UBC Mathematics 320(101)—Assignment 7
Due by PDF upload to Canvas at 18:00, Saturday 28 Oct 2023

References: Rudin pp. 58b–69a; Thomson-Bruckner-Bruckner, Sections 3.1–3.6 [but skip 3.3]

1. Use $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ and a splitting argument to evaluate $S = 1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \cdots$.

2. Test the following series for convergence. Treat all real values of the constant parameter p.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p(\log n)}$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$.

3. Consider the set ℓ^2 consisting of all real sequences $x = (x_1, x_2, \ldots)$ enjoying the special property that $\displaystyle\sum_n |x_n|^2$ converges. Define an inner product on ℓ^2 as follows:

$$\forall x, y \in \ell^2, \quad \langle x\,,\, y\rangle \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} x_n y_n.$$

(a) Prove that the series in this definition converges.

Informally, this is the natural generalization of Euclidean k-space to the case $k = \aleph_0$; the inner product $\langle x\,,\, y\rangle$ in ℓ^2 is analogous to the dot product $\mathbf{x} \bullet \mathbf{y}$ in $\mathbb{R}^k$. It's only a small stretch to call the elements of ℓ^2 "vectors". Add further credibility to this interpretation by defining $\|x\| = \sqrt{\langle x\,,\, x\rangle}$ for each $x \in \ell^2$, and then proving

(b) $|\langle x\,,\, y\rangle| \le \|x\|\,\|y\|$ for all $x, y \in \ell^2$,

(c) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \ell^2$.

This generalization has some limitations, however. In $\mathbb{R}^k$, any sequence of vectors $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$, whose component sequences converge must be a convergent sequence of vectors, and its limit can be identified by taking the limit in each component separately. Show that this fails in ℓ^2, as follows:

(d) Construct a sequence $x^{(1)}, x^{(2)}, \ldots$, of vectors in ℓ^2 such that $\left\|x^{(n)}\right\| = 1$ for all n, and yet for every $p \in \mathbb{N}$ the 'p-th component sequence' $\left\langle \mathbf{e}_p\,,\, x^{(n)}\right\rangle$ converges to 0 as $n \to \infty$. Here, just as in $\mathbb{R}^k$, $\mathbf{e}_p$ denotes the "standard unit vector" with exactly one nonzero entry, which is a 1 in position p.

4. Given that the sequence $(s_n + 2s_{n+1})$ converges, prove that the sequence (s_n) converges.

5. Prove that if $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n^q}$ converges for any constant $q > \frac{1}{2}$.

6. In parts (a)–(c) below, suppose $a_n > 0$ and $b_n > 0$ for all n, and define

$$A = \sum_{n=1}^{\infty} a_n, \qquad B = \sum_{n=1}^{\infty} b_n.$$

(a) Prove the Limit Comparison Test: If b_n/a_n converges to a real number $L > 0$, then series A converges if and only if series B converges.

(b) Prove the Ratio Comparison Test: If $a_{n+1}/a_n \le b_{n+1}/b_n$, convergence of series B implies convergence of series A. What if $a_{n+1}/a_n \le b_n/b_{n-1}$ instead?
[Clue: Start by finding upper and lower bounds for the sequence $r_n = a_n/b_n$.]

(c) Use (b) with $\zeta(p)$ to prove Raabe's Test: if $p > 1$ and $a_{n+1}/a_n \le 1 - p/n$ for all n sufficiently large, then series A converges.
[Clue: First show that $1 - px < (1-x)^p$ for all $x \le 1$. Just use calculus.]

(d) Test $\sum_{n} a_n$ for convergence, where $\quad a_n = \dfrac{1 \cdot 4 \cdot 7 \cdots (3n+1)}{n^2\, 3^n\, n!}$.

7. Prove: If each $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges. Does the converse hold?

8. (a) Prove: Given any $D \in \mathbb{R}$ and $\delta > 0$, there is a finite collection of numbers $a_1, a_2, \ldots, a_N$ such that $D = a_1 + a_2 + \cdots + a_N$ and

$$\delta > |a_1| > |a_2| > \cdots > |a_N| > 0.$$

(b) Let $(\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Explain how to construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying, simultaneously,

(i) $|x_n| > |x_{n+1}|$ for all n, and $x_n \to 0$ as $n \to \infty$, and

(ii) the sequence $(s_N)_{N\in\mathbb{N}}$ defined by $s_N = \sum_{n=1}^{N} x_n$ has $(\sigma_n)_n$ as a subsequence.

Discussion: This shows how badly the converse of the Crude Divergence Test can fail: the series $\sum_n x_n$ has terms tending to 0, yet its sequence of partial sums can be wild enough to hit all elements of the preassigned $(\sigma_n)_n$.