Math 220 Homework 4

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Question 1. We will prove this by induction on n.

Base case (n=0): When $n = 0$, $n^3 + (n+1)^3 + (n+2)^3 = 1^3 + 2^3 = 9$ which is divisible by 9.

Inductive step: Assume the result holds for n, i.e. $9|n^3 + (n+1)^3 + (n+2)^3$. Then plugging in $n + 1$, we get

$$(n+1)^3+(n+2)^3+(n+3)^3 = n^3+(n+2)^3+9n^2+27n+27 \equiv 9n^2+27n+27 \mod 9 = 0 \mod 9$$

Thus by induction the result holds for every n. □

Question 2. Let $a, b, c \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and assume $a|bc$. Then by Bézout's identity we have that $\exists x, y \in \mathbb{Z}$ s.t. $ax + by = 1$. Also, based on the assumption of divisility we have that $\exists m \in \mathbb{Z}$ s.t. $bc = ma \Rightarrow b = \frac{ma}{c}$. Substituting we get

$$ax + \frac{may}{c} = 1 \Rightarrow axc + may = c = a(xc + my)$$

This means that c is divisible by a since $xc + my$ is an integer. □

Question 3a. This statement is false, which can be shown with a counterexample. Let $x = 3 \in P, y = 3 \in P$, so $x + y = 6 \notin P$ which contradicts the statement. □

Question 3b. This statement is false. To show this let $x = 7 \in P$, and let $y \in P$ be arbitrary. There are two cases: either P is odd or even. If it is even then the only even prime is two, so $x + y = 7 + 2 = 9 \notin P$. If y is odd, then it can be expressed as $y = 2m + 1$ and we have $x + y = 7 + 2m + 1 = 2(4m + 1)$ which can't be prime, since it's divisible by two. In either case the result can't be prime, so the original statement is false.

Question 3c. This statement is false. Let $x \in P$. If $x = 2$ then let $y = 2$, so $x + y = 2 + 2 = 4 \notin P$. If $x \neq 2$ then x is odd and let $y = 3$. Then we can express $x = 2m + 1, m \in \mathbb{Z}$ and $x + y = 2m + 1 + 3 = 2(m + 2)$ which isn't prime since it's divisible by two. Thus the original statement was false.

Question 3d. This statement is true. To show this choose $x = 2$ and $y = 3$, so $x, y \in P$ and $x + y = 3 + 2 = 5 \in P$ and we're done. □

Question 4. Let $\epsilon > 0$, and let $M = \frac{2}{\sqrt{\epsilon}}$. Then for $x \geq M = \frac{2}{\sqrt{\epsilon}}$, we have

$$\left|\frac{2x^2}{x^1+1} - 2\right| = \left|\frac{-2}{x^2+1}\right| < 2\left|\frac{1}{x^2}\right| \leq 2\left|\frac{\epsilon}{4}\right| = \frac{\epsilon}{2} < \epsilon$$

This matches the result so we're done. □

Question 5. f is continuous at $x = 0$. Let $\epsilon > 0$ and choose $\delta = \sqrt{\epsilon}$. Then $\forall |x| < \delta$, we have

$$|x^2 \sin(\frac{1}{x}) - 0| \leq |x^2| < |(\sqrt{x})^2| = |\epsilon| = \epsilon$$

This means that $\lim_{x\to a} f(x) = 0 = f(0)$ so f is continuous. $\square$

Question 6. By definition if (x_n) converges to 0, then $\forall\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|x_n| < \epsilon \forall n > N$. Let $\epsilon = 1$, and using the previously stated definition $\exists N \in \mathbb{N}$ s.t. $|x_n| < 1 \forall n > N$. Since N is finite we can define $M' = \max(x_0, x_1, \ldots, x_N)$. Let $M = \max(M', 1)$. (x_n) is bounded by M since $\forall n \in \mathbb{N}$, either $n < N$ and $|x_n| \leq M' \leq M$ or $n > N$ and $|x_n| < \epsilon = 1 \leq M$. Thus (x_n) is bounded. $\square$

Question 7a. Let $M \in \mathbb{R}$. If $M \leq 1$ then let $t = \frac{1}{e^2} \in (0,1)$, so $|f(t)| = |\log e^{-2}| = 2 > 1 \geq M$. Otherwise let $t = e^{-M-1} \in (0,1)$ and we have $|f(t)| = |\log e^{-M-1}| = |-M-1| = M+1 > M$. In either case it is not bounded so we're done. $\square$

Question 7b. Let $c = \frac{3}{4} \in (1/2, 3/1)$ and let $M \in \mathbb{R}$. If $M \leq 0$ then the result follows trivially so assume $M > 0$. Next let $t = \frac{3}{4} - 1/(4\sqrt{M+1})$. Then we have

$$|f(t)| = \left|\frac{(1-t)^2}{(\frac{3}{4} - 1/(4\sqrt{M+1}) - \frac{3}{4})^2}\right| \geq \left|\frac{(1/4)^2}{1/(16(M+1))}\right| = |M+1| > M$$

This fulfills the definition so we're done. $\square$