Math 406 Homework 5

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Question 1a. For all functions v in some class, the following must be true:

$$\int_0^1 v(u'' + k^2 u - f)dx = 0.$$

Integrating by parts:

$$\int_0^1 -u'v' + k^2 uv - fvdx + v(1)\beta - v(0)u'(0) = 0 \implies \int_0^1 u'v'dx = k^2 \int_0^1 uvdx - \int_0^1 fvdx + \beta v(1).$$

Thus the weak form is to find $u \in H^1_\alpha$ such that $\int_0^1 u'v'dx = k^2 \int_0^1 uvdx - \int_0^1 fvdx + \beta v(1) \forall v \in H^1_0$.

Question 1b. Since we used the strong form to derive the weak form while relaxing constraints, the strong form always implies the weak form. For the other direction:

$$\int_0^1 u'v'dx = k^2 \int_0^1 uvdx - \int_0^1 fvdx + \beta v(1) \forall v.$$

Assuming sufficient differentiability, we can go backwards using integration by parts:

$$\implies u'v\Big|_0^1 - \int_0^1 v\left(u'' + k^2 u - f\right)dx - \beta v(1) = 0 = v(1)\left(u'(1) - \beta\right) - \int_0^1 v\left(u'' + k^2 u - f\right)dx.$$

Since this holds for all v, we have $u'' + k^2 u = f$, $u'(1) = \beta$ and $u(0) = \alpha$ (since $u \in H^1_\alpha$) as required.

Question 1c. Let $u^h(x) = \sum_{n=0}^N u_n N_n(x)$, where $N_n(x)$ are the basis functions. Similarly let $v^h(x) = \sum_{n=1}^N v_n N_n(x)$. Using the weak form derived above, we need to find $u^h \in V^h_\alpha$ such that:

$$\int_0^1 \left(\alpha N_0' + \sum_{n=1}^N u_n N_n'\right)\left(\sum_{m=1}^N v_m N_m'\right)dx = k^2 \int_0^1 \left(\alpha N_0 + \sum_{n=1}^N u_n N_n\right)\left(\sum_{m=1}^N v_m N_m\right)dx$$

$$- \int_0^1 f\left(\sum_{m=1}^N v_m N_m\right)dx + \sum_{m=1}^N v_m N_m(1)\beta = 0.$$

$$\implies \sum_{m=1}^N v_m \Bigg[\alpha \int_0^1 N_0' N_m' dx + \sum_{n=1}^N u_n \int_0^1 N_n' N_m' dx - k^2 \left(\alpha \int_0^1 N_0 N_m dx + \sum_{n=1}^N u_n \int_0^1 N_n N_m dx\right)$$

$$+ \int_0^1 f N_m dx - N_m(1)\beta\Bigg] = 0.$$

$$\implies \sum_{n=1}^{N} u_n(K_{mn} - k^2 M_{mn}) = \delta_{mn}\beta - \alpha(K_{0m} - k^2 M_{0m}) + \int_0^1 f N_m dx \implies (K - k^2 M) = b.$$

Here K is the stiffness matrix and M is the mass matrix. This derivation was done in class and in the notes so most of the algebra is taken from there.

Question 1d. To use piecewise linear basis functions with discretization size h, all we have to do is construct the stiffness and mass matrices. Consider just a single element e along with the change of coordinates $x(\xi) = x_{e-1}N_1(\xi) + x_e N_2(\xi)$ where N_1 and N_2 are the two linear basis functions in $[-1,1]$ (with corresponding $N_a(x), N_b(x)$ in the x domain):

$$\int_{x_{e-1}}^{x_e} N_m'(x) N_n'(x) dx = \int_{-1}^{1} \frac{dN_a}{d\xi}\frac{d\xi}{dx}\frac{dN_b}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi} d\xi = \frac{2}{h}\int_{-1}^{1} \frac{\xi_a \xi_b}{4} d\xi = \frac{\xi_a \xi_b}{h}$$

$$\implies K_{ab}^e = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Similarly for the mass matrix:

$$M_{mn}^e = \int_{x_{e-1}}^{x_e} N_m N_n dx = \int_{-1}^{1} \frac{1}{4}(1 + \xi_a \xi)(1 + \xi_b \xi)\frac{dx}{d\xi} d\xi = \frac{h}{4}\left(1 + \frac{\xi_a \xi_b}{3}\right)$$

$$\implies M^e = \frac{h}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Putting these together and summing over all the elements (and being careful about summing the correct conditions at the boundary, we get the final matrices to be:

$$K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & 0 & \cdots & & \\ 0 & 0 & -1 & 2 & -1 & \cdots & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & \cdots & & \\ 1 & 4 & 1 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 4 & 1 & \cdots & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

Finally we have also have:

$$b = \begin{pmatrix} -\left(\frac{1}{h} - \frac{k^2 h}{6}\right)\alpha - (f, N_1) \\ -(f, N_2) \\ -(f, N_3) \\ \vdots \\ \beta - (f, N_N) \end{pmatrix} \quad \text{where } (f, N_m) = \int_0^1 f N_m dx.$$

Thus the finite element discretization is to solve the equation $(K - k^2 M)u = b$ with K, M and b defined as above.

Question 1e. The solution to the homogeneous equation $u'' + 10^2 u = 0$ is $u = A\sin 10x + B\cos 10x$. For the nonhomogeneous equation assume that $u = a_1 + a_2 x + a_3 x^2 + a_4 x^3$, then we have $a_3 + a_4 x + 100a_1 + 100a_2 x + 100a_3 x^2 + 100a_4 x^3 = x^3 \implies u = -\frac{6x}{10^4} + \frac{x^3}{10^2}$. Thus plugging in boundary conditions, the analytic solution is $u(x) = \frac{\sin 10x}{10\cos 10}\left(1 + \frac{6}{10^4} - \frac{3}{10^2}\right) - \frac{6x}{10^4} + \frac{x^3}{10^2}$.

As for the numerical solution, the matrices described above were construction for $N = 10, 20, 30$ and solved. The results can be seen in figure 1. The code used was:

```
a=0;b=1;
k=10;
f=@(x) x.^3;
alpha = 0; beta = 1;

exact = @(x) (sin(10.*x) ./ (10.*cos(10))) .* (1 + 6./10^4 - 3./10^2) - (6.*x
    ↪ /10^4) + (x.^3./10^2);

Ns = [10,20,30];

for i=1:3
    n = Ns(i);
    h=(b-a)/n;

    K = full(gallery('tridiag',n,-1,2,-1));
    K(n, n-1) = -1;
    K(n, n) = 1;
    K = (1/h) * K;

    M = full(gallery('tridiag',n,1,4,1));
    M(n, n) = 2;
    M = (h/6) * M;

    B = zeros(1,n);
    for m=1:n
        % B(m) = -integral(f, (m-1)*h, m*h);
        x1 = (m-1)*h;
        x2 = m*h;
        x3 = (m+1)*h;
        N1 = @(x) (x-x1)./h;
        N2 = @(x) (x3-x)./h;
        B(m) = -integral(@(x) f(x).*N1(x), (m-1)*h, m*h);
        % B(m) = B(m) - integral(@(x) f(x).*N1(x), (m-1)*h, m*h)
        if m<n
            B(m) = B(m)-integral(@(x) f(x).*N2(x), m*h, (m+1)*h);
        end
    end
    B(1) = B(1) - (1/h-k^2*h/6)*alpha;
    B(n) = B(n) + beta;
```

```
    u = [0, B / (K-k^2*M)];

    x = linspace(a, b, n+1);
    xe = linspace(a, b, 1000);
    figure;
    plot(x, u, xe, exact(xe));
    title("N=" + n);
    saveas(gcf, "q1aN="+n+".png")
end
```

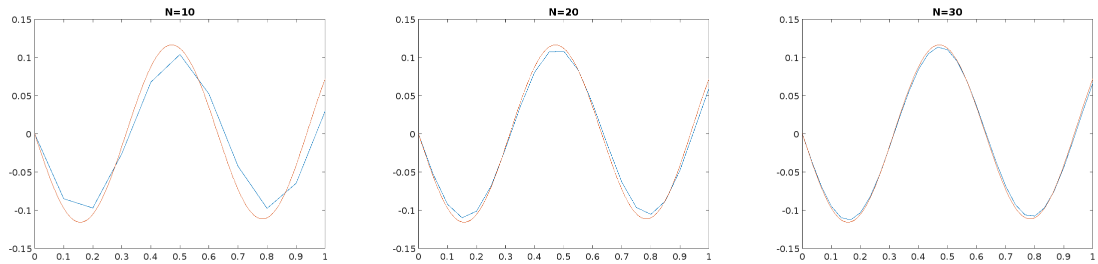

Figure 1: Graphs for question 1e. Orange is exact and blue is numerical.

Question 1f. All of the work for the previous parts have been to reduce the problem to one in the given form. Here $f = \alpha = \beta = 0$ so $b = 0$, and thus $Ku = -k^2 Mu$. Letting $x = u, A = K, B = -M$ and $\lambda = k^2$, we see that this is the desired eigenvalue problem.

For finite differences, we discretize the equation $u'' = -\lambda u$ to get $u_{n+1} - 2u_n + u_{n-1} = -\lambda h^2 u_n$. The two boundary conditions give us that $u_2 - 2u_1 = \lambda u_1$ and $u_N - u_{N-1} = h(0) = 0$. Putting this in matrix form, we have

$$Au = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & \cdots & & & \\ -1 & 2 & -1 & \cdots & & \\ \vdots & & & & & \\ & & \cdots & -1 & 2 & -1 \\ & & & \cdots & -1 & 1 \end{pmatrix} u = \lambda u.$$

The eigenvalues (technically $\sqrt{\lambda}$ to keep it consistent with the later plot of k_j) for both methods can be seen in table 1.

The approximate expression for the eigenvalues is a solved problem, see e.g. here for a derivation (I don't have time to rewrite it here):

$$\lambda_k = -\frac{4}{h^2}\sin^2\left(\frac{\pi(k-0.5)}{(2N+1)}\right).$$

For the plots, see figure 2.

The code for this question is as follows:

```
n = 10;
a = 0; b = 1;
h=(b-a)/n;
```

$k_{FE} = \sqrt{\lambda}$	$k_{DE} = \sqrt{\lambda}$
1.5724	1.4946
4.7561	4.4504
8.0571	7.3068
11.5542	10.0000
15.3203	12.4698
19.4002	14.6610
23.7547	16.5248
28.1465	18.0194
31.9858	19.1115
34.3236	19.7766

Table 1: Values of k for finite element and difference equation.

```
alpha = 0;
beta = 0;

K = full(gallery('tridiag',n,-1,2,-1));
K(n, n-1) = -1;
K(n, n) = 1;
K = (1/h) * K;

M = full(gallery('tridiag',n,1,4,1));
M(n, n) = 2;
M = (h/6) * M;

[V,D] = eig(K,M);
lambdaFE = diag(D);
eigenvecFE = V;

A = full(gallery('tridiag',n,-1,2,-1));
A(n,n) = 1;
A = A/(h^2);

[V,D] = eig(A);
lambdaDE = diag(D);
eigenvecDE = V;

figure;
scatter(1:n, lambdaFE.^0.5);
hold on;
scatter(1:n, lambdaDE.^0.5);
hold off;
title('Question 1f, k_j vs j')

x = linspace(a, b, n+1);
for i=1:3
```

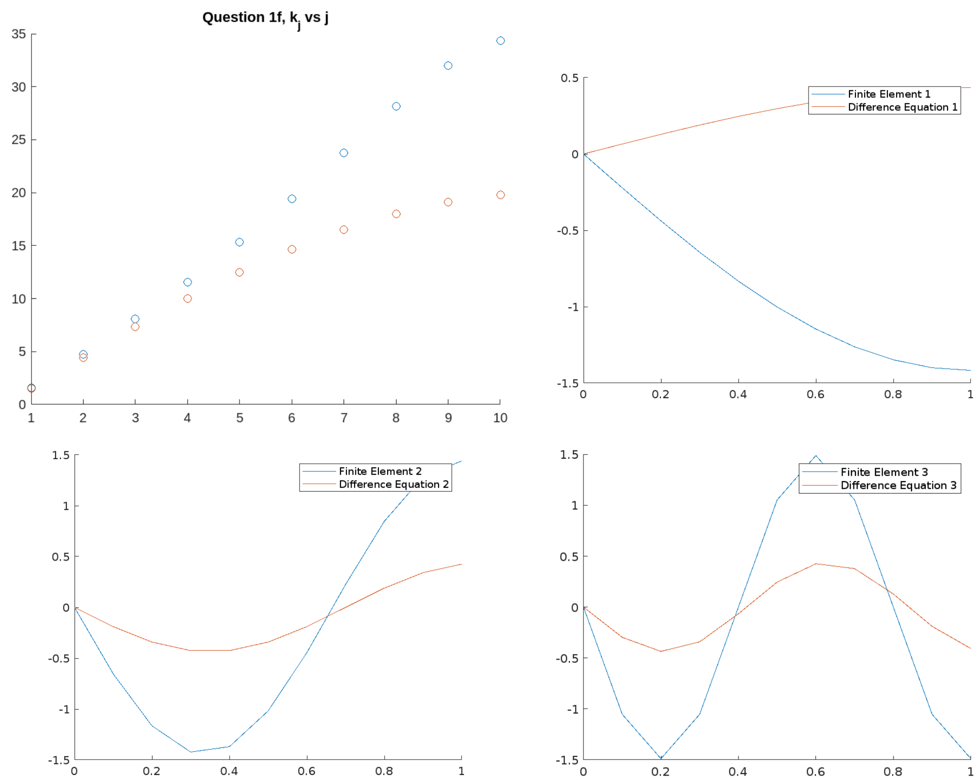

Figure 2: Plots for question 1f. In k_j plot, blue is finite element and orange is difference equation.

```
    figure;
    hold on;
    plot(x, [0;eigenvecFE(:,i)], 'DisplayName', "Finite Element " + i);
    plot(x, [0;eigenvecDE(:,i)], 'DisplayName', "Difference Equation " + i);
    hold off;
    legend();
end
```

Question 2a. From the main boundary value problem, we have that $\frac{D}{r}(rp_r)_r = 0$, which has solutions in the form $p(r) = A + B \log r$. The boundary conditions forces $\lim_{r\to 0} r\frac{\partial p}{\partial r} = B = -\frac{Q_0}{2\pi D}$, and $p(R(t), t) = 0 \implies A = \frac{Q_0}{2\pi D} \log R(t)$. Putting this together we have $p(r, t) = \frac{Q_0}{2\pi D} \log \frac{R(t)}{r}$.

Using the Stefan condition for front velocity:

$$\dot{R}(t) = -\frac{w_0^2}{\mu'} \frac{dp}{dr}\bigg]_{r=R(t)} = \frac{w_0^2}{\mu'} \frac{Q_0}{2\pi D R(t)}.$$

Finally using separation of variables we can solve this differential equation:

$$\int R(t) dR(t) = \int_0^t \frac{w_0^2}{\mu'} \frac{Q_0}{2\pi D} dt \implies R(t) = \sqrt{\frac{w_0^2 Q_0 t}{\mu' \pi D}}.$$

Question 2bi. Expanding, the equation to solve is $Lp = p_{rr} + \frac{1}{r} p_r = g(r, t)$. We can find the adjoint operator:

$$\int_0^{R(t)} Gfrdr = \int_0^{R(t)} G\cdot(Lu)\cdot rdr = \int_0^{R(t)} G\left(ru'' + u'\right)dr.$$

From this point the self adjoint operator and associated boundary terms were derived in class (lecture 14), they are:

$$= \left[Ga_0u' + Ga_1u - (a_0G)'u\right]_0^{R(t)} + \int_0^{R(t)} u\left((a_0G)'' - (a_1G)' + a_2G\right)dr$$

$$= \left[ru'G - rG'u\right]_0^{R(t)} + \int_0^{R(t)} u\left(rG'' + G'\right)dr.$$

Thus under the radial integration L acts as self adjoint (there's probably some deeper reason for the way the algebra works out with the extra r, but from the algebra it works out at least). Thus to eliminate unknowns we want to find $G(s,r)$ such that $sG_{ss}(s,r) + G_s(s,r) = \delta(s-r)$ with $G(R(t),r) = 0$ and $G(0,r) < \infty$. We already determined that the solution to the homogeneous equation is $p(r) = A + B\log r$. Thus we can split the Green's function into two parts:

$$G(s,r) = \begin{cases} A_- + B_-\log s & 0 < s < r \\ A_+ + B_+\log s & r < s < R(t) \end{cases}.$$

The boundary conditions force $B_- = 0$ and $A_+ = -B_+\log R(t)$. Continuity forces that $A_- = B_+\log\frac{r}{R(t)}$. Thus we can write G once again as

$$G(s,r) = \begin{cases} B_+\log\frac{r}{R(t)} & 0 < s < r \\ B_+\log\frac{s}{R(t)} & r < s < R(t) \end{cases}$$

$$G_s(s,r) = \begin{cases} 0 & 0 < s < r \\ \frac{B_+}{s} & r < s < R(t) \end{cases}.$$

The only condition left is the jump condition:

$$\int_{r-\epsilon}^{r+\epsilon} sG_{ss} + G_s ds = \int_{r-\epsilon}^{r+\epsilon} (sG_s)_s ds = sG_s\Big|_{r-\epsilon}^{r+\epsilon} = r\left(G_s(r_+,r) - G_s(r_-,r)\right) = \int_{r-\epsilon}^{r+\epsilon} s\delta(r-s)ds = r$$

$$\implies 1 = (B_+ - 0) \implies B_+ = 1.$$

We can then get a final formulation for G:

$$G(s,r) = \begin{cases} \log\frac{r}{R(t)} & 0 < s < r \\ \log\frac{s}{R(t)} & r < s < R(t) \end{cases}.$$

The only boundary term is $\lim_{s\to 0} su'(s)G(s,r) = -\frac{Q_0}{2\pi D}\log\frac{r}{R(t)}$. Putting this together we can determine an expression for $p(r,t)$ in terms of $R(t)$ and $g(r,t)$:

$$p(r,t) = -\frac{Q_0}{2\pi D}\log\frac{r}{R(t)} + \int_0^r \log\frac{r}{R(t)}g(s)sds + \int_r^{R(t)} \log\frac{s}{R(t)}g(s)sds.$$

Finally, using the explicit expression for $f(r,t)$ given in the question we get

$$p(r,t) = -\frac{Q_0}{2\pi D}\log\frac{r}{R(t)} + \int_0^r \log\frac{r}{R(t)}\frac{C'H(t-t_0(s))}{D\sqrt{t-t_0(s)}}sds + \int_r^{R(t)} \log\frac{s}{R(t)}\frac{C'H(t-t_0(s))}{D\sqrt{t-t_0(s)}}sds.$$

Question 2bii. Plugging in $r = R(t)$:

$$p_r(r,t) = -\frac{Q_0}{2\pi Dr} + \left(\frac{1}{r}\right)\int_0^r g(s)sds + rg(r)\log\frac{r}{R(t)} - rg(r)\log\frac{r}{R(t)}$$

$$p_r(r,t) = -\frac{Q_0}{2\pi Dr} + \int_0^r g(s)\frac{s}{r}ds$$

$$p_r(R(t),t) = -\frac{Q_0}{2\pi DR(t)} + \int_0^{R(t)} g(s)\frac{s}{R(t)}ds.$$

Plugging this in to the Stefan condition:

$$\dot{R}(t) = -\frac{w_0^2}{\mu'}\frac{dp}{dr}\bigg|_{r=R} = -\frac{w_0^2 Q_0}{2\mu'\pi DR(t)} + \frac{w_0^2}{\mu' R(t)}\int_0^{R(t)}\frac{C'H(t-t_0(s))}{D\sqrt{t-t_0(s)}}sds.$$

Make the suggested change of variables $s = R(\tau), ds = \dot{R}(\tau)d\tau$. Then $\tau = R^{-1}(s) = t_0(s)$, $s = 0 \implies \tau = 0$ and $s = R(t) \implies \tau = t$. Then we get

$$\dot{R}(t)R(t) = \phi(R,\dot{R}) = -\frac{Q_0}{2\pi w_0} + \frac{1}{w_0}\int_0^t \frac{C'H(t-\tau)\dot{R}(\tau)R(\tau)}{\sqrt{t-\tau}}d\tau.$$

Note that $\tau < t$ throughout the whole integral, so the Heavyside function is always 1. Then we have:

$$\phi(R,\dot{R}) = -\frac{Q_0}{2\pi w_0} + \frac{C'}{w_0}\int_0^t \frac{\phi(R,\dot{R})}{\sqrt{t-\tau}}d\tau.$$

Question 2biii. Since the integral given is in the form of a convolution integral, the Laplace transform of the whole thing will be the product of the Laplace transforms of each one. Let $F(s) = \mathcal{L}\{\phi(R,\dot{R})\}$. Then we have

$$\mathcal{L}\{\phi(R,\dot{R})\} = F(s) = \mathcal{L}\left\{-\frac{Q_0}{2\pi w_0} + \frac{C'}{w_0}\int_0^\infty \phi(R,\dot{R})\frac{H(t-\tau)}{\sqrt{t-\tau}}d\tau\right\} = F(s) = -\frac{Q_0}{2\pi w_0 s} + \frac{C'}{w_0}F(s)\sqrt{\frac{\pi}{s}}$$

$$\implies F(s) = -\frac{Q_0}{2\pi w_0 s}\left(1 - \frac{C'}{w_0}\sqrt{\frac{\pi}{s}}\right)^{-1} = -\frac{Q_0\sqrt{s}}{2\pi s(w_0\sqrt{s} - C'\sqrt{\pi})} = \frac{Q_0}{2\pi^{3/2}C'}\frac{1}{s - w_0\sqrt{s}/(C'\sqrt{\pi})}.$$

Question 2biv. We can invert the Laplace transform using the given transform. Let $\alpha = -\frac{w_0}{C'\sqrt{\pi}}$ and $A = \frac{Q_0}{2\pi^{3/2}C'}$ (I'd give it 50/50 odds I've dropped a constant by this point), and compute:

$$\phi(R,\dot{R}) = \mathcal{L}^{-1}\{F(s)\} = Ae^{\alpha^2 t}\text{erfc}\left(\alpha\sqrt{t}\right).$$

We can then use separation of variables to solve for R:

$$\int RdR = \frac{1}{2}R^2 = \int_0^t \phi(R,\dot{R})dt = \frac{A}{\alpha^2}\left(e^{\alpha^2 t}\text{erfc}\left(\alpha t^{1/2}\right) - 1\right) + \frac{2At^{1/2}}{\alpha\pi^{1/2}}$$

$$\implies R(t) = \sqrt{\frac{2A}{\alpha^2}\left(e^{\alpha^2 t}\mathrm{erfc}\left(\alpha t^{1/2}\right) - 1\right) + \frac{4At^{1/2}}{\alpha \pi^{1/2}}}.$$

The question asks to find $\dot{R}$ in the process of finding R, but given I didn't need to and the expression for R seems lengthy to differentiate I assume it's not necessary. Our expression for $p(r,t)$ is the same except now $R(t)$ is known using the above equation:

$$p(r,t) = -\frac{Q_0}{2\pi D}\log\frac{r}{R(t)} + \int_0^r \log\frac{r}{R(t)}\frac{C'}{D\sqrt{t-t_0(s)}}sds + \int_r^{R(t)}\log\frac{s}{R(t)}\frac{C'}{D\sqrt{t-t_0(s)}}sds.$$

Question 2bva. Given $F(R) = \frac{1}{2}R^2$ in my case this seems too simple and I may have made a mistake at this point, but I can't seem to find it and I'll continue nonetheless. See the radius in figure 3. The code used was:

```
Q=1;C=1;mu=1;w0=1;

alpha = -w0/C/pi^(0.5);
A = Q/(2*pi^(3/2)*C);

tmax = 500;
t = linspace(0,tmax,1000);

R = (2*A/alpha^2*(exp(alpha^2.*t).*erfc(alpha*t.^(1/2))-1)+4*A*t.^(1/2)/alpha/pi
    ↪ ^(1/2)).^(0.5);

plot(t, R);
title("Radius over time")
```

Question 2bvb. Ran out of time, sorry.

Question 3a. Let v be an arbitrary perturbation, then we can integrate by parts:

$$\int_0^{R(t)} v\frac{1}{r}\left(rp_r\right)_r rdr = v(rp_r)\Big|_0^{R(t)} - \int_0^{R(t)} v'p'rdr = 0.$$

Thus the weak form is to find u such that the above equation is satisfied for all $v \in H_0^1$.

Question 3b. Let $p(r) = \sum_{n=0}^{N} p_n N_n(r) = \sum_{n=0}^{N-1} p_n N_n(r)$ and $v(r) = \sum_{n=0}^{N-1} v_n N_n(r)$. Plugging this into the equation:

$$\int_0^{R(t)}\left(\sum_{n=0}^{N-1} p_n N_n'\right)\left(\sum_{m=0}^{N-1} v_m N_m'\right) rdr = v_0\frac{Q_0}{2\pi D}$$

$$\implies \sum_{m=0}^{N-1} v_m\left(\sum_{n=0}^{N-1} u_n\left(\int_0^{R(t)} N_n' N_m' rdr\right)\right) = v_0\frac{Q_0}{2\pi D} \implies Ku = b,$$

where $b_0 = \frac{Q_0}{2\pi D}$ and $b_i = 0 \forall i > 0$, and K is the stiffness matrix. To construct K, consider an individual element e. Then similar to question 1 by parameterizing by ξ and using linear basis functions $N_a(\xi), N_b(\xi)$, we get

$$r(\xi) = r_{e-1}N_1(\xi) + r_e N_2(\xi) = \frac{x_e + x_{e-1}}{2} + \frac{h\xi}{2}.$$

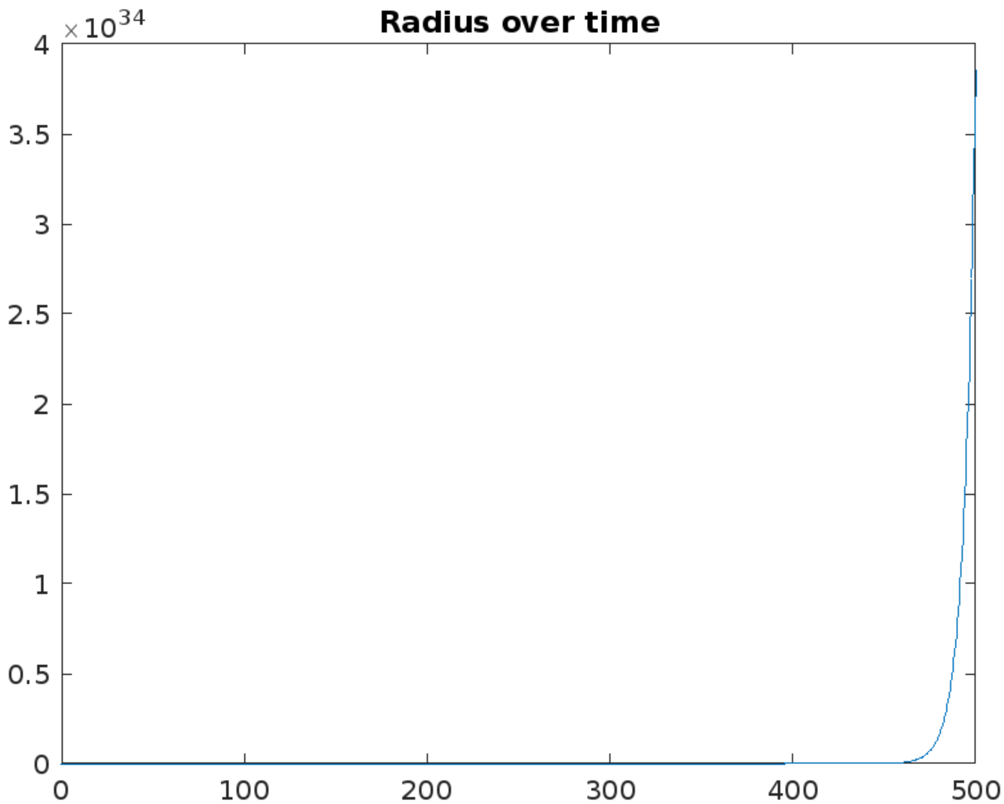

Figure 3: Graph for question 2bva. It grows exponentially which doesn't seem reasonable given the version of the ODE without the sink didn't, but I don't have time to figure out what's wrong.

$$\implies \int_{x_{e-1}}^{x_e} N_m' N_n' r dr = \int_{-1}^{1} \frac{dN_a}{d\xi}\frac{d\xi}{dr}\frac{dN_b}{d\xi}\frac{d\xi}{dr}\frac{dr}{d\xi}\left(\frac{x_e + x_{e-1}}{2} + \frac{h\xi}{2}\right) d\xi$$

$$= \frac{2}{h}\int_{-1}^{1} \frac{\xi_a \xi_b}{4}\left(\frac{x_e + x_{e-1}}{2} + \frac{h\xi}{2}\right) d\xi = \frac{2}{h}\int_{-1}^{1} \frac{\xi_a \xi_b \left(x_e + x_{e-1}\right)}{8} d\xi$$

$$\implies K_{ab}^e = \frac{x_e + x_{e-1}}{2h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The final matrix K can be constructed by adding these individual element matrices together. Unlike the Hemoltz case this doesn't sum in a clean way since each of these components has radial dependence that makes it tricky to write explicitly.

Question 3c. See figure 4 for the plot of the finite element and exact solution. As can be seen, they are extremely close and the finite element provides a good approximation of the real value. The code used was:

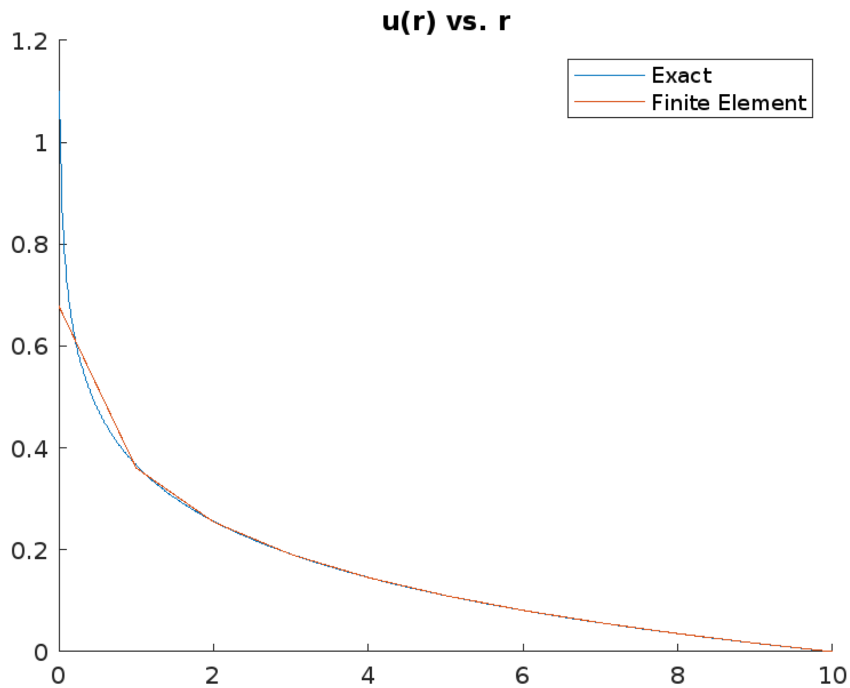

Figure 4: Finite element and exact solution for question 3c.

```
n = 10;
a=0; b=10;
h = (b-a)/n;
Q=1;D=1;

exact = @(x) Q./(2*pi*D)*log(b./x);

x = linspace(a,b,n+1);
xe = linspace(a,b,1000);
```

```
K = zeros(n);
for i=2:n
    tmp = (x(i-1)+x(i))/2/h;
    K(i,i) = K(i,i) + tmp;
    K(i-1,i-1) = K(i-1,i-1) + tmp;
    K(i,i-1) = K(i,i-1) - tmp;
    K(i-1,i) = K(i-1,i) - tmp;
end

K(n,n) = K(n,n) + (x(n)+x(n+1))/2/h;

b = zeros(1,n);
b(1)=Q/(2*pi*D);

p = [b / K, 0];

figure;
hold on;
plot(xe, exact(xe), 'DisplayName', "Exact")
plot(x, p, 'DisplayName', "Finite Element")
hold off;
title("u(r) vs. r")
legend;
```

Question 3d+e. Ran out of time.