

Math 318 Homework 5

Xander Naumenko

02/03/23

Question 1a. The meaning of $1 - F_M(x)$ is the probability that the minimum is above a given value x . This occurs when each of the X_i are above x , which happens with probability each $1 - x$. Thus we have

$$1 - F_M(x) = (1 - x)^n \implies F_M(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - (1 - x)^n & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}.$$

Question 1b. The PDF is just the derivative of the CDF:

$$f_M(x) = \frac{d}{dx}F_M(x) = n(1 - x)^{n-1}1_{0 \leq x \leq 1}.$$

Question 1c. To find the mean we can integrate:

$$\mu = \int_0^1 nx(1 - x)^{n-1}dx = x(1 - x)^n \Big|_0^1 + \int_0^1 (1 - x)^n dx = \frac{1}{n+1}.$$

Similarly for the variance, omitting the boundary terms since as in the previous calculation they obviously go to zero:

$$\begin{aligned} \text{Var}(M) &= \int_0^1 nx^2(1-x)^{n-1}dx - \frac{1}{(n+1)^2} = \int_0^1 2x(1-x)^n dx - \frac{1}{(n+1)^2} = \int_0^1 \frac{2}{n+1}(1-x)^{n+1}dx - \frac{1}{(n+1)^2} \\ &= \frac{2}{(n+1)(n+2)} - \frac{1}{(n+1)^2}. \end{aligned}$$

Question 1d. Let $y = xn$. Then the CDF of Y is

$$F_Y(y) = F_M\left(\frac{y}{n}\right) = 1 - \left(1 - \frac{y}{n}\right)^n.$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \left(1 - \frac{y}{n}\right)^{n-1}1_{0 \leq y \leq n}.$$

Question 1e. Taking the limit and using a definition of e stated in class:

$$\lim_{n \rightarrow \infty} = \left(1 - \frac{y}{n}\right)\left(1 - \frac{y}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{y}{n}\right)e^{-y} = e^{-y}.$$

Question 2. Clearly $E[X] = 0.5$ and $E[Y] = -0.5$ since they're uniform distributions. We also have

$$E[X^2] = E[Y^2] = \int_0^1 x^2 dx = \frac{1}{3}.$$

Now computing the covariance using the definition

$$\begin{aligned} \text{cov}(X, X^2) &= E[(X - E[X])(X^2 - E[X^2])] = E[XX^2] - E[X]E[X^2] \\ &= \int_0^1 x \frac{d}{dx'} P(X^3 < x') dx - \frac{1}{6} = \int_0^1 \frac{1}{3} x^{\frac{1}{3}} dx - \frac{1}{6} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}. \end{aligned}$$

Doing the same calculation for Y :

$$\begin{aligned} \text{cov}(Y, Y^2) &= E[(Y - E[Y])(Y^2 - E[Y^2])] = E[YY^2] - E[Y]E[Y^2] \\ &= \int_{-1}^0 \frac{1}{3} yy^{\frac{2}{3}} dy - \frac{1}{6} = -\frac{1}{4} - \frac{1}{6} = -\frac{1}{12}. \end{aligned}$$

Question 3a. Note that $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Computing the Jacobian:

$$J(x, y) = \begin{vmatrix} \frac{\partial \frac{u+v}{2}}{\partial u} & \frac{\partial \frac{u+v}{2}}{\partial v} \\ \frac{\partial \frac{u-v}{2}}{\partial u} & \frac{\partial \frac{u-v}{2}}{\partial v} \end{vmatrix} = \frac{1}{4} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{2}.$$

Thus we have that

$$f_{U,V}(u, v) = f_{X,Y}\left(\frac{1}{2}(u + v), \frac{1}{2}(u - v)\right) J(x, y) = \frac{1}{2\pi} e^{-\frac{1}{4}((u+v)^2 - (u-v)^2)} = \frac{1}{4\pi} e^{-\frac{1}{4}(u^2 + v^2)}.$$

Question 3b. Consider the characteristic functions of U and V . Then we have

$$\phi_U(t) = \phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{-t^2} \implies f_U(u) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2}.$$

A similar calculation for V :

$$\phi_V(t) = \phi_{X-Y}(t) = \phi_X(t)\phi_{-Y}(t) = e^{-t^2} \implies f_V(v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}v^2}.$$

Therefore we have that

$$f_U(u)f_V(v) = \frac{1}{4\pi} e^{-\frac{1}{4}(u^2 + v^2)}.$$

Since is the same as our result from 3a, they are independent.

Question 3c. We already calculated the marginal distribution in part b using the characteristic functions, where we found it to be

$$f_U(u) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2} = N(0, 2).$$

Question 4a. Integrating to find the marginal distribution:

$$f(x) = \int_0^\infty \lambda^2 e^{-\lambda z} 1_{0 \leq x \leq z} dz = \int_x^\infty \lambda^2 e^{-\lambda z} dz = \lambda e^{-\lambda x} 1_{0 \leq x}.$$

Question 4b. Doing same as previous question:

$$f(z) = \int_0^\infty \lambda^2 e^{-\lambda z} 1_{0 \leq x \leq z} dx = \int_0^z \lambda^2 e^{-\lambda z} dx = \lambda^2 z e^{-\lambda z} 1_{0 \leq z}.$$

Question 4c. As the hint suggests, we can take the derivative of the joint cumulative distribution function of X and Y . To find what this is, we can first find the joint cumulative distribution of X, Z :

$$\begin{aligned} F_{X,Z}(x, z) &= \int_0^z \int_0^x \lambda^2 e^{-\lambda z'} 1_{0 \leq x' \leq z'} dx' dz' = \int_0^z \lambda^2 e^{-\lambda z'} \min(x, z') dz' \\ &= \int_0^x \lambda^2 z' e^{-\lambda z'} dz' + x \int_x^z \lambda^2 e^{-\lambda z'} dz' = -\lambda z' e^{-\lambda z'} \Big|_0^x + \int_0^x \lambda e^{-\lambda z'} dz' + \lambda x e^{-\lambda x} - \lambda x e^{-\lambda z}. \\ &= 1 - e^{-\lambda x} - \lambda x e^{-\lambda z}. \end{aligned}$$

We can then do the calculation to find the joint density

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{d}{dx} \frac{d}{dy} P(X \leq x, Y \leq y) = \frac{d}{dx} \frac{d}{dy} P(X \leq x, Z \leq x + y) = \frac{d}{dx} \frac{d}{dy} F_{X,Z}(x, x + y) \\ &= \frac{d}{dx} \frac{d}{dy} \left(1 - e^{-\lambda x} - \lambda e^{-\lambda(x+y)} \right) = \frac{d}{dx} \lambda^2 e^{-\lambda(x+y)} \\ &= \lambda^2 e^{-\lambda(x+y)} 1_{0 \leq x \leq z}. \end{aligned}$$

Question 4d. Finding the marginal pdf:

$$f_Y(y) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)} dx = \lambda e^{-\lambda y}.$$

Question 5a. The following python code was used to produce the output. The result was $P(A) \approx 0.0271$.

```
from math import comb
n = 100
p = 0.6
P = sum([comb(n, k) * p**k * (1-p)**(n-k) for k in range(0, 51)])
print(P)
```

Question 5b. Let X_1, X_2, \dots, X_{100} be a set of independent Bernoulli variables with parameter $p = 0.6$, so $X = \sum_{i=1}^{100} X_i$. The mean of X is clearly $np = 60$ and the variance of each X_i is $0.6 \cdot 0.4 = 0.24$, so by the central limit theorem we have

$$Z = \frac{X - 60}{\sqrt{24}} \approx N(0, 1).$$

Question 5c. Using the fact that $X = \sqrt{24}Z + 60$, we get:

$$P(X \leq 50) = P(\sqrt{24}Z + 60 \leq 50) = P(Z \leq -\frac{10}{\sqrt{24}}) \approx \Phi(-\frac{10}{\sqrt{24}}) \approx 0.0206.$$

Question 5d. Using the exact same approach as part c except with 51 instead of 50:

$$P(X < 51) = P(\sqrt{24}Z + 60 < 51) = P(Z < -\frac{9}{\sqrt{24}}) \approx \Phi(-\frac{9}{\sqrt{24}}) \approx 0.0331.$$

Question 5e. Taking the pattern from parts c and d, we get that

$$P(X < 50.5) = P(\sqrt{24}Z + 60 < 50.5) = P(Z < -\frac{9.5}{\sqrt{24}}) \approx \Phi(-\frac{9.5}{\sqrt{24}}) \approx 0.0262.$$

Question 6a. We can compute the probability distribution of each S_n by taking convolutions of the pdfs of S_{n-1} X_n . The following code does this, the resultant graphs can be seen in figure 1

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

ns = (1,2,3,4,5,10,50)

u = np.array([1/3, 1/3, 1/3])
S = [u]

for i in range(1, max(ns)+1):
    s = np.convolve(u, S[-1])
    S.append(s)

fig, ax = plt.subplots(nrows=4, ncols=2, sharey=True)
for i, n in enumerate(ns):
    a = ax[i//2, i%2]
    a.plot(S[n-1], label='$S_n$')
    a.title.set_text(f'n={n}')
    mu = n
    var = 2/3 * (n)

    x = np.arange(len(S[n-1]))
    a.plot(norm.pdf(x, mu, var**0.5), label=f'$N(\{\mu\}, \{\text{round}(var, 1)\})$')
    a.legend()

plt.show()
```

Question 6b. The resulting plots can be seen in figure 2. The code very similar to the previous part, it is as follows:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

ns = (1,2,3,4,5,10,50)

x = np.array([-1, 0, 1, 2, 3, 4])
```

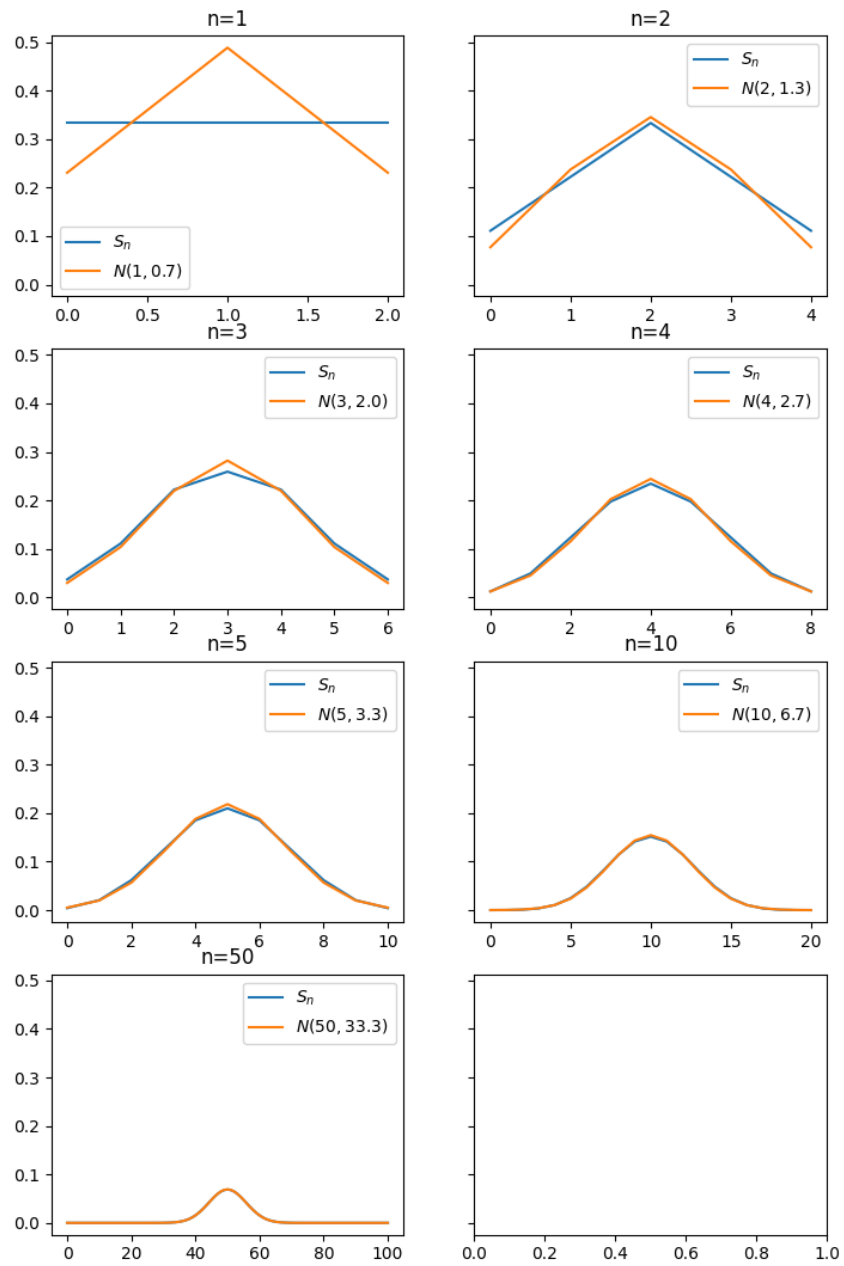


Figure 1: Graphs for question 6a

```

y = np.array([1/15, 1/15, 11/15, 1/15, 0, 1/15])
X = [x]
T = [y]

for i in range(1, max(ns)+1):
    first = X[-1][0]
    last = X[-1][-1]
    X.append(np.insert(np.append(X[-1], [last+1, last+2, last+3, last+4])\
        , 0, first-1))
    T.append(np.convolve(y, T[-1]))

fig, ax = plt.subplots(nrows=4, ncols=2, sharey=True)
for i, n in enumerate(ns):
    a = ax[i//2, i%2]
    a.plot(X[n-1], T[n-1], label='$S_n$')
    a.title.set_text(f'n={n}')
    x = X[n-1]
    t = T[n-1]
    mu = sum([x[i] * y for i, y in enumerate(t)])
    var = sum([(x[i]-mu)**2 * y for i, y in enumerate(t)])

    a.plot(X[n-1], norm.pdf(X[n-1], mu, var**0.5), label=f'$N(\{\mu\}, \{\text{round}(\text{var}, 1)\})$')
    a.legend()

plt.tight_layout()
plt.show()

```

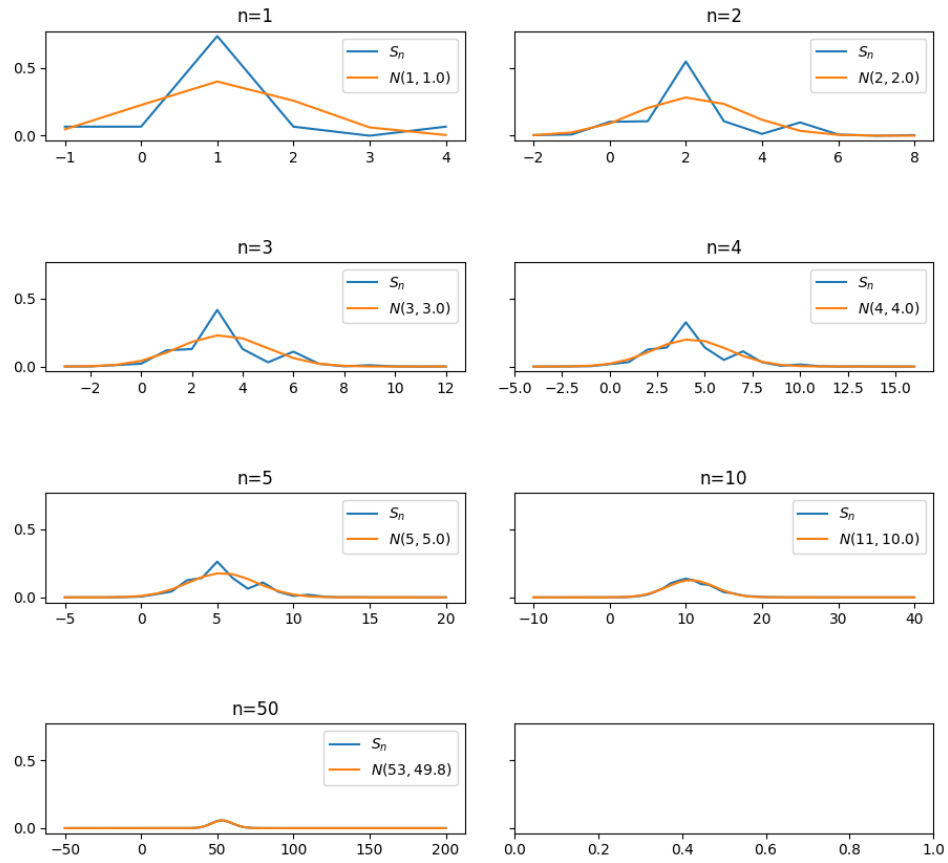


Figure 2: Graphs for 6b.