Math 305 Homework 2

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10pts each

1. Find all values of the following equation

(a)$z^3 = i - 1$

$$i - 1 = \sqrt{2}e^{i\frac{3\pi}{4}} = (2^{\frac{1}{6}}e^{ni\frac{\pi}{4}} + i\frac{2k\pi}{3})^3, k \in \{0,1,2\}.$$

(b) $z^5 = \frac{2i}{1-\sqrt{3}i}$

$$\frac{2i}{1-\sqrt{3}} = \frac{-2\sqrt{3}+2i}{4} = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i = e^{i\frac{5\pi}{6}} = e^{i\left(\frac{\pi}{6}+2k\frac{\pi}{5}\right)}, k \in \{0,\dots,5\}.$$

(c) $(z-i)^2 = i$

$$(z-i)^2 = i = e^{i\frac{\pi}{2}} \implies z - i = e^{i\frac{\pi}{4}+i\pi k} \implies z = e^{i\frac{\pi}{4}+i\pi k} + e^{i\frac{\pi}{2}}, k \in \{0,1\}.$$

(d) $z^2 + 2iz + 1 = 0$

$$z^2 + 2iz - 1 = (z+i)^2 = -2 = 2e^{i\pi} \implies z + i = \sqrt{2}e^{i\left(\frac{\pi}{2}+\pi k\right)} \implies z = \sqrt{2}e^{i\left(\frac{\pi}{2}+\pi k\right)} - i, k \in \{0,1\}.$$

*2. Let m and n be positive integers that have no common factor and z_0 be a complex number. Let $z_0^{\frac{1}{n}}$ denote the set of all complex numbers such that $z^n = z_0$, i.e., $z_0^{\frac{1}{n}} = \{z \mid z^n = z_0\}$. Prove that the set of numbers $(z_0^{1/n})^m$ is the same as the set of numbers $(z_0^m)^{1/n}$. Use this result to find all values of $(1-i)^{3/2}$. Here $(z_0^{1/n})^m = \{z^m \mid z^n = z_0\}$.

Hint: since m and n have no common factor, for any integer k, we can write it as $k = mk_1 + nk_2$ where k_1, k_2 are two integers.

*: An extra 10points will be awarded to Problem 2 if your answer is correct.

Assume that $z_0 = re^{i\phi}$. Then we have that $z_0^{1/n} = e^{i\left(\frac{\phi}{n}+\frac{2\pi k}{n}\right)}, k \in \mathbb{Z}$ and $A = (z_0^m)^{1/n} = e^{i\left(\frac{\phi m}{n}+\frac{2\pi k_1}{n}\right)}, k_1 \in \mathbb{Z}$. Taking the first term to the power of m gives $B = \left(z_0^{1/n}\right)^m = e^{i\left(\frac{\phi m}{n}+\frac{2\pi k_2 m}{n}\right)}, k_2 \in \mathbb{Z}$. To prove that these are equivalent sets, we will show that $\forall a \in A, a \in B$ and that $\forall b \in B, b \in A$, which is enough to show set equality.

Let $b \in B$. It can thus be written as $b = e^{i\left(\frac{\phi}{n}+\frac{2\pi k_2 m}{n}\right)}$ for some $k_2 \in \mathbb{Z}$. Let $k_1 = k_2 m$. Then we have that

$$b = e^{i\left(\frac{\phi}{n}+\frac{2\pi k_2 m}{n}\right)} = e^{i\left(\frac{\phi}{n}+\frac{2\pi k_1}{n}\right)} \in A,$$

meaning $B \subseteq A$.

For the other direction, let $a \in A$. Then it can be written as $e^{i\left(\frac{\phi m}{n}+\frac{2\pi k_1}{n}\right)}$ for some $k_1 \in \mathbb{Z}$. Using the hint (sorry for the awkward choice of variable names), there exist $x, y \in \mathbb{Z}$ such that $k_1 = mx + ny$. Plugging this in, we get

$$a = e^{i\left(\frac{\phi}{n}+\frac{2\pi k_1}{n}\right)} = e^{i\left(\frac{\phi}{n}+\frac{2\pi(mx+ny)}{n}\right)} = e^{i\left(\frac{\phi}{n}+\frac{2\pi mx}{n}+2\pi y\right)} = e^{i\left(\frac{\phi}{n}+\frac{2\pi mx}{n}\right)}.$$

Letting $k_2 = x$ it is clear that $a \in B \implies A \subseteq B$.

Since we have shown that $A \subseteq B$ and $B \subseteq A$, it must be the case that $A = B$ as required. $\square$

To find the values of $(1-i)^{3/2}$ we can expand, which is justified since the we just found that the order doesn't matter:

$$(1-i)^{3/2)}\left((1-i)^{1/2}\right)^3 = \left(\sqrt[4]{2}e^{i\left(\frac{7}{8}\pi+\pi k\right)}\right)^3, k \in \{0,1\} = 2^{\frac{3}{4}}e^{i\left(\frac{7}{8}\pi+\pi k\right)}, k \in \{0,1\}.$$

3. Write the following functions in the form $w = u(x,y) + iv(x,y)$.

(a) $f(z) = \frac{z+i}{z+1}$

$$f(z) = \frac{z+i}{z+1} = \frac{x+i(y+1)}{(x+1)+iy} = \frac{(x+i(y+1))(x+1-iy)}{(x+1)^2+y^2}.$$

$$= \frac{x^2+x+y^2+y+i((y+1)(x+1)-xy)}{(x+1)^2-y^2} = \frac{x^2+x+y^2+y}{(x+1)^2+y^2} + i\frac{x+y+1}{(x+1)^2+y^2}.$$

(b) $f(z) = \frac{e^z}{z}$

$$f(z) = \frac{e^z}{z} = \frac{e^{x+iy}}{x+iy} = \frac{e^x}{x^2+y^2}(x\cos y + y\sin y) + i\frac{e^x}{x^2+y^2}(x\sin y - y\cos y).$$

(c) $f(z) = \frac{z^2+3}{|z-1|^2}$

$$f(z) = \frac{x^2-y^2+2ixy+3}{(x-1)^2+y^2} = \frac{x^2-y^2+3}{(x-1)^2+y^2} + i\frac{2xy}{(x-1)^2+y^2}.$$

4. Describe the image of the following sets under the following maps

(a) $f(z) = (1-i)z + 5$ for $S = \{Re(z) > 0\}$

The multiplication by $1-i$ rotates the original set by $-\frac{\pi}{2}$ and scales by a factor of $\sqrt{2}$, then adding 5 shifts the set 5 unites in the real axis. Thus the final set is

$$f(S) = \{w \mid Re(w) > 5 + Im(w)\}.$$

(b) $f(z) = \frac{z-i}{z+i}$ for $S = \{|z| < 3\}$

Expanding f out we get

$$w = f(z) = \frac{z-i}{z+i} \implies z(1-w) - i = iw \implies z = \frac{i(w+1)}{1-w}$$

$$|z| = \frac{|w+1|}{|w-1|} = \sqrt{\frac{(u+1)^2+v^2}{(u-1)^2+v^2}} < 3$$

$$\implies (u+1)^2+v^2 < 9(u-1)^2+9v^2 \implies 1 < 8u^2-20u+9+8v^2 = 8\left(u-\frac{5}{4}\right)^2 - \frac{7}{2} + 8v^2$$

$$\implies \frac{9}{2} < 8\left(u - \frac{5}{4}\right)^2 + 8v^2.$$

This is the equation of a circle of radius $\frac{9}{2}$, with an offset of $\frac{5}{4}$ in the real direction:

$$f(S) = \{w \mid \left|w - \frac{5}{4}\right| > \frac{9}{2}\}.$$

(c) $f(z) = -2z^5$ for $S = \{|z| < 1, 0 < Argz < \frac{\pi}{2}\}$

The image is contained in the circle around the origin of radius 2 because $|z| < 1$ and f only scales it by 2. For the argument, consider that $arg(w) = 5arg(z)$ spans the interval $[0, 2\pi]$ since $0 < arg(z) < \frac{\pi}{2}$. Thus the image is all the points contained in the circle of radius 2 centered at the origin:

$$f(S) = \{w \mid |w| < 2\}.$$

5. Describe the image of the following sets under the given map

(a) $S = \{Re(z) = 1\}$, $w = e^z$

Since $Re(z) = 1$, it must be in the form $z = 1 + iy, y \in \mathbb{R}$. Thus we have that

$$w = e^{1+iy} = e \cdot e^{iy}$$

which is just the equation for a circle of radius e:

$$f(S) = \{w \mid |w| = e\}.$$

(b) $S = \{0 \leq Im(z) \leq \frac{\pi}{4}\}$, $w = e^z$

Since $0 \leq Im(z) \leq \frac{\pi}{4}$, the output $w = e^z = e^{x+iy} = e^x e^{iy}$ is constrained to the outputs with argument $0 \leq Arg(w) \leq \frac{\pi}{4}$. Since there is no restriction on $Re(z)$ it encompasses the entire quadrant, so

$$f(S) = \{w \mid 0 \leq Arg(w) \leq \frac{\pi}{4}\}.$$

(c) $S = \{0 \leq Re(z) \leq 1, Im(z) = 1\}$, $w = z^2$

Based on the restrictions to z it must be in the form $z = x + i, 0 \leq x \leq 1$. Thus we have that

$$w = z^2 = (x+i)^2 = x^2 + 2xi - 1 = (x^2 - 1) + 2xi = u + iv.$$

Using these definitions for u, v, we know that $u = x^2 - 1 = \frac{v^2}{4} - 1$. This is the equation for a parabola oriented along the real axis offset by along the real axis, with only the line segment corresponding to $0 \leq x \leq 1 \implies 0 \leq v \leq 2$ included.

$$f(S) = \{w \mid Re(w) = \frac{Im(w)^2}{4} - 1, 0 \leq Im(w) \leq 2\}.$$

6. The Joukowski map is defined by

$$w = f(z) = \frac{1}{2}(z + \frac{1}{z})$$

Show that J maps the circle $S = \{|z| = r_0\}$ $(r_0 > 0, r_0 \neq 1)$ onto an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the unit circle $S = \{|z| = 1\}$ onto the real interval $[-1, 1]$.

Hint: use polar form of z.

For the first part let $z = r_0 e^{i\theta}$. Then we get that

$$f(z) = \frac{1}{2}\left(r_0 e^{i\theta} + r_0^{-1} e^{-i\theta}\right) = \frac{r_0}{2}\left((1+\frac{1}{r_0^2})\cos\theta + i(1-\frac{1}{r_0^2})\sin\theta\right)$$

$$= \frac{1}{2}\left((1+\frac{1}{r_0^2})x + i(1-\frac{1}{r_0^2})y\right) = u + iv.$$

Using the fact that $x^2 + y^2 = r_0^2$, we get that

$$1 = r_0^{-2}\left(4\left(1+\frac{1}{r_0^2}\right)^{-2} u^2 + 4\left(1-\frac{1}{r_0^2}\right)^{-2} v^2\right)$$

$$\implies 1 = \frac{4u^2}{(r_0 + r_0^{-1})^2} + \frac{4v^2}{(r_0 - r_0^{-1})^2}.$$

Thus the output is an ellipse of the form $1 = \frac{u^2}{a^2} + \frac{v^2}{b^2}$, with $a = \frac{r_0^2 + r_0}{2r_0}$ and $b = \frac{r_0^2 - r_0}{2r_0}$.

For the unit circle, we can no longer divide out by r_0 as we did in the previous part. Since it's on the unit circle $z = e^{i\theta}$, which means that

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}(2\cos\theta) = \cos\theta.$$

As θ ranges over $\mathbb{R}$, $\cos\theta$ goes over the interval $[-1, 1]$ as required.

7. Prove that $|e^{-z^4}| \leq 1$ for all z with $-\frac{\pi}{8} \leq Arg(z) \leq \frac{\pi}{8}$.

Let $z = re^{i\theta}$ with $Arg(z) = \theta$ between $-\frac{\pi}{8}$ and $\frac{\pi}{8}$. Then we have that

$$\left|e^{-z^4}\right| = \left|e^{-r\cos(4\theta) - ir\sin(4\theta)}\right| = \left|e^{-r\cos(4\theta)}\right|.$$

From the way we defined it we know that $-\frac{\pi}{8} \leq \theta \leq \frac{\pi}{8} \implies -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Since $\cos\theta > 0$ for all such θ, we have that

$$\left|e^{-z^4}\right| = \left|e^{-r\cos(4\theta)}\right| \leq \left|e^{-r}\right| \leq 1.$$

8. Show that the function $f(z) = \bar{z}$ is continuous everywhere but not differentiable anywhere.

Let $\epsilon > 0$, and select $\delta = \epsilon$. Then $\forall z \in \mathbb{C}$ with $|z - w| < \delta$, we have that

$$|f(z) - f(w)| = |\overline{z} - \overline{w}| = |\overline{z - w}| = |z - w| < \delta = \epsilon.$$

By the definition of the limit this implies that implies that

$$\lim_{z \to w} z = w$$

for all w, i.e. $\overline{z}$ is continuous everywhere.

To show it is not differentiable anywhere, consider the the two paths approaching any point w from both the real direction and the imaginary direction. Then we get that

$$\lim_{h \to 0} \frac{\overline{z + h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

On the other hand if you approach from the imaginary axis, you get

$$\lim_{h\to 0} \frac{\overline{z+ih}-\overline{z}}{ih} = \lim_{h\to 0} \frac{\overline{ih}}{ih} = \lim_{h\to 0} -\frac{ih}{ih} = -1.$$

Since the derivative doesn't agree depending on the different paths, it must be that the function is differentiable nowhere.

9. Discuss the differentiability and analyticity of the following functions

(a) $(x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$

To be differentiable the function must satisfy the Cauchy-Riemann equations:

$$u_x = 1 + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$v_y = 1 + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x.$$

As can be seen $u_y = -v_x$ for all x, y, and $u_x = v_y$ when $x^2 = y^2 \implies |x| = |y|$. Thus the function is differentiable on the lines $x = y$ and $x = -y$. However since these are only lines, there are no neighborhoods over which the function is analytic so it is nowhere analytic.

(b) $|z|^2 + 2z$

Rewritting we get that $|z|^2 + 2z = z\overline{z} + 2z$. The Cauch-Riemann equations are equivalent to having no derivative with respect to $\overline{z}$, in this case we have

$$\frac{\partial}{\partial \overline{z}}(z\overline{z} + 2z) = z.$$

This is only equal to 0 when $z = 0$, so the function is only differentiable at $z = 0$. Similarly to before, since there is only one point at which it is differentiable it is nowhere analytic.

10. Let

$$f(z) = \begin{cases} (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2+y^2), \text{if } z \neq 0; \\ 0 \text{ if } z = 0 \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$ but f is not differentiable at $z = 0$.

Hint: consider the limit with $\Delta z = (1+i)h, h \to 0$.

First we show the Cauchy Riemann equations (I assume we don't have show differentiability of u and v):

$$u_x = \frac{\frac{4}{3}x^{1/3}y^{5/3}}{(x^2+y^2)} + \frac{2x^{7/3}y^{5/3}}{(x^2+y^2)^2}.$$

$$v_y = \frac{\frac{4}{3}y^{1/3}x^{5/3}}{(x^2+y^2)} + \frac{2y^{7/3}x^{5/3}}{(x^2+y^2)^2}.$$

$$u_y = \frac{\frac{5}{3}x^{4/3}y^{2/3}}{x^2+y^2} + \frac{2y^{8/3}x^{4/3}}{x^2+y^2}.$$

$$v_x = \frac{\frac{5}{3}y^{4/3}x^{2/3}}{x^2+y^2} + \frac{2x^{8/3}y^{4/3}}{x^2+y^2}.$$

Note that at $z = 0$, $u_x = u_y = 0$ and $u_y = -u_x = 0$, so the Cauchy Riemann equations are obeyed. However, as the hint suggested consider the limit of the derivative with $\Delta z = (1+i)h$ as h goes to zero.

$$\lim_{h\to 0} \frac{f(z+(1+i)h) - f(z)}{(1+i)h} = \lim_{h\to 0} \frac{h^{4/3}h^{5/3} + ih^{5/3}y^{4/3}}{(h^2+h^2)(1+i)h} = \lim_{h\to 0} \frac{h^3 + ih^3}{(h^3+h^3)(1+i)} = \frac{1}{2}.$$

Now consider the limit as h approaches from the real axis. Then we have that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0.$$

The two limits disagree, so it must be the case that the function is not differentiable at $z = 0$.