Math 322 Homework 6

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Question 2. Consider G, H, K as in the question. Then by theorem 1.5 in the textbook we have that $|G| = |H|[G : H]$, $|H| = |K|[H : K]$ and $|G| = |K|[G : K]$. Multiplying these three identities together, we get $|K|[G : K]|G||H| = |H|[G : H]|K|[H : K] \implies [G : K] = [G : H][H : K]$.

Question 3. Let $x \in G$ and $y \in (H_1 \cap H_2)x$. Then there exists $h \in H_1 \cap H_2$ s.t. $hx = y$, so h also witness that both $y \in H_1x$ and $y \in H_2x$. Since this is true of any y we have that $(H_1 \cap H_2)x \subseteq H_1x \cap H_2x$.

For the other direction, let $y \in H_1x \cap H_2x$. Then there exist $h_1 \in H_1, h_2 \in H_2$ with $y = h_1x$ and $y = h_2x$. But every element in a group is invertible so $h_1 = h_2 = yx^{-1}$, so in particular $h_1 \in H_1 \cap H_2 \implies (H_1 \cap H_2)x \supseteq H_1x \cap H_2x$. Putting the two last paragraphs together we get that $(H_1 \cap H_2)x = H_1x \cap H_2x$.

To prove Poincaré's theorem, since we have that $[G : H_1] < \infty$ and $[G : H_2] < \infty$, we can write $G = H_1x_1 \cup \ldots \cup H_1x_m$, $G = H_2y_1 \cup \ldots \cup H_2y_n$ for some $x_i \in G, y_i \in G$ with $H_1x_i \cap H_1x_j = \emptyset, H_2y_i \cap H_2y_j = \emptyset$ for $i \neq j$. By our previously proven result, every coset $(H_1 \cap H_2)z$ can be written as $H_1z \cap H_2z$, but there are only m and n unique cosets for H_1 and H_2 in G respectively, so there are at most $mn < \infty$ unique cosets generated this way.

Question 4. Let $G = \langle s_1, s_2, \ldots, s_n \rangle$ and assume that $H \subseteq G$ with finite index. Since H has finite index we can write $G = Hx_1 \cup Hx_2 \cup \ldots \cup Hx_{n-1}$ with $x_1 = 1$. Thus for every combination x_i, s_j, we have that there exists $h_{ij}, x_{k_{ij}}$ such that $x_is_j = h_{ij}x_{k_{ij}}$. I claim that the finite set of all these h_{ij}s generate H. To see why, let $h \in H$. Since G is finitely generated we can write $h = s_{i_1} \cdot \ldots s_{i_m}$. Since $x_1 = 1$, we can write $s_{i_1} = x_1s_{i_1} = h_{1i_1}x_{k_{1i_1}}$. We've thus converted our previous expression for h into $h = h_{1i_1}x_{k_{1i_1}}s_{i_2} \cdots s_{i_m}$. Now considering $x_{k_{1i_1}}s_{i_2}$, we can repeat this process repeatedly to convert each element in this product to purely elements of H, to arrive at a product of the form $h = h_{1i_1} \cdots h_{mi_m}x_{k_{mi_m}}$. I claim that $x_{k_{mi_m}} = x_1 = 1$. Since $h_{1i_1} \cdots h_{mi_m} \in H$, if $x_{k_{mi_m}} \neq x_1$ then the right side of the equality wouldn't be in H, but since $h \in H$ it must be that the last element is x_1. Thus we have that $h = h_{1i_1} \cdots h_{mi_m}$ is a finite combination of the h_{ij}s.

Question 5. Denote $f_{hk}(x) = hxk$ be the elements of the group described, and let F be the set of all such maps. Clearly f_{hk} permutes elements of G, so we just need to show that it is indeed a group. For closure, let f_{hk} and $f_{h'k'}$ be maps and note that $f_{hk}f_{h'k'}x = hh'xk'k = f_{(hh')(k'k)}x$ which is in F (since H, K are subgroups $hh' \in H$ and $k'k \in K$). Note that $f_{h^{-1}k^{-1}}f_{hk} = h^{-1}hxkk^{-1}x = x$, so invertibility is fulfilled. Finally since they are subgroups $1 \in H, 1 \in K$, so $f_{11}x = 1x1 = x$ for identity. Since F is a group and it permutes elements of G, it is a group of transformations.

Consider an arbitrary combination of these maps, $f_{h_1k_1}f_{h_1k_1} \ldots f_{h_mk_m}x = h_1h_2 \ldots h_mxk_m \ldots k_1$. Since H, K are groups, by closure $h_1h_2 \ldots h_m \in H$ and $k_m \ldots k_1 \in K$, so $f_{h_1k_1}f_{h_1k_1} \ldots f_{h_mk_m}x \in HxK$. But also every element $y = hxk \in HxK$ is reachable from x via f_{hk}, so we have that the orbit of x is exactly HxK.

Now suppose G is finite. I will prove the first equality, the second follows by the exact same argument except with right multiplication replaced with left and vice versa. Let $A = x^{-1}Hx \cap K$. I claim that there is a bijection between K/A to HxK/H, more specifically the mapping $Ak \to Hxk$. To show that it is well defined, consider k, k' such that $Ak = Ak'$. Then we have that $k(k')^{-1} \in A \implies k(k')^{-1} \in x^{-1}Hx$, which implies that $xk(k')^{-1}x^{-1} \in H \implies Hxk = Hxk'$.

To show one-to-one, assume that for some k, k' we have that $Hxk = Hxk'$. Then just applying the same logic we just used in reverse, $xk(k')^{-1}x^{-1} \in H \implies k(k')^{-1} \in x^{-1}Hx \implies k(k')^{-1} \in A$ (since also $k, k' \in K$) $\implies Ak = Ak'$. The mapping is clearly onto, since for any coset Hxk of H in HxK, Mk maps to it. Thus $|A|$ is the cardinality of the number of cosets of H in HxK and each one has size $|H|$, so putting this together gives $|HxK| = |H||A| = |H|\left|K : x^{-1}Hx \cap K\right|$.

Question 3. Let $g = (a, b) \in G$ and $k = (1, c) \in K$. Note that as proven in homework 2, $g^{-1} = (\frac{1}{a}, -\frac{b}{a})$. Then we have that

$$g^{-1}kg = \left(\frac{1}{a}, -\frac{b}{a}\right)(1, c)(a, b) = \left(\frac{1}{a}, -\frac{b}{a}\right)(a, b + c) = (1, \frac{c}{a} - \frac{b}{a}) \in K.$$

Thus K is normal. For the second part, define a map $\phi : G/K \to (\mathbb{R}^*, \cdot, 1)$ as $(a, b)K \to a$. Since multiplication by $(1, c)$ scales the second element arbitrarily, this is a well defined function as $a \in \mathbb{R}$ is the only free parameter in both sides. It is also injective and onto, since for different a on the left produce different outputs and for any real a, choosing $(a, 0)$ produces it. Thus $G/K \cong (\mathbb{R}^*, \cdot, 1)$.

Question 4. Let H be a subgroup of G with index 2, for any $h \in H$, $hH = Hh$. Since $[G : H] = 2$, $H' = G \setminus H$ is also a group. For any $h' \in H'$, we also have that $h'H = H'$ and $Hh' = H'$, since otherwise any element $h \in H$ with $hh' \in H$ would imply that $h' \in H'$, contradiction. Since $x \in H$ or $x \in H'$ are the only possibilities, we thus have that in general $xH = Hx \forall x \in G$. Applying x^{-1} on both sides give $xHx^{-1} = H$, so H is normal.

To see that A_n is normal in S_n, all we must do is show that $[S_n : A_n] = 2$, then by the previously proven property the result follows. By the previously shown result in the textbook in section 1.7 we know that $|S_n| = 2|A_n|$, but we also know that by theorem 1.5 $|S_n| = [S_n : A_n]|A_n|$, which when put together give $[S_n : A_n] = 2$ as required.

Question 5. Consider normal subgroups H_1, H_2 and let $H = H_1 \cap H_2$. Let $x \in G$. Then for all $h_1 \in H_1$, $xh_1x^{-1} \in H_1$ and for all $h_2 \in H_2$, $xh_2x^{-1} \in H_2$. But then for any $h \in H$ using these facts we have that $xhx^{-1} \in H_1$ and $xhx^{-1} \in H_2$, i.e. $xhx^{-1} \in H_1 \cap H_2 = H$, so H is normal. If instead of just two normal subgroups we had a list of normal subgroups $H_1, H_2 \ldots$ with $H = H_1 \cap H_2 \cap \ldots$, we can repeatedly apply the version just shown to reduce the problem until only a single normal subgroup remains.

Let H, K be normal subgroups of G. Let $hk \in HK$. Then for any $x \in G$ we have $xhkx^{-1} = xhx^{-1}xkx^{-1}$. Both $xhx^{-1} \in H$ and $xkx^{-1} \in K$ by hypothesis, so we have that $xhkx^{-1} \in HK$, the requirement for HK to be normal.