TU Delft

Delft Center for Systems and Control

Homework exercise I

Filtering and Identification (SC42025)

Hand in pictures / scans of your hand-written solutions as a pdf for exercise one and two. For the MATLAB exercise, please export your live script as a pdf (instructions in template). Then, **merge all files and upload them through Brightspace** on **November 22nd 2024** before **18:00**. You are allowed and encouraged to discuss the exercises together but must hand in individual solutions.

Please highlight your final answer!

Exercise 1

For the system $y = F\theta + \epsilon$, with $\epsilon \sim \mathcal{N}(0, I)$ and a full rank matrix F,

a) Please write down the expression of least squares estimate $\hat{\theta}$ and its covariance (represented by F, y).

Based on a), given three measurements of the unknown vector $\theta \in \mathbb{R}^2$:

$$F = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \tag{1}$$

b) Please compute the least-square solution $\hat{\theta}$ and the residual $||y - F\hat{\theta}||_2^2$ (specific values are required) and show the different steps in your derivation.

Assume that, very rarely, it is possible to obtain very accurate measurements. Consider the case that N noisy measurements are made that can be used to identify the unknown parameter vector θ. Then, two *perfect* (without any error) measurements are made. The researchers are glad to include the new *perfect* measurements which results in the following least-square problem:

$$\min_{\theta} \epsilon^T \epsilon, \qquad \begin{bmatrix} y \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} F \\ F_a \\ F_b \end{bmatrix} \theta + \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Here, $\theta \in \mathbb{R}^n$, $F \in \mathbb{R}^{N \times n}$, $N \gg n$, full column rank. Furthermore, $F_a \in \mathbb{R}^{1 \times n}$, $F_b \in \mathbb{R}^{1 \times n}$. Let's partition F_a and F_b as $F_a = [1 \quad 1 \quad f^T]$, $F_b = [2 \quad 1 \quad 3f^T]$, for $f \in \mathbb{R}^{(n-2) \times 1}$. Similarly, F can be partitioned as $F = [f_a \quad f_b \quad M]$ for $M \in \mathbb{R}^{N \times (n-2)}$, $f_a \in \mathbb{R}^{N \times 1}$, $f_b \in \mathbb{R}^{N \times 1}$.

c) Please give the expression for the least-square solution of θ using all the information above. You can assume that all necessary (pseudo-) inverses exist in this exercise.
Hint: you may also want to partition the unknown θ.

Exercise 2

Assume that you are interested in the posterior Gaussian distribution

$$p(\theta \mid y) = \mathcal{N}(\mu_{\theta|y}, P_{\theta|y}), \tag{3}$$

where $\mu_{\theta|y}$ and $P_{\theta|y}$ are unknown, given the measurements

$$y = F\theta + L\epsilon, \quad \epsilon \sim \mathcal{N}(0, I),$$

with F, L known deterministic matrices and $W^{-1} = LL^\top$ square and invertible. Before obtaining any measurements we define the prior distribution of θ as

$$p(\theta) = \mathcal{N}(\mu_\theta, P_\theta),$$

where $P_\theta > 0$ is a known symmetric covariance matrix.

a) Write out the prior $p(\theta)$, likelihood $p(y|\theta)$ and posterior $p(\theta|y)$ in terms of their explicit multivariate normal distributions (with the exponent). For the posterior, write it as a function of the unknown mean and covariance from Equation (3). Then show that $p(\theta|y) \propto p(y|\theta)p(\theta)$ can be written as

$$\exp\left(-\frac{1}{2}\theta^\top P_{\theta|y}^{-1}\theta + \theta^\top P_{\theta|y}^{-1}\mu_{\theta|y} - \frac{1}{2}\mu_{\theta|y}^\top P_{\theta|y}^{-1}\mu_{\theta|y}\right)$$

$$\propto \exp\left(-\frac{1}{2}\theta^\top(F^\top WF + P_\theta^{-1})\theta + \theta^\top(F^\top Wy + P_\theta^{-1}\mu_\theta) - \frac{1}{2}y^\top Wy - \frac{1}{2}\mu_\theta^\top P_\theta^{-1}\mu_\theta\right).$$

b) Assuming that

$$\mathbb{E}[(\theta - \mu_\theta)\epsilon^\top] = 0,$$

please prove (4),(5) by deriving an unbiased estimate $\mu_{\theta|y} = [M \quad N]\begin{bmatrix} y \\ \mu_\theta \end{bmatrix}$ such that $E[(\theta - \mu_{\theta|y})(\theta - \mu_{\theta|y})^T]$ is minimized.

$$\mu_{\theta|y} = \mu_\theta + P_\theta F^T\left(FP_\theta F^T + W^{-1}\right)^{-1}(y - F\mu_\theta) \tag{4}$$

$$P_{\theta|y} = P_\theta - P_\theta F^T\left(FP_\theta F^T + W^{-1}\right)^{-1}FP_\theta \tag{5}$$

Hint: you might need to use the Schur complement (Lemma 2.3 on page 19 of book Verhaegen)

c) Show that

$$\begin{aligned} \mu_{\theta|y} =& \mu_\theta + P_\theta F^T\left(FP_\theta F^T + W^{-1}\right)^{-1}(y - F\mu_\theta) \\ =& \mu_\theta + (P_\theta^{-1} + F^T WF)^{-1}F^T W(y - F\mu_\theta), \end{aligned} \tag{6}$$

using Lemma 2.2 on page 19 of the book by Verhaegen and Verdult.

MATLAB exercise

See the MATLAB live script `Matlab_1_template.mlx`.