VIII. Introduction to Continuity

A. The Big Picture—Continuity on a HTS

Throughout this section, $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ are Hausdorff topological spaces (HTS's).

Definition. Let $f\colon X \to Y$ be a [single-valued] mapping. To say, f **is continuous [on** $(X, \mathcal{T}_X)$**]** means

$$\forall G \in \mathcal{T}_Y, \quad f^{-1}(G) \in \mathcal{T}_X.$$

In words, for every open subset G of Y, the preimage $f^{-1}(G) = \{x \in X \,:\, f(x) \in G\}$ is an open subset of X. [Shorthand: cts = "continuous"; cty = "continuity".]

This definition is easy to state and to work with, but not very easy to check. We'll fix that in Section C, below.

Lemma. *$f\colon X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed for every closed set C in Y.*

Proof. For any $W \subseteq Y$, we have $f^{-1}(W^c) = f^{-1}(W)^c$. (This takes a little reasoning: try it!) Since a set $C \subseteq Y$ is closed if and only if C^c is open, the result follows. A more detailed argument appears below.

($\Rightarrow$) Let C be any closed subset of Y. Then C^c is open, so continuity implies that $f^{-1}\left(C^c\right) = f^{-1}(C)^c$ is open, i.e., $f^{-1}(C)$ is closed.

($\Leftarrow$) Let G be any open subset of Y. Then G^c is closed, so the assumption implies that $f^{-1}\left(G^c\right) = f^{-1}(G)^c$ is closed, i.e., $f^{-1}(G)$ is open. ////

Example. *In any metric space (X, d), for any point $p \in X$, the function $f\colon X \to \mathbb{R}$ defined below is continuous:*

$$f(x) = d(x, p), \qquad x \in X.$$

Proof. (Outline.) For any open interval (a, b) in $\mathbb{R}$ with $a \geq 0$

$$f^{-1}((a,b)) = \{x \in X \,:\, a < d(x,p) < b\} = \mathbb{B}[p;b) \setminus \mathbb{B}[p;a] = \mathbb{B}[p;b) \cap \left(\mathbb{B}[p;a]^c\right).$$

The set on the right, being an intersection of two open sets, is open. If $a < 0$ and $b > 0$, the calculation gets simpler but the right side is still open:

$$f^{-1}((a,b)) = \{x \in X \,:\, a < d(x,p) < b\} = \mathbb{B}[p;b).$$

If $b \leq 0$, $f^{-1}((a,b)) = \emptyset$ is open again. Since every point of a given open set G in $\mathbb{R}$ can be wrapped in an open interval inside G, this is enough to show that every point of $f^{-1}(G)$ is an interior point, as required. ////

Continuity and Density. A continuous function is characterized by its behaviour on a dense set. This is the content of the Theorem below, for which we warm up by recalling Separation Axiom (HTS4):

$$y_1 \neq y_2 \iff \exists U_1 \in \mathcal{N}(y_1),\ U_2 \in \mathcal{N}(y_2) \,:\, U_1 \cap U_2 = \emptyset.$$

The statements are equivalent iff their negations are equivalent, so we have

$$y_1 = y_2 \iff \forall U_1 \in \mathcal{N}(y_1),\ U_2 \in \mathcal{N}(y_2),\ U_1 \cap U_2 \neq \emptyset. \tag{$\dagger$}$$

Theorem. *Suppose both $f_1, f_2 \colon X \to Y$ are continuous. Then for any set $Q \subseteq X$*

$$\Big[f_1(q) = f_2(q) \quad \forall q \in Q \Big] \implies \Big[f_1(x) = f_2(x) \quad \forall x \in \overline{Q} \Big].$$

Proof. Pick any $x \in \overline{Q}$ and define $y_j = f_j(x)$ in Y. We must show $y_1 = y_2$. We'll use (†) to do this.

So pick any open sets $U_j \in \mathcal{N}(y_j)$. Then both $\Omega_j = f^{-1}(U_j)$ are open sets in X containing x. Let $\Omega = \Omega_1 \cap \Omega_2$: clearly $\Omega \in \mathcal{N}(x)$, so $\Omega \cap Q \neq \emptyset$. (Because $x \in \overline{Q}$.) Pick any $q \in Q \cap \Omega$. Then $f_1(q) = f_2(q)$ by assumption, but also $f_j(q) \in U_j$, so

$$f_1(q) = f_2(q) \in U_1 \cap U_2.$$

In particular, $U_1 \cap U_2 \neq \emptyset$. Since the neighbourhoods $U_j \in \mathcal{N}(y_j)$ are arbitrary, (†) forces $y_1 = y_2$, as required. ////

This result implies, e.g., that any two continuous functions that agree on all points of the form $m/2^n$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ must actually agree on the whole real line.

B. Continuity and Compactness

Compactness of the domain space X has many important consequences.

Theorem (Continuous Image of a Compact Set). *Suppose $f \colon X \to Y$ is continuous, and X is compact. Then $f(X) = \{f(x) : x \in X\}$ is compact in Y. (In particular, $f(X)$ is closed as a subset of Y.)*

Proof. Suppose $\mathcal{G}$ is an open cover for $f(X)$. Use it to define $\mathcal{G}_0 = \left\{ f^{-1}(G) \,:\, G \in \mathcal{G} \right\}$: by continuity, each set in $\mathcal{G}_0$ is open. And clearly, every point of X belongs to some set in $\mathcal{G}_0$, so $\mathcal{G}_0$ is an open cover of X. By compactness, there exists a finite subcover. That is, there exist $G_1, G_2, \ldots, G_N$ such that

$$X \subseteq f^{-1}(G_1) \cup f^{-1}(G_2) \cup \cdots \cup f^{-1}(G_N).$$

Hence

$$f(X) \subseteq f\,(\mathrm{RHS}) \subseteq G_1 \cup G_2 \cup \cdots G_N.$$

(Notes: $f(A \cup B) = f(A) \cup f(B)$ and $f(f^{-1}(C)) \subseteq C$ are true for any f and any sets A, B, C.) Thus $\mathcal{G}$ contains a finite subcover of $f(X)$. ////

Corollary (Boundedness). *Suppose $(X, \mathcal{T}_X)$ is a HTS and (Y, d) is a metric space. If $f \colon X \to Y$ is continuous and X is compact, then $f(X)$ is a bounded subset of Y. That is, there exist $y_0 \in Y$ and $R > 0$ such that*

$$d(f(x), y_0) \leq R \qquad \forall x \in X.$$

Proof. Pick any $y_0 \in Y$ and observe that $\mathcal{G} = \{\mathbb{B}[y_0, r) \,:\, r > 0\}$ is an open cover for Y, hence for the compact set $f(X)$. Extract a finite subcover and take its union: this will be an open ball with centre at y_0 and finite radius. Call the radius R. ////

Corollary (Existence of Extrema). *Suppose $(X, \mathcal{T}_X)$ is a HTS. If $f\colon X \to \mathbb{R}$ is continuous, and X is compact, then there exist $p, q \in X$ such that*

$$\begin{aligned} & f(p) \le f(x)\ \forall x \in X, \quad \text{i.e., } p \text{ is an absolute minimizer for } f \text{ over } X, \\ \text{and} \quad & f(x) \le f(q)\ \forall x \in X, \quad \text{i.e., } q \text{ is an absolute maximizer for } f \text{ over } X. \end{aligned}$$

Proof. [Note that $X \neq \emptyset$: this assumption is built into our definition for an HTS.] Since X is compact and f is continuous, $f(X)$ is compact. This has two consequences:

(1) $f(X)$ is bounded and nonempty. In particular, both quantities below are finite:

$$m \stackrel{\text{def}}{=} \inf\{f(x) \,:\, x \in X\}, \qquad M \stackrel{\text{def}}{=} \sup\{f(x) \,:\, x \in X\}.$$

(2) $f(X)$ is closed. In particular, both $m, M \in f(X)$. To prove this for M, observe that for any $\varepsilon > 0$, the interval $(M-\varepsilon, M+\varepsilon)$ must contain points of both $f(X)$ and $f(X)^c$. (Reason: M is an upper bound for $f(X)$, but $M - \varepsilon$ is not.) This shows $M \in \partial f(X)$, and we know $\partial f(X) \subseteq f(X)$ because $f(X)$ is a closed set. The situation for m is similar.

Now since $m, M \in f(X)$, there must be points $p, q \in X$ such that $m = f(p)$ and $M = f(q)$. These points have the properties announced in the statement. ////

In many practical optimization problems, a compact domain is unavailable, but the growth properties of the function to be extremized provide enough compactness to apply the previous corollary. Here is one situation fitting this description.

Corollary. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous. Suppose further that there exists a function $g\colon [0, +\infty) \to \mathbb{R}$ such that both*

$$\lim_{t\to\infty} g(t) = +\infty \qquad \text{and} \qquad f(x) \ge g(|x|) \quad \forall x \in \mathbb{R}^n.$$

Then f attains its global minimum over $\mathbb{R}^n$. That is, there exists $p \in \mathbb{R}^n$ such that $f(p) \le f(x)$ for all x in $\mathbb{R}^n$.

Proof. Let $c = f(0)$ and consider $K = \{x \in \mathbb{R}^n \,:\, f(x) \le c\}$. Clearly $0 \in K$, so K is not empty. Also, since $K = f^{-1}((-\infty, c])$ is the preimage of a closed set under a continuous function, K is itself a closed set. What is more, K is bounded. To see this, use the constant c to challenge the definition of $g(t) \to \infty$ as $t \to \infty$: this produces some number $R > 0$ so large that

$$\forall t \ge R, \quad g(t) > c. \tag{$*$}$$

Now for each $x \in K$, the condition defining K and the growth condition linking f and g give

$$c \ge f(x) \ge g(|x|).$$

According to $(*)$, this implies $|x| < R$. In other words, K is a subset of the ball $\mathbb{B}[0, R)$. Since K is both closed and bounded, it is compact, so there is some point p in K such that $f(p) \le f(x)$ for all $x \in K$. In particular, $f(p) \le f(0) = c$. Every $x \notin K$ obeys $f(x) > c \ge f(p)$. Thus we have $f(p) \le f(x)$ for all x in $K \cup K^c = \mathbb{R}^n$, as required. ////

Example. The function $f(x) = (x-1)^2(1+\frac{1}{2}\sin(x^2)) - 1$ has an absolute minimum over $\mathbb{R}$ because

$$f(x) \geq \tfrac{1}{2}(x-1)^2 - 1 = x^2 - 2x \geq |x| - 3.$$

The argument above formalizes the idea that the minimum *value* must be less than $f(0) = 0$, so it's safe to look for it in the set K of all x where $f(x) \leq 0$. The inequality involving $g(t) = t - 3$ shows that $K \subseteq [-3, 3]$. (Here it's clear that $p = 1$ gives the minimum for f: the point is just to illustrate the flow of logic above.)

Theorem (Inverse Mappings). *Suppose X is compact and $f\colon X \to Y$ is one-to-one, onto, and continuous. Then $f^{-1}\colon Y \to X$ is well-defined and continuous.*

Proof. It is clear that f^{-1} is well-defined and single-valued: these observations require no topology. To keep the notation simple, define $h = f^{-1}$, so that $h\colon Y \to X$. To prove that h continuous, it suffices to show that $h^{-1}(C)$ is a closed subset of Y for every closed C in X. So pick such a set C. Then, since f is 1-1,

$$\begin{aligned} h^{-1}(C) &= \{y \in Y \,:\, h(y) \in C\} \\ &= \left\{y \in Y \,:\, f^{-1}(y) \in C\right\} \\ &= \{y \in Y \,:\, y \in f(C)\} \\ &= f(C). \end{aligned}$$

Any closed subset C in X inherits compactness from X. This makes $f(C)$ compact by the important result above. And every compact set is closed, so $f(C) = h^{-1}(C)$ is closed, as required. ////

Proposition. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both continuous, then $g \circ f\colon X \to Z$ is continuous too.*

Proof. (Practice) Given any open subset $G \subseteq Z$, consider the preimage:

$$\begin{aligned} (g \circ f)^{-1}(G) &= \{x \in X \,:\, g(f(x)) \in G\} \\ &= \left\{x \in X \,:\, f(x) \in g^{-1}(G)\right\} \\ &= f^{-1}\left(g^{-1}(G)\right). \end{aligned}$$

Since g is continuous, $U = g^{-1}(G)$ is open in Y; since f is continuous, $f^{-1}(U)$ is open in X. ////

C. The Local Picture—Continuity at a Point

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Definition. Let HTS's $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be given, along with a mapping $f\colon X \to Y$ and a specific input point $x \in X$. To say "f is continuous at x" means this: for each $W \in \mathcal{N}_Y(f(x))$, one has $f^{-1}(W) \in \mathcal{N}_X(x)$. Equivalently,

$$\forall W \in \mathcal{T}_Y \text{ containing } f(x),\ \exists U \in \mathcal{T}_X \text{ containing } x \,:\, f(U) \subseteq W.$$

Lemma. *For $f\colon X \to Y$ as above, TFAE:*

(a) *f is continuous on X (i.e., $f^{-1}(W)$ is open for every open W in Y);*

(b) *for each $x \in X$, f is continuous at x.*

Proof. (a⇒b) Obvious. (Fix an arbitrary $x \in X$, and let W be any open set containing $f(x)$. According to (a), $U \stackrel{\text{def}}{=} f^{-1}(W)$ is open. Clearly $x \in U$ and $f(U) \subseteq W$.)

(b⇒a) Let $W \neq \emptyset$ be any open subset of Y, and consider $U = f^{-1}(W)$. To prove U is open, pick any x in U: then W is an open set containing $f(x)$, so (b) says that $f^{-1}(W) = U$ contains an open superset of x. That is, $x \in U^\circ$. This proves $U \subseteq U^\circ$: since $U^\circ \subseteq U$ is true for any U, we have $U = U^\circ$. Thus U is open. ////

Lemma. *Suppose $f\colon X \to Y$ is continuous at x_0, and $g\colon Y \to Z$ is continuous at $y_0 \stackrel{\text{def}}{=} f(x_0)$. Then $h = g \circ f$ is continuous at x_0.*

Proof. Let V be an open set containing $h(x_0)$. then $g^{-1}(V)$ contains an open set containing y_0: call this open set Ω. Then $f^{-1}(\Omega)$ contains an open set (call it U) containing x_0. As noted before,

$$\begin{aligned} h^{-1}(V) = \{x \in X \,:\, g(f(x)) \in V\} &= \left\{x \in X \,:\, f(x) \in g^{-1}(V)\right\} \\ &\supseteq \{x \in X \,:\, f(x) \in \Omega\} = f^{-1}(\Omega) \supseteq U. \end{aligned}$$

Hence $h(U) \subseteq V$, as required. ////

Euclidean Range Spaces. For functions $f\colon X \to \mathbb{R}^n$, we have the following standard facts.

Theorem. *Let $f\colon X \to \mathbb{R}^n$; write $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Suppose $x_0 \in X$.*

(a) *The function f is continuous if and only if each component function $f_i\colon X \to \mathbb{R}$ is continuous.*

(b) *If both $f, g\colon X \to \mathbb{R}^n$ are continuous at the point x_0, then so are $f + g$ and $f \cdot g$. If in addition $n = 1$ and $g(x_0) \neq 0$, then f/g is continuous at x_0.*

The Metric Case. If (Y, d_Y) is a metric space, then every open set V containing y must contain $\mathbb{B}_Y[y; \varepsilon)$ for some $\varepsilon > 0$; if (X, d_X) is a metric space, then every open set U containing x must contain $\mathbb{B}_X[x; \delta)$ for some $\delta > 0$. Thus line (b) of the proposition below is often used to *define* continuity for a mapping between metric spaces.

Proposition. *Let (X, d_X) and (Y, d_Y) be metric spaces, with $x \in X$ and $f\colon X \to Y$. Then TFAE:*

(a) *$f\colon X \to Y$ is continuous at x.*

(b) $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x' \in \mathbb{B}[x;\delta),\ d_Y(f(x'), f(x)) < \varepsilon$.

(c) *For every sequence (x_n) in X obeying $x_n \to x$, one has $f(x_n) \to f(x)$ in Y.*

Proof. (a⇒b) Assume (a). Given any $\varepsilon > 0$, let $V = \mathbb{B}_Y[f(x);\varepsilon)$. Since f is continuous at x, $f^{-1}(V) \in \mathcal{N}(x)$. This means that $f^{-1}(V)$ must contain $\mathbb{B}_X[x;\delta)$ for some $\delta > 0$. In other words, $\exists \delta > 0$ such that

$$\forall x' \in \mathbb{B}_X[x;\delta), \qquad f(x') \in \mathbb{B}_Y\left(f(x), \varepsilon\right).$$

(b⇒c) Let (x_n) be an arbitrary sequence in X satisfying $x_n \to x$. Given $\varepsilon > 0$, apply (b) to get some $\delta > 0$ so small that

$$d_Y\left(f(t), f(x)\right) < \varepsilon \qquad \text{whenever} \quad t \in \mathbb{B}_X[x;\delta).$$

Then apply the definition of $x_n \to x$ using this tolerance δ to obtain some $N \in \mathbb{N}$ so large that

$$\forall n > N, \quad d_X(x_n, x) < \delta.$$

For this N, we have

$$\forall n > N, \quad x_n \in \mathbb{B}_X[x;\delta) \quad \text{so} \quad d_Y\left(f(x_n), f(x)\right) < \varepsilon.$$

This confirms the definition of $f(x_n) \to f(x)$ in Y.

(c⇒a) Use contraposition. That is, suppose (a) is false. This means that there exists an open set $V \in \mathcal{N}_Y\left(f(x)\right)$ with the property that for each $U \in \mathcal{N}_X(x)$, one has $f(U) \not\subseteq V$. By shrinking V if necessary, we may retain this bad behaviour while assuming $V = \mathbb{B}_Y\left(f(x), \varepsilon\right)$ for some constant $\varepsilon > 0$. In particular, for each $n \in \mathbb{N}$ we must have $f\left(\mathbb{B}_X[x;1/n)\right) \not\subseteq \mathbb{B}_Y\left(f(x), \varepsilon\right)$. This implies that some point x_n satisfying $d_X(x_n, x) < 1/n$ must obey $f(x_n) \notin \mathbb{B}_Y\left(f(x), \varepsilon\right)$. This construction produces a sequence x_n that clearly satisfies $x_n \to x$, yet for which

$$d_Y\left(f(x_n), f(x)\right) \geq \varepsilon \qquad \forall n \in \mathbb{N}.$$

For this sequence, we have $f(x_n) \not\to f(x)$, so statement (c) is false. Contrapose. ////

Remark. To prove that f **is continuous** at x, line (b) is most useful; to prove that f **is not continuous** at x, line (c) can be ideal.

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Examples. (a) $f(x) = 1/x$ is continuous on $(0, +\infty$ because the preimage of every open interval is an open interval. Pointwise continuity is done directly in the next section.

(b) Dirichlet's Function, discontinuous at every point: $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$

(c) The function $g(x) = xf(x)$ for Dirichlet's function f is continuous at exactly one point.

(d) Thomae's Function, which is discontinuous at each rational but continuous at each irrational:

$$f(x) = \begin{cases} 1/q, & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ for } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with no common factors,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof of these properties is on Wikipedia, and makes a nice exercise. We will do a more ambitious one a little later.

D. Uniform Continuity

In this section, (X, d_X) and (Y, d_Y) are always **metric spaces.**

Definition. A function $f\colon X \to Y$ is **uniformly continuous on** X if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \,:\, \forall s \in X,\ \forall t \in \mathbb{B}_X[s;\delta),\ d_Y(f(s), f(t)) < \varepsilon. \tag{UC}$$

(The same $\delta = \delta(\varepsilon)$ must work for any point $s \in X$.)

For contrast, recall that $f\colon X \to Y$ is [merely] continuous on X if and only if

$$\forall s \in X,\ \forall \varepsilon > 0,\ \exists \delta > 0 \,:\, \forall t \in \mathbb{B}_X[s;\delta),\ d_Y(f(x_0), f(t)) < \varepsilon.$$

(Here $\delta = \delta(s, \varepsilon)$ is allowed to depend on s.)

Example. *Let X be the real interval $(0, +\infty)$, and define $f\colon X \to \mathbb{R}$ by $f(x) = 1/x$. This f is continuous on X, but not uniformly continuous on X.*

Proof. For any $s, t \in X$, estimate

$$\left| \frac{1}{t} - \frac{1}{s} \right| = \frac{|t-s|}{t\,s}.$$

For all t in $(s/2, 3s/2)$, we have $t^{-1} < 2/s$, so

$$\left| \frac{1}{t} - \frac{1}{s} \right| \leq \left(\frac{2}{s^2} \right) |t-s|.$$

It follows that for any $\varepsilon > 0$, choosing

$$\delta = \min\left\{ \frac{s}{2}, \frac{s^2}{2}\varepsilon \right\} \tag{Δ}$$

will imply that every $t \in \mathbb{B}[s, \delta)$ obeys $|f(t) - f(x_0)| < \varepsilon$. Thus f is continuous at s; since this works for every $s \in X$, f is continuous on X.

But f is not uniformly continuous on X. Indeed, consider $\neg$(UC):

$$\neg(\mathrm{UC}) \iff \exists\varepsilon>0 \,:\, \forall\delta>0,\ \exists s\in X \,:\, \exists t\in\mathbb{B}_X[s;\delta) \,:\, d_Y(f(x_0),f(t))\geq\varepsilon.$$

This holds with $\varepsilon=1$. For any $\delta>0$, pick $s\in(0,\min\{1,\delta\})$ and $t\in(0,s/2)$: then certainly $0<s-t<s<\delta$ [so $t\in\mathbb{B}_X[s;\delta)$], and

$$\frac{1}{t}-\frac{1}{s}>\frac{2}{s}-\frac{1}{s}=\frac{1}{s}>1.$$

A sketch shows how strong and unavoidable is the s-dependence is in line (Δ) above. ////

$\boxed{\text{01 Dec 2023}}$

Theorem. *Let X and Y be metric spaces, and suppose $f\colon X\to Y$ is continuous on X. If the input space X is compact, then f is uniformly continuous on X.*

Proof. [Option 1: Sequences and Contraposition.] Suppose f is not uniformly continuous: we'll show that f has a point of discontinuity in X. Indeed, using $\neg$(UC) we have some $\varepsilon>0$ such that for every choice $\delta=1/n$, there are points $s_n\in X$ and $t_n\in\mathbb{B}[s_n,1/n)$ such that $d_Y(f(s_n),f(t_n))\geq\varepsilon$. Since X is compact, the sequence (s_n) has a convergent subsequence, say $(s_{n_k})_k$, with limit $\widehat{s}\in X$. Since $d_X(t_{n_k},s_{n_k})<1/n_k\to 0$ as $k\to\infty$, we also have $t_{n_k}\to\widehat{s}$. But

$$\varepsilon\leq d_Y(f(s_{n_k}),f(t_{n_k}))\leq d_Y(f(s_{n_k}),f(\widehat{s}))+d_Y(f(\widehat{s}),f(t_{n_k}))\qquad\forall k.$$

In particular, we cannot have both $f(s_{n_k})\to f(\widehat{s})$ and $f(t_{n_k})\to f(\widehat{s})$. Thus at least one of these sequences demonstrates that f is discontinuous at the point $\widehat{s}$. ////

Proof. [Option 2: Classic compactness.] Given any $\varepsilon>0$, apply the definition of continuity at each point x in X with tolerance $\varepsilon'=\varepsilon/5$ to get some $\delta=\delta(x)>0$ such that

$$\forall x'\in\mathbb{B}[x;\delta(x)),\quad d(f(x'),f(x))<\frac{\varepsilon}{5}.$$

Build an open cover of X using balls with smaller radii:

$$\mathcal{G}=\{B[x;\delta(x)/7)\,:\,x\in X\}.$$

Use compactness to get a finite subcover, indexed by the selected centre points $x_1,x_2,\ldots,x_N$. Then define $\delta=\frac{1}{7}\min\{\delta(x_1),\ldots,\delta(x_N)\}$. We can prove that this δ has the desired property. Indeed, let any $x,x'\in X$ be given with $d(x,x')<\delta$. By the finite subcovering property, x must lie in the ball $\mathbb{B}[x_k;\delta(x_k)/7)$ for one of the centres listed above. And x' is not far from x, so

$$d(x',x_k)\leq d(x',x)+d(x,x_k)<\delta+\frac{\delta(x_k)}{7}\leq 2\frac{\delta(x_k)}{7}<\delta(x_k).$$

From continuity at the point x_k, we deduce

$$d(f(x'),f(x))\leq d(f(x'),f(x_k))+d(f(x_k),f(x))<\frac{\varepsilon}{5}+\frac{\varepsilon}{5}<\varepsilon,$$

as required.

[Choosing different prime denominators in the presentation above makes it easy to track their roles in the flow of logic above. Denominators smaller than 5 and 7 can certainly be used. The writeup shown here provides a glimpse of a situation where the logic is correct but a final step of polishing has not been applied.]

Discussion. Lipschitz continuity implies uniform continuity, but $f(x) = \sqrt{x}$ on $[0, +\infty)$ shows that some uniformly continuous functions are not Lipschitz. (Details ...)

E. Examples in Euclidean Spaces

Here are three examples where $X = \mathbb{R}^n$, $Y = \mathbb{R}$, with the usual (Euclidean) metric.

Example. *$f(x, y) = xy$ is continuous at $(0, 0)$.*

Proof. Observe (since $0 \le (|x| - |y|)^2 = x^2 - 2|xy| + y^2$) that

$$|f(x, y) - f(0, 0)| = |xy| \le \tfrac{1}{2}\left(x^2 + y^2\right) = \tfrac{1}{2}|(x, y) - (0, 0)|^2.$$

So given $\varepsilon > 0$, choose $\delta = \sqrt{2\varepsilon}$ to obtain

$$|(x, y) - (0, 0)| < \delta \implies |f(x, y) - f(0, 0)| < \varepsilon.$$

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Example. *This $f\colon \mathbb{R}^2 \to \mathbb{R}$ is discontinuous at $(0, 0)$:*

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Proof. Consider the sequence $(x_n, y_n) = (1/n, 0)$. Clearly $(x_n, y_n) \to (0, 0)$, but $f(x_n, y_n) = 1$ does not converge to $0 = f(0, 0)$. ////

Example. *This function has exactly one point of continuity:*

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ -x, & \text{otherwise.} \end{cases}$$

Proof. This f is continuous at $x_0 = 0$: indeed, given any $\varepsilon > 0$, choose $\delta = \varepsilon$. If $|x - 0| < \delta$, then we have $|f(x) - f(0)| < \varepsilon$ whether x is rational or not. But f is continuous nowhere else. For example, suppose $x_0 \ne 0$ lies in $\mathbb{Q}$: then there is a sequence of irrationals $x_n \to x_0$, and along this sequence $f(x_n) = -x_n$ converges to $-x_0$. But $f(x_0) = x_0 \ne -x_0$, so f is discontinuous at x_0. The case where $x_0 \in \mathbb{Q}^c$ is similar. ////

Example. *There is a function $f\colon \mathbb{R} \to \mathbb{R}$ which is strictly increasing, continuous at every irrational point, and discontinuous at every rational.*

Proof. Enumerate the rationals: $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$. Then define $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \sum_{i \in I(x)} \frac{1}{2^i},$$
$$\text{where} \quad I(x) = \{i \in \mathbb{N} \,:\, q_i < x\}.$$

Notice that $0 < f(x) < 1$ for all real x, and that $a < b$ implies $I(a)$ is a proper subset of $I(b)$, since $\mathbb{Q} \cap (a,b) \neq \emptyset$.

We considered this function in HW11 Question 6.

1: *f is strictly increasing.*

Whenever $a < b$, we have $f(b) - f(a) = \displaystyle\sum_{i \in I(b) \setminus I(a)} \frac{1}{2^i} > 0$, since $I(b) \setminus I(a) = \{i \in \mathbb{N} : a \leq q_i < b\} \neq \emptyset$.

2: *f is discontinuous at each $q \in \mathbb{Q}$.*

Fix any $m \in \mathbb{N}$ and consider q_m. Consider the sequence $x_n = q_m + 1/n$. The index m lies in every set $I(x_n)$, but it is not in $I(q_m)$. Thus

$$f(x_n) - f(q_m) = \sum_{i \in I(x_n) \setminus I(q_m)} \frac{1}{2^i} \geq \frac{1}{2^m}.$$

So as $n \to \infty$, we have $x_n \to q_m$ but $f(x_n) \not\to f(q_m)$. Therefore f is discontinuous at q_m.

3: *f is continuous at each $x_0 \in \mathbb{R} \setminus \mathbb{Q}$.*

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so large that $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \varepsilon$. Then let $\delta = \min\{|x_0 - q_1|, |x_0 - q_2|, \ldots, |x_0 - q_N|\}$. This choice guarantees that for any $x \in \mathbb{B}[x_0, \delta)$, none of the first N rational numbers in the given enumeration lies between x and x_0 (inclusive): thus

$$|f(x) - f(x_0)| \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \varepsilon.$$

////

Remark. The Thomae Function mentioned earlier is also continuous at all irrational points and discontinuous at all rational points, but it's not increasing. See Rudin Problem 4.18.

Monotonicity. The increasing function just discussed is about as bad as an increasing function can be. Indeed,

Proposition. *If $f\colon (a,b) \to \mathbb{R}$ is nondecreasing, then f is discontinuous at a point $c \in (a,b)$ if and only if $\alpha(c) < \beta(c)$, where*

$$\alpha(c) \overset{\text{def}}{=} \sup\{f(x) : x < c\} =: \sup A(c), \qquad \beta(c) \overset{\text{def}}{=} \inf\{f(x) : c < x\} =: \inf B(c).$$

In particular, the set of points where f is discontinuous is at most countable.

Proof. Clearly, if $a \in A(c)$ and $b \in B(c)$, we have $f(a) \leq f(c) \leq f(b)$. Consequently $\alpha(c) \leq \beta(c)$.

If $\alpha(c) = \beta(c)$ then f is continuous at c. To see this, let any $\varepsilon > 0$ be given. Since $\alpha(c) - \varepsilon/2 < \alpha$, there exists some $x_0 < c$ such that $f(x_0) > \alpha(c) - \varepsilon/2$. Likewise, $\beta(c) + \varepsilon/2 > \beta$ implies that some $x_1 > c$ must obey $f(x_1) < \beta(c) + \varepsilon/2$. Choose $\delta = \min\{x_1 - c, c - x_0\}$. Then $\delta > 0$, and $|x - c| < \delta$ implies $x_0 < x < x_1$, which in turn gives

$$\alpha(c) - \varepsilon/2 < f(x_0) \leq f(x) \leq f(x_1) < \beta(c) + \varepsilon/2 = \alpha(c) + \varepsilon/2.$$

In particular, $|f(x) - \alpha(c)| < \varepsilon/2$. Since this holds true also for $x = c$, we have for all x with $|x - c| < \delta$ that

$$|f(x) - f(c)| \leq |f(x) - \alpha(c)| + |\alpha(c) - f(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence f is continuous at c.

If, on the other hand, $\alpha(c) < \beta(c)$, then consider the two sequences $a_n = c - 1/n$ and $b_n = c + 1/n$. Clearly $a_n \to c$ and $b_n \to c$, while the definitions of $\alpha(c)$ and $\beta(c)$ imply

$$f(a_n) \leq \alpha(c) < \beta(c) \leq f(b_n) \qquad \text{for all } n.$$

These inequalities make it clear that the real sequences $f(a_n)$ and $f(b_n)$ cannot have the same limit, so at least one of these sequences must fail to converge to $f(c)$. Hence f is discontinuous at c.

Finally, let $D = \{c \in (a,b) \,:\, f \text{ is not continuous at } c\}$. For each c in D, choose a rational number $\phi(c)$ in the open interval $(\alpha(c), \beta(c))$. Clearly $\phi\colon D \to \mathbb{Q}$. In fact, ϕ is $1-1$ [why?], so it follows that $D \sim \phi(D) \subseteq \mathbb{Q}$. Hence D is at most countable. ////

F. Connectedness and the IVT

Proposition. *Let X be a HTS. Given any continuous $f\colon X \to \mathbb{R}$ and number $q \in \mathbb{R}$, let*

$$\Omega(q) = \{x \in X \,:\, f(x) < q\}.$$

Then $\partial\Omega(q) \subseteq \{x' \in X \,:\, f(x') = q\}$.

Proof. For q as in the setup, pick any $x' \in \partial\Omega(q)$ and define $y' = f(x')$. We must show that $y' = q$. To do this, fix any $\varepsilon > 0$ and consider $U_\varepsilon = f^{-1}((y' - \varepsilon, y' + \varepsilon))$. By continuity, this set is open; clearly $x' \in U_\varepsilon$. Now the definition of boundary points gives two facts:

(i) $U_\varepsilon \cap \Omega(q) \neq \emptyset$. That is, some $u \in U$ satisfies

$$\text{both} \quad y' - \varepsilon < f(u) < y' + \varepsilon \quad \text{and} \quad f(u) < q.$$

Chaining these together gives $y' - \varepsilon < q$.

(i) $U_\varepsilon \cap \Omega(q)^c \neq \emptyset$. That is, some $w \in U$ satisfies

$$\text{both} \quad y' - \varepsilon < f(w) < y' + \varepsilon \quad \text{and} \quad f(w) \geq q.$$

Chaining these together gives $y' + \varepsilon > q$.

Taken together, these facts give $y' - \varepsilon < q < y' + \varepsilon$, and this conclusion holds for arbitrary $\varepsilon > 0$. It follows that $y' = q$, as required. ////

Remark. Take $X = \mathbb{R}$, $q = 0$, and $f(x) = (x+1)x(x-1)^2$ to see that the inclusion stated in the proposition can be strict. (Draw $y = f(x)$. Note that the factored-form presentation makes it easy to locate the points where $f(x) = 0$ and use them to show $\Omega(0) = (-1, 0)$ whereas $\partial\Omega(0) = \{-1, 0, 1\}$.)

Corollary. *In the setup of the proposition, if $\Omega(q) \neq \emptyset$ and $f(x') \neq q$ for every $x' \in X$, then the set $\Omega(q)$ is closed in X.*

Proof. In this situation, the proposition above implies that $\partial\Omega(q)$ is a subset of the empty set. In particular $\overline{\Omega(q)} = \Omega(q) \cup \partial\Omega(q) = \Omega(q)$. ////

Corollary (Intermediate Value Theorem). *Suppose $[a, b]$ is a real interval and $f\colon [a, b] \to \mathbb{R}$ is continuous at each point. Then for each real q between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ for which $f(c) = q$.*

Proof. Contraposition: Suppose $f(x') \neq q$ for each $x' \in [a, b]$. Then the set $\Omega(q)$ is simultaneously open and closed in $[a, b]$. This requires either $\Omega(q) = \emptyset$ (so that $f(x') > q$ for all x') or $\Omega(q) = [a, b]$ (so that $f(x') < q$ for all x'). In either case, q is not "between $f(a)$ and $f(b)$".

Detail: Showing that the only open subsets of $X = [a, b]$ with no boundary points are $\emptyset$ and X takes some work. Most of this was done on HW12 Q6. ////

G. Limits of Functions

04 Dec 2023

Given a function $f\colon X \to Y$ and a point $x_0 \in X$, think of "$\lim\limits_{x \to x_0} f(x)$" as an element $y_0 \in Y$ that provides a reasonable constant approximation to f near the point $x = x_0$, based on observations of $f(x)$ for x near to **but distinct from** x_0. To guarantee that there are some such observations to make, we must require that x_0 is a limit point of X. Here is a definition in metric space.

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f\colon X \to Y$. To say "$\lim\limits_{x \to x_0} f(x) = y_0$" means both

(i) $x_0 \in X'$, and

(ii) $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x \in \mathbb{B}_X(x_0; \delta),\ d_Y(f(x), y_0) < \varepsilon$.

Thus we have

$$f \text{ continuous at } x_0 \iff \lim_{x \to s} f(x) = f(x_0) \text{ OR } s \notin X'.$$

For example, consider $X = [0, 1] \cup \{2\}$ as a subspace of $(\mathbb{R}, |\cdot|)$, and define $f\colon X \to \mathbb{R}$ by $f(x) = x^2$. Then f is continuous on X, in particular continuous at $s = 2$. However, $\lim_{x \to 2} f(x)$ is not defined, because $2 \notin X'$. (Sketch.)

Alternatively, one could introduce limits as a theoretical means of removing discontinuities. So, let x_0 be a point of X and let $f\colon X \setminus \{x_0\} \to Y$ be given. If x_0 is an isolated point of X, any definition of $f(x_0)$ will make f continuous at x_0, but if $x_0 \in X'$ this is not the case. Thus we might say (as a definition) $\lim_{x \to x_0} f(x) = y_0$ if and only if both

(i) $s \in X'$ and

(ii) the function $\phi(x) = \begin{cases} f(x), & \text{if } x \neq x_0, \\ y_0, & \text{if } x = x_0, \end{cases}$ is continuous at x_0.

Once the uniqueness issue is settled, this definition makes a tautology of the statement that if f is continuous at x_0, then $f(x) \to f(x_0)$ as $x \to x_0$, and facilitates a discussion of removable discontinuities.