MATH 305 Homework 4

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05/02/22

1. Find a conformal mapping from the following set onto the upper half plane $S^{'} = \{(u, v) \mid v > 0\}$:
   (a) $S = \{x > 0, -\frac{\pi}{2} < y < \frac{\pi}{2}\}$
   Let $f(z) = \sin(iz) = \sin(-y + ix) = -\cosh x \sin y + i \sinh x \cos y$. Then for $x > 0, -\frac{\pi}{2} < y < \frac{\pi}{2}$, we have that $\sinh x \cos y > 0$ and $-\cosh x \sin y$ spans the reals.
   (b) $S = \{-1 < x < 3, y > 0\}$
   Let $f(z) = \sin\left(\frac{\pi}{4}(z-1)\right)$. Then the real part of the argument of the sin function goes between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ and the imaginary part varies over all the positive reals, which we already saw in class maps to the upper half plane as required.
2. Evaluate the following
   (a) $log(i)$

$$\log(i) = \ln 1 + iArg(i) + 2\pi k i = i\frac{\pi}{2} + 2\pi k i, k \in \mathbb{Z}.$$

   (b) $Log(\sqrt{3} - i)$

$$\text{Log}(\sqrt{3} - i) = \ln 2 + iArg(z) = \ln 2 - \frac{\pi}{6}i.$$

   (c) $log(e^{1+i})$

$$\log(e^{1+i}) = \ln e + iArg(1+i) + i2\pi k = 1 + i\frac{\pi}{4} + 2\pi k i, k \in \mathbb{Z}.$$

   (d) $e^{log(1+i)}$

$$e^{\log(1+i)} = e^{\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)} = \sqrt{2}e^{\frac{\pi}{4}} = 1 + i.$$

3. Find all values of
   (a) $e^z = -1 - i$

$$\log e^z = z + 2\pi k i = \log(-1-i) = \ln\sqrt{2} - i\frac{3\pi}{4} + 2\pi k i \implies z = \ln\sqrt{2} - i\frac{3\pi}{4} + 2\pi k i, k \in \mathbb{Z}.$$

   (b) Principal Values of $(1+i)^i$

$$(1+i)^i = e^{i\text{Log}(1+i)} = e^{-\frac{\pi}{4} + i\ln\sqrt{2}}.$$

   (c) $i^{\frac{1}{3}}$

$$i^{\frac{1}{3}} = e^{\frac{1}{3}\log i} = e^{\frac{1}{3}i\left(\frac{\pi}{2} + 2\pi k\right)}, k \in \mathbb{Z}.$$

4. Solve the following equations

(a) $Log(z^2-1) = \frac{i\pi}{2}$

$$z^2 - 1 = e^{i\frac{\pi}{2}} = i \implies z^2 = \sqrt{2}e^{i\frac{\pi}{4}} \implies z = \sqrt[4]{2}e^{i\left(\frac{\pi}{8}\right)} \text{ or } \sqrt[4]{2}e^{i\left(\frac{9\pi}{8}\right)}.$$

(b) $e^{2z} + e^z + 1 = 0$

$$(e^z)^2 + e^z + 1 = 0 \implies e^z = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = e^{\pm i\frac{2\pi}{3}} \implies z = e^{\frac{2\pi}{3}+2i\pi k} \text{ or } z = e^{\frac{2\pi}{3}+2i\pi k}, k \in \mathbb{Z}.$$

(c) $z^{\frac{1}{2}} + 1 - i = 0$ (here $z^{\frac{1}{2}}$ denotes the principal branch)

There are not solutions. The equation requires that $Re(z^{\frac{1}{2}} = -1 < 0$ but this is not possible in the principal branch, so there are no solutions.

5. Determine the domain of analyticity (branch cut) of

(a) $Log(1+z^2)$

The roots of $1+z^2$ are $\pm i$. From those points, the argument of the Log function is negative when $Re(z) = 0, |Im(z)| > 1$. Therefore we have that the domain of analyticity is $D = \mathbb{C} \setminus \{z \mid Re(z) = 0, |Im(z)| > 1\}$.

(b) $Log(\frac{1-z}{1+z})$

Simplifying the argument of the log assuming $z \neq -1$, this is equivalent to Log $\left(\frac{(1-z)^2}{1-z^2}\right)$. This is not a negative real if $Im(z) \neq 0$, and if it is then the argument is negative only when $|z| \geq 0$. Thus the domain of analyticity is $D = \mathbb{C} \setminus \{z \mid Im(z) = 0, |Re(z)| >= 0\}$

6. Which of the followings are true statements? For the ones that are false find a counterexample

(a) $e^{log(z)} = z$

This is true.

(b) $e^{Log(z)} = z$

This is true.

(c) $Log(e^z) = z$

This is not true. For example $\text{Log}(e^{3\pi i}) \neq 3\pi i$

(d) $log(e^z) = z$

This is not true. For example $\log\left(e^{i\frac{\pi}{2}}\right) = i\frac{\pi}{2} + 2\pi ki \neq i\frac{\pi}{2}$. Is suppose one could argue that the right hand side is contained in the left, although it's a bit like comparing apples to oranges since one is a set and the other is a number so they're not equal.

(e) $log(z_1 z_2) = log z_1 + log z_2$

This is true.

(f) $log(z) = -log(\frac{1}{z})$

This is true.

(g) $log(z^{\frac{1}{2}}) = \frac{1}{2}log(z)$

This is not true. For example consider $z = i$: $\log(i^{\frac{1}{2}}) = i\frac{\pi}{4} + 2\pi ki$ or $i\frac{5\pi}{4} + 2\pi ki \neq i\frac{\pi}{4} + \pi ki = \frac{1}{2}\log(z)$

7. Find a branch cut of $log(z-1)$ that is analytic at all points in the plane except those on the following rays.

(a) $\{x \leq 1, y = 0\}$

Consider the branch cut $(-\infty, 1]$ with $-\pi < \phi < \pi$. Then $\log(z-1)$ is analytic on $D = \mathbb{C} \setminus \{x \leq 1, y = 0\}$

(b) $\{x \geq 1, y = 0\}$

Consider the branch cut $[1, -\infty)$ with $0 < \phi < 2\pi$. Then $\log(z-1)$ is analytic on $D = \mathbb{C} \setminus \{x \geq 1, y = 0\}$

(c) $\{x = 1, y \geq 0\}$

Consider the branch cut $\{z \mid Re(z) = 1, Im(z) \geq 0\}$ with $\frac{\pi}{2} < \phi < \frac{5\pi}{2}$. Then $\log(z-1)$ is analytic on $D = \mathbb{C} \setminus \{x = 1, y \geq 0\}$

8. Find a branch cut for $\sqrt{z(z-1)}$ that is analytic in $C \backslash [0, 1]$ and takes value $\sqrt{2}$ at $z = 2$.

Consider the principle branch as it was defined in class. Then we have that

$$\sqrt{z(z-1)} = |z(z-1)|^{\frac{1}{2}} e^{i\frac{1}{2}Arg(z(z-1))}.$$

$z(z-1)$ is only negative for $z \in [0, 1]$, so this branch satisfies the analyticity requirement. It also satisfies the requirement that $\sqrt{2(2-1)} = |2(2-1)|^{\frac{1}{2}} e^{i\frac{1}{2}Arg(2(2-1))} = \sqrt{2}$, so we're done.

9. Determine a branch of $log(z^2 + 2z + 2)$ that is analytic at $z = -1$ and takes value 0 at $z = -1$, and find its derivative there.

Factoring we get that the given expression is $\log(z^2+2z+2) = \log(z+1+i)(z+1-i)$. Consider simply the principal branch of log, $f(z) = \text{Log}(z^2 + 2z + 2)$. Then since $z^2 + 2z + 2 > 0 \forall z \in \mathbb{R}$, f is analytic at $z = -1$ (in this case f is analytic over $D = \mathbb{C} \setminus \{z \mid Re(z) = -1, |Im(z)| >= 0\}$). In addition we have that $f(-1) = 0$. Finally we can compute the derivative using the chaing rule to be

$$f'(-1) = \frac{2z + 2}{z^2 + 2z + 2} = 0.$$

10. Determine a branch of $log(1 + z^2)$ that is analytic at $z = 0$ and takes the value $2\pi i$ there.

Consider the same branch cut as the principal branch except with angles measured from $\pi < \phi < 3\pi$. In equation this is $f(z) = \text{Log}(1 + z^2) + 2\pi i$. Thus clearly f is analytic at $z = 0$, and by computing we get $f(0) = \text{Log}(1) + 2\pi i = 2\pi i$ as required.