Math 318 Homework 7

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Question 1a. The definition of a characteristic function is $\phi_X(t) = E[e^{itX}]$. Expanding this, we get:

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} f_X(x)e^{itx}dx = \sum_{x=-\infty}^{\infty} p_X(x)e^{itx} = \sum_{x=-\infty}^{\infty} p_X(x)\left(\cos(tx) + i\sin(tx)\right).$$

The only t dependence is in the $\sin(tx)$ and $\cos(tx)$ terms which are both 2π periodic (since x is an integer), so the characteristic function of discrete variables is 2π-periodic.

Question 1b. Let $Y =$

$$\phi_X(0) = \int_{-\infty}^{\infty} f_X(x)dx = \phi_X(2\pi) = \int_{-\infty}^{\infty} f_X(x)e^{2\pi ix}dx.$$

Question 2. Expressing as an integral:

$$E[X^3] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u+v)^3 f_U(u)f_V(v)dudv = \iint (u^3 + 3u^2v + 3uv^2 + v^3)f_U(u)f_V(v)dudv$$

$$= E[U^3] + E[V^3] + 3E[U^2]E[V] + 3E[U]E[V^2] = E[U^3] + E[V^3].$$

Similarly for $E[X^4]$:

$$E[X^4] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u+v)^4 f_U(u)f_V(v)dudv = \iint (u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^3)f_U(u)f_V(v)dudv.$$

$$= E[U^4] + E[V^4] + 6E[U^2]E[V^2] + 4E[U^3]E[V] + 4E[U]E[V^3] = E[U^4] + E[V^4] + 6E[U^2]E[V^2].$$

Question 3a. Let the entries of A be denoted by A_{ik}, where i is the row and k is the column. Then by the definition of matrix multiplication,

$$Y_i = \sum_{k=1}^{n} A_{ik}X_k.$$

The distribution of a sum of normal variables is also a normal variable with the sum of mean and variance added, so $Y_i = N(0,n)$.

Question 3b. Computing covariance

$$\mathrm{Cov}(Y_i, Y_j) = E[Y_iY_j] - E[Y_i]E[Y_j] = E\left[\left(\sum_{k=1}^{n} A_{ik}X_k\right)\left(\sum_{k=1}^{n} A_{jk}X_k\right)\right]$$

$$= E\left[\sum_{k=1}^{n}\sum_{l=1}^{n} A_{ik}A_{jl}X_kX_l\right] = \sum_{k=1}^{n}\sum_{l=1}^{n} A_{ik}A_{jl}E[X_kX_l] = \sum_{k=1}^{n} A_{ik}A_{jk}E[X_k^2]$$

$$= A_i \cdot A_j$$

where A_i and A_j are the ith and jth row vector of A respectively.