Math 220 Homework 8

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Question 1. This does not need to be the case. For example let R be the equivalence class modulo 2 and S be the equivalence class modulo 3. Then if $Q = R \cup S$, then we have $0Q2$ since $0R2$ and $2Q4$ since $2S4$ but $0 \not{Q} 4$, since neither $0R4$ nor $0S4$ is true. Thus Q is not a equivalence relation since it is not transitive. $\square$

Question 2. The statement is false. To show this simply let $A = \mathbb{R}$ and $\mathcal{R} = \emptyset$. Then $\mathcal{R}$ is symmetric and transitive, but since no element is related to itself it is not reflexive.

Question 3. The relationship is an equivalence relation. For reflexivity, if $a = b$, then $5a - 8b = 5a - 8a = a(5-8) = -3a \equiv 0 \mod 0$. For symmetry, suppose aRb, i.e. $5a - 8b \equiv 0 \mod 3$. It follows that

$$5b - 8a \equiv 5b - 8a + 5a - 8b \equiv -3b - 3a \equiv 0 \mod 3$$

Finally for transitivity, assume that $5a - 8b \equiv 0 \mod 3$ and $5b - 8c \equiv 0 \mod 3$. Then we have

$$5a - 8c \equiv 5a - 8c - 5b + 8c - 5a + 8b \equiv 3b \equiv 0 \mod 3$$

Thus the relationship is an equivalence relation since it is transitive, reflexive and symmetric. $\square$

Question 4-1. $\mathcal{R}$ is reflexive and symmetric but not transitive. To show symmetric, note that if $f\mathcal{R}g \implies \exists x \in \mathbb{R}$ s.t. $f(x) = g(x)$ then $g(x) = f(x)$ as well, so $f\mathcal{R}g \implies g\mathcal{R}f$. For reflexive, assume that $f\mathcal{R}g$. Then let $x = 0$ and $f(0) = f(0)$ so $f\mathcal{R}f$.

To show it is not transitive, let $f(x) = x^2$, $g(x) = x$ and $h(x) = -x^2 - 2$. Then $f\mathcal{R}g$ by choosing $x = 0$ and $g\mathcal{R}h$ by choosing $x = -1$. However $f \not{\mathcal{R}} h$ since f is positive for all x and h is negative for all x. $\square$

Question 4-2. R is symmetric but not symmetric or transitive. To show symmetry, if xRy, then $xy \equiv yx \equiv 0 \mod 4$, so yRx. To show R isn't symmetric, note that $1 \not{R} 1$ since $1 \cdot 1 \equiv 1 \mod 4$.

To show R isn't transitive, let $x = 1, y = 4$ and $z = 3$. Then xRy and yRz since $1 \cdot 4 \equiv 0 \mod 4$ and $3 \cdot 4 \equiv 0 \mod 4$, but $x \not{R} z$ since $1 \cdot 3 = 3 \mod 4$. Therefore R isn't transitive or reflexive, but is symmetric. $\square$

Question 5. Let $x \in A$. To show R is a partition we must show that it is contained in exactly one element of R. Since P and Q are partitions, $\exists S, T$ s.t. $x \in S$ and $x \in T$. Also since each are partitions these are the only S, T that contain x. Then $x \in S \cap T \implies x \in R$ and for all other elements of P, Q, x is not contained in at least one of them. Therefore all elements of x are contained in exactly one element of R, which is the definition of a partition. $\square$

Question 6. First we will show reflexive. If $x \in A \cap B$ or $x \in \bar{A} \cap \bar{B}$, then either $x \in B \cap A$ or $x \in \bar{B} \cap \bar{A}$ since both operators work the same both ways. For symmetric, if $A\mathcal{R}B$, then either $x \in A \cap B$ or $x \in \hat{A} \cap \hat{B}$. Again since the intersect operator is symmetric, in either case it also works for A and B in reverse order. Thus $A\mathcal{R}B \implies B\mathcal{R}$ which is symmetry as required.

To show transitive, assume $A\mathcal{R}B$ and $B\mathcal{R}C$. Either $x \in B$ or $x \notin B$. In the first case then since $A\mathcal{R}B$, then $x \in A$ (since $A\mathcal{R}B$ implies either x is in both or neither of them, and x being in B implies it must be the former case), and since $B\mathcal{R}C$, $x \in C$ using the exact same logic. Then $A\mathcal{R}C$ if $x \in B$ since $x \in A$ and $x \in C$. In the second case where $x \notin B$, this means that $x \notin A$ since $A\mathcal{R}B$ (again since if x were contained in A then A wouldn't be related to B) and similarly $x \notin C$ because $B\mathcal{R}C$. Then it follows that $A\mathcal{R}C$ since $x \notin A$ and $x \notin C$. In either case $A\mathcal{R}C$, so the relation must be transitive. $\square$

Question 7-1. We will use proof by contradiction, so suppose not. Then $\exists x \in \mathbb{Z}$ s.t. either $x \notin S$ or $x \in X_a$ and $x \in X_b$, $a \neq b$. The first case is not possible, since by euclidean division by n, there exists $p \in \mathbb{Z}, r \in \mathbb{Z}$ with $0 \leq r < n$ such that $x = pn + r$, which implies $x \in X_r$. The second case would imply that $x = nk + a = nk' + b$ with $a \neq b$ and $a, b < n$. Clearly $k \neq k'$ since this would imply $a = b$, but if $k \neq k'$ then means that $n(k - k') = a - b$. However $|n(k - k')| \geq n > a > |a - b|$. Since it is a strict inequality this contradicts our assumption that $x = nk + a$ and $x = nk' + b$, so our original assumption must have been wrong and S forms a partition of $\mathbb{Z}$. $\square$

Question 7-2. R is clearly reflexive, since as we just showed in the previous part $\forall x \in \mathbb{Z}, \exists i$ s.t. $x \in X_i$. Then $aRa \forall a \in \mathbb{Z}$. For symmetric, assume that aRb. Then $\exists X_i$ s.t. $a, b \in X_i \implies bRa$. Finally for transitive, assume that aRb and bRc and $b \in X_i$ for some j. Then $a \in X_i$ and $c \in X_i$, which means aRc as required. Then since R is reflexive, symmetric and transitive it is an equivalence relation as required. $\square$

Question 7-3. First we will show that the elements of S are equivalence classes. For every pair of elements $a, b \in X_i$, aRb by the definition of R. Next we will show that S is the set of all equivalence classes of R. We proved in the first part that S forms a partition of $\mathbb{Z}$, so for every $x \in \mathbb{Z} \exists X \in S$ s.t. $x \in X$. Also since R is an equivalence relation every x can only belong to a single equivalence class. Combining these two facts we see that every integer x belongs to exactly one $X \in S$, which means that S is the set of all equivalence classes of R. $\square$

Because the series $a_1 + a_2 + \ldots a_\infty$ converges a^x $a_1 + a_2^2 + a_3^3 + \ldots + a_\infty$