

Math 437 Homework 3

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Question 1. We are trying to find instances of when $a_n \equiv 0 \pmod{2023}$, so consider the recurrence relation $\pmod{2023}$. All future values of the sequence are determined by the 5-tuple $(a_n, a_{n+1}, a_{n+2}, n^2, 5^n)$, each value within being modulo 2023. There are 2023 possibilities for the first 4 and $\text{ord}_{2023} 5 = 816$ possibilities for the last, so by the pigeonhole principle the sequence must repeat after at most $2023^4 \cdot 816 \approx 1.367 \cdot 10^{16}$ steps. Since it repeats infinitely from then on out, all we must do is show that there is at least one 0 within the repeating section. I claim that $a_0 = 0$ is in the repeating section which fulfills this requirement.

To see why, let p be the period of repetition (i.e. the smallest number such that $a_m \equiv a_{m+p}, 5^m \equiv 5^{m+p}, m^2 \equiv (m+p)^2 \pmod{2023}$ for all m sufficiently large, this is guaranteed to exist as shown above) and n be the smallest number such that $a_m \equiv a_{m+p}, 5^m \equiv 5^{m+p}, m^2 \equiv (m+p)^2 \pmod{2023} \forall m \geq n$. In particular, $a_n \equiv a_{n+p}, a_{n+1} \equiv a_{n+p+1}, a_{n+2} \equiv a_{n+p+2} \pmod{2023}$. Note that $816|p$ and $2023|p$ due to the 5^m and m^2 requirement and the fact that 2023 isn't a perfect square. Then consider a_{n-1} , using the fact that $11 \cdot 184 \equiv 1 \pmod{2023}$:

$$\begin{aligned} a_{n-1} &\equiv 184(a_{n+2} - 5^n a_{n+1} - n^2 a_n) \equiv 184(a_{n+p+2} - 5^{n+p} a_{n+p+1} - (n+p)^2 a_{n+p}) \pmod{2023} \\ &\equiv a_{n+p-1} \pmod{2023}. \end{aligned}$$

Since $5^{n-1} \equiv 1214 \cdot 5^n \equiv 1214 \cdot 5^{n+p} \equiv 5^{n+p-1} \pmod{2023}$ and $(n-1)^2 \equiv n^2 - 2n + 1 \equiv (n+p)^2 - 2(n+p) + 1 \equiv (n+p-1)^2 \pmod{2023}$, this contradicts our assumption that n was chosen to be minimal. The only way this doesn't lead to a contradiction is if $n = 0$ as $a_{n-1} = a_{-1}$ isn't defined. Thus $a_0 \equiv 0 \pmod{2023}$ is in the repetition and $2023|a_n$ infinitely many times. \square

While this concludes the proof, alternatively to carefully proving that a_0 is in the repeating section one also could have just brute force searched for a repeating 5-tuple in the form above and checked that the repeating section contains a zero. Here's some python code to do so, it turns out to repeat with period $p = 4660992$ and $n = 0$ as expected.

```
N = 10000000
a = [0,1,2] + [0]*(N-3)

seen = {}
repeat_n = -1
repeat_h = ()

for n in range(0,N-3):
    a[n+3] = (5**(n%816)*a[n+2]+n**2*a[n+1]+11*a[n])%2023
    h = (n%816, (n**2)%2023, a[n+2], a[n+1], a[n])
    if h in seen:
        print(f'Found at n={n}')
```

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        repeat_n = n
        repeat_h = h
        break
    seen[h] = n

if repeat_n == -1:
    print('No repeat found')
else:
    n1 = seen[repeat_h]
    n2 = repeat_n
    print(f'Found repeat at n1={n1}, n2={n2}')
    print(repeat_h)
    print('Searching for zeros...')
    for n in range(n1, n2+1):
        if a[n] == 0:
            print(f'Found zero at n={n}')
            print(f'Sanity check: n1={n1}, a[{n1}]=a[{n1}], a[{n1}+1]=a[{n1+1}], a
                ↳ [{n1}+2]=a[{n1+2}], (5^{n1})%2023={(5**n1)%2023}, ({n1}^2)
                ↳ %2023={(5**n1)%2023}')
            print(f'Sanity check: n2={n2}, a[{n2}]=a[{n2}], a[{n2}+1]=a[{n2+1}], a
                ↳ [{n2}+2]=a[{n2+2}], (5^{n2})%2023={(5**n2)%2023}, ({n2}^2)
                ↳ %2023={(5**n2)%2023}')
            break

```

Question 2. Factoring the equation, we get $(x-1)(x^2+x+1) \equiv 0 \pmod{n}$. Thus $x=1$ is always a solution

Question 3. I will construct the sequence $\{n_k\}_{k \geq 1}$ and show that the sequence is valid. They will be in the form $n_k = \prod_{i=1}^k p_i^{a_i}$, p_i is the i th prime and $a_i \in \{0, 1\}$ are shared between all n_k (i.e. a_i isn't a function of k), so the sequence a_i uniquely defines the sequence n_k .

Let $\alpha \in [0, 1]$ and $a_1 = 1$. Note that for n_k in the form above we have $n_k = p_k^{\alpha} n_{k-1}$.