Math 406 Homework 5

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Question 1a. For all functions v in some class, the following must be true:

$$\int_0^1 v(u'' + k^2u - f)dx = 0.$$

Integrating by parts:

$$\int_0^1 -u'v' + k^2uv - fvdx + v(1)\beta - v(0)u'(0) = 0 \implies \int_0^1 u'v'dx = k^2\int_0^1 uvdx - \int_0^1 fvdx + \beta v(1).$$

Thus the weak form is to find $u \in H^1_\alpha$ such that $\int_0^1 u'v'dx = k^2\int_0^1 uvdx - \int_0^1 fvdx + \beta v(1) \forall v \in H^1_0$.

Question 1b. Since we used the strong form to derive the weak form while relaxing constraints, the strong form always implies the weak form.

Question 1c. Let $u^h(x) = \sum_{n=0}^N u_nN_n(x)$, where $N_n(x)$ are the basis functions. Similarly let $v^h(x) = \sum_{n=1}^N v_nN_n(x)$. Using the weak form derived above, we need to find $u^h \in V^h_\alpha$ such that:

$$\int_0^1 \left(\alpha N_0' + \sum_{n=1}^N u_nN_n'\right)\left(\sum_{m=1}^N v_mN_m'\right)dx = k^2\int_0^1 \left(\alpha N_0 + \sum_{n=1}^N u_nN_n\right)\left(\sum_{m=1}^N v_mN_m\right)dx$$

$$-\int_0^1 f\left(\sum_{m=1}^N v_mN_m\right)dx + \sum_{m=1}^N v_mN_m(1)\beta = 0.$$

$$\implies \sum_{m=1}^N v_m\Bigg[\alpha\int_0^1 N_0'N_m'dx + \sum_{n=1}^N u_n\int_0^1 N_n'N_m'dx - k^2\left(\alpha\int_0^1 N_0N_mdx + \sum_{n=1}^N u_n\int_0^1 N_nN_mdx\right)$$

$$+\int_0^1 fN_mdx - N_m(1)\beta\Bigg] = 0.$$

$$\implies \sum_{n=1}^N u_n(K_{mn} - k^2M_{mn}) = \delta_{mn}\beta - \alpha(K_{0m} - k^2M_{0m}) + \int_0^1 fN_mdx \implies (K - k^2M) = b.$$

Here K is the stiffness matrix and M is the mass matrix. This derivation was done in class and in the notes so most of the algebra is taken from there.

Question 1d. To use piecewise linear basis functions with discretization size h, all we have to do is construct the stiffness and mass matrices. Consider just a single element e along with the change of coordinates $x(\xi) = x_{e-1}N_1(\xi) + x_e N_2(\xi)$ where N_1 and N_2 are the two linear basis functions in $[-1, 1]$ (with corresponding $N_a(x), N_b(x)$ in the x domain):

$$\int_{x_{e-1}}^{x_e} N_m'(x)N_n'(x)dx = \int_{-1}^{1} \frac{dN_a}{d\xi}\frac{d\xi}{dx}\frac{dN_b}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}d\xi = \frac{2}{h}\int_{-1}^{1}\frac{\xi_a\xi_b}{4}d\xi = \frac{\xi_a\xi_b}{h}$$

$$\implies K_{ab}^e = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Similarly for the mass matrix:

$$M_{mn}^e = \int_{x_{e-1}}^{x_e} N_m N_n dx = \int_{-1}^{1}\frac{1}{4}(1+\xi_a\xi)(1+\xi_b\xi)\frac{dx}{d\xi}d\xi = \frac{h}{4}\left(1 + \frac{\xi_a\xi_b}{3}\right)$$

$$\implies M^e = \frac{h}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Putting these together and summing over all the elements (and being careful about summing the correct conditions at the boundary, we get the final matrices to be:

$$K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & 0 & \cdots & & \\ 0 & 0 & -1 & 2 & -1 & \cdots & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & \cdots & & \\ 1 & 4 & 1 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 4 & 1 & \cdots & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

Finally we have also have:

$$b = \begin{pmatrix} -\left(\frac{1}{h} - \frac{k^2h}{6}\right)\alpha - (f, N_1) \\ -(f, N_2) \\ -(f, N_3) \\ \vdots \\ \beta - (f, N_N) \end{pmatrix} \quad \text{where } (f, N_m) = \int_0^1 f N_m dx.$$

Thus the finite element discretization is to solve the equation $(K - k^2M)u = b$ with K, M and b defined as above.

Question 1e. The solution to the homogeneous equation $u'' + 10^2u = 0$ is $u = A\sin 10x + B\cos 10x$. For the nonhomogeneous equation assume that $u = a_1 + a_2x + a_3x^2 + a_4x^3$, then we have $a_3 + a_4x + 100a_1 + 100a_2x + 100a_3x^2 + 100a_4x^3 = x^3 \implies u = -\frac{6x}{10^4} + \frac{x^3}{10^2}$. Thus plugging in boundary conditions, the analytic solution is $u(x) = \frac{\sin 10x}{10\cos 10}\left(1 + \frac{6}{10^4} - \frac{3}{10^2}\right) - \frac{6x}{10^4} + \frac{x^3}{10^2}$.

As for the numerical solution, the matrices described above were construction for $N = 10, 20, 30$ and solved. The results can be seen in figure .

Question 1f. All of the work for the previous parts have been to reduce the problem to one in the given form. Here $f = \alpha = \beta = 0$ so $b = 0$, and thus $Ku = k^2 Mu$. Letting $x = u, A = K, B = M$ and $\lambda = k^2$, we see that this is the desired eigenvalue problem.

Question 2a. From the main boundary value problem, we have that $\frac{D}{r}(rp_r)_r = 0$, which has solutions in the form $p(r) = A + B\log r$. The boundary conditions forces $\lim_{r\to 0} r\frac{\partial p}{\partial r} = B = -\frac{Q_0}{2\pi D}$, and $u(R(t), t) = 0 \implies A = \frac{Q_0}{2\pi D}\log R(t)$. Putting this together we have $p(r,t) = \frac{Q_0}{2\pi D}\log\frac{R(t)}{r}$.

Using the Stefan condition for front velocity:

$$\dot{R}(t) = -\frac{w_0^2}{\mu'}\frac{dp}{dr}\Big]_{r=R(t)} = \frac{w_0^2}{\mu'}\frac{Q_0}{2\pi D R(t)}.$$

Finally using separation of variables we can solve this differential equation:

$$\int R(t)dR(t) = \int_0^t \frac{w_0^2}{\mu'}\frac{Q_0}{2\pi D}dt \implies R(t) = \sqrt{\frac{w_0^2 Q_0 t}{\mu' \pi D}}.$$

Question 2bi. Expanding, the equation to solve is $Lp = p_{rr} + \frac{1}{r}p_r = g(r,t)$. We can find the adjoint operator:

$$\int_0^{R(t)} Gfrdr = \int_0^{R(t)} G\cdot(Lu)\cdot rdr = \int_0^{R(t)} G\left(ru'' + u'\right)dr.$$

From this point the self adjoint operator and associated boundary terms were derived in class (lecture 14), they are:

$$= \left[Ga_0u' + Ga_1u - (a_0G)'u\right]_0^{R(t)} + \int_0^{R(t)} u\left((a_0G)'' - (a_1G)' + a_2G\right)dr$$

$$= - + \int_0^{R(t)} u\left(rG'' + G'\right)dr.$$

Thus under the radial integration L acts as self adjoint (there's probably some deeper reason for the way the algebra works out with the extra r, but from the algebra it works out at least). Thus we want to find $G(s,r)$ such that $sG_{ss}(s,r) + G_s(s,r) = \delta(s-r)$. We already determined that the solution to the homogeneous equation is $p(r) = A + B\log r$. Thus we can split the Green's function into two parts:

$$G(s,r) = \begin{cases} A_- + B_-\log s & 0 < s < r \\ A_+ + B_+\log s & r < s < R(t) \end{cases}.$$

The boundary conditions force $B_- = -\frac{Q_0}{2\pi D}$ and $A_+ = -B_+\log R(t)$. Continuity forces that $A_- - \frac{Q_0}{2\pi D}\log r = B_+\log\frac{r}{R(t)} \implies A_- = B_+\log\frac{r}{R(t)} + \frac{Q_0}{2\pi D}\log r$. Thus we can write G once again as

$$G(s,r) = \begin{cases} B_+\log\frac{r}{R(t)} + \frac{Q_0}{2\pi D}\log\frac{r}{s} & 0 < s < r \\ B_+\log\frac{s}{R(t)} & r < s < R(t) \end{cases}$$

$$G_s(s,r) = \begin{cases} -\frac{Q_0}{2\pi Ds} & 0 < s < r \\ \frac{B_+}{s} & r < s < R(t) \end{cases}.$$

The only condition left is the jump condition:

$$\int_{r-\epsilon}^{r+\epsilon} sG_{ss} + G_s ds = \int_{r-\epsilon}^{r+\epsilon} (sG_s)_s ds = sG_s \Big|_{r-\epsilon}^{r+\epsilon} = r\left(G_s(r_+, r) - G_s(r_-, r)\right) = \int_{r-\epsilon}^{r+\epsilon} s\delta(r-s)ds = r$$

$$\implies 1 = \frac{1}{r}B_+ + \frac{Q_0}{2\pi D r} \implies B_+ = r - \frac{Q_0}{2\pi D}.$$

We can then get a final formulation for G:

$$G(s,r) = \begin{cases} r\log\frac{r}{R(t)} + \frac{Q_0}{2\pi D}\log\frac{R(t)}{s} & 0 < s < r \\ \left(r - \frac{Q_0}{2\pi D}\right)\log\frac{s}{R(t)} & r < s < R(t) \end{cases} = -\frac{Q_0}{2\pi D}\log\frac{s}{R(t)} + \begin{cases} r\log\frac{r}{R(t)} & 0 < s < r \\ r\log\frac{s}{R(t)} & r < s < R(t) \end{cases}.$$

Putting this together we can determine an expression for $p(r,t)$ in terms of $R(t)$ and $g(r,t)$:

$$p(r,t) = \int_0^r r\log\frac{r}{R(t)}g(s)sds + \int_r^{R(t)} r\log\frac{s}{R(t)}g(s)sds - \int_0^{R(t)} \frac{Q_0}{2\pi D}\log\frac{s}{R(t)}g(s)sds.$$

Question 2bii. Plugging in $r = R(t)$:

$$p(R(t),t) = .$$