MATH 305 Homework 9

Xander Naumenko

04/04/22

1. (20) Compute the Laurent series for
   (a) $\frac{1}{z(z+2)}, 1 < |z-1| < 3$
   Partial fractions:

$$\frac{1}{z(z+2)} = \frac{1}{2z} - \frac{1}{2(z+2)} = \frac{1}{2}\left(\frac{1}{(z-1)}\frac{1}{1+1/(z-1)} - \frac{1}{3}\frac{1}{1+(z-1)/3}\right).$$

$$= \frac{1}{2(z-1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n} - \frac{1}{3}\sum_{n=0}^{\infty}\left((-1)\frac{z-1}{3}\right)^n = -\frac{1}{6} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2(z-1)^n} - \frac{(z-1)^n}{2(-3)^{n+1}}.$$

   (b) $\frac{1}{z^2+4}, |z-2i| > 4$

$$\frac{1}{z^2+4} = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{(z-2i)\left(1+4i/(z-2i)\right)}\right).$$

$$= \frac{1}{4i(z-2i)} + \frac{1}{4i}\sum_{n=0}^{\infty}\frac{(-4i)^n}{(z-2i)^{n+1}} = (\frac{1}{4i}-1)\frac{1}{z-2i} + \sum_{n=1}^{\infty}\frac{(-1)^n(4i)^{n-1}}{(z-2i)^{n+1}}.$$

2. (20) Determine the types of all the isolated singularities of the following functions and compute the residue at each isolated singularity
   (a) $\frac{z}{\tan z}$

$$\frac{z}{\tan z} = \frac{z\cos z}{\sin z}.$$

This function has singularities for $z = n\pi$. For $z = 0$, this is a removable singularity since $\lim_{z\to 0}\frac{z\cos z}{\sin z} = 1$, so $Res[\frac{z}{\tan z}; 0] = 0$. For $z = n\pi, n \neq 0$ the function has simple poles, which gives:

$$Res[\frac{z}{\tan z}; n\pi] = \frac{n\pi}{\sec^2 nz} = n\pi.$$

   (b) $\frac{\cos z}{z^3}$
   The only singularity is $z = 0$, which is a pole of order 3 (since $\cos(z) \neq 0$, which gives that

$$Res[\frac{\cos z}{z^3}; 0] = \frac{1}{2}\frac{d^2}{dz^2}\cos z = -\frac{1}{2}.$$

   (c) $\frac{\mathrm{Log}(z)}{(z^2+1)^2}$
   The two isolated singularities are at $z = \pm i$, which are simple poles of order 2. Since $\mathrm{Log}(\pm i) \neq 0$, we get that the residue is

$$Res[\frac{\mathrm{Log}(z)}{(z^2+1)^2}; \pm i] = \frac{d}{dz}\left(\frac{\mathrm{Log}(z)}{(z\pm i)^2}\right) = \left.\frac{\frac{1}{z}(z\pm i)^2 - 2(z\pm i)\mathrm{Log}(z)}{(z\pm i)^4}\right|_{z=\pm i}.$$

$$= \frac{\mp i(-2i)^2 \pm 4i(\mp \frac{\pi}{2} i)}{16} = \pm\frac{1}{4}i + \frac{\pi}{8}.$$

(d) $\frac{e^z}{1-\sqrt{z}}$

Since we're using the principle branch the only pole is that $z = 1$. Consider the function as follows:

$$\frac{e^z}{1-\sqrt{z}} = \frac{e^z(1+\sqrt{z})}{1-z}.$$

Then $z = 1$ is clearly a simple pole, so the residue is

$$Res[\frac{e^z(1+\sqrt{z})}{1-z}; 1] = -2e.$$

3. (20) Evaluate the following integrals by Cauchy residue Theorem

(a) $\int_{|z|=3} \frac{e^z}{(z-1)^2 z^3}$

Calculating residue:

$$Res[f(z); 1] = \frac{d}{dx}\left(\frac{e^z}{z^3}\right) = \left.\frac{z^3 e^z - 3z^2 e^z}{z^6}\right|_{z=1} = -2e.$$

$$Res[f(z); 0] = 2\frac{d^2}{dx^2}\left(\frac{e^z}{(z-1)^2}\right) = \frac{1}{2}\frac{d}{dz}\left(\frac{(z-1)^2 e^z - 2(z-1)e^z}{(z-1)^4}\right).$$

$$= \frac{1}{2}\left.\left(\frac{\left(2(z-1)e^z + (z-1)^2 e^z - 2e^z - 2(z-1)e^z\right)(z-1)^4 - \left((z-1)^2 e^z - 2(z-1)e^z\right)4(z-1)^3}{(z-1)^8}\right)\right|_{z=0}.$$

$$= \frac{1}{2}\left((-2+1-2+2) + 4\left(1+2\right)\right) = \frac{11}{2}.$$

$$\implies \int_{|z|=3} \frac{e^z}{(z-1)^2 z^3} = (\frac{11}{2} - 2e)2\pi i.$$

(b) $\int_{|z|=1} \frac{1}{z^2 \sin z} dz$

The one singularity is $z = 0$.

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3}\frac{1}{1 - z^2/6 + O(z^4)} = \frac{1}{z^3}\left(1 + \left(\frac{z^2}{6} - O(z^4)\right) + \dots\right) \implies Res[\frac{1}{z^2 \sin z}; 0] = \frac{1}{6}.$$

$$\implies \int_{|z|=1} \frac{1}{z^2 \sin z} dz = \frac{\pi i}{3}.$$

(c) $\int_{|z|=1} e^{\frac{1}{z}} \cos(z) dz$

Calculating residue:

$$e^{\frac{1}{z}} \cos(z) = \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots\right)\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right) \implies Res[e^{\frac{1}{z}} \cos(z); 0] = \sum_{n=1} \frac{1}{n((n-1)!)^2}.$$

$$\implies \int_{|z|=1} e^{\frac{1}{z}} \cos(z) dz = 2\pi i \sum_{n=1} \frac{(-1)^{n+1}}{n((n-1)!)^2}.$$

(d) $\int_{|z|=1} \frac{e^z}{\sin^3 z} dz$

Computing residue:

$$\frac{e^z}{\sin^3 z} = \frac{1+z+z/2}{(z-z^3/6+\ldots)^3} = \frac{1}{z^3}\left(1+z+z^2/2\right)\left(1-3\left(1-z^2/6+\ldots\right)+\ldots\right).$$

$$\implies Res[\frac{e^z}{\sin^3 z};0] = \frac{1}{2}+\frac{3}{3!} = 1 \implies \int_{|z|=1}\frac{e^z}{\sin^3 z}dz = 2\pi i.$$

4. Computing the following integrals

(a) $\int_0^\pi \frac{1}{1+\sin^2\theta}d\theta$

Let $z = e^{i\theta} \implies dz = ie^{i\theta}d\theta$.

$$\int_0^\pi \frac{1}{1+\sin^2\theta}d\theta = \frac{1}{2}\int_{|z|=1}\frac{-iz^{-1}}{1-\left(z-z^{-1}\right)^2/4}dz = \int_{|z|=1}\frac{-2iz^{-1}}{6-z^2-z^{-2}}dz.$$

$$= -8\pi Res\left[\frac{z}{z^4-6z^2+1};\sqrt{2}-1\right] = \frac{\pi}{\sqrt{2}}.$$

(b) $\int_0^{2\pi}\frac{\sin^2\theta}{3+\cos\theta}d\theta$

Let $z = e^{i\theta} \implies dz = ie^{i\theta}d\theta$.

$$\int_0^{2\pi}\frac{\sin^2\theta}{3+\cos\theta}d\theta = \int_{|z|=1}\frac{z^2-2+z^{-2}}{-4iz(3+(z+z^{-1})/2)}dz = \frac{i}{2}\int_{|z|=1}\frac{z^4-2z^2+1}{z^2(z^2+6z+1)}dz.$$

$$= -\pi Res\left[\frac{z^4-2z^2+1}{z^2(z^2+6z+1)};0\right] - \pi Res\left[\frac{z^4-2z^2+1}{z^2(z^2+6z+1)};2\sqrt{2}-3\right] = \left(6-4\sqrt{2}\right)\pi.$$

5. (30) Using contour integrals to compute the following integrals

(a) $\int_0^\infty \frac{x^2}{(x^2+4)^2}dx$

Taking the contour to be a disk of radius R in the upper half plane as $R\to\infty$ and noting that the function is even:

$$2\int_0^\infty \frac{x^2}{(x^2+4)^2}dx = 2\pi i Res[\frac{z^2}{(z-2i)^2(z+2i)^2};2i] = 2\pi i\frac{d}{dz}\frac{z^2}{(z+2i)^2}\bigg|_{z=2i}.$$

$$= 2\pi i\frac{2z(z+2i)^2-2z^2(z+2i)}{(z+2i)^4} = -2\pi i\frac{i}{8}.$$

$$\implies \int_0^\infty \frac{x^2}{(x^2+4)^2}dx = \frac{\pi}{8}.$$

(b) $\int_0^\infty \frac{1}{x^4+x^2+1}dx$

Same integration contour as last time with the outside going to zero, and the function is again even:

$$2\int_0^\infty \frac{1}{x^4+x^2+1}dx = 2\pi i Res[\frac{1}{z^4+z^2+1};e^{\frac{\pi}{3}i}] + 2\pi i Res[\frac{1}{z^4+z^2+1};e^{\frac{2\pi}{3}i}].$$

$$= \frac{2\pi i}{2\left(e^{i\frac{\pi}{3}}-2\right)} - \frac{2\pi i}{2\left(2+e^{i\frac{2\pi}{3}}\right)} = \frac{\pi}{3\sqrt{3}}.$$

(c) $\int_0^\infty \frac{1}{x^3+1}dx$.

Let the contour be the wedge with angle $\frac{2\pi}{3}$. Then the boundary term goes to zero and the integral is

$$\int_0^\infty \frac{1}{x^3+1}dx - \int_0^\infty \frac{e^{\frac{2\pi}{3}i}}{\left(re^{\frac{2\pi}{3}i}\right)^3+1}dr = \left(1-e^{\frac{2\pi i}{3}}\right)\int_0^\infty \frac{1}{x^3+1}dx = 2\pi i Res[\frac{1}{z^3+1}; e^{i\frac{\pi}{3}}] = 2\pi i \frac{1}{3\left(e^{i\frac{2\pi}{3}}\right)}.$$

$$\implies \int_0^{\frac{1}{x^3}+1} dx = \frac{2\pi}{3\sqrt{3}}.$$

(d) $\int_0^\infty \frac{\cos x}{x^4+1}dx$

Expand the given integral into the z plane on the upper half disk of radius R as $R \to \infty$.

$$2\int_0^\infty \frac{\cos x}{x^4+1}dx = Re\left(2\pi i Res\left[\frac{e^{iz}}{z^4+1}; e^{i\frac{\pi}{4}}\right] + 2\pi i Res\left[\frac{e^{iz}}{z^4+1}; e^{i\frac{3\pi}{4}}\right]\right).$$
$$= \frac{\pi}{2} Re\left(ie^{i\frac{\pi}{4}-(-1)^{3/4}} + ie^{i\frac{3\pi}{4}-(-1)^{1/4}}\right).$$

Computing this numerically, since it doesn't simplify nicely algebraically:

$$\int_0^\infty \frac{\cos x}{x^4+1}dx \approx 0.772138.$$

(e) $\int_{-\infty}^\infty \frac{\sin x}{x^2+2x+2}dx$

Taking the same contour as the previous part:

$$\int_{-\infty}^\infty \frac{\sin x}{x^2+2x+2}dx = Im\left(2\pi i Res\left[\frac{e^z}{z^2+2z+2}; -1+i\right]\right) = Im\left(\pi e^{-1-i}\right) = \frac{-\pi\sin(1)}{e} \approx -0.972551.$$