Math 437 Homework 3

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Question 1. We are trying to find instances of when $a_n \equiv 0 \mod 2023$, so consider the recurrence relation $\mod 2023$. All future values of the sequence are determined by the 5-tuple $(a_n, a_{n+1}, a_{n+2}, n^2, 5^n)$, each value within being modulo 2023. There are 2023 possibilities for the first 4 and $\text{ord}_{2023} 5 = 816$ possibilities for the last, so by the pigeonhole principle the sequence must repeat after at most $2023^4 \cdot 816 \approx 1.367 \cdot 10^{16}$ steps. Since it repeats infinitely from then on out, all we must do is show that there is at least one 0 within the repeating section. I claim that $a_0 = 0$ is in the repeating section which fulfills this requirement.

To see why, let p be the period of repetition (i.e. a number such that $a_m \equiv a_{m+p}, 5^m \equiv 5^{m+p}, m^2 \equiv (m+p)^2 \mod 2023$ for all m sufficiently large, this is guaranteed to exist as shown above) and n be the smallest number such that $a_m \equiv a_{m+p}, 5^m \equiv 5^{m+p}, m^2 = (m+p)^2 \mod 2023 \forall m \geq n$. In particular, $a_n \equiv a_{n+p}, a_{n+1} \equiv a_{n+p+1}, a_{n+2} \equiv a_{n+p+2} \mod 2023$. Note that $816|p$ and $2023|p$ due to the 5^m and m^2 requirement and the fact that 2023 isn't a perfect square. Then consider a_{n-1}, using the fact that $11 \cdot 184 = 1 \mod 2023$:

$$a_{n-1} \equiv 184\left(a_{n+2} - 5^n a_{n+1} - n^2 a_n\right) \equiv 184\left(a_{n+p+2} - 5^{n+p} a_{n+p+1} - (n+p)^2 a_{n+p}\right) \mod 2023$$
$$\equiv a_{n+p-1} \mod 2023.$$

Since $5^{n-1} \equiv 1214 \cdot 5^n \equiv 1214 \cdot 5^{n+p} \equiv 5^{n+p-1} \mod 2023$ and $(n-1)^2 \equiv n^2 - 2n + 1 \equiv (n+p)^2 - 2(n+p) + 1 \equiv (n+p-1)^2 \mod 2023$, this contradicts our assumption that n was chosen to be minimal. The only way this doesn't lead to a contradiction is if $n = 0$ as $a_{n-1} = a_{-1}$ isn't defined. Thus $a_0 \equiv 0 \mod 2023$ is in the repetition and $2023|a_n$ infinitely many times. $\square$

While this concludes the proof, alternatively to carefully proving that a_0 is in the repeating section one also could have just brute force searched for a repeating 5-tuple in the form above and checked that the repeating section contains a zero. Here's some python code to do so, it turns out to repeat with period $p = 4660992$ and $n = 0$ as expected.

```python
N = 10000000
a = [0,1,2] + [0]*(N-3)

seen = {}
repeat_n = -1
repeat_h = ()

for n in range(0,N-3):
    a[n+3] = (5**(n%816)*a[n+2]+n**2*a[n+1]+11*a[n])%2023
    h = (n%816,(n**2)%2023,a[n+2],a[n+1],a[n])
    if h in seen:
        print(f'Found at n={n}')
```

```
        repeat_n = n
        repeat_h = h
        break
    seen[h] = n

if repeat_n == -1:
    print('No repeat found')
else:
    n1 = seen[repeat_h]
    n2 = repeat_n
    print(f'Found repeat at n1={n1}, n2={n2}')
    print(repeat_h)
    print('Searching for zeros...')
    for n in range(n1,n2+1):
        if a[n] == 0:
            print(f'Found zero at n={n}')
            print(f'Sanity check: n1={n1}, a[{n1}]={a[n1]}, a[{n1}+1]={a[n1+1]}, a
                ↪ [{n1}+2]={a[n1+2]}, (5^({n1}))%2023={(5**n1)%2023}, ({n1}^2)
                ↪ %2023={(5**n1)%2023}')
            print(f'Sanity check: n2={n2}, a[{n2}]={a[n2]}, a[{n2}+1]={a[n2+1]}, a
                ↪ [{n2}+2]={a[n2+2]}, (5^({n2}))%2023={(5**n2)%2023}, ({n2}^2)
                ↪ %2023={(5**n2)%2023}')
            break
```

Question 2. Factoring the equation, we get $(x-1)(x^2+x+1) \equiv 0 \mod n$. Thus $x = 1$ is always a solution