UBC Mathematics 320(101)—Assignment 4
Due by PDF upload to Canvas at 18:00, Saturday 07 Oct 2023

Readings: Loewen, lecture notes on Sequences and Series (2023-09-27 or newer); Rudin, pages 11b–12a, 47–58.

1. Prove or disprove, giving plenty of detail:

(i) If (x_n) is a real sequence obeying $x_n \to +\infty$, then $x_n \leq x_{n+1}$ for all n sufficiently large.

(ii) If $(x_n)_{n=1}^{\infty}$ is a real sequence obeying $x_n \to +\infty$, then (x_n) has a subsequence $(x_{n_k})_{k=1}^{\infty}$ satisfying $x_{n_k} \leq x_{n_{k+1}}$ for all k.

2. Decide whether these sequences converge or diverge. Then present detailed ε,N proofs confirming your decisions.

(a) $a_n = n\left(\sqrt{1+\dfrac{1}{n}} - 1\right)$ (b) $b_n = \dfrac{(-1)^n\, n}{n+1}$

3. Let us extend our familiar idea of addition by defining a generalized sum, Σ, that assigns a value in $\mathbb{R} \cup \{+\infty\}$ to every subset of the real interval $[0, +\infty)$. The first step is easy: let $\Sigma(\emptyset) = 0$, and for any nonempty finite set $F = \{a_1, a_2, \ldots, a_n\}$ in $[0, +\infty)$, let

$$\Sigma(F) = a_1 + a_2 + \cdots + a_n.$$

Now suppose A is any nonempty subset of $[0, +\infty)$: define

$$\Sigma(A) = \sup\left\{\Sigma(F) \,:\, F \text{ is a finite subset of } A\right\}.$$

(This is clearly consistent with the previous setup when A is finite.) Prove:

(a) If $\Sigma(A)$ is defined and finite, then A is finite or countable.

(b) If $A = \{a_1, a_2, \ldots\}$ with all $a_n > 0$, then $\Sigma(A) = \displaystyle\lim_{N\to\infty} \sum_{n=1}^{N} a_n$. (Work in $\mathbb{R} \cup \{+\infty\}$.)

4. Nonempty sets X and Y and a function $f\colon X \times Y \to \mathbb{R}$ are given. Assume $f(X \times Y)$ is bounded. Define $M_1\colon X \to \mathbb{R}$ and $W_2\colon Y \to \mathbb{R}$ as follows:

$$M_1(x) = \sup\left\{f(x,y) \,:\, y \in Y\right\}, \qquad W_2(y) = \inf\left\{f(x,y) \,:\, x \in X\right\}.$$

(a) Prove that $\displaystyle\sup_Y W_2 \leq \inf_X M_1$. Note: This is shorthand for

$$\sup\left\{W_2(y) \,:\, y \in Y\right\} \leq \inf\left\{M_1(x) \,:\, x \in X\right\}.$$

A nice restatement of the result in original notation is worth remembering:

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) \leq \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

(b) Show by example that strict inequality is possible in (a).

5. Prove: For every nonempty set S of positive real numbers,

$$\text{either} \quad \bigcap_{s\in S}[0,s) = [0,\inf(S)) \quad \text{or} \quad \bigcap_{s\in S}[0,s) = [0,\inf(S)].$$

Include, with proof, a simple test involving the number $\inf(S)$ and the set S that predicts exactly which outcome will occur.

6. All of the sequences in this problem have rational elements. Give direct proofs of the following:

(a) If (x_n) and (y_n) are Cauchy sequences, then $s_n = x_n + y_n$ defines a Cauchy sequence.

(b) If (x_n) and (y_n) are Cauchy sequences, then $p_n = x_n y_n$ defines a Cauchy sequence.

(c) If (x_n) is a Cauchy sequence and (y_n) is a sequence satisfying $(y_n - x_n) \to 0$ as $n \to \infty$, then (y_n) is a Cauchy sequence.

(Work entirely in $\mathbb{Q}$: do not mention $\mathbb{R}$ or use any of its distinctive properties.)

7. Given some $\lambda \in (0,1)$ and $a_0, a_1 \in \mathbb{R}$, define a sequence $(a_n)_{n\geq 0}$ recursively as follows:

$$a_n = (1-\lambda)a_{n-1} + \lambda a_{n-2}, \qquad n = 2,3,4,\ldots.$$

(a) Prove that the sequence (a_n) must converge. (Try for a method that does not rely on part (b).)

(b) Express $\alpha = \lim_{n\to\infty} a_n$ in terms of λ, a_0, a_1. (Hint available, on Tue or Wed only.)

Note: This question is inspired by its special case $\lambda = \frac{1}{2}$, which appeared (with no hint) on the final exam for MATH 120 in December 2009.

8. For any nonempty set S of *positive* real numbers, define $S^{-1} = \left\{x^{-1} : x \in S\right\}$. Prove:

(a) $\inf(S) = 0 \iff \sup\left(S^{-1}\right) = +\infty$;

(b) $0 < \inf(S) < +\infty \iff 0 < \sup\left(S^{-1}\right) < +\infty$,
and when these are true one has $\sup\left(S^{-1}\right) = [\inf(S)]^{-1}$.

Taken together, items (a) and (b) provide some rationale for the symbolic equations "$1/0^+ = +\infty$" and "$1/(+\infty) = 0^+$". (These are "symbolic" because the usual rules of algebra are not available: we cannot infer a value for $(0^+)(+\infty)$!) Using these symbolic equations, prove

(c) If $x_n > 0$ for each n, then $\limsup_{n\to\infty}\left(x_n^{-1}\right) = \left(\liminf_{n\to\infty} x_n\right)^{-1}$.