PHYS 408 Homework 5

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Question 1a. From in class, we have that the density of cavity modes is $\frac{8\pi}{\lambda^3\nu}$. For a spectral range of 0.1nm, we that the number of cavity modes is approximately

$$\#\text{ Modes} = \frac{8\pi}{\lambda^3\nu} \cdot 120 \cdot 10^9 \cdot 0.2 \cdot 0.1 = 8.04 \cdot 10^{14}.$$

Question 1b. The spectral width range goes from 499.95 to 500.05nm, so it is of width $\Delta\nu = c\left(\frac{1}{499.95\cdot 10^{-9}} - \frac{1}{500.05\cdot 10^{-9}}\right) = 120\text{GHz}$. The number of $\text{TEM}_{q,0,0}$ modes will be approximately the spectral width divided by the FSR, where $\nu_F = \frac{c}{2d} = 750\text{MHz}$, so $2\frac{\Delta\nu}{\nu} = 320$.

Question 1c. Assuming each mode is equally likely, we get that the probability is $\frac{320}{8.04\cdot 10^{14}} = 3.98 \cdot 10^{-13}$.

Question 1d. Using the cross section formula:

$$\sigma = \frac{\gamma_{sp}\lambda^2}{4\Delta\omega} = 8.3 \cdot 10^{-19}\text{m}.$$

Question 2a. From in class, we have the following relation:

$$\log\frac{I_2}{I_1} + \frac{I_2 - I_1}{I_{sat}} = g_0 l_g.$$

Since $g_0 l_g$ are the same in both cases, we can solve for I_{sat}:

$$\log 10 + \frac{9}{I_{sat}} = \log 7.5 + \frac{13}{I_{sat}} \implies I_{sat} = 13.9\text{W/cm}^2.$$

Question 2b. Using this same equation to solve for g_0:

$$g_0 = \frac{1}{l_g}\left(\log\frac{I_2}{I_1} + \frac{I_2 - I_1}{I_{sat}}\right) = 2.95\text{cm}^{-1}.$$

Question 2c. Above we saturation, we have $\frac{d}{dz}I_{out}(z) = g_0 I_{sat}$. Integrating this over the gain medium, we get

$$I_2 - I_1 = g_0 I_{sat} l_g = 41\text{W/cm}^2.$$

Question 2d. Using $I_2 - I_1 = \frac{41}{2} = 20.5\text{W/cm}^2$ and solving for I_1, we can solve the following equation numerically:

$$\log\frac{20.5 + I_1}{I_1} + \frac{20.5}{I_{sat}} = g_0 l_g \implies I_1 = 6.08\text{W/cm}^2.$$

Question 2e. Again, we derived expressions for both of these values in class:

$$T_{oc}^{(opt)} = \sqrt{g_0 l_g a} - a = 0.162.$$

$$P^{(opt)} = I_{out}^{(opt)} \cdot A = 10^{-6} \cdot I_{sat} \left(\sqrt{g_0 l_g} - \sqrt{a}\right)^2 = 0.364\text{W}.$$

Question 3a. For question 3, as question suggests we use the weak pump approximation. This means that $R_p \tau \ll 1$, and so $g(I) = \frac{g_{0,weak}}{1+I/I_{sat}}$. Note that $g \propto N_2$ and $P \propto N_2$, so $\frac{g(I)}{g_{0,weak}} = \frac{P_l}{P_0}$ (since $g_{0,weak}$ and P_0 are when no lasing occurs, while $g(I)$ and P_0 are for when it does). Plugging this into the original equation:

$$P_l = \frac{P_0}{1 + I/I_{sat}}.$$

Question 3b. We can relate I_{out} with I_2 with $I_2 = \frac{I_{out}}{T_{oc}} = 100\text{W/cm}^2$. Plugging into part a:

$$1 = \frac{2}{1 + 100/I_{sat}} \implies I_{sat} = I_{sat} = 100\text{W/cm}^2.$$

Question 4a. Rate equations:

$$\dot{N}_1 = A_{21} N_2 + R_{13} \left(N_3 - N_1\right)$$

$$\dot{N}_2 = A_{32} N_3 - A_{21} N_2$$

$$\dot{N}_3 = -A_{32} N_3 - R_{13} \left(N_3 - N_1\right).$$

Using $N = N_3 + N_2 + N_1$:

$$\dot{N}_1 = A_{21} N_2 + R_{13} \left(N - 2N_1 - N_2\right)$$

$$\dot{N}_2 = A_{32}(N - N_1 - N_2) - A_{21} N_2.$$

Substituting given constants:

$$\dot{n}_1 = n_2 + R\left(1 - 2n_1 - n_2\right)$$

$$\dot{n}_2 = aR\left(1 - n_1 - n_2\right) - n_2.$$

Question b. Setting $\dot{n}_{1,2} = 0$ and solving is a system of two linear equations with two unknowns, which gives:

$$n_1 = \frac{a+1}{aR + a + 2}, n_2 = \frac{aR}{aR + a + 2} \implies \Delta n = \frac{a(R-1) - 1}{a(R+1) + 2}.$$

Question c. It is a similar form as $\frac{R-1}{R+1}$, although with extra terms of -1 and $+2$. When a is large it reduces to the form we saw in class though. To calculate $a_{10\%}$, we can solve directly (because the new form makes the numerator smaller and the denominator bigger we know that the real value will be lower than the in class estimate):

$$\frac{a(R-1) - 1}{a(R+1) + 2} = 0.9 \frac{R-1}{R+1} \implies \frac{a-1}{3a+2} = 0.3 \implies a_{10\%} = 16.$$

Question d. See figure 1, the parameters used were $R = 2$ and $a = 16$ as calculated. As expected, the equilibrium settles to within 10% of the theoretical value. The code used for it was:

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

R = 2
a = 16

F = lambda t,n: np.array([n[1]+R*(1-2*n[0]-n[1]), a*R*(1-n[0]-n[1])-n[1]])
t_eval = np.arange(0, 4.01, 0.01)

sol = solve_ivp(F, [0,4], [1, 1], t_eval=t_eval)

plt.plot(t_eval, sol.y.T[:,0], label='$n_1$')
plt.plot(t_eval, sol.y.T[:,1], label='$n_2$')
plt.axhline(y=(R-1)/(R+1)+sol.y.T[:,0][-1], color='g', linestyle='-', label='$\\frac{R-1}{R+1}$')
plt.axhline(y=(R-1)/(R+1)*0.9+sol.y.T[:,0][-1], color='y', linestyle='-', label='$\\pm 10\%$')
plt.legend()
plt.title('Numerical Solution to Three Level Lasing System')
plt.xlabel('Time (s)')
plt.ylabel('Scaled number in level')
plt.show()
```

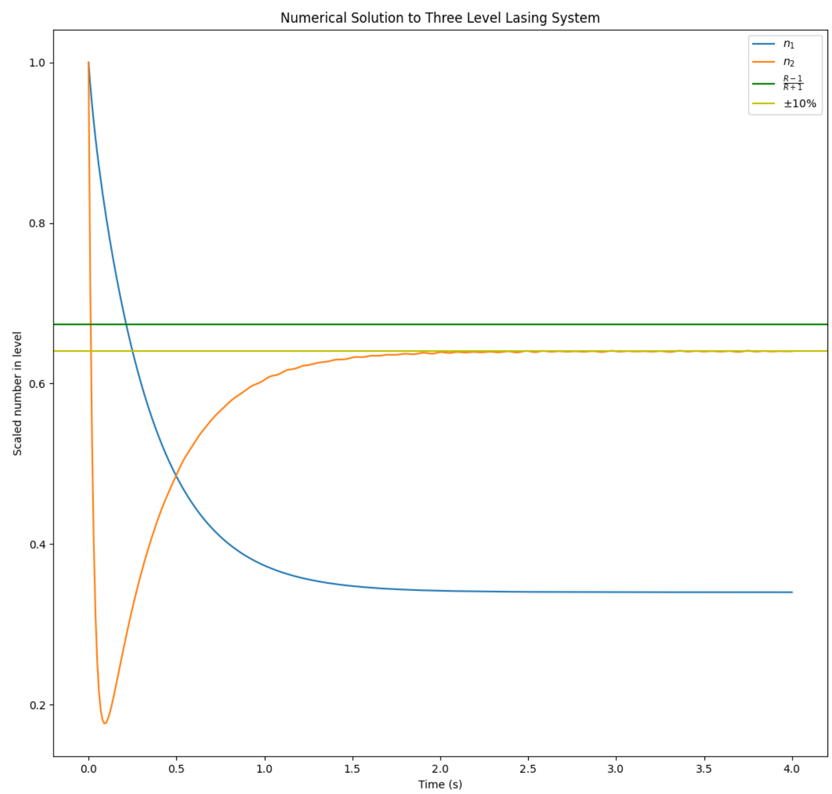

Figure 1: Numerical solution for question 4d. Arbitrary scaling for time and number, parameters used were $R = 2$ and $a = 16$.