Math 320 Homework 7

Xander Naumenko

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Question 1. Consider the following sum of series:

$$\sum_{n=1}^{\infty}\frac{1}{n^4}+2^{-4}\sum_{n=1}^{\infty}\frac{1}{n^4}=1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots-\frac{1}{2^4}-\frac{1}{(2\cdot 2)^4}-\frac{1}{(2\cdot 3)^4}=1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots.$$

However using the given information we know how to evaluate the sums on the left and the expression on the right is what we're after, so we have that

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{90}-\frac{\pi^4}{16\cdot 90}=\frac{\pi^4}{96}.$$

Question 2a. The series diverges for all $p \in \mathbb{R}$. We know that the logarithm grows slower than any positive power, in specific

$$\lim_{n\to\infty} \frac{\log n}{n^{\frac{1}{p}}} = 0$$

(it was specifically stated on Piazza 282 that we could use this fact). Thus applying the comparison test, for N sufficiently large:

$$\sum_{n=N}^{\infty} \frac{1}{(\log n)^p} > \sum_{n=N}^{\infty} \frac{1}{(n^{1/p})^p} = \sum_{n=N}^{\infty} \frac{1}{n} = \infty.$$

Question 2b. Applying the root test:

$$(a_n)^{\frac{1}{n}} = \left((\log n)^{-n}\right)^{\frac{1}{n}} = \frac{1}{\log n} \to 0.$$

Thus the series converges.

Question 2c. For $p > 1$ clearly the sum converges since $\zeta(p)$ converges. For $p = 1$, apply Cauchy's condensation test. Then we have:

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^k \log 2^k} = \sum_{k=0}^{\infty} \frac{1}{k \log 2} = \infty.$$

Finally for $p < 1$ each term is strictly greater than when $p = 1$ which diverges, so for $p \leq 1$ the sums diverge.

Question 2d. Applying Cauchy's condensation test:

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^k (\log 2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{k^p (\log 2)^p}.$$

Thus since $\zeta(p)$ converges if and only if $p > 1$, the given sum also converges if and only if $p > 1$.

Question 3a. Since $x \in \ell^2, y \in \ell^2$, there exist some upper bound X and Y for $\sum_{n=1}^{\infty} |x_n|^2$ and $\sum_{n=1}^{\infty} |y_n|^2$ respectively. Then we have that

$$0 \leq \sum_{n=1}^{\infty} |x_n y_n| \leq \sum_{n=1}^{\infty} \max(|x_n|^2, |y_n|^2) \leq \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2 \leq X + Y.$$

Thus $\langle x, y \rangle$ converges, since it converges absolutely.

Question 3b. Let $N \in \mathbb{N}$ and consider the partial sums:

$$\sqrt{x_1^2 + \ldots + x_N^2}\sqrt{y_1^2 + \ldots + y_N^2} \geq (|x_1| + \ldots + |x_N|)(|y_1| + \ldots + |y_N|)$$

$$\geq x_1 y_1 + x_2 y_2 + \ldots + x_N y_N = \sum_{n=1}^{N} x_n y_n.$$

Since the inequality is true for all the partial sums, it must also be true in the limit (this isn't necessarily true of strict inequalities, but here we only need $\leq$), so $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Question 3c. Again for $N \in \mathbb{N}$ and considering the partial sums:

$$\sum_{n=1}^{N} |x_n + y_n|^2 \leq \sum_{n=1}^{N} \left(|x_n|^2 + |y_n|^2\right) = \sum_{n=1}^{N} |x_n|^2 + \sum_{n=1}^{N} |y_n|^2.$$

Again, since the inequality holds for every partial sum, it holds in the limit so $\|x + y\| \leq \|x\| + \|y\|$.

Question 3d. Define $x^{(n)} = e_n$. Then we have $\|x^{(n)}\| = \sum_{i=1}^{\infty} x_i^{(n)} = 0 + \ldots + 0 + 1 \cdot 1 + 0 + \ldots = 1$, but for any $p \in \mathbb{N}$, we have that for $n > p$,

$$\langle e_p, x^{(n)} \rangle = 0 + 0 + \ldots + \underbrace{1 \cdot 0}_{p\text{th component}} + 0 + \ldots + \underbrace{0 \cdot 1}_{n\text{th component}} + 0 + \ldots = 0.$$

Thus $\langle e_p, x^{(n)} \rangle \to 0$ as required.

Question 4. Assume that $(s_n + 2s_{n+1}) \to L$. Changing the sequence to $s'_n = s_n - \frac{L}{3}$ doesn't change the convergence of either s'_n or $2s'_{n+1} + s'_n$ but does cause $(2s'_{n+1} + s'_n) \to 0$, so to make the algebra later a bit simpler from now on assume that $L = 0$. Let $x_n = 2s_{n+1} + s_n$, then we have $s_n = \frac{1}{2}x_{n-1} + \frac{1}{2}s_{n-1}$ and $x_n \to 0$. Expanding out this recursive expression for s_n as far as possible, we get:

$$s_n = \frac{1}{2}x_{n-1} - \frac{1}{4}x_{n-2} + \frac{1}{8}x_{n-3} - \ldots - \frac{1}{2^{n-2}}x_2 + \frac{1}{2^{n-1}}x_1 - \frac{1}{2^{n-1}}s_1 = \sum_{i=1}^{n} \frac{1}{2^i}x_{n-i} + \frac{(-1)^{n-1}}{2^{n-1}}s_1.$$

Let $\epsilon > 0$. Since $x_n \to 0$, choose N s.t. $n > N \implies |x_n| < \epsilon$. Splitting the expression above for s_n:

$$s_n < \sum_{i=1}^{n-N-1} \frac{\epsilon}{2^i} + \sum_{i=N}^{n} \frac{1}{2^i}x_{n-i} + \frac{(-1)^{n-1}}{2^{n-1}}s_1.$$

Now take the $\limsup_{n\to\infty}$ on both sides of that inequality:

$$\limsup_{n\to\infty} s_n \le \epsilon \sum_{i=i}^{\infty} \frac{1}{2^i} + 0 + 0 = \epsilon \left(\frac{1}{1 - \frac{1}{4}} - 1 \right) = \frac{\epsilon}{3} < \epsilon.$$

Note that the $\frac{1}{3}$ means that the limit of s_n is a third of that of x_n without the assumption that $L = 0$, although since the question only asks for convergence I won't prove that rigorously. Similarly repeating this whole process for $\liminf_{n\to\infty}$ just involves replacing $\epsilon \to -\epsilon$ and so gives $\liminf_{n\to\infty} s_n > -\epsilon$. Since this is true of all $\epsilon > 0$, it must be that $\lim_{n\to\infty} s_n = 0$, and in particular it converges.

Question 5. This is a straightforward application of question 3a. By hypothesis $a \in l^2$, and as proved in class, $\zeta(2p) = \sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges, so $\frac{1}{n^q} \in l^2$. Thus by question 3a we have that

$$\langle a, \frac{1}{n^q} \rangle = \sum_{n=1}^{\infty} \frac{a_n}{n^q}$$

converges.

Question 6a. Since $\frac{b_n}{a_n}$ converges, we can find N s.t. $n > N \implies |\frac{b_n}{a_n} - L| < \frac{L}{2}$.Using this, we get the following two inequalities:

$$\sum_{n=N+1}^{\infty} a_n > \frac{L}{2} \sum_{n=N+1}^{\infty} b_n \text{ and } \sum_{n=N+1}^{\infty} a_n < \frac{3L}{2} \sum_{n=N+1}^{\infty} b_n.$$

Since we assume $a_n > 0, b_n > 0$, the sums are increasing sequences. Thus the first inequality proves the forward direction by showing the bound on $\sum_{n=1}^{\infty} b_n < \sum_{n=1}^{N} b_n + \frac{2}{L} \sum_{n=N+1}^{\infty} a_n$, and the second inequality proves the backwards direction by putting a bound on $\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \frac{3L}{2} \sum_{n=N+1}^{\infty} b_n$.

Question 6b. Rearranging the given inequality, we have that $\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n}$, i.e. $\frac{a_n}{b_n}$ is a decreasing sequence. Then we have that

$$A = \sum_{n=1}^{\infty} \frac{a_n}{b_n} b_n \leq \frac{a_1}{b_1} \sum_{n=1}^{\infty} b_n = \frac{a_1}{b_1} B < \infty.$$

For the second inequality given, again rearranging it we get $\frac{a_{n+1}}{b_n} \leq \frac{a_n}{b_{n-1}}$, again implying that the sequence of ratios is a decreasing sequence. Then we get:

$$A = a_1 + \sum_{n=2}^{\infty} \frac{a_n}{b_{n-1}} b_{n-1} \leq a_1 + \frac{a_2}{b_1} \sum_{n=1}^{\infty} b_n < \infty.$$

Question 6c. As the clue suggests, consider $1 - px$ and $(1-x)^p$ for $0 \leq x \leq 1$. At $x = 0$ we have $1 - px = 1 = (1-x)^p$. Also $(1-px)' = -p < -p(1-x)^{p-1} = ((1-x)^p)'$ for all $x \in (0,1)$, so since they start at the same point but $1 - px$ decreases faster over the whole interval, $1 - px < (1-x)^p \forall x \in [0,1]$. Now use this to extend the given inequality for n sufficiently large:

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n} < \left(1 - \frac{1}{n}\right)^p = \frac{\frac{1}{n^p}}{\frac{1}{(n-1)^p}}.$$

Thus applying part b with $B = \zeta(p)$, we conclude that a_n converges.

Question 6d. Applying Raabe's test:

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{3n(n+1)^2}{(3n+4)\,n^2} - 1\right) = \frac{3n(n+1)^2 - (3n+4)n^2}{(3n+4)n}$$

$$= \frac{2n^2 + 3n}{(3n+4)n} \to \frac{2}{3} \implies \sum_{n=1}^{\infty} a_n = \infty.$$

Question 7. I will prove the contrapositive, so suppose that $S = \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = L$. Since S converges, there must exist an upper bound a s.t. $a_n \leq a \forall n \in \mathbb{N}$. Then we get

$$\sum_{n=1}^{\infty} a_n = (1+a) \sum_{n=1}^{\infty} \frac{a_n}{1+a} \leq (1+a) \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = (1+a)L < \infty.$$

Thus the sum $\sum_{n=1}^{\infty} a_n$ converges. Since the contrapositive holds, the original statement is true. The converse also holds by the comparison test. Now assume that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges, then

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = \infty.$$

Question 8a. Let $N = \frac{2D}{\delta}$, note that this means that $\frac{2D}{\delta} < N+1 \implies \delta > \frac{2D}{N+1}$. For $1 \leq n \leq N$, define

$$a_n = \frac{2nD}{N(N+1)}.$$

Summing these, we get

$$a_1 + a_2 + \ldots + a_N = \frac{2D}{N(N+1)} \sum_{n=1}^{N} n = D.$$

Also since $\delta > \frac{2D}{N+1} = a_1$ and $a_1 > 0$, the inequality requirement is also satisfied.

Question 8b. Begin with $\delta = 1$ and $D = \sigma_1$. Using the method described in part a, find a finite sequence $a_1, a_2, \ldots, a_N$ with $\delta > |a_1| > \ldots > |a_n| > 0$ with $D = a_1 + a_2 + \ldots + a_N$ and assign $x_1 = a_1, \ldots, x_N = a_N$. Note that this means that $\sum_{n=1}^{N} x_n = \sigma_1$. Now set $\delta' = x_{\frac{N}{2}}$ and $D' = \sigma_2 - \sum_{n=1}^{N} x_n = \sigma_2 - \sigma_1$. We can repeat the process described to get a sequence of $a_1', \ldots, a_{N'}'$ with the $a_1' + \ldots + a_{N'}' = D'$, assign them to $x_N + 1 = a_1', \ldots, x_{N+N'} + a_{N'}'$. By construction $\sum_{n=1}^{N+N'} x_n = \sigma_2$. We can repeat this process infinitely to define the whole sequence (x_n).

Clearly by the way we defined it s_N has $(\sigma_n)_n$ as a subsequence and $|x_{n+1}| < |x_n|$. Also since each time we redefine δ we divide the previous value by 2, $x_n \to 0$.