

Phys 403 Homework 1

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Question I1.

$$\begin{aligned} I_0^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy = \int_{-\infty}^{\infty} \int_0^{2\pi} r e^{-r^2/2} d\phi dr \\ &= 2\pi \int_0^{\infty} e^{-u} du = 2\pi \implies I_0 = \sqrt{2\pi}. \end{aligned}$$

Question I2.

$$\begin{aligned} I(\mu, \sigma) &= \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{\infty} e^{-(y/\sigma)^2/2} dy = \int_{-\infty}^{\infty} = \sigma \int_{-\infty}^{\infty} e^{-z^2/2} dz \\ &= \sigma I_0 = \sigma \sqrt{2\pi}. \end{aligned}$$

Question I3.

$$\begin{aligned} Z(\kappa) &= \int_{-\infty}^{\infty} e^{\kappa x} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} dx = \int_{-\infty}^{\infty} e^{\kappa x} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} e^{((x-\sigma^2\kappa)^2 - \sigma^4\kappa^2)/(2\sigma^2)} \frac{1}{\sqrt{2\pi\sigma^2}} dx = \int_{-\infty}^{\infty} e^{\sigma^2\kappa^2/2} e^{(x-\sigma^2\kappa)^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi\sigma^2}} dx = e^{\sigma^2\kappa^2/2}. \end{aligned}$$

Question II1.

$$\int_0^{\infty} \frac{1}{\mathcal{N}} e^{-\lambda x} dx = -\frac{1}{\lambda \mathcal{N}} e^{-\lambda x} \Big|_0^{\infty} = \frac{1}{\lambda \mathcal{N}} = 1 \implies \mathcal{N} = \frac{1}{\lambda}.$$

Question II2.

$$\begin{aligned} \tilde{p}(k) &= \langle e^{-ikx} \rangle = \int_0^{\infty} \lambda e^{-\lambda x} e^{-ikx} dx = \frac{\lambda}{\lambda + ik} = \frac{1}{1 - (-\frac{ki}{\lambda})} = \sum_{j=0}^{\infty} \frac{(-ik)^j}{j!} \left(\frac{j!}{\lambda^j} \right) \\ \implies \langle x^j \rangle &= \frac{j!}{\lambda^j} \implies \langle x^1 \rangle = \frac{1}{\lambda}, \langle x^2 \rangle = \frac{2}{\lambda^2}, \langle x^3 \rangle = \frac{6}{\lambda^3}, \langle x^4 \rangle = \frac{24}{\lambda^4}. \end{aligned}$$

Question II3.

$$\begin{aligned} \log \tilde{p}(k) &= -\log \left(1 - \left(-\frac{ki}{\lambda} \right) \right) = \sum_{j=1}^{\infty} \frac{(-ki)^j}{j!} \frac{(j-1)!}{\lambda^j} \\ \implies \langle x^j \rangle_c &= \frac{(j-1)!}{\lambda^j} \implies \langle x^1 \rangle_c = \frac{1}{\lambda}, \langle x^2 \rangle_c = \frac{1}{\lambda^2}, \langle x^3 \rangle_c = \frac{2}{\lambda^3}, \langle x^4 \rangle_c = \frac{6}{\lambda^4}. \end{aligned}$$

Question III.

$$\begin{aligned}
h[p] = \langle -\log p \rangle &= \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \left((x-\mu)^2/2\sigma^2 + \log \sqrt{2\pi\sigma^2} \right) dx = \frac{1}{2\sigma^2} \langle (x-\mu)^2 \rangle + \frac{1}{2} \log 2\pi\sigma^2 \\
&= \frac{1}{2} \log 2\pi\sigma^2 + \frac{1}{2}.
\end{aligned}$$

Shannon coding theorem says that that optimal storage size depends linearly with entropy, and the above expression shows that entropy is quadratic with standard deviation (linear with variance). Thus a quadrupling of standard deviation would result in a 16 fold increase in storage space, so approximately 16GB.

Question IV. From the central limit theorem the standard error is proportional to $\frac{1}{\sqrt{N}}$ ($\epsilon = \sqrt{\frac{\langle X \rangle_c}{N}}$), so to reduce it by a factor of 10, we would need to increase the number of repetitions by a factor of 100. Thus 10000 repetitions would be necessary.