

Math 318 – homework 7 – due 2023-03-17

Problem 1. (a) If X is an integer valued random variable, show that the characteristic function of X has period 2π .

- (b) Prove the converse: if $\phi(t) = \phi(t + 2\pi)$ for every t , then show that X takes only integer values. (Hint: If a random variable Y satisfies $Y \geq 0$ and $E[Y] = 0$ then $P(Y = 0) = 1$. Use this for a carefully chosen function of X . Hint: consider $\phi(2\pi)$.)

Problem 2. Let U, V be two independent random variables, with $E[U] = E[V] = 0$, and let $X = U + V$. Assume also that U and V have finite moments. Find an expression for $E[X^3]$ and for $E[X^4]$, in terms of moments of U, V .

Problem 3. Let A be an $n \times n$ matrix, and let $X = (X_1, \dots, X_n)^T$ be a vector of i.i.d. $N(0, 1)$ random variables. Let $Y = AX$, and Y_i its coordinates.

- (a) What is the distribution of Y_i ?
(b) Find $\text{Cov}(Y_i, Y_j)$.
(c) If $n = 2$ and A is invertible, show that Y_1, Y_2 have joint probability density function

$$\frac{1}{(2\pi)^{n/2} |\det A|} e^{-(y^T A^T A^{-1} y)/2}$$

on \mathbb{R}^2 , where $y = (y_1, y_2)^T$ is a vector.) (Hint: What is the Jacobian of the mapping from X to Y ?)
bonus For any n , if A is invertible, show that Y has probability density function with the same formula.

Problem 4. Consider the random walk in one dimension \mathbb{Z} with bias. We will consider $p = 0.5$, $p = 0.51$, and $p = 0.502$. Let $X_i = \pm 1$ with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ be independent. Let $S_n = X_1 + X_2 + \dots + X_n$. For each of these, do the following.

- (a) Simulate and plot a random walk with 10^6 steps. Submit your code and a plot of S_0, \dots, S_n .
(b) Simulate 1000 independent random walks, each for up to 10^6 steps. For each walk, let $T > 0$ be the first time the walk returns to 0. If a walk does not return to 0, let $T = 10^6$. For each of the p 's, how many of the 1000 walks did not return to 0? Submit histograms of the values of T observed.
(c) Make a log-log plot of the fraction of times $T > k$ for $k = 0, \dots, 10^6$.
(d) Based on the previous plot, guess the asymptotics of $P(T > n)$ as $n \rightarrow \infty$. What do you think is the mean of T if there was no limit of 10^6 ?

Problem 5. This problem investigates the recurrence of 2-dimensional random walk. Let (X_n, Y_n) be a two dimensional random walk. At each step pick either X or Y and either increase or decrease it by 1, all with equal probabilities, and independent of other steps.

- (a) Simulate this process for 10^6 steps. Submit code and a plot of the walk in the $X - Y$ plane (there is no coordinate for time, though you could use colour to indicate the time a point is visited).
(b) Simulate 1000 random walks for up to 10^6 steps, and for each keep track of the number of steps before it returns to 0 for the first time. How many of the walks failed to return to 0? Is this consistent with the theorem that the random walk is recurrent?
(c) Make a histogram of the resulting return times for the walks that did return, on a logarithmic scale. Can you guess how fast $P(T > n)$ decays from this?

Note. it is much faster to only generate each random walk until it returns to 0, and stop there. Otherwise this make take a long time to run.

Extra practice problems