade (T, T') is connected if and only if for any $t_1, t_2 \in T \cup T'$ it is possible to get t_2 from t_1 by consequent application of $\sigma_R, \sigma_C, \sigma_S$.*

Proof. Simple, see the comment above. □

Lemma 1.14. *Let $\{X, Y\} \subset \{R, C, S\}$. Then the mapping $\sigma_Y \sigma_X : T \cup T' \to T \cup T'$ is a permutation without a fixed point.*

Proof. The mapping is a bijection with inverse $\sigma_X \sigma_Y$ on a finite set, thus it is a permutation. It changes two coordinates of its argument, and therefore has no fixed points. □

As a motivation for our last definition, consider this question: When is it possible to reconstruct a latin bitrade from its graph? Clearly we can do that only up to isotopy and conjugacy, as the graph representation forgets any orderings. Also, in every component of the graph we might switch roles of T and T'.

The graph of a bitrade is edge 3-colorable. By excluding edges of one color, say corresponding to R, the graph splits into cycles, in which all elements have the same R-coordinate. If the coordinates are in different cycles different, the bitrade is called *R-separated*, analogously for C and S.

Definition 1.15. A latin bitrade is *separated* if it is R-, C- and S-separated.

Every latin bitrade can be transformed into a separated one – for a symbol spanning multiple cycles, it suffices to introduce a new symbol for each of them and relabel accordingly. Clearly, this new bitrade yields the same graph as the original one.

Example 1.16. The bitrade from Example 1.12 is separated. Figure 1.4 illustrates the cycles after deletion of edges corresponding to S.

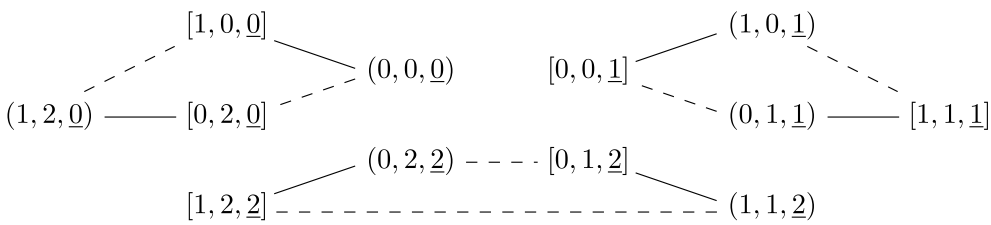


Figure 1.4: 2-color cycles in a separated latin bitrade.

Theorem 1.17. *A connected separated spherical latin bitrade (T, T') can be reconstructed from its graph G up to isotopy, conjugacy, and switch of the roles of T and T'.*

Proof. See Cavenagh and Lisoněk [3]. □

2. Embedding latin trades into groups

In this chapter we define gdist(n) and present a proof for the lower bound in Conjecture 1. We proceed as Drápal and Kepka in [11] and [10]. Since their work is a bit older, they use old terminology and their proof seems to be difficult to understand. We attempted to redo the proof using modern terminology and make it more accessible.

Moreover, we were able to improve the constant in the lower bound from $e \approx 2.718$ to $3\log_3(e) \approx 2.731$, which is the best constant known so far. The key part in doing so is Lemma 2.4.

There are other proofs of the lower bound available, but they are not as general as the one of Drápal and Kepka. The proof of Cavenagh and Wanless [6] considers only planar latin bitrades and abelian groups. (On the other hand, it exhibits an interesting connection between determinants and permanents of certain matrices, which we omit.) Another proof by Cavenagh [4] restricts itself to cyclic groups only.[1] The proof of Drápal and Kepka works with all latin bitrades and all finite groups.

2.1 Sparse matrices

Definition 2.1. A matrix is *sparse* if its elements are from $\{0, 1\}$ and there is at most one occurrence of 1 in every row.

We denote by $\overline{M}_r^c$ the matrix obtained from M by excluding row r and column c.

Lemma 2.2. *Let M_1, M_2 be sparse matrices with the same number of rows such that $M = (M_1, M_2)$ is a square block matrix. Then* $\det(M) \in \{-1, 0, 1\}$.

Proof. Let c_1, c_2 be the number of columns in M_1 and M_2 respectively. The proof is by induction on $n := c_1 + c_2$. The case with $n = 1$ is trivial. Therefore assume $n > 1$.

There are at most two ones in every row of M.

- If there is a row with zeros only, then $\det(M) = 0$.
- If every row contains exactly two ones, then

  $$v := (\underbrace{1, \ldots, 1}_{c_1}, \underbrace{-1, \ldots, -1}_{c_2})$$

  is such that $Mv^T = 0$. Thus M is singular and $\det(M) = 0$.
- Otherwise there is a row r which contains only a single one in column c. Then expanding the determinant by row r yields

  $$\det(M) = \pm \det(\overline{M}_r^c). \tag{2.1}$$

  The matrix $\overline{M}_r^c$ consists also from two sparse matrices, thus the result follows by induction.

[1] The author suspects that the proof of Cavenagh is not complete – he silently assumes that determinant of a certain matrix is nonzero, but the proof is missing.

□

Lemma 2.3. *Let M_1, M_2, M_3 be sparse matrices of sizes $n \times c_1, n \times c_2, n \times c_3$ such that $M = (M_1, M_2, M_3)$ is a square block matrix. Let k_i denote the number of ones in column i. Then*

$$|\det(M)| \leq \prod_{i \in [c_1]} k_i. \tag{2.2}$$

Proof. The proof is by induction on c_1. Denote $M = (m_{i,j})_{i,j \in [n]}$.

1. If $c_1 = 0$, then $|\det(M)| \leq \prod_{i \in \emptyset} k_i = 1$ holds by Lemma 2.2.
2. Otherwise expand by the first column:

$$|\det(M)| \leq \sum_{i \in [n]} m_{i,1} |\det(\overline{M}_i^1)| \leq k_1 \prod_{i \in [2,c_1]} k_i. \tag{2.3}$$

   The last inequality holds since there are k_1 nonzero summands and the product majorizes each subdeterminant from induction.

□

For our final result we need the following technical lemma:

Lemma 2.4. *Let n be a positive integer and $k_1 + \cdots + k_m = n$ for an integer m. Then*

$$\prod_{i \in [m]} k_i \leq 3^{n/3}. \tag{2.4}$$

Proof. For $n = 1$ it holds trivially, let us assume $n > 1$. Let $k_1 \leq \cdots \leq k_m$ be lexicographically smallest such that the maximum is attained. Observe:

- $2 \cdot (k - 2) \geq k$ for $k \geq 4$, therefore $k_i \leq 3$;
- $(1 + k) > 1 \cdot k$, therefore $k_i > 1$;
- $3 \cdot 3 > 2 \cdot 2 \cdot 2$, therefore there are at most two twos amongst k_i.

Thus there are 3 possibilities:

$$\begin{aligned}
n &= 3k & &\Rightarrow k_1 = \cdots = k_m = 3 & &\Rightarrow \prod k_i = 3^{n/3} \\
n &= 3k + 2 & &\Rightarrow k_1 = 2,\ k_2 = \cdots = k_m = 3 & &\Rightarrow \prod k_i = 2 \cdot 3^{(n-2)/3} \\
n &= 3k + 4 & &\Rightarrow k_1 = k_2 = 2,\ k_3 = \cdots = k_m = 3 & &\Rightarrow \prod k_i = 4 \cdot 3^{(n-4)/3}
\end{aligned}$$

Where the products are over $i \in [m]$. Each of them is less than or equal to $3^{n/3}$, thus we are done. □

Lemma 2.5. *Let M_1, M_2, M_3 be sparse matrices such that $M = (M_1, M_2, M_3)$ is a square block $n \times n$ matrix. Then*

$$|\det(M)| \leq 3^{n/3}. \tag{2.5}$$

Proof. Let c_1 be the number of columns of M_1 and k_i the number of ones in the column i. Since M_1 is sparse, surely $\sum_{i \in [c_1]} k_i \leq n$. The proof is finished by combining Lemmas 2.3 and 2.4:

$$|\det(M)| \leq \prod_{i \in [c_1]} k_i \leq 3^{n/3}. \tag{2.6}$$

□

2.2 Trade matrix

Recall that

- $R = \{r_1, \dots, r_{|R|}\}$ denotes the set of rows,
- $C = \{c_1, \dots, c_{|C|}\}$ denotes the set of columns and
- $S = \{s_1, \dots, s_{|S|}\}$ denotes the set of symbols.

Without loss of generality assume that these sets are disjoint and let

$$X = R \cup C \cup S = \{r_1, \dots, r_{|R|}, c_1, \dots, c_{|C|}, s_1, \dots, s_{|S|}\}. \tag{2.7}$$

Definition 2.6. Let T be a latin trade. Fix ordering of elements of X as above and choose an ordering on T.

We define a matrix $M = (m_{i,j})_{i \in T, j \in X}$ of size $|T| \times |X|$ such that

$$t = (r, c, s) \in T \Rightarrow \begin{cases} m_{t,r} = m_{t,c} = m_{t,s} = 1, & \\ m_{t,x} = 0 & \text{for } x \in X \setminus \{r, c, s\}. \end{cases} \tag{2.8}$$

We call it the *trade matrix* and denote by M_T.

Lemma 2.7. *Suppose that all elements of X are used in a connected latin trade T. Then*

$$\operatorname{Ker} M_T = \{(\underbrace{x, \dots, x}_{|R|}, \underbrace{y, \dots, y}_{|C|}, \underbrace{z, \dots, z}_{|S|}) \mid x + y + z = 0\} \tag{2.9}$$

where M_T is considered as a matrix over $\mathbb{Q}$.

Proof. The proof is divided into several steps.

Step 1. Denote the set in (2.9) by V. Any vector $v \in V$ is surely a solution for $M_T v^T = 0$. Let $f : X \to \mathbb{Q}$ be defined by

$$v = (f(r_1), \dots, f(c_1), \dots, f(s_1), \dots). \tag{2.10}$$

Choose $(x_0, y_0, z_0) \in T$ such that $f(z_0)$ is maximal. By resetting

$$v := v + (-f(x_0), \dots, -f(y_0), \dots, f(x_0) + f(y_0), \dots) \tag{2.11}$$

we obtain a solution in which $f(x_0) = f(y_0) = f(z_0) = 0$. Now it suffices to prove that $f(x) = 0$ for any $x \in X$. To shorten notation, we denote $f(x, y, z) = f((x, y, z)) = (f(x), f(y), f(z))$.

Step 2. For all $(x, y, z) \in T$ holds $f(x) + f(y) + f(z) = 0$. Since 0 is the largest element of $\{f(z) \mid z \in S\}$ and T' occupies the same cells as T, for all $(x, y, z) \in T \cup T'$

$$f(x) + f(y) \geq 0. \tag{2.12}$$

In steps 3 and 4 we prove that if $t \in T \cup T'$ and $f(t) = (0, 0, 0)$, then

$$f(\sigma_Y(t)) = (0, 0, 0) \tag{2.13}$$

for $Y \in \{R, C, S\}$. Since the bitrade is connected, Lemma 1.13 implies that $f(t) = (0, 0, 0)$ for all $t \in T \cup T'$. Because all symbols are used in the bitrade, from that we have the desired $f(x) = 0$ for all $x \in X$.

Step 3. Let $(x, y, z) \in T'$ such that $f(x, y, z) = (0, 0, 0)$. Then

$$f(\sigma_R(x, y, z)) = f(x', y, z) = (f(x'), 0, 0) \tag{2.14}$$

for some $x' \in R$ such that $(x', y, z) \in T$. Therefore $f(x') + 0 + 0 = 0$. Similarly for σ_C and σ_S.

Step 4. Now let $(x_1, y_1, z) \in T$ such that $f(x_1, y_1, z) = (0, 0, 0)$. Consider a chain of elements in $T \cup T'$:

$$\begin{array}{llllll}
(x_1, y_1, z) \in T & \xrightarrow{\sigma_C} & (x_1, y_2, z) \in T' & \xrightarrow{\sigma_R} \\
(x_2, y_2, z) \in T & \xrightarrow{\sigma_C} & (x_2, y_3, z) \in T' & \xrightarrow{\sigma_R} \\
(x_3, y_3, z) \in T & \xrightarrow{\sigma_C} & (x_3, y_4, z) \in T' & \dots
\end{array}$$

From $f(x) + f(y) \geq 0$ we have

$$\begin{array}{ll}
f(x_1) + f(y_1) = 0, & f(x_1) + f(y_2) \geq 0, \\
f(x_2) + f(y_2) = 0, & f(x_2) + f(y_3) \geq 0, \\
f(x_3) + f(y_3) = 0, & f(x_3) + f(y_4) \geq 0, \ \dots,
\end{array}$$

which yields

$$\begin{array}{l}
0 = f(x_1) \geq f(x_2) \geq f(x_3) \geq \dots \\
0 = f(y_1) \leq f(y_2) \leq f(y_3) \leq \dots
\end{array}$$

According to Lemma 1.14, the chain is a non-trivial cycle, and thus all terms in the inequalities equal to zero. Especially $f(\sigma_C(x_1, y_1, z)) = (0, 0, 0)$.

The result for σ_R can be obtained by changing the roles of σ_R and σ_C.

For σ_S, consider a cycle generated by alternating σ_C and σ_S. We already know that $f(\sigma_S\sigma_C(x_1, y_1, z)) = \sigma_S(0, 0, 0) = (0, 0, 0)$. Therefore all elements in the cycle are mapped to $(0, 0, 0)$. By reversing the cycle we get the result.

□

Corollary 2.8. *Suppose that all elements of X are used in a connected latin trade T. Then the trade matrix M_T has rank $|X| - 2$ over $\mathbb{Q}$.*

Note. Suppose that M is a square submatrix of M_T of rank $|X| - 2$. It must have been obtained from M by deleting two columns and some rows. These two columns cannot be both from R. If they were, then

$$(\underbrace{0, \dots, 0}_{|R|-2}, \underbrace{y, \dots, y}_{|C|}, \underbrace{-y, \dots, -y}_{|S|}) \tag{2.15}$$

would be solutions of $Mv^T = 0$, which contradicts the regularity of M. Therefore the deleted columns must be from two different sets from R, C, S.

Corollary 2.9. $|T| \geq |X| - 2$.

2.3 Homotopies from latin trades to groups

Definition 2.10. *A Cayley table* of a group G is a latin square $L \subset G \times G \times G$ such that $(r, c, s) \in L$ if and only if $r \cdot c = s$. We denote it by G when no confusion can arise.

For abelian groups, we will use different definition:

Definition 2.11. *A Cayley table* of an abelian group $(G, +)$ is a latin square $L \subset G \times G \times G$ such that $(r, c, s) \in L$ if and only if $r + c + s = 0$.

It easily follows that the Cayley tables by first and second definition are isotopic. Because we will be interested in existence of homotopies into Cayley tables, it does not matter which definition we use.

Definition 2.12. Let $z(G)$ denote the size of the smallest trade T such that there exist a non-trivial homotopy from T to G. Let $z(n)$ be the minimum across all groups of order n.

Lemma 2.13. *Let p be a prime, T connected latin trade using all symbols in X, $h : T \to \mathbb{Z}_p$ non-trivial homotopy and M square submatrix of M_T of rank $|X - 2|$. Then $p \leq \det(M)$.*

Proof. Let $h = (h_R, h_C, h_S)$. Then

$$v = \big(\underbrace{h_R(r_1), \ldots, h_R(r_{|R|})}_{|R|}, \underbrace{h_C(c_1), \ldots, h_C(c_{|C|})}_{|C|}, \underbrace{h_S(s_1), \ldots, h_S(s_{|S|})}_{|S|}\big)$$

is a solution of $M_T v^T = 0$ over $\mathbb{Z}_p$. Without loss of generality assume that columns r_1, c_1 were deleted from M_T to obtain M (see note after Corollary 2.8). Also suppose that $h_R(r_1) = 0 = h_C(c_1)$, otherwise we can set

$$h'_R := h_R - h_R(r_1), \quad h'_C := h_C - h_C(c_1), \quad h'_S := h_S + h_R(r_1) + h_C(c_1).$$

Then

$$w = \big(\underbrace{h_R(r_2), \ldots, h_R(r_{|R|})}_{|R|-1}, \underbrace{h_C(c_2), \ldots, h_C(c_{|C|})}_{|C|-1}, \underbrace{h_S(s_1), \ldots, h_S(s_{|S|})}_{|S|}\big)$$

is a solution of $M w^T = 0$ over $\mathbb{Z}_p$ which is non-trivial, since h is non-trivial.

Therefore $\det(M) = 0$ in $\mathbb{Z}_p$, which means $p \mid \det(M)$. Because M is reqular, $\det(M) \neq 0$ and we are done. □

Lemma 2.14. $3 \log_3(p) \leq z(p)$.

Proof. Let T be a latin trade such that $|T| = z(p)$ and there exists a non-trivial homotopy $T \to \mathbb{Z}_p$. From Lemma 2.13 we know that $p \leq \det(M)$ for a submatrix of M_T of rank $|X| - 2$.

We can write $M_T = (M_1, M_2, M_3)$ for sparse matrices M_1, M_2, M_3, and therefore any submatrix M of M_T is of the same type. Thus from Lemma 2.5 and Corollary 2.9

$$p \leq \det(M) \leq 3^{(|X|-2)/3} \leq 3^{z(p)/3}. \tag{2.16}$$

□

Lemma 2.15. *Let H be a normal subgroup of G and $h : T \to G$ is a non-trivial homotopy. Then there exists a non-trivial homotopy $h_1 : T \to H$ or $h_2 : T \to G/H$.*

Proof. Let $\pi : G \times G \times G \to G/H \times G/H \times G/H$ be the natural projection. If $h_2 := \pi h$ is non-trivial, we are done. Otherwise $\mathrm{Im}(h) \subset H \times H \times H$ and we can set $h_1 := h$. □

Lemma 2.16. *Let G be a group of odd order. Then*

$$z(G) = \min\{z(p) \mid \text{ prime } p \text{ divides } |G|\}. \tag{2.17}$$

Proof. Lemma 2.15 together with odd order theorem imply the "$\geq$" inequality. The other one follows from Cauchy's theorem. □

2.4 Lower bound for gdist(n)

Definition 2.17. A latin trade T *can be embedded* (or *is embeddable*) in a group G if there exists an injective homotopy from T to G.

Let $\mathrm{gdist}(G)$ denote the size of the smallest trade embeddable in G and let $\mathrm{gdist}(n)$ be the minimum across all groups of order n.

From the tabular point of view, a latin trade T can be embedded in a group G if we can find an isotopic copy of the partial latin square T inside of the Cayley table of G. The next lemma states that $\mathrm{gdist}(G)$ is the smallest number of cells in the Cayley table of G that have to be changed in order to get another latin square. Therefore $\mathrm{gdist}(n)$ is the minimal "Hamming distance" between groups of order n and latin squares.

Lemma 2.18. *Let G be a group. Then*

$$\mathrm{gdist}(G) = \min\{|G \setminus L| : L \subset G \times G \times G \text{ is a latin square}, L \neq G\}. \tag{2.18}$$

Proof. $(G \setminus L, L \setminus G)$ is a latin bitrade, hence "$\leq$" holds. On the other hand, if (T, T') is a latin bitrade embeddable in G via injective homotopy h, then $G \setminus h(T) \cup h(T')$ is a latin square. □

Example 2.19. See Figure 2.1.

Theorem 2.20. *Let p be the smallest prime factor of $n \geq 2$. Then*

$$3\log_3(p) \leq \mathrm{gdist}(n). \tag{2.19}$$

Proof. If n is even, then $p = 2$, $\mathrm{gdist}(n) = 4$ and the inequality holds. Otherwise by Lemmas 2.14 and 2.16 there is a prime factor p_0 of $|G|$ such that $\mathrm{gdist}(2k) = 4$

$$3\log_3(p_0) \leq z(p_0) = z(G) \leq \mathrm{gdist}(G), \tag{2.20}$$

where the last inequality is trivial from definition. The fact that $p \leq p_0$ finishes the proof. □

$$
\begin{array}{|cccc|}
\hline
0 & & 3 & 4 \\
 & 3 & 4 & 0 \\
3 & 0 & & \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|cccc|}
\hline
3 & & 4 & 0 \\
 & 0 & 3 & 4 \\
0 & 3 & & \\
\hline
\end{array}
$$

$$T \qquad\qquad\qquad T'$$

$$
\begin{array}{|ccccc|}
\hline
\boxed{0} & 1 & 2 & \boxed{3} & \boxed{4} \\
1 & 2 & \boxed{3} & \boxed{4} & \boxed{0} \\
2 & 3 & 4 & 0 & 1 \\
\boxed{3} & 4 & \boxed{0} & 1 & 2 \\
4 & 0 & 1 & 2 & 3 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|ccccc|}
\hline
\boxed{3} & 1 & 2 & \boxed{4} & \boxed{0} \\
1 & 2 & \boxed{0} & \boxed{3} & \boxed{4} \\
2 & 3 & 4 & 0 & 1 \\
\boxed{0} & 4 & \boxed{3} & 1 & 2 \\
4 & 0 & 1 & 2 & 3 \\
\hline
\end{array}
$$

$$\mathbb{Z}_5 \qquad\qquad\qquad L$$

Figure 2.1: Latin trade T embedded in $\mathbb{Z}_5$ and the corresponding latin square L.

3. Dissections of equilateral triangles

The study of dissections was initiated by the paper *The dissection of rectangles into squares* by Brooks, Smith, Stone and Tutte [2]. They answered the question whether it is possible to dissect a square into some number of unequal squares (yes, it is), and developed methods to study such dissections.[1]

Inspired by a puzzle called *Mrs Perkins's quilt* by Dudeney [12], Conway [7] considered the case where the dissecting squares can be equally large. He proposed a question about the minimal number of integer-sided squares needed to dissect a square of side n. It is easy to observe that when n is divisible by d, then it is possible to use a scaled up dissection of a square of side d. Therefore he considered only dissections where the dissecting squares do not have a common factor.

Conway proved that at least $c \log(n)$ squares are needed. A year later Trustrum [16] proved that $O(\log(n))$ is sufficient, and thus established that the answer is asymptotically logarithmic. However, the best constants in the estimates do not appear to be known.

In this chapter we prove that it is possible to dissect an equilateral triangle of side n into $O(\log(n))$ equilateral triangles. We do so by modifying a dissection of a rectangle into squares. We explain the connection to latin bitrades and prove the upper bound in Conjecture 1. The last section of this chapter contains a generalization of the dissection method, which might be useful for further improvements of the upper bound.

Note that the first one to study dissections of equilateral triangles was Tutte [17].

3.1 Definitions

Unless specified otherwise, in the rest of this chapter we use *triangle* instead of *equilateral triangle* for brevity.

Definition 3.1. *A dissection of size m of a rectangle* is a set of m squares of integral side which cover the rectangle and overlap at most on their boundaries. Such a dissection is *$\oplus$-free* if no four squares share a common point, it is *prime* if their sides do not have a common factor.

We denote by $r_d(n)$ the minimal size of a dissection of an $n \times (n+d)$ rectangle.

Definition 3.2. *A dissection of size m of a triangle* is a set of m triangles of integral side which cover the original triangle and overlap at most on their boundaries. Such a dissection is *$\circledast$-free* if no six triangles share a common point, it is *prime* if their sides do not have a common factor.

We denote by $t(n)$ (respectively, $\hat{t}(n)$) the minimal size of a dissection (respectively, prime dissection) of a triangle of side n. Δ_n denotes a triangle of side n.

[1] They showed, for example, that dissections into squares are related to electrical circuits obeying Kirchhoff's laws.

We use terms *dissection* and *tiling* interchangeably. Also by *rectangle* or *triangle dissection* we mean *dissection of a rectangle* or *triangle* respectively. Moreover, for squares and triangles we mean the same by *side* and *size*.

Note that in a triangle dissection only 2, 3, 4 or 6 triangles can share a common point. Therefore $\circledast$-freeness implies that actually no more than 4 triangles meet.

Lemma 3.3. *For a positive integer n and a prime p holds $t(n) \leq \hat{t}(n)$ and $t(p) = \hat{t}(p)$.*

Proof. Trivial. □

3.2 Logarithmic dissection of a rectangle

Let us describe an algorithm to dissect an $n \times (n+3)$ rectangle for $n \geq 2$. Fix the orientation of the rectangle with the shorter side on the left. For convenience, we say that a dissection is *padded* if it has a square of side at least 2 in the upper left corner. Then the algorithm is as follows:

Algorithm 3.4.

(A1) For $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$ dissect into $4, 2, 5, 5, 3, 6, 6, 4, 7$ squares respectively such that the dissection is $\oplus$-free and padded;

(A2) for n of form $4k + z$ with $k \geq 2, z \in \{3, 4, 5, 6\}$, depending on z dissect into 3 or 5 squares and a rectangle of size $2k \times 2(k + 3)$. Then dissect this rectangle with two times larger tiles recursively. Figure 3.1 illustrates the method.

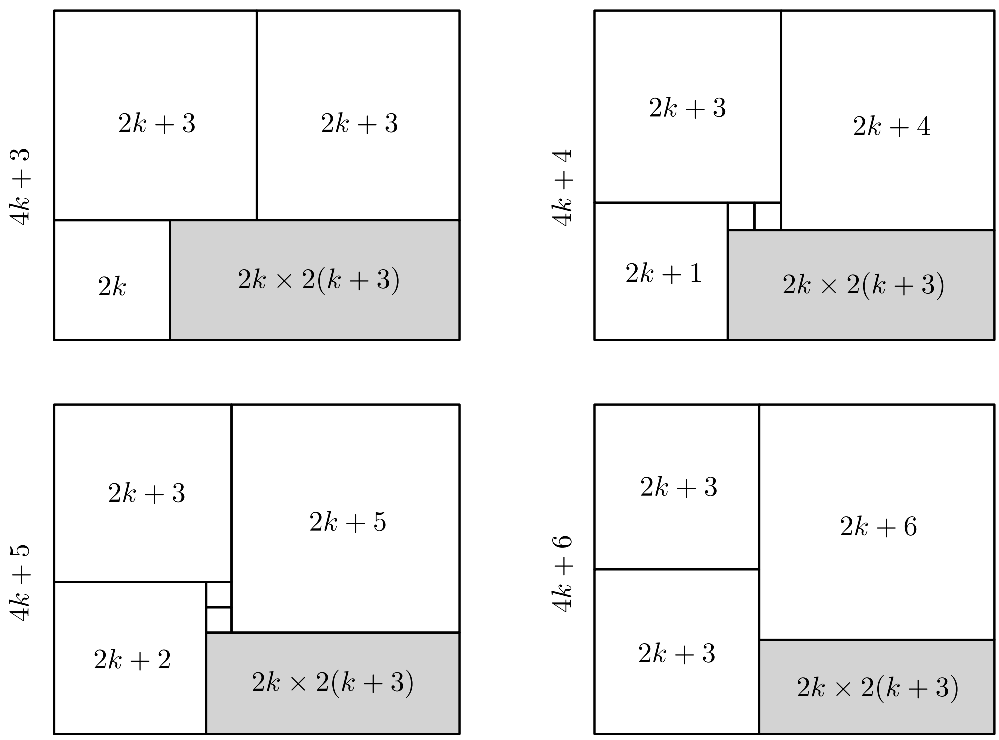


Figure 3.1: Dissecting a rectangle of size $n \times (n + 3)$

Recall that by $r_3(n)$ we denote the smallest size of a $\oplus$-free dissection of an $n \times (n+3)$ rectangle. Note that $r_3(1) = 4$, and let us estimate the remaining values using the algorithm:

Lemma 3.5. *Let $n \geq 2$ be an integer. Then the algorithm results in a padded $\oplus$-free dissection of an $n \times (n+3)$ rectangle. Furthermore $r_3(n) \leq 5\log_4(n) + \frac{3}{2}$.*

Proof. The proof is by induction on n; for n in (A1) the claim holds.

Let $n = 4k + z$ where $k \geq 2, z \in \{3, 4, 5, 6\}$. By (A2) we clearly get a padded rectangle dissection. The inside of the recursively dissected rectangle $2k \times 2(k+3)$ is $\oplus$-free by the induction hypothesis, and the outside is such by design. Therefore the only points where $\oplus$-freeness might be broken lie on its border.

However, the recursive dissection is padded and tiled with two times larger tiles, therefore there is a square of size at least 4 in the upper left corner which covers all possible problematic points.

Finally,

$$r_3(4k+z) \leq 5 + r_3(k) \leq 5 + 5\log_4(k) + \tfrac{3}{2} \leq 5\log_4(4k+z) + \tfrac{3}{2}. \tag{3.1}$$

□

3.3 Logarithmic dissection of a triangle

Lemma 3.6. *Let $5 \leq n = 2k + 3$ be an odd integer not divisible by 3. Then $\hat{t}(n) \leq 2\,r_3(k) + 2$.*

Proof. Consider a triangle of side n. We can cut off triangles of sides k and $(k+3)$ from two of its corners, which leaves us with a parallelogram of sides k and $(k+3)$. By a linear mapping f we can transform it into a $k \times (k+3)$ rectangle (see Figure 3.2), which can be dissected into $r_3(k)$ squares.

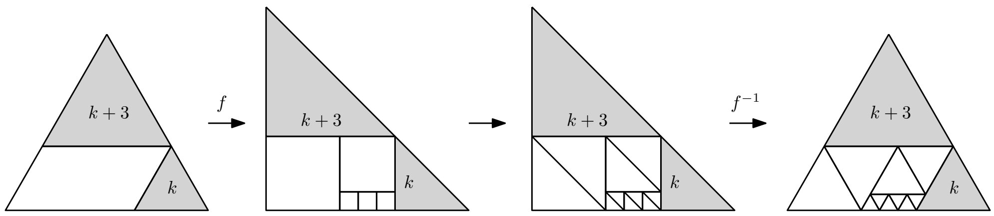


Figure 3.2: Dissecting a triangle using a dissection of a rectangle.

Now, every square in the dissection can be diagonally cut into two right-angled triangles, such that after application of f^{-1} they transform into equilateral triangles. This gives us a dissection of the original triangle into $2\,r_3(k) + 2$ triangles. Moreover $\gcd(k+3, 3) = \gcd(k, 3)$ and $3 \nmid k$, therefore the dissection is prime.

It remains to prove $\circledast$-freeness. Clearly, the condition cannot be violated on the sides of the parallelogram.

Note that all the diagonal cuts have to be parallel, which means that there is at most one of them adjacent to every square corner (the rectangle dissection is $\oplus$-free). Thus we increase the number of shapes incident with every point at most by one and the resulting dissection is $\circledast$-free. □

Corollary 3.7. *Let $n > 1$ be an odd integer not divisible by 3. Then $\hat{t}(n) < 5\log_2(n)$.*

Proof. The conditions imply $n \geq 5$. Now by plugging $k = \frac{n-3}{2}$ into Lemma 3.5:

$$2r_3(\tfrac{n-3}{2}) + 2 \leq 10\log_4(\tfrac{n-3}{2}) + 5 = 5\log_2(n-3) < 5\log_2(n). \tag{3.2}$$

□

Theorem 3.8. *Let $n \geq 2$ be an integer. Then $\hat{t}(n) < 5\log_2(n)$.*

Proof. Set $n = 2^p 3^q r$, where p, q, r are nonnegative integers such that $\gcd(r, 6) = 1$. Use the following algorithm to get a dissection of a triangle of side n:

(B1) If $p > 0$, dissect into 4 triangles of size $n/2$ and repeat for one of them recursively;

(B2) If $q > 0$, dissect into 6 triangles and repeat for one of size $n/3$ recursively;

(B3) If $r = 1$ then finish, otherwise dissect into at most $5\log_2(r)$ triangles as in Corollary 3.7.

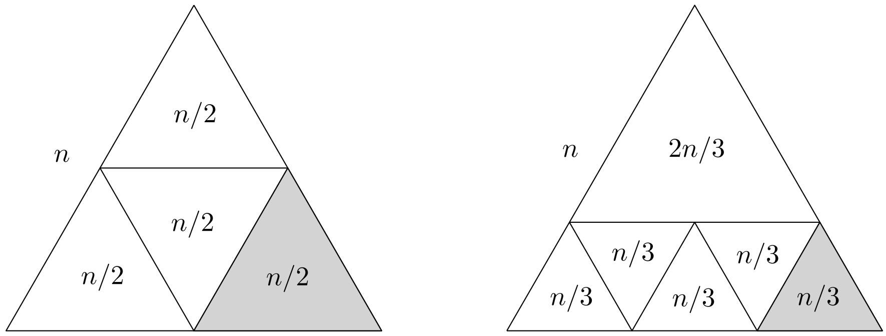


Figure 3.3: Dissecting a triangle of side divisible by 2 and 3.

Steps (B1) and (B2) are illustrated on Figure 3.3. In (B3) we always use a prime dissection, therefore the resulting dissection is also prime. Clearly it is also ⊛-free.

Let us count the number of used triangles. If $r > 1$, then

$$\begin{aligned}\hat{t}(n) &< 3p + 5q + 5\log_2(r)\\ &< 5p\log_2(2) + 5q\log_2(3) + 5\log_2(r)\\ &= 5\log_2(2^p 3^q r).\end{aligned}$$

If $r = 1$, then $\hat{t}(n) \leq 3p + 5q + 1$ and

$$\begin{aligned}3p + 5q + 1 < 5\log_2(2^p 3^q) \quad &\Leftrightarrow\\ 5q + 1 < 2p + 5q\log_2(3) \quad &\Leftrightarrow\\ 1 < 2p + (5\log_2(3) - 5)q,&\end{aligned}$$

which holds every time at least one of p, q is nonzero. □

3.4 Triangle dissections and latin bitrades

There is an interesting connection between triangle dissections and latin bitrades, first noted by Drápal [8]. Let us begin with parametrization of a triangle dissection.

Consider a plane ρ defined by $x+y+z=0$ in 3-dimensional Euclidean space. The planes with one fixed integer coordinate $x=k$, $y=k$, $z=k$ intersect with ρ, and lines of the intersections form a triangular grid.

We identify vector (x_0, y_0, z_0) with the triangle bounded by lines $x = x_0$, $y = y_0$ and $z = z_0$. The number $|x_0 + y_0 + z_0|$ is the size (or side) of the triangle. Degenerate triangles of size 0 are points in the plane ρ.

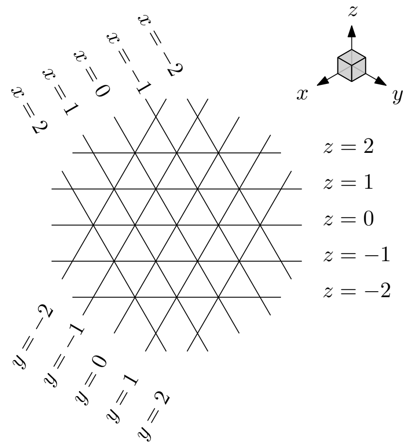


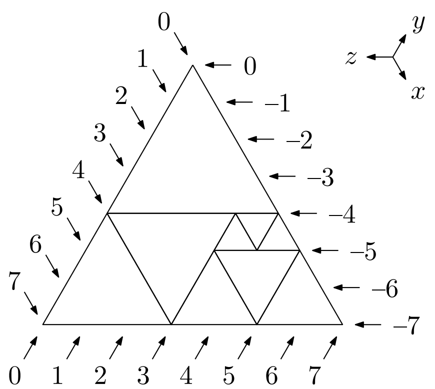
Figure 3.4: Triangular grid in plane $x+y+z=0$.

Now, we can embed a triangle dissection into the grid by choosing its position and orientation.

Definition 3.9. For a dissection of a triangle Δ embedded in the plane $x+y+z = 0$ define sets of vectors T^*, $T^\triangle$ such that

- T^* is the set of vertices of the triangles in the dissection, with the vertices of Δ excluded and Δ itself included;
- $T^\triangle$ is the set of triangles in the dissection.

Example 3.10. See Figure 3.5.



T^*	$T^\triangle$
$(0,0,-7)$	$(0,0,-4)$
$(4,0,-4)$	$(4,0,-7)$
$(1,3,-4)$	$(1,3,-5)$
$(0,4,-4)$	$(0,4,-5)$
$(1,4,-5)$	$(1,4,-4)$
$(2,3,-5)$	$(2,3,-7)$
$(0,5,-5)$	$(0,5,-7)$
$(4,3,-7)$	$(4,3,-4)$
$(2,5,-7)$	$(2,5,-5)$

Figure 3.5: Construction of T^* and $T^\triangle$ from a triangulation.

Lemma 3.11. *Let us have a $\circledast$-free dissection of a triangle Δ. Then $(T^*, T^\triangle)$ is a latin bitrade for any embedding of Δ into the plane. All such latin bitrades are from the same class.*

Proof. $(T^*, T^\triangle)$ is a bitrade straightforwardly from $\circledast$-freeness. Translation of the triangle corresponds to isotopy and rotation to conjugacy, therefore the bitrades are from the same class. □

Recall that Δ_n denotes a triangle of side n.

Lemma 3.12. $\mathrm{gdist}(n) \le t(n)$.

Proof. Let us have a $\circledast$-free dissection of Δ_n into $t(n)$ triangles. We claim that the map

$$h : (x, y, z) \mapsto \big((x \bmod n), (y \bmod n), (z \bmod n)\big) \tag{3.3}$$

is embedding of T^* into $\mathbb{Z}_n$, which would prove the statement.

Since the size of the triangle is n, it follows easily that h is injective. Because $|x + y + z| \in \{0, n\}$ for $(x, y, z) \in T^*$, also $x + y + z \equiv 0 \pmod n$ holds and h is a homotopy into $\mathbb{Z}_p$. □

Lemma 3.13. *If p is a prime factor of n, then* $\mathrm{gdist}(n) \le \mathrm{gdist}(p)$.

Proof. Clearly if H is a subgroup of G, then $\mathrm{gdist}(G) \le \mathrm{gdist}(H)$. The rest follows from Cauchy's theorem. □

Finally, we have proved everything needed for our main result – the proof of Conjecture 1:

Theorem 3.14. *Let $n \ge 2$ and p be the smallest prime factor of n. Then*

$$3\log_3(p) \le \mathrm{gdist}(n) < 5log_2(p). \tag{3.4}$$

Proof. The lower bound is Theorem 2.20. For the upper bound, combine Lemma 3.13, Lemma 3.12 and Theorem 3.8 to get

$$\mathrm{gdist}(n) \le \mathrm{gdist}(p) \le t(p) = \hat{t}(p) < 5\log_2(n). \tag{3.5}$$

□

Corollary 3.15. *Let $n \ge 2$ and p be the smallest prime factor of n. Then*

$$3\log_3(e) \le \frac{\mathrm{gdist}(n)}{\log(p)} < 5\log_2(e). \tag{3.6}$$

3.5 Families of logarithmic dissections

In previous sections we have seen how to use a logarithmic dissection into squares to get a logarithmic dissection into triangles. While the method presented gives the best results that we are aware of, in this section we show how it can be generalized, as it can possibly lead to ideas, which might be helpful in improving the upper bound in Corollary 3.15.

Let us sketch the method first. A convex hexagon, which we call *core*, defines a dissection of a parallelogram into the core, 6 triangles and a smaller parallelogram.

The sizes of the parallelograms depend on the shape of the core, and if chosen appropriately, the smaller parallelogram can be dissected recursively.

In the following, all shapes considered are aligned in a grid formed by unit equilateral triangles, i.e. all lengths are integer and all angles are multiples of $\pi/3$.

For this section we redefine $t(n)$ – **we relax the $\circledast$-freeness condition** on the dissections. We also do not concern ourselves with prime dissections.

Definition 3.16. A convex hexagon H in a unit triangular grid is *a core*. Let us denote its side lengths consecutively by $a_1, \dots, a_6 \in \mathbb{Z}_0^+$. We allow the hexagon to be degenerate, i.e. some of its sides can be zero. From the properties of such a hexagon, the following holds:

$$a_1 + a_2 = a_4 + a_5 =: \alpha \tag{3.7}$$

$$a_2 + a_3 = a_5 + a_6 =: \beta \tag{3.8}$$

$$a_3 + a_4 = a_6 + a_1 =: \gamma \tag{3.9}$$

Therefore the hexagon is uniquely specified by a 4-tuple $(a_1, \alpha, \beta, \gamma)$. We will often identify $H = (a_1, \alpha, \beta, \gamma)$.

Note that not every 4-tuple specifies a valid hexagon. Also note that

$$a_1 + \cdots + a_6 = \alpha + \beta + \gamma \tag{3.10}$$

is perimeter of a core.

Definition 3.17. *A shape* S is a union of finitely many unit triangles in the triangular grid. Let us denote by $t(S)$ the minimal number of triangles needed to dissect the shape S, and let $t_d(n)$ denote $t(S)$ for a parallelogram S of size $n \times (n + d)$.

We kindly ask the reader to extrapolate the formal definition of a dissection from Definition 3.2. As an example, $t(\Delta_n) = t(n)$.

Lemma 3.18. *Let $H = (a_1, \alpha, \beta, \gamma)$ be a core and k a positive integer. Set $n = 2k + a_1 + \alpha + \beta$ and denote by P and P' parallelograms of sizes $n \times (n + \gamma)$ and $k \times (k + \alpha + \beta + \gamma)$. Then there exists a dissection of P into H, P' and six triangles. Therefore*

$$t_\gamma(n) \leq 6 + t(H) + t_{\alpha+\beta+\gamma}(k) \tag{3.11}$$

Proof. See Figure 3.6. □

Now, let us set the variables such that we can use the tiling recursively. First, fix γ and $\alpha + \beta + \gamma$, so that P and P' are always of sizes $n \times (n + \gamma)$ and $k \times (k + \alpha + \beta + \gamma)$. Next, we would like to tile P with tiles of sides which are multiples of an integer d. Therefore reset $k := dk$ and set $\alpha + \beta + \gamma = d\gamma$. In this setting,

$$P \text{ is of size } n \times (n + \gamma), \text{ and}$$
$$P' \text{ is of size } dk \times (dk + d\gamma)$$

with $n = 2dk + (d - 1)\gamma + a_1$.

Finally, if n can be of any integer value (possibly for $n > n_0$ for some n_0), we can use the dissection recursively. Since k can be any integer, it suffices for

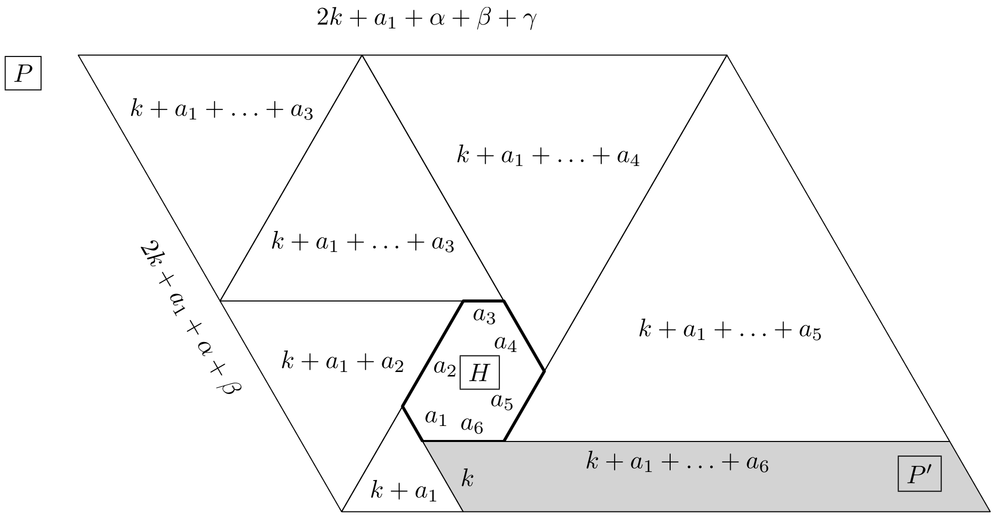


Figure 3.6: Dissection of a parallelogram into convex hexagon, six triangles and a parallelogram.

$(d-1)\gamma + a_1$ to go through all remainders modulo $2d$. The term $(d-1)\gamma$ is a constant, therefore a_1 must be such. Because a_1 is nonnegative and $a_1 \le \gamma = a_1 + a_6$, this gives us the final requirement $2d - 1 \le \gamma$.

Lemma 3.19. *Let $d, \gamma \ge 2$ be integers such that $2d - 1 \le \gamma$. Then there exists n_0 and a constant T such that*

$$t_\gamma(n) \le 6 + T + t_\gamma(k) \tag{3.12}$$

for $n > n_0$ and some $k < n/(2d)$.

Proof. For $a \in [0, 2d)$ denote

$$\begin{aligned} T_a = \min\{t(H) \mid H = (a_1, \alpha, \beta, \gamma) & \text{ is a core,} \\ \alpha + \beta + \gamma & = d\gamma, \\ a_1 \equiv a & \pmod{2d}\} \end{aligned}$$

and define $T = \max\{T_a \mid a \in [0, 2d)\}$. T_a is well-defined for every a, since it can be easily seen that there always exists a core with required parameters.

Set $n_0 = 2d + d\gamma$ and take $n > n_0$. Then there is $a \in [0, 2d)$ such that $n \equiv (d-1)\gamma + a \pmod{2d}$ and a core $H = (a_1, \alpha, \beta, \gamma)$ which we have chosen such that $t(H) = T_a$.

Now, n can be written as $2dk + (d-1)\gamma + a_1$ for a positive integer k. Plugging into Lemma 3.18 we get

$$t_\gamma(n) \le 6 + t(H) + t_{d\gamma}(dk) \le 6 + T + t_\gamma(k). \tag{3.13}$$

Clearly $k < n/(2d)$, which completes the proof. ☐

Corollary 3.20. *Let d, γ be as in Lemma 3.19. Then there exist constants T, C such that*

$$t_\gamma(n) \le (6 + T)\log_{2d}(n) + C. \tag{3.14}$$

Example 3.21. Let us choose $d = 2$ and $\gamma = 3$, they meet the conditiion $2d - 1 \le \gamma$. Consider the cores on Figure 3.7, they have to have perimeter $d\gamma = 6$.

We chose $a_1 \in \{0, 1, 2, 3\}$ as this is the only choice such that $a_1 \leq \gamma$ and a_1 runs through all remainders modulo $2d = 4$. We can set $T = 4$ and from Corollary 3.20 we have

$$t_3(n) \leq 10 \log_4(n) + C = 5 \log_2(n) + C \tag{3.15}$$

for a constant C. The resulting tiling is in fact the tiling from Section 3.2 with every square diagonally cut in halves.

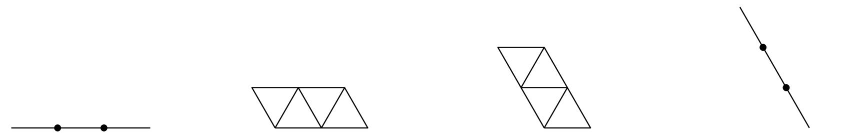


Figure 3.7: Cores for $d = 2$, $\gamma = 3$. We denote the tiling of the corresponding core briefly by $t(a_1, \alpha, \beta, \gamma)$.

It would be desirable to construct a chain of better and better dissections that converge to the expected bound proposed in Chapter 4. However, the following lemmas show that using this method, this is not possible.

Lemma 3.22. *Let $H = (a_1, \alpha, \beta, \gamma)$ be a core of perimeter $d\gamma$ and $a_1 \neq 0 \neq a_6$. Then $t(H) \geq d$.*

Proof. Let us denote by $a_1, \ldots, a_6$ the corresponding sides instead of their lengths. Distance between the pair of parallel lines a_2, a_5 is $\frac{\sqrt{3}}{2}\gamma$, and therefore the largest triangle that can fit in H can be of side γ. Therefore to cover the sides a_2 and a_5 we have to use at least $(a_2 + a_5)/\gamma = (d\gamma - 2\gamma)/\gamma = d - 2$ triangles.

Since $a_1 \neq 0 \neq a_6$, we have to use one more triangle to cover each of these sides. These triangles have to be distinct from those lying on sides a_2 and a_5, hence $t(H) \geq d$.

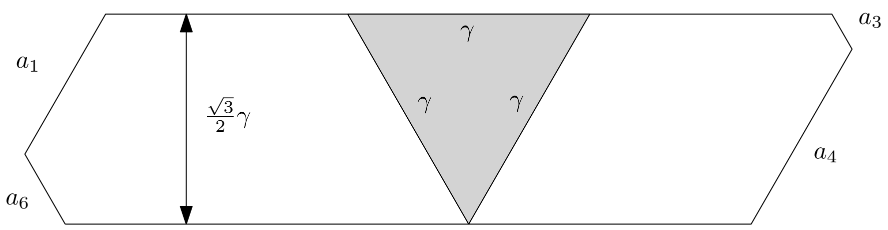


Figure 3.8: Tiling a core of perimeter $d\gamma$.

□

Lemma 3.23. *Let d, γ be as in Lemma 3.19 and $\bar{t}_\gamma(n)$ denote the size of the dissection constructed in the same lemma. Then*

$$\bar{t}_\gamma(n) \geq (6 + d) \log_{2d}(n). \tag{3.16}$$

Proof. Let us have T, d as in the proof of Lemma 3.19. By Lemma 3.22, $T \geq d$. The result now follows by plugging into Corollary 3.20. $\square$

Let us compare with the dissection into $5\log_2(n)$ triangles. Because we compare dissections into asymptotically logarithmically many triangles, we are interested in the ratio over $\log(n)$. Therefore the necessary condition for the method to be better is

$$\frac{\bar{t}_\gamma(n)}{\log(n)} < \frac{5\log_2(n)}{\log(n)} \tag{3.17}$$

$$\Rightarrow \qquad \frac{6+d}{\log(2d)} < \frac{5}{\log(2)} \tag{3.18}$$

$$\Leftrightarrow \qquad 2^{d+1} < d^5. \tag{3.19}$$

The last inequality has integer solutions only for $d \leq 20$. Therefore there can be only finitely many better dissections, as for fixed d the constant in the dissection depends on the value of T, which is a positive integer.

It was not our primary goal to establish that the dissection into $5\log(n)$ triangles is the best in any sense. However, we conjecture that it actually is among all "$\bar{t}_\gamma$" dissections.

4. Refining the bounds

In previous chapters we have established that gdist(p) is asymptotically logarithmic, or, more precisely, that

$$2.73 \approx 3\log_3(e) < \frac{\text{gdist}(p)}{\log(p)} < 5\log_2(e) \approx 7.21 \tag{4.1}$$

for all primes p. The obvious question is – what are the best possible constants in these estimates?

While the question is open, in this chapter we provide evidence which suggests that the following might be true:

Conjecture 4.1. Let P be a real such that $P^3 = P + 1$. Then for primes p:

$$\lim_{p\to\infty} \frac{\text{gdist}(p)}{\log(p)} = 1/\log(P) \approx 3.56. \tag{4.2}$$

Our argument is based on the connection of gdist(p) to dissections of triangles. Recall that the key to establish the upper bound was the fact that gdist$(p) \leq t(p)$. However, it seems that the following conjecture might be true:

Conjecture 4.2. Let p be a prime. Then gdist$(p) = t(p)$.

If that was the case, we could find bounds of gdist(n) by examining triangle dissections only. In this chapter we present computational data of Rosendorf [14] and of our own which support the following:

Conjecture 4.3. Let P be a real such that $P^3 = P + 1$. Then

$$\lim_{n\to\infty} \frac{\hat{t}(n)}{\log(n)} = 1/\log(P). \tag{4.3}$$

Note that Conjectures 4.3 and 4.2 imply Conjecture 4.1.

In some sense, as we will see, it is easier to approach Conjecture 4.3. One of the reasons is that we do not have to restrict ourselves only to primes, but can regard all numbers equally.

4.1 Padovan sequence

Definition 4.4. *Padovan sequence* is a linear recurring sequence $(a_k)_{k\geq 1}$ defined by

$$a_1 = a_2 = a_3 = 1, \quad a_{k+3} = a_{k+1} + a_k \quad \text{for } n \geq 1. \tag{4.4}$$

The first few terms are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, \ldots$

For more information about the sequence see e.g. [13].

Let P, λ_1, λ_2 be roots of the polynomial $x^3 - x - 1$, where P is the only real root. Then we can write explicitly

$$a_k = c_0 P^k + c_1 \lambda_1^k + c_2 \lambda_2^k \tag{4.5}$$

for some complex constants c_0, c_1, c_2. Enumerating the values, we get $|\lambda_1|, |\lambda_2| < 1$ and $c_0 \approx 0.545$ is a real. Therefore $a_k \sim c_0 P^k$, or $\log_P(a_k) \sim k$.

The number $P \approx 1.325$ is called *the plastic constant*. As a side note, along with its mathematical properties, it has also its application in architecture [15].

Definition 4.5. Let n be a positive integer. By $\operatorname{spb}(n)$ we denote an integer such that

$$a_{\operatorname{spb}(n)-1} < n \leq a_{\operatorname{spb}(n)}. \tag{4.6}$$

Note that $a_{\operatorname{spb}(n)}$ is the nearest term in Padovan sequence which is larger or equals to n. Also $\operatorname{spb}(n) \sim \log_P(n)$.

Consider a trapezoid consisting of three unit triangles. In each step, we can attach a triangle to the longest side of the shape to get a pentagon. This way we get a spiral-like tiling. By adding two more triangles to the pentagon we obtain a ⊛-free dissection of a triangle. (Figures 4.1, 4.2.)

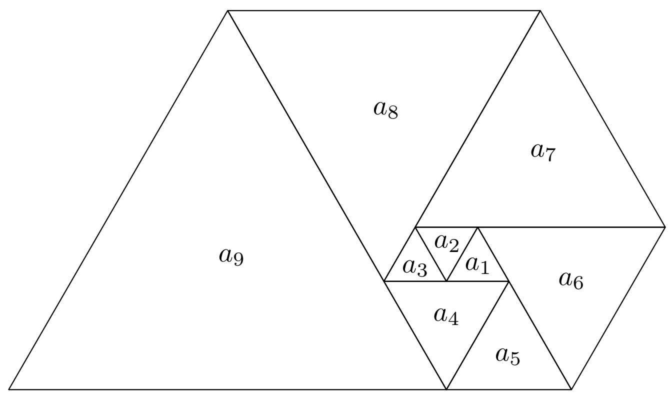


Figure 4.1: Spiral tiling.

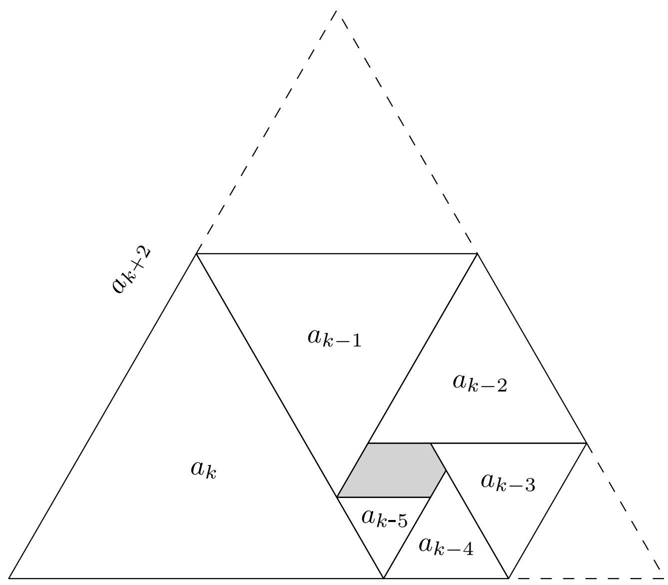


Figure 4.2: Completion of a pentagon into a triangle.

It is easy to derive that the sizes of the triangles are exactly the terms of Padovan sequence. Therefore we can construct a dissection of a triangle of side a_{k+2} into $k+2$ triangles. Since in such a dissection there is always a triangle of side 1, we obtain

$$\hat{t}(a_k) \leq k = \operatorname{spb} a_k \sim \log_P(a_k). \tag{4.7}$$

4.2 Computational results of Rosendorf

Based on Drápal's suggestion, Rosendorf studied in his master's thesis [14] a modification of the spiral tiling, which can be applied to triangles of any size. Consider the following algorithm:

Algorithm 4.6.

- In the beginning, from two corners of the original triangle cut off two triangles to get a pentagon or a parallelogram;
- Then, until the remaining shape is a triangle, cut off a triangle from the current shape to get either a pentagon, a parallelogram, a trapezoid or a triangle.

The algorithm is nondeterministic – if the current shape is not a pentagon, we can choose the placement and the size of the triangle to be cut off. Rosendorf proved that these dissections are exactly those which are $\circledast$-free and do not contain a subset of triangles forming a proper convex hexagon. Following his notation, let us denote such a dissection as *(M6)*, standing for "missing hexagon".

Rosendorf enumerated all minimal (M6) dissections of triangles of side less than 10252. The data show that

$$\hat{t}(n) \leq \mathrm{spb}(n) + 2 \qquad \text{for } n < 10252. \tag{4.8}$$

On the other hand, he also proved that at least $\mathrm{spb}(p)$ triangles are needed in an (M6) dissection of a triangle of prime side p. That, however, is not true when we allow all $\circledast$-free dissections.

4.3 Enumerations of minimal dissections

To support Conjecture 4.3, we generated all triangle dissections up to size 23 and therefore established values of $t(n)$ and $\hat{t}(n)$ for $n \leq 416$. For comparison, in [9] Drápal and Hämäläinen were able to generate dissections up to size 20, which corresponds to $n \leq 160$. They were, however, primarily interested in the number of triangulations of given size and perfect dissections (no two triangles of the same orientation have the same size), not in the values of $t(n)$ and $\hat{t}(n)$.

We use essentially the same algorithm as in [9].[1] As we have shown, from a triangle dissection it is possible to construct a latin bitrade, and thus also a graph of the bitrade. All such graphs can be effectively generated by plantri software [1]. Our algorithm computes all triangle dissections which correspond to a given graph.

@ Potrbujem najprv odseparovat??

Lemma 4.7. *Let us have a dissection of Δ_n and $(T^*, T^\vartriangle)$ the corresponding bitrade. Then*

$$@M_{T^*}x^T = (n, 0, \ldots, 0)^T \tag{4.9}$$

has the only solution $(x_1, \ldots, x_?, \ldots)$.

[1] Although it is not absolutely clear from their paper, our implementation has probably better time complexity.

Proof. @ Spherical ... dimension ... full rank.
@ The solution is solution ☐

Lemma 4.8. *Let D be a prime dissection of Δ_n and $(T^*, T^\vartriangle)$ the corresponding bitrade. Then*

$$@M_{T^*}x^T = (1, 0, \dots, 0)^T \tag{4.10}$$

has the only solution in which $n = \mathrm{lcm}(blah)$.

Proof. @ proof ☐

Corollary 4.9. *The vector x is k-th column of M_T^{-1}.*

The algorithm to generate all triangle dissections is as follows:

Algorithm 4.10.

1. Use plantri to generate planar Eulerian triangulation.
2. Construct corresponding separated bitrade (T, T').
3. Find M_T^{-1}. Every row describes a homotopy of T into $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
4. Check that the homotopy is embedding. Then it must correspond to a triangle dissection.
5. Repeat for T'.

The generated values of $\hat{t}(n)$ are listed in Table. Because we generated all dissections up to size 23, we also know that $\hat{t}(n) = 24$ for those n, for which we know at least one dissection into 24 triangles. The values suggestively keep close to $\mathrm{spb}(n)$. We conjecture the following:

Conjecture 4.11. $\mathrm{spb}(n) - 1 \le \hat{t}(n) \le \mathrm{spb}(n)$.

Conjencture 4.3 then follows by $\mathrm{spb}(n) \sim \log_P(n) = \log(n)/\log(P)$.

Conclusion

Appendix A: Values of $\hat{\mathbf{t}}(\mathbf{n})$

Table A.1 contains sizes of minimal $\circledast$-free prime dissections of Δ_n for $n \leq 465$, compared with corresponding spiral bounds. Our algorithm generated only data with $\hat{t}(n) \leq 23$. The last three rows in the table rely on data by Rosendorf [14]. He found dissections into 24 and 25 triangles for n in those rows.

The table lists all n with $\mathrm{spb}(n) \leq 24$.

n	$\hat{t}(n)$	spb(n)
2	4	4
3	6	6
4	7	7
5	8	8
6-7	9	9
8-9	10	10
10-12	11	11
13-16	12	12
17-21	13	13
22-28	14	14
29-37	15	15
39	15	16
38, 40-49	16	16
50-65	17	17
66-67	17	18
68-86	18	18
87, 90-91, 93	18	19
88-89, 92, 94-114	19	19
115-117, 120, 122, 130	19	20
118-119, 121, 123-129, 131-151	20	20
152-160, 162, 165	20	21
161, 163-164, 166-200	21	21
201-220, 225-226, 235	21	22
221-224, 227-234, 236-265	22	22
266-295, 300-301, 304-306, 315, 319	22	23
296-299, 302-303, 307-314, 316-318, 320-351	23	23
352-382, 384-388, 390-395, 397, 400-401, 404, 408-412, 414, 433	23	24
383, 389, 396, 398-399, 402-403, 405-407, 413, 415-416,	24	24
418-429, 431-432, 435, 437, 441, 444-445, 447, 449, 465	24	24
417, 430, 434, 436, 438-440, 442-443, 446, 448, 450-464	24-25	24

Table A.1: Values of $\hat{t}(n)$ and spb(n) for $n \leq 465$.

Appendix B: Minimal triangle dissections

Figure B.1 shows examples of minimal ⊛-free prime dissections for triangles of sizes up to 37. The full list is attached as Appendix C.

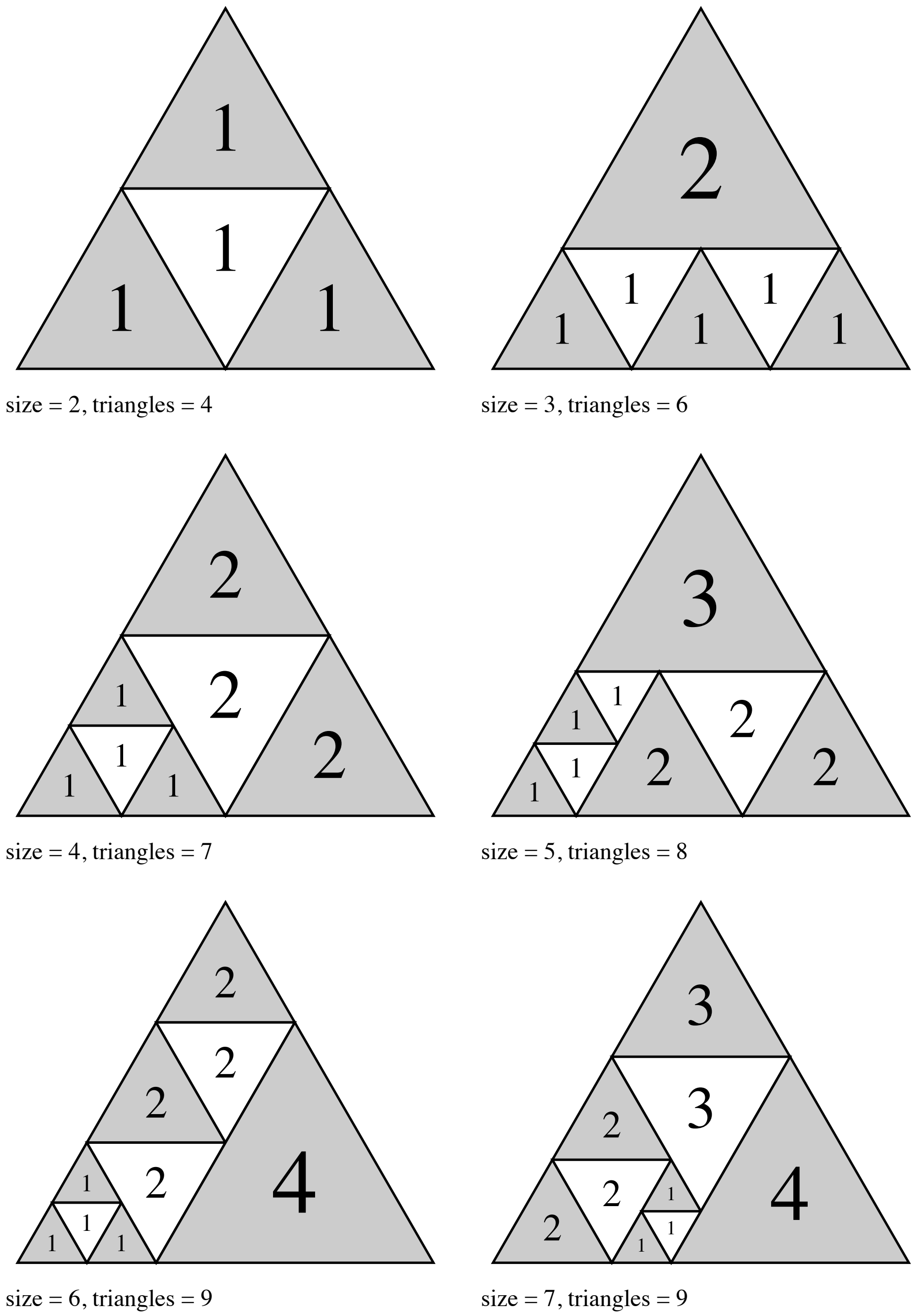
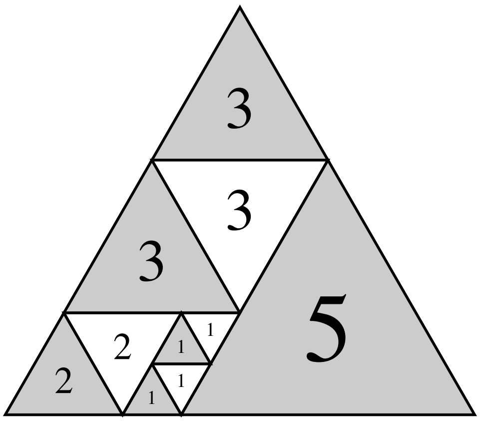
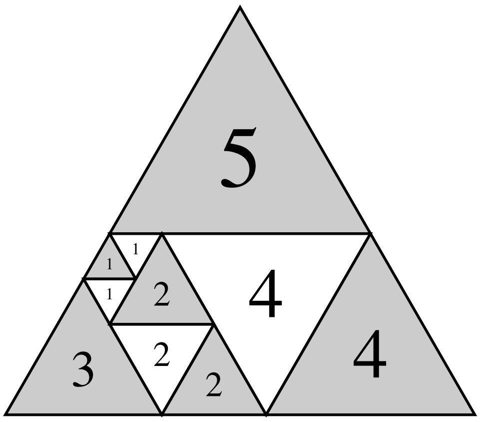


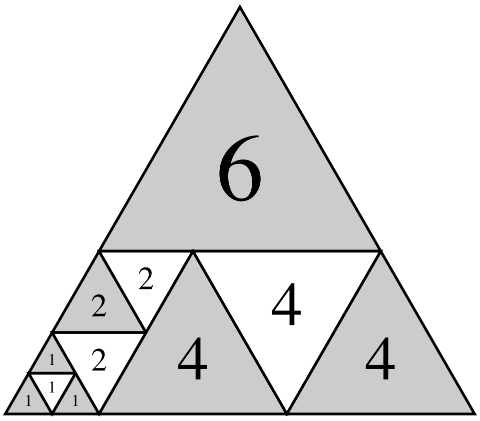
Figure B.1: Minimal triangle dissections.



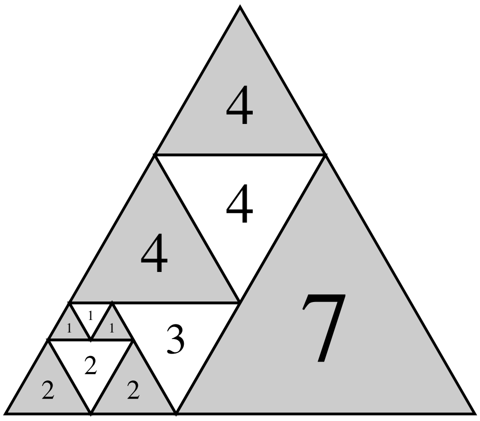
size = 8, triangles = 10



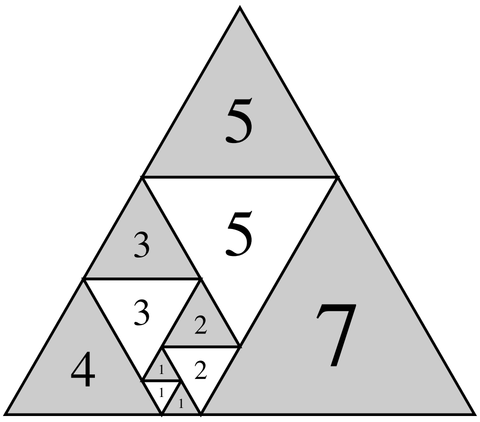
size = 9, triangles = 10



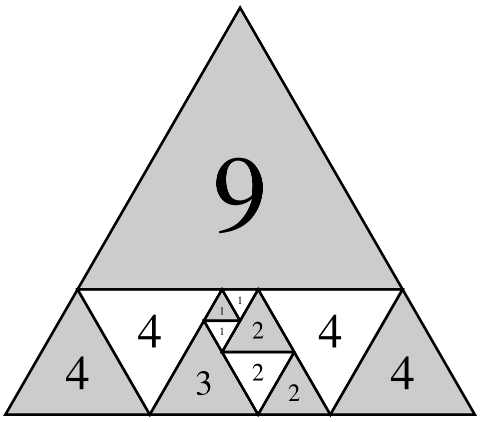
size = 10, triangles = 11



size = 11, triangles = 11



size = 12, triangles = 11



size = 13, triangles = 12

Figure B.1: Minimal triangle dissections.

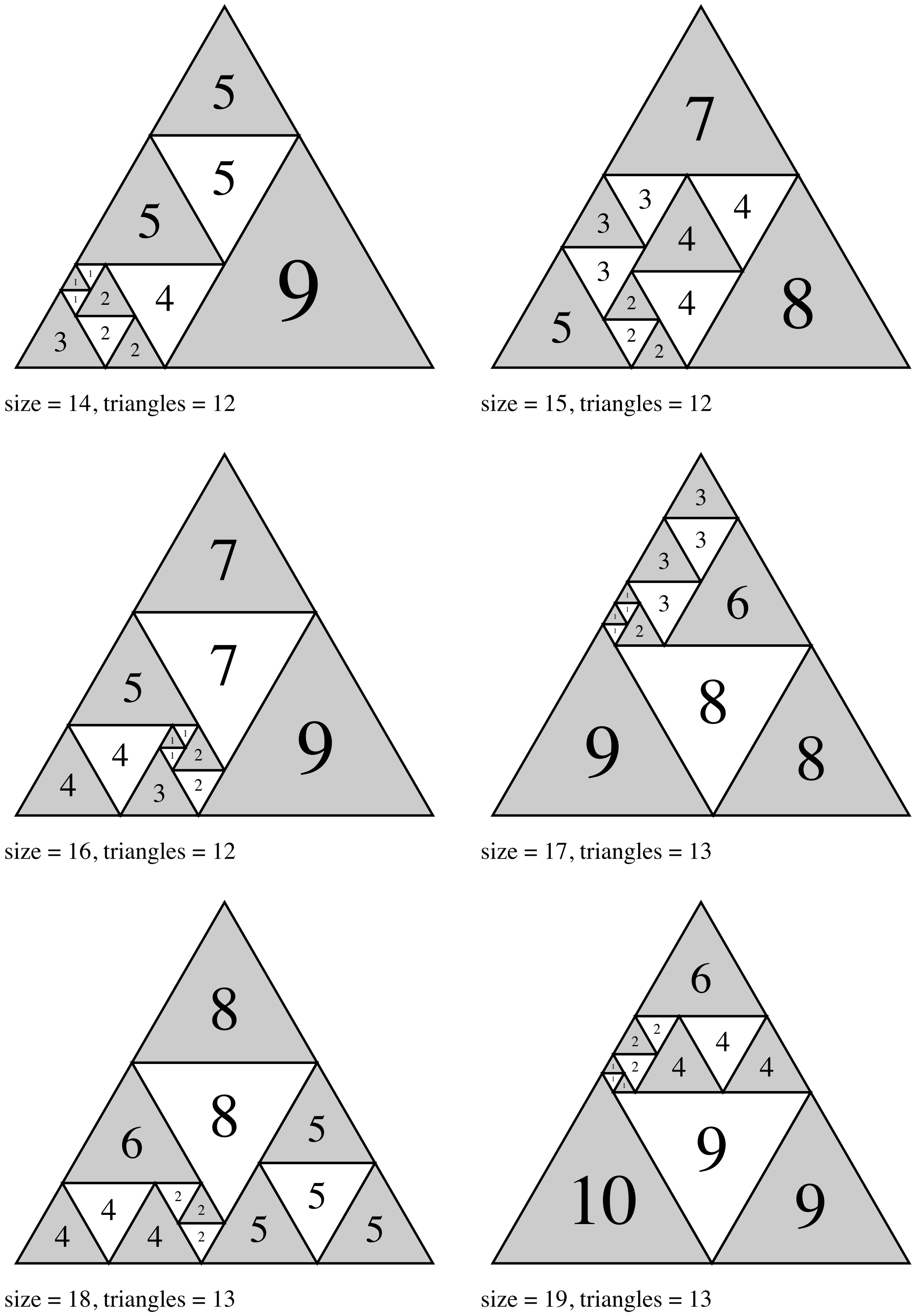
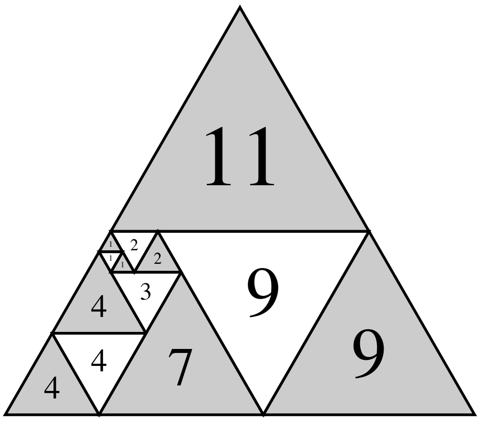
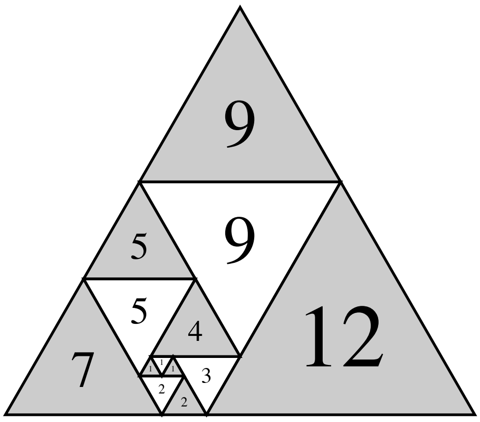


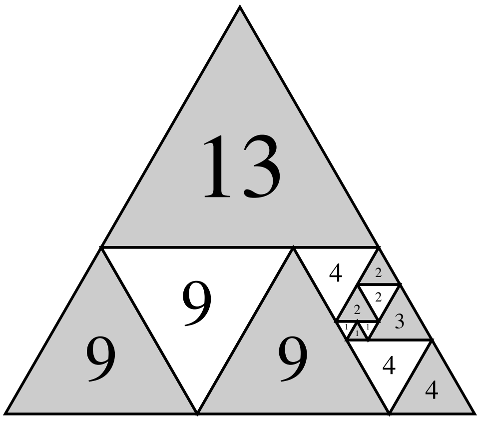
Figure B.1: Minimal triangle dissections.



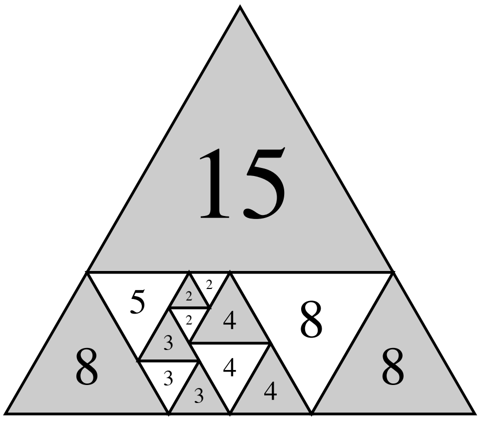
size = 20, triangles = 13



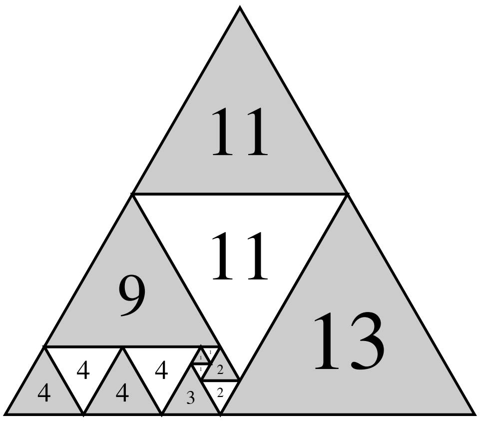
size = 21, triangles = 13



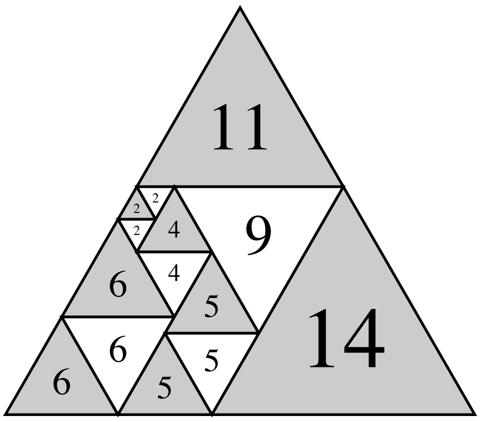
size = 22, triangles = 14



size = 23, triangles = 14

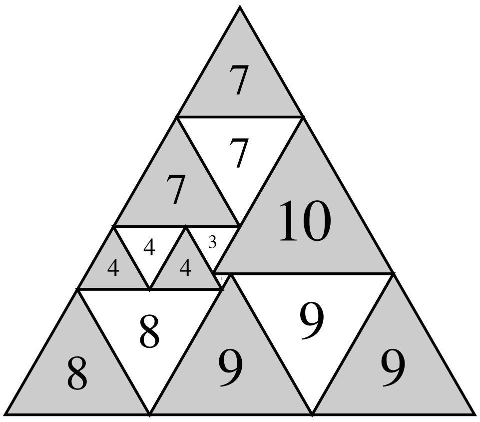


size = 24, triangles = 14

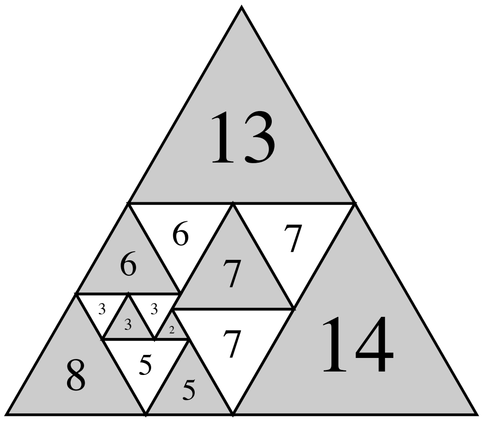


size = 25, triangles = 14

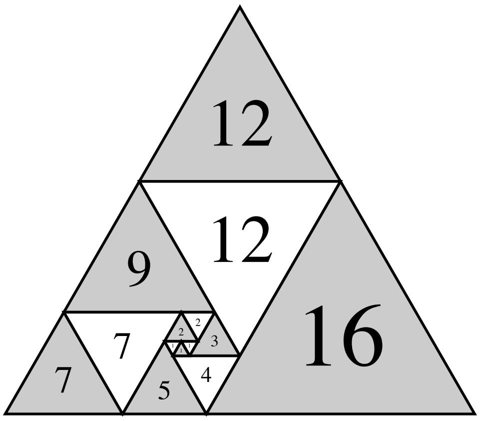
Figure B.1: Minimal triangle dissections.



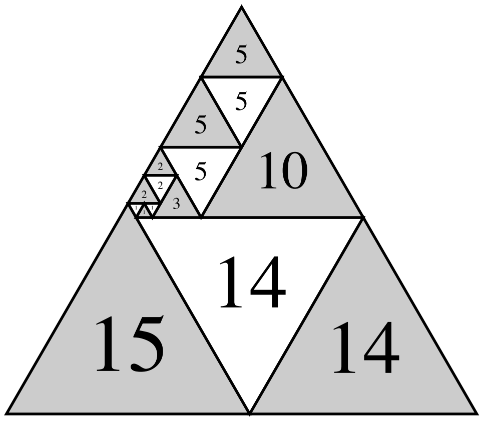
size = 26, triangles = 14



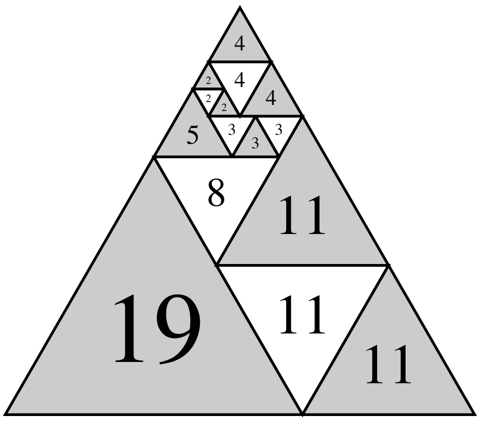
size = 27, triangles = 14



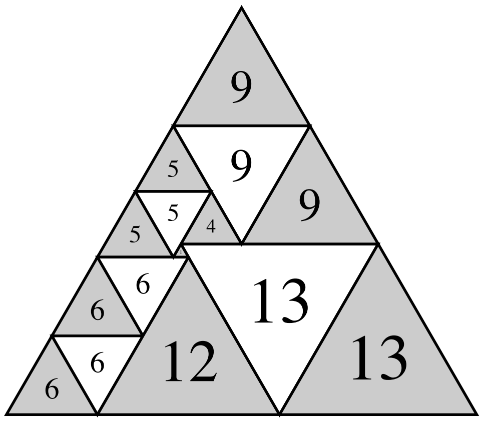
size = 28, triangles = 14



size = 29, triangles = 15



size = 30, triangles = 15



size = 31, triangles = 15

Figure B.1: Minimal triangle dissections.

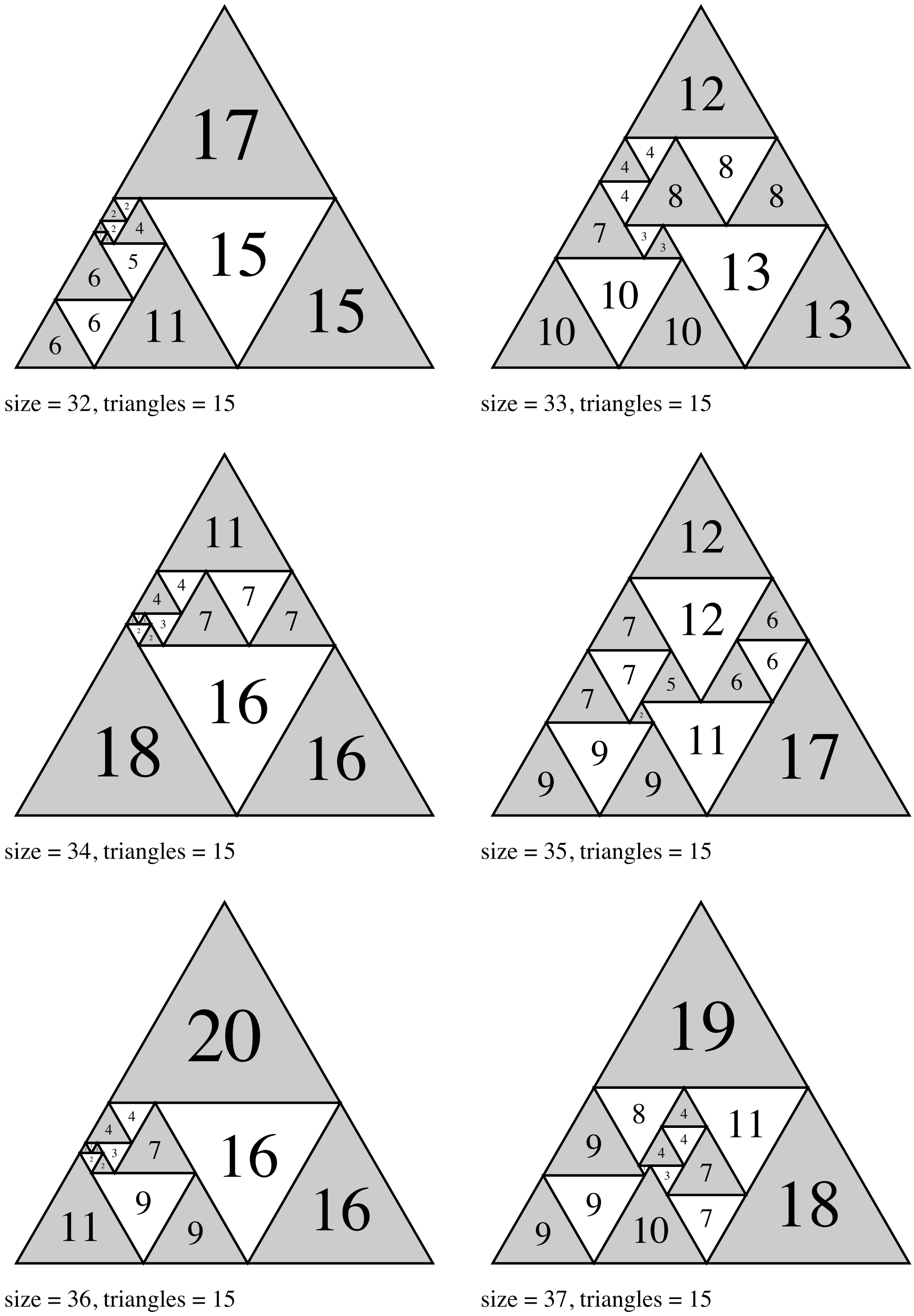


Figure B.1: Minimal triangle dissections.

Appendix C: Program in C++

The attached CD contains a C++ program which was used to generate data presented in this thesis.

Benchmark?

Description?

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