V. Series in $\mathbb{R}$

UBC M320 Lecture Notes by Philip D. Loewen

A. Basic Definitions

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Given a sequence $a_1, a_2, \ldots$ in $\mathbb{R}$, the corresponding **series** is another sequence $s_1, s_2, \ldots$, defined by building **partial sums**:

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N \qquad \forall N \in \mathbb{N}.$$

We are interested in the convergence properties of this new sequence, i.e., in

$$S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n. \qquad \text{Notation: } S = \sum_{n=1}^{\infty} a_n.$$

The series *converges* when S has a value in $\mathbb{R}$, and *diverges* otherwise. For some divergent series the extended values $-\infty$ and $+\infty$ may be appropriate.

Example (Geometric Series). *For a fixed real number r, consider $S = \sum_{n=0}^{\infty} r^n$. (Use $r^0 = 1$ for all r.)*

(a) *If $|r| < 1$, S converges: $S = \dfrac{1}{1-r}$.*

(b) *If $|r| \geq 1$, S diverges.*

Proof. For any N,

$$s_N = 1 + r + r^2 + \ldots + r^N.$$

If $r \geq 1$, $s_N \geq N+1$ diverges to $+\infty$ as $N \to \infty$. ("$S = +\infty$.")
If $r \neq 1$, then

$$s_N = \frac{1 - r^{N+1}}{1-r} \qquad \forall N \geq 0.$$

If $|r| < 1$, then sending $N \to \infty$ gives

$$S = \lim_{N\to\infty} s_N = \frac{1}{1-r}.$$

If $r \leq -1$, the sequence (s_N) diverges by the Crude Divergence Test below. ////

Remark. In different words, consider $f(x) = \sum_{n=0}^{\infty} x^n$. Then the domain of f is $(-1, 1)$, and $f(x) = \dfrac{1}{1-x}$ on that set. Geometric series foreshadow general *power series.*

Example (Telescoping Series). *For any $N \geq 2$,*

$$\sum_{n=1}^{N} \frac{2}{4n^2-1} = 1 - \frac{1}{2N-1}.$$

Consequently

$$\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \lim_{N\to\infty} \left(1 - \frac{1}{2N-1}\right) = 1.$$

Proof. This starts with the partial-fractions style identity

$$\frac{1}{2n-1} - \frac{1}{2n+1} = \frac{(2n+1)-(2n-1)}{(2n)^2-(1)^2} = \frac{2}{4n^2-1}.$$

The similarity in successive terms is key to massive cancellation ("telescoping"):

$$\begin{aligned}\sum_{n=1}^{N} \frac{2}{4n^2-1} &= \sum_{n=1}^{N} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2N-1} - \frac{1}{2N+1}\right) \\ &= \frac{1}{1} - \frac{1}{2N+1}.\end{aligned}$$

Our main question: does S converge? (Other courses [Applied math/numerical analysis] deal with, "Calculate the limit".) To decide convergence, the first few million terms are irrelevant, so we sometimes adopt the lazy notation $S = \sum_n a_n$. All convergence tests adapt accordingly.

We know two ways of proving a limit exists without knowing its value in advance.

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Theorem (Monotone Convergence Criterion). *If $a_n \geq 0$ for all n, then the series $S = \sum_n a_n$ converges if and only if the sequence of partial sums is bounded.*

Proof. Since $s_{n+1} - s_n = a_n \geq 0$ for all n, the partial sums form a nondecreasing sequence. We have dealt with these earlier. ////

Theorem (Cauchy's Convergence Criterion). *The series $S = \sum_n a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\forall m \geq N,\ \forall p \geq 0, \quad |a_m + a_{m+1} + \cdots + a_{m+p}| < \varepsilon.$$

Proof. Apply Cauchy's criterion to the sequence of partial sums: note that

$$\begin{aligned}|s_{m+p} - s_{m-1}| &= |(a_1 + \cdots + a_{m+1} + a_m + \cdots + a_{m+p}) - (a_1 + \cdots + a_{m-1})| \\ &= |a_m + a_{m+1} + \cdots + a_{m+p}|.\end{aligned}$$

Cauchy's criterion says the sequence (s_n) converges if and only if this quantity can be made small (regardless of p) by choosing m sufficiently large. ////

Thm (Crude Divergence Test). *If "$\lim_n a_n = 0$" is false, then $\sum_n a_n$ diverges.*

Proof. [Contraposition] Suppose $\sum_n a_n$ converges. Given any $\varepsilon > 0$, Cauchy (above) supplies $N \in \mathbb{N}$ such that (taking $p = 0$)

$$\text{each } m > N \text{ obeys } |a_m| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows $\lim_n a_n = 0$. ////

Theorem (Comparison Test). *Suppose $0 \le |a_n| \le b_n$ for all n. Then ...*

(a) *If $\sum_n b_n$ converges, then $\sum_n a_n$ converges.*

(b) *If $\sum_n |a_n| = +\infty$, then $\sum_n b_n = +\infty$.*

Proof. (a) Apply the Cauchy criterion.

(b) The partial sums $s_N = \sum_{n=1}^{N} |a_n|$ form an unbounded sequence by hypothesis, and the partial sums $t_N = \sum_{n=1}^{N} b_n$ are even bigger. ////

Corollary. *If $\sum_n |a_n| < +\infty$, then $\sum_n a_n$ converges. In words, "Absolute Convergence implies Convergence."*

Proof. Use $b_n = |a_n|$ above. ////

Example (Harmonic Series). $\displaystyle\sum_n \frac{1}{n}$ *diverges to $+\infty$. (However, $\dfrac{1}{n} \to 0$.)*

Proof. The negation of Cauchy's criterion is

$$\begin{aligned} &\neg\left(\forall\varepsilon > 0, \exists N \in \mathbb{N} : \forall m \ge N,\ \forall p \ge 0, \quad |a_m + a_{m+1} + \cdots + a_{m+p}| < \varepsilon\right) \\ \text{i.e.,}\quad &\exists\varepsilon > 0 : \forall N \in \mathbb{N},\ \exists m \ge N,\ p \in \mathbb{N} : |a_m + a_{m+1} + \cdots + a_{m+p}| \ge \varepsilon. \end{aligned}$$

This holds with $\varepsilon = 1/2$. Indeed, for any $N \in \mathbb{N}$, pick $m = N$ and $p = N$: then

$$\begin{aligned} a_m + \cdots + a_{m+p} &= \frac{1}{N} + \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{N+N} \\ &\ge \frac{1}{2N} + \frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N} = (N+1) \times \left(\frac{1}{2N}\right) > \frac{1}{2}. \end{aligned}$$

This is interesting because the series diverges, but the crude divergence criterion above is not sharp enough to detect this. Informally, this is because series diverges rather slowly. A quick sketch and an informal integral show that $\sum_{n=1}^{N} \frac{1}{n} < 1 + \log(N)$, so to get a partial sum of 200 or more requires at least $N = e^{200}$ terms. That's more than 10^{86}. Reputable online sources use 10^{82} as a generous estimate of the number of atoms in the known universe. ////

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Dangerous Nonsense. *We know*

$$1 = \sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right).$$

This does not split to give

$$1 = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{2n+1}.$$

Indeed, by comparison, both series on the right diverge to $+\infty$, and the expression $\infty - \infty$ is undefined.

Example. $\sum \sin\left(\frac{100}{n}\right)$ diverges, because $\sin\theta \geq 2\theta/\pi$ for $\theta \in [0, \pi/2]$ and the harmonic series diverges.

Theorem (Root Test). *Consider $S = \sum_n a_n$. Define $\alpha = \limsup_n |a_n|^{1/n}$.*

(a) *If $\alpha < 1$, S converges absolutely.*

(b) *If $\alpha > 1$, S diverges.*

Proof. (a) Choose any $r \in (\alpha, 1)$. Since $r > \alpha$, there exists $N \in \mathbb{N}$ so large that

$$\forall n \geq N, \quad |a_n|^{1/n} < r, \quad \text{i.e.,} \quad |a_n| < r^n.$$

Hence $\sum_{n=N}^{\infty} |a_n|$ converges by comparison with the geometric series $\sum_{n=N}^{\infty} r^n$.

(b) Suppose $\alpha > 1$. Choose $R \in (1, \alpha)$. Since $R < \alpha$, there is a subsequence $(a_{n_k})_k$ of (a_n) satisfying $|a_{n_k}| \geq R > 1$ for all k. Clearly $a_{n_k} \not\to 0$, so "$\lim_n a_n = 0$" is false: divergence follows from the Crude Test. ////

Remark. When applying the Root Test, it's useful to know (Rudin, Thm. 3.20) that

$$\lim_{n\to\infty} n^{1/n} = 1, \qquad \lim_{n\to\infty} x^{1/n} = 1 \ \forall x > 0.$$

Theorem (Ratio Test). *Consider $S = \sum_n a_n$, where all $a_n \neq 0$.*

(a) *If $\overline{\alpha} \stackrel{\text{def}}{=} \limsup_n \left| \frac{a_{n+1}}{a_n} \right| < 1$, then S converges absolutely.*

(b) *If $\underline{\alpha} \stackrel{\text{def}}{=} \liminf_n \left| \frac{a_{n+1}}{a_n} \right| > 1$, then S diverges.*

Proof. (a) Choose $r \in (\overline{\alpha}, 1)$. Since $r > \overline{\alpha}$, there exists $N \in \mathbb{N}$ so large that

$$\forall n \geq N, \quad \left| \frac{a_{n+1}}{a_n} \right| < r, \quad \text{i.e.,} \quad |a_{n+1}| < r|a_n|.$$

It follows that $|a_{N+k}| < r^k |a_N|$, so by comparison

$$\sum_k |a_{N+k}| \le |a_N| \sum_k r^k < +\infty.$$

Convergence of S follows.

(b) Choose $r \in (1, \underline{\alpha})$. Then $r < \underline{\alpha}$, so there exists $N \in \mathbb{N}$ such that

$|a_{n+1}| \ge r|a_n| \ge |a_n|$ for all $n \ge N$. Thus "$\lim_n a_n = 0$" is false. Divergence follows from the Crude Test. ////

Remark. The ratio test is easier to try, but the root test is more discriminating. In both tests, certain values of $\alpha, \overline{\alpha}, \underline{\alpha}$ leave you with no useful conclusion.

Summary. For $S = \sum_{n=1}^{\infty} a_n$, with all $a_n \ne 0$, define

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}, \qquad \underline{\alpha} = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \qquad \overline{\alpha} = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

(i) $\overline{\alpha} < 1 \overset{(a)}{\Longrightarrow} \alpha < 1 \overset{(b)}{\Longrightarrow} \sum_n |a_n|$ converges $\implies S$ converges;

(ii) $\underline{\alpha} > 1 \implies \alpha > 1 \implies S$ diverges;

(iii) If $\alpha = 1$ (which implies $\underline{\alpha} \le 1 \le \overline{\alpha}$) any outcome is possible.

Some implications here remain to be proved. Focus on line (i): implication (b) is the Root Test, proved earlier.

To prove (a), we show $\alpha \le \overline{\alpha}$ [Rudin Thm. 3.37].

Case $\overline{\alpha} = +\infty$: Stmt "$\alpha \le +\infty$" is obvious, so desired result holds.

Case $\overline{\alpha} < +\infty$. Thanks to Archimedes, we can show $\alpha \le \overline{\alpha}$ by proving

$$(*) \qquad \forall \varepsilon > 0, \quad \alpha \le \overline{\alpha} + \varepsilon.$$

So let $\varepsilon > 0$ be given; define $\beta = \overline{\alpha} + \varepsilon$. [Note that $\beta > 0$ since $\overline{\alpha} \ge 0$ and $\varepsilon > 0$.] Deduce the existence of some $N \in \mathbb{N}$ such that

$$\forall n \ge N, \quad \left|\frac{a_{n+1}}{a_n}\right| < \beta, \text{ i.e., } |a_{n+1}| < \beta |a_n|.$$

(Used $\varepsilon = \beta - \overline{\alpha}$ in Rudin 3.17.) Then for any $p \in \mathbb{N}$,

$$|a_{N+p}| < \beta |a_{N+p-1}| < \beta^2 |a_{N+p-2}| < \cdots < \beta^p |a_N|.$$

In other words, for any $m > N$,

$$|a_m| < \beta^{m-N} |a_N| = \left[\beta^{-N} |a_N|\right] \beta^m.$$

Thus

$$|a_m|^{1/m} < \beta \left[\beta^{-N} |a_N|\right]^{1/m} \qquad \forall m > N.$$

Take $\limsup_{m\to\infty}$ both sides: strict inequality degrades to give

$$\alpha \le \beta = \overline{\alpha} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $(*)$ holds—proof complete! ////

Example. [Rudin 3.35.] $S = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ evidently converges, with

$$S = \frac{1}{1-(1/2)} + \frac{1}{1-(1/3)} = \frac{7}{2}.$$

How do the tests work out?

Here

$$a_{2n} = \frac{1}{2^n}, \qquad a_{2n+1} = \frac{1}{3^n}, \qquad n = 0, 1, 2, \ldots,$$

so

$$|a_{2n}|^{1/2n} = (1/2^n)^{1/2n} \to 1/\sqrt{2},$$

$$\text{while } |a_{2n+1}|^{1/(2n+1)} = (1/3^n)^{1/(2n+1)} \to 1/\sqrt{3}.$$

This gives

$$\alpha = \limsup_n |a_n|^{1/n} = \frac{1}{\sqrt{2}} < 1,$$

so the root test predicts convergence. However,

$$\left|\frac{a_{2n+2}}{a_{2n+1}}\right| = \frac{1/2^{n+1}}{1/3^n} = \frac{1}{2}\left(\frac{3}{2}\right)^n \to +\infty.$$

Fortunately, however,

$$\left|\frac{a_{2n+1}}{a_{2n}}\right| = \frac{1/3^n}{1/2^n} = \left(\frac{2}{3}\right)^n \to 0$$

so $\underline{\alpha} = 0$: the ratio test is *inapplicable,* but at least *not wrong*! ////

The next test resolves some cases where a geometric comparison is too demanding.

Theorem (Cauchy Condensation Test). *If $a_n \geq a_{n+1} \geq 0$ for all n, TFAE:*

(a) $S = \sum_{n=1}^{\infty} a_n < +\infty$,

(b) $T = \sum_{k=0}^{\infty} 2^k a_{2^k} < +\infty$.

Proof. (b)⇒(a) The key idea is illustrated by this sandwich of inequalities:

$$\begin{aligned} &a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \cdots + a_{15}) \\ \leq &a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \cdots + a_8) \end{aligned}$$

Suppose $T < +\infty$. For any n, choose k so $n < 2^{k+1}$: then

$$\begin{aligned} s_n &\overset{\text{def}}{=} \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \\ &\leq T. \end{aligned}$$

Hence (s_n) is bounded above; clearly $s_n \uparrow$, so (s_n) converges.

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(a)⇒(b) Suppose $S < +\infty$. Consider a typical partial sum associated with T:

$$\begin{aligned}
t_n &= \sum_{k=0}^{n} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n} \\
&\leq 2\left[\tfrac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n}\right] \\
&\leq 2\left[\ a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n})\right] \\
&\leq 2S.
\end{aligned}$$

Hence (t_n) is bounded above; clearly $t_n \uparrow$, so (t_n) converges. ////

p-Series. The notation below comes from the famous *Riemann zeta function.*

Proposition. *The series* $\zeta(p) \overset{\text{def}}{=} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p \in \mathbb{R})$ *converges if and only if* $p > 1$.

Proof. If $p \leq 0$, the series diverges by the Crude Test. When $p > 0$, we can apply Cauchy's Condensation Test. The sequence $a_n = 1/n^p$ is decreasing and nonnegative, so $\zeta(p)$ converges if and only if this series does:

$$T = \sum_{k=0}^{\infty} 2^k \left(\frac{1}{(2^k)^p}\right) = \sum_{k=0}^{\infty} \left(2^k\right)^{1-p} = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k.$$

This T is geometric, with ratio $r = 2^{1-p}$. It converges if and only if $r < 1$, i.e., $p > 1$. ////

Examples.

(i) For $\zeta(1) = \displaystyle\sum_n \frac{1}{n}$ (harmonic series), we have

$$\begin{aligned}
\alpha &= \limsup_n \left(\frac{1}{n}\right)^{1/n} = \lim_n \frac{1}{n^{1/n}} = 1, \\
\overline{\alpha} = \underline{\alpha} &= \lim_n \left(\frac{1/(n+1)}{1/n}\right) = \lim_n \left(\frac{n}{n+1}\right) = 1.
\end{aligned}$$

Both the root and the ratio tests are inconclusive; the series diverges.

(ii) For $\zeta(2) = \displaystyle\sum_n \frac{1}{n^2}$, we have

$$\begin{aligned}
\alpha &= \limsup_n \left(\frac{1}{n^2}\right)^{1/n} = \lim_n \left(\frac{1}{n^{1/n}}\right)^2 = 1, \\
\overline{\alpha} = \underline{\alpha} &= \lim_n \left(\frac{1/(n+1)^2}{1/n^2}\right) = \lim_n \left(\frac{n}{n+1}\right)^2 = 1.
\end{aligned}$$

Both the root and the ratio tests are inconclusive; the series converges. (In fact, $\zeta(2) = \pi^2/6$; $\zeta(4) = \pi^4/90$; $\zeta(6) = \pi^6/945$; ... and $\pi^{2n}/\zeta(2n)$ is a known rational number for all $n \in \mathbb{N}$. The first of these rational numbers that is not actually an integer is $\pi^{12}/\zeta(12) = 638512875/691$.)

Euler's Number. Please read about the number e in Text, paragraphs 3.30–3.32.

Question. Suppose $p(n) > 1$ for each $n \in \mathbb{N}$. Does this guarantee convergence for $\sum_{n=1}^{\infty} \frac{1}{n^{p(n)}}$? No! When $p = 1 + 1/n$, this series diverges. Try showing this with the Cauchy Condensation Test.

Theorem (Kummer's Test–TBB pp. 115ff). *Consider $S = \sum_{n=1}^{\infty} a_n$, where $a_n > 0$ for each n. Let (D_n) be any sequence of positive numbers. Define*

$$\underline{L} = \liminf_{n\to\infty} \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}, \qquad \overline{L} = \limsup_{n\to\infty} \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}.$$

(a) If $\underline{L} > 0$ then S converges.

(b) If $\overline{L} < 0$ and $\sum_n \frac{1}{D_n} = +\infty$, then S diverges.

Proof. (a) If $\underline{L} > 0$ then we can choose some $r \in (0, \underline{L})$. The definition of lim inf implies that for some $N \in \mathbb{N}$,

$$\forall k \geq N, \qquad r < \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}, \quad \text{i.e.,} \quad r a_{k+1} < D_k a_k - D_{k+1}a_{k+1}.$$

This is a telescoping-sum opportunity:

$$\begin{aligned} r a_{N+1} &< D_N a_N - D_{N+1}a_{N+1} \\ r a_{N+2} &< D_{N+1}a_{N+1} - D_{N+2}a_{N+2} \\ &\vdots \\ r a_{N+p} &< D_{N+p-1}a_{N+p-1} - D_{N+p}a_{N+p} \end{aligned}$$

Add these, then remember that all $D_n > 0$ and all $a_n > 0$:

$$r(a_{N+1} + \ldots + a_{N+p}) \leq D_N a_N - D_{N+p}a_{N+p} \leq D_N a_N - 0.$$

This shows that the partial sums for S are bounded (by $D_N a_N$), which implies that S converges.

(b) If $\overline{L} < 0$, then there exists $N \in \mathbb{N}$ such that

$$\forall k \geq N, \qquad \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}} \leq 0, \quad \text{i.e.,} \quad D_k a_k \leq D_{k+1}a_{k+1}.$$

Chaining together inequalities like this shows that for all $p \in \mathbb{N}$,

$$D_N a_N \leq \cdots \leq D_{N+p} a_{N+p}, \quad \text{i.e.,} \quad a_{N+p} \geq (D_N a_N) \frac{1}{D_{N+p}}.$$

Since $\displaystyle\sum_p \frac{1}{D_{N+p}} = +\infty$ by hypothesis, we conclude $\displaystyle\sum_p a_{N+p} = +\infty$ also, as required. ////

Example. Take $D_k = 1$ for each k in Kummer's Test. Then, in the notation introduced earlier (with extended-real interpretations),

$$\underline{L} = \liminf_{n\to\infty} \left(\frac{a_k}{a_{k+1}} - 1 \right) = \frac{1}{\overline{\alpha}} - 1, \quad \text{so} \quad \underline{L} > 0 \iff \overline{\alpha} < 1;$$
$$\overline{L} = \limsup_{n\to\infty} \left(\frac{a_k}{a_{k+1}} - 1 \right) = \frac{1}{\underline{\alpha}} - 1, \quad \text{so} \quad \overline{L} > 0 \iff \underline{\alpha} > 1.$$

Thus Kummer's Test extends the Ratio Test.

Theorem (Raabe's Test). *Let $S = \sum_k a_k$, with each $a_k > 0$. Suppose this limit exists:*

$$R = \lim_{k\to\infty} k \left(\frac{a_k}{a_{k+1}} - 1 \right).$$

Then

(a) *If $R > 1$, the series S converges;*

(b) *If $R < 1$, the series S diverges.*

Proof. Choose $D_k = k$ and apply Kummer's Test. That result involves ratios like

$$\frac{D_k a_k - D_{k+1} a_{k+1}}{a_{k+1}} = \frac{k a_k - (k+1) a_{k+1}}{a_{k+1}} = k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1.$$

By hypothesis, the right side converges, so we have $\underline{L} = \overline{L} = R - 1$ in Kummer's Test.

(a) If $R > 1$ then $\underline{L} > 0$, so S converges.

(b) If $R < 1$ then $\overline{L} < 0$, so S diverges. ////

Enrichment. Check out the lovely further story about Gauss's Ratio Test in Section 3.6.11 of TBB.

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Alternating Series. Intuitively, it is easier for a series whose terms alternate in sign to converge than for a series of positive terms. For example, the "alternating harmonic series" $S = \displaystyle\sum_n \frac{(-1)^n}{n}$ converges, as a consequence of the following result.

Theorem (Alternating Series Test—AST). *If* $S = \sum_n (-1)^n a_n$ *and*

$$\text{(i) } a_0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad\qquad \text{(ii) } \lim_n a_n = 0,$$

then S converges.

Proof. Sketch $s_0, s_1, s_2, \ldots$ on a number line. It looks like $s_2 \geq s_4 \geq s_6 \geq s_8 \geq \cdots$, while $s_1 \leq s_3 \leq s_5 \leq \cdots$. To prove this, use condition (i): for any $n \in \mathbb{N}$,

$$s_{n+2} - s_n = (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} = (-1)^{n+1}\left[a_{n+1} - a_{n+2}\right] \begin{cases} \leq 0, & \text{if } n \text{ even}, \\ \geq 0, & \text{if } n \text{ odd}. \end{cases}$$

Furthermore, for any $m \in \mathbb{N}$,

$$s_{2m+1} - s_{2m} = (-1)^{2m+1} a_{2m+1} \leq 0, \qquad \text{i.e.,} \quad s_{2m+1} \leq s_{2m}.$$

Given any $k, \ell \in \mathbb{N}$, choose $m \geq \max\{k, \ell\}$ to get

$$s_{2k+1} \leq s_{2m+1} \leq s_{2m} \leq s_{2\ell}.$$

So every odd-index s_n is no larger than any even index s_n:

$$s_1 \leq s_3 \leq s_5 \leq \cdots \leq s_6 \leq s_4 \leq s_2.$$

It follows that both sequences $(s_{2k+1})_k$ and $(s_{2k})_k$ are bounded and monotonic, so they both converge. Now use (ii): Since $|s_{2k+1} - s_{2k}| = a_{2k} \to 0$ as $k \to \infty$, these two sequences must have the same limit. It follows that the entire sequence (s_n) converges to this common limit. ////

Remarks. 1. The inequality $s_{2n+1} \leq S \leq s_{2n}$ in this proof is useful in estimating S.

2. The textbook proof (Thm. 3.43) is dramatically different, and based on an interesting analogue of integration by parts called "summation by parts". It deserves careful reading.

3. Alternative method: test Cauchy's criterion directly.

Summation by Parts. The analogy with integration by parts is emphasized when we use notation suggested by Folland's *Advanced Calculus*. The goal is to simplify

$$\sum_{k=0}^{n} A_k b_k.$$

So we define $A'_k = A_k - A_{k-1}$ and $B_k = b_0 + b_1 + \cdots + b_k$. It's consistent to note $b_k = B'_k = B_k - B_{k-1}$. Then

$$\begin{aligned} &A_0 b_0 + A_1 b_1 + A_2 b_2 + \cdots + A_n b_n \\ &\quad = A_0 B_0 + A_1(B_1 - B_0) + A_2(B_2 - B_1) + \ldots + A_n(B_n - B_{n-1}) \\ &\quad = (A_0 - A_1)B_0 + (A_1 - A_2)B_1 + \cdots + (A_{n-1} - A_n)B_{n-1} + A_n B_n \\ &\quad = -A'_1 B_0 - A'_2 B_1 - \cdots - A'_n B_{n-1} + A_n B_n \end{aligned}$$

In compact form,

$$\sum_{k=0}^{n} A_k B'_k = A_n B_n - \sum_{k=1}^{n} A'_k B_{k-1}.$$

This supports the following generalization of the AST, which can be recovered by choosing $b_n = (-1)^n$.

Theorem (Dirichlet). *Consider the series* $S = \sum_{k=0}^{\infty} a_k b_k$. *If*

- $a_0 \geq a_1 \geq a_2 \geq \cdots$ *and* $\lim_{n\to\infty} a_n = 0$, *and*
- $B_n = b_0 + b_1 + \cdots + b_n$ *is a bounded sequence,*

then the series S *converges.*

Proof. Think of $A_k = a_k$ in the summation by parts formula above. For each $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n} a_k' B_{k-1}, \qquad \text{where } a_k' = a_k - a_{k-1}.$$

Both RHS terms converge as $n \to \infty$. Indeed, the boundedness hypothesis guarantees that $C = \sup_n |B_n|$ is a real number, so

$$|a_n B_n| \leq C a_n \to 0.$$

And the monotonicity assumption gives (by telescoping)

$$\sum_{k=1}^{n} |a_k' B_{k-1}| \leq C \sum_{k=1}^{n} |a_k'| = C \sum_{k=1}^{n} (a_{k-1} - a_k) = C(a_0 - a_n) \leq C a_0. \qquad (\dagger)$$

Absolute convergence implies convergence, so $\sum_{k=1}^{n} a_k' B_{k-1}$ has a real-valued limit as $n \to \infty$. This completes the proof. ////

Remark. Dirichlet's Theorem does *not* assert absolute convergence for the series $\sum_k a_k b_k$. Indeed that would be wrong, because this theorem generalizes the AST, and therefore asserts convergence for series like $\sum_k (-1)^k/\sqrt{k}$ that do not converge absolutely. It's true that the proof relies on the absolute convergence of a certain series, but *this is a different series* from the one in the statement.

Application (Home Practice). Use geometric series methods to prove

$$\sum_{k=1}^{n} \left(e^{ix}\right)^k = e^{i(n+1)x/2} \frac{\sin(nx/2)}{\sin(x/2)}.$$

Then if $b_k = \sin(kx)$, deduce $|B_n| \leq \frac{1}{\sin(x/2)}$. It follows that whenever $a_n \downarrow 0$, the Fourier Sine Series

$$S(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$$

converges for each x where $\sin(x/2) \neq 0$. But the only x not covered here have the form $x = 2n\pi$ for some $n \in \mathbb{Z}$, and for all such x we have $\sin(kx) = 0$ for each $k \in \mathbb{N}$. So $S(2n\pi) = 0$ for each $n \in \mathbb{N}$, and thus $S(x)$ is defined for all real x. ////

Absolute vs Conditional Convergence. Recall:

- If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges.
- $S = \sum_n a_n$ is said to **converge absolutely** if $\sum_n |a_n|$ converges.

Now $\displaystyle\sum_n \frac{(-1)^n}{\sqrt{n}}$ converges by the AST, but $\displaystyle\sum_n \left|\frac{(-1)^n}{\sqrt{n}}\right| = \zeta(1/2) = +\infty$. Series like this one, where $\sum_n a_n$ converges but $\sum_n |a_n|$ does not, are called **nonabsolutely** or **conditionally** convergent.

Rearrangements. For absolutely convergent series, shuffling the terms does not change the limit. (For series whose terms are positive, this is a consequence of HW04 Question 3; for more general series, see the short proof of Rudin's Theorem 3.55.)

Conditional convergence is full of horrors. Start with the alternating harmonic series,

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots.$$

This converges by the AST. But if we re-order the terms by picking up 2 negative terms after each positive one, we get

$$S' = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots.$$

Inserting parentheses reveals something rather unsettling:

$$\begin{aligned} S' &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots \\ &= \left(\frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{10}\right) - \frac{1}{12} + \ldots \\ &= \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots\right]. \end{aligned}$$

Yes, $S' = \frac{1}{2}S$! Innocent-looking operations like re-ordering the terms of the series can change the number it converges to. In fact, according to a theorem of Riemann, for every conditionally convergent series $\displaystyle\sum_{n=1}^{\infty} a_n$ and every real number L, there exists a bijection $\phi\colon \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} a_{\phi(n)} = L$. We will not dwell on such matters; TBB explain everything in Section 3.7 and the associated exercises.

F. Power Series

Here are some things worth knowing, not covered in class.

Series involving a variable parameter (a.k.a. "series of functions") have many uses in pure and applied mathematics. Typically the series will converge for some x and not for others, and we want to know what happens where. For example, the set of real x where the series $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges is precisely the interval $(1, +\infty)$.

The simplest series of functions are power series, which have the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

for given constants c_n and x_0. (Shorthand: $(x - x_0)^0 = 1$ for all x, including $x = x_0$ ... a slight offence against our usual refusal to define 0^0.) For these, the set of x giving convergence has a simple shape.

Theorem. *For any power series $\sum c_n (x - x_0)^n$, there exists $R \in [0, +\infty) \cup \{+\infty\}$ such that $|x - x_0| < R$ implies absolute convergence and $|x - x_0| > R$ implies divergence.*

Remarks. 1. The series obviously converges (to c_0) when $x = x_0$, even when $R = 0$. This does not contradict the statement, "$|x - x_0| < 0$ implies convergence."

2. This same result is valid for complex c_n, x_0, and x. In this case, the inequality $|x - x_0| < R$ describes an open disk in $\mathbb{C}$, centred at x_0, called the **disk of convergence.** (If $R = 0$ the disk is empty; if $R = +\infty$ it is the whole plane.) This explains why the number R is called the **radius of convergence** for the given series.

3. You can find R using the root test (or sometimes the ratio test) as in proof below: don't memorize a special formula for power series.

4. The theorem gives no information about points x where $|x - x_0| = R$: for these, use one of the many convergence tests developed previously.

Proof. For fixed $x \neq x_0$, this is an ordinary series with summands

$$a_n = c_n (x - x_0)^n.$$

Apply the Root Test, computing

$$\begin{aligned} \alpha &= \limsup_n |a_n|^{1/n} \\ &= \limsup_n |c_n (x - x_0)^n|^{1/n} \\ &= |x - x_0| \limsup_n |c_n|^{1/n} \\ &\stackrel{\text{def}}{=} |x - x_0| \, \gamma. \end{aligned}$$

The series is certain to converge if $\alpha < 1$, i.e., either $\gamma = 0$ or else $|x - x_0| < 1/\gamma$; and to diverge if $\alpha > 1$, i.e., either $\gamma = +\infty$ or else $|x - x_0| > 1/\gamma$. Hence the statement holds for $R = 1/\gamma$ (extended interpretation in $[0, +\infty]$). ////

Example. For $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (in which $x_0 = 0$), apply the Ratio Test:

$$\overline{\alpha} = \limsup_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \limsup_{n\to\infty} \left| \frac{x}{(n+1)} \right| = 0 \qquad \forall x \in \mathbb{R}.$$

This series converges for all real x: $R = +\infty$.
[Corollary: $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all real x, by the Crude Test for Divergence.]

For $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$, the Ratio test gives

$$\overline{\alpha} = \limsup_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}/(n+1)^{n+1}}{n!x^n/n^n} \right| = \limsup_{n\to\infty} \left| \frac{x}{\left(1+\frac{1}{n}\right)^n} \right| = \frac{|x|}{e} \qquad \forall x \in \mathbb{R}.$$

Convergence is assured if $|x| < e$. Similarly,

$$\underline{\alpha} = \liminf_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}/(n+1)^{n+1}}{n!x^n/n^n} \right| = \frac{|x|}{e} \qquad \forall x \in \mathbb{R},$$

so divergence is assured if $|x| > e$. Hence the radius of convergence is $R = e$. When $x = e$, divergence follows from the Crude Test. Indeed, the power series definition gives

$$\forall x \geq 0,\ \forall n \in \mathbb{N},\ e^x > \frac{x^n}{n!}.$$

In particular, when $x = n \in \mathbb{N}$, $e^n > n^n/n!$, so the terms of the given series obey

$$\frac{n!e^n}{n^n} > 1 \qquad \forall n \in \mathbb{N}.$$

When $x = -e$, terms of the same size show up with alternating signs. The Crude Test still applies, and shows divergence. The series converges if and only if $|x| < e$.