

Phys 408 Homework 1

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25/01/24

Question 1a. From the Gaussian beam derivation, we have that $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$, where $w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$. The smallest spot on the screen occurs when $w(z)$ is minimized, which occurs when $\sqrt{\frac{\lambda}{\pi} \left(z_0 + \frac{d^2}{z_0}\right)}$ is minimized, which is equivalent to minimizing $z_0 + \frac{d^2}{z_0}$:

$$\frac{d}{dz_0} \left(z_0 + \frac{d^2}{z_0} \right) = 0 \implies 1 - \frac{d^2}{z_0^2} = 0 \implies z_0 = d.$$

This corresponds to a beam waist parameter of $w_0 = \sqrt{\frac{\lambda d}{\pi}}$. This is a $1/e^2$ beam radius of $\sqrt{2}w_0 = \sqrt{\frac{2\lambda d}{\pi}}$.

Question 1b. Plugging into the above expression:

$$\sqrt{\frac{2\lambda d}{\pi}} = \sqrt{\frac{2 \cdot 600 \cdot 10^{-9} \cdot 0.1}{\pi}} \approx 0.1954 \text{ mm}.$$

Question 2. I'm not entirely clear what the question is asking, I interpret it to mean that we should prove the first two equations given, as the third one is already given to us. If the intention is to prove the equation $R(z) = z \left(1 + \left(\frac{z_0}{z}\right)^2\right)$ using these relations, then you can just follow the algebra given backwards. To derive the first equation, use the aforementioned relation $R(z) = z \left(1 + \left(\frac{z_0}{z}\right)^2\right)$:

$$\begin{aligned} R_1 &= z_1 \left(1 + \left(\frac{z_0}{z_1}\right)^2\right), R_2 = (z_1 + d) \left(1 + \left(\frac{z_0}{z_1 + d}\right)^2\right) \\ \implies R_1 z_1 - z_1^2 &= R_2 (z_1 + d) - (z_1 + d)^2 \\ \implies z_1 (R_2 - R_1 - 2d) &= -d(R_2 - d) \implies z_1 = \frac{-d(R_2 - d)}{R_2 - R_1 - 2d}. \end{aligned}$$

For the second, plug this result back into the original:

$$\begin{aligned} z_0^2 &= z_1(R_1 - z_1) = \frac{-d(R_2 - d)}{R_2 - R_1 - 2d} \left(R_1 + \frac{d(R_2 - d)}{R_2 - R_1 - 2d} \right) \\ \implies z_0^2 &= \frac{-d(R_1 + d)(R_2 - d)(R_2 - R_1 - d)}{(R_2 - R_1 - 2d)^2}. \end{aligned}$$

Question 3. Substituting:

$$\nabla_T^2 E(r) + 2ik \frac{\partial E(r)}{\partial z} = (-k^2 x^2 + 2ik - k^2 y^2) \frac{1}{z} E(r) + 2ik \left(-\frac{1}{z} + ik(x^2 + y^2) \frac{1}{2z^2} \right) E(r) = (0)E(r) = 0.$$

Question 4. We can find the final transfer function by combining the matrices of the parts:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} -1 & 2f \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

We can try seeing what the result of an incoming beam with a given slope is:

$$\begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ m \end{pmatrix} = \begin{pmatrix} -y + 2fm \\ -m \end{pmatrix}.$$

As expected, the resulting beam is parallel.

Question 5. I assume that all angles/distances involved are small, so $\sin(\theta) \approx \theta$. Then from Snell's law, we have $n_1 \left(\theta_1 + \frac{y}{R} \right) = n_2 \left(-\theta_2 - \frac{y}{R} \right)$. Rearranging, we get

$$\theta_2 = \frac{n_1}{n_2} \left(\theta_1 + \frac{y}{R} \right) - \frac{y}{R} = -\frac{n_2 - n_1}{n_2 R} y + \frac{n_1}{n_2} \theta_1.$$

This is exactly the second line of the given matrix, and the first line is obvious since it's just a material boundary so doesn't translate the beam.