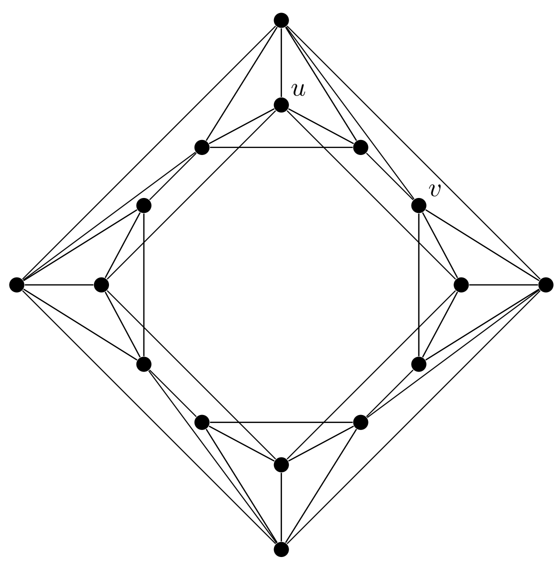
Homework 6, Math 443

Due Wednesday, March 15, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (6 points) For all three parts of this question, create graphs where the minimum number of vertices in a uv-separating set is at least 2. You may specify which vertices are u and v.
   (a) In Case 1 of our proof of Menger's Theorem, we discussed a graph G with distinct nonadjacent vertices u, v and a minimum uv-separating set containing a mutual neighbour of u and v. Find such a graph.
   (b) In Case 2 of our proof of Menger's Theorem, we discussed a graph G with distinct nonadjacent vertices u, v and a minimum uv-separating set with a nonneighbour of u and a nonneighbour of v. Find such a graph.
   (c) In Case 3 of our proof of Menger's Theorem, we discussed a graph G with distinct nonadjacent vertices u, v where for **every** minimum uv-separating set W, $W \subseteq N(u) - N(v)$ or $W \subseteq N(v) - N(u)$. Find such a graph.

2. (4 points) Determine, with proof, the minimum size of a $u - v$ separating set in the graph below.



3. (4 points) Prove the following.

   If G is a k-connected graph and $u, v_1, v_2, \ldots, v_k$ are $k + 1$ distinct vertices of G, then there exists a set of paths $\{P_i : i \in [k]\}$ where P_i is a $u - v_i$ path and for every distinct $i, j \in [k]$, $V(P_i) \cap V(P_j) = \{u\}$.

4. (5 points) For a graph G and a whole number r, let G_r be the graph obtained from G by adding r new vertices, each of them incident to every other new vertex and every vertex of G. (That is, G_r is the join of G and K_r.)

   Show that $\kappa(G_r) = \kappa(G) + r$.

5. (5 points) Theorems 5.21 and 5.22 from the text are given below. Prove Theorem 5.22 as a corollary of Theorem 5.21.

   Theorem 5.21 For distinct vertices u and v in a graph G, the minimum number of edges of G that separate u and v equals the maximum number of pairwise edge-disjoint uv paths in G.

   Theorem 5.22 A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint uv-paths for each pair u, v of distinct vertices of G.

6. (6 points) Using strong induction on k, prove the following theorem (Dirac, 1960):

   Let G be a k-connected graph, $k \geq 2$, and let $S = \{e_1, e_2\} \cup \{v_i : 1 \leq i \leq k-2\}$ be a collection of 2 edges and $k-2$ vertices of G. There exists a cycle in G containing every element of S.

7. (6 points) DON'T DO THIS ONE

8. (4 points) Prove that every circuit contains a cycle.

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	4	4	5	5	6	6	4	40