UBC Mathematics 320(101)—Assignment 8
Due by PDF upload to Canvas at 18:00, Saturday 04 Nov 2023

References: Loewen, lecture notes on Series (2023-10-27 or newer—see Canvas); Rudin pp. 58b–78a [but skip items 3.48–3.51 and 3.54]; Thomson-Bruckner-Bruckner, Sections 3.1–3.6 [but skip 3.3]

Presentation: To qualify for full credit, submissions must satisfy the detailed specifications provided on Canvas.

1. Prove: If $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges. Is this statement still true if the word "absolutely" is removed?

2. For each series below, find the set of $x \in \mathbb{R}$ where the series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} c^{n^2}(x-1)^n \quad (c > 0 \text{ const.})$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\left[\frac{x+1}{2x+1}\right]^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{(2n)!}{n(n!)^2}\right](x-e)^n$

3. Discuss the series whose nth terms are shown below:

$$a_n = (-1)^n \frac{n^n}{(n+1)^{n+1}}, \qquad b_n = \frac{n^n}{(n+1)^{n+1}},$$

$$c_n = (-1)^n \frac{(n+1)^n}{n^n}, \qquad d_n = \frac{(n+1)^n}{n^{n+1}}.$$

4. Suppose $x_1 \geq x_2 \geq x_3 \geq \cdots$ and $\lim_{n\to\infty} x_n = 0$. Show that the following series converges:

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \pm \cdots.$$

5. (a) Prove: if $a_n \geq a_{n+1} \geq 0$ for all n, and $\sum a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

(b) Prove: If $\sum \left(b_n^2/n\right)$ converges, then $\displaystyle\frac{1}{N}\sum_{j=1}^{N} b_j \to 0$ as $N \to \infty$.

[Hint: In part (a), it's enough to prove that $\frac{1}{2} n a_n \to 0$.]

6. Define $f(\theta) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin((2k-1)\theta)$. Determine the domain of f, namely, the set of all real θ where the series converges, by completing the steps below.

(a) Obtain the following identities, valid for each $n \in \mathbb{N}$ at all points where $\sin\theta \neq 0$:

$$C_n(\theta) = \cos(\theta) + \cos(3\theta) + \cos(5\theta) + \ldots + \cos((2n-1)\theta) = \frac{\sin(2n\theta)}{2\sin\theta},$$

$$S_n(\theta) = \sin(\theta) + \sin(3\theta) + \sin(5\theta) + \ldots + \sin((2n-1)\theta) = \frac{1-\cos(2n\theta)}{2\sin\theta}.$$

[Suggestion: Use geometric sums of complex numbers, with $e^{it} = \cos(t) + i\sin(t)$.]

(b) Prove that the domain of f is the interval $(-\infty, +\infty)$.

(c) Find a sequence (θ_n) such that $\theta_n \to 0$ and $S_n(\theta_n) \to +\infty$ as $n \to \infty$. Explain why your solution in part (b) is correct in spite of the evident unboundedness of the sequence $(S_n(\theta_n))$.