Problem Set 7 -The Ising model and Mean-Field theory
(Dated: PHYS403, Spring 2024)

I. 1d Ising model: Exact solution by transfer matrix method [20pts]

Consider a 1d Ferromagnetic Ising chain with N sites and periodic boundary conditions:

$$H = -J\sum_{i=1}^{N} s_i s_{i+1\mathrm{mod}N} \tag{1}$$

1. Show that the partition function can be written as $\mathcal{Z} = \mathrm{tr}M^N$ where M is a 2×2 matrix:
$$M = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}$$
.

2. To take powers of a Hermitian matrix, it is useful to work in the eigenbasis, for example $\mathrm{tr}\left[M^N\right] = \sum_\alpha \lambda_\alpha^N$ where λ_α are the eigenvalues of M and α labels the different eigenvalues. Find the eigenvalues and eigenvectors of M. Make sure to normalize your eigenvectors. Label the larger eigenvalue as λ_+ and the smaller one as λ_-, and the corresponding eigenvectors as $\vec{v}_\pm$.

3. Show that the two-spin correlation function: $\langle s_{i+x}s_i\rangle$ can be written as
$$\langle s_{i+x}s_i\rangle = \frac{1}{Z}\mathrm{tr}\left[M^{N-x}\sigma^z M^x \sigma^z\right],$$
where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and evaluate this expression.
Hint: the trace is independent of basis, so it's useful to work in the eigenbasis of M. Note that multiplying the eigenvectors by σ^z has a particularly simple effect.

4. Show that for $N\to\infty$, the correlation function takes the form: $\langle s_{i+x}s_i\rangle = e^{-x/\xi}$. Find ξ in terms of the parameters of the model, and evaluate its asymptotic form for low temperatures ($k_BT \ll J$). What does this result imply about the critical temperature for magnetic ordering for this model?
Hint: by definition $\lambda_- < \lambda_+$ so $\lambda_-/\lambda_+ < 1$ so $\lim_{N\to\infty}\left(\frac{\lambda_-}{\lambda_+}\right)^N = 0$

5. Find the average excitation energy per spin, $\frac{\Delta E}{N} \equiv \frac{\langle E\rangle - E_0}{N}$ in the large N limit, where $E_0 = -JN$ is the ground-state ($T=0$) energy. For low-temperatures, I argued that we can picture the low-temperature thermal state of th $1d$ Ising model as a dilute gas of domain walls (DWs) between regions ferromagnetic regions with $s=\pm 1$. From your expression for $\Delta E/L$ estimate the density of domain walls, n_{DW}, and typical distance, ℓ_{DW} between domain walls. Compare these results to the low-temperature asymptotic value of the correlation length ξ appearing in your answer above. Can you explain, physically, why ℓ_{DW}, ξ are related?

II. Spin-1 Heisenberg model (Mean-Field) [20pts]

Consider a spin-1 Ferromagnetic Heisenberg model on a $3d$ cubic lattice:

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \tag{2}$$

where $\vec{S}_i$ are (quantum) spin-1 operators for each site i on the cubit lattice. In this problem set $\hbar = 1$, so that the possible eigenvalues of spin along a given axis, say the z-axis are $S^z \in -1, 0, +1$. Consider $J > 0$, i.e. the interactions are ferromagnetic.

1. What are the symmetries of this model?
2. What are ground-state(s) of the system, how many are there, and which if any symmetries are spontaneously broken at $T = 0$?
3. Write down the mean-field Hamiltonian for this model.
4. Find the mean-field free energy as a function of the magnetization $\vec{m} = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle$ (N is the total number of spins in the system).
5. Find the mean-field self-consistency equation for the magnitude of the magnetization $m = |\vec{m}|$ by minimizing $F(\vec{m})$ with respect to $\vec{m}$.
6. Within the mean-field approximation, find the critical temperature for magnetic ordering.
7. What do you expect $F(\vec{m})$ to look like near the ordering transition? *Hint: you could compute this by expanding T near T_c from your previous answer, but try, instead, to argue for the form based simply on symmetry considerations.*

III. First order spontaneous symmetry-breaking transition [15 pts]

Consider an order parameter ϕ, in a system with that is symmetric under the transformation: $\phi \to -\phi$. When ϕ is small, the free-energy density, $f(\phi)$ can be expanded in a Taylor series in ϕ, because of the symmetry, only even powers of ϕ arise in the expansion:

$$f(\phi) \approx a\phi^2 + b\phi^4 + c\phi^6 + \dots \tag{3}$$

Where $a \sim (T - T_c)$ changes signs at the critical temperature. Assume b, c are independent of temperature for simplicity.

In class, we examined the case where $b > 0$, and found a continuous phase transition at $a = 0$, near this transition we could ignore c and higher order terms since they were less important than b when ϕ was small.

Suppose, instead, that the quartic were negative, $b < 0$. In this case, we need to consider the ϕ^6 term to stabilize the free energy (physically, f must be lower bounded). Show that the transition tuned by a becomes a discontinuous, first order transition and no longer occurs at $a = 0$. *hint: since all the terms in f contain even powers of ϕ, work with $x = \phi^2$ instead of ϕ itself.*