VI. Hausdorff Topological Spaces

A. Open Sets – Three Steps

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Step One: Euclidean k-space. Recall that $\mathbb{R}^k = \{x = (x_1, \ldots, x_k) : x_n \in \mathbb{R}\}$ is the set of k-element tuples of real numbers. (When $k = 1$, we ignore the difference in types between a 1-element tuple (x) and the ordinary real number x inside it, and casually pretend $\mathbb{R}^1 = \mathbb{R}$.)

Definition. Given $x \in \mathbb{R}^k$ and $r > 0$, the **open ball of centre x and radius r** is

$$\mathbb{B}[x; r) = \left\{ y \in \mathbb{R}^k \, : \, |y - x| < r \right\}.$$

A subset U of $\mathbb{R}^k$ is **open** if and only if

$$\forall x \in U, \ \exists r > 0 \ \ : \ \ \mathbb{B}[x; r) \subseteq U. \tag{$*$}$$

(Intuition: From every point x in U, moving in any direction is allowed in U, at least for distances below r.)

Remark. For fixed $\overline{x}$ in $\mathbb{R}^k$ and $\overline{r} > 0$, the set $U = \mathbb{B}[\overline{x}; \overline{r})$ is an "open set" (so the name "open ball" is well-deserved).

> Indeed, pick any x in U. Then $|x - \overline{x}| < \overline{r}$, so $\varepsilon \overset{\text{def}}{=} \overline{r} - |x - \overline{x}| > 0$. To show that $\mathbb{B}[x; \varepsilon) \subseteq U$, pick any $z \in \mathbb{B}[x; \varepsilon)$: then $|z - x| < \varepsilon$, so
>
> $$|z - \overline{x}| = |z - x + x - \overline{x}| \le |z - x| + |x - \overline{x}| < \overline{r}.$$
>
> That is, $z \in \mathbb{B}[\overline{x}; \overline{r}) = U$. So $\mathbb{B}[x; \varepsilon) \subseteq U$, as required.

Remark. In Euclidean 2-space, $U = \{(x, y) : y > 0\}$ is open; $C = \{(x, y) : y \ge 0\}$ is not open but $C^c \overset{\text{def}}{=} \{(x, y) : y < 0\}$ is open; $L = \{(x, y) : y > 0 \text{ or } x \ge 0\}$ is not open and L^c is not open either.

Notation.
1. $\mathcal{T} = \left\{ U \subseteq \mathbb{R}^k : U \text{ is an open set} \right\}$, the usual "topology" on $\mathbb{R}^k$.
2. $\mathcal{N}(x) = \{S \in \mathcal{P}(X) : \text{ some } U \in \mathcal{T} \text{ obeys } x \in U \subseteq S\}$, the set of "neighbourhoods of x". [Some writers insist on using only open sets as "neighbourhoods". There is no good reason for this restriction.]

Convergence. Convergence $x_n \to \widehat{x}$ in $\mathbb{R}^k$ can be expressed equivalently as

(a) $\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ \ : \ \ \forall n > N, \ x_n \in \mathbb{B}[\widehat{x}; \varepsilon)$.

(b) $\forall S \in \mathcal{N}(\widehat{x}), \ \exists N \in \mathbb{N} \ \ : \ \ \forall n > N, \ x_n \in S$.

(b⇒a): Assume (b). Given $\varepsilon > 0$, take $S = \mathbb{B}[\widehat{x}; \varepsilon)$: then $S \in \mathcal{N}(x)$, so conclusion of (a) follows.

(a⇒b): Assume (a). Given $S \in \mathcal{N}(\widehat{x})$, choose $\varepsilon > 0$ such that $\mathbb{B}[\widehat{x}; \varepsilon) \subseteq S$. Apply (a) to this ε. Get back N such that all $n > N$ give $x_n \in \mathbb{B}[\widehat{x}; \varepsilon)$. For these same n, we have $x_n \in S$, by choice of ε above.

Properties of Open Sets. The numbering here anticipates properties (HTS1)–(HTS4) below.

(i) Both $\emptyset$ and $\mathbb{R}^k$ are open. In symbols,

$$\emptyset \in \mathcal{T} \quad \text{and} \quad \mathbb{R}^k \in \mathcal{T}.$$

(ii) Any union of open sets is open. That is, if $\mathcal{G}$ is any collection of open sets, then defining $U = \bigcup \mathcal{G}$ produces an open set. Recall that $\bigcup \mathcal{G} = \bigcup_{G \in \mathcal{G}} G$. In symbols,

$$\mathcal{G} \subseteq \mathcal{T} \implies \bigcup \mathcal{G} \in \mathcal{T}.$$

(iii) Any intersection of **finitely many** open sets is open. In symbols,

$$\Big[n \in \mathbb{N}, \quad U_1, \ldots, U_n \in \mathcal{T} \Big] \implies U_1 \cap \cdots \cap U_n \in \mathcal{T}.$$

(iv) Distinct points can be given disjoint neighbourhoods. In symbols,

$$\forall x, y \in \mathbb{R}^k, \qquad x \neq y \implies \Big[\exists U, V \in \mathcal{T} \,:\, x \in U,\ y \in V,\ U \cap V = \emptyset \Big].$$

[Sketch justifications; note that finiteness is essential in (iii), because $\bigcap_n \left(-\infty, 1/n \right) = (-\infty, 0]$ is not open.]

(i) Clearly $X = \mathbb{R}^k$ is open; and $U = \emptyset \in \mathcal{T}$ because there are no points x in U to falsify $(*)$.

(ii) Let $\mathcal{G}$ be any subset of $\mathcal{T}$, and consider $U = \bigcup \mathcal{G} = \bigcup_{G \in \mathcal{G}} G$. If $x \in U$, then there exists $G \in \mathcal{G}$ such that $x \in G \in \mathcal{T}$. By $(*)$ for G, $\exists \varepsilon > 0$ s.t. $\mathbb{B}[x; \varepsilon) \subseteq G \subseteq U$.

(iii) Use induction, defining

$$P(n) : \qquad \mathcal{T} \text{ contains any intersection of } n \text{ members of } \mathcal{T}.$$

Start with $n = 2$. If $U_1, U_2 \in \mathcal{T}$, let $U = U_1 \cap U_2$. For any x in U, we have

$$\begin{aligned} x \in U_1 &\implies \exists \varepsilon_1 > 0 \,:\, \mathbb{B}[x; \varepsilon_1) \subseteq U_1, \\ x \in U_2 &\implies \exists \varepsilon_2 > 0 \,:\, \mathbb{B}[x; \varepsilon_2) \subseteq U_2. \end{aligned}$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ to see $\mathbb{B}[x; \varepsilon) \subseteq U_1 \cap U_2 = U$, as required.
Now assume $P(n)$ is true, and let $U_1, \ldots, U_{n+1}$ be given members of $\mathcal{T}$. Then

$$U \stackrel{\text{def}}{=} \bigcap_{k=1}^{n+1} U_k = [U_1 \cap U_2 \cap \cdots U_n] \cap U_{n+1}$$

displays U as an intersection of two sets: the first is open by $P(n)$, and the second (U_{n+1}) is open by hypothesis. Hence U is open by $P(2)$. This establishes $P(n+1)$, and it follows from the principle of mathematical induction that $P(k)$ is true for all $k \in \mathbb{N}$.

(iv) If $x, y \in \mathbb{R}^k$ obey $x \neq y$, then $|y-x| > 0$. So let $\varepsilon = |y-x|/2$: then

$$\mathbb{B}[x;\varepsilon) \cap \mathbb{B}[y;\varepsilon) = \emptyset.$$

The triangle inequality shows this. Indeed, any specific $z \in \mathbb{B}[x;\varepsilon)$ obeys $|z-x| < \varepsilon$, so

$$|z-y| \geq |y-x| - |z-x| > 2\varepsilon - \varepsilon = \varepsilon.$$

That is, $z \notin \mathbb{B}[y;\varepsilon)$. Remember that both $\mathbb{B}[x;\varepsilon)$ and $\mathbb{B}[y;\varepsilon)$ are open. ////

Step 2: Metric Spaces.

Definition. Given a nonempty set X, a function $d\colon X \times X \to \mathbb{R}$ is called a **metric** if

(a) $d(x,y) \geq 0\ \forall x, y \in X$, with equality iff $x = y$ [positivity];

(b) $d(x,y) = d(y,x)\ \forall x, y \in X$ [symmetry];

(c) $d(x,z) \leq d(x,y) + d(y,z)\ \forall x, y, z \in X$ [triangle inequality].

In this case the pair (X,d) is called a **metric space**.

Examples. • For any nonempty set X (including $\mathbb{R}^k$!),

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

defines **the discrete metric**.

- For $X = \mathbb{R}^k$, $d(x,y) = |y-x|$ defines **the Euclidean metric**. Properties (a)–(b) are clear; we proved (c) earlier this term. Unless otherwise specified, this is the metric we mean whenever we write just "$\mathbb{R}^k$".
- For $X = \mathbb{R}^k$, other famous metrics include

$$d_1(x,y) = \sum_{n=1}^{k} |x_n - y_n|,$$
$$d_\infty(x,y) = \max\left\{|x_n - y_n| \,:\, n = 1, 2, \ldots, k\right\},$$
$$d_p(x,y) = \left(\sum_{n=1}^{k} |x_n - y_n|^p\right)^{1/p} \qquad p \geq 1.$$

Properties (a)–(c) are obvious for the first two above; for general $p > 1$, the triangle inequality for d_p is known as Minkowski's Inequality. (The fact that it is named after somebody suggests, correctly, that it is not obvious.)

- For $X = \ell^2$, the set of sequences $x = (x_1, x_2, \ldots)$ obeying $\displaystyle\sum_{n=1}^{\infty} |x_n|^2 < +\infty$,

$$d(x,y) = \sqrt{\sum_{n=1}^{\infty} |y_n - x_n|^2}$$

  defines a metric that shares many features with the Euclidean metric on $\mathbb{R}^k$. Students proved various supporting facts on HW.

- For the set X of bounded functions $x\colon [0,1] \to \mathbb{R}$,

$$d(x,y) = \sup\left\{|y(t) - x(t)| \,:\, t \in [0,1]\right\}$$

  defines a metric (home practice!).

Notation. In a metric space (X,d), we define various "balls" for any $x \in X$ and $r > 0$:

$$\begin{aligned}
\mathbb{B}[x;r) &= \{y \in X \,:\, 0 \le d(x,y) < r\}, \\
\mathbb{B}[x;r] &= \{y \in X \,:\, 0 \le d(x,y) \le r\}, \\
\mathbb{B}(x;r) &= \{y \in X \,:\, 0 < d(x,y) < r\}, \\
\mathbb{B}(x;r] &= \{y \in X \,:\, 0 < d(x,y) < r\}.
\end{aligned}$$

The inequalities requiring $d(x,y) \ge 0$ in the first two lines are redundant. Showing them here is supposed to help explain the interpretation of various parenthesis/bracket shapes.

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The Open Sets. In a metric space (X,d), let $\mathcal{T}$ denote the collection of all subsets $G \subseteq X$ with this property:

$$\forall x \in G,\ \exists r > 0 \,:\, \mathbb{B}[x;r) \subseteq G.$$

A set G is called *open* if and only of $G \in \mathcal{T}$.

Proposition. *In any metric space (X,d), properties (i)–(iv) enumerated earlier for open sets in $\mathbb{R}^k$ remain valid.*

Proof. Just reread the sketchy justifications above, changing $|y-x|$ to $d(x,y)$ wherever the former appears. ////

Convergence. In a metric space (X,d), a sequence (x_n) and an element $\widehat{x}$ are related by saying $\widehat{x} = \lim_{n\to\infty} x_n$ (or $x_n \to \widehat{x}$ as $n \to \infty$) exactly when

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \,:\, \forall n > N,\ d(x_n, \widehat{x}) < \varepsilon.$$

This is perfectly analogous with the familiar situation in $\mathbb{R}^k$. A given sequence (x_n) is called *convergent* if and only of there exists $\widehat{x} \in X$ such that $x_n \to \widehat{x}$ as $n \to \infty$.

Note that the limiting statement $x_n \to \widehat{x}$ in (X, d) is logically equivalent to the limiting statement $d(x_n, \widehat{x}) \to 0$ in $\mathbb{R}$. So any method for showing that a nonnegative sequence of real numbers converges to 0 may be useful in the study of metric spaces. (The Squeeze Theorem is a leading candidate for this.)

In the presence of a metric, most topological concepts admit simple (and useful!) characterizations built from sequences. Here is a first result of this type.

Proposition. *In a metric space (X, d), with subset A, the following are equivalent:*

(a) *A is an open set.*

(b) *For every $x \in A$, and every sequence (x_n) obeying $x_n \to x$, one has $x_n \in A$ for all n sufficiently large. That is,*

$$\exists N \in \mathbb{N} : \forall n > N,\ x_n \in A.$$

(*Constant sequences are allowed.*)

Proof. (a⇒b) Suppose A is open. Pick any $x \in A$ and any sequence (x_n) converging to x. Since A is open and $x \in A$, there exists $\varepsilon > 0$ so small that $\mathbb{B}[x; \varepsilon) \subseteq A$. Use this $\varepsilon > 0$ as the tolerance in the convergence definition: this gives some $N \in \mathbb{N}$ such that for all $n > N$, one has $d(x_n, x) < \varepsilon$. This inequality means precisely that $x_n \in \mathbb{B}[x; \varepsilon) \subseteq A$, as required.

(b⇒a) Contraposition: Suppose A is not open. This means that there exists some point $x \in A$ such that

$$\begin{aligned} &\forall \varepsilon > 0,\ \mathbb{B}[x; \varepsilon) \not\subseteq A, \\ \text{i.e.,}\qquad &\forall \varepsilon > 0,\ \mathbb{B}[x; \varepsilon) \cap A^c \neq \emptyset. \end{aligned}$$

For each $n \in \mathbb{N}$, use this statement with $\varepsilon = 1/n$ to see that $\mathbb{B}[x; 1/n)$ contains a point $x_n \notin A$. This produces a sequence x_n for which $x_n \notin A$ for all n and yet $d(x_n, x) < 1/n$ so that $x_n \to x$. We have shown "$\neg$(a) $\Longrightarrow$ $\neg$(b)", which is logically equivalent to "(a) $\Longrightarrow$ (b)". ////

Step Three: Hausdorff Topological Spaces.

Definition. A **topological space** has two ingredients: a set X, and a family $\mathcal{T}$ of subsets of X called "the open sets", related by properties (HTS1)–(HTS3) below:

(HTS1) Both $\emptyset$ and X are open (i.e., $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$).

(HTS2) Any union of open sets is open. That is, for any subset $\mathcal{G}$ of $\mathcal{T}$, one has $\bigcup \mathcal{G} \in \mathcal{T}$.

(HTS3) Any intersection of **finitely many** open sets is open. That is, if $N \in \mathbb{N}$ and $U_1, \ldots, U_N \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_N \in \mathcal{T}$.

We will deal only with **Hausdorff** topological spaces, which also obey

(HTS4) Whenever x and y are distinct points of X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

We write HTS instead of Hausdorff Topological Space.

We have already verified conditions (HTS1)–(HTS4) for the topology generated by any metric space. These notes depart slightly from the textbook presentation by avoiding explicit references to metrics wherever possible. This perspective is useful because it takes almost no extra work, and produces results that can be used in HTS's that involve topologies that do not originate with a metric. (Such HTS's actually exist and have uses in the wider world!)

IMPORTANT
All HTS definitions and theorems are available in any metric topology.

Notation. For $x \in X$, write $\mathcal{N}(x) = \{S \in \mathcal{P}(X) \,:\, \text{ some } U \in \mathcal{T} \text{ obeys } x \in U \subseteq S\}$. This is the set of "neighbourhoods of x".

Example. (a) Discrete Topology. For any X one may consider

$$\mathcal{T} = \{U \,:\, U \subseteq X\}\,.$$

In this topology, every set is open, and conditions (i)–(iv) are obvious.

(b) A non-Hausdorff topology (the only one we'll ever consider!). Take $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1\}, X\}$. Conditions (i)–(iii) clearly hold, but (iv) fails on $x = 2$, $y = 3$.

(c) The usual topology on $\mathbb{R}^k$. Take $X = \mathbb{R}^k$. Define $\mathcal{T}$ by saying that $U \in \mathcal{T}$ iff every point x in U has this property: for some $\varepsilon > 0$ (depending on x),

$$U \supseteq \mathbb{B}[x;\varepsilon] \stackrel{\text{def}}{=} \left\{y \in \mathbb{R}^k \,:\, |y - x| < \varepsilon\right\}. \qquad (*)$$

Conditions (i)–(iv) are demonstrated in Section A.

B. Neighbourhoods and Interior

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Definition. Let $(X, \mathcal{T})$ be a HTS. For each $x \in X$, let $\mathcal{N}(x)$ denote *the set of all neighbourhoods of x,* defined by

$$S \in \mathcal{N}(x) \iff \exists U \in \mathcal{T} \,:\, x \in U \subseteq S.$$

(Every open set containing x belongs to $\mathcal{N}(x)$. Often some non-open sets do, too.)

Lemma. *In a HTS $(X,\mathcal{T})$, with $A \subseteq X$, TFAE:*

(i) *A is open;*

(ii) *For each $x \in A$, one has $A \in \mathcal{N}(x)$.*

Proof. (i⇒ii) For any x in A, we obviously have $x \in A \subseteq A$. When A is open, we have $A \in \mathcal{T}$, so the definition of $A \in \mathcal{N}(x)$ stated above holds with $U = A$.

(ii⇒i) For each x in A, choose an open set U_x obeying both $x \in U_x$ and $U_x \subseteq A$. Then let

$$U = \bigcup_{x \in A} U_x.$$

As a union of open sets, U is open; since $U_x \subseteq A$ for all x, we have $U \subseteq A$; and for every x in A, we have $x \in U_x \subseteq U$, so $A \subseteq U$. Thus $A = U$, and A is open. ////

Definition. Let A be any set in a HTS $(X,\mathcal{T})$. The **interior of** A is the set

$$A^\circ = \{x \in A \,:\, x \in U \text{ and } U \subseteq A \text{ for some } U \in \mathcal{T}\}.$$

(Alternative notation: $A^\circ = \operatorname{int} A = \{x \in A \,:\, A \in \mathcal{N}(x)\}$.)

Clearly, whenever $A \subseteq B$ then $A^\circ \subseteq B^\circ$. Intuitively, A° is the largest open subset of A, in the sense captured by parts (a)–(b) of the following result.

Proposition. (a) *A° is open, and $A^\circ \subseteq A$.*

(b) *If G is open and $G \subseteq A$, then $G \subseteq A^\circ$.*

(c) *A is open if and only if $A = A^\circ$.*

Proof. (a) Suppose $z \in A^\circ$. Then $\exists U \in \mathcal{N}(z) \cap \mathcal{T}$ such that $U \subseteq A$. Every x in U obeys

$$U \in \mathcal{N}(x) \text{ and } U \subseteq A, \qquad \text{i.e.,} \qquad x \in A^\circ.$$

Hence $U \subseteq A^\circ$. Since z is arbitrary in A°, the Lemma above implies A° is open.

(b) For any open $G \subseteq A$, pick any $z \in G$. Then $G \in \mathcal{N}(z)$ and $G \subseteq A$, so $z \in A^\circ$. Hence $G \subseteq A^\circ$.

(c) (⇐) Obvious from (a). (⇒) $A \supseteq A^\circ$ holds for any A. When A is open, choosing $G = A$ in part (b) gives the reverse inclusion. Hence equality holds. ////

Example. If $a < b$ in $\mathbb{R}$, then $[a,b]^\circ = [a,b)^\circ = (a,b]^\circ = (a,b)^\circ = (a,b)$; $\mathbb{Q}^\circ = \emptyset$.

Remark. A° is the largest open subset of A. The smallest open subset is $\emptyset$; the largest open superset is X. The smallest open superset may fail to exist. For example, any open superset U of $[a,b]$ must contain an open interval of the form $(a - 1/n, b + 1/n)$ for some $n \in \mathbb{N}$; boosting n to $n+1$ defines a new open set that is smaller than U, but still covers $[a,b]$. Informally, every open superset is shrinkable: it is impossible to uniquely specify one to call "the smallest". Note also that

$$\bigcap_{n \in \mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a,b].$$

This shows that an infinite intersection of open sets may fail to be open.

C. Closed Sets and Closure

Definition. In a HTS $(X, \mathcal{T})$, a set $A \subseteq X$ is **closed** iff A^c is open.

Taking the set-complement in the definition above is different from taking the logical negation. A set can be "not open" at the same time that its complement is "not open", and in this case it is *neither open nor closed*. Even on the real line, all four possibilities below arise:

(1) the interval $(0,1)$ is open,

(2) the interval $[0,1]$ is closed,

(3) the interval $[0,1)$ is not open, and simultaneously not closed,

(4) the interval $(-\infty,+\infty)$ is open (obviously), and simultaneously closed (because its complement is $\emptyset$, which is open).

Lemma. *In a HTS $(X, \mathcal{T})$, with $A \subseteq X$, TFAE:*

(i) *A is closed.*

(ii) *For every $x \notin A$, some neighbourhood $U \in \mathcal{N}(x)$ obeys $U \subseteq A^c$.*

Proof. Immediate from out similar lemma on open sets. ////

Prop. *In a HTS $(X, \mathcal{T})$,*

(a) *Any intersection of closed sets is closed;*

(b) *Any* **finite** *union of closed sets is closed.*

Proof. (a) Given a collection of closed sets C_i, $i \in I$, consider $C = \bigcap_{i\in I} C_i$. Observe

$$C^c = \left[\bigcap_{i\in I} C_i\right]^c = \bigcup_{i\in I} C_i^c.$$

Each set C_i^c is open, so C^c is a union of open sets. Thus C^c is open.

(b) If $C = C_1 \cup \cdots \cup C_n$ is a finite union of closed sets, then

$$C^c = C_1^c \cap \cdots \cap C_n^c$$

is a finite intersection of open sets. This makes C^c open. ////

Definition. Let A be a set in the HTS $(X, \mathcal{T})$. The **closure of** A is

$$\overline{A} = \left((A^c)^\circ\right)^c.$$

(Alternative notation: $\overline{A} = \operatorname{cl} A$.)

The definition implies monotonicity for this operation:

$$A \subseteq B \implies \overline{A} \subseteq \overline{B}.$$

Intuitively, $\overline{A}$ is the smallest closed superset of A, as expressed in parts (a)–(b) of the next result.

Prop. (a) *$\overline{A}$ is closed, and $\overline{A} \supseteq A$.*

(b) *If F is closed and $F \supseteq A$, then $F \supseteq \overline{A}$.*

(c) *A is closed if and only if $\overline{A} = A$.*

Proof. (a) As the complement of the open set $(A^c)^\circ$, the set $\overline{A}$ is closed. Also,

$$(A^c)^\circ \subseteq A^c \implies \left((A^c)^\circ\right)^c \supseteq (A^c)^c, \text{ i.e., } \overline{A} \supseteq A.$$

(b) Suppose F closed and $F \supseteq A$. Then F^c is open, so $F^c = (F^c)^\circ$, and

$$\begin{aligned} F \supseteq A &\implies F^c \subseteq A^c \\ &\implies F^c = (F^c)^\circ \subseteq (A^c)^\circ \\ &\implies F = (F^c)^c \supseteq \left((A^c)^\circ\right)^c = \overline{A}. \end{aligned}$$

(c) ($\Leftarrow$) Obvious, from (a). ($\Rightarrow$) If A is closed, then choosing $F = A$ in (b) gives $A \supseteq \overline{A}$. Hence, by (a), $A = \overline{A}$. ////

Example. If $a < b$ in $\mathbb{R}$, then $\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$; $\overline{\mathbb{Q}} = \mathbb{R}$.

Practice. Earlier we saw that it is impossible to make sense of the phrase "smallest open superset of A". For practice, use the open interval $A = (0,1)$ to explain why the idea of a "largest closed subset of A" cannot be given a trustworthy definition.

D. Boundary Points

Definition. Given a HTS $(X, \mathcal{T})$, let $A \subseteq X$. A point z in X is a **boundary point of A** iff

$$\forall G \in \mathcal{N}(z), \qquad \text{both } A \cap G \neq \emptyset \quad \text{and} \quad A^c \cap G \neq \emptyset.$$

The set of boundary points of A is denoted ∂A.

Remark. Boundary points may lie in either A or A^c. In fact, interchanging A and A^c in the definition above makes no logical difference, so we have

$$\partial A = \partial(A^c).$$

Example. In $\mathbb{R}$, $\partial(a,b) = \partial(a,b] = \partial[a,b) = \partial[a,b] = \{a,b\}$; $\partial\mathbb{Q} = \mathbb{R}$; $\partial\mathbb{Z} = \mathbb{Z}$; $\partial\mathbb{R} = \emptyset$. Notice that $\mathbb{Q} \subseteq \mathbb{R}$ but $\partial\mathbb{Q}$ is a strict superset of $\partial\mathbb{R}$. There is no "monotonicity" relation for the boundary operation.

For a set A in a HTS $(X,\mathcal{T})$,

(a) $\partial A = \overline{A} \cap \overline{A^c}$.

(b) *A is closed if and only if $\partial A \subseteq A$; also, $\overline{A} = A \cup \partial A$.*

(c) *A is open if and only if $\partial A \subseteq A^c$; also, $A^\circ = A \setminus \partial A$.*

Proof. Home Practice. ////

E. Limit Points and Isolated Points

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Definition. Let A be a set in a HTS $(X,\mathcal{T})$. A point z in X is a **limit point of A** exactly when

$$\forall U \in \mathcal{N}(z), \quad (U \setminus \{z\}) \cap A \neq \emptyset.$$

The set of limit points for A is denoted A'. Other authors call these *cluster points* or *accumulation points* of A, and call A' the *derived set for A.*

Notes. (i) If $A \subseteq B$, then $A' \subseteq B'$.

(ii) $z \notin A' \iff \exists U \in \mathcal{N}(z) \,:\, (U \setminus \{z\}) \cap A = \emptyset$.

Proposition. *In a metric space (X,d), let $A \subseteq X$. TFAE:*

(a) *$x \in A'$;*

(b) *one has $x = \lim_{n\to\infty} x_n$ for some sequence of distinct points, with $x_n \in A$ for each n.*

Proof. (a$\Rightarrow$b) Given any $x \in A'$, construct a suitable sequence as follows. Let $r_1 = 1$ and use the definition of $x \in A'$ to select some $x_1 \in \mathbb{B}(x;r_1) \cap A$. Note that $x_1 \neq x$, so we can define $r_2 = d(x_1,x)/2$ and be sure $r_2 > 0$. Since $x \in A'$, the set $\mathbb{B}(x;r_2) \cap A$ is nonempty, so we can select some x_2 in there; clearly $x_2 \neq x_1$. Let $r_3 = d(x_2,x)/2$. Choose $x_3 \in \mathbb{B}(x;r_3) \cap A$, and note that $x_3 \notin \{x_1,x_2\}$. Define $r_4 = d(x_3,x)/2$, and continue. This produces a sequence of distinct elements of A for which $d(x_n,x) < r_n \leq 2^{-n}$, so indeed $x_n \to x$, as required.

(b$\Rightarrow$a) Given a sequence as in (b), suppose U is an arbitrary neighbourhood of x. Then there must be some $\varepsilon > 0$ such that $\mathbb{B}[x;\varepsilon) \subseteq U$. The definition of convergence gives some $N \in \mathbb{N}$ such that $x_n \in \mathbb{B}[x;\varepsilon)$ for all $n > N$. This gives an infinite number of distinct points in the set $\mathbb{B}[x;\varepsilon) \cap A$. Passing to the subset $\mathbb{B}(x;\varepsilon) \cap A$ excludes at most one of these points (namely, x), so certainly $\mathbb{B}(x;\varepsilon) \cap A \neq \emptyset$. This is a subset of $(U \setminus \{x\}) \cap A$, so the latter must be nonempty too. ////

Example. In $\mathbb{R}$, $a < b$ implies $(a,b)' = [a,b]' = [a,b]$; $\mathbb{Q}' = \mathbb{R}$ and $\mathbb{Z}' = \emptyset$.

Prop. *For any set G in a HTS $(X,\mathcal{T})$,*

$$G \text{ is open} \iff G \cap (G^c)' = \emptyset.$$

Proof. ($\Rightarrow$) Pick an arbitrary $x \in G$. Get some $U \in \mathcal{N}(x)$ such that $U \subseteq G$. Then $U \cap G^c = \emptyset$, so

$$\emptyset = (U \setminus \{x\}) \cap G^c.$$

That is, $x \notin (G^c)'$.

($\Leftarrow$) Pick any $x \in G$. By hypothesis, $x \notin (G^c)'$, so $\exists U \in \mathcal{N}(x)$ satisfying

$$\emptyset = (U \setminus \{x\}) \cap G^c = U \cap G^c. \qquad \text{(Recall } x \in G.\text{)}$$

That is, $U \subseteq (G^c)^c = G$. Since $x \in G$ is arbitrary, this shows G is open. ////

Cor. *For any set F in a HTS $(X, \mathcal{T})$,*

$$F \text{ is closed} \iff F \supseteq F'.$$

Proof. Apply the previous result to F^c:

$$F \text{ is closed} \iff F^c \text{ is open} \iff F^c \cap F' = \emptyset \iff F' \subseteq F. \qquad ////$$

Prop. *For any set A in a HTS $(X, \mathcal{T})$, the set A' is closed.*

Proof. Home Practice. ////

Theorem. *For any set A, one has $\overline{A} = A \cup A'$.*

Proof. ($\supseteq$) Let $F = \overline{A}$. Then F is closed, so $F \supseteq F'$. Also, $F \supseteq A$, so $F' \supseteq A'$. Hence $F \supseteq F' \supseteq A'$, and $F \supseteq A \cup A'$.

($\subseteq$) Let $C = A \cup A'$. An arbitrary $z \in C^c$ will obey both $z \notin A$ and $z \notin A'$, so $\exists U \in \mathcal{N}(z) \cap \mathcal{T}$ such that

$$\emptyset = (U \setminus \{z\}) \cap A = U \cap A, \quad \text{i.e.,} \quad U \subseteq A^c.$$

Hence $C^c \subseteq (A^c)^\circ$, giving $C \supseteq \left((A^c)^\circ\right)^c = \overline{A}$. ////

Definition. For a set A in a HTS $(X, \mathcal{T})$, the collection of **isolated points** is $A \setminus A'$. I.e., x is an isolated point of A iff both

$$x \in A \quad \text{and} \quad \exists U \in \mathcal{N}(x) \,:\, A \cap U = \{x\}\,.$$

Example. In $(\mathbb{R}, \text{usual})$,

$\mathbb{Z}$ consists only of isolated points (i.e., $\mathbb{Z} = \mathbb{Z} \setminus \mathbb{Z}'$, because $\mathbb{Z}' = \emptyset$);

$\mathbb{Q}$ has no isolated points (i.e., $\mathbb{Q} \setminus \mathbb{Q}' = \mathbb{Q} \setminus \mathbb{R} = \emptyset$);

$A = [\mathbb{Q} \cap (-\infty, 0)] \cup \mathbb{N}$ has $A' = (-\infty, 0]$, so the isolated points of A form the set $\mathbb{N}$.

Set Extraction Summary. Given a set A in a HTS $(X,\mathcal{T})$,

$$\begin{aligned} A^c &= X\setminus A, \qquad \text{the complement of } A\\ A^\circ &= \text{the interior of } A,\\ \overline{A} &= \text{the closure of } A,\\ A' &= \text{the limit points of } A,\\ \partial A &= \text{the boundary of } A,\\ A\setminus A' &= \text{the isolated points of } A. \end{aligned}$$

E.g., in $\mathbb{R}^k$, the set $A=\mathbb{B}[0;1)\cup\{2\mathbf{e}_1\}=\left\{x\in\mathbb{R}^k\,:\,|x|<1\right\}\cup\{(2,0,\ldots,0)\}$ has

$$\begin{aligned} A^c &= \left\{x\in\mathbb{R}^k\,:\,|x|\ge 1\right\}\cap\left\{x\in\mathbb{R}^k\,:\,x\ne 2\mathbf{e}_1\right\},\\ A^\circ &= \mathbb{B}[0;1),\\ \overline{A} &= \left\{x\in\mathbb{R}^k\,:\,|x|\le 1\right\}\cup\{2\mathbf{e}_1\},\\ A' &= \left\{x\in\mathbb{R}^k\,:\,|x|\le 1\right\},\\ \partial A &= \left\{x\in\mathbb{R}^k\,:\,|x|= 1\right\}\cup\{2\mathbf{e}_1\},\\ A\setminus A' &= \{2\mathbf{e}_1\}=\{(2,0,\ldots,0)\}. \end{aligned}$$

F. Sequential Characterizations

In any HTS where the topology comes from a metric, every single one of the concepts and symbols introduced above can be described in terms of sequences. Earlier sections have presented sequential characterizations of open sets and limit points in metric spaces. Students are urged to formulate sequential descriptions also for closed sets, boundary points, and isolated points.

G. Subspaces; Relative Topologies; Bases

Definition. Let $(X,\mathcal{T})$ be a HTS. For any subset $Y\subseteq X$, the collection

$$\mathcal{T}_Y=\{G\cap Y\,:\,G\in\mathcal{T}\}.$$

makes $(Y,\mathcal{T}_Y)$ into a HTS all by itself. (Practice: Check axioms (HTS1)–(HTS4).) It's typical to call $(Y,\mathcal{T}_Y)$ a **(topological) subspace of** $(X,\mathcal{T})$**.** (There is no simple relationship with the concept of "subspace" in linear algebra.)

In the context above, it's hard to completely forget about X, so we must be especially careful with the words we use. For example, we will call a set $A\in\mathcal{T}_Y$ **open relative to** Y. To see why this is important, let X be the real line with the usual topology for $\mathcal{T}$, and consider the subset $Y=[-1,1]$. The set $A=(0,1]$ is open relative to Y, because $A=(0,2)\cap Y$ belongs to $\mathcal{T}_Y$ … but clearly A is not open relative to X.

If the subset $Y\subseteq X$ happens to be open to start with, then $\mathcal{T}_Y\subseteq\mathcal{T}$. In this case a subset of Y is open relative to $\mathcal{T}_Y$ iff it is open in the original sense from X. Thus the terms "open relative to Y" and simply "open" mean the same thing. But, as indicated above, if we start with a Y that is not open in X, extra care is needed.

Example. Let $X = \mathbb{R}$. If $Y = \mathbb{Q}$, the set $W = (0,1) \cap \mathbb{Q}$ is open relative to Y, but it is not open in X.
Or, let $X = \mathbb{R}^2$ and let $Y = \mathbb{R} \times \{0\} = \{(x,0) : x \in \mathbb{R}\}$. The segment $I = \{(x,0) : 0 < x < 1\}$ is open relative to Y, but not open in X. ////

A topological base. Given any nonempty set X and a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, try using sets in $\mathcal{B}$ to define neighbourhoods and reverse-engineer a topology. That is, for each x in X, define

$$\mathcal{N}(x) = \{S \subseteq X : \exists B \in \mathcal{B} \text{ with } x \in B \subseteq S\}.$$

Then declare a set $G \subseteq X$ to be "open" if and only if $G \in \mathcal{N}(x)$ holds for each $x \in G$. We know that this linkage between open sets and neighbourhoods must hold in any HTS. What properties do we need from the set $\mathcal{B}$ to make all this work out? The answer is embedded in the next result.

Proposition. *Given a nonempty set X, let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy the following conditions (a)–(c). Then the construction above defines a Hausdorff topology $\mathcal{T}$ on X. In this case, the set $\mathcal{B}$ is called a* **base** *for $\mathcal{T}$.*

(a) $\bigcup \mathcal{B} = X$ *[every point of X belongs to at least one set B in $\mathcal{B}$];*

(b) whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ satisfying $x \in B \subseteq B_1 \cap B_2$; and

(c) whenever $x_1, x_2 \in X$ obey $x_1 \neq x_2$, there exist $B_1, B_2 \in \mathcal{B}$ satisfying $x_1 \in B_1$, $x_2 \in B_2$, and $B_1 \cap B_2 = \emptyset$.

Examples. In $\mathbb{R}^k$, let $\mathcal{B}$ denote the set of balls $\mathbb{B}[p;r)$ for which the centre point p lies in $\mathbb{Q}^k$ and the radius $r > 0$ is rational. This is a base for the usual topology on $\mathbb{R}^k$. It is interesting because $\mathcal{B}$ is a *countable* collection of sets. A topological space for which a countable base exists is called "second-countable": see "Second-countable space" on Wikipedia if you would like to know more.