Math 220 Homework 5

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October 18, 2021

Question 1. We will use induction on n.

Base Case (n=1): Computing the sum we get

$$\sum_{k=1}^{1}(2k-1)2^k = (2-1)2 = 2 = 6 + 2(4-6) = 6 + 2^n(4n-6)$$

Induction Step: Assume that $\sum_{n=1}^{n}(2k-1)2^k = 6 + 2^n(4n-6)$. Then we have

$$\sum_{k=1}^{n+1}(2k-1)2^k = 6 + 2^n(4n-6) + (2n+1)2^{n+1} = 6 + 2^n(4n-6) + (4n+2)2^n$$

$$= 6 + 2^n(4n-6) + (4(n+1)-6)2^n = 6 + (4(n+1)-6)2^{n+1}$$

This matches the result so we're done. $\square$

Question 2. We will use strong induction on n for $n \geq 3$.

Base Case (n=3): Using our definition for a_n we have

$$a_3 = 5a_2 - 6a_1 = 25 - 6 = 19 = 27 - 8 = 3^3 - 2^3$$

Induction Step: Suppose that the result holds for all natural numbers less than $n+1$. Then we have

$$a_{n+1} = 5a_n - 6a_{n-1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) = (3^{n+1}) - (-2^{n+1}) = 3^{n+1} + 2^{n+1}$$

This is what we are trying to prove so we're done. $\square$

Question 3. We will use strong induction on n.

Base Case (n=3): Using our definition we have

$$a_3 = a_2 + a_1 + a_0 = 1 + 3 + 9 = 14 \leq 27 = 3^n$$

Inductive Step: Assume that $a_m \leq 3^m$ for all $m \leq n$. Then We have

$$a_{n+1} = a_n + a_{n-1} + a_{n-2} \leq 3^n + 3^{n-1} + 3^{n-2} \leq 3^n + 3^n + 3^n = 3^{n+1}$$

Thus the result holds for all $n \geq 3$. $\square$

Question 4. We will use strong induction on n.

Base Case (n=1): By explicit definition we have that

$$f_{n+1}f_{n-1} - f_n^2 = (1+0)0 - 1 = -1 = (-1)^1 = (-1)^n$$

Inductive Step: Assume that the result holds for all values less than or equal to n, which gives $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$. Then we get

$$f_{n+2}f_n - f_{n+1}^2 = (f_{n+1} + f_n)f_n - f_{n+1}(f_n + f_{n-1}) = f_{n+1}f_n + f_n^2 - f_{n+1}f_n - f_{n+1}f_{n-1}$$

$$= -(f_{n+1}f_{n-1} - f_n^2) = -(-1)^n = (-1)^{n+1}$$

Thus by induction the result holds for all $n \geq 1$. $\square$

Question 5. We will prove the result by induction on n.

Base Case ($n = 0$): Plugging in we get $7^{0+3} + 2 = 343 + 2 = 345 = 0 \mod 5$.

Induction Step: Assume that the result holds for n. Then we have that $\exists m \in \mathbb{Z}$ s.t. $7^{4n+3} + 2 = 5m$ and so

$$7^{4(n+1)+3} + 2 = 7^3 7^{4n+3} + 2 = 7^4(5m - 2) + 2 = 7^4 \cdot 5m - 4802 + 2 = 5(7^4 m - 960) = 0 \mod 5$$

Thus by induction the result holds for all nonnegative integers n.

Question 6. We will use induction to prove that a_n is increasing and greater than 2 for all n.

Base Case (n=1): $a_1 = 3 < 6 = 9 - 3 = a_1^2 - a_1 = a_2$, and $a_1 = 3 > 2$.

Induction Step: Assume that $a_n > a_{n-1}$ and $a_n > 2$. Then because $a_n > 2$ then $a_n - 1 > 1$, so

$$a_{n+1} = a_n^2 - a_n = a_n(a_n - 1) > a_n$$

Thus the result holds for all n. $\square$

Question 7. We will use strong induction on n.

Base Cases ($n = 0$ and $n = 1$): For $n = 0$ then we have $u_0 = 1 = \cos(0) = \cos(nx)$. For $n = 1$ we get $u_1 = \cos x = \cos(nx)$ so the base case is fulfilled.

Inductive Step: Assume that the result holds for all numbers less than $n + 1 > 1$. Then

$$u_{n+1} = 2u_1 u_n - u_{n-1} = 2\cos(x)\cos(nx) - \cos((n-1)x)$$

$$= 2\cos(x)\cos(nx) - \cos(nx)\cos(x) - \sin(nx)\sin(x) =$$

$$\cos(x)\cos(nx) - \sin(nx)\sin(x) = \cos((n+1)x)$$

Thus the result holds for all n. $\square$