Math 305 Homework 1

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1a. Expanding:

$$(1+i)(3-2i)(2+3i) = (3+2+3i-2i)(2+3i) = (5+i)(2+3i)$$

$$= 10-3+2i+15i = 7+17i.$$

1b. Expanding:

$$(\frac{1+i}{2+i})^2 = (\frac{(1+i)(2-i)}{4+1})^2 = \frac{1}{25}(3+i)^2 = \frac{8}{25} + \frac{6i}{25}.$$

1b. Expanding:

$$(1+i)^8 = (\sqrt{2}e^{\frac{\pi}{4}})^8 = (\sqrt{2})^8 = 16.$$

2. Let $z = a+bi$ with $|z| = 1$. Then we have that

$$Re(\frac{1}{1-z}) = Re(\frac{1-a+bi}{(1-a-bi)(1-a+bi)}) = Re(\frac{1-a+bi}{1+a^2-2a+b^2}) = Re(\frac{1-a}{1-2a+1}) = \frac{1}{2}.$$

Note that the second to last step used the fact that $a^2+b^2 = 1$ since $|z| = 1$.

3a. Note that $\sqrt{3}+i = \frac{1}{4}e^{\frac{\pi}{6}}$, so $|\sqrt{3}+i| = |\sqrt{3}-i| = \frac{1}{4}$. Thus we have that

$$\left|\frac{(\sqrt{3}+i)^{100}}{(\sqrt{3}-i)^{100}}\right| = |\frac{4}{4}| = 1.$$

3b. Using the formula for the (capital A) argument with the appropriate quadrant, we get

$$Arg(-1-\sqrt{3}i) = -\pi + \arctan(\frac{\sqrt{3}}{1}) = -\frac{2\pi}{3}.$$

3c. Using the same process as previous question except adding an additional periodic term:

$$arg(1-\sqrt{3}i) = \arctan(\frac{\sqrt{3}}{1}) + 2\pi n = -\frac{2\pi}{3} + 2\pi n, n \in \mathbb{Z}.$$

3d. Expanding:

$$arg(-1+2i) = \arctan(-2) + 2\pi n, n \in \mathbb{Z}.$$

4a. The principal argument of the number geometrically is $\frac{3\pi}{4}$. Writing it in polar form uses this angle along with its magnitude to get $3\sqrt{2}e^{\frac{3}{4}\pi i}$.

4b. Writing the numerator and denominator in polar coordinates and dividing, we get

$$\frac{1-i}{-\sqrt{3}+i} = \frac{\sqrt{2}e^{-\frac{\pi}{4}}}{4e^{\frac{5\pi}{6}}} = \frac{1}{\sqrt{2}}e^{\pi i\frac{13}{12}} = \frac{1}{\sqrt{2}}e^{-\pi i\frac{11}{12}}.$$

Thus the principal argument is $\frac{11}{12}$.

4c. Writing the number first in polar form then squaring, we get

$$(\sqrt{3}-i)^2 = (2e^{-\frac{\pi}{6}i})^2 = 4e^{-\frac{\pi}{3}i}.$$

Thus the principal argument is $-\frac{\pi}{3}$

5a. This is note true, for example consider the case that $Arg(z_1) = Arg(z_2) = \frac{3\pi}{4}$. Then it can't be that $Arg(z_1z_2) = \frac{3\pi}{2}$ so the identity can't hold.

5b. The statement is true. Taking the conjugate reflects the original number accross the real axis, which corresponds with the negative of the angle (barring the case where the number is a negative real number, which wasn't included in the question).

5c. This statement is true. For all non real numbers this is true as a consequence of the previous part. For real numbers, this is true since $arg(z) = \pi + 2\pi n = -\pi + 2\pi n = -arg(\overline{z})$ for the case where z is negative and real.

5d. This is true by the definition of $arg(z)$.

6a. Expanding with De Moivre's formula, we get

$$i\sin(3\theta) = ((\cos\theta + i\sin\theta)^3 - \cos(3\theta)).$$

The real part must go to zero, so we only care about the imaginary part:

$$i\sin(3\theta) = 3i\sin\theta\cos^2\theta - i\sin^3\theta = 3i\sin\theta - 3i\sin^2\theta - i\sin^3\theta.$$

$$= 3i\sin\theta - 4i\sin^3\theta.$$

Dividing by i gives the required result.

6b. Rewriting as exponentials, we have

$$\sum_{k=0}^{n}\cos k\theta = \sum_{k=0}^{n}\frac{1}{2}\left(e^{ik\theta}+e^{-ik\theta}\right).$$

Applying the geometric series, this becomes

$$= \frac{1}{2}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} + \frac{1-e^{-i(n+1)\theta}}{1-e^{-i\theta}}\right) = \frac{1}{2}\left(-\frac{e^{-i\frac{1}{2}\theta}-e^{i(n+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta}+e^{i\frac{1}{2}\theta}} + \frac{e^{i\frac{1}{2}\theta}-e^{-i(n+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta}-e^{i\frac{1}{2}\theta}}\right).$$

$$= \frac{1}{2} + \frac{e^{i(n+\frac{1}{2}\theta}-e^{-i(n+\frac{1}{2}\theta}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}} = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})}.$$

7a. Expanding using the formula derived in class with $m = 3$:

$$\int_0^{2\pi}\cos^6\theta d\theta = \frac{1}{2^{2m}}C_{2m}^{m}2\pi = \frac{1}{8\cdot 8}20\cdot 2\pi = \frac{5}{8}\pi.$$

7b. First do a change of variables, using $\theta' = 2\theta$:

$$\int_0^{4\pi} \sin^6(2\theta)d\theta = \frac{1}{2}\int_0^{4\pi} \sin^6(\theta')d\theta' = \frac{1}{2^{2m}}C^m_{2m}2\pi = \frac{5}{8}\pi.$$

8a. Let $z = x + iy$. Then using the definition of magnitude, we get

$$|z - 1 - i|^2 = (x-1)^2 + (y-1)^2 = x^2 + (y+2)^2.$$

$$-2x + 1 - 6y - 3 = 0 \implies y = -\frac{1}{3}x - \frac{1}{3}.$$

Thus the resultant shape is a line.

8b. Let $z = x + iy$. Then we get that

$$|z|^2 = x^2 + y^2 = (2|z+1|)^2 = 4(x+1)^2 + 4y^2 \implies 0 = 3x^2 + 3y^2 - 8x + 1.$$

$$\implies 3(x - \frac{4}{3})^2 + 3y^2 = \frac{10}{3} \implies (x - \frac{4}{3})^2 + y^2 = \frac{10}{9}.$$

This is the equation for a circle shifted in the x direction by $\frac{4}{3}$ with radius $\frac{10}{9}$.

8c. Geometrically, note that this describes the set of all points for whom the sum of the distance between $(1,0)$ and $(-1,0)$ is equal to 4. This is the definition of an ellipse, from which simple geometry shows that the two major dimensions are $a = \sqrt{3}$ and $b = 2$. Thus the equation for the resulting shape is

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1.$$

9. To find the final upper bound, we first find the lower bound of $|z-5|$. To do this first note that $|z - 5| \geq ||z| - 5|$. Also, using the condition given to use, we have that

$$|z - 1| \leq 1 \implies ||z| - 1| \leq 1 \implies |z| \leq 2$$

(we can get rid of the absolute value in the last step since the only way that that wouldn't be valid is if $|z| < 1$ which is an even stronger statement than $|z| \leq 2$). Putting these two statements together we get that $|z - 5| \geq ||z| + 5| \geq 5 - 3 = 3$. Using this we have that

$$\left|\frac{1}{z-5}\right| \leq \frac{1}{3}.$$

10a. Expanding with Euler's identity, we get

$$z(t) = 2e^{it} = 2\cos t + 2i \sin t.$$

Because $|z(t)| = 2\forall t$ of the above definition showing it goes through the entire span of the circle over $0 \leq t \leq 2\pi$, it must be a circle of radius 2. You can see the sketches in figure 1. part a is shifted upwards by 1 in the imaginary direction since you add i to it, while the second one spirals since if you expand you get an exponential e^t term in front which causes the line to grow as rotates around.

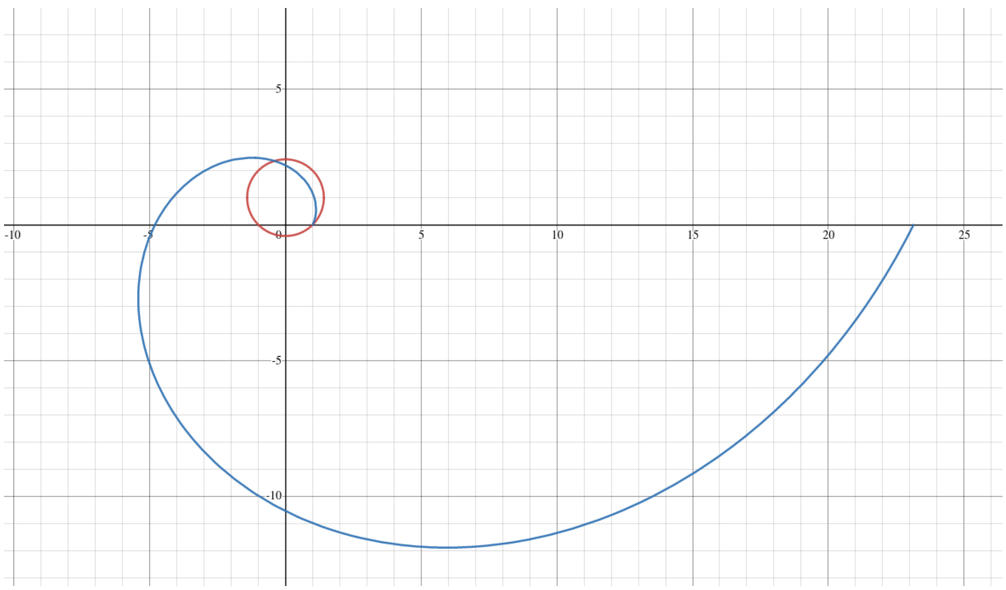

Figure 1: Sketchs for question 10. The red circle is part a and the blue line is part b.