PHYS 304 Assignment 4

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Question 1a.

$$\int_{-\infty}^{\infty} \Phi^* \Phi dx = A^2 \int_{-\infty}^{\infty} (9|\phi_0| + 16|\phi_1| + 12\phi_0^*\phi_1 + 12\phi_0\phi_1^*)\, dx = 25 \implies A = \frac{1}{5}.$$

Question 1b. Since the given intial conditions is just the superposition of eigenstates, it will continue to be. Then we have that:

$$\Psi(x,t) = \frac{1}{5}(3\psi_0(x)e^{-i(\frac{1}{2}\omega t)} + 4\psi_1(x)e^{-i(\frac{3}{2})\omega t})$$

$$= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(3e^{-\xi^2/2}e^{-i(\frac{1}{2}\omega t)} + 4\frac{1}{\sqrt{2}}2\xi e^{-\xi^2/2}e^{-i(\frac{3}{2})\omega t}), \xi = \sqrt{\frac{m\omega}{\hbar}}x.$$

Next calculate the absolute value:

$$|\Psi(x,t)|^2 = \frac{1}{5}\left(9(\psi_0(x))^2 + 16(\psi_1(x))^2 + 12\psi_1\psi_0 e^{-i\omega t} + 12\psi_1\psi_0 e^{i\omega t}\right)$$

$$= \frac{1}{25}\left(9(\psi_0(x))^2 + 16(\psi_1(x))^2 + 24\psi_0\psi_1\cos(\omega t)\right).$$

Question 1c.

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) dx = \frac{9}{25}\langle \psi_0 \rangle + \frac{16}{25}\langle \psi_1 \rangle + \int_{-\infty}^{\infty} \frac{24}{25} x\psi_1\psi_2 \cos(\omega t) dx$$

$$= 0 + 0 + \frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{m\omega\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{2}}\xi^2 e^{-\xi^2}d\xi = \frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi$$

$$= \frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{4}} = \sqrt{\frac{\hbar}{2m\omega}}\frac{24}{25}\cos(\omega t).$$

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t).$$

Verifying Ehrenfest's theorem:

$$\frac{d\langle p \rangle}{dt} = \frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\cos(\omega t) = m\omega\langle x \rangle = \left\langle -\frac{dV}{dx}\right\rangle$$

as expected.

Question 1d. The probabilities are the square of the coefficients, associated with that eigenstate. Thus the probability of having energy $E_0 = \frac{1}{2}\hbar\omega$ is

$$(\frac{3}{5})^2 = \frac{9}{25}.$$

The probability of having energy $E_1 = \frac{3}{2}\hbar\omega$ is

$$(\frac{4}{5})^2 = \frac{16}{25}.$$

Question 2. The probability is the integral of the square of the wavefunction, and because the wavefunction is symmetric about 0 we have:

$$P = 2\int_{\sqrt{\frac{\hbar}{m\omega}}}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega x}{\hbar}} dx$$

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}} \int_{1}^{\infty} e^{-\xi^2} d\xi.$$

Solving this numerically with WolframAlpha we get that this integral is equal to $P = 0.157$.

Question 3a.

$$\int_{-\infty}^{\infty} A^2 e^{-2ax^2} dx = A^2\sqrt{\frac{\pi}{2a}} = 1 \implies A = \left(\frac{2a}{\pi}\right)^{1/4}.$$

Question 3b. From the textbook the general expression for the wavefunction of a free particle is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)e^{ikx}dk, \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)e^{-ikx}dx$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-ikx}dx.$$

Let $y = \sqrt{a}(x + \frac{b}{2a}) = \sqrt{a}(x + \frac{ik}{2a})$. Then the integral becomes

$$\phi(k) = A\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2+\frac{k}{4a}} dy = \frac{e^{-k^2/4a}}{\sqrt{2a}}\left(\frac{2a}{\pi}\right)^{1/4} = \frac{e^{-k^2/4a}}{(2a\pi)^{1/4}}.$$

Then the expression for the wavefunction is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}(2a\pi)^{1/4}}\int_{-\infty}^{\infty} e^{-\left(k^2\left(1/4a+\frac{i\hbar}{2m}t\right)-ixk\right)}dk.$$

Let $j = \frac{1}{\sqrt{1/4a+\frac{i\hbar}{2m}t}}\left(k + \frac{1/4a+\frac{i\hbar}{2m}t}{2ix}\right) \implies k^2\left(1/4a + \frac{i\hbar}{2m}t\right) - ikx = j^2 + \left(\frac{x^2}{1/a+\frac{2i\hbar}{m}t}\right)$. Plugging this back into the integral:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}(2a\pi)^{1/4}}\frac{1}{\sqrt{1/4a + \frac{i\hbar}{2m}t}}\int_{-\infty}^{\infty} e^{-j^2-\left(\frac{x^2}{1/a+\frac{2i\hbar}{m}t}\right)}dj.$$

Let $\gamma = \sqrt{1 + (2i\hbar at/m)}$. Then the above integral reduces to (taking the normal distribution integral known to be $\sqrt{\pi}$):

$$\Psi(x,t) = \frac{1}{\sqrt{2}(2a\pi)^{1/4}}\frac{2\sqrt{a}}{\gamma}e^{-ax^2/\gamma^2}$$

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}.$$

Question c. Multiplying the above expression with its conjugate:

$$|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{(1+(2i\hbar at/m)(1-(2i\hbar at/m)}} e^{-ax^2/(1+(2i\hbar at/m))} e^{-ax^2/(1-(2i\hbar at/m))}$$

$$= \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}.$$

As can be seen in figure 1 and 2, the wavepacket spreads out over time. Note that the values used in the graphs are arbitrary and are just meant to show the trend over time.

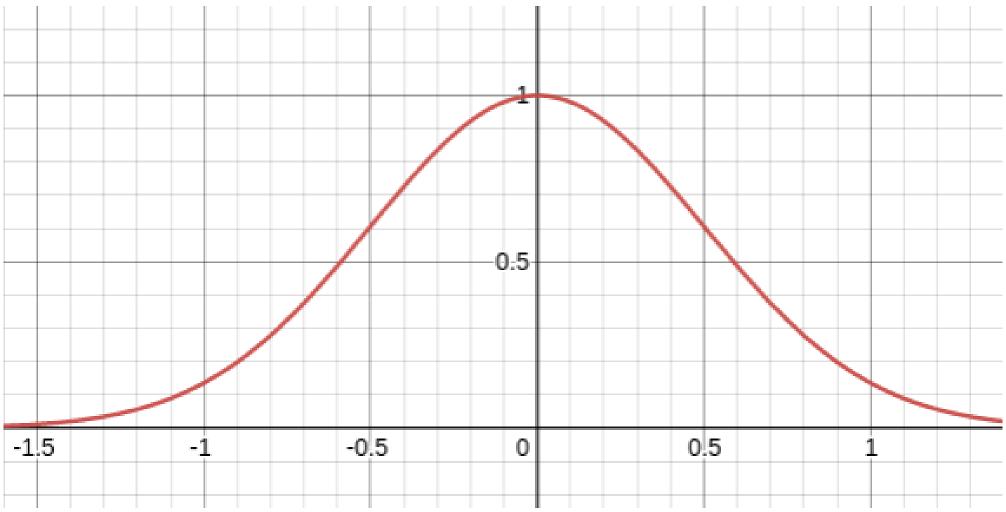

Figure 1: Question 3 graph at $t = 0$.

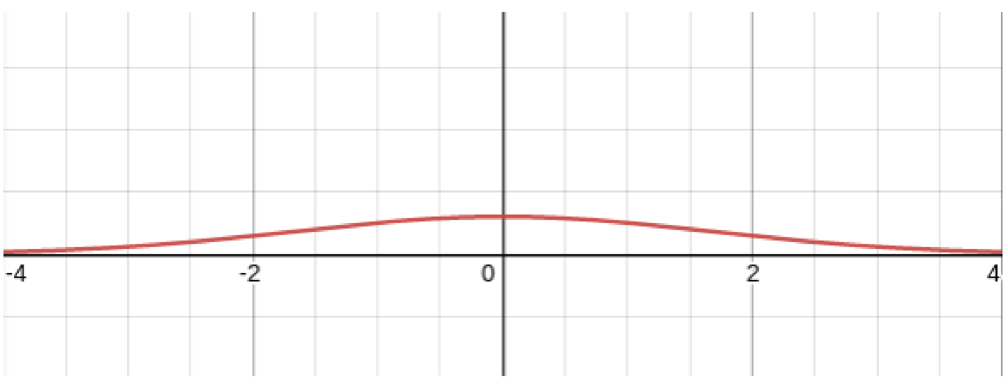

Figure 2: Question 3 graph at $t \to \infty$.

Question 3d. The expressian is a Gaussian centered at $x = 0$, so $\langle x \rangle = 0$.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} w x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi}{2w^2}} \frac{w}{4w^2} = \frac{1}{4w^2}.$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0.$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \frac{2}{\pi} \left(we^{-2w^2x^2} \right)^* \frac{d^2}{dx^2} \left(we^{-2w^2x^2} \right) dx = h^2 a.$$

$$\sigma_x = \frac{1}{2w}.$$

$$\sigma_p = h\sqrt{h\sqrt{a}}.$$

Question 3e. The uncertainty principle does hold, and in fact is exactly equal to the limit when $t = 0$:

$$\sigma_x \sigma_p = \frac{h\sqrt{a}}{2w} = \sqrt{1 + (2\hbar ta/m)^2}\frac{\hbar}{2} \geq \hbar/2.$$

Question 4. The given potential gives the boundary condition that $\psi(0) = 0$. Thus the allowed energies are those corresponding to odd wavefunctions, i.e. $E = (2(n-1) + \frac{1}{2})\hbar\omega, n \in \mathbb{N}$.