Math 320 Homework 12

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03/12/23

Question 1. Since the closure only adds points $f(E^\circ) \subseteq f(\overline{E})$ and $f(E^\circ) \subseteq \overline{f(E)}$, so we can just consider the boundary points. Let $x \in \partial E$. Let $V \subseteq Y$ be an open set with $f(x) \in V$, then $x \in f^{-1}(V)$. Since $x \in \partial E$ and f being continuous means $f^{-1}(V)$ is open, there exists $y \in f^{-1}(V)$ with $y \in E \implies V \cap f(E) \neq \emptyset$. Since this is true of all such V, $f(x) \in \overline{f(E)}$, and as x was chosen arbitrarily, $f(\overline{E}) \subseteq \overline{f(E)}$.

To witness a proper subset, consider $f : (1, \infty) \to \mathbb{R}$ defined as $f(x) = \frac{1}{x}$. Then $f(\overline{(1,\infty)}) = f([1,\infty)) = (0,1] \neq \overline{f((1,\infty))} = \overline{(0,1)} = [0,1]$.

Question 2a. Both directions:

(i) $\implies$ (ii): Let $\epsilon > 0$, and using uniform continuity find a δ that satisfies the continuity definition for every $x \in X$. Let (x_n) and (x'_n) satisfy $d_X(x_n, x'_n) \to 0$. Let N be sufficiently large so that $n > N \implies d_X(x_n, x'_n) < \delta$. Then by the uniform continuity condition we have that $d_Y(f(x_n), f(x'_n)) = d_Y(y_n, y'_n) < \epsilon$. This is exactly the definition of convergence so $d_Y(y_n, y'_n) \to 0$ as required.

(i) $\impliedby$ (ii): Contrapositive, so assume f is not uniformly continuous and let $\epsilon > 0$. Since f isn't uniformly continuous, for every $n \in \mathbb{N}$ there exists x_n, x'_n s.t. $d_X(x_n, x'_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(x'_n)) \geq \epsilon$. These $(x_n), (x'_n)$ thus contradict (ii), so not being uniformly continuous implies that statement (ii) is false. By contrapositive (ii) $\implies$ (i).

Question 2b. I claim that $p \in [0, 1]$ are the only reals that work. For $p < 0$, let $\epsilon = 1$. For any $\delta > 0$, choose $s = \min\{1, \delta\}$ and $t \in (0, 2^{1/p}s)$. Then $|s - t| = s - t < \delta$, but

$$|t^p - s^p| = t^p - s^p > 2s^p - s^p = s^p > 1.$$

Thus x^p is not uniformly continuous for $p < 0$. For $p > 1$, let $\epsilon = 1$ also. For any $\delta > 0$, choose $s = \left(\frac{2}{p\delta}\right)^{1/(p-1)}$ and $t = s + \frac{\delta}{2}$. Then by their definition $|t - s| < \delta$, but we have

$$|t^p - s^p| = t^p - s^p > (t - s)\,(x^p)'\,\big|_s = \frac{\delta}{2}p\left(\frac{2}{p\delta}\right) = 1.$$

Note the derivative part uses the fact that $(x^p)'' = p(p-1)x^{p-2} > 0 \forall x > 0$, so the largest derivative in the range (s, t) occurs at t. Thus x^p is not uniformly continuous for $p > 1$.

Finally, for $p \in [0, 1]$, let $\epsilon > 0$ and choose $\delta = \epsilon$. Then for $s, t \in (0, \infty)$ assuming without loss of generality that $t > s$ with $s - t < \delta$, we have

$$|t^p - s^p| = t^p - s^p < (t - s)(x^p)'\big|_{x=t} < \epsilon p t^{p-1} < \epsilon \cdot 1 \cdot 1 = \epsilon.$$

Again the derivative part uses the fact that $(x^p)'' = p(p-1)x^{p-2} > 0 \forall x > 0$. Thus $f(x) = x^p$ is uniformly continuous for $p \in [0, 1]$ and nowhere else.

Question 3a. A set being closed is equivalent to its complement being open, so let $y \in \mathbb{B}[x;r)$ and let $z \in \mathbb{B}[y;\frac{r}{2})$. Then using the ultrametric we have

$$d(x,y) = r \leq \max\{d(x,z), d(z,y)\} \leq \max\left\{d(x,z), \frac{r}{2}\right\}.$$

The only way this equation is satisfies is if $d(x,z) \geq r$, so $z \notin \mathbb{B}[x;r)$. Thus $\mathbb{B}[y;\frac{r}{2}) \subseteq \mathbb{B}[x;r)^c$. Therefore $\mathbb{B}[x;r)^c$ is open and thus the original ball is closed.

Question 3b. Let $z \in \mathbb{B}[y;r)$, then by the ultrametric we get

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \leq \max\{r,r\} = r \implies z \in \mathbb{B}[x;r).$$

This gives $\mathbb{B}[y;r) \subseteq \mathbb{B}[x;r)$. Next let $w \in \mathbb{B}[x;r)$. Then

$$d(w,y) \leq \max\{d(w,x), d(x,y)\} \leq \max\{r,r\} = r \implies w \in \mathbb{B}[y;r).$$

Since both sets contain each other, we then get $\mathbb{B}[y;r) = \mathbb{B}[x;r)$.

Question 3c. Suppose without loss of generality that $r_1 \leq r_2$. I claim that $\mathbb{B}[x;r_1) \subseteq \mathbb{B}[y;r_2)$. By the intersection hypothesis select $z \in X$ that is contained in both balls and let $w \in \mathbb{B}[x;r_1)$. Then by the distance ultrametric we have

$$d(w,y) \leq \max\{d(w,x), d(x,y)\} \leq \max\{d(w,x), \max\{d(x,z), d(z,y)\}\} \leq \max\{r_1, r_1, r_2\} = r_2.$$

Thus $w \in \mathbb{B}[y;r)$. Since w was arbitrary this gives the desired inclusion relationship.

Question 4. Let $x \in E^c$. Use the separation property of Y to find $U, V \in \mathcal{T}_Y$ with $f(x) \in U, f(y) \in V$ and $U \cap V = \emptyset$. Since f and g are continuous $f^{-1}(U)$ and $f^{-1}(V)$ are also open, let $W \in \mathcal{T}_X$ be their intersection. Since U and V don't intersect, $f(W) \cap g(W) = \emptyset$, i.e. every $x \in W$ is also in E^c, implying $W \in E^c$. Since we can find an open set containing x that is itself contained in E^c for every $x \in E^c$, E^c is open and E is closed.

Question 5a. Plugging in $x = y = 0$ gives $f(0) = 2f(0) \implies f(0) = 0$. Let $m = f(1)$. Note that $f(kx) = f(x)+f((k-1)x) = \ldots = f(x)+f(x)+\ldots+f(x) = kf(x)$ for $k \in \mathbb{N}$, so $f(k) = km$ for the natural numbers at least. Using the previous identity we also have $f(1) = m = nf(\frac{1}{n}) \implies f(\frac{1}{n}) = \frac{m}{n}, n \in \mathbb{N}$. Putting these two facts together gives $f(\frac{k}{n}) = kf(\frac{1}{n}) = \frac{km}{n}$, so the function is determined for all positive rationals $\frac{k}{n}$. $f(x - x) = f(0) = 0 = f(x) + f(-x) \implies f(x) = -f(-x)$, so we've determined $f(q) = mq$ for $q \in \mathbb{Q}$. A continuous function is completely determined by its behavior on a dense subset of its domain however which $\mathbb{Q}$ is, so $f(x) = mx$ for all $x \in \mathbb{R}$ (to be precise to the theorem given in the notes, if f^* is another continuous function fulfilling $f(x+y) = f(x)+f(y)$, then $f^*(x) = f(x) = mx \forall x \in \mathbb{R}$).

Question 5b. Consider swapping x and y:

$$f(x + y) = g(x) + h(y) = f(y + x) = g(y) + h(x) \implies g(x) - g(0) = h(x) - h(0).$$

Let $g(0) = b$ and $c = h(0)$, then we have $h(x) = g(x) - b + c$. Plugging this into the identity gives $f(x + y) = g(x) + g(y) - b + c$. Computing $f(x + y)$ two different ways gives

$$f(x + y) = g(x) + g(y) + b - c = g(x + y) + g(0) - b + c \implies g(x + y) = g(x) + g(y) - g(0).$$

This is almost identical to the identity we saw in part a, so we can solve it in similar manner. Let $g(0) = b$. First note that $g(kx) = g(x) + g((k - 1)x) - b = \ldots = kg(x) - (k - 1)b$. Using this gives $g(1) = g(n\frac{1}{n}) = ng(\frac{1}{n}) - (n - 1)b \implies g(\frac{1}{n}) = \frac{1}{n}(g(1) - b) + b$. Let $m = g(1) - b$. Applying the previous identity once again to this new equation gives

$$g\left(\frac{k}{n}\right) = k\left(\frac{m}{n} + b\right) - (k - 1)b = m\frac{k}{n} + b.$$

Also $g(x - x) = g(0) = g(x) + g(-x) + g(0) \implies g(x) = -g(-x)$, so we've specified $g(x)$ on the rationals. Since it's continuous, by the same logic as in part a we've also determined it to be $g(x) = mx + b$ for all $x \in \mathbb{R}$. Building the other functions back, the final most generalized form is $g(x) = mx + b$, $h(x) = g(x) - b + c = mx + c$ and $f(x) = g(x) + g(0) - b + c = mx + b + c$.

Question 6a. Since $x \in U$ and U is open, there exists an open interval contained in U that contains x, so $I(x)$ is nonempty. Since it's nonempty and $\alpha(x), \beta(x)$ are defined in such a way that $\alpha(x) < x < \beta(x)$, we must have that $x \in I(x)$. Let $y \in I(x)$, suppose for now that $\alpha(x) < y < x$. Since $y > \alpha(x)$ there exists an interval $\left(y - \frac{y-\alpha(x)}{2}, b\right) \subseteq U$ if $\alpha(x) > -\infty$ or $(y-1, b) \subseteq U$ otherwise for some $b \in \mathbb{R}$ with $b > x$. y is in that interval, so $y \in U$. If instead $y > x$, the exact same argument works in reverse by symmetry, so $I(x) \subseteq U$.

To show $\alpha(x) \notin U$, by contradiction suppose that it was. Since this statement wouldn't make sense if $|\alpha(x)| = \infty$, assume $|\alpha(x)| < \infty$. Then since U is open, $\exists r \in \mathbb{R}$ s.t. $(\alpha(x)-r, \alpha(x)+r) \subseteq U$. Let z be in this interval such that $z < \alpha(x)$. Then the interval $(z, \beta(x))$ is contained in U with $x \in (z, \beta(x))$, so $\alpha(x)$ wasn't chosen to be minimal, contradiction. Thus $\alpha(x) \notin U$. The exact same argument works with signs flipped to show $\beta(x)$ also isn't contained in U.

Question 6b. By contradiction suppose that it wasn't true. Then there exists two intervals, $I(x), I(y)$, such that $I(x) \cap I(y) \neq \emptyset$ but $I(x) \neq I(y)$. Let $\alpha = \min\{\alpha(x), \alpha(y)\}$ and $\beta = \max\{\beta(x), \beta(y)\}$. Then $x, y \in (\alpha, \beta)$, I claim also $(\alpha, \beta) \subseteq U$. Let $z \in I(x) \cap I(y)$. Then we get

$$(\alpha, \beta) = (\alpha, z] \cup [z, \beta) \subseteq (\alpha(x), \beta(x)) \cup (\alpha(y), \beta(y)) \subseteq U$$

Thus $(\alpha, \beta) \subseteq U$ with $x, y \in (\alpha, \beta)$. But by hypothesis either $\alpha(x) \neq \alpha(y)$ or $\beta(x) \neq \beta(y)$, so one of those wasn't chosen to be maximal. This gives a contradiction, so $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$ after all.

Question 6c. By its definition $\mathcal{G}$ is a set of disjoint open intervals whose union is U, so all that remains is to prove that it is countable or finite. Let $S = \left\{\frac{\alpha+\beta}{2} : (\alpha, \beta) \in \mathcal{G}\right\}$, i.e. the midpoints of all the intervals. $|S| = |\mathcal{G}|$, so it's sufficient to prove that S is countable. Let $x = \frac{\alpha+\beta}{2} \in S$, and consider whether it's a limit point or not. Since x is a midpoint for the interval (α, β) and each interval in $\mathcal{G}$ is disjoint, we have that $(\alpha, \beta) \cap S = \emptyset$. Thus $x \notin S' \implies S \cap S' = \emptyset$. By the contrapositive of homework 11 question 4, this gives us that S isn't uncountable, i.e. it is countable or finite as required.