Math 322 Homework 10

Xander Naumenko

19/11/23

Question 4.6.1. The backwards direction is clear, as stated in the textbook (top of page 249) every finite group has a composition series so in particular finite abelian groups do too. For the forward direction, suppose by contradiction that G is an infinite abelian group and $G = G_1 \rhd \ldots \rhd G_{s+1} = \{1\}$ is a composition series. Since this is a composition series each of the G_i/G_{i+1} are simple, and since they are also abelian they must be of prime order. But then we have:

$$|G| = [G_1 : G_2]|G_2| = |G_1/G_2|\,|G_2| = |G_1/G_2| \ldots |G_s/G_{s+1}| < \infty.$$

This contradicts our assumption that G was infinite though, so no such composition series can exist.

Question 4.6.2. To show that p_i is a prime, by contradiction assume that it isn't, i.e. assume that $n_i/n_{i+1} = aq$ for some $i \in \mathbb{N}, a, q > 1$ and q prime. G_i/G_{i+1} is cyclic and thus abelian, so every subgroup is normal. Also by Cauchy's theorem, there exists a subgroup H of order q in G_i/G_{i+1} since it has order $n_i/n_{i+1} = aq$. But then $G_i \supset HG_{i+1} \supset G_{i+1}$ contradiction our assumption that G_{i+1} is maximal, so in fact p_i was a prime.

Let $n = n_1, n_2, \ldots, n_{s+1} = 1$ be a sequence of integers such that $p_i = n_i/n_{i+1}$ is prime. Construct the composition series $G_1 = G = \langle a \rangle$, $G_2 = \langle a^{n/n_2} \rangle$, $G_3 = \langle a^{n/n_3} \rangle, \ldots, G_s = \langle a^{n/n_s} \rangle, G_{s+1} = \{1\}$. Then each G_i/G_{i+1} has order $\frac{n_{i+1}}{n_i} = p_i$ prime, so every subgroup of G_i/G_{i+1} is either the identity or the whole group, which means that G_{i+1} is maximal normal. Also $G_i \rhd G_{i+1}$ for all i, so this is a valid composition series with the required properties.

Question 4.6.3. Expanding and using the fact that $(g, h)^{-1} = (h, g)$:

$$(g, hk) = g^{-1}k^{-1}h^{-1}ghk = (g, k)(gk)^{-1}h^{-1}ghk = (g, k)(g^{-1}h^{-1}gh)^k = (g, k)(g, h)^k.$$

$$(gh, k) = h^{-1}g^{-1}k^{-1}ghk = h^{-1}g^{-1}k^{-1}g(kh)(h, k) = (g, k)^h(h, k).$$

$$(g^h, (h, k))(h^k, (k, g))(k^g, (g, h)) = (g^h)^{-1}(k, h)g^h(h, k)(h^k)^{-1}(g, k)h^k(k, g)(k^g)^{-1}(h, g)k^g(g, h)$$
$$= (h^{-1}g^{-1}hk^{-1}h^{-1}kgk^{-1}hk)\left(k^{-1}h^{-1}kg^{-1}k^{-1}ghg^{-1}kg\right)\left(g^{-1}k^{-1}gh^{-1}g^{-1}hkh^{-1}gh\right) = 1.$$

Question 4.6.4. Let $h \in H, k \in K$. Then since (K, H) is a group $(k, h)^{-1}(k^{-1}h^{-1}kh)^{-1} = h^{-1}k^{-1}hk = (h, k) \in (K, H)$. Since h, k were arbitrary we have that (K, H) contains all the generators of (H, K), so $(H, K) \subseteq (K, H)$. By symmetry the exact same argument works in reverse to show $(H, K) \supseteq (K, H)$, so $(H, K) = (K, H)$.

To show normality, let $g \in G$ and $x \in (H, K)$. Since (H, K) is generated by commutators (h, k) it can be broken up as $x = (h_1, k_1) \cdots (h_n, k_n)$. Then we have

$$g^{-1}xg = (g^{-1}(h_1, k_1)g)(g^{-1}(h_2, k_2)g) \cdots (g^{-1}(h_n, k_n)g).$$

Thus it suffices to show that $g^{-1}(h,k)g \in (H,K) \forall h \in H, k \in K$. Since H and K are normal we have $g^{-1}hg \in H, g^{-1}kg \in K$. Then:

$$g^{-1}(h,k)g = g^{-1}h^{-1}gg^{-1}k^{-1}gg^{-1}hgg^{-1}kg = (g^{-1}hg)^{-1}(g^{-1}kg)^{-1}(g^{-1}hg)(g^{-1}kg) \in (H,K).$$

Since all the generators are expressible in terms of one another $(H,K) = (K,H)$ are normal in G.

Herstein, Question 2.13.11a. There is exactly one subgroup of order q, since Sylow II says that n_q (the number of q-Sylow subgroups) divides the index of q-Sylow subgroups which is p, but since $p < q$ this forces $n_q = 1$. As for p, Sylow II forces $n_p \equiv 1 \mod p \implies n_p = pk + 1, k \in \mathbb{N} \cup \{0\}$. But then $pk + 1 | q$, and since q is a prime either $pk + 1 = q$ or $k = 0$. The first case can't happen though since then $pk = q - 1 \implies p | q - 1$ which it doesn't by hypothesis, so $n_p = p(0) + 1 = 1$ as well.

Therefore there are unique p, q-Sylow subgroups. Since $p < q$ it can't be that $p = q = 2$, so $p + q < pq$. Therefore there's at least one element $g \in G$ s.t. g isn't in either the p-Sylow or q-Sylow subgroup. g can't have order 1 since then it would be the identity and be in both, it can't have order p since then $\langle g \rangle$ would be a p-Sylow subgroup (and g was chosen not to be in any), and similarly it can't have order q since then $\langle g \rangle$ would be a q-Sylow subgroup. Therefore the only possibility is that $|g| = pq$, so $G = \langle g \rangle$ is cyclic.