Xander Naumenko

29/09/23

Question 1a. Let $a, b, c, d \in \mathbb{Z}$ with either $a \neq c$ or $b \neq d$. By contradiction suppose that $f(a,b) = f(c,d) \implies a + b\sqrt{2} = c + d\sqrt{2} \implies a - c = \sqrt{2}\,(d-b) \implies \sqrt{2} = \frac{a-c}{d-b}$ or $d - b = 0$. However $\sqrt{2}$ isn't rational, so in the former case $a - c = 0$. However $a - c = 0 \implies a - d = 0$ and vice versa, but this implies that both $a = c$ and $b = d$ which contradicts the definition of a, b, c, d. Thus $f(a,b) \neq f(c,d)$ and f is one-to-one.

Question 1b. To show this I will prove that for any $M \in \mathbb{Z}$, there exists $m, n \in \mathbb{Z}$ with $m \geq M$ with $m + n\sqrt{2} \in (0,1)$. If $S \cap (0,1)$ was finite then there would be a maximum M for which this is no longer possible, so proving it is sufficient.

Let $M \in \mathbb{Z}$, and consider $m_1 = M, n_1 = -\left[\frac{M}{\sqrt{2}}\right]$, where $[x]$ represents the integer part (or floor) of x. Then $m_1 + n_1\sqrt{2} = m_1 - \left[\frac{m_1}{\sqrt{2}}\right]\sqrt{2} > 0$. Also note that $m_1 - \left[\frac{m_1}{\sqrt{2}}\right]\sqrt{2} \leq m_1 - \frac{m_1}{\sqrt{2}} m_1 + \sqrt{2} < \sqrt{2}$. If it is less than 1 then we're done, since $0 < m_1 + n_1\sqrt{2} < 1$. Otherwise, note that the pair $m_2 = 2m_1, n_2 = 2n_1 + 1$ works, since:

$$m_2 - n_2\sqrt{2} = 2\left(m_1 + n_1\sqrt{2}\right) - \sqrt{2} \geq 2 \cdot 1 - \sqrt{2} > 0$$

$$m_2 - n_2\sqrt{2} = 2\left(m_1 + n_1\sqrt{2}\right) - \sqrt{2} \geq 2 \cdot 1 - \sqrt{2} > 0.$$