MATH 305 Homework 7

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1. Let f be analytic inside and on the simple closed loop C and let z_0 lie outside C. What is the value of $\int_C \frac{f(z)}{z-z_0}dz$?

By Cauchy's integral theorem, the integral is 0.

2. Evaluate

(a) $\int_{|z|=3} \frac{e^{iz}}{(z^2+1)^2}dz$.

There are singularities at $\pm i$:

$$\int_{|z|=3} \frac{e^{iz}}{(z-i)^2(z+i)^2}dz = 2\pi i \frac{d}{dx}\left(\frac{e^{iz}}{(z+i)^2}\right)\bigg|_{z=i} + 2\pi i \frac{d}{dx}\left(\frac{e^{iz}}{(z-i)^2}\right)\bigg|_{z=-i}.$$

$$= 2\pi i \frac{ie^{iz}(z+i)^2 - 2e^{iz}(z+i)}{(z+i)^4}\bigg|_{z=i} + 2\pi i \frac{ie^{iz}(z-i)^2 - 2e^{iz}(z-i)}{(z-i)^4}\bigg|_{z=-i}.$$

$$= 2\pi\left(\frac{4e^{-1}+4e^{-1}}{16} + \frac{4e-4e}{16}\right).$$

$$= \frac{\pi}{e}.$$

(b) $\int_{|z|=2} \frac{\cos z}{z^2(z-1)}dz$

Singularities at $0, 1$:

$$\int_{|z|=2} \frac{\cos z}{z^2(z-1)}dz = 2\pi i \frac{\cos z}{z^2}\bigg|_{z=1} + 2\pi i \frac{-\sin z(z-1) - \cos z}{(z-1)^2}\bigg|_{z=0}.$$

$$= 2\pi i\left(\cos 1 - 1\right).$$

3. Evaluate

(a) $\int_{|z|=2} \frac{z^2+1}{(z-1)^3}dz$.

$$\int_{|z|=2} \frac{z^2+1}{(z-1)^3}dz = 2\pi i \frac{d^2}{dx^2}\left(z^2+1)\right)\bigg|_{z=1} = 2\pi i.$$

(b) $\int_{|z|=2} \frac{\sin z}{z^2(z-3)}dz$

$$\int_{|z|=2} \frac{\sin z}{z^2(z-3)}dz = 2\pi i \frac{\cos z(z-3) - \sin z}{(z-3)^2}\bigg|_{z=0} = -\frac{2}{3}\pi i.$$

4. Evaluate

(a) $\int_{|z|=5} \frac{z^2+1}{z^4+z+1}dz$.

Note that for $|z| \geq 5$, we have $\left|z^4+z+1\right| \geq |z^4| - |z| - 1 \geq 5^4 - 5 - 1 = 619 > 0$. Then the integral around the contour is the same as the integral around $|z|=5$ to $|z|=R$.

$$\left|\int_{|z|=5} \frac{z^2+1}{z^4+z+1} dz\right| = \left|\int_{|z|=R} \frac{z^2+1}{z^4+z+1} dz\right| \leq \lim_{R\to\infty} \left|\frac{R^2+1}{R^4+R+1}\right| 2\pi R = 0.$$

Thus the integral is zero.

(b) $\int_{|z|=2} \frac{z}{(z-3)(z^4+z+1)} dz$

$$\int_{|z|=2} \frac{z}{(z-3)(z^4+z+1)} dz = \int_{|z|=R} \frac{z}{(z-3)(z^4+z+1)} dz - 2\pi i \frac{z}{z^4+z+1}\bigg|_{z=3}.$$

Taking the limit:

$$\left|\int_{|z|=R} \frac{z}{(z-3)(z^4+z+1)} dz\right| \leq \lim_{R\to\infty} \left|\frac{z}{(z-3)(z^4+z+1)}\right| 2\pi R = 0.$$

$$\implies \int_{|z|=2} \frac{z}{(z-3)(z^4+z+1)} dz = -\frac{2\pi i 3}{3^4+3+1} = -2\pi i \frac{3}{85}.$$

5. Evaluate

(a) $\int_0^{2\pi} \frac{1}{2+\sin\varphi} d\varphi$.

Let $z = e^{i\varphi}$. Then the integral becomes:

$$\int_C \frac{2iz}{4iz+z^2-1} dz = \int_C \frac{2iz}{(z+2i+i\sqrt{3})(z+2i-i\sqrt{3})} \frac{1}{iz} dz = 2\pi i \frac{2}{z+2i+i\sqrt{3}}\bigg|_{z=-2i+i\sqrt{3}}.$$

$$= \frac{4\pi}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

(b) $\int_0^{\pi} \frac{1}{2-\cos\varphi} d\varphi$.

Let $z = e^{2i\varphi} \implies d\varphi = \frac{1}{2iz}$

$$\int_0^{\pi} \frac{1}{2-\cos\varphi} d\varphi = \int_C \frac{1}{(4z - z^{3/2} - z^{1/2})i} dz = \int_C \frac{1}{i\left(z^{1/2} - 2 + \sqrt{3}\right)\left(z^{1/2} - 2 - \sqrt{3}\right)} dz.$$

$$= \frac{2\pi i}{i(z-2-\sqrt{3})}\bigg|_{z=2-\sqrt{3}} = \frac{\pi}{\sqrt{3}}.$$

(c) $\int_0^{2\pi} \sin^{10}\varphi d\varphi$.

Let $z = e^{i\varphi}$:

$$\int_0^{2\pi} \sin^{10}\varphi d\varphi = \int_C -\frac{1}{1024}\left(z+z^{-1}\right)^{10} \frac{1}{iz} dz.$$

$$\int_C -\frac{1}{1024}\left(z^{-10} + \frac{10}{z^8} + \frac{45}{z^6} + \frac{120}{z^4} + \frac{210}{z^2} + 252\right) \frac{1}{iz} dz = \frac{\pi}{128}(63).$$

6. Suppose that $f(z)$ is entire and $|f(z)| \leq 2(1+|z|)^3$. Show that $f(z)$ is a polynomial of degree at most three.

By Cauchy's integral formula, with C being a ring of arbitrary radius around z_0:

$$|f^{(4)}(z_0)| = \left|\frac{2}{\pi i}\int_C \frac{f(z)}{(z-z_0)^5} dz\right| = |\frac{f(z_0)}{(z-z_0)^5}| 2\pi R \leq |\frac{2(1+|z|)^3}{R^5}| 2\pi R \to 0.$$

Since the fourth derivative is zero and f is analytic, the only possibility is that f is a polynomial of degree at most four.

7. Let f be entire and suppose that $Re(f(z)) \leq 2Im(f(z))$. Show that $f(z)$ must be a constant function.

Hint: consider $g = e^{\alpha f}$ for some complex number α.

Consider $g = e^{(1+2i)f(z)}$. Then if $f(z) = u + iv$, we have that $Re((1+2i)f(z)) = u - 2v \leq 2v - 2v = 0$. Then we get that

$$|g| = |e^{(1+2i)f(z)}| = e^{Re(1+2i)f(z))} \leq e^0 = 1.$$

Since this function is bounded is must be analytic by Liouville's theorem, so f must also be constant.

8. Let f be analytic in $D = \{|z| \leq 1\}$. Assume that $|f(z)| \leq M$ for $|z| = 1$. Show that

(a) $|f''(0)| \leq 2M$.

Applying Cauchy's integral formula and the maximum principle:

$$\pi i f''(0) = \int \frac{f(z)}{z^3} dz \leq 2\pi M.$$

$$\implies |f''(0)| \leq 2M.$$

(b) $|f^{('')}(\frac{1}{2})| \leq 16M$

Again applying Cauchy's integral formula:

$$\pi i f''(\frac{1}{2}) = \int \frac{f(z)}{(z-\frac{1}{2})^3} dz \leq 2\pi M \left(\frac{1}{2}\right)^{-3}.$$

$$\implies f''(\frac{1}{2}) \leq 16M.$$

9. Find the maximum value of $|z^2 + 3z - 1|$ in the disk $|z| \leq 1$.

Let $p(z) = z^2 + 3z - 1$. Because p is analytic it's maximum must lie on the boundary, i.e. with $|z| = 1$. Then we have:

$$\left|z^2 + 3z - 1\right| = \left|z - z^{-1} + 3\right| = |2i \sin\theta + 3| = \sqrt{13}.$$

10. Show that $\max_{|z|\leq 1} |4z^{100} - 5z| = 9$.

Let $z = -1$. Then we have $|z| \leq 1$ and

$$|4(-1)^{100} - 5(-1)| = |4 + 5| = 9.$$