Phys 408 Homework 1

Xander Naumenko

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Question 1a. From the Gaussian beam derivation, we have that $w(z) = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2}$, where $w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$. The smallest spot on the screen occurs when $w(z)$ is minimized, which occurs when $\sqrt{\frac{\lambda}{\pi}\left(z_0+\frac{d^2}{z_0}\right)}$ is minimized, which is equivalent to minimizing $z_0 + \frac{d^2}{z_0}$:

$$\frac{d}{dz_0}\left(z_0+\frac{d^2}{z_0}\right) = 0 \implies 1-\frac{d^2}{z_0^2} = 0 \implies z_0 = d.$$

This corresponds to a beam waist parameter of $w_0 = \sqrt{\frac{\lambda d}{\pi}}$. This is a $1/e^2$ beam radius of $\sqrt{2}w_0 = \sqrt{\frac{2\lambda d}{\pi}}$.

Question 1b. Plugging into the above expression:

$$\sqrt{\frac{2\lambda d}{\pi}} = \sqrt{\frac{2\cdot 600\cdot 10^{-9}\cdot 0.1}{\pi}} \approx 0.1954\text{mm}.$$

Question 2. To derive the first equation, use the relation $R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right)$:

$$R_1 = z_1\left(1+\left(\frac{z_0}{z_1}\right)^2\right), R_2 = (z_1+d)\left(1+\left(\frac{z_0}{z_1+d}\right)^2\right)$$

$$\implies R_1 z_1 - z_1^2 = R_2(z_1+d) - (z_1+d)^2$$

$$\implies z_1(R_2-R_1-2d) = -d(R_2-d) \implies z_1 = \frac{-d(R_2-d)}{R_2-R_1-2d}.$$

For the second, plug this result back into the original:

$$z_0^2 = z_1(R_1-z_1) = \frac{-d(R_2-d)}{R_2-R_1-2d}\left(R_1+\frac{d(R_2-d)}{R_2-R_1-2d}\right)$$

$$\implies z_0^2 = \frac{-d(R_1+d)(R_2-d)(R_2-R_1-d)}{(R_2-R_1-2d)^2}.$$

Question 3. Substituting:

$$\nabla_T^2 E(r) + 2ik\frac{\partial E(r)}{\partial z} = (-k^2x^2+2ik-k^2y^2)\frac{1}{z}E(r) + 2ik\left(-\frac{1}{z}+ik(x^2+y^2)\frac{1}{2z^2}\right)E(r) = (0)E(r) = 0.$$

Question 4. We can find the final transfer function by combining the matrices of the parts:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} -1 & 2f \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix}.$$

We can try seeing what the result of an incoming beam with a given slope is:

$$\begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ m \end{pmatrix} = \begin{pmatrix} 2fm \\ -m \end{pmatrix}.$$

As expected, the resulting beam is parallel.

Question 5. I assume that all angles/distances involved are small, so $\sin(\theta) \approx \theta$. Then from Snell's law, we have $n_1 \left(\theta_1 + \frac{y}{R}\right) = n_2 \left(-\theta_2 - \frac{y}{R}\right)$. Rearranging, we get

$$\theta_2 = \frac{n_1}{n_2} \left(\theta_1 + \frac{y}{R}\right) - \frac{y}{R} = -\frac{n_2 - n_1}{n_2 R} y + \frac{n_1}{n_2} \theta_1.$$

This is exactly the second line of the given matrix, and the first line is obvious since it's just a material boundary so doesn't translate the beam.