UBC Mathematics 320(101)—Assignment 3
Due by PDF upload to Canvas at 18:00, Saturday 30 Sep 2023

Readings: Loewen, lecture notes for Week 3; Rudin, pages 47–58.

1. A set $S \subseteq \mathbb{R}$ is called **dense in** $\mathbb{R}$ whenever this property holds:

$$\text{for each nonempty real interval } (a,b), \text{ one has } S \cap (a,b) \neq \emptyset.$$

(a) Define $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by $f(m,n) = m + n\sqrt{2}$. Prove that f is one-to-one.

(b) Let $S = \left\{ m + n\sqrt{2} \,:\, m,n \in \mathbb{Z} \right\}$. Prove that $S \cap (0,1)$ is infinite.

(c) Prove that for each $\varepsilon > 0$, $S \cap (0,\varepsilon) \neq \emptyset$. [Hint available. Ask on Piazza.]

(d) Prove that S is dense in $\mathbb{R}$.

2. For each $x \in \mathbb{R}$, evaluate

$$f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{1+nx}.$$

Use the ε, N definition of a limit to prove your answer.

3. Given a real sequence $(a_n)_n$ with $a_n \to A$ as $n \to \infty$, present direct ε, N-proofs that that $a_n^3 \to A^3$ and $a_n^{1/3} \to A^{1/3}$ as $n \to \infty$. (Assume $A \in \mathbb{R}$.)

4. (a) Prove: For any real M, m, and b obeying $M > m$, there is some real R for which

$$Mx > mx + b \qquad \forall x > R.$$

(This is easy, but it sets the conceptual stage for the next part.)

(b) Suppose $(y_n)_{n\in\mathbb{N}}$ is a real sequence with the property that (y_n/n) converges to some number M. Prove that for every real $m \in (-\infty, M)$ and $b \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$y_n > mn + b \qquad \forall n > N.$$

(c) True or False (with proof or counterexample):

$$\textit{If } \frac{y_n}{n} \to M \textit{ as } n \to \infty, \textit{ then } |y_n - Mn| \to 0.$$

5. Let α and β be positive real numbers. Prove that $\lim\limits_{n\to\infty} (\alpha^n + \beta^n)^{1/n} = \max\{\alpha, \beta\}$.

6. (a) Let (x_n) be a sequence of positive real numbers obeying

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1.$$

Show that there exist $r \in (0,1)$ and $C > 0$ for which $0 < x_n < Cr^n$ holds for all n sufficiently large. Use this to prove that $\lim\limits_{n\to\infty} x_n = 0$.

(b) Prove that if $x_n \to 0$, then the sequence $y_n = 1/x_n$ cannot converge.

(c) Use (a) and (b) to test for convergence: $\left(\dfrac{10^n}{n!}\right)$, $\left(\dfrac{2^n}{n}\right)$, and $\left(\dfrac{2^{3n}}{3^{2n}}\right)$.

[Detailed ε-N arguments are expected in (a)–(b), but not in (c).]

7. Let (x_n) and (y_n) be real sequences. Prove: If $(x_n y_n)$ converges, and $y_n \to +\infty$ as $n \to \infty$, then $x_n \to 0$ as $n \to \infty$.

[For "$y_n \to +\infty$," see Rudin, Definition 3.15, p. 55; note also the following paragraph.]

8. Given a real-valued sequence $a_1, a_2, \ldots$, consider the corresponding sequence of averages

$$s_n = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad n = 1, 2, 3, \ldots.$$

(a) Prove: If $a_n \to a$ as $n \to \infty$ (with $a \in \mathbb{R}$), then also $s_n \to a$ as $n \to \infty$.

(b) Proof or Counterexample: If $s_n \to a$ as $n \to \infty$ (with $a \in \mathbb{R}$), then also $a_n \to a$ as $n \to \infty$.

(c) Repeat parts (a)–(b), after changing "(with $a \in \mathbb{R}$)" to "(with $a = +\infty$)" in both parts.