Math 406 Assignment 4

Must be submitted in .pdf form to Canvas before midnight on Wednesday November 1

1. (a) Show that $a(x)\delta(x) = a(0)\delta(x)$

   (b) Express $x^2\delta^{(3)}(x)$ in terms of $\delta^{(1)}(x)$. Use this to determine the general solution to $x^2 g(x) = 0$.

   (c) Write $\delta(\cos(x))$ as a sum of δ-functions.

   (d) Show that $f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x)$.

2. Consider the boundary value problem

$$Lu = 2x^2u'' + 3xu' - u = f, \; u(0) \text{ is finite and } u(1) = 0 \tag{1}$$

   (a) Determine the adjoint operator L^* and appropriate boundary conditions associated with L and the boundary conditions above. Determine the Green's function G so that u has an integral representation in terms of G and f.

   (b) Multiply the equation in (1) by an appropriate function to make L formally self-adjoint. Now determine the Green's function for the new problem and use this to obtain an integral representation for u in terms of f. How does this compare to the integral representation found in part (a)?

3. Use the method of Green's functions to find an integral representation to the solution of the initial value problem

$$Lu = u'' + u = f(x), \; x > 0, \; u(0) = u_0 \text{ and } u'(0) = v_0$$

4. In class we discussed the discrete analogue of the Green's function in terms of the matrix inverse. Consider the difference equations obtained by discretizing the differential operator

$$Lu = u'' = f \text{ subject to } u(0) = 0 = u(1)$$

   by finite differences on a uniform mesh with spacing $h = 1/N$, namely

$$A_N u = f : u_{i+1} - 2u_i + u_{i-1} = h^2 f_i, \; u_0 = 0 = u_N \tag{2}$$

   The Green's function G_{ij} (equivalently the matrix inverse A_N^{-1}) is the solution to the corresponding difference equation boundary value problem

$$G_{i+1j} - 2G_{ij} + G_{i-1j} = \delta_{ij}, G_{0j} = 0 = G_{Nj} \tag{3}$$

   where δ_{ij} is the Kroneker delta.

   (a) Assuming a solution of the form $G_{ij} = r^i$ determine an explicit expression for the Green's function defined by the solution to (3).

   (b) Compare your results with the numerical inverse for A_5 obtained using MATLAB.