PHYS 304 Homework 2

Xander Naumenko

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Question 1a. To normalize, integrate:

$$1 = \int_{-\infty}^{\infty} \psi^* \psi dx = \int_{-a}^{a} A^2(a^2 - x^2)^2 dx = \int_{-a}^{a} A^2(x^4 - 2a^2x^2 + a^4)dx = A^2\left(\frac{2}{5}a^5 - \frac{4}{3}a^5 + 2a^5\right) = A^2\frac{16}{15}a^5$$

$$\implies A = \frac{\sqrt{15}}{4a^5}.$$

Question 1b. To find the average value, apply the $\langle x \rangle$ operator:

$$\int_{-\infty}^{\infty} \psi^* x \psi dx = \int_{-a}^{a} A^2(a^2 - x^2)^2 x dx = \int_{-a}^{a} A^2(x^5 - 2a^2x^3 + a^4x)dx.$$

Because all the x terms are odd and we're integrating around 0, the integral goes to zero and the expectation value is $\langle x \rangle = 0$.

Question 1c. To calculate the momentum we use the momentum operator:

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} dx = -i\hbar \int_{-a}^{a} A^2(a^2 - x^2)2x dx.$$

Because all the terms in the integral are once again odd, the integral goes to zero so $\langle p \rangle = 0$.

Question 1d. Taking the integral:

$$\int_{-\infty}^{\infty} \psi^* x^2 \psi dx = \int_{-a}^{a} A^2(x^6 - 2a^2x^4 + a^4x^2)dx = A^2a^7\left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) = A^2a^7\frac{16}{105} = \frac{1}{7}a^2.$$

Question 1e. Again taking the integral:

$$\langle p \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial x^2} \psi dx = \hbar^2 \int_{-a}^{a} 2A^2(a^2 - x^2)dx = 2\hbar^2 A^2 a^3\left(2 - \frac{2}{3}\right) = \frac{8}{3}A^2a^3\hbar^2 = \frac{5}{2}a^{-2}\hbar^2.$$

Question 1f. The uncertainty is the standard deviations, so

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{7}}a.$$

Question 1g. Same as last question:

$$\sigma_y = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\sqrt{5}}{\sqrt{2}}a^{-1}\hbar.$$

Question 1h. As can be seen the results confirm the uncertainty principle:

$$\sigma_x \sigma_y = \sqrt{\frac{5}{14}}\hbar \geq \frac{1}{2}\hbar.$$

Question 2a. Setting the wavelength equal to the characteristic size of the system, we can solve:

$$d = 0.3nm = \lambda = \frac{h}{\sqrt{3mk_bT}} \implies T = \frac{h^2}{3d^2mk_b} = 129342K.$$

Thus for any reasonable temperatures they act quantum mechanically. For the nucleus it is the exact same calculation except with a different mass:

$$T = \frac{h^2}{3d^2mk_b} = 2.5K.$$

Thus the nucleus generally behaves classically.
Question 2b. First we find d. To do so assume that the gas is spread out in a lattice structure, with density of $\frac{N}{V} = \frac{P}{k_bT}$. Thus the intermolecular spacing is $\sqrt[3]{\frac{V}{N}} = \left(\frac{k_bT}{P}\right)^{1/3}$. Putting this into our expression for T:

$$T = \frac{h^2}{3d^2mk_b} = \frac{h^2P^{2/3}}{3mT^{2/3}k_b^{5/3}} \implies T = \left(\frac{1}{k_b}\right)\left(\frac{h^2}{3m}\right)^{\frac{3}{5}} P^{\frac{2}{5}}$$

as expected. Putting the numbers in, for hydrogen the requisite temperature is

$$T = 2.937K.$$

For Helium in the atmosphere, we can solve for what the wavelength:

$$\lambda = \frac{h}{\sqrt{3mk_bT}} = 1.45 \cdot 10^{-9}m.$$

Since this is clearly much smaller than 1cm, the Hydrogen acts classically.
Question 3. As the hint suggests, rewrite equation 2.5 as

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\left(V(x) - E\right)\psi.$$

Consider the case where E never exceeds the minimum value of V, and we will show that this is impossible. In this case then we have that $V(x) - E > 0 \forall x$. Then from the differential equation we just wrote out this means that the second derivative and the function have the same sign for all x.

Next note that for any ψ that satisfies our differential equation, we can assume that $\psi(0) \geq 0$. This is because if this wasn't true, we could consider the new function $\psi' = -\psi$ which fulfills the differential equation and doesn't affect the square integrability of ψ. Similarly, we can assume that $\frac{d\psi}{dx}(0) \geq 0$, since if this wasn't the case then we could consider $\psi'(x) = \psi(-x)$ which would also fulfill the differential equation and have no effect on the integrability. Let $\frac{d\psi}{dx}(0) = m$. Since $\frac{d^2\psi}{dx^2} > 0$ as long as $\psi > 0$, $\psi(x) \geq \psi(0) + mx \forall x \geq 0$. This is linear function which clearly doesn't converge even in the limit, which means that ψ is not square integrable.
Question 4. Take the time derivative inside the integral:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = \int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial\Psi_2}{\partial t}+\frac{\partial\Psi_1^*}{\partial t}\Psi_2\right)dx.$$

The Schrödinger equation, after being manipulated is:

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi.$$

Plugging this into the first equation and noticing that the potential terms drop out due to the taking of the conjugate, we get

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = \int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}-\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right) = \frac{i\hbar}{2m}\left(\Psi_1^*\frac{\partial\Psi_2}{\partial x}-\frac{\partial\Psi_1^*}{\partial x}\Psi_2\right)\Big|_{-\infty}^{\infty}.$$

Since the wavefunctions are presumably normalizable given that they are solutions to the Schrödinger, both Ψ_1, Ψ_2 must go to zero at infinity. Thus the integral goes to zero and we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = 0$$

as required.