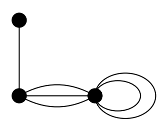
Homework 2, Math 443

Due Wednesday, February 1, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (4 points)

   A **multigraph** consists of a finite set V and a multiset E, where every element of E has the form $\{x, y\}$ for some (not necessarily distinct) $x, y \in V$.
   Multigraphs are like graphs, but we allow parallel edges (that is, multiple edges with the same endpoints) and 'loops' (edges of the form xx).
   The degree of a vertex in a multigraph is the number of times it shows up in E: so it's the number of edges it's incident to (including multiplicity), plus twice the number of loops it's in. So, for example, a degree sequence of the multigraph below is $1, 4, 7$.

   Prove that a nonnegative integer list $d_1, \ldots, d_n$ $(n \geq 1)$ is the degree sequence of a multigraph if and only if

   $$\sum_{i=1}^{n} d_i \text{ is even.}$$

2. (4 points) Consider the non-increasing sequence of non-negative integers

   $$s\colon\, d_1, d_2, d_3, \cdots, d_n$$

   and use it to define

   $$s_1\colon\, \underbrace{d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1}_{d_1}, d_{d_1+2}, d_{d_1+3}, \cdots d_n$$

   Suppose some entry in s_1 is negative. Using first principals (in particular, not using Havel-Hakimi), prove that s is not graphical.

3. (6 points) The Erdős-Gallai Theorem [1960] states:

   > The nonnegative integer list $d_1, \ldots, d_n$, with $d_1 \geq d_2 \geq \cdots \geq d_n$, is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and
   > $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$
   > for every $k \in [n]$.

Suppose the sequence from the theorem is graphic. Show that $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for every $k \in [n]$.
(That is: prove the necessity of the condition of the theorem. We won't prove sufficiency.)

4. (4 points) A graph G has adjacency matrix A, and

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

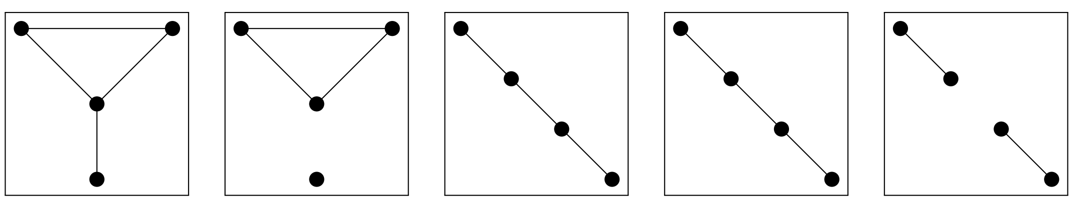
Give a degree sequence of G.

5. (4 points) A **principal submatrix** of a square matrix A is a submatrix selecting rows and columns with the same indices.

Suppose B is a principal submatrix of A, and A is the adjacency matrix of a graph G. Let I be the set of indices of rows/columns that were deleted from A to make B.

Describe the graph with adjacency matrix B, in terms of G and I.

6. (5 points) Find all graphs G that have the following deck of vertex-deleted subgraphs:



7. (4 points) Prove the following statement, or provide a counterexample:

Let G be a regular graph with $|G| > 2$. Then G is reconstructible.

8. Given graphs G and Q, define
$$s_Q(G)$$
to be the number of subgraphs of G isomorphic to Q.
   (a) (2 points) To get used to this definition, explain why $s_{K_3}(K_7) = \binom{7}{3}$, and $s_{P_3}(K_7) = 3\binom{7}{3}$.
   (b) (5 points) Let Q and G be graphs with $|Q| < |G|$. Show that $s_Q(G)$ is recognizable.
   (Hint: we showed this in class for $Q = P_2$.)
   (c) (2 points) Show that if $s_Q(G)$ is recognizable for all graphs Q with $|Q| = |G|$, then the reconstruction conjecture is true.

Question:	1	2	3	4	5	6	7	8	Total
Points:	4	4	6	4	4	5	4	9	40