1) a)

$$x_1(k+1) = -a_1 x_1(k) + x_2(k) + b_1 u(k)$$

$$x_2(k+1) = -a_2 x_1(k) + \qquad b_2 u(k)$$

$$x_3(k+1) = -(a_1+d_1) x_3(k) + x_4(k) - (a_1+d_1) e(k)$$

$$x_4(k+1) = -(a_2+d_2+a_1d_1) x_3(k) + x_5(k) - (a_2+d_2+a_1d_1) e(k)$$

$$x_5(k+1) = -(a_1d_2+a_2d_1) x_3(k) + x_6(k) - (a_1d_2+a_2d_1) e(k)$$

$$x_6(k+1) = -(a_2d_2) x_3(k) - (a_2d_2) e(k)$$

$y(k) = x_1(k) + x_3(k) + e(k)$, so y is related to (x_1, x_3)

the equations for x_1, x_2 form an observable canonical form for a transfer fn. (ARMAX)

$\Downarrow$

$$A = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow x_1(k) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1} + a_2 q^{-2}} u(k)$$

same way for x_3 : $A = \begin{bmatrix} -(a_1+d_1) & 1 & 0 & 0 \\ -(a_2+d_2+a_1d_1) & 0 & 1 & 0 \\ -(a_1d_2+a_2d_1) & 0 & 0 & 1 \\ -a_2d_2 & 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -(a_1+d_1) \\ -(a_2+d_2+a_1d_1) \\ -a_1d_2+a_2d_1 \\ -(a_2d_2) \end{bmatrix}$

$$F^e(q^{-1}) = \frac{-(a_1+d_1) q^{-1} - (a_2+d_2+a_1d_1) q^{-2} - (a_1d_2+a_2d_1) q^{-3} - (a_2d_2) q^{-4}}{1 + (a_1+d_1) q^{-1} + (a_2+d_2+a_1d_1) q^{-2} + (a_1d_2+a_2d_1) q^{-3} + (a_2d_2) q^{-4}} =$$

$$= \frac{-(a_1q^{-1} + a_2q^{-2} + d_1q^{-1} + d_2q^{-2} + (a_1q^{-1} \cdot b_1q^{-1}) + (a_1q^{-1} \cdot d_2q^{-2} + a_2q^{-2} \cdot d_1q^{-1}) + a_2q^{-2} d_2 q^{-2})}{1 + a_1q^{-1} + a_2q^{-2} + d_1q^{-1} \cdot (1 + a_1q^{-1} + a_2q^{-2}) + d_2q^{-2}(1 + a_1q^{-1} + a_2q^{-2})} =$$

$$= \frac{1 - (1 + a_1q^{-1} + a_2q^{-2})(1 + d_1q^{-1} + d_2q^{-2})}{(1 + a_1q^{-1} + a_2q^{-2})(1 + d_1q^{-1} + d_2q^{-2})} = \frac{1 - A(q^{-1}) \cdot D(q^{-1})}{A(q^{-1}) \cdot D(q^{-1})}$$

$$F^u(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \cdot u(t) \quad F^e = \frac{1 - A(q^{-1})B(q^{-1})}{A(q^{-1})B(q^{-1})} \cdot e(t)$$

$$y(t) = F^u \cdot u(t) + F^e e(t) + e(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) - \frac{A(q^{-1})D(q^{-1})}{A(q^{-1})D(q^{-1})} e(t) +$$

$$+ \frac{1}{A(q^{-1})D(q^{-1})} e(t) + e(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) + \frac{1}{A(q^{-1})D(q^{-1})} e(t)$$

$$\Downarrow$$

$$\underline{A(q^{-1})\, y(t) = B(q^{-1})\, u(t) + \frac{1}{D(q^{-1})} e(t)}, \text{ where}$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2}$$

$$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2}$$

$$D(q^{-1}) = 1 + d_1 q^{-1} + d_2 q^{-2}$$

b, transfer fn predictor for model $y(t) = G(q^{-1})u(t) + H(q^{-1})e(t)$

$$\hat{y} = H^{-1}(q^{-1})\, G(q^{-1})\, u(t) + (1 - H^{-1}(q^{-1}))\, e(t)$$

$$\hat{y}(t) = \underbrace{\frac{D(q^{-1})A(q^{-1})B(q^{-1})}{A(q^{-1})}}_{\hat{G}} u(t) + \underbrace{(1 - D(q^{-1})A(q^{-1}))}_{\hat{H}} e(t)$$

$$\hat{y}(t) = (1 + d_1 q^{-1} + d_2 q^{-2})(b_1 q^{-1} + b_2 q^{-2})\, u(t) + (1 - (1 + d_1 q^{-1} + d_2 q^{-2}) \ldots$$

$\downarrow$

$\hat{G}$ | $\hat{H}$

no poles | no poles

2 zeros for D, B each → 4 zeros | 2 zeros

predictor for state space form:

$$\hat{x}(k+1) = A\hat{x}(k) + B u(k) + K(y_k - C\hat{x}(k) - D u(k))$$

$$\hat{y}(k) = C\hat{x}(k) + D u(k)$$

Q1 a)

$$y(k) = \frac{b_1 q^{-1}}{1+a_1 q^{-1}} u(k) + \frac{1-c_1 q^{-1}}{1+a_1 q^{-1}} e(k)$$

predictor:

$$\hat{y}_1 = H^{-1} G u(k) + (1 - H^{-1}) y(k) =$$

$$= \frac{(1+a_1 q^{-1}) b_1 q^{-1}}{(1+a_1 q^{-1})(1-c_1 q^{-1})} u(k) + \left(1 - \left(\frac{1+a_1 q^{-1}}{1-c_1 q^{-1}}\right)\right) y(k) =$$

$$= \frac{b_1 q^{-1}}{1-c_1 q^{-1}} u(k) + \frac{1 - c_1 q^{-1} - 1 - a_1 q^{-1}}{1-c_1 q^{-1}} y(k) = \frac{b_1 q^{-1}}{1-c_1 q^{-1}} u(k) + \frac{-c_1 q^{-1} - a_1 q^{-1}}{1-c_1 q^{-1}} y(k)$$

$$(1-c_1 q^{-1}) \hat{y}_1 = b_1 q^{-1} u(k) + (-c_1 q^{-1} - a_1 q^{-1}) y(k)$$

$$\hat{y}(k) - c_1 \hat{y}(k-1) = b_1 u(k-1) + (-c_1 - a_1) y(k-1)$$

$$\hat{y}(k) = c_1 \hat{y}(k-1) + b_1 u(k-1) + (-c_1 - a_1) y(k-1)$$

given $\Theta = (b_1 \;\; -(c_1+a_1) \;\; c_1)^T$

$$\hat{y}(k) = (u(k-1) \;\; y(k-1) \;\; \hat{y}(k-1)) \cdot \Theta$$

if sufficiently many measurements $y(k)$ exist, then ϕ has full rank, so the problem is a well defined least-squares b,

$\frac{1}{1-c_1 q^{-1}}$ let $q^{-1} = x$

Taylor expansion: $f(x) = f(x_0) + \frac{f'(x_0)}{1!}\cdot(x-a) + \frac{f''(x_0)}{2!}\cdot(x-a)^2 + \dots$

$f(x) = \frac{1}{1-c_1 x}$ $f'(x) = \frac{c_1}{(1-c_1 x)^2}$ $f''(x) = c_1 \cdot \frac{-2\cdot(1-c_1 x)\cdot(-c_1)}{(1-c_1 x)^4} = \frac{-2c_1^2}{(1-c_1)^3}$

suppose $f^{n}(x) = a \cdot \frac{1}{(1-c_1 x)^{n+1}}$, then

$f^{n+1}(x) = a \cdot \frac{-(1-c_1 x)^{n+1'}}{(1-c_1 x)^{2(n+1)}} = a \cdot \frac{-(n+1)(1-c_1 x)^{n}\cdot(-c_1)}{(1-c_1 x)^{2(n+1)}} =$

$= \frac{a\cdot(n+1)(-c_1)}{(1-c_1 x)^{n+2}} = f^{n}(x) \cdot \frac{-(n+1)(-c_1)}{(1-c_1 x)}$

since f^{0} fits supposition, by induction all subsequent f^{n} will be $\frac{(n!)\cdot(c_1)^n}{(1-c_1 x)^{n+1}}$

centering the series expansion at $x=0$, $\frac{f^{n}(0)}{n!}$ simplifies to

c_1^n, so $m_n = c_1^n$ $m_0 = c_1$ $m_2 = c_1^2$ $m_3 = c_1^3$

if $|c_1 x| < 1$ the series coverges, otherwise diverges to infinity, so $|x| \leq \frac{1}{|c_1|}$

c)

$$\frac{1-c_1 q^{-1}}{1} = \lim_{p\to\infty} \frac{1}{\sum_{n=0}^{p} c_1^n q^{-n}}$$

$$y^{ARX}(k) = \frac{B(q^{-1})}{A(q^{-1})} u(k) + \frac{1}{(1+a_1 q^{-1}) \sum_{n=0}^{p} c_1^n q^{-n}} e(k)$$, so $\lim_{p\to\infty} y_k^{ARX} = y_k$

$$\lim_{p\to\infty} \frac{1}{(1+a_1q^{-1})\sum_{n=0}^{p} c_1^n q^{-n}} = \frac{1}{A(q^{-1})}$$

$$A(q^{-1}) = \lim_{p\to\infty}(1+a_1q^{-1})\sum_{n=0}^{p} c_1^n q^{-n}$$

$$\frac{B(q^{-1})}{A(q^{-1})} = \frac{b_1 q^{-1}}{(1+a_1q^{-1})} \Rightarrow B(q^{-1}) = \lim_{p\to\infty} b_1 q^{-1}\left(\sum_{n=0}^{p} c_1^n q^{-n}\right)$$

$\frac{B(q^{-1})}{A(q^{-1})}$ has p poles and zeros at 0

⇓

p cancellations ($p\to\infty$)

d) $$\lim_{p\to\infty} \hat{y}^{ARX}(k) = A(q^{-1}) \cdot \frac{B(q^{-1})}{A(q^{-1})} u(k) + (1-A(q^{-1}))\cdot y^{ARX}(k)$$

$$\lim_{p\to\infty}\hat{y}^{ARX}(k) = \sum_{n=1}^{p} c_1^{n-1} b_1 u(k-n) + y^{ARX}(k) - \sum_{n=0}^{p} c_1^n (1+a_1q^{-1}) q^{-n} =$$

$$= \sum_{n=1}^{p} c_1^{n-1} b_1 u(k-n) + y^{ARX}(k) - \sum_{n=0}^{p} c_1^n y(k-n) - \sum_{n=1}^{p} c_1^{n-1} a_1 y^{ARX}(k-n) =$$

$$= \sum_{n=1}^{p} c_1^{n-1} b_1 u(k-n) + \sum_{n=1}^{p} c_1^n y^{ARX}(k-n) - \sum_{n=1}^{p} c_1^{n-1} a_1 y^{ARX}(k-n)$$

$$\theta = \begin{pmatrix} b_1 & c_1 & -a_1 & c_1 b_1 & c_1 c_1 & -c_1 a_1 & c_1^2 b_1 & c_1^2 c_1 & -c_1^2 a_1 & \dots \end{pmatrix}^T$$

$$\varphi = \begin{pmatrix} u(k-1) & y^{ARX}(k-1) & y^{ARX}(k-1) \\ \vdots \end{pmatrix}$$

both θ and φ are infinite

e_1

$\lim_{p\to\infty}(y_k - y_k^{ARX})$ the values of $B(q^{-1})$ and $A(q^{-1})$

were derived such, that $\lim_{p\to\infty} y_k^{ARX} = y_k$, what to prove here?

$$E(y_k) = E\left(\frac{b q^{-1}}{1+a_1 q^{-1}} u(k) + \frac{1-c_1 q^{-1}}{1+a_1 q^{-1}} e(k)\right)$$

$$\Downarrow \qquad \Downarrow$$

$$(1+a_1 q^{-1})\, y_k = b q^{-1} u(k) + (1-c_1 q^{-1}) e(k)$$

$$y_k + a_1 y_{k-1} = b u(k-1) + e(k) - c_1 e(k-1)$$

$$y_{k-1} = b u(k-2) + e(k-1) - c_1 e(k-2) - a_1 y_{k-2}$$

$$y_k = b_1 u(k-1) + e(k) - c_1 e(k-1) - a_1 \cdot \left(b_1 u(k-2) + e(k-1) - c_1 e(k-2) - a_1 (b_1 u(k-3) \ldots\right.$$

$$\Downarrow$$

$$y_k = \sum_{n=1}^{p} b_1 \cdot a_1^{n-1} u(k-n) + \sum_{n=0}^{p} a_1^{n} e(k-n) - \sum_{n=1}^{p} a_1^{n-1} e(k-1) - a^{p} y_{k-p}$$

$$\underset{p\to\infty}{E[y_k]} = b_1 \cdot \frac{1}{1-a_1} \cdot \underset{\bar{u}}{E[u(k-n)]} - a^{p} \cdot \underset{0}{E[y_{k-p}]} = \frac{b_1 \bar{u}}{1-a_1}$$