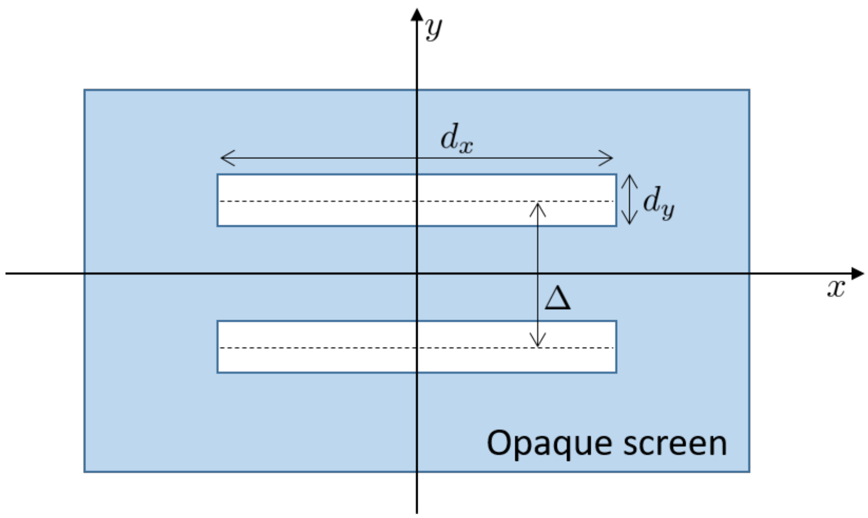
PHYS 408, 2023W2

Problem Set 2: Fourier Optics

Posted: Fri, January 26 $\longrightarrow$ *Due*: Wed, February 14.

1. In Lecture 7, we have considered the Fraunhofer diffraction of a plane wave on a single one-dimensional slit of size d. Using the results from class, extend them to the case of two slits (the famous double-slit geometry) of finite size $d_x \times d_y$, separated by a distance Δ, as shown in the figure below.



   (a) Derive the expression for the two-dimensional intensity distribution of the Fraunhofer diffraction pattern in the far field. Explain every step in your derivation.

   (b) Plot the normalized cross sections of this pattern along the $(x', y' = 0)$ and $(x' = 0, y')$ axes in the far-field observation plane located $z = 50$ m away from the screen, assuming $d_x = 1$ cm, $d_y = 1$ mm, $\Delta = 5$ mm, and the wavelength of light $\lambda = 500$ nm.

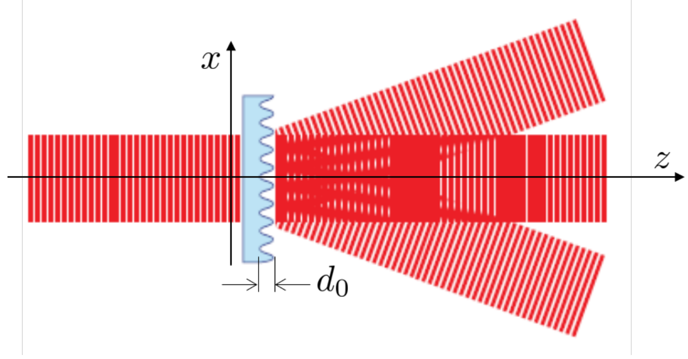
   (c) Can one use the Fraunhofer approximation, i.e. are the above results valid?

   (d) *Bonus*: Confirm your answer in (c) by calculating the same cross sections numerically by means of the Fresnel integral (i.e. without using the far-field Fraunhofer approximation), and comparing these results with the analytical results in (b).

2. In Lecture 7, we have described the transmission of a plane wave through an *amplitude grating*. Most real gratings, however, are based on the periodic *phase modulation*. One way to implement such a periodic phase mask is with a thin transparent glass plate, whose thickness varies sinusoidally in space, as shown below. Assume that the glass thickness depends on x as $d(x) = \bar{d} + d_0/2\sin(2\pi x/\Lambda)$, with $\bar{d}$ the average thickness of the plate, d_0 the groove depth and Λ the groove period of the grating. Note the slight difference between the suggested $d(x)$ and the one used in the text.

   (a) First, derive the transfer function of such phase grating in one dimension, $t(x)$, assuming that its overall length along x is d_x.

   (b) Show that an incident plane wave traveling at a small angle θ_i with respect to the z axis is transmitted in the form of a sum of plane waves traveling at angles $\theta_q \approx \theta_i + q\lambda/\Lambda$, where $q = 0, \pm 1, \pm 2, \ldots$ is the diffraction order. For simplicity, here assume that the grating is infinitely long, $d_x \to \infty$. *Hint:* use the mathematical identity $\exp[im\sin(x)] = \sum_{q=-\infty}^{\infty} J_q(m)\exp[iqx]$, where J_q is the Bessel function of the first kind and order q.



3. In Lecture 8, we found out that the E-field distribution in the *focal plane* of a lens corresponds to the Fourier transform of the object field. This proved the Fourier-transforming property of lenses. On the other hand, from the basic geometric optics, you know that a lens will also create an image of the object in the *image plane*. The distances between the lens and the object/image, d_o and d_i respectively, are connected through the well known lens equation:

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}. \tag{1}$$

   Prove this equation using the methods of Fourier optics.

4. In Lecture 8, we showed that if an object is placed against the lens (as shown below), the Fourier transform of the object's transfer function appears at the focal "Fourier plane" located at $z = f$. How large can the misalignment shift Δ away from that plane be so that the Fourier transform still provides an accurate description of the intensity distribution at $z = f - \Delta$?