UBC Mathematics 320(101)—Assignment 9
Due by PDF upload to Canvas at 18:00, Saturday 11 Nov 2023

References: Loewen, lecture notes on Point-Set Topology (2023-11-04 or newer—see Canvas); Rudin pp. 30b–36a; Thomson-Bruckner-Bruckner, Sections 13.1–13.5

Presentation: To qualify for full credit, submissions must satisfy the detailed specifications provided on Canvas.

1. Solve in either order:

(a) Construct, with justification, a subset A of $\mathbb{R}$ such that every point of A is isolated and $A' \neq \emptyset$.

(b) Rudin Chapter 2, problem 5, page 43.

2. (a) Give an example of two sets A and B in some HTS satisfying

$$\operatorname{int}(A \cup B) \neq \operatorname{int}(A) \cup \operatorname{int}(B).$$

(b) Give an example of two sets A and B in some HTS satisfying

$$\overline{A \cap B} \neq \overline{A} \cap \overline{B}.$$

(c) Working in $\mathbb{R}^k$ with the usual topology, express the open ball $\mathbb{B}[0; 1)$ as a union of closed sets. Can $\mathbb{B}[0; 1)$ be expressed as an intersection of closed sets?

3. Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ as follows:

> A set $G \subseteq \mathbb{R}$ belongs to $\mathcal{T}$ if and only if for every x in G, there exists $r > 0$ such that $[x, x + r) \subseteq G$.

(a) Prove that $(\mathbb{R}, \mathcal{T})$ is a HTS. (It is called the *Sorgenfrey line.*)

All our terminology—open set, closed set, boundary point, limit point, convergence—depends on what topology we use. Use the Sorgenfrey topology in parts (b)–(d):

(b) Show that the interval $[0, 1)$ is open.

(c) Find all boundary points of the interval $(0, 1)$.

(d) Let $s_n = -1/n$ and $t_n = 1/n$. Prove that one of these sequences converges to 0, and the other does not. Use the definition given in class, i.e., $x_n \to \widehat{x}$ means that for every open set U containing $\widehat{x}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$.

4. Let A be a subset of a HTS $(X, \mathcal{T})$. The **boundary of** A is a set denoted ∂A: we say $z \in \partial A$ if and only if every open U containing z satisfies both $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. Prove:

(a) $\partial A = \overline{A} \cap \overline{A^c}$.

(b) A is closed if and only if $\partial A \subseteq A$.

(c) A is open if and only if $A \cap \partial A = \emptyset$.

5. Prove: For every set A in a HTS $(X, \mathcal{T})$, A' is closed.

6. Recall the sequence space ℓ^2 from HW07 Q3. Given a specific $M = (M_1, M_2, \ldots)$ in ℓ^2, let

$$S = \left\{ x \in \ell^2 \,:\, \forall n \in \mathbb{N},\ |x_n| \leq M_n \right\}.$$

Prove: every sequence $\left(x^{(n)}\right)$ in S has a convergent subsequence, whose limit lies in S.

Practice Problems—Not for Credit

These are not to be handed in. Solutions will be provided.

7. Let A be a subset of a HTS $(X, \mathcal{T})$. Prove:

(a) $\overline{A} = A \cup \partial A$.

(b) $A^\circ = A \setminus \partial A$.

8. Let A and B be subsets of $\mathbb{R}$. Write

$$A + B = \{a + b \,:\, a \in A,\ b \in B\}.$$

If A and B are closed, does it follow that $A + B$ is closed? Does this follow if in addition one of the two is bounded?