

Math 443 Homework 5

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01/03/23

Question 1. Let k be a positive integer, and let A_1, A_2 be two copies of K_{k+1} . Let G_k be created by taking a new vertex, v , and connecting it to k vertices in each of A_1, A_2 . Clearly $\kappa(G_k) = 1$ since if v is removed G_k gets separated into A_1, A_2 as components. To see that $\lambda(G_k) \leq k$, note that by removing all edges between v and A_1 results in a disconnect graph, which is an edge set of size k .

To see why this is a minimum edge cut, let $E \subset E(G_k)$ s.t. $|E| < k$ (since E is an edge set the $||$ syntax denotes size of set, not number of vertices). Then E couldn't have removed all the edges between A_1 and v , since there are k edges between them. By symmetry the same applies for A_2 and v . $A_1 - E$ and $A_2 - E$ are still connected since they are both copies of K_k , so the total graph is also connected. Thus $\lambda(G_k) = k$ and $\kappa(G_k) = 1$ as required.

Question 2a. The statement is true. Let E be a separating edge set of G , and let A be a smallest resulting component of $G - E$. Clearly $|A| \leq \frac{n}{2}$ since it the smaller of at least two components whose total vertices is n . Note that $||A|| \leq K_{|A|} = |A|(|A| - 1)$. Also note that the total number of edges of the vertices of A in G is $|A| \cdot \delta(G)$. The difference between these numbers is at least the number of vertices taken away by E , i.e. $|E| \geq |A|\delta(G) - |A|(|A| - 1) = |A|(\delta(G) - |A| + 1)$. This is a downward parabola, so from calculus its minimum must lie on one of the two endpoints, i.e. $|A| = 1$ or $|A| = \frac{n}{2}$. These two values are:

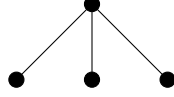
$$|E| \geq \delta(G) - 1 + 1 = \delta(G).$$

$$|E| \geq \frac{n}{2}(\delta(G) - \frac{n}{2} + 1) \geq \delta(G).$$

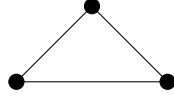
We conclude that $\lambda(G) \geq \delta(G)$. In class we also proved that $\lambda(G) \leq \delta(G)$, so these two inequalities together tell us that $\lambda(G) = \delta(G)$ as required. \square

Question 2b. The statement is false. As a counterexample let $k \geq 3$ and consider two copies of K_k , A_1, A_2 . Form G by adding two vertices v_1, v_2 connected to all vertices in A_1, A_2 . Each vertex in A_1, A_2 has $k - 1$ neighbors from the complete graph as well as v_1, v_2 for a total of $k - 1 + 2 = k + 1$. v_1, v_2 each have $2k$ neighbors since their connected to each vertex in A_1, A_2 . The total vertices is $2k + 2$, so $\delta(G) = k + 1 \geq \frac{|G|}{2}$. We can disconnect the graph by removing v_1, v_2 , so $\kappa = 2$ (it clearly can't be less than that). Let E be an edge set with $|E| \leq 2$. A_1 remain connected in $G - E$ since they are $k \geq 3$ complete, and there is at least one edge from A_1 to v_1 and v_2 in $G - E$, since there were at least 3 edges before and E removed at most 2. By symmetry the same is true for A_2 , so all 4 of A_1, A_2, v_1, v_2 are internally connected and connected together. Thus E is not an edge cut, so $\lambda(G) > 2$ and $\lambda(G) \neq \kappa(G)$.

Question 3. The statement is not true. Consider the following graph G :



The three groups are three copies of K_2 , and they are each connected to each other since they each share a vertex, so this is \mathcal{G} :



This obviously isn't a tree since it contains a cycle, so we're done and the statement is false.

Question 4a. The statement is true. Let $v \in G[\mathcal{E}]$. Let $x, y \in G[\mathcal{E}] - v$. Let C be a cycle in $G[\mathcal{E}]$ containing x, y (this exists since you can choose any two edges attached to x, y and they're guaranteed to share a cycle). Let P be a path along C that doesn't include v , since there are two options (each way around C) this will always exist. Then x, y are connected in $G[\mathcal{E}] - v$ through P . This works for any v , so there are no cut vertices in $G[\mathcal{E}]$ so it is nonseparable. \square

Question 4b. The statement is true. From the definition of $G[\mathcal{E}]$, $E(G[\mathcal{E}]) \subset E(G)$. The vertex set of $G[\mathcal{E}]$ is the vertex set of all endpoints of \mathcal{E} in G , since these are the vertices that don't become isolated vertices when $E(G) - \mathcal{E}$ are removed from G . This is exactly the definition of an induced subgraph though, so $G[\mathcal{E}] \leq G$. \square

Question 4c. The statement is true. We will use proof by contradiction. From part a $G[\mathcal{E}]$ is nonseparable, so the only way $G[\mathcal{E}]$ could not be a block is if it isn't maximal. By way of contradiction suppose that $\exists B \subset G, v \in G, v \notin G[\mathcal{E}]$ s.t. B is nonseparable and $V(G[\mathcal{E}]) \subset V(B), v \in B$. It is asserted that there exists paths P_1, P_2 from v to $G[\mathcal{E}]$ in B with the endpoints of $u_1, u_2 \in G[\mathcal{E}], u_1 \neq u_2$. At least one path P_1 must exist for v and $G[\mathcal{E}]$ to be connected, and since removing u_1 shouldn't disconnect $G[\mathcal{E}] + v$ a second path must exist with a different endpoint in $G[\mathcal{E}]$. Choose P_1, P_2 in such a way that u_1, u_2 are their only member in $G[\mathcal{E}]$. Consider the first common ancestor between P_1 and P_2 starting from u_1, u_2 , call it x . $G[\mathcal{E}]$ is connected so there exists a path in it between u_1 and u_2 , call it P_3 . Then $u_1 P_3 u_2 P_1 x P_2 u_1$ is a cycle in G containing an edges not in $G[\mathcal{E}]$, namely the edges of P_1, P_2 before x . This is impossible since we assumed $G[\mathcal{E}]$ was formed by the entire equivalence class, so it must be that $G[\mathcal{E}]$ is a block of G . \square

Question 4d. The statement is false. Consider the extr