Phys 403 Homework 1

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19/01/24

Question I1.

$$I_0^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dxdy = \int_{-\infty}^{\infty}\int_0^{2\pi} re^{-r^2/2} d\phi dr$$
$$= 2\pi \int_0^{\infty} e^{-u} du = 2\pi \implies I_0 = \sqrt{2\pi}.$$

Question I2.

$$I(\mu,\sigma) = \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{\infty} e^{-(y/\sigma)^2/2} dy = \int_{-\infty}^{\infty} = \sigma \int_{-\infty}^{\infty} e^{-z^2/2} dz$$
$$= \sigma I_0 = \sigma\sqrt{2\pi}.$$

Question I3.

$$Z(\kappa) = \int_{-\infty}^{\infty} e^{\kappa x} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} dx = \int_{-\infty}^{\infty} e^{\kappa x} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} dx$$
$$= \int_{-\infty}^{\infty} e^{\left(\left(x-\sigma^2\kappa\right)^2-\sigma^4\kappa^2\right)/(2\sigma^2)} \frac{1}{\sqrt{2\pi\sigma^2}} dx = \int_{-\infty}^{\infty} e^{\sigma^2\kappa^2/2} e^{\left(x-\sigma^2\kappa\right)^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi\sigma^2}} dx = e^{\sigma^2\kappa^2/2}.$$

Question II1.

$$\int_0^{\infty} \frac{1}{\mathcal{N}} e^{-\lambda x} dx = -\frac{1}{\lambda\mathcal{N}} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda\mathcal{N}} = 1 \implies \mathcal{N} = \frac{1}{\lambda}.$$

Question II2.

$$\tilde{p}(k) = \langle e^{-ikx} \rangle = \int_0^{\infty} \lambda e^{-\lambda x} e^{-ikx} dx = \frac{\lambda}{\lambda + ik} = \frac{1}{1-(-\frac{ki}{\lambda})} = \sum_{j=0}^{\infty} \frac{(-ik)^j}{j!} \left(\frac{j!}{\lambda^j}\right)$$
$$\implies \langle x^j \rangle = \frac{j!}{\lambda^j} \implies \langle x^1 \rangle = \frac{1}{\lambda}, \langle x^2 \rangle = \frac{2}{\lambda^2}, \langle x^3 \rangle = \frac{6}{\lambda^3}, \langle x^4 \rangle = \frac{24}{\lambda^4}.$$

Question II3.

$$\log \tilde{p}(k) = -\log\left(1 - \left(-\frac{ki}{\lambda}\right)\right) = \sum_{j=1}^{\infty} \frac{(-ki)^j}{j!} \frac{(j-1)!}{\lambda^j}$$
$$\implies \langle x^j \rangle_c = \frac{(j-1)!}{\lambda^j} \implies \langle x^1 \rangle_c = \frac{1}{\lambda}, \langle x^2 \rangle_c = \frac{1}{\lambda^2}, \langle x^3 \rangle_c = \frac{2}{\lambda^3}, \langle x^4 \rangle_c = \frac{6}{\lambda^4}.$$

Question III.

$$h[p] = \langle -\log p \rangle = \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \left((x-\mu)^2/2\sigma^2 + \log \sqrt{2\pi\sigma^2} \right) dx = \frac{1}{2\sigma^2} \langle (x-\mu)^2 \rangle + \frac{1}{2} \log 2\pi\sigma^2$$

$$= \frac{1}{2} \log 2\pi\sigma^2 + \frac{1}{2}.$$

Shannon coding theorem says that that optimal storage size depends linearly with entropy, and the above expression shows that entropy is proportional to the log of standard deviation. Calculating explicitly:

$$\frac{\frac{1}{2}\log_2 2\pi + \frac{5}{2}}{\frac{1}{2}\log 2\pi + \frac{1}{2}} \approx 2.095\text{GB}.$$

Question IV. From the central limit theorem the standard error is proportional to $\frac{1}{\sqrt{N}}$ ($\epsilon = \sqrt{\frac{\langle X \rangle_c}{N}}$), so to reduce it by a factor of 10, we would need to increase the number of repetitions by a factor of 100. Thus 10000 repetitions would be necessary.