Math 443 Homework 6

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Question 1.

Question 2.

Question 3. Consider a new graph G' that is a copy of G with an additional vertex v that is adjacent to each of the v_i. I claim that G' is k-connected. Let $x, y \in V(G')$. If x and y are both in G then a vertex set of size less than k clearly can't separate them due to the k-connectivity of G, so assume $y = v$. Let $S \subset V(G)$ be a set of vertices on $G' - x - v$ with $|S| < k$. G is k-connected so there still exists a path from x to v_i in $G - S$ for all v_i that are in $G - S$. $|S| < k$ so at least on such v_i is still in $G' - S$, so there exists a path from x to v_i and an edge from v_i to v, so v and x are connected in $G' - S$. Thus G' is k-connected.

Now apply Menger's Theorem to u and v on G'. G' is k-connected as justified above, so $\lambda(u,v) \geq k \implies \kappa(u,v) \geq k$ (although by our construction we know equality holds, it's not important). Let $P_1', \dots, P_k'$ be k disjoint paths from u to v which are guaranteed to exist by our bound of κ. The only neighbors of v for the k paths to go through are each of the v_i of which there are k, so each path must go through exactly one. Letting $P_i = P_i' - v$, these now fill the requirements of the P_i asked for in the question and we're done. $\square$

Question 4. We will establish bounds on both sides of κ and show that they are equal. Let $u \in V(G_r)$. Note that

Question 5. Both directions will be proven separately.

($\Rightarrow$) Assume G is k-edge-connected, and let $u, v \in V(G)$. By hypothesis a minimum uv separating edge set is of size at least k, so the maximum number of pairwise edge-disjoint uv paths is at least k by Theorem 5.21 which is what we needed to prove.

($\Leftarrow$) Assume that G contains k pairwise edge-disjoint uv-paths for each $u, v \in V(G)$. Let $u, v \in V(G)$. The maximum number of pairwise edge-disjoint uv paths in G is always at least k, so the maximum number of such paths is greater than or equal to k. Thus by Theorem 5.21, a minimum uv separating edge set is of size at least k. Since this is true of each $u, v \in V(G)$, G is k-edge-connected. $\square$

Question 6. As the hint suggests we will use strong induction on k.
Base case (k=2): Let G be 2-connected and let $e_1, e_2 \in E(G)$. G contains no cut vertices so by definition it's a block. From homework 5, question 4, all edges in a block share a cycle. Thus e_1 and e_2 share a cycle in G as required.
Inductive step: Let G be a $(k+1)$-connected graph, and assume the theorem holds for all graphs of connectivity k or less. Let $e_1, e_2 \in E(G)$ and $v_1, \dots v_{k-1} \in V(G)$. By the inductive hypothesis there exists a cycle C containing e_1, e_2 and $v_i \forall i \in [k_2]$. Let $k' = \min(|C|, k+1)$. Since there are $k-2$ v_is, $k' >= k-2$. Let $V \subset V(C)$ be a set of k' vertices in C. By question 3 of this homework, there exist k' disjoint paths from v_{k-1} to C with separate endpoints (if $k' < k+1$ we can just

choose arbitrary vertices outside of C and ignore that paths resulting to them). For each of these paths, consider the shortened version, starting from its first intersection of C, to v_{k-1}, call these paths $P_i, i \in [k']$. Since the paths are disjoint their endpoints in C are also still distinct.

There are at most $k' - 1$ gaps between $e_1, e_2, v_1, \ldots, v_{k-2}$

Question 7. $\square$ (pretty concise proof, huh)

Question 8. Let c be a circuit in a graph G with ordered vertices $v_1, v_2, \ldots, v_n$ (potentially some repeats). Let v_j be the first repeated vertex (i.e. minimal j), and let i be the index of the vertex that v_j first occurred in. Then $v_i, v_{i+1}, \ldots, v_j$ is a closed walk with no repeated vertices by the minimally of j which is exactly the definition of a cycle so we're done. $\square$