# MATH 443 Homework 4

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**Question 1.** Suppose by way of contradiction that the root of a DRT $T$ had in-degree nonzero. Let $r$ be the root of $T$ and $v$ be a vertex such that $vr \in E(G)$. Then $vr$ is a path from $v$ to $r$ in the underlying tree of $T$. Since $T$ is a tree it is the only path between $r$ and $v$, so there doesn't exist a directed path from $r$ to $v$, which is a contradiction of our assumption that $T$ is a DRT. Therefore $r$ has in-degree zero since no vertices can lead into it.

To see why each other vertex $v \neq r$ must have in-degree 1, first note that they clearly can't have in-degree 0 since there exists a directed path from $r$ to $v$, and the last edge in this path will contribute 1 to the in-degree of that vertex. To see why the number of in vertices can't be more than one, supposed by contradiction that it was, i.e. suppose there exist $u_1, u_2 \in V(T)$ s.t. $u_1v \in E(T), u_2v \in E(T)$. Since $u_1, u_2 \in V(T)$ there exists ordered paths $P_1, P_2$ such that the first vertex of both is $r$ and the last is $u_1$ and $u_2$ respectively. Let $w$ be the last vertex of $P_1$ that is also in $P_2$. Let $Q_1$, $Q_2$ be the sub paths of $P_1, P_2$ that go from $w$ to $v$. $Q_1 \cap Q_2 = \emptyset$ since $w$ was chosen to be the last shared vertex. However then $wQ_1vQ_2w$ is a cycle which is impossible in a tree. Therefore the in-degree number can't be 0 and can't be 2 or greater, so it must be 1. $\square$

**Question 2.** Let $T_1, T_2$ be disjoint DRTs and let $e$ be a directed edge with one endpoint in $T_1$ and the other in $T_2$.

($\Rightarrow$) Assume $(T_1 \cup T_2) + e$ is a DRT, we will prove that the second vertex of $e$ is the root of $T_1$ or $T_2$, call them $r_1, r_2$. To see why suppose by contradiction that that the second vertex of $e = uv$ is not $r_1, r_2$, and WLOG assume $u \in E(T_1), v \in E(T_2), v \neq r_2$. Let $P_1$ be a directed path from $r_1$ to $u$ and let $P_2$ be a directed path from $r_2$ to $v$. Then $P_1 + e$ is a directed path from $r_1$ to $v$ in the new graph. However this implies that in the new graph $v$ has an in-degree of at least 2 (since the second last vertex of $P_1$ is in $T_1$ and the second last vertex of $P_2$ is in $T_2$). However problem 1 in this homework proved that no vertex on a DRT has in-degree 2 or greater, so it must be that $e$ was the root of $T_1$ or $T_2$.

($\Leftarrow$) Assume the endpoint of $e$ is the root of $T_1$ or $T_2$. WLOG assume $e = ur_2$ where $r_2$ is the root of $T_2$, and let $r_1$ be the root of $T_1$. Let $T = (T_1 \cup T_2) + e$ and let $x \in V(T)$. If $x \in T_1$ then since $T_1$ is a DRT there exists a directed path from $r_1$ to $x$. If $x \in T_2$ then let $P_1$ be a directed path from $r_1$ to $u$ and $P_2$ be a directed path from $r_2$ to $x$. Then $r_1P_1eP_2x$ is a directed path from $r_1$ to $x$, so $r_1$ fulfills all the root requirements for $T$. Also note that since $T_1, T_2$ were disconnected before adding $e$, $e$ is a bridge. $T_1$, $T_2$ were both trees and we added a bridge to connect them so $T$ is a tree. Finally since $T_1, T_2$ were both DRTs and we added an edge $e = ur_2$ such that $r_2u \notin E(T)$, the second requirement for a DRT is also satisfied. We've shown that $T$ fulfills all the requirements for a DRT, so it is one and we're done. $\square$

**Question 3.** The statement is true. Let $P$ be a longest path of a connected graph $G$, and let $u, v$ be its two endpoints. The statement will be shown for $u$, although since $u, v$ were arbitrary it also holds for $v$. By way of contradiction let $x, y \in G - u$ such that $x, y$ are disconnected in $G - u$.

$G$ is connected so there exists a path $Q \subset G$ from $x$ to $y$. Given that $x, y$ became disconnected after removing $u$ it must be that $u \in V(Q)$. $x \neq u, y \neq u$, so there exist paths $Q_1$ between $x$ and $u_1$ and $Q_2$ between $u_2$ and $y$, where $u_1, u_2 \in N(u)$. We proved in class that $N(u) \in V(P)$, since otherwise you could extend $P$ by including a vertex of $N(u)$ not already in $P$. Let $P_1 \subset P$ be the portion of $P$ between $u_1, u_2$. Note that $u \notin P_1$ since $u$ is an endpoint of $P$, so it's not a midpoint of any subpath. But then $xQ_1u_1P_1u_2Q_2y$ is a path of $G - u$ between $x$ and $y$ which contradicts our assumption that $x, y$ disconnected. Therefore $u$ couldn't have been a cut vertex, and by the symmetry of the argument $v$ couldn't have been either. $\square$

**Question 4.** The flaw is that the distance between the endpoints of a longest path of a graph $G$ are not necessarily the farthest from each other. To see why consider $C_4$. Then the longest path includes all 4 vertices, but the distance between two endpoints of such a path is just 1 whereas the actual farthest vertices are distance 2. Therefore it's not valid to assume that the endpoints of a longest path are the vertices that are the farthest from one another.

**Question 5.** The statement is true. Let $C$ be a component of $G$. Clearly $C$ is connected, since otherwise it wouldn't be a component. Let $V$ be a set of vertices of size smaller than 1. The only possibility is that $|V| = 0 \implies V = \emptyset$. However $C - V = C - \emptyset = C$ is connected, so $C$ is 1-connected. $\square$

**Question 6.** Let $v$ be the vertex added to $G$ and $V$ be a minimal separating set of $G'$. We will consider two cases: $v \in V$ and $v \notin V$.
**Case 1 ($v \in V$):** Note that $V - v$ must be a separating set for $G$ (since $V - v \subset G$ and $V$ separates $G'$). Therefore $|V - v| \geq k \implies |G| \geq k + 1 > k$ as required.
**Case 2 ($v \notin V$):** If $v$ is a trivial component in $G' - V$, $N(v) \subset V$ (in fact $N(v) = V$ by $V$'s minimality but this isn't required) and $|V| \geq |N(v)| \geq k$ which is what we're trying to show. If $v$ isn't its own component, it is part of a component $C \subset G', |C| \geq 2$ and $G' - V$ is disconnected. Then $C - v \neq \emptyset$ so $G - V = G' - V - v$ separates $G$, so $|V| \geq k$. Either way we've shown that $|V| \geq k$, which is the definition of being $k$-connected. $\square$

**Question 7.** Consider the number of edges that must be removed to generate three components. Let $x$ be the number of vertices of the first component $G_1$, and $y$ be the number of vertices of the second component $G_2$. Then the last vertex has $3n - x - y$ vertices, call it $G_3$. The number of edges to disconnect $G_1$ is $x(3n - x)$, since each of the $x$ vertices is attached to $3n - x$ vertices. $G_2$ has $y$ vertices attached to $3n - y$ vertices, but $x$ of the edges attached to the second of those vertices was already removed in the first step. Therefore you must remove $y(3n - x - y)$ vertices to separate the second component. After removing both of these the last will be disconnected, since neither of the $G_1, G_2$ are connected to it by design. Thus the total number of edges required to disconnect is:

$$f(x, y) = x(3n - x) + y(3n - x - y) = 3n(x + y) - (x^2 + xy + y^2).$$

Fix $x$ and consider optimizing $y$. The optimum occurs either on the edges $(y = 1, y = 3n - 2)$ or where the derivative is zero. Note however that this is a inverted parabola with its vertex at $y = \frac{-b}{2a} = \frac{3n-x}{2}$ which is its maximum point, so if we were to take the derivative and set it to zero we would find a maximum, not am minimum. Thus $y = 1$ or $y = 3n - 2$.

If $y = 3n - 2$, then the only option for $x$ would be $x = 1$ and the total number of edges is $f(1, 3n - 2) = 6n - 3$. If instead $y = 1$, then using the same logic as we used to optimize $y$, $x = 3n - 2$ or $x = 1$. Since the remaining component will be the opposite of whatever choice of $x$ we use (and switching them doesn't change $f$), WLOG assume $x = 1$ as well. Then the total number of removed edges is still $f(1, 1) = 6n - 3$. We've covered all possible cases, so the minimum number of edges required is $|X| = 6n - 3$.