MATH 443 Homework 4

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Question 1. Suppose by way of contradiction that the root of a DRT T had in-degree nonzero. Let r be the root of T and v be a vertex such that $vr \in E(G)$. Then vr is a path from v to r in the underlying tree of T. Since T is a tree it is the only path between r and v, so there doesn't exist a directed path from r to v, which is a contradiction of our assumption that T is a DRT. Therefore r has in-degree zero since no vertices can lead into it.

To see why each other vertex $v \neq r$ must have in-degree 1, first note that they clearly can't have in-degree 0 since there exists a directed path from r to v, and the last edge in this path will contribute 1 to the in-degree of that vertex. To see why the number of in vertices can't be more than one, supposed by contradiction that it was, i.e. suppose there exist $u_1, u_2 \in V(T)$ s.t. $u_1v \in E(T), u_2v \in E(T)$. Since $u_1, u_2 \in V(T)$ there exists ordered paths P_1, P_2 such that the first vertex of both is r and the second is u_1 and u_2 respectively. Let w be the last vertex of P_1 that is also in P_2. Let Q_1, Q_2 be the sub paths of P_1, P_2 that go from w to v. $Q_1 \cap Q_2 = \emptyset$ since w was chosen to be the last shared vertex. However then wQ_1vQ_2w is a cycle which is impossible in a tree. Therefore the in-degree number can't be 0 and can't be 2 or greater, so it must be 1. $\square$

Question 2. Let T_1, T_2 be disjoint DRTs and let e be a directed edge with one endpoint in T_1 and the other in T_2.

($\Rightarrow$) Assume $(T_1 \cup T_2) + e$ is a DRT, we will prove that the second vertex of e is the root of T_1 or T_2, call them r_1, r_2. To see why suppose by contradiction that that the second vertex of $e = uv$ is not r_1, r_2, and WLOG assume $u \in E(T_1), v \in E(T_2), v \neq r_2$. Let P_1 be a directed path from r_1 to u and let P_2 be a directed path from r_2 to v. Then $P_1 + e$ is a directed path from r_1 to v in the new graph. However this implies that in the new graph v has an in-degree of at least 2 (since the second last vertex of P_1 is in T_1 and the second last vertex of P_2 is in T_2). However problem no vertex on a DRT has in-degree 2 or greater, so it must be that e was the root of T_1 or T_2.

($\Leftarrow$) Assume the endpoint of e is the root of T_1 or T_2. WLOG assume $e = ur_2$ where r_2 is the root of T_2, and let r_1 be the root of T_1. Let $T = (T_1 \cup T_2) + e$ and let $x \in V(T)$. If $x \in T_1$ then since T_1 is a DRT there exists a directed path from r_1 to x. If $x \in T_2$ then let P_1 be a directed path from r_1 to u and P_2 be a directed path from r_2 to x. Then $r_1P_1eP_2x$ is a directed path from r_1 to x, so r_1 fulfills all the root requirements for T. Also note that since T_1, T_2 were disconnected before adding e, e is a bridge. T_1, T_2 were both trees and we added a bridge to connect them so T is a tree. Finally since T_1, T_2 were both DRTs and we added an edge $e = ur_2$ such that $r_2e \notin E(T)$, the second requirement for a DRT is also satisfied. We've shown that T fulfills all the requirements for a DRT, so it is one and we're done. $\square$

Question 3. The statement is true. Let P be a longest path of a connected graph G, and let u, v be its two endpoints. The statement will be shown for u, although since u, v were arbitrary it also holds for v. By way of contradiction let $x, y \in G - u$ such that x, y are disconnected in $G - u$. G is connected so there exists a path $Q \subset G$ from x to y. Given that x, y became disconnected after removing u it must be that $u \in V(Q)$. $x \neq u, y \neq u$, so there exist paths Q_1 between x and u_1 and Q_2 between u_2 and y, where $u_1, u_2 \in N(u)$. We proved in class that $N(u) \in V(P)$, since otherwise you could extend P by including a vertex of $N(u)$ not already in P. Let $P_1 \subset P$ be the portion of P between u_1, u_2. Note that $u \notin P_1$ since u is an endpoint of P, so it's not a midpoint of any subpath. But then $xQ_1u_1P_1u_2Q_2y$ is a path of $G - u$ between x and y which contradicts our assumption that x, y disconnected. Therefore u couldn't have been a cut vertex, and by the symmetry of the argument v couldn't have been either. □

Question 4. The flaw is that the distance between the endpoints of a longest path of a graph G are not necessarily the farthest from each other. To see why consider C_4. Then the longest path includes all 4 vertices, but the distance between two endpoints of such a path is just 1 whereas the actual farthest vertices are distance 2. Therefore it's not valid to assume that the endpoints of a longest path are the vertices that are the farthest from one another.

Question 5.

Question 6.

Question 7. Consider the number of edges that must be removed to generate three components. Let x be the number of vertices of the first component G_1, and y be the number of vertices of the second component G_2. Then the last vertex has $3n - x - y$ vertices, call it G_3. The number of edges to disconnect G_1 is $x(3n - x)$, since each of the x vertices is attached to $3n - x$ vertices. G_2 has y vertices attached to $3n - y$ vertices, but x of the second of those vertices was already removed in the first step. Therefore you must remove $y(3n - x - y)$ vertices to separate the second component. After removing both of these the last will be disconnected, since neither of the G_1, G_2 are connected to it by design. Thus the total number of edges required to disconnect is:

$$f(x, y) = x(3n - x) + y(3n - x - y) = 3n(x + y) - (x^2 + xy + y^2).$$

Fix x and consider optimizing y. The optimum occurs either on the edges $(y = 1, y = 3n - 2)$ or where the derivative is zero. Note however that this is a inverted parabola with its vertex at $y = \frac{-b}{2a} = \frac{3n-x}{2}$ which is its maximum point, so if we were to take the derivative and set it to zero we would find a maximum, not am minimum. Thus $y = 1$ or $y = 3n - 2$.

If $y = 3n - 2$, then the only option for x would be $x = 1$ and the total number of edges is $f(1, 3n - 2) = 6n - 3$. If instead $y = 1$, then using the same logic as before $x = 3n - 2$ or $x = 1$. Since the remaining component will be the opposite of whatever choice of x we use (and switching them doesn't change f), WLOG assume $x = 1$ as well. Then the total number of removed edges is still $f(1, 1) = 6n - 3$. We've covered all possible cases, so the minimum number of edges required is $|X| = 6n - 3$.