Math 437 Homework 2

Xander Naumenko

16/10/23

Question 1. Since $1|n \forall n$, we have that $3|n+2 \implies n \equiv 1 \mod 3$. Clearly any prime of the form $3k+1$ works and no other prime does, since for such numbers 1 is the only factor. I claim that primes of that form are the only solution for n.

Proof by contradiction, assume that $n = pa$ where $p < n$ is the smallest prime divisor of n, with $p+2|n+2$ and $a+2|n+2$. If $2|n$ then we have $\frac{n}{2}+2|n+2$, which is impossible since $\frac{n+2}{2} < \frac{n}{2}+2 < n+2$ so $p \neq 2$. Next consider the set of congruence relations:

$$\begin{cases} n' \equiv 0 & \mod p \\ n' \equiv -2 & \mod p+2 \end{cases}.$$

p is odd so $\gcd(p, p+2) = 1$, so by the chinese remainder theorem the solution n' is unique up to multiples of $p(p+2)$. $n' = p$ fulfills both criteria, so we can express $n = p + kp(p+2), k \in \mathbb{Z}$.

Now consider $a = \frac{n}{p} = 1+k(p+2)$. By hypothesis $a+2|n+2 \implies (3+k(p+2))|(p+kp(p+2)+2) \implies (1+k(p+2))|2$. Clearly this is impossible for $p > 1$ which it is, so this is a contradiction suggesting n can't in fact have more factors than 1 and itself. Since we've shown that primes of the form $3k+1$ work and any composite numbers don't, all primes of that form are the only numbers that fulfill the requirements. $\square$

Question 2. Consider the equation mod 3:

$$2^m \equiv 1 \mod 3 \implies m = 2k, k \in \mathbb{N}.$$

Now consider the same equation mod 4:

$$4^k - 3^n \equiv -3^n \equiv 3 \mod 4 \implies n = 2l, l \in \mathbb{N}.$$

But then the equation reduces to $4^k - 9^l = (2^k + 3^l)(2^k - 3^l) = 7$. Since 7 is prime this means that $2^k + 3^l = 7$, $2^k - 3^l = 1$. Since $2^k + 3^l$ is clearly increasing in k, l it's trivial to check the possibilities $k = 1, 2, l = 1$ and see that the only solutions correspond to $m = 4, n = 2$. $\square$

Question 3. By theorem 13.4, we know that for a number n, it is expressible as $a^2 + b^2$ if and only if the exponent its prime factors in the form $4l+3$ is even. There are infinitely prime numbers of the form $4l+3$, as if there were finitely many of them $4k_1+3, 4k_2+3 \ldots, 4k_m+3$, then we would have that $4(4k_1+3)\cdots(4k_m+3)+3$ isn't divisible by any of them but is of the form $4l+3$. It's prime factors can't be just of the form $4l+1$ as $(4l_1+1)(4l_2+1) = 4(4l_1l_2+l_1+l_2)+1$, so at least on of its prime factors wasn't included on our supposedly complete list, implying there are infinitely many.

Using the fact that there are infinitely many take $q_0, \ldots, q_{k-1}$ to be arbitrary distinct primes of the form $4l + 3$. Using the chinese remainder theorem, there exists a unique solution to the following system of equations:

$$\begin{cases} x \equiv 0 & \mod q_0 \\ x \equiv -1 & \mod q_1 \\ \vdots & \\ x \equiv -k+1 & \mod q_{k-1} \end{cases}$$

up to mod $q_1 \cdots q_{k-1}$. Let $m_i = 1$ if $\exp_{q_i}(x+i) \equiv 0 \mod 2$ and $m_i = \exp_{q_i}(x+i) + 1$ otherwise. I claim that the following sequence of k integers satisfies the required properties, where n ranges from 0 to $k-1$:

$$x_n = x + n + \prod_{i=0}^{k-1} q_i^{m_i}.$$

Note that the product term does not conflict with the congruence relations found above, since it is a multiple of $q_1 \cdots q_{k-1}$. Consider any individual sequence element x_n. If $\exp_{q_n}(x+n) \equiv 0 \mod 2$, then we can write $x + n = q_n^2 l$ (it can't be that $\exp_{q_n}(x_n) = 0$ since x was the solution to $x \equiv -n \mod q_n$) and $x_n = q_n(q_n l + q_0^{m_0} \cdots q_{k-1}^{m_{k-1}})$. Importantly q_n does not divide the second part of the addition but does the first, so $\exp_{q_n}(x_n) = 1$.

If instead $\exp_{q_n}(x+n) \equiv 1 \mod 2$, then we can write $x + n = q_n^{m_n} l$ for $q_n \nmid l$, and $x_n = q_n^{m_n}(l + q_n q_0^{m_0} \cdots q_{k-1}^{m_{k-1}})$. In reverse from the previous case here the first term is not divisible by l and the second is, so $\exp_{q_n}(x_n) \equiv m_n \equiv 1 \mod 2$. In either case we have that $\exp_{q_n}(x_n) \equiv 1 \mod 2$, so by theorem 13.4 none of the x_n are expressible as $a^2 + b^2$. $\square$

Question 4a. I claim that the limit is equal to 0. Writing $n! = \prod_{i=1}^{r} p_i^{\alpha_i}$ with $p_1 < p_2 < \ldots < p_r$, using identities proven in class we have that

$$d(n!)\phi(n!) = \left(\prod_{i=1}^{r}(\alpha_i + 1)\right) n! \left(\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right)\right) = n! \left(\prod_{i=1}^{r}(\alpha_i + 1)\left(1 - \frac{1}{p_i}\right)\right).$$

Since $n! | (n+1)!$, each individual term in the product above only increases as n increases. Also since $\alpha_i \geq 1$ and $1 - \frac{1}{p_i} \geq \frac{1}{2}$, each individual term in the product is greater or equal to 1. Thus:

$$\frac{n!}{d(n!)\phi(n!)} \leq \frac{n!}{n! \frac{\exp_2(n!)}{2}} = \frac{2}{\exp_2(n!)} \to 0.$$

Question 4b.