Math 220 Homework 2

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Question 1. Prove that if $a \in \mathbb{Z}$, then $4 \not| a^2 + 1$.

There are two cases: either a is even or it is odd. If it is even, then it can be represented as $a = 2m$, $m \in \mathbb{Z}$ and we get

$$a^2 + 1 = 4m^2 + 1$$

Since $m^2 \in \mathbb{Z}$, $a^2 + 1$ is not divisible by four in this case. In the other case, a being odd, it can be represented as $a = 2n + 1$ and we get

$$a^2 + 1 = (2n + 1)^2 + 1 = 4n^2 + 4n + 1 + 1 = 4(n^2 + n) + 2$$

Since $n^2 + n \in \mathbb{Z}$ this also isn't divisible by four. Thus in all cases $4 \not| a^2 + 1$ and so we are done. $\square$

Question 2. Let x be a positive real number. Prove that if $x - \frac{3}{x} > 2$, then $x > 3$.

Rearranging, we get

$$x - \frac{3}{x} > 2 \Rightarrow x^2 - 2x - 3 > 0 \Rightarrow (x - 3)(x + 1) > 0$$

Since $x + 1$ is always positive (since $x > 0$), the only way this is true is if $x - 3 > 0$, i.e. $x > 3$ and we're done. $\square$

Question 3. Prove that if $k \in \mathbb{Z}$, then $3|k(2k + 1)(4k + 1)$.

By Euclidean division there are three possible cases: $k = 3m, k = 3m + 1$ or $k = 3m + 2, m \in \mathbb{Z}$.
Case $k = 3m$: In this case, we have that the expression in the question becomes

$$k(2k + 1)(4k + 1) = 3m(2k + 1)(4k + 1)$$

This is divisible by 3 so we're done with this case.
Case $k = 3m + 1$: Again writing the expression we get

$$k(2k + 1)(4k + 1) = k(4k + 1)(6m + 3) = 3k(4k + 1)(2m + 1)$$

This is divisible by 3 so we're done with this case.
Case $k = 3m + 2$: Rewriting the expression one last time gives

$$k(2k + 1)(4k + 1) = k(2k + 1)(12m + 9) = 3k(2k + 1)(4m + 3)$$

This is divisible by 3 so we're done with this case.
Since all cases were consistent with the result the given expression must be divisible by 3. $\square$

Question 4. Let $n \in \mathbb{Z}$. **a.** Show that if $3|n$ and $4|n$, then $12|n$. **b.** Use the previous part to show that if $n > 3$ is a prime, then $n^2 \equiv 1 (\text{mod } 12)$.

Question 4a. Using the hypotheses we can write $n = 3a, a \in \mathbb{Z}$. Since n is also divisible by 4 and $4 \not| 3$, a must be divisible by 4, i.e. $a = 4b, a \in \mathbb{Z}$. Thus $n = 12b$ and we're done. □
Question 4b. Since n is prime it is not divisible by either 2 or 3. Thus we have $n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1, a \in \mathbb{Z}$. Define $m = n^2 - 1 = 4(m^2 + m)$ Since it also isn't divisible by 3 we either get $n = 3b + 1$ or $n = 3c + 2, b, c \in \mathbb{Z}$. In the first case, we get

$$n^2 = 9b^2 + 6b + 1 = 3(3b^2 + 2b) + 1 = m + 1$$

In the second we get that

$$n^2 = 9c^2 + 12c + 4 = 3(3c^2 + 4c + 1) + 1 = m + 1$$

Either way m is divisible by both 3 and 4, so by the first part of the question $12|m$. Since $n^2 = m + 1$ we have that $n^2 \equiv 1 (\text{mod } 12)$. □
Question 5. Prove that if $n^3 + n^2 - n + 3$ is a multiple of three, then n is a multiple of three.

We will use proof by contradiction, so assume that $n^3 + n^2 - n + 3$ is divisible by 3 but n isn't divisible by 3. We will consider the cases that $n = 3m + 1$ and $n = 3m + 2$.
Case $n = 3m + 1$: Expanding we get

$$n^3 + n^2 - n + 3 = 27m^3 + 27m^2 + 9m + 1 + 9m^2 + 6m + 1 - 3m - 1 + 3$$

$$= 3(9m^3 + 18m^2 + 5m) + 1$$

Thus the expression isn't divisible by 3 which contradicts our assumption it was.
Case $n = 3m + 2$: Expanding we get

$$n^3 + n^2 - n + 3 = 27m^3 + 54m^2 + 36m + 8 + 9m^2 + 12m + 4 - 3m + 2 + 3 = 3(18m^3 + 21m^2 + 17m) + 2$$

It is not divisible by 3, which contradicts our assumption it was. In both cases a contradiction arises, so our original assumption must have been false and n is divisible by 3. □
Question 6. Let $x \in \mathbb{R}$. Then, prove that $x^2 + |x - 6| > 5$.

Consider three possibilities: $x < 1$, $1 \leq x \leq 2$, $2 < x \leq 3$ and $x > 3$.
Case $x < 1$: In this case we have

$$x^2 + |x - 6| > 0 + |1 - 6| = 5$$

Case $1 \leq x \leq 2$: In this case we have

$$x^2 + |x - 6| > 1 + |2 - 6| = 5 > 5$$

Case $2 < x \leq 3$: In this case we have

$$x^2 + |x - 6| > 4 + |3 - 6| = 7 > 5$$

Case $x > 3$: In this case we have

$$x^2 + |x - 6| > 9 + 0 = 0 > 5$$

Since this covers all cases we're done. $\square$

Question 7. Let $x, y \in \mathbb{Z}$. Prove that $3 \not| (x^3 + y^3)$ if and only if $3 \not| (x + y)$.

We will prove the contrapositive, i.e. that $3|(x + y)$ if and only if $3|(x^3 + y^3)$. For the forward direction of this statement, assume that $3|(x + y)$. Then we have that $x + y = 3m, m \in \mathbb{Z}$ and

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 3m(x^2 - xy + y^2)$$

This is divisible by 3 so this direction is done. For the reverse direction, assume $x^3 + y^3 = 3m, m \in \mathbb{Z}$. Expanding we get that the following is divisible by 3 also:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = (x + y)((x + y)^2 - 3xy)$$

This means that either $x + y$ is divisible by three or $(x + y)^2 - 3xy$ is divisible case (or both, in which the argument for either case works). In the former case the result is shown automatically, and in the latter this implies that $\exists n \in \mathbb{Z}$ s.t. $(x + y)^2 - 3xy = 3n \Rightarrow (x + y)^2 = 3(n + xy)$. Since $(x + y)^2$ is divisible by 3 and 3 is prime this means that $x + y$ must also be divisible by 3, and we're done in both cases. $\square$