UBC Mathematics 320(101)—Assignment 6
Due by PDF upload to Canvas at 18:00, Saturday 21 Oct 2023

References: Loewen, lecture notes on Sequences and Series (2023-10-12 or newer—see Canvas); Rudin, pages 1–12, 22–23, 47–58; Thomson-Bruckner-Bruckner, Chapter 2.

1. Let $0 < a_1 < b_1$ and define $\qquad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \dfrac{a_n + b_n}{2}, \quad n \in \mathbb{N}.$

(a) Prove that the sequences (a_n) and (b_n) both converge.
(Suggestion: Use induction to prove $0 < a_n < a_{n+1} < b_{n+1} < b_n$.)

(b) Prove that the sequences (a_n) and (b_n) have the same limit.

2. (a) Suppose $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence with integer values.

(i) Prove that both numbers below are integers:

$$\lambda = \liminf_{n\to\infty} z_n, \qquad \mu = \limsup_{n\to\infty} z_n.$$

(ii) Prove that there are infinitely many integers n for which $z_n = \lambda$.

(b) Let $d_n = p_{n+1} - p_n$ ($n \in \mathbb{N}$) denote the sequence of prime differences, built from the sequence of primes

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ \ldots.$$

(i) Prove that $\limsup\limits_{n\to\infty} d_n = +\infty$.

(ii) Some mathematicians believe that

$$\delta \stackrel{\text{def}}{=} \liminf_{n\to\infty} d_n = 2. \tag{$*$}$$

However, the best estimate of δ known to date is $2 \le \delta \le 246$. Identify by name a famous unsolved problem in mathematics that is equivalent to proving or disproving line $(*)$. (After giving the name, clearly explain the required relationship.)

3. (a) Prove: For any sequences (a_n) and (b_n) of nonnegative real numbers,

$$\limsup_{n\to\infty} (a_n b_n) \le \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right),$$

provided the right side does not involve the product of 0 and ∞.

(b) Give an example in which the result of part (a) holds with a strict inequality.

4. (a) Show that for any $r \ge 1$, one has

$$n(r-1) \le r^n - 1 \le nr^{n-1}(r-1) \qquad \forall n \in \mathbb{N}.$$

(b) Prove that for each real $a \ge 1$, the following sequence converges:

$$x_n = n\left(a^{1/n} - 1\right), \qquad n \in \mathbb{N}.$$

(c) Prove that the sequence in part (b) also converges for each real $a \in (0, 1)$.

(d) Let $L(x) = \lim\limits_{n\to\infty} n\left(x^{1/n} - 1\right)$ for $x > 0$. Prove that $L(ab) = L(a) + L(b)$ for all $a > 0, b > 0$.

Note: Present solutions that use only methods discussed in MATH 320. No calculus, please!