UBC Mathematics 320(101)—Assignment 12
Due by PDF upload to Canvas at 23:00, Sunday 03 Dec 2023

References: Loewen, lecture notes on CCC and Continuity (2023-11-22 or newer—see Canvas); Rudin Chapter 4; Thomson-Bruckner-Bruckner, Sections 5.4–5.5, 13.6.

Presentation: To qualify for full credit, submissions must satisfy the detailed specifications provided on Canvas.

1. If $f\colon X \to Y$ is a continuous mapping between Hausdorff topological spaces X and Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

2. (a) Let X and Y be metric spaces. Prove that for $f\colon X \to Y$, TFAE:
- (i) f is uniformly continuous on X;
- (ii) for any sequences (x_n) and (x'_n) in X satisfying $d_X(x_n, x'_n) \to 0$, one has $d_Y(y_n, y'_n) \to 0$, where $y_n = f(x_n)$, $y'_n = f(x'_n)$.

(b) Identify, with proof, all real numbers p for which the function $f(x) = x^p$ is uniformly continuous on $X = (0, +\infty)$. [It's OK to use a little calculus to support your findings.]

3. A metric space (X, d) is called an *ultrametric space* if d satisfies the condition

$$\forall x, y, z \in X, \qquad d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

(This makes d itself "an ultrametric".) Show that in any ultrametric space (X, d), ...
- (a) every open ball $\mathbb{B}[x;r)$ is a closed set;
- (b) one has $y \in \mathbb{B}[x;r)$ if and only if $\mathbb{B}[y;r) = \mathbb{B}[x;r)$; and
- (c) if $\mathbb{B}[x;r_1) \cap \mathbb{B}[y;r_2) \neq \emptyset$, then one of these balls must contain the other, i.e.,

$$\mathbb{B}[x;r_1) \subseteq \mathbb{B}[y;r_2) \neq \emptyset \quad \text{or} \quad \mathbb{B}[x;r_1) \supseteq \mathbb{B}[y;r_2) \neq \emptyset.$$

[The "p-adic numbers" form an ultrametric space of interest in number theory.]

4. Given Hausdorff Topological Spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and continuous functions $f, g\colon X \to Y$, consider the *equalizer*:

$$E = \{x \in X \,:\, f(x) = g(x)\}.$$

Prove that E is closed in X.

5. Three continuous functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ are related by the identity

$$f(x+y) = g(x) + h(y).$$

- (a) In the special case where $f = g = h$, show that there must be a real number m such that $f(t) = mt$ for all real t.
- (b) Drop the hypothesis that f, g, h are identical. Describe the most general trio of continuous functions compatible with the given identity.

6. Here's a key fact every math student should know:

> *Every nonempty open set in $\mathbb{R}$ can be expressed as a finite or countable union of disjoint open intervals.*

Prove this, referring to a given open set $U \neq \emptyset$, by following these steps:

(a) For each $x \in U$, let $I(x) = (\alpha(x), \beta(x))$, where

$$\alpha(x) = \inf \{a \,:\, \text{one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\},$$
$$\beta(x) = \sup \{b \,:\, \text{one has } x \in (a,b) \text{ for some } (a,b) \subseteq U\}.$$

Prove that $x \in I(x)$ and $I(x) \subseteq U$, while $\alpha(x) \notin U$ and $\beta(x) \notin U$.
[Argue carefully, since both $\alpha(x) = -\infty$ and $\beta(x) = +\infty$ are possible.]

(b) Let $\mathcal{G} = \{I(x) \,:\, x \in U\}$. Show that any two intervals in $\mathcal{G}$ must be either disjoint or identical.

(c) Explain why the key fact stated above must hold.

Practice Problems—Not for Credit

These are not to be handed in. Solutions will be provided.

7. If f is defined on E, the *graph of* f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.
Suppose E is compact. Prove that f is continuous on E if and only if its graph is compact.

8. Let X be a compact metric space. Let $\mathcal{F}$ be a set of real-valued functions on X. Suppose that if $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $fg \in \mathcal{F}$, where fg denotes the "product function", $(fg)(x) \stackrel{\text{def}}{=} f(x)g(x)$. Suppose further that for any $x \in X$, there exists a function $f \in \mathcal{F}$ that is zero throughout some neighbourhood of x. Prove that $\mathcal{F}$ contains the constant function $f(x) = 0$.