HOMEWORK EXERCISE II

FILTERING AND IDENTIFICATION (SC42025)

Hand in pictures / scans of your hand-written solutions as a pdf for exercise one and two (the warm up will not be graded). For the MATLAB exercise, please export your live script as a pdf (instructions in template). Then, **merge all files as a single pdf and upload them through Brightspace** on **November 29th before 18:00. You are allowed and encouraged to discuss the exercises together but you need to hand in individual solutions.**

Please highlight your final answer!

Warm Up

The diagram in Figure 1 visualises the different distributions that are used in the time update and the measurement update of the Kalman filter. Complete the diagram by filling in the following distributions into the empty boxes: prediction pdf, dynamics, measurement, by following the example with the filtering pdf / the prior.

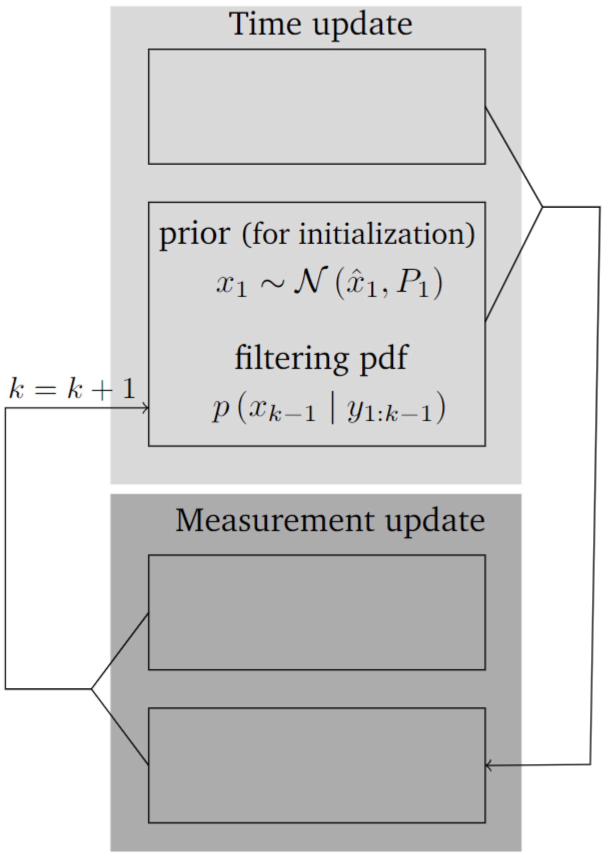

Figure 1: Recursive Updating Scheme of Kalman Filter.

Exercise 1

We are given the following state space model

$$
\begin{aligned}
x_{k+1} &= f(x_k) + w_k, \qquad & w_k &\sim \mathcal{N}(0, Q) \\
y_k &= C(x_k)\theta + v_k, \qquad & v_k &\sim \mathcal{N}(0, R) \\
\theta &\sim \mathcal{N}(0, P_0)
\end{aligned}
\tag{1}
$$

where $f(\cdot)$ is a nonlinear function and $C(x_k)$ is a matrix where the entries depend on x_k. Furthermore, w_k, v_k are zero-mean white noise signals with joint covariance matrix:

$$
E\left[\begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_k^T & v_k^T \end{bmatrix}\right] = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} > 0 \tag{2}
$$

We are interested in estimating both $x_k, k = 1, \ldots, K$ and θ. We are interested in recursively estimating $p(x_{1:k}, \theta \mid y_{1:k})$. The state space model is linear conditioned on $x_{1:K}$.

a) Using the fundamental probability relations found in Section 2 of the Lecture Notes, write $p(x_{1:k}, \theta \mid y_{1:k})$ as a multiplication of two terms. One of these terms needs to be computed using nonlinear estimation, while the other can be computed using linear estimation. Explicitly discuss which term is which and why.

Now let us assume that we have an algorithm that can sample a state trajectory from $p(x_{1:k} \mid y_{1:k})$. A type of algorithm that can do this is called a particle filter. We use the notation $\bar{x}_{1:k}$ to refer to this sampled state trajectory. The fact that it is a sample implies that it is *deterministic*. Using this sampled state trajectory, we can recursively estimate $p(\theta \mid \bar{x}_{1:k}, y_{1:k})$,

$$
p(\theta \mid \bar{x}_{1:k}, y_{1:k}) = \frac{p(y_k \mid \theta, \bar{x}_k) p(\theta \mid \bar{x}_{1:k-1}, y_{1:k-1})}{p(y_k \mid \bar{x}_{1:k}, y_{1:k-1})} \tag{3}
$$

This is the focus of exercise b. In part c, we will subsequently compute the denominator $p(y_k \mid \bar{x}_{1:k}, y_{1:k})$, which is used in the particle filter.

b) Let us assume that from the previous time step we know that $p(\theta \mid \bar{x}_{1:k-1}, y_{1:k-1}) = \mathcal{N}(\hat{\theta}_{k-1}, P_{k-1})$. First we write down the distribution $p(y_k \mid \theta, \bar{x}_k)$ explicitly. Now use Theorems 1 and 2 from the lecture notes to show that $p(\theta \mid \bar{x}_{1:k}, y_{1:k}) = \mathcal{N}(\hat{\theta}_k, P_k)$ with

$$
\hat{\theta}_k = \hat{\theta}_{k-1} + P_{k-1} C(\bar{x}_k)^T \left(C(\bar{x}_k) P_{k-1} C(\bar{x}_k)^T + R\right)^{-1} (y_k - C(\bar{x}_k)\hat{\theta}_{k-1}) \tag{4a}
$$

$$
P_k = P_{k-1} - P_{k-1} C(\bar{x}_k)^T \left(C(\bar{x}_k) P_{k-1} C(\bar{x}_k)^T + R\right)^{-1} C(\bar{x}_k) P_{k-1} \tag{4b}
$$

Also justify why we can use Theorems 1 and 2 to compute $p(\theta \mid \bar{x}_{1:k}, y_{1:k})$.

c) Assume now that we are interested in computing $p(y_k \mid \bar{x}_{1:k}, y_{1:k-1})$. This distribution is Gaussian. Compute its mean $\hat{y}_k$ and its covariance $E[(y_k - \hat{y}_k)(y_k - \hat{y}_k)^T]$.
Hint: Carefully distinguish between random variables z, their estimates $\hat{z}$ and the time steps that you use in the subscripts. The derivation requires you to compute $p(y_k \mid \bar{x}_{1:k}, y_{1:k-1})$ and $p(\theta \mid \bar{x}_{1:k}, y_{1:k-1})$.

Exercise 2

Consider the following model:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \qquad & w_k &\sim \mathcal{N}(0, Q), \\ y_k &= Cx_k + v_k, & v_k &\sim \mathcal{N}(0, R), \end{aligned} \tag{5}$$

Where the measurement are $y \in \mathbb{R}^N$ and the unknown states $x_k \in \mathbb{R}^m$. We assume that we have an unbiased prior on x_k given by $x_k \sim \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$ and that this prior and the process noise w_k are correlated. In other words,

$$E\left[\begin{bmatrix} x_k - \hat{x}_{k|k-1} \\ w_k \\ v_k \end{bmatrix} \begin{bmatrix} (x_k - \hat{x}_{k|k-1})^T & w_k^T & v_k^T \end{bmatrix}\right] = \begin{bmatrix} P_{k|k-1} & S & 0 \\ S^T & Q & 0 \\ 0 & 0 & R \end{bmatrix} > 0, \tag{6}$$

where the zero's are matrices of appropriate size filled with zeros. We are interested in using this information to obtain an unbiased, minimum variance one step ahead prediction of x_{k+1}. In other words, we are interested in deriving the linear estimator

$$\hat{x}_{k+1|k} = \begin{bmatrix} M_1 & M_2 & N \end{bmatrix} \begin{bmatrix} y_k \\ Bu_k \\ \hat{x}_{k|k-1} \end{bmatrix} \tag{7}$$

for the model from Equation 5 and the available prior information.

a) Give the condition on the matrices M_1, M_2, and N such that the estimate $\hat{x}_{k+1|k}$ is unbiased.

b) Derive an explicit expression for the matrices M_1, M_2, and N such that the covariance matrix of the unbiased estimate $\hat{x}_{k+1|k}$ is minimal. Furthermore, give the resulting expressions for the linear estimator and for the minimum variance.
Hint: Use the COS Lemma from slide 8 of the lecture slides of video 3.

MATLAB exercise

See the MATLAB live script `Matlab_2_template.mlx`.