PHYS 304 Homework 8

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Question 1a. Let $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$. Then we have:

$$[r_i, r_j] = r_i r_j f - r_j r_i f = 0.$$

$$[p_i, p_j] = p_i p_j f - p_j p_i f = 0.$$

$$[r_i, p_j] = -[p_i, r_j] = -i\hbar r_i \frac{d}{dr_j} f + i\hbar \frac{d}{dr_j}(r_i f) = i\hbar \delta_{ij}.$$

Question 1b. The derivation for the generalized Ehrenfest theorem is the exact same as for three dimensions, so we can use it in the three dimensional case. Then we have that

$$[\hat{H}, x] = \frac{1}{2m}[p_x^2, x] = -i\frac{\hbar}{m} p_x.$$

Then by the generalized Ehrenfest theorem and applying it in all three Cartesian coordinates:

$$\frac{d}{dt}\langle \vec{r} \rangle = \frac{1}{m}\langle \vec{p} \rangle.$$

Next for momentum:

$$[\hat{H}, p] = \frac{1}{2m}[p^2 + V(x), p] = \frac{1}{2m}[V(x), p_x] = i\hbar \frac{\partial V}{\partial x}.$$

$$\implies \frac{d}{dt}\langle \vec{p} \rangle = \langle -\nabla V \rangle.$$

Question 1c. Using the result of part a we know that $[r_i, p_j] = i\hbar\delta_{ij}$. Thus using the generalized uncertainty principle:

$$\sigma_{r_i}\sigma_{p_j} \geq i\hbar\delta_{ij}.$$

Question 2. Assume that we can write the solution to $\psi(x, y, z) = X(x)Y(y)Z(z)$. Then the time independent schrödinger equation for inside the box is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(XYZ) = EXYZ.$$

$$\implies \hat{H}X = E_x X, \text{ etc..}$$

This is exactly the same as the one dimensional case, which we solved to get:

$$X(x) = \sqrt{\frac{2}{a}}\sin(kx), k = \frac{\sqrt{2mE_x}}{\hbar}.$$

The boundary conditions imposed require that the energy levels are discrete:

$$E_{xn} = \frac{\pi^2\hbar^2}{2ma^2}n^2.$$

Once all the equations are put together the final solution is:

$$\psi(x,y,z) = \left(\frac{2}{a}\right)^{\frac{3}{2}} \sin(\frac{n_x\pi}{a}x)\sin(\frac{n_y\pi}{a}y)\sin(\frac{n_z\pi}{a}z)$$

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}\left(n_x^2+n_y^2+n_z^2\right).$$

Question 2b. This is a combinatorics problem, since there are three possible variables providing the degeneracy.

n	E_n	Degeneracy
1	$3\frac{\pi^2\hbar^2}{2ma^2}$	1
2	$6\frac{\pi^2\hbar^2}{2ma^2}$	3
3	$9\frac{\pi^2\hbar^2}{2ma^2}$	3
4	$11\frac{\pi^2\hbar^2}{2ma^2}$	3
5	$12\frac{\pi^2\hbar^2}{2ma^2}$	1
6	$14\frac{\pi^2\hbar^2}{2ma^2}$	6

Question 2c. E_{14} has both the $n_x = n_y = n_z = 3$ state and the $n_x = n_y = 1, n_z = 5$ states, which gives a total degeneracy of 4.

Question 3a. Time independent radial equation:

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right)\psi = 0.$$

$$\frac{d}{dr}\left(\frac{-r^2}{a}Ae^{-r/a}\right) - \frac{2mr^2}{\hbar^2}(V(r)-E)Ae^{-r/a} = 0.$$

$$\implies \frac{r^2}{a^2} - \frac{2r}{a} = \frac{2mr^2}{\hbar^2}(V(r)-E).$$

$$\implies V(r) = -\frac{\hbar^2}{mra}.$$

$$\implies E = -\frac{\hbar^2}{2ma^2}.$$

Question 3b. Using the exact same approach:

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right)\psi = 0.$$

$$\frac{d}{dr}\left(\frac{-2r^3}{a^2}Ae^{-r^2/a^2}\right) - \frac{2mr^2}{\hbar^2}(V(r)-E)Ae^{-r^2/a^2} = 0.$$

$$\implies \frac{4r^4}{a^4} - \frac{6r^2}{a^2} = \frac{2mr^2}{\hbar^2}(V(r)-E).$$

$$\implies V(r) = \frac{2\hbar^2 r^2}{mr^2 a^4}.$$

$$\implies E = \frac{3\hbar^2}{ma^2}.$$

Question 4a. One situation that resulted in degenerate eigenstates was the free particle. The energy is the same for both the left and right travelling free particles.

Question 4b. Because the potential is completely symmetric and the energy eigenstates is a identical in the different directions, switching a direction under consideration yields an identical energy which is the same as degeneracy.

Question 4c. The symmetry you could consider is the symmetry of the real axis, and how the schrödinger equation is symmetric under the transformation $x' = -x$. Thus left and right travelling waves are effectively identical from the perspective of their energies.