Math 220 Assignment 7

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Question 1. Note that the power set given expands to $\mathcal{P}(\{1,2\}) = \{\{1,2\},\{1\},\{2\},\emptyset\}$. This is only 4 elements, so writing out all possible combinations that fulfill the requirements we get

$$\mathcal{R} = \{(\{1\},\emptyset),(\emptyset,\{1\}),(\{2\},\emptyset),(\emptyset,\{2\}),(\{1,2\},\emptyset),(\emptyset,\{1,2\}),(\emptyset,\emptyset),(\{1\},\{2\}),(\{2\},\{1\})\}$$

Question 2-1. The statement is false. Since R is reflexive, $\forall a \in A, (a,a) \in R$. By the definition of set subtraction this means that $(a,a) \notin A \times A - R = \bar{R}$. However being a reflexive relation requires $(a,a) \in R \forall A$, so $\bar{R}$ is not reflexive. $\square$

Question 2-2. The statement is true. We will use proof by contradiction, so suppose not. Then $\exists a, b \in A$ s.t. $(a,b) \in \bar{R}$ but $(b,a) \notin \bar{R}$. By the definition of $\bar{R}$ though that means that $(a,b) \notin R$ and $(b,a) \in R$, which can't be the case due to the assumption that R is symmetric. Therefore by contradiction $\bar{R}$ must be symmetric. $\square$

Question 2-3. The statement is false. Choose $A = \{1,2,3,4\}$ with $R = \{(1,2),(2,3),(1,3)\}$. Then R is transitive by simple inspection ($1R2$ and $2R3 \Rightarrow 1R3$) However $\bar{R} = A \times A - R$ is not transitive since $1\bar{R}4$ and $4\bar{R}3$ but $(1,3) \notin \bar{R}$. $\square$

Question 3. First we will prove symmetric. Let $a, b \in A$ with aRb. Next let $c = a$. Then from the given fact about R we have that $(aRa \wedge bRa) = bRa \Rightarrow aRb$, which is the definition of symmetric.

For transitive, we can use the fact that we just proved that R is reflexive. Let $a, b, c \in A$ and then we have $(aRc \wedge bRc) = (aRc \wedge cRb) = aRb$, which is the definition of transitive so we're done. $\square$

Question 4a. This statement is false. Let $f = x^2$, and we have $f : \mathbb{R} \to \mathbb{R}$. Then let $b = -1 \in B = \mathbb{R}$, but since $f(a) = a^2 \neq -1 \forall a \in A$ then $(-1,-1) \notin \mathcal{R}'$, which means that $\mathcal{R}'$ is not reflexive.

We are using the fact that here B is the codomain of f not the image of f. If f was surjective it then the original statement would be true, but since it is possible that f is not surjective $\mathcal{R}$ is not necessarily reflexive. $\square$

Question 4b. The statement is true. Let $(a,b) \in \mathcal{R}'$. Then $\exists x, y \in \mathbb{R}$ s.t. $(x,y) \in \mathcal{R}$ and $f(x) = a, f(y) = b$. By assumption of symmetry we also have $(y,x) \in \mathcal{R}$, which implies that $(f(y), f(x)) = (b,a) \in \mathcal{R}'$ which is the definition of symmetry. $\square$

Question 5a. The statement is true. By assumption $x_1 \mathcal{R} y_1 \implies \exists n_1 \in \mathbb{N}$ s.t. $y_1 = x_1 + n_1$ and $x_2 \mathcal{R} y_2 \implies \exists n_2 \in \mathbb{N}$ s.t. $y_2 = x_2 + n_2$. It follows that $y_1 + y_2 = x_1 + x_2 + (n_1 + n_2)$. Let $n_3 = n_1 + n_2$. Then $y_1 + y_2 = (x_1 + x_2) + n_3 \implies (x_1 + x_2)\mathcal{R}(y_1 + y_2)$ as required. $\square$

Question 5b. The statement is not true. Let $x_1 = 0$, $y_1 = 1$, $x_2 = \frac{1}{2}$ and $y_2 = \frac{1}{2}$. Then $x_1 \mathcal{R} y_1$ and $x_2 \mathcal{R} y_2$ as required, but $(x_1 \cdot y_1, x_2 \cdot y_2) = (0, \frac{1}{2}) \notin \mathcal{R}$. $\square$

Question 6. For reflexive, note that aTa since $\frac{a}{a} = 1 \in \mathbb{Q} \forall a \in \mathbb{R} - \{0\}$. For symmetric, assume aTb. Then $\frac{a}{b} \in \mathbb{Q}$, so $\frac{b}{a} = (\frac{a}{b})^{-1}$ is also a rational and $(b,a) \in T$ as well. Finally for transitive assume that aTb and bTc. Then $\frac{a}{b} \in \mathbb{Q}$ and $\frac{b}{c} \in \mathbb{Q}$. This means that $\frac{a}{c} = \frac{a}{b}\frac{b}{c}$ must also be rational since it is just two rationals multiplied together, which fulfills the definition of transitive and we're done. $\square$

Question 7a.

$$\mathcal{R} = \{(0,0),(0,3),(3,0),(1,2),(2,1),(3,3)\}$$

Question 7b. The relation is not reflexive, since for example $(1,1) \notin \mathcal{R}$ since $3 \not| 1+1$. $\square$

Question 7c. The relation is symmetric. If $(a,b) \in \mathcal{R}$, then $3|(a+b) \implies 3|(b+a) \implies (b,a) \in \mathcal{R}$. $\square$

Question 7d. The only elements of $\mathcal{R}$ that make it non-transitive is the fact that $1\mathcal{R}2 \wedge 2\mathcal{R}1$ but $1 \not\mathcal{R} 1$ and $2\mathcal{R}1 \wedge 1\mathcal{R}2$ but $2 \not\mathcal{R} 2$. Thus to make it transitive all we have to do is add $(1,1)$ and $(2,2)$ to $\mathcal{R}$. $\square$