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MASTER THESIS

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Dissections of triangles and distances of groups

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@ Dedication.

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Preface

Introduction

Let us begin with introducing two combinatorial problems:

Problem 1. Consider a table of addition modulo n; it is a latin square $n \times n$. What is the smallest number of cells we have to change in order to get another latin square?

$$\begin{array}{|ccccccc|}\hline \boxed{0} & 1 & 2 & \boxed{3} & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ \boxed{3} & 4 & \boxed{5} & \boxed{6} & 0 & 1 & 2 \\ 4 & 5 & \boxed{6} & \boxed{0} & 1 & 2 & 3 \\ \boxed{5} & 6 & \boxed{0} & 1 & 2 & 3 & 4 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|ccccccc|}\hline \boxed{3} & 1 & 2 & \boxed{0} & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ \boxed{5} & 4 & \boxed{6} & \boxed{3} & 0 & 1 & 2 \\ 4 & 5 & \boxed{0} & \boxed{6} & 1 & 2 & 3 \\ \boxed{0} & 6 & \boxed{5} & 1 & 2 & 3 & 4 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

Figure 1: Caption.

Problem 2. Let Δ be an equilateral triangle of side n. What is the smallest number of integer-sided equilateral triangles, into which Δ can be dissected, such that no six of them share a common point?

Figure

Though it is not obvious at first glance, these two problems are fundamentally related. Both triangle dissections and pairs of latin squares describe a combinatorial structure called *latin bitrade*. This structure will be of central interest throughout this work.

Let us denote by $\mathrm{gdist}(n)$ and $t(n)$ the minimal numbers described in Problems 1 and 2 respectively. Our main result is a solution to the twenty-year-old conjecture of Drápal, Cavenagh and Wanless:

Conjecture 1. There exist positive constants c_1 and c_2 such that

$$c_1 \log(p) \leq \mathrm{gdist}(p) \leq c_2 \log(p) \tag{1}$$

for sufficiently large primes p.

In other words, the conjecture states that $\mathrm{gdist}(n)$ is asymptotically logarithmic, the condition for n to be a prime is only a technical requirement. We also prove the same statement for $t(n)$ in place of $\mathrm{gdist}(n)$.

The lower bound in (1) was already established before. In 1989 Drápal and Kepka [7] proved the inequality for $c_1 = e$, and later Cavenagh [2] found an alternative proof of the same estimate. Yet another proof was given in a paper [3] by Cavenagh and Wanless, but with a slightly smaller constant.

All of these proofs are dealing with another structure which defines a latin bitrade – certain kind of 0-1 matrices. The lower bound is then determined by establishing an upper bound for determinant of such a matrix. In Chapter 1 we present a modified proof which leads to $c_1 = 3\log_3(e)$, the best estimate known so far.

The previously known best upper bound is due to Drápal [6] and states that $\text{gdist}(p) = O(\log^2(p))$. He discovered the connection between latin bitrades and dissections of equilateral triangles, and proved that $\text{gdist}(n) \le t(n)$. However, he was only able to construct triangle dissections with $O(\log^2(n))$ triangles.

In Chapter 2 we prove Conjecture 1 by constructing dissections into logarithmically many triangles. The method used is inspired by Trustrum's method [9] to dissect a square of side n into logarithmically many integer-sided squares. To be more precise, we show how to dissect an $n \times (n+3)$ rectangle into $5\log_4(n) + \frac{3}{2}$ squares and how to adapt the construction to get a dissection of an equilateral triangle of side n into $5\log_2(n)$ triangles.

In fact, our construction belongs to a wider class of logarithmic dissections of triangles. In Section 2.2

Now that the asymptotic behavior of $\text{gdist}(n)$ and $t(n)$ is known, it is natural to ask about the constants in the estimates. Putting our results together, we get

$$2.73 \approx 3\log_3(e) \le \frac{\text{gdist}(p)}{\log(p)} \le \frac{t(p)}{\log(p)} \le 5\log_2(e) \approx 7.21. \tag{2}$$

That, however, do not seem to be the best estimates. The following is conjectured:

Conjecture 2. Let P be a real such that $P^3 = P + 1$. Then

$$\lim_{p\to\infty} \frac{\text{gdist}(p)}{\log(p)} = \lim_{p\to\infty} \frac{t(p)}{\log(p)} = 1/\log(P) \approx 3.56. \tag{3}$$

In Chapter 3 we gathered evidence which supports this claim. We expose a connection between certain triangle dissections and an integer sequence satisfying the recurrence relation $a_{n+3} = a_{n+1} + a_n$. We also describe a computer algorithm with which we generated the exact values of $t(n)$ for $n \le 416$. The data, together with corresponding triangulations, are listed in Appendix A.

1. Latin bitrades

An $n \times n$ table such that every row and column contains every number in $[n]$ exactly once is a well-known combinatorial object called *latin square*. In this chapter we define *latin bitrade*, which can be thought of as an object of differences between two latin squares.

To describe a table of elements formally, we use ordered triples (r, c, s) to represent the fact that the cell in row r and column c contains the symbol s. For that we use the following notation. Let

- $R = \{r_0, \ldots, r_{|R|-1}\}$ denote the set of rows,
- $C = \{c_0, \ldots, c_{|C|-1}\}$ denote the set of columns and
- $S = \{s_0, \ldots, s_{|S|-1}\}$ denote the set of symbols.

We consider only the case when R, C, and S are finite. As an example, a latin square is formally a subset of $R \times C \times S$ with $R = C = S = [n]$. We shall see this in more detail in a moment.

In this chapter we define only necessary notions for our purposes. For a more comprehensive introduction to latin bitrades we refer the reader to a survey by Cavenagh [4].

1.1 Partial latin squares

Definition 1.1. *A partial latin square* L is a subset of ordered triples from $R \times C \times S$, such that for any two given coordinates of $(r, c, s) \in L$ the third one is determined uniquely.

A partial latin square is usually interpreted as a partially filled $|R| \times |C|$ table. The uniqueness condition implies that the table is well defined (there is at most one symbol in every cell), and that no symbol repeats itself within a column or a row.

Definition 1.2. *A latin square* L is a partial latin square such that $R = C = S$ and every cell in the table is filled. Equivalently, for every $a_1, a_2 \in R$ there are $r, c, s \in R$ such that

$$(a_1, a_2, s), (a_1, c, a_2), (r, a_1, a_2) \in L. \tag{1.1}$$

There are two important maps from partial latin squares to partial latin squares: *isotopy* and *conjugacy*.

Definition 1.3. Let $A \subset R_A \times C_A \times S_A$ and $B \subset R_B \times C_B \times S_B$ be partial latin squares. *A homotopy* is a triple of maps $h = (h_R, h_C, h_S)$ such that

$$\begin{aligned} h_R &: R_A \to R_B, \\ h_C &: C_A \to R_C, \\ h_S &: S_A \to R_S \end{aligned}$$

and $(r, c, s) \in A \Rightarrow \big(h_R(r), h_C(c), h_S(s)\big) \in B$. *An isotopy* is a homotopy with homotopic inverse.

Figure isotopic PLSs.

Definition 1.4. Let $A \subset R \times C \times S$ be a partial latin square and σ be a permutation of the 3-element set $\{R, C, S\}$. Then the partial latin square

$$\{(a_{\sigma(R)}, a_{\sigma(C)}, a_{\sigma(S)}) \mid (a_R, a_C, a_S) \in A)\} \tag{1.2}$$

is said to be *conjugated* with A.

Note that there are six conjugacies, each one corresponding to a permutation of $\{R, C, S\}$.

Definition 1.5. Two partial latin squares belong to the same *class* if one can be obtained from the other by composition of conjugacy and isotopy.

1.2 Latin bitrades

Now we can define latin bitrade.

Definition 1.6. *A latin bitrade* is a pair (T, T') of partial latin squares on $R \times C \times S$ which are disjoint and for every $(r, c, s) \in T$ (respectively, T') there exist unique r', c', s' such that

$$(r', c, s), (r, c', s), (r, c, s') \in T' \text{ (respectively, } T). \tag{1.3}$$

Let us call T and T' *latin trades*. Elements in T and T' can be paired with respect to the first two coordinates. Therefore $|T| = |T'|$ and we shall call this number the *size* of the bitrade (or a trade).

From the tabular point of view, a latin bitrade is a pair of partial latin squares such that they occupy the same cells, but the symbols in corresponding rows and columns are permuted. Moreover no symbol is at the same position in both of the tables.

Figure of a latin bitrade of size X.

Note that two latin squares L, L' defined on the same set specify a latin bitrade $(L \setminus L', L' \setminus L)$.

Definition 1.7 (Graph representation of a latin bitrade)**.** Let (T, T') be a latin bitrade. We can associate it with a graph $G = (V, E)$ such that

$$V = T \cup T'$$
$$E = \{(t, t') \mid t \in T, t' \in T' : \ t \text{ and } t' \text{ differ at exactly one coordinate}\}.$$

We shall call it the *graph of a latin bitrade* (T, T').

Clearly, the graph is bipartite with partitions T and T'. It is also 3-regular from the definition of latin bitrade.

A latin bitrade (T, T') is *connected* if there do not exist two non-empty disjoint latin bitrades (T_0, T_0'), (T_1, T_1') such that $T = T_0 \cup T_1$ and $T' = T_0' \cup T_1'$. Equivalently, a bitrade is connected if and only if its graph is connected.

Let $\sigma_r R\sigma_C, \sigma_S : T \to T'$ be such that for $(r, c, s) \in T$

$$\sigma_R(r, c, s) = (r', c, s) \in T', \tag{1.4}$$
$$\sigma_C(r, c, s) = (r, c', s) \in T', \tag{1.5}$$
$$\sigma_S(r, c, s) = (r, c, s') \in T'. \tag{1.6}$$

The definition of the latin bitrade implies that these are bijections.

We can relate $\sigma_R, \sigma_C, \sigma_S$ to the graph of a latin bitrade. They correspond to the edges – a vertex $t \in T$ is connected with $\sigma_R(t), \sigma_C(t)$ and $\sigma_S(t)$ and a vertex $t' \in T'$ is connected with $\sigma_R^{-1}(t'), \sigma_C^{-1}(t')$ and $\sigma_S^{-1}(t')$. Hence the edges of the graph are 3-colorable, let us denote the colors by R, C, S.

For later use, we need the following two simple observations:

Lemma 1.8. *The bitrade is connected if for any two vertices t_0, t_1 it is possible to get one from the other by application of $\sigma_R, \sigma_C, \sigma_S$.*

Now, if we delete edges of one color, every vertex will become of degree two and the graph splits into cycles. Therefore the map which corresponds to traversal of the graph by alternating edges of two colors is a permutation:

Lemma 1.9. *Let $X, Y \in \{R, C, S\}$. Then the mapping $\tau_{XY} : T \cup T' \to T \cup T'$:*

$$\tau_{XY}(t) \mapsto \begin{cases} \sigma_Y(t) & \text{if } t \in T \\ \sigma_X^{-1}(t) & \text{if } t \in T' \end{cases}$$

is a permutation.

@ planar? separated?

@ planar graphs <=> separated planar bitrades

2. Embedding latin trades into groups

@ Only finite groups
@ Credit to Drápal

2.1 Sparse matrices

Definition 2.1. A matrix is *sparse* if its elements are from $\{0, 1\}$ and there is at most one occurrence of 1 in every row.

Lemma 2.2. *Let M_0, M_1 be sparse matrices of the same number of rows such that $M = (M_0, M_1)$ is a square block matrix. Then* $\det(M) \in \{-1, 0, 1\}$.

Proof. Let c_0, c_1 be the number of columns in M_0 and M_1 respectively. The proof is by induction on $n := c_0 + c_1$. The case with $n = 1$ is trivial. Therefore assume $n > 1$.

There are at most two ones in every row of M.

- If there is a row with zeros only, then $\det(M) = 0$.
- If every row contains exactly two ones, then
$$v := (\underbrace{1, \ldots, 1}_{c_0}, \underbrace{-1, \ldots, -1}_{c_1})$$
is such that $Mv^T = 0$. Thus M is singular and $\det(M) = 0$.
- Otherwise there is a row r which contains only a single one in column c. Then expanding the determinant by row r yields
$$\det(M) = \pm \det(\overline{M}_r^c). \tag{2.1}$$
The matrix $\overline{M}_r^c$ consists also from two sparse matrices, thus the result follows by induction.

□

Lemma 2.3. *Let M_0, M_1, M_2 be sparse matrices of sizes $n \times c_0, n \times c_1, n \times c_2$ such that $M = (M_0, M_1, M_2)$ is a square block matrix. Let k_i denote the number of ones in column i. Then*
$$|\det(M)| \le \prod_{i \in [c_0]} k_i. \tag{2.2}$$

Proof. The proof is by induction on c_0.

1. If $c_0 = 0$, then $|\det(M)| \le \prod_{i=0}^{0} k_i = 1$ holds by Lemma 2.2.
2. Otherwise expand by the first column:
$$|\det(M)| \le \sum_{i \in [n]} M_{i,0} \, |\det(\overline{M}_i^1)| \le k_0 \prod_{i \in [1, c_0)} k_i. \tag{2.3}$$
The last inequality holds since there are k_0 nonzero summands and the product majorizes each subdeterminant from induction.

□

@ Technical lemma

Lemma 2.4. *Let $n \in \mathbb{N}$ and $k_0 + \cdots + k_{m-1} = n$ for some $m \in \mathbb{N}$. Then*

$$\prod_{i \in [m]} k_i \leq 3^{n/3}. \tag{2.4}$$

Proof. For $n = 1$ it holds trivially, let us assume $n > 1$. Let $k_0 \leq \cdots \leq k_{m-1}$ be lexicographically smallest such that the maximum is attained. Observe:

- $2 \cdot (k-2) \geq k$ for $k \geq 4$, therefore $k_i \leq 3$;
- $(1+k) > 1 \cdot k$, therefore $k_i > 1$;
- $3 \cdot 3 > 2 \cdot 2 \cdot 2$, therefore there are at most two twos amongst k_i.

Thus there are 3 possibilities:

$$\begin{aligned}
n &= 3k && \Rightarrow k_0 = \cdots = k_{m-1} = 3 && \Rightarrow \prod k_i = 3^{n/3} \\
n &= 3k+2 && \Rightarrow k_0 = 2, k_1 = \cdots = k_{m-1} = 3 && \Rightarrow \prod k_i = 2 \cdot 3^{(n-2)/3} \\
n &= 3k+4 && \Rightarrow k_0 = k_1 = 2, k_2 = \cdots = k_{m-1} = 3 && \Rightarrow \prod k_i = 4 \cdot 3^{(n-4)/3}
\end{aligned}$$

Where the products are over $i \in [m]$. Each of them is less than or equal to $3^{n/3}$, thus we are done. □

Lemma 2.5. *Let M_0, M_1, M_2 be sparse matrices such that $M = (M_0, M_1, M_2)$ is a square block $n \times n$ matrix. Then*

$$|\det(M)| \leq 3^{n/3} \tag{2.5}$$

and thus $3 \log_3(|\det(M)|) \leq n$.

Proof. Let c_0 be the number of columns of M_0 and k_i the number of ones in the column i. Since M_0 is sparse, surely $\sum_{i \in [c_0]} k_i \leq n$. The proof is finished by combining Lemmas 2.3 and 2.4:

$$|\det(M)| \leq \prod_{i \in [c_0]} k_i \leq 3^{n/3}. \tag{2.6}$$

□

2.2 Trade matrices

Recall that

- $R = \{r_0, \ldots, r_{|R|-1}\}$ is the set of rows,
- $C = \{c_0, \ldots, c_{|C|-1}\}$ is the set of columns and
- $S = \{s_0, \ldots, s_{|S|-1}\}$ is the set of symbols.

Without loss of generality assume that these sets are disjoint and let $X = R \cup C \cup S$.

Definition 2.6. Let T be a latin trade. We define a matrix M of size $|T| \times |X|$ such that if we index the rows by elements of T and columns by elements of X, then

$$t = (r, c, s) \in T \Rightarrow \begin{cases} M_{t,r} = M_{t,c} = M_{t,s} = 1, & \\ M_{t,x} = 0 & \text{for } x \in X \setminus \{r, c, s\}. \end{cases} \tag{2.7}$$

We call it the *trade matrix* and denote by M_T.

Lemma 2.7. *Suppose that all elements of X are used in a connected latin trade T. Then the trade matrix M_T has rank $|X| - 2$ over $\mathbb{Q}$.*

Proof. @ Prepisat dokaz! (x,y,z)

The vector

$$v = (\underbrace{a, \ldots, a}_{|R|}, \underbrace{b, \ldots, b}_{|C|}, \underbrace{c, \ldots, c}_{|S|}) \tag{2.8}$$

is surely a solution for $M_T v^T = 0$ if $a + b + c = 0$. It suffices to show that these are the only solutions.

Let v be any solution and $f : X \to \mathbb{Q}$ is defined by

$$v = (f(r_0), \ldots, f(c_0), \ldots, f(s_0), \ldots). \tag{2.9}$$

Let $\bar{s} \in S$ be such that $v_{\bar{s}}$ is maximal. $\bar{s}$ is surely a coordinate of a triple $(\bar{r}, \bar{c}, \bar{s}) \in T$. Without loss of generality assume that $(\bar{r}, \bar{c}, \bar{s}) = (0, 0, 0)$, otherwise we can shift the coordinates. Now we want to prove that $a = b = c = 0$.

Since 0 is the largest element of $f(s)$ and T' occupies the same cells as T, then for any $(r, c, s) \in T \cup T'$

$$f(r) + f(c) \geq 0. \tag{2.10}$$

We prove that for $(r, c, s) \in T \cup T'$ such that $f(t) = (0, 0, 0)$ holds that $f(\sigma_Y(r, c, s)) = (0, 0, 0)$ and $f(\sigma_Y^{-1}(r, c, s)) = (0, 0, 0)$ for $Y \in \{R, C, S\}$.

Let $(r, c, s) \in T'$ such that $f(r, c, s) = (0, 0, 0)$. Then $\sigma_R^{-1}(r, c, s) = (r', c, s) \in T$ and $0 = f(r') + f(c') + f(s') = f(r')$. Therefore $f(\sigma_R^{-1}(r, c, s)) = (0, 0, 0)$. For σ_C and σ_S analogously.

Now let $(r, c, s) \in T$ such that $f(r, c, s) = (0, 0, 0)$. Consider a chain of elements in $T \cup T'$:

$$\begin{array}{cccc}
(r, c, s) \in T & \xrightarrow{\sigma_C} & (r, c^1, s) \in T' & \xrightarrow{\sigma_R^{-1}} \\
(r^1, c^1, s) \in T & \xrightarrow{\sigma_C} & (r^1, c^2, s) \in T' & \xrightarrow{\sigma_R^{-1}} \\
(r^2, c^2, s) \in T & \xrightarrow{\sigma_C} & (r^2, c^3, s) \in T' & \ldots
\end{array}$$

The inequality gives us

$$f(r) + f(c) = 0, f(r) + f(c^1) \geq 0,$$
$$f(r^1) + f(c^1) = 0, f(r^1) + f(c^2) \geq 0, \ldots$$

which yields

$$0 = f(r) \geq f(r^1) \geq f(r^2) \geq \ldots$$
$$0 = f(c) \leq f(c^1) \leq f(c^2) \leq \ldots$$

But it is a cycle, therefore all in the cycle are equal to zero. So $f(\sigma_C(r, c, s)) = (0, 0, 0)$. The result for σ_R can be obtained by reversing the cycle.

For σ_S, consider a cycle by alternating maps σ_C and σ_S^{-1}. We already know that $f(\sigma_S^{-1}\sigma_C(r,c,s)) = \sigma_S^{-1}(0,0,0) = (0,0,0)$. Therefore all elements in the cycle are mappe to $(0,0,0)$. By reversing the cycle we get the result.

$\square$

2.3 Definition of gdist(n)

A Cayley table of a group G can be formally defined as a latin square $L \subset G \times G \times G$ such that $(r,c,s) \in L$ if and only if $rc = s$.

Definition 2.8. A latin trade T *can be embedded* in a group G if there exists an injective homotopy from T to the Cayley talbe of G. Let $\mathrm{gdist}(G)$ denote the size of the smallest trade embeddable in G and let $\mathrm{gdist}(n)$ be the minimum across all groups of order n.

We shall explain this definition in more detail. The latin trade T can be embedded in a group G if we can find an isotopic copy of the partial latin square T inside of the Cayley table of G (see Figure 1).

Without loss of generality now assume that (T, T') is a bitrade on $G \times G \times G$ (i.e. that the isotopy is the identity function). In that case we can change the symbols in the cells of T with the corresponding symbols in T' to get a latin square. This can be also reversed – a latin square $L \subset G \times G \times G$ uniquely defines a latin bitrade $(C_G \setminus L, L \setminus C_G)$ such that $C_G \setminus L$ is embedded in G, where C_G denotes the Cayley table of G.

Therefore $\mathrm{gdist}(G)$ is the smallest number of cells we have to change in a Cayley table of G to get another latin square.

Clearly if H is a subgroup of G, then $\mathrm{gdist}(G) \leq \mathrm{gdist}(H)$. Hence from Cauchy's theorem:

Lemma 2.9. *If p is a prime factor of n, then* $\mathrm{gdist}(n) \leq \mathrm{gdist}(p)$.

3. Dissections of equilateral triangles

The study of dissections was initiated by the paper *The dissection of rectangles into squares* by Brooks, Smith, Stone and Tutte [1]. They answered the question whether it is possible to dissect a square into some number of unequal squares (yes, it is), and developed methods to study such dissections.[1]

Inspired by a puzzle called *Mrs Perkins's quilt* by Dudeney [8], Conway [5] considered the case where the dissecting squares can be equally large. He proposed a question about the minimal number of integer-sided squares needed to dissect a square of side n. Because when n is divisible by d then it is possible to use a scaled up dissection of a square of side d, he considered only dissections where the dissecting squares do not have a common factor.

Conway proved that at least $c\log(n)$ squares are needed. A year later Trustrum [9] proved that $O(\log(n))$ is sufficient, and thus established that the answer is asymptotically logarithmic. However, the best constants in the estimates do not appear to be known.

In this chapter we prove that it is possible to dissect an equilateral triangle of side n into $O(\log(n))$ triangles. To do so, @

@ After publishing the paper [1] on squares, Tutte generalized the theorems for dissections of equilateral triangles in [10].

3.1 Definitions

In the rest of this chapter, for brevity we use *triangle* instead of *equilateral triangle*, unless specified otherwise.

Definition 3.1. *A dissection of size m of a rectangle* is a set of m squares of integral side which cover the rectangle and overlap at most on their boundaries. Such a dissection is *$\oplus$-free* if no four squares share a common point, it is *prime* if their sides do not have a common factor.

We denote by $r_d(n)$ the minimal size of a dissection of an $n \times (n+d)$ rectangle.

Definition 3.2. *A dissection of size m of a triangle* is a set of m triangles of integral side which cover the original triangle and overlap at most on their boundaries. Such a dissection is *$\circledast$-free* if no six triangles share a common point, it is *prime* if their sides do not have a common factor.

We denote by $t(n)$ (respectively, $\hat{t}(n)$) the minimal size of a dissection (respectively, prime dissection) of a triangle of side n.

We use terms *dissection* and *tiling* interchangeably. Similarly, for squares and triangles we mean the same by *side* and *size*.

Note that in a triangle dissection only 2,3,4 or 6 triangles can share a common point. Therefore $\circledast$-freeness implies that actually no more than 4 triangles meet.

@ Lemma? $t(n) \le \hat{t}(n)$, $t(p) = \hat{t}(p)$.

[1] They showed, for example, that dissections into squares are related to electrical circuits obeying Kirchhoff's laws.

3.2 Logarithmic dissection of a rectangle

Let us describe an algorithm to dissect an $n \times (n+3)$ rectangle for $n \geq 2$. Fix the orientation of the rectangle with the shorter side on the left. For convenience, we say that a dissection is *padded* if it has a square of side at least 2 in the upper left corner. Then the algorithm is as follows:

(A1) For $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$ dissect into $4, 2, 5, 5, 3, 6, 6, 4, 7$ squares respectively such that the dissection is $\oplus$-free and padded;

(A2) for n of form $4k + z$ with $k \geq 2, z \in \{3, 4, 5, 6\}$, depending on z dissect into 3 or 5 squares and a rectangle of size $2k \times 2(k + 3)$. Then dissect this rectangle with two times larger tiles recursively. Figure 3.1 illustrates the method.

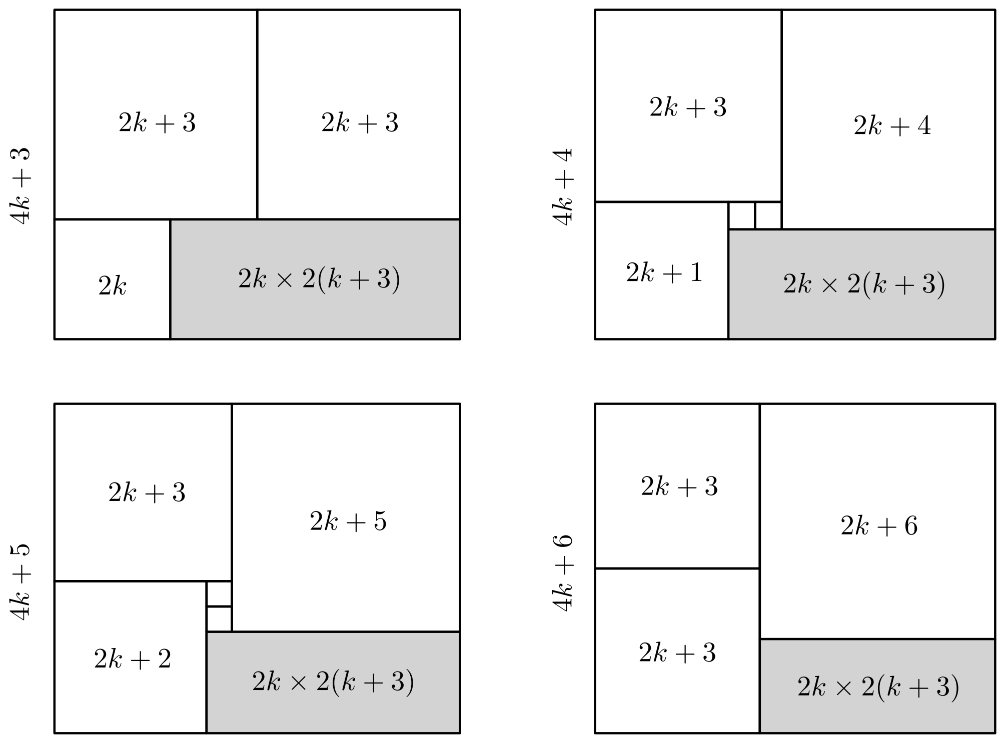


Figure 3.1: Dissecting a rectangle of size $n \times (n + 3)$

Recall that by $r_3(n)$ we denote the smallest size of a $\oplus$-free dissection of an $n \times (n + 3)$ rectangle. Note that $r_3(1) = 4$, and let us estimate the remaining values using the algorithm:

Lemma 3.3. *Let $n \geq 2$ be an integer. Then the algorithm results in a padded $\oplus$-free dissection of an $n \times (n + 3)$ rectangle. Furthermore $r_3(n) \leq 5 \log_4(n) + \frac{3}{2}$.*

Proof. The proof is by induction on n; for n in (A1) the claim holds.

Let $n = 4k + z$ where $k \geq 2, z \in \{3, 4, 5, 6\}$. By (A2) we clearly get a padded rectangle dissection. The inside of the recursively dissected rectangle $2k \times 2(k+3)$ is $\oplus$-free by the induction hypothesis, and the outside is such by design. Therefore the only points where $\oplus$-freeness might be broken lie on its border.

However, the recursive dissection is padded and tiled with two times larger tiles, therefore there is a square of size at least 4 in the upper left corner which covers all possible problematic points.

Finally,

$$r_3(4k+z) \le 5 + r_3(k) \le 5 + 5\log_4(k) + \tfrac{3}{2} \le 5\log_4(4k+z) + \tfrac{3}{2}. \tag{3.1}$$

□

3.3 Logarithmic dissection of a triangle

Lemma 3.4. *Let $5 \le n = 2k+3$ be an odd integer not divisible by 3. Then $\hat{t}(n) \le 2\,r_3(k) + 2$.*

Proof. Consider a triangle of side n. We can cut off triangles of sides k and $(k+3)$ from two of its corners, which leaves us with a parallelogram of sides k and $(k+3)$. By a linear mapping f we can transform it into a $k \times (k+3)$ rectangle (see Figure 3.2), which can be dissected into $r_3(k)$ squares.

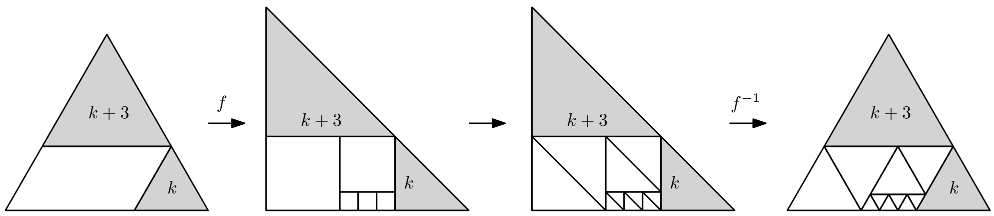


Figure 3.2: Dissecting a triangle using a dissection of a rectangle

Now, every square in the dissection can be diagonally cut into two right-angled triangles, such that after application of f^{-1} they transform into equilateral triangles. This gives us a dissection of the original triangle into $2\,r_3(k) + 2$ triangles. Moreover $\gcd(k+3, 3) = \gcd(k, 3)$ and $3 \nmid k$, therefore the dissection is prime.

It remains to prove $\circledast$-freeness. Clearly, the condition cannot be violated on the sides of the parallelogram.

Note that all the diagonal cuts have to be parallel, which means that there is at most one of them adjacent to every square corner (the rectangle dissection is $\oplus$-free). Thus we increase the number of shapes incident with every point at most by one and the resulting dissection is $\circledast$-free. □

Corollary 3.5. *Let $n > 1$ be an odd integer not divisible by 3. Then $\hat{t}(n) < 5\log_2(n)$.*

Proof. The conditions imply $n \ge 5$. Now by plugging $k = \frac{n-3}{2}$ into Lemma 3.3:

$$2r_3(\tfrac{n-3}{2}) + 2 \le 10\log_4(\tfrac{n-3}{2}) + 5 = 5\log_2(n-3) < 5\log_2(n). \tag{3.2}$$

□

Theorem 3.6. *Let $n \ge 2$ be an integer. Then $\hat{t}(n) < 5\log_2(n)$.*

Proof. Set $n = 2^p 3^q r$, where p, q, r are nonnegative integers such that $\gcd(r, 6) = 1$. Then use the following algorithm to get a dissection of a triangle of side n:

(B1) If $p > 0$, dissect into 4 triangles of size $n/2$ and repeat for one of them recursively;

(B2) If $q > 0$, dissect into 6 triangles and repeat for one of size $n/3$ recursively;

(B3) If $r = 1$ then finish, otherwise dissect into at most $5\log_2(r)$ triangles as in Corollary 3.5.

Steps (B1) and (B2) are illustrated on Figure. In (B3) we always use a prime dissection, therefore the resulting dissection is also prime. Clearly it is also $\circledast$-free.

Let us count the number of used triangles. If $r > 1$, then

$$\begin{aligned}\hat{t}(n) &< 3p + 5q + 5\log_2(r)\\ &< 5p\log_2(2) + 5q\log_2(3) + 5\log_2(r)\\ &= 5\log_2(2^p 3^q r).\end{aligned}$$

If $r = 1$, then $\hat{t}(n) \le 3p + 5q + 1$ and

$$\begin{aligned}3p + 5q + 1 < 5\log_2(2^p 3^q) \quad &\Leftrightarrow\\ 5q + 1 < 2p + 5q\log_2(3) \quad &\Leftrightarrow\\ 1 < 2p + (5\log_2(3) - 5)q,&\end{aligned}$$

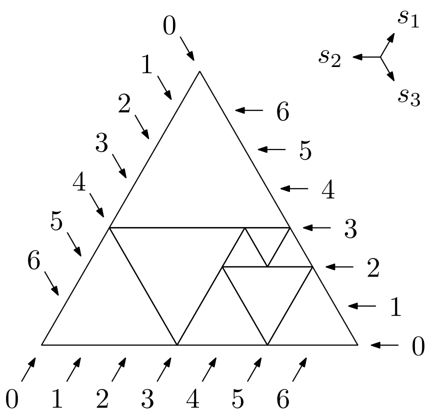
which holds every time at least one of p, q is nonzero. $\square$

3.4 Upper bound for gdist(n)

The following lemma by Drápal [6] reveals the interesting connection between triangle dissections and embeddings of latin bitrades into groups:

Theorem 3.7. $\mathrm{gdist}(n) \le t(n)$.

Proof. Set $G = \mathbb{Z}_n$ and take a triangle of side n with its $\circledast$-free dissection into $t(n)$ triangles. All the boundary lines of the triangles can be divided into three groups s_1, s_2, s_3 depending on which side of the original triangle they are parallel to. Let us label the lines in each group consecutively by $0, 1, \ldots, n-1$ such that the sides of the original triangle are labeled by 0, as illustrated on Figure 3.3.



$\triangle$	0	1	2	3	4	5	6
0	$\boxed{3}$	1	2	$\boxed{0}$	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	$\boxed{5}$	4	$\boxed{6}$	$\boxed{3}$	0	1	2
4	4	5	$\boxed{0}$	$\boxed{6}$	1	2	3
5	$\boxed{0}$	6	$\boxed{5}$	1	2	3	4
6	6	0	1	2	3	4	5

Figure 3.3: Construction of $(G, \triangle)$ from a triangulation for $n = 7$

Consider triples of lines (p, q, r) such that they belong to groups s_1, s_2, s_3 respectively. Then p, q and r meet at one point if and only if $p + q + r = n$, in the other case they determine a triangle bounded by these three lines. For our purposes, let us consider $(0,0,0)$ as a special case and not as a triple determining a triangle.

Computing modulo n, define a new operation $\triangle$ on G by

- $p \triangle q = -r$ if (p,q,r) determines a triangle in our tiling,
- $p \triangle q = p + q$ otherwise.

Since the tiling is $\circledast$-free, for every p, q there exists at most one r such that (p,q,r) forms a triangle. Therefore the operation is well defined and differs from $+$ on exactly $t(n)$ inputs.

If we show that the Cayley table of $(G, \triangle)$ is a latin square, we are finished. That follows almost immediately from the $\circledast$-freeness, a more detailed proof can be found in [6], Theorem 2.5. $\square$

@ The following corollary states Conjecture 1.

Corollary 3.8. *Let $n \geq 2$ and p be the smallest prime factor of n. Then*

$$3\log_3(p) \leq \mathrm{gdist}(n) < 5log_2(p). \tag{3.3}$$

Proof. The lower bound is Theorem. For the upper bound, combine Lemma 2.9, Theorem 3.7 and Theorem 3.6 to get

$$\mathrm{gdist}(n) \leq \mathrm{gdist}(p) \leq t(p) = \hat{t}(p) < 5\log_2(n). \tag{3.4}$$

$\square$

Corollary 3.9. *Let $n \geq 2$ and p be the smallest prime factor of n. Then*

$$3\log_3(e) \leq \frac{\mathrm{gdist}(n)}{\log(p)} < 5\log_2(e). \tag{3.5}$$

3.5 Families of logarithmic dissections

In previous sections we have seen how to use a logarithmic dissection into squares to get a logarithmic dissection into triangles. While the method presented gives the best results that we are aware of, in this section we show how it can be generalized as it can possibly lead to ideas, which might be able to improve the upper bound in Corollary 3.9.

Let us sketch the method first. A convex hexagon, which we call *core*, defines a dissection of a parallelogram into the core, 6 triangles and a smaller parallelogram. The sizes of the parallelograms depend on the shape of the core, and if chosen appropriately, the smaller parallelogram can be dissected recursively.

In the following, all shapes considered are aligned in a grid formed by unit equilateral triangles, i.e. all lengths are integer and all angles are multiples of $\pi/3$.

For this section we redefine $t(n)$ – **we relax the $\circledast$-freeness condition** on the dissections. We also do not concern ourselves with prime dissections.

Definition 3.10. A convex hexagon H in a unit triangular grid is *a core*. Let us denote its sides consecutively by $a_0, \ldots, a_5 \in \mathbb{Z}_0^+$. We allow the hexagon to be degenerate, i.e. some of its sides can be zero. From the properties of such a hexagon, the following holds:

$$a_0 + a_1 = a_3 + a_4 =: \alpha \tag{3.6}$$

$$a_1 + a_2 = a_4 + a_5 =: \beta \tag{3.7}$$

$$a_2 + a_3 = a_5 + a_0 =: \gamma \tag{3.8}$$

Therefore the hexagon is uniquely specified by a 4-tuple $(a_0, \alpha, \beta, \gamma)$. We will often identify $H = (a_0, \alpha, \beta, \gamma)$.

Note that not every 4-tuple specifies a valid hexagon. Also note that

$$a_0 + \cdots + a_5 = \alpha + \beta + \gamma \tag{3.9}$$

is the perimeter of a core.

Definition 3.11. *A shape S is a union of finitely many unit triangles in the grid. Let us denote by $t(S)$ the minimal number of triangles needed to dissect the shape S, and let $t_d(n)$ denote $t(P)$ for a parallelogram P of size $n \times (n + d)$.*

We kindly ask the reader to extrapolate the formal definition of a dissection from Definition 3.2. As an example, if we denote a triangle of side n by Δ_n, then $t(\Delta_n) = t(n)$.

Lemma 3.12. *Let $H = (a_0, \alpha, \beta, \gamma)$ be a core and k a positive integer. Set $n = 2k + a_0 + \alpha + \beta$ and denote by P_0 and P_1 parallelograms of sizes $n \times (n + \gamma)$ and $k \times (k + \alpha + \beta + \gamma)$. Then there exists a dissection of P_0 into H, P_1 and six triangles. Therefore*

$$t_\gamma(n) \le 6 + t(H) + t_{\alpha+\beta+\gamma}(k) \tag{3.10}$$

Proof. See Figure 3.4. □

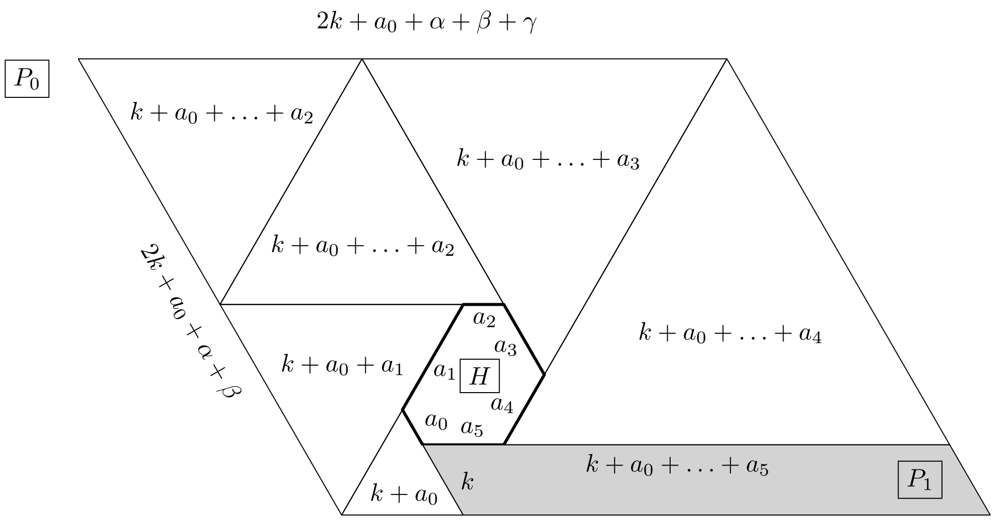


Figure 3.4: Caption

Now, let us set the variables such that we can use the tiling recursively. First, fix γ and $\alpha + \beta + \gamma$, so that P_0 and P_1 are always of sizes $n \times (n + \gamma)$ and $k \times (k + \alpha + \beta + \gamma)$.

Next, we would like to tile P_1 with tiles of sides which are multiples of an integer d. Therefore reset $k := dk$ and set $\alpha + \beta + \gamma = d\gamma$. In this setting,

$$P_0 \text{ is of size } n \times (n + \gamma), \text{ and}$$
$$P_1 \text{ is of size } dk \times (dk + d\gamma)$$

with $n = 2dk + (d-1)\gamma + a_0$.

Finally, if n can be of any integer value (possibly for $n > n_0$ for some n_0), we can use the dissection recursively. Since k can be any integer, it suffices for $(d-1)\gamma + a_0$ to go through all remainders modulo $2d$. The term $(d-1)\gamma$ is a constant, therefore a_0 must be such. Because a_0 is nonnegative and $a_0 \leq \gamma = a_0 + a_1$, this gives us the final requirement $2d - 1 \leq \gamma$.

Lemma 3.13. *Let $d, \gamma \geq 2$ be positive integers such that $2d - 1 \leq \gamma$. Then there exists n_0 and a constant T such that*

$$t_\gamma(n) \leq 6 + T + t_\gamma(k) \tag{3.11}$$

for $n > n_0$ and some $k < n/(2d)$.

Proof. For $a \in [2d]$ denote

$$\begin{aligned} T_a = \min\{t(H) \mid\ & H = (a_0, \alpha, \beta, \gamma) \text{ is a core}, \\ & \alpha + \beta + \gamma = d\gamma, \\ & a_0 \equiv a \pmod{2d}\} \end{aligned}$$

and define $T = \max\{T_a \mid a \in [2d]\}$. T_a is well-defined for every a, since it can be easily seen that there always exists a core with required parameters.

Set $n_0 = 2d + d\gamma$ and take $n > n_0$. Then there is $a \in [2d]$ such that $n \equiv (d-1)\gamma + a \pmod{2d}$ and a core $H = (a_0, \alpha, \beta, \gamma)$ which we have chosen such that $t(H) = T_a$.

Now, n can be written as $2dk + (d-1)\gamma + a_0$ for a positive integer k. Plugging into Lemma 3.12 we get

$$t_\gamma(n) \leq 6 + t(H) + t_{d\gamma}(dk) \leq 6 + T + t_\gamma(k). \tag{3.12}$$

Clearly $k < n/(2d)$, which completes the proof. □

Corollary 3.14. *Let d, γ be as in Lemma 3.13. Then there exist constants T, C such that*

$$t_\gamma(n) \leq (6 + T)\log_{2d}(n) + C. \tag{3.13}$$

Example 3.15. Let us choose $d = 2$ and $\gamma = 3$, they meet the conditiion $2d - 1 \leq \gamma$. Consider the cores on Figure 3.5, they have to have perimeter $d\gamma = 6$.

We chose $a_0 \in \{0, 1, 2, 3\}$ as this is the only choice such that $a_0 \leq \gamma$ and a_0 runs through all remainders modulo $2d = 4$. We can set $T = 4$ and from Corollary 3.14 we have

$$t_3(n) \leq 10\log_4(n) + C = 5\log_2(n) + C \tag{3.14}$$

for a constant C. The resulting tiling is in fact the tiling from Section 3.2 with every square diagonally cut in halves.

It would be desirable to construct a chain of better and better dissections that converge to the expected bound proposed in Chapter 4. However, the following lemmas show that using this method, this is not possible.

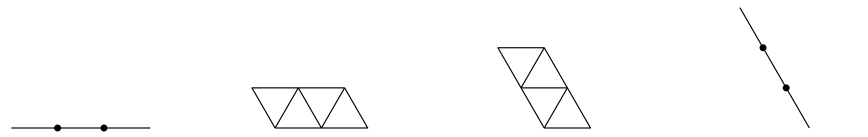


Figure 3.5: Cores for $d = 2$, $\gamma = 3$. We denote the tiling of the corresponding core briefly by $t(a_0, \alpha, \beta, \gamma)$.

Lemma 3.16. *Let $H = (a_0, \alpha, \beta, \gamma)$ be a core of perimeter $d\gamma$ and $a_0 \neq 0 \neq a_5$. Then $t(H) \geq d$.*

Proof. The distance between the pair of parallel lines a_1, a_4 is $\frac{\sqrt{3}}{2}\gamma$, and therefore the largest triangle that can fit in H can be of side γ. Therefore to cover the sides a_1 and a_4 we have to use at least $(a_1 + a_4)/\gamma = (d\gamma - 2\gamma)/\gamma = d - 2$ triangles.

Since $a_0 \neq 0 \neq a_5$, we have to use one more triangle to cover each of these sides. These triangles have to be distinct from those lying on sides a_4 and a_1, hence $t(H) \geq d$.

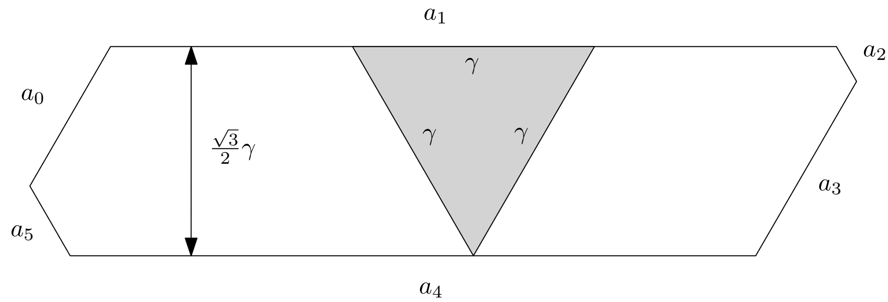


Figure 3.6: Tiling a core of perimeter $d\gamma$.

□

Lemma 3.17. *Let d, γ be as in Lemma 3.13 and $\bar{t}_\gamma(n)$ denote the size of the dissection constructed in the same lemma. Then*

$$\bar{t}_\gamma(n) \geq (6 + d)\log_{2d}(n). \tag{3.15}$$

Proof. By Lemma 3.16, $T \geq d$ in the proof of Lemma 3.13. The result follows by plugging into Corollary 3.14. □

Let us compare with the dissection into $5\log_2(n)$ triangles. For large n, if the method is better, then

$$\frac{\bar{t}_\gamma(n)}{\log(n)} < \frac{5\log_2(n)}{\log(n)} \tag{3.16}$$

$$\Rightarrow \quad \frac{6+d}{\log(2d)} < \frac{5}{\log(2)} \tag{3.17}$$

$$\Leftrightarrow \quad 2^{d+1} < d^5. \tag{3.18}$$

The last inequality has integer solutions only for $d \leq 20$. Therefore there can be only finitely many better dissections, as for fixed d the constant in the dissection depends on the value of T, which is a positive integer.

@ Not our main goal to settle this

@ Conjecture: our is the best

4. Refining the bounds

Blah.

Conclusion

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List of Tables

List of Abbreviations

Attachments