PHYS 403 Homework 4

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Question I. As the hint suggests, first compute $\langle N_{\vec{p}} \rangle$. Since the polarization only changes things by a constant factor 2 as we saw in class, this factor is multiplied at the beginning and the polarization is ignored.

$$\langle N_{\vec{p}} \rangle = 2V \frac{\sum_{N=0}^{\infty} N e^{-\beta c p N}}{\sum_{N=0}^{\infty} e^{-\beta c p N}} = -2V \frac{\partial}{\partial(\beta c p)} \log\left(\frac{1}{1-e^{-\beta c p}}\right) = \frac{2V}{e^{\beta c p}-1}.$$

We can convert the sum given in the hint to an integral similarly to how we did in class, by adding a factor of $(2\pi\hbar)^3$ and integrating over continuous momentum:

$$\langle N \rangle = 2V \int \frac{dp^3}{(2\pi\hbar)^3} \frac{1}{e^{\beta c p}-1} = \int \frac{\omega^2 V}{\pi^2 c^3 \hbar} \frac{1}{e^{\beta\hbar\omega}-1} d\omega = \frac{(k_B T)^3 V}{\pi^2 c^3 \hbar^4} \int \frac{u^2}{e^u - 1} du.$$

The value of the integral doesn't matter, it's clearly bounded so it's just a constant factor. Thus the T^3 dependence in $\langle N \rangle$ is clear.

Question II1. From the geometry of the planets, the sun absorbs $\frac{\pi(6.371\cdot 10^6)^2}{4\pi(1.5\cdot 10^{11})^2} = 4.51 \cdot 10^{-10}$ of the sun's radiation.

$$\frac{dE_{earth}}{dt} = P_{in} - P_{out} = \sigma \cdot \left(4\pi \cdot (7\cdot 10^8)^2\right) T_s^4 \cdot 4.51 \cdot 10^{-10} - \sigma \cdot \left(4\pi \cdot (6.371 \cdot 10^6)^2\right) T_e^4.$$

$$= 1.5696 \cdot 10^2 T_s^4 - 2.892 \cdot 10^7 T_e^4.$$

Question II2. The steady state corresponds to when the left side the above equation is 0, so plugging in $T_s = 6000\text{K}$ and solving for T_e gives $T_e = 289.6\text{K}$.

Question III1. Recall from in class, the expression for energy in three dimensions is (similarly to question 1):

$$\langle E \rangle = 2V(4\pi) \int \frac{p^2 dp}{(2\pi\hbar)^3} \frac{cp}{e^{\beta c p}-1} = AT^4.$$

Here A is just an arbitrary constant since we just care about functional dependence. Note that 2 of the powers of T came from the p^2 in front which is from the conversion from the 3 dimensional to the 1 dimensional integral, 1 is from the change of variables of $u = \beta c p$, and the last one is from the additional factor of p. Thus for d dimensions this expression is identical except with the p^2 replaced with p^{d-1}, so $\langle E \rangle \propto T^{d-1} \cdot T \cdot T = T^{d+1}$.

Question III2. The key to converting the sums to integrals like we did in question 1 and in the derivation of $\langle E \rangle$ was to send $V \to \infty$ so that the spacing of the momenta were small. However for very small dimensions this isn't valid as $\frac{2\pi\hbar}{l_p} \approx 10$, so for such dimensions there is only one momentum state and there is no contribution to the integral.