Math 318 Homework 7

Xander Naumenko

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Question 1a. The definition of a characteristic function is $\phi_X(t) = E[e^{itX}]$. Expanding this, we get:

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} f_X(x)e^{itx}dx = \sum_{x=-\infty}^{\infty} p_X(x)e^{itx} = \sum_{x=-\infty}^{\infty} p_X(x)\left(\cos(tx) + i\sin(tx)\right).$$

The only t dependence is in the $\sin(tx)$ and $\cos(tx)$ terms which are both 2π periodic (since x is an integer), so the characteristic function of discrete variables is 2π-periodic.

Question 1b. Let $Y =$

$$\phi_X(0) = \int_{-\infty}^{\infty} f_X(x)dx = \phi_X(2\pi) = \int_{-\infty}^{\infty} f_X(x)e^{2\pi ix}dx.$$

Question 2. Expressing as an integral:

$$E[X^3] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u+v)^3 f_U(u)f_V(v)dudv = \iint (u^3 + 3u^2v + 3uv^2 + v^3)f_U(u)f_V(v)dudv$$

$$= E[U^3] + E[V^3] + 3E[U^2]E[V] + 3E[U]E[V^2] = E[U^3] + E[V^3].$$

Similarly for $E[X^4]$:

$$E[X^4] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u+v)^4 f_U(u)f_V(v)dudv = \iint (u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^3)f_U(u)f_V(v)dudv.$$

$$= E[U^4] + E[V^4] + 6E[U^2]E[V^2] + 4E[U^3]E[V] + 4E[U]E[V^3] = E[U^4] + E[V^4] + 6E[U^2]E[V^2].$$

Question 3a. Let the entries of A be denoted by A_{ik}, where i is the row and k is the column. Then by the definition of matrix multiplication,

$$Y_i = \sum_{k=1}^{n} A_{ik}X_k.$$

The distribution of a sum of normal variables is also a normal variable with the sum of mean and variance added, so $Y_i = N(0, n)$.

Question 3b. Computing covariance

$$\mathrm{Cov}(Y_i, Y_j) = E[Y_iY_j] - E[Y_i]E[Y_j] = E\left[\left(\sum_{k=1}^{n} A_{ik}X_k\right)\left(\sum_{k=1}^{n} A_{jk}X_k\right)\right]$$

$$= E\left[\sum_{k=1}^{n}\sum_{l=1}^{n} A_{ik}A_{jl}X_kX_l\right] = \sum_{k=1}^{n}\sum_{l=1}^{n} A_{ik}A_{jl}E[X_kX_l] = \sum_{k=1}^{n} A_{ik}A_{jk}E[X_k^2]$$
$$= A_i \cdot A_j$$

where A_i and A_j are the ith and jth row vector of A respectively.

Question 3c.

Question 4. For the following sections this code was used to generate the figures:

```
import numpy as np
import random
import matplotlib.pyplot as plt
from tqdm import tqdm

n = 1000000
sims = 1000
ps = (0.5, 0.51, 0.502)

figa, axa = plt.subplots(3)
for j, p in enumerate(ps):
    X = [0]
    for i in range(n-1):
        X.append(X[i] + (-1 if random.random() > p else 1))

    axa[j].plot(list(range(n)), X)
    axa[j].set_title(f'p={p}')

plt.show()

figb, axb = plt.subplots(3)
figc, axc = plt.subplots(3)
for j, p in enumerate(ps):
    T = np.array([n]*sims)
    for sim in tqdm(range(sims)):
        X = 0
        for t in range(n):
            X += -1 if random.random() > p else 1
            if X == 0:
                T[sim] = t
                break

    axb[j].hist(T, bins=np.arange(0, n + 10000, 10000))
    axb[j].set_title(f'p={p}')

    T.sort()
    F = []
    s = 0
    for t in range(n):
```

```
        while s < len(T) and T[s] <= t:
            s += 1
        F.append((sims-s)/sims)

    axc[j].loglog(np.arange(n), F)
    axc[j].set_title(f'p={p}')

plt.show()
```

Question 4a. See figure 1.

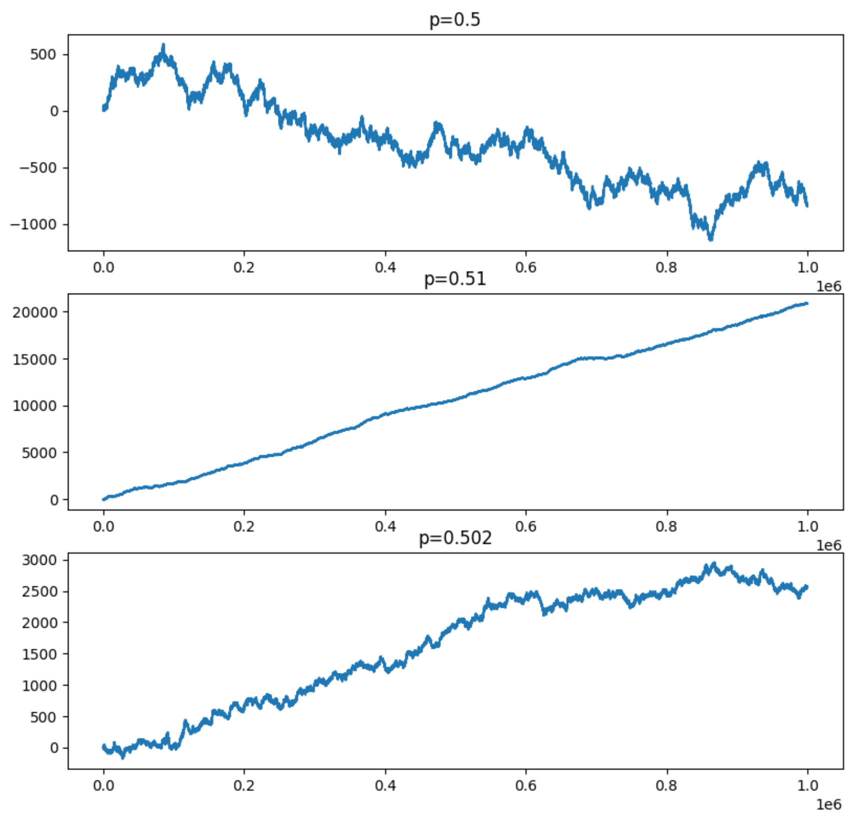

Figure 1: Graph for question 4a.

Question 4b. See figure 2.

Question 4c. See figure 3.

Question 4d. From the plots it seems that $P(T > n) = 0$ as $n \to \infty$.

Question 5a.

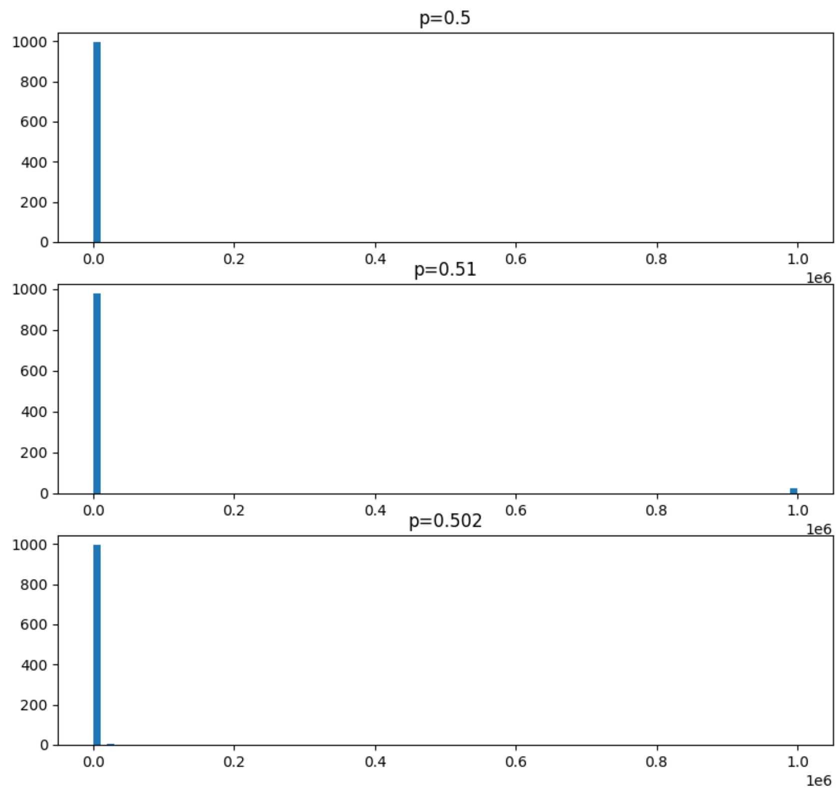

Figure 2: Histogram for question 4b.

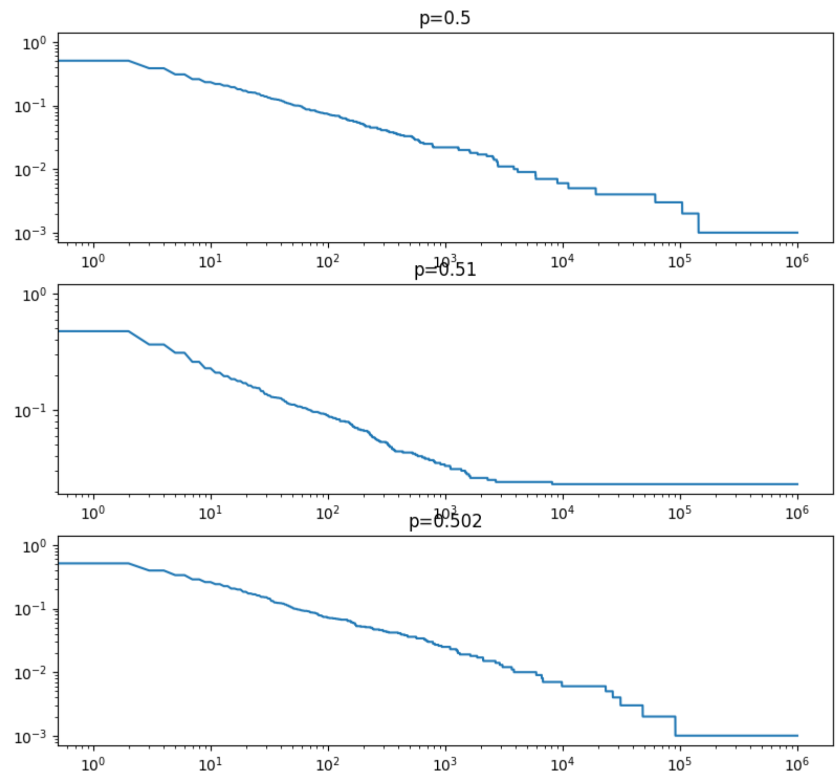

Figure 3: Graphs for question 4c.