Math 322 Homework 2

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Question 1. Simply using the definition of the maps and manually carrying through where each number gets mapped:

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}.$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}.$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}.$$

Question 4. It is clearly closed, since by definition the operation always produces a tuple of reals, and since the first entry can never be zero the product can't either. For associativity, let $(a,b),(c,d),(e,f) \in G$. Then $((a,b)(c,d))(e,f) = (a,b)((c,d)(e,f)) = (ace, ad+b+acf)$. The inverse of $(a,b) \in G$ is just $(\frac{1}{a}, -\frac{b}{a})$, since $(a,b)(\frac{1}{a}, -\frac{b}{a}) = (1,0) = I$. Finally for any $(a,b) \in G$ we have $(a,b)(1,0) = (1,0)(a,b) = (a,b)$. Thus G is a group.

Question 7. If we apply c to both sides of $ab = 1$, we get $cab = 1 \cdot b = b = c$, as required. Since $b = c$ is a left and right inverse of a, we have $a^{-1} = b$.

For the forward direction of the second part, let $b = a^{-1}$. Then we have $aba = aa^{-1}a = a$ and $ab^2a = a(a^{-1})^2a = 1$ as required. For the backward direction, assume that $aba = a$ and $ab^2a = 1$. Then we have that ab^2 is a left inverse of a and b^2a is a right inverse, so by the first part of the question a is invertible and we have we have $ab^2 = b^2a = a^{-1}$. Applying the inverse:

$$aba = a \implies ab = 1 \text{ and } ba = 1 \implies a^{-1} = b.$$

Question 8. Since transformations of the plane can be written as matrices, from linear algebra we have:

$$\alpha = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using these to compute the given values:

$$\rho\alpha\rho^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} = \alpha^{-1}.$$

Question 11. In a group every element has an inverse, we have that $\left(ax = b \implies x = a^{-1}b\right)$ and $\left(ya = b \implies y = ba^{-1}\right)$. By closure both of these solutions are themselves in the group, so both equations always have solutions.

For the second part, assume that $ax = b$ and $ya = b$ have solutions for all $a, b \in G$. Let $a, b \in G$. By hypothesis $ax = a$ has a solution $x \in G$. Also $ya = b$ has a solution $y \in G$, and note that $bx = yax = ya = b$, which is true of all $b \in G$. By symmetry there similarly exists $x' \in G$ such that $\forall b \in G, x'b = b$. However these together give that $x'x = x$ and $xx' = x'$. Let $c \in G$, then $(xx')c = xc = x'c = c \implies xc = c$, together with the fact we found previously $cx = x$ giving us $x = 1$.

Since G has a unit, the equations $ax = 1$ and $xa = 1$ have solutions for all $a \in G$, and applying question 7 to x and y tells us that a is invertible and $x = y = a^{-1}$. Since G is a semigroup that has a unit and each element has an inverse, it is a group.

Question 13. Let G be a group for which there is no $a \in G$ with $a^2 = 1$. Note that $a^2 = 1 \implies a = a^{-1}$, so each element is distinct from its inverse except of course the identity. Enumerating all the elements in G, each pair (a, a^{-1}) adds two to the total, plus the unit for one additional element. However an even number plus an odd one is odd, so G is of odd order. Thus by contrapositive any group of even order has at least one non-1 element that is its own inverse.