Math 220 Homework 8 Question 8

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Question 8a. To show it is an equivalence relation we must show it is reflexive, symmetric and transitive. For reflexive, let $u = 1$. Then $[u]_n$ is obviously invertible since $1 \cdot 1 = 1 \mod n$, and $\forall x \in \mathbb{Z}$, $xu = x \cdot 1 = x \implies xRx$, i.e. R is reflexive.

For symmetric, assume xRy. Then $\exists u$ s.t. $[u]_n$ is invertible and $xu = y$. Let u^{-1} be this inverse. The $xuu^{-1} = x = yu^{-1}$. Since u^{-1} is invertible (it's inverse is simply u itself) then yRx as symmetry requires. Thus R is symmetric as well.

Finally for transitivity, assume xRy and yRz. Then $\exists u_1, u_2$ s.t. $xu_1 = y$ and $yu_2 = z$ and $[u_1]_n, [u_2]_n$ are invertible. If follows that $\exists u_1^{-1}, u_2^{-1}$ s.t. $u_1 u_1^{-1} \equiv 1 \mod n$ and $u_2 u_2^{-1} \equiv 1 \mod n$. Then we get that

$$x \cdot u_1 \cdot u_2 = y \cdot u_2 = z$$

Note that $[u_1 u_2]_n$ is invertible with inverse $u_1^{-1} u_2^{-1}$. Thus xRy and yRz implies that xRz, so R is transitive. Since we have shown that R is reflexive, symmetric and transitive it is an equivalence relation. $\square$

Question 8b. The invertible elements of $\mathbb{Z}_6$ are

- $[1]_6$ with inverse $[1]_6$
- $[5]_6$ with inverse $[5]_6$

Using these we find that $[2]_6 R [4]_6$, since $[2]_6 [5]_6 = [4]_6$. Also $[1]_6 R [5]_6$ since $[1]_6 [5]_6 = [5]_6$. Also note that $[3]_6 [5]_6 = [3]_6$ and $[3]_6 [1]_6 = [3]_6$ so it is only related to itself. Thus the equivalence classes of R are $\{[1]_6, [5]_6\}, \{[2]_6, [4]_6\}$ and $\{[3]_6\}$. $\square$