PHYS408 Homework 2

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Question 1a. In class, we derived the following result in the Fraunhofer limit:

$$E(x',y') \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{x'^2+y'^2}{2z}} \tilde{E}(k_x,k_y).$$

Thus all we have to do is compute the Fourier transform of the two slits. We can describe the slits as two shifted rectangles (assuming the input field E is uniform and has magnitude 1):

$$E(x,y) = \text{rect}\left(\frac{x}{d_x}\right)\left(\text{rect}\left(\frac{y-\Delta/2}{d_y}\right) + \text{rect}\left(\frac{y+\Delta/2}{d_y}\right)\right).$$

Also recall the following transform:

$$\mathcal{F}(\text{rect}(ax)) = \frac{1}{|a|}\text{sinc}\left(\frac{k_x}{a}\right).$$

Using this we get the following:

$$\tilde{E}(k_x,k_y) = d_x d_y \text{sinc}\left(\frac{d_x k_x}{2\pi}\right)\text{sinc}\left(\frac{d_y k_y}{2\pi}\right)\left(e^{i\Delta k_y/2} + e^{-i\Delta k_y/2}\right).$$

Intensity is proportional to electric field squared:

$$I(x',y') \propto |E(x',y')|^2 = \frac{1}{\lambda z}\text{sinc}\left(\frac{d_x k_x}{2\pi}\right)\text{sinc}\left(\frac{d_y k_y}{2\pi}\right)\cos\left(\frac{\Delta k_y}{2}\right)$$

where as usual $k_x = \frac{kx'}{z}$ and $k_y = \frac{ky'}{z}$.

Question 1b. See figure 1. The code used to produced the graphs is here:

```
import numpy as np
import matplotlib.pyplot as plt

dx = 0.01
dy = 0.001
delta = 0.005
lam = 500e-9
z = 50

k = 2*np.pi/lam
```

```
c = 3e8
eps = 8.85e-12

x = np.linspace(-0.05, 0.05, 1000)
y = np.linspace(-0.05, 0.05, 1000)

kx = k*x/z
ky = k*y/z

Ix = c*eps*dx*dy/lam/z*np.sinc(dx*kx/(2*np.pi))
Iy = c*eps*dx*dy/lam/z*np.sinc(dy*ky/(2*np.pi))*np.sin(delta*ky/2)

plt.plot(x, Ix)
plt.title("Intensity of Double Slit for y=0 Axis")
plt.xlabel("x' (m)")
plt.ylabel("I (W/m$^2$)")
plt.show()

plt.plot(y, Iy)
plt.title("Intensity of Double Slit for x=0 Axis")
plt.xlabel("y' (m)")
plt.ylabel("I (W/m$^2$)")

plt.show()
```

Question 1c. The Fraunhofer limit applies only if $\frac{x^2}{\lambda}, \frac{y^2}{\lambda} \ll \frac{z}{\pi}$. In our case, these are $\frac{(d_x/2)^2}{\lambda} = 50$ and $\frac{(d_y/2)^2}{\lambda} = \frac{1}{2}$. In comparison to $\frac{z}{\pi} \approx 15.9$, we see that the x-axis is no in the Fraunhofer approximation, so our results above are not totally valid.

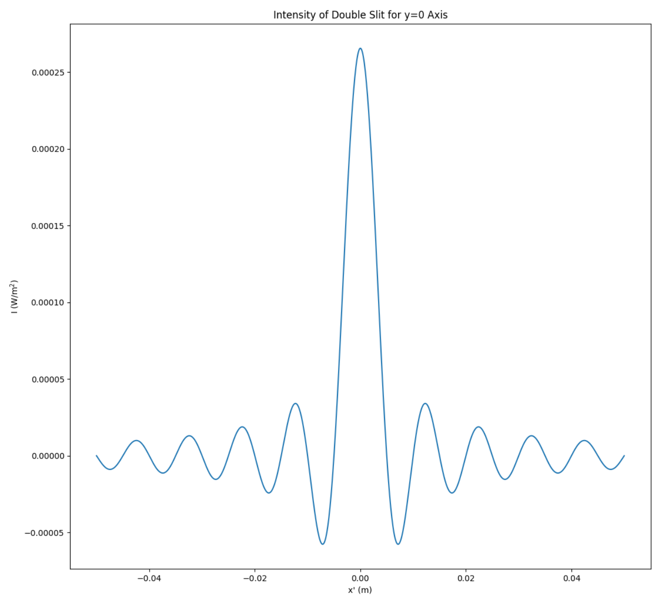
Question 1d.

Question 2a. The phase function is going to be proportional to the thickness, so assuming no transmission occurs outside of d_x, we have

$$t(x) = \text{rect}\left(\frac{x}{d_x}\right) e^{ikn\left(\bar{d}+\frac{d_0}{2}\sin(2\pi x/\Lambda)\right)} e^{-ik\frac{d_0}{2}\sin(2\pi x/\Lambda)} = \text{rect}\left(\frac{x}{d_x}\right) e^{ikn\bar{d}+ik(n-1)\frac{d_0}{2}\sin(2\pi x/\Lambda)}.$$

Question 2b. The incoming plane wave can be represented as $E = E_0 e^{ik\theta_i x}$. Thus the transmitted waves are:

$$\begin{aligned}
E_t(x) &= E(x) \cdot t(x) E_0 e^{ik\theta_i x} e^{ikn\bar{d}+ik(n-1)\frac{d_0}{2}\sin(2\pi x/\Lambda)} \\
&= E_0 \sum_{q=-\infty}^{\infty} J_q\left(k(n-1)\frac{d_0}{2}\right) e^{i2\pi q x/\Lambda + ik\theta_i x + ikn\bar{d}} \\
&= E_0' \sum_{q=-\infty}^{\infty} J_q\left(k(n-1)\frac{d_0}{2}\right) e^{i2\pi q x/\Lambda + ik\theta_i x} \\
&= E_0' \sum_{q=-\infty}^{\infty} J_q\left(k(n-1)\frac{d_0}{2}\right) e^{ikx(\lambda q x/\Lambda + \theta_i)}.
\end{aligned}$$



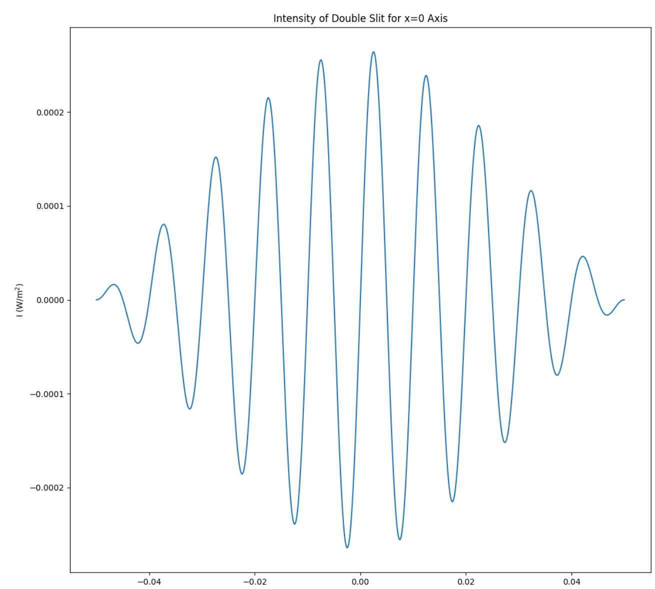


Figure 1: Graphs for question 1b.

This is exactly an infinite combination of plane waves with angles $\theta_q = \theta_i + \frac{q\lambda}{\Lambda}$, as required.

Question 3. For the image plane to be a reproduction of the original object, the transfer function of the whole system between the object plane and the image plane must be be a constant (there can potentially be a constant factor scaling, but there can't be any x, y dependent phase terms). There are three optical components in play: the free space before the lens, the lens, and the free space after the lens. Ignoring the constant factors since they're not relevant, the total transfer function is then:

$$H(x', y') \propto e^{-i\frac{k}{2}(x'^2+y'^2)\frac{1}{d_o}} e^{-i\frac{k}{2}(x'^2+y'^2)\frac{1}{d_i}} e^{i\frac{k}{2}(x'^2+y'^2)\frac{1}{f}} = e^{-i\frac{k}{2}(x'^2+y'^2)\left(\frac{1}{d_0}+\frac{1}{d_i}-\frac{1}{f}\right)}.$$

The only way that there is no x', y' dependence is if the rightmost bracketed term is 0, which gives:

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}.$$

Question 4. From lecture, we know that the $E(x', y') \propto \mathcal{F}(\overline{E}(x, y))$, so if we want $E(x', y')$ to be proportional to $\tilde{t}(x, y)$, we need $\overline{E}$ to have a small phase factor. The expression for $\overline{E}$ is

$$\overline{E} = E_0 t_0(x, y) e^{-i\frac{k}{2}(x^2+y^2)\left(\frac{1}{f}-\frac{1}{f-\Delta}\right)}.$$

Since $\Delta \ll f$, we can say $\frac{1}{f} - \frac{1}{f-\Delta} = -\frac{\Delta}{f(f-\Delta)} \approx \frac{\Delta}{f^2}$. The Fourier transform of $\overline{E}$ only turns into $\tilde{t}(x, y)$ if the phase factor is very small while $x^2 + y^2 < D^2$, which occurs when $\frac{k}{2}(x^2 + y^2)\frac{\Delta}{f^2} \ll 1 \implies \Delta \ll \frac{2f^2}{kD^2}$.