Math 320 Homework 9

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Question 1a. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\forall x = \frac{1}{n} \in A$, let $\epsilon = \frac{1}{(n+1)^2}$. Then $\forall y = \frac{1}{m} \in A$ with $x \neq y$, we have $|x - y| = \left|\frac{1}{n} - \frac{1}{m}\right| \geq \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| > \frac{1}{(n+1)^2}$, so $\mathbb{B}[x; \epsilon) \cap A = \emptyset$ implying x is isolated. However $0 \in A'$. To see why, let U be an open set containing 0, and let $\mathbb{B}(0; \epsilon)$ be contained in U. Then using Archimedes there exists $n \in \mathbb{N}$ with $n > \frac{1}{\epsilon} \implies \epsilon > \frac{1}{n}$, so $\frac{1}{n} \in \mathbb{B}[0; \epsilon) \cap A \implies \frac{1}{n} \in U \cap A$. Since this is true of any open set U containing 0, 0 is a limit point and so $A' \neq \emptyset$.

Question 1b: Construct a bounded set of real numbers with exactly three limit points.

The construction for part a gives a set with at least one limit point at 0, I will first prove that this limit point is unique, so let A be as defined in part a. Let $x \in \mathbb{R}$ with $x \neq 0$. Consider the set $S = \{|\frac{1}{n} - x| : n \in \mathbb{N}, \frac{1}{n} \neq x\}$, i.e. the set of distances to each member of A from x except itself if x happens to be in A. Let $\epsilon = \inf S$. Note that since $x \neq 0$, $\epsilon \neq 0$. Thus $\mathbb{B}(x; \epsilon) \cap S = \emptyset \implies x \notin S'$.

Let $X = S \cup (S + 2) \cup (S + 4)$, where $S + r = \{x + r : x \in S\}$. Clearly X is bounded above by 5 and below by 0. Since $S \subset (0, 1]$, the three components $S, S + 2$ and $S + 4$ are disjoint and separated by distance 1. Thus X has exactly three limit points at $0, 2$ and 4.

Question 2a. Let $A = (0,1]$ and $B = [1,2)$. Then $\text{int}(A \cup B) = \text{int}((0,2)) = (0,2)$, while $\text{int}(A) \cup \text{int}(B) = (0,1) \cup (1,2) \neq (0,2)$ as required.

Question 2b. Let $A = \mathbb{Q}$ and $B = \sqrt{2}Q = \{\sqrt{2}q : q \in \mathbb{Q}\}$. As stated in the notes $\overline{\mathbb{Q}} = \mathbb{R}$, and also $\overline{B} = \mathbb{R}$ since it is just scaled by a constant. Then we have $\overline{A \cap B} = \overline{\mathbb{Q} \cap \sqrt{2}\mathbb{Q}} = \overline{\{0\}} = \{0\}$ but $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

Question 2c. Let $C_n = \mathbb{B}[0; 1 - \frac{1}{n}]$. Then $\mathbb{B}[0;1) = \bigcup_{n=1}^{\infty} C_n$, as for any $x \in \mathbb{B}[0;1)$, using Archimedes there exists n such that $1 - \frac{1}{n} > |x| \implies x \in C_n$. It is not possible to express $\mathbb{B}[0;1)$ as the intersection of closed sets as being able to do so would imply that it A is closed. But $\mathbb{B}[0;1)^c$ is not open as $\mathbb{B}[0;1) \cap e_1 \neq \emptyset$ (where $e_1 = (1, 0, \ldots, 0)$) while $e_1 \in \mathbb{B}[0;1)^c$, so such a representation couldn't have been possible.

Question 3a. $\emptyset$ and $\mathbb{R}$ are clearly open, the former is because there are no points to pick and the latter because every interval sits in $\mathbb{R}$. Let G be a collection of open sets, and let $S = \bigcup G$ and $x \in S$. Choose $U \in S$ arbitrary with $x \in U$. Then $\exists r > 0$ s.t. $[x, x + r) \subseteq U \subseteq S$, and since this is true of all $x \in S$, S is open.

Next let $H = \{U_1, U_2, \ldots, U_N\}$ be a finite collection of open sets, and $T = \bigcap H$. Let $x \in T$, and let $r_n > 0$ be such that $[x, x+r_n) \in U_n$, this exists since $x \in T \subseteq U_n$. Let $r = \min\{r_n : 1 \leq n \leq N\}$. Then $[x, x + r) \in T$ by construction and thus T is open.

Finally, for any $x, y \in \mathbb{R}$, without loss of generality assume that $x < y$. Choose $U_x = [x, x + \frac{y-x}{2})$ and $U_y = [y, y + 1)$. Both U_x and U_y are open, as for any $z \in U_x$, $[z, x + \frac{y-x}{2}) \subseteq U_x$ and likewise for U_y. Then $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$ as required for a Hausdorff space.

Question 3b. Let $x \in [0, 1)$ and choose $r = \frac{1-x}{2}$. Then $[x, x + r) \subseteq [x, 1) \subseteq [0, 1)$, so it is open.

Question 3c. 0 is a boundary point, as for any open set U with $0 \in U$, we have that for some $r > 0$, $[0, r) \in U$. Since $[0, r) \cap (0, 1) = (0, r) \neq \emptyset$ but $0 \notin (0, 1)$, 0 meets the definition of boundary point. For any $x \in (0, 1)$ we have $[x, \frac{1-x}{2}) \subseteq (0, 1)$ so they can't be boundary points. For any $x \in [0, \infty)$ we have that $\forall r > 0, [x, x+r) \cap (0, 1) = \emptyset$ so they aren't boundary points either. Finally for $x < 0$ we have $[x, \frac{x}{2}) \cap (0, 1) = \emptyset$, so the only boundary point is $x = 0$.

Question 3d. First I will show s_n doesn't converge. Let $U = [0, 1)$, then $s_n \notin U \forall n$ since $s_n < 0 \forall n$. Thus s_n doesn't converge to 0. Next to show t_n converges, let U be open with $0 \in U$. Then by the definition of open sets there exists $r > 0$ with $[0, r) \subseteq U$. By Archimedes there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n} < r$, so $t_n \in [0, r) \subseteq U$ and t_n converges to 0.

Question 4a. Let $x \in \partial A$. Then $x \notin (A^c)^\circ$ (since every open subset containing x intersects with both A and it's complement), so $x \in ((A^c)^\circ)^c = \overline{A}$. By symmetry between A and A^c the exact same argument works for A^c so $x \in \overline{A^c}$. Thus $x \in \overline{A} \cap \overline{A^c}$.

For the other direction, let $y \in \overline{A} \cap \overline{A^c}$. Since $y \in \overline{A}$, by definition this means that every open set containing y intersects with A. Similarly since $y \in \overline{A^c}$, this means that every open set containing y intersects with A^c. This is exactly the definition for ∂A though, so $y \in \partial A$. Since both sets contain one another, we have $\partial A = \overline{A} \cap \overline{A^c}$.

Question 4b. The two directions will be proven separately:

($\Longrightarrow$) Suppose A is closed. Then A^c is open. Consider $x \in \partial A$. Since A^c is open then for every point in A^c there must exist an open subset containing it that is contained in A^c, but by definition this isn't possible for x. Thus $x \notin A^c \implies x \in A$, so $\partial A \subseteq A$.

($\Longleftarrow$) Suppose that $\partial A \subseteq A$. Let $x \in A^c$. Since $x \notin \partial A$ and any open set containing x intersects with A^c (namely x itself), there must exist a open set U with $x \in U$ and $U \cap A = \emptyset$. Since this is true of any $x \in A^c$ we have that A^c is open implying that A is closed.

Question 4c. Note that A being open and $A \cap \partial A = \emptyset$ are logically equivalent to A^c being closed and $\partial A \subseteq A^c$. Since the definition of ∂A was completely symmetric in A, A^c, we have $\partial A = \partial A^c$, so the ∂A in the second equivalent statement can be replaced with ∂A^c. Thus applying part b to A^c we see that the A is open if and only if $A \cap \partial A = \emptyset$.

Question 5. By contradiction assume that $(A')^c$ isn't open, so assume there exists $x \in (A')^c$ such that for all open sets U containing x, $U \cap A' \neq \emptyset$. For every such U, let $y \in U \cap A'$. Then by the definition of A' applied to the fact that $y \in U$, $U \cap A \neq \emptyset$. Since this is true of every open set containing x though, by definition this means that $x \in A'$. This contradicts the fact that $x \in (A')^c$, so it must be that $(A')^c$ is open implying A' is closed.

Question 6. I will first prove the Bolzano Weierstrass theorem for $\mathbb{R}$: every bounded sequence has a convergent subsequence. Let x be a bounded sequence, I claim that it either has a increasing or decreasing subsequence. If it does, then that subsequence converges and we're done since it is monotone and bounded. If x has a decreasing subsequence then we're done, so assume that it doesn't. There exists $n_1 \in \mathbb{N}$ with $x_{n_1} = \inf(x)$, as otherwise one could construct a decreasing subsequence by repeatedly taking closer and closer values to the infimum. Let $y_1 = x_{n_1}$, and repeat this process except for $x_{n_2} = \inf\{x_n : n > n_1\}$, which gets us a $y_2 = x_{n_2}$ and $y_1 \leq y_2$. Applying this process repeatedly generates an infinite increasing subsequence y. Thus either x has an increasing or decreasing subsequence, and since it is bounded this subsequence converges.

Now to the question at hand, Apply Bolazano Weierstrass as proven above to $x_1^{(n)}$, this is possible since $-M_1 \leq x_1^{(n)} \leq M_1$ (I will assume that $M_k > 0 \forall k$ since if they ever aren't then that term is trivial). Then there exists a subsequence of $x^{(n)}$, call it $x^{(n),1}$, with $x_1^{(n),1}$ convergent. Applying the same theorem to $x_2^{(n),1}$, we can get a subsequence $x^{(n),2}$ of $x^{(n),1}$ that has $x_2^{(n),2}$ convergent, and since it's a subsequence $x_1^{(n),2}$ is still convergent and converges to the same value. Applying this repeatedly, for any $k \in \mathbb{N}$ we can get a subsequence of $x^{(n)}$ that has $x_i^{(n),k}$ convergent for each $i = 1, \dots, k$. Let a_k be the value that $x_k^{(n),k}$ converges to, since taking subsequences doesn't change the limit value this we also have $x_k^{(n),i} \to a_k \forall i \geq k$. Finally, let $y^{(n)} = x^{(1),n}$. I claim $y^{(n)}$ is the subsequence that fulfills the desired properties.

First, I will show that $y^{(n)} \to a$. For $k \in \mathbb{N}$, we have that for $n > k$, $y_k^{(n)}$ is a subsequence of $x_k^{(n),k}$ which by definition converges to a_k. Thus $y_k^{(n)} \to a_k$. To show that $y_k^{(n)} \to a$, first let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n^2 < \infty$, choose N_1 such that $\sum_{n=N_1}^{\infty} M_n^2 < \frac{\epsilon^2}{8}$. Since $y_k^{(n)} \to a_k$ for each $k = 1, \dots, N_1$, choose N_2 such that $\forall k \in \{1, \dots, N_1\}, n > N_2 \implies |y_k^{(n)} - a_k| < \frac{\epsilon}{2\sqrt{N_1}}$. Then we have that for $n > \max(N_1, N_2)$, we have

$$\|y^{(n)} - a\| = \left(\sum_{k=1}^{\infty}(y_k^{(n)} - a_k)^2\right)^{1/2} = \left(\sum_{k=1}^{N_1}(y_k^{(n)} - a_k)^2 + \sum_{k=N_1+1}^{\infty}(y_k^{(n)} - a_k)^2\right)^{1/2}$$

$$< \left(N_1 \frac{\epsilon^2}{2N_1} + \sum_{k=N_1+1}^{\infty}(2M_k)^2\right)^{1/2} < \left(\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}\right) = \epsilon.$$

Also, since $\left|y_k^{(n)}\right| \leq M_k$, we have $|a_k| \leq M_k$. Thus $\sum_{k=1}^{\infty} |a_k|^2 \leq \sum_{k=1}^{\infty} M_k < \infty \implies a \in S$. Thus $y^{(n)}$ is a convergent subsequence of $x^{(n)}$ whose limit a lies in S, as required.