# IV. Constructing the Real Numbers

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11 Oct 2023

Introduce the following notation:

$CS(\mathbb{Q})$: the set of all Cauchy sequences with rational elements.

$x, y, z$: typical symbols for elements of $CS(\mathbb{Q})$. Thus, e.g., $x = (x_1, x_2, \ldots)$.

$R[x]$: the subset of $CS(\mathbb{Q})$ associated with a given $x \in CS(\mathbb{Q})$ as follows:

$$R[x] = \left\{ x' \in CS(\mathbb{Q}) \,:\, \lim_{n\to\infty} |x'_n - x_n| = 0 \right\}.$$

$\Phi$: the function that takes each rational number $q$ into the subset of $CS(\mathbb{Q})$ containing the corresponding constant sequence, i.e.,

$$\Phi(q) = R\left[(q, q, \ldots)\right] \qquad \forall q \in \mathbb{Q}.$$

Let $\mathcal{R} = \{R[x] \,:\, x \in CS(\mathbb{Q})\}$. This will be our model for $\mathbb{R}$. Note that each set $R[x]$ contains *infinitely many Cauchy sequences.*

## A. Equality

For $x, x' \in CS(\mathbb{Q})$, define "$x \sim x'$" to mean $(x'_n - x_n) \to 0$, i.e.,

$$\forall \varepsilon > 0 \ (\varepsilon \in \mathbb{Q}),\ \exists N \in \mathbb{N} \,:\, \forall n \geq N,\ |x'_n - x_n| < \varepsilon.$$

This is an equivalence relation ($x \sim x$; $x \sim y$ iff $y \sim x$; $x \sim y$ and $y \sim z$ imply $x \sim z$). For each $x \in CS(\mathbb{Q})$, $R[x]$ is the equivalence class. This means that $R[x] = \{x' \in CS(\mathbb{Q}) \,:\, x' \sim x\}$, and it implies (see HW05) that, for any $x, y \in CS(\mathbb{Q})$,

$$\begin{aligned} &x \in R[x], \qquad \text{and} \\ &R[x] \cap R[y] \neq \emptyset \iff R[x] = R[y]. \end{aligned}$$

## B. Addition

Extend "+" to $\mathcal{R}$ by first extending it to $CS(\mathbb{Q})$. Let's say that for $x = (x_n)$ and $y = (y_n)$ in $CS(\mathbb{Q})$,

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots).$$

The result belongs to $CS(\mathbb{Q})$ [HW04 Q6(a)].

**Proposition.** *The following binary operation on $\mathcal{R}$ is well-defined:*

$$R[x] + R[y] = R[x + y], \qquad x, y \in CS(\mathbb{Q}).$$

*That is, whenever $R[x'] = R[x]$ and $R[y'] = R[y]$, one has $R[x' + y'] = R[x + y]$.*

*Proof.* For $x, x', y, y'$ as in the setup, we have $x' \in R[x]$ and $y' \in R[y]$, so

$$x'_n - x_n \to 0, \qquad y'_n - y_n \to 0.$$

The sum rule for limits implies that

$$(x'_n + y'_n) - (x_n + y_n) \to 0.$$

This shows that $(x' + y') \in R[x + y]$. But of course $(x' + y') \in R[x' + y']$ too, so $R[x + y] \cap R[x' + y'] \neq \emptyset$. Therefore $R[x + y] = R[x' + y']$. ////

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**Proposition.** $(\mathcal{R}, +)$ *is an Abelian group, i.e., for ever* $x, y, z \in CS(\mathbb{Q})$,

(A1) $R[x] + R[y]$ *is a well-defined element of* $\mathcal{R}$.

(A2) $R[x] + R[y] = R[y] + R[x]$.

(A3) $(R[x] + R[y]) + R[z] = R[x] + (R[y] + R[z])$.

(A4) *The element* $\Phi(0) = R[(0, 0, \ldots)]$ *obeys* $\Phi(0) + R[x] = R[x]$.

(A5) $\mathcal{R}$ *contains a set, denoted "$-R[x]$", such that*

$$R[x] + (-R[x]) = \Phi(0).$$

*Proof.* (A1) This takes some thinking. We did it in the previous proposition.

(A2) For the underlying Cauchy sequences $x, y \in CS(\mathbb{Q})$, $x + y = y + x$ follows from the properties of addition for rational numbers. (Apply these component-by-component.)

(A3) For the underlying Cauchy sequences $x, y, z \in CS(\mathbb{Q})$, $(x+y)+z = (x+y)+z$ follows from the properties of addition for rational numbers. (Apply these component-by-component.)

(A4) For the underlying Cauchy sequences $x \in CS(\mathbb{Q})$ and $(0, 0, 0, \ldots)$, $x + (0, 0, \ldots) = x$ follows from the properties of addition for rational numbers. (Apply these component-by-component.)

(A5) Given any $x \in CS(\mathbb{Q})$, invent the related sequence "$-x$" by defining

$$-x = (-x_1, -x_2, -x_3, \ldots).$$

Clearly $-x \in CS(\mathbb{Q})$, and $x + (-x) = (0, 0, \ldots)$ in $CS(\mathbb{Q})$. Therefore $R[x] + R[-x] = R[(0, 0, \ldots)] = \Phi(0)$. So the set $R[-x]$ (which clearly lies in $\mathcal{R}$) can be used to define $-R[x]$. ////

## C. Multiplication

Extend "$\cdot$" to $\mathcal{R}$ by first extending it to $CS(\mathbb{Q})$. Let's say that if $x = (x_n)$ and $y = (y_n)$ then

$$x \cdot y = (x_1 y_1, x_2 y_2, \ldots).$$

The result belongs to $CS(\mathbb{Q})$ [HW04 Q6(b)].

**Proposition.** *The following binary operation on $\mathcal{R}$ is well-defined:*

$$R[x]\cdot R[y] = R[x\cdot y], \qquad x,y\in CS(\mathbb{Q}).$$

*That is, whenever $R[x'] = R[x]$ and $R[y'] = R[y]$, one has $R[x'\cdot y'] = R[x\cdot y]$.*

*Proof.* For $x, x', y, y'$ as in the setup, we have $x' \in R[x]$ and $y' \in R[y]$, so

$$x'_n - x_n \to 0, \qquad y'_n - y_n \to 0.$$

Let's rearrange

$$x'_n y'_n - x_n y_n = [(x'_n - x_n) + x_n]y'_n - x_n y_n = [x'_n - x_n]y'_n + x_n[y'_n - y_n].$$

Every Cauchy sequence is bounded, so there are numbers $M_{y'}$ and $M_x$ such that

$$|x'_n y'_n - x_n y_n| \le M_{y'}|x'_n - x_n| M_x |y'_n - y_n|, \qquad \forall n\in\mathbb{N}.$$

The right side converges to 0, so the left side does too. This shows that $(x'\cdot y') \in R[x\cdot y]$. Of course $(x'\cdot y')\in R[(x'\cdot y')]$ also, so $R[x\cdot y]\cap R[x'\cdot y'] \ne \emptyset$, and this gives $R[x\cdot y] = R[x'\cdot y']$. ////

**Proposition.** $(\mathcal{R}\setminus\{\Phi(0)\},\cdot)$ *is an Abelian group, i.e., for any $x,y,z\in CS(\mathbb{Q})$ such that $\Phi(0)\notin\{R[x],R[y],R[z]\}$, …*

- (M1) $R[x]\cdot R[y]$ *is a well-defined element of* $\mathcal{R}$.
- (M2) $R[x]\cdot R[y] = R[y]\cdot R[x]$.
- (M3) $(R[x]\cdot R[y])\cdot R[z] = R[x]\cdot(R[y]\cdot R[z])$.
- (M4) *The element $\Phi(1) = R[(1,1,\ldots)]$ obeys $\Phi(1)\ne\Phi(0)$ and $\Phi(1)\cdot R[x] = R[x]$.*
- (M5) $\mathcal{R}$ *contains a set, denoted "$1/R[x]$", such that*

$$R[x]\cdot(1/R[x]) = \Phi(1).$$

*Proof.* (M1) This takes some thinking. We did it in the previous proposition.

- (M2) For the underlying Cauchy sequences $x,y\in CS(\mathbb{Q})$, $x\cdot y = y\cdot x$ follows from the properties of multiplication for rational numbers. (Apply these component-by-component.)
- (M3) For the representative Cauchy sequences $x,y,z\in CS(\mathbb{Q})$, $(x\cdot y)\cdot z = (x\cdot y)\cdot z$ follows from the properties of multiplication for rational numbers. (Apply these component-by-component.)
- (M4) For the representative Cauchy sequences $x\in CS(\mathbb{Q})$ and $(1,1,1,\ldots)$, the identity $x\cdot(1,1,\ldots) = x$ follows from the properties of multiplication for rational numbers. (Apply these component-by-component.)
- (M5) This takes some thinking. It's HW05 Question 6. ////

Often we simplify notation by assuming the "·" between elements of $\mathcal{R}$ written side-by-side.

## D. Distribution

**Proposition.** *For any $x, y, z \in CS(\mathbb{Q})$, we have*

$$R[x] \cdot (R[y] + R[z]) = (R[x] \cdot R[y]) + (R[x] \cdot R[z]).$$

*Proof.* Carefully apply the definitions for $\cdot$ and $+$ to condense both sides:

$$\begin{aligned} &R[x] \cdot (R[y] + R[z]) = R[x] \cdot R[y+z] = R[x \cdot (y+z)], \\ &(R[x] \cdot R[y]) + (R[x] \cdot R[z]) = R[x \cdot y] + R[x \cdot z] = R[x \cdot y + x \cdot z]. \end{aligned}$$

So a Cauchy sequence belonging to the set on the left has terms $x_n(y_n + z_n)$. These terms, taken one by one, equal $(x_n y_n + x_n z_n)$, so this same Cauchy sequence belongs to $R[x \cdot y + x \cdot z]$. Therefore the sets on the left and right in the desired identity have an element (which is a Cauchy sequence) in common. Therefore the sets must be identical. (HW05 Q3) ////

## E. Order

**Definition.** For $x, y$ in $\mathcal{R}$, say $R[x] < R[y]$ if and only if

$$\exists r > 0 \ (r \in \mathbb{Q}), \ \exists N \in \mathbb{N} : \forall n > N, \ x_n + r < y_n.$$

**Proposition.** *The relation "$<$" is well-defined on $\mathcal{R}$. Also, ...*

(i) *For every $x, y \in CS(\mathbb{Q})$, exactly one of the following holds:*

$$R[x] < R[y], \quad R[x] = R[y], \quad R[y] < R[x].$$

(ii) *Whenever $x, y, z \in CS(\mathbb{Q})$, one has*

$$(R[x] < R[y] \text{ and } R[y] < R[z]) \implies R[x] < R[z].$$

*Proof.* HW05, Problems 4–5. ////

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**Proposition OC (Order Components).** *Let $a, b \in CS(\mathbb{Q})$.*

(a) *If $R[a] > R[b]$, then there exists $N \in \mathbb{N}$ such that $R[a] > \Phi(b_N)$.*

(b) *If $R[a] \leq \Phi(b_k)$ for all $k \in \mathbb{N}$, then $R[a] \leq R[b]$.*

*Proof.* It suffices to prove (a), since (b) is the contrapositive.

If $R[a] > R[b]$, then there exist $N_0 \in \mathbb{N}$ and $r > 0$ in $\mathbb{Q}$ such that

$$a_n > b_n + r \qquad \forall n \geq N_0.$$

Also, the sequences $a$ and $b$ are Cauchy, so there exist $N_a, N_b \in \mathbb{N}$ such that

$$\begin{aligned} |a_m - a_n| &< \frac{r}{3} \qquad \forall n \geq N_a, \\ |b_m - b_n| &< \frac{r}{3} \qquad \forall n \geq N_b. \end{aligned}$$

Let $N = \max\{N_0, N_a, N_b\}$: for every $m \geq N$,

$$b_N = b_m + (b_N - b_m) < b_m + \frac{r}{3} < (a_m - r) + \frac{r}{3} = a_m - \frac{2r}{3}.$$

This shows $\Phi(b_N) < R[a]$, as required. ////

**Pause for Reflection.** Note that $\Phi\colon \mathbb{Q} \to \mathcal{R}$ provides a working copy of $\mathbb{Q}$ in the set of equivalence classes of rational-valued Cauchy sequences. For any rational numbers $p, q$, the corresponding constant sequences satisfy (among other things)

$$\begin{aligned} &\Phi(p+q) = \Phi(p) + \Phi(q), && \Phi(-q) = -\Phi(q), \\ &\Phi(pq) = \Phi(p)\Phi(q), && \Phi(1/p) = 1/\Phi(p) \text{ [for } p \neq 0], \\ &p < q \iff \Phi(p) < \Phi(q). \end{aligned}$$

The promise and potential of this whole project is that the set $\mathcal{R}$ contains an equivalence class for *every* Cauchy sequence with rational entries, not only the constant sequences. This is a definite step forward.

## F. Completeness

As suggested above, the embedding $\Phi\colon \mathbb{Q} \to \mathcal{R}$ falls far short of being surjective. Many of the equivalence classes in $\mathcal{R}$ contain no constant sequence at all. Indeed, there are enough additional equivalence classes in $\mathcal{R}\backslash\Phi(\mathbb{Q})$ to make $\mathcal{R}$ order-complete.

**Notation Shift.** Let's start using single Greek letters for the elements of $\mathcal{R}$, with typical correspondences like $\alpha = R[a]$, $\beta = R[b]$, ..., for $a, b, \ldots \in CS(\mathbb{Q})$.

**Theorem.** *Let $A \neq \emptyset$ be a subset of $\mathcal{R}$ with an upper bound, i.e., suppose there exists $\mu \in \mathcal{R}$ such that*

$$\forall \alpha \in A, \qquad \alpha \leq \mu.$$

*Then there exists an element $\beta \in \mathcal{R}$ such that*

(i) *$\forall \alpha \in A$, $\alpha \leq \beta$ (i.e., $\beta$ is an upper bound for $A$), and*

(ii) *for each $\gamma \in \mathcal{R}$ with $\gamma < \beta$, there exists $\alpha \in A$ such that $\alpha > \gamma$ (i.e., $\beta$ is the least upper bound for $A$).*

*Proof.* For each $n$, consider the rational number

$$b_n = \min(S_n), \quad \text{where} \quad S_n = \left\{ \frac{k}{2^n} \,:\, k \in \mathbb{Z}, \alpha \leq \Phi\left(\frac{k}{2^n}\right) \ \forall \alpha \in A \right\}.$$

To see that the minimum on the right makes sense, start with the assumption that $\mu \in \mathcal{R}$ is an upper bound for the set $A$. According to HW05 Q5(b), there must be some rational number $r$ such that $\mu < \Phi(r)$, and this implies that the set of upper bounds for $A$ in $\mathcal{R}$ includes $\Phi(s)$ for every rational $s \geq r$. So indeed the set $S_n$ is not empty. On the other hand, for any $\alpha \in A$ (note $A \neq \emptyset$), HW05 Q5(b) provides a rational number $q$ such that $\Phi(q) < \alpha$. This $q$ provides a lower bound for $S_n$. As a nonempty discrete set with a lower bound, $S_n$ must have a smallest element.

For each $n$, the definition of $b_n$ implies two things:

$$\forall \alpha \in A,\ \alpha \leq \Phi(b_n) \qquad \text{and} \qquad \exists \alpha_n \in A : \alpha_n > \Phi\left(b_n - \frac{1}{2^n}\right). \tag{$*$}$$

Further, since $S_n \subseteq S_{n+1}$ for all $n$, we have

$$b_n \geq b_{n+1} \geq b_n - \frac{1}{2^{n+1}}, \quad \text{so} \quad 0 \leq b_n - b_{n+1} \leq \frac{1}{2^{n+1}}.$$

Iterating this leads to an inequality valid for all $p, n \in \mathbb{N}$:

$$\begin{aligned} 0 \leq b_n - b_{n+p} &= (b_n - b_{n+1}) + (b_{n+1} - b_{n+2}) + \ldots + (b_{n+p-1} - b_{n+p}) \\ &\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^{n+p}} < \frac{1}{2^n}. \end{aligned}$$

It follows that the sequence $b = (b_1, b_2, b_3, \ldots)$ is Cauchy. (Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $2^{-N} < \varepsilon$. Then any $m \geq n \geq N$ make $|b_m - b_n| = b_n - b_m < 2^{-n} < \varepsilon$.)

Let $\beta = R[b]$. Recall from $(*)$ that every $\alpha \in A$ obeys $\alpha \leq \Phi(b_n)$ for every $n \in \mathbb{N}$. According to Propostion OC above, it follows that $\alpha \leq R[b] = \beta$. This shows (i).

To confirm (ii), fix any $\gamma = R[c]$ such that $\gamma < \beta$. There must be some rational $r > 0$ and index $N \in \mathbb{N}$ such that $c_n + r < b_n$ for all $n > N$. By increasing $N$ if necessary, we may assume also that $1/2^N < r/2$. Recalling that the sequence $(b_n)$ is nonincreasing, we have

$$c_n + \frac{r}{2} = c_n + r - \frac{r}{2} < b_n - \frac{r}{2} < b_N - \frac{1}{2^N} \qquad \forall n > N.$$

This element-by-element inequality gives $R[c] < \Phi\left(b_N - \dfrac{1}{2^N}\right)$. The second assertion in $(*)$ above (use $n = N$ and transitivity of $>$) now establishes (ii). ////

Our construction is finished. The set $\mathcal{R}$ with the operations of $+$ and $\cdot$ and $<$ is an ordered field, in the sense defined by Rudin in Definitions 1.5–1.8 and 1.12. Further, it is order-complete, in the sense of Rudin's Definition 1.10. This proves Rudin's Theorem 1.19.

## G. Consequences

**Theorem (Archimedes).** *The set $\mathbb{N}$ has no upper bound in $\mathbb{R}$. In particular, for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.*

*Proof.* Suppose, seeking a contradiction, that $\mathbb{N}$ has an upper bound. Let $\alpha \stackrel{\text{def}}{=} \sup(\mathbb{N})$. Then $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that $n > \alpha - \frac{1}{2}$. But then $n + 1$ is an integer satisfying $n + 1 > \alpha + \frac{1}{2} > \alpha$, and this contradicts the defining property of $\alpha$. ////

*Remark.* The Archimedean property of $\mathbb{Q}$ was never in doubt. But not every ordered field containing a work-alike copy of $\mathbb{Q}$ inherits the property. See Rechnitzer's M319 notes for a reasonably simple example in which the elements are rational functions with rational coefficients. (`https://arechnitzer.gitlab.io/m319text/`)