Homework 8, Math 443

Due Wednesday, March 29, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (4 points) Generate (and justify) a version of Euler's identity for a planar graph with precisely k components.

2. (6 points) Prove the following statement, or provide a counterexample:

   If the girth of a planar graph G is 5, then $\|G\| \leq \frac{5}{3}(|G|-1)$.

3. (5 points) Prove that any planar graph on at least 4 vertices has at least 4 vertices of degree 5 or less.

   For this question, you may use without proof the observation below.

   Observation: Let G be a maximal planar graph on at least 3 vertices. Every edge has a triangle as its border, so every vertex has degree at least 2. Suppose a vertex v has degree precisely two, with $N(v) = \{u, w\}$. Then v is contained in precisely one triangle: $uvwu$.
   The edge uv is not a bridge, so it is on the boundary of precisely two faces, F_1 and F_2. Since G is a triangulation, F_1 and F_2 both have triangles as their boundaries. Then v is on the boundary of both of these faces, so both of them have $uvwu$ as their boundary. Then $G = K_3$.

4. (6 points) Given a graph G, we define its square G^2 to be the graph with the same vertex set as G, and

   $$E(G^2) = E(G) \cup \{xy : \exists z \in V(G) \text{ st } xz, zy \in E(G)\}$$

   That is: we add all edges between vertices at distance two.

   Determine, with proof, whether C_n^2 is planar or nonplanar, for each integer $n \geq 3$.

5. (6 points) Determine (up to isomorphism) all maximal planar graphs with $3k$ vertices where k vertices have degree 3, k have degree 4, and k have degree 5.

6. (3 points) Suppose both a graph and its complement are planar. Show the graph has at most 10 vertices.

7. We will build a family of graphs $\{G_i : i \in \mathbb{N}\}$ as follows:

   - $G_1 = K_4$
   - For $i \geq 1$, G_{i+1} is constructed from G_i as follows:
     - Subdivide every edge of G_i.
     - For every pair of (distinct) edges e and f incident to the same face in G_i, add an edge between the vertex subdividing e and the vertex subdividing f.

   You may assume without proof that all graphs in the family are triangulations, i.e. planar graphs where every face is bounded by a triangle.

(a) (2 points) Give a plane drawing of G_2.

(b) (3 points) What do the graphs in these family tell you about the result of Question 3?

(c) (5 points) Give $|G_i|$ as a function of i.

Question:	1	2	3	4	5	6	7	Total
Points:	4	6	5	6	6	3	10	40