Math 318 Homework 5

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Question 1a. The meaning of $1 - F_M(x)$ is the probability that the minimum is above a given value x. This occurs when each of the X_i are above x, which happens with probability each $1 - x$. Thus we have

$$1 - F_M(x) = (1 - x)^n \implies F_M(x) = 1 - (1 - x)^n.$$

Question 1b. The PDF is just the derivative of the CDF:

$$f_M(x) = \frac{d}{dx} F_M(x) = n(1 - x)^{n-1}.$$

Question 1c. To find the mean we can integrate:

$$\mu = \int_0^1 nx(1 - x)^{n-1} dx = x(1 - x)^n \Big|_0^1 + \int_0^1 (1 - x)^n dx = \frac{1}{n + 1}.$$

Similarly for the variance, omitting the boundary terms since as in the previous calculation they obviously go to zero:

$$\mathrm{Var}(M) = \int_0^1 nx^2(1-x)^{n-1} dx - \frac{1}{(n + 1)^2} = \int_0^1 2x(1-x)^n dx - \frac{1}{(n + 1)^2} = \int_0^1 = \int_0^1 \frac{2}{n + 1}(1-x)^{n+1} dx$$

$$= \frac{2}{(n + 1)(n + 2)} - \frac{1}{(n + 1)^2}.$$

Question 1d. Let $y = xn$. Then the CDF of Y is

$$F_Y(y) = F_M(\frac{y}{n}) = 1 - (1 - \frac{y}{n})^n.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = (1 - \frac{y}{n})^{n-1}.$$

Question 1e. Taking the limit and using a definition of e stated in class:

$$\lim_{n\to\infty} = (1 - \frac{y}{n})(1 - \frac{y}{n})^n = \lim_{n\to\infty} (1 - \frac{y}{n}) e^{-y} = e^{-y}.$$

Question 2. Clearly $E[X] = 0.5$ and $E[Y] = -0.5$ since they're uniform distributions. We also have

$$E[X^2] = E[Y^2] = \int_0^1 x^2 dx = \frac{1}{3}.$$

Now computing the covariance using the definition

$$\operatorname{cov}(X, X^2) = E\left[(X - E[X])(X^2 - E[X^2])\right] = E[XX^2] - E[X]E[X^2]$$

$$= \int_0^1 x \frac{d}{dx'} P(X^3 < x')dx - \frac{1}{6} = \int_0^1 \frac{1}{3} x^{\frac{1}{3}} dx - \frac{1}{6} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Doing the same calculation for Y:

$$\operatorname{cov}(Y, Y^2) = E\left[(Y - E[Y])(Y^2 - E[Y^2])\right] = E[YY^2] - E[Y]E[Y^2].$$

$$= \int_{-1}^0 \frac{1}{3} y y^{\frac{-2}{3}} dy - \frac{1}{6} = -\frac{1}{4} - \frac{1}{6} = -\frac{1}{12}.$$

Question 3a. Note that $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Computing the Jacobian:

$$J(x, y) = \begin{vmatrix} \frac{\partial \frac{u+v}{2}}{\partial u} & \frac{\partial \frac{u+v}{2}}{\partial v} \\ \frac{\partial \frac{u-v}{2}}{\partial u} & \frac{\partial \frac{u-v}{2}}{\partial v} \end{vmatrix} = \frac{1}{4} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{2}.$$

Thus we have that

$$f_{U,V}(u, v) = f_{X,Y}(\frac{1}{2}(u + v), \frac{1}{2}(u - v))J(x, y) = \frac{1}{2\pi} e^{-\frac{1}{4}\left((u+v)^2 - (u-v)^2\right)} = \frac{1}{4\pi} e^{-\frac{1}{4}\left(u^2 + v^2\right)}.$$

Question 3b. Consider the characteristic functions of U and V. Then we have

$$\phi_U(t) = \phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{-t^2} \implies f_U(u) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2}.$$

A similar calculation for V:

$$\phi_V(t) = \phi_{X-Y}(t) = \phi_X(t)\phi_{-Y}(t) = e^{-t^2} \implies f_V(v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}v^2}.$$

Therefore we have that

$$f_U(u) f_V(v) = \frac{1}{4\pi} e^{-\frac{1}{4}\left(u^2 + v^2\right)}.$$

Since is the same as our result from 3a, they are independent.

Question 3c. We already calculated the marginal distribution in part b using the characteristic functions, where we found it to be

$$f_U(u) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2} = N(0, 2).$$

Question 4a. Integrating to find the marginal distribution:

$$f(x) = \int_0^\infty \lambda^2 e^{-\lambda z} 1_{0 \le x \le z} dz = \int_x^\infty \lambda^2 e^{-\lambda z} dz = \lambda e^{-\lambda x} 1_{0 \le x}.$$

Question 4b. Doing same as previous question:

$$f(z) = \int_0^\infty \lambda^2 e^{-\lambda z} 1_{0 \le x \le z} dx = \int_0^z \lambda^2 e^{-\lambda z} dx = \lambda^2 z e^{-\lambda z} 1_{0 \le z}.$$

Question 4c. As the hint suggests, we can take the derivative of the joint cumulative distribution function of X and Y. To find what this is, we can first find the joint cumulative distribution of X, Z:

$$F_{X,Z}(x,z) = \int_0^z \int_0^x \lambda^2 e^{-\lambda z'} 1_{0 \le x' \le z'} dx' dz' = \int_0^z \lambda^2 e^{-\lambda z'} \min(x, z') dz'$$

$$= \int_0^x \lambda^2 z' e^{-\lambda z'} dz' + x \int_x^z \lambda^2 e^{-\lambda z'} dz' = -\lambda z' e^{-\lambda z'} \Big|_0^x + \int_0^x \lambda e^{-\lambda z'} dz' + \lambda x e^{-\lambda x} - \lambda x e^{-\lambda z}.$$

$$= 1 - e^{-\lambda x} - \lambda x e^{-\lambda z}.$$

We can then do the calculation to find the joint density

$$f_{X,Y}(x,y) = \frac{d}{dx}\frac{d}{dy} P(X \le x, Y \le y) = \frac{d}{dx}\frac{d}{dy} P(X \le x, Z \le x + y) = \frac{d}{dx}\frac{d}{dy} F_{X,Z}(x, x+y)$$

$$= \frac{d}{dx}\frac{d}{dy}\left(1 - e^{-\lambda x} - \lambda e^{-\lambda(x+y)}\right) = \frac{d}{dx} \lambda^2 e^{-\lambda(x+y)}$$

$$= \lambda^2 e^{-\lambda(x+y)} 1_{0 \le x \le z}.$$

Question 4d. Finding the marginal pdf:

$$f_Y(y) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)} dx = \lambda e^{-\lambda y}.$$

Question 5a. The following python code was used to produce the output. The result was $P(A) \approx 0.0271$.

```
from math import comb
n = 100
p = 0.6
P = sum([comb(n, k) * p**k * (1-p)**(n-k) for k in range(0, 51)])
print(P)
```

Question 5b. Let $X_1, X_2, \ldots, X_{100}$ be a set of independent Bernoulli variables with parameter $p = 0.6$, so $X = \sum_{i=1}^{100} X_i$. The mean of X is clearly $np = 60$ and the variance of each X_i is $0.6 \cdot 0.4 = 0.24$, so by the central limit theorem we have

$$Z = \frac{X - 60}{\sqrt{24}} \approx N(0,1).$$

Question 5c. Using the fact that $X = \sqrt{24}Z + 60$, we get:

$$P(X \le 50) = P(\sqrt{24}Z + 60 \le 50) = P(Z \le -\frac{10}{\sqrt{24}}) \approx \Phi(-\frac{10}{\sqrt{24}}) \approx 0.0206.$$

Question 5d. Using the exact same approach as part c except with 51 instead of 50:

$$P(X < 51) = P(\sqrt{24}Z + 60 < 51) = P(Z < -\frac{9}{\sqrt{24}}) \approx \Phi(-\frac{9}{\sqrt{24}}) \approx 0.0331.$$

Question 5e. Taking the pattern from parts c and d, we get that

$$P(X < 50.5) = P(\sqrt{24}Z + 60 < 50.5) = P(Z < -\frac{9.5}{\sqrt{24}}) \approx \Phi(-\frac{9.5}{\sqrt{24}}) \approx 0.0262.$$

Question 6a. We can compute the probability distribution of each S_n by taking convolutions of the pdfs of S_{n-1} X_n. The following code does this, the resultant graphs can be seen in figure 1

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

ns = (1,2,3,4,5,10,50)

u = np.array([1/3, 1/3, 1/3])
S = [u]

for i in range(1, max(ns)+1):
    s = np.convolve(u, S[-1])
    S.append(s)

fig,ax = plt.subplots(nrows=4, ncols=2, sharey=True)
for i,n in enumerate(ns):
    a = ax[i//2,i%2]
    a.plot(S[n-1], label='$S_n$')
    a.title.set_text(f'n={n}')
    mu = n
    var = 2/3 * (n)

    x = np.arange(len(S[n-1]))
    a.plot(norm.pdf(x, mu, var**0.5), label=f'$N({mu},{round(var,1)})$')
    a.legend()

plt.show()
```

Question 6b. The resulting plots can be seen in figure 2. The code very similar to the previous part, it is as follows:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

ns = (1,2,3,4,5,10,50)

x = np.array([-1, 0, 1, 2, 3, 4])
y = np.array([1/15, 1/15, 11/15, 1/15, 0, 1/15])
X = [x]
T = [y]

for i in range(1, max(ns)+1):
```

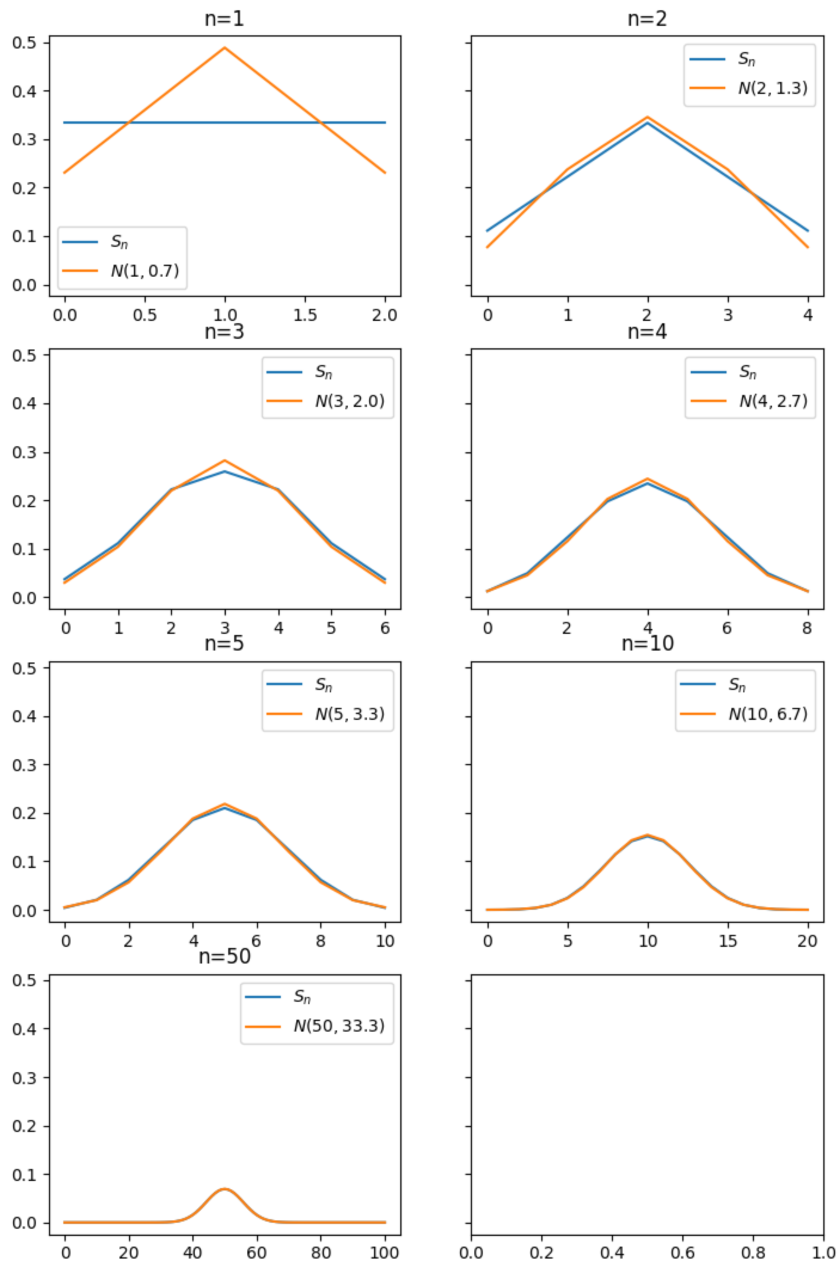

Figure 1: Graphs for question 6a

```
    first = X[-1][0]
    last = X[-1][-1]
    X.append(np.insert(np.append(X[-1], [last+1, last+2, last+3, last+4])\
        , 0, first-1))
    T.append(np.convolve(y, T[-1]))

fig,ax = plt.subplots(nrows=4, ncols=2, sharey=True)
for i,n in enumerate(ns):
    a = ax[i//2,i%2]
    a.plot(X[n-1], T[n-1], label='$S_n$')
    a.title.set_text(f'n={n}')
    x = X[n-1]
    t = T[n-1]
    mu = sum([x[i] * y for i,y in enumerate(t)])
    var = sum([(x[i]-mu)**2 * y for i,y in enumerate(t)])

    a.plot(X[n-1], norm.pdf(X[n-1], mu, var**0.5), label=f'$N({mu},{round(var,1)})$
    a.legend()

plt.tight_layout()
plt.show()
```

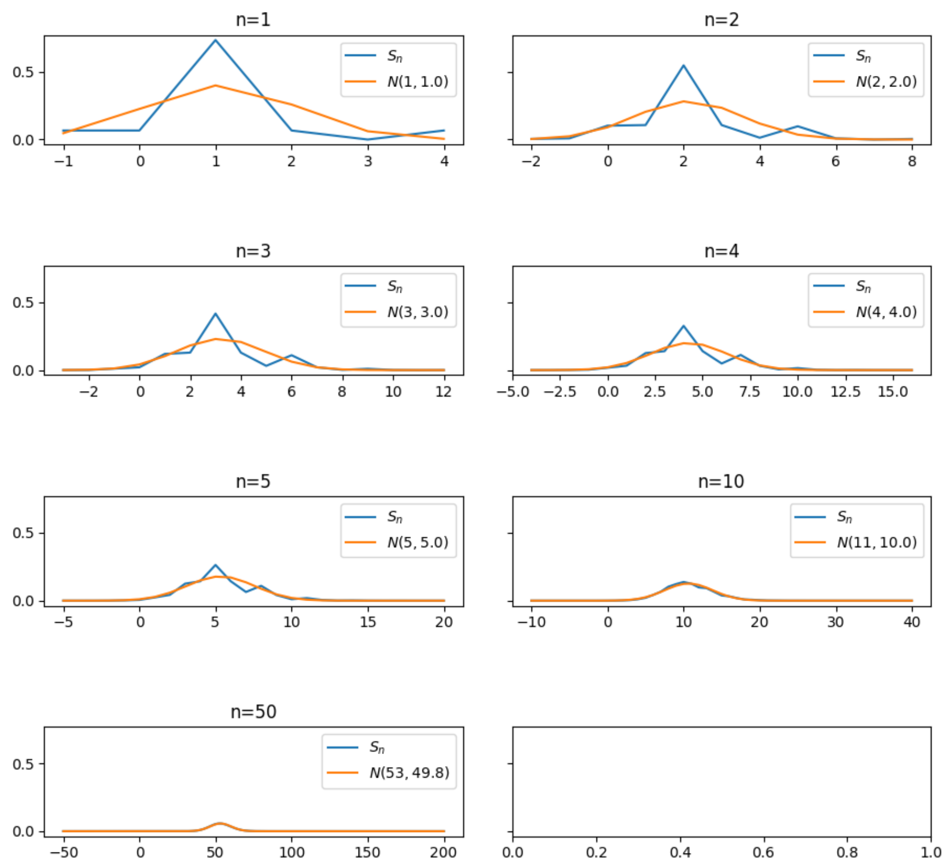

Figure 2: Graphs for 6b.