PHYS 403 Homework 3

Xander Naumenko

18/03/24

Question I. Taking the derivative with respect to a specific p_k:

$$\frac{\partial}{\partial p_k}\left(S[p_n]-\alpha\left(\sum_n p_n-1\right)-\beta\left(\sum_n p_n-\bar{E}\right)-\gamma\left(\sum_n p_nN_n-\bar{N}\right)\right)$$
$$=-\log p_k-1-\alpha-\beta E_n-\gamma N_n=0 \implies p_k=e^{-(1+\alpha+\beta E_n+\gamma N_n)}.$$

By comparing this expression to what we derived in class ($p_k=\frac{1}{Z}e^{-\beta(E_n-\mu N_n)}$), we conclude that $e^{-1-\alpha}=\frac{1}{Z}$, $\beta=\frac{1}{k_BT}$ and $\gamma=-\frac{\mu}{k_BT}$.

Question II. Assume the balloon initially at a radius R, and consider a small change dR in the radius. Let P_i be the pressure inside the balloon and P_o to be that outside. Then we have:

$$P_o dA=P_i dA-\sigma dV \implies P_i-P_o=\sigma\frac{dA}{dV}.$$

We have that $V=\frac{4}{3}\pi R^3$, so $dV=\frac{4}{3}\pi((R+dR)^3-R^3)=\frac{4}{3}\pi\left(3R^2dR+3RdR^2+dR^3\right)\approx 4\pi R^2dR$. Similarly, we have $dA=4\pi\left((R+dR)^2-R^2\right)\approx 8\pi RdR$. Plugging this into the above expression, we get $P_i-P_o=\frac{2\sigma}{R}$ as required.

Question III.1.

$$Z^{(d)}_{G.C.}(\mu,T)=\sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^3r_1\cdots d^3r_Nd^3p_1\cdots d^3p_N}{(2\pi\hbar)^{dN}}e^{-\beta\left(\sum_{i=1}^N\frac{p_i^2}{2m}-(\mu-\epsilon_d)N\right)}$$
$$=\sum_{N=0}^{\infty}e^{\beta(\mu-\epsilon_d)N}\frac{1}{N!}\left(\frac{R^{dN}}{\lambda^d}\right)^N=\exp\left(\frac{e^{\beta(\mu-\epsilon_d)}R^{dN}}{\lambda^d}\right).$$

Question III.2. Using the derivative of the partition function:

$$\langle N\rangle=\frac{1}{\mu}\frac{\partial}{\partial\beta}\log Z=\frac{e^{\beta\mu}R^{3N}}{\lambda^3}=\frac{e^{\beta\mu}V}{\lambda^3}$$
$$\implies n_V=\frac{\langle N\rangle}{V}=\frac{e^{\beta\mu}}{\lambda^3}.$$

Question III.3. Using same technique as before:

$$\langle N\rangle=\frac{1}{\mu}\frac{\partial}{\partial\beta}\log Z=\frac{A}{\lambda^2}e^{\beta(\mu+\epsilon_0)}.$$
$$\implies n_S=\frac{e^{\beta(\mu+\epsilon_0)}}{\lambda^2}.$$

Question III.4. The total change to surface tension, as stated in the question of part 3, is $\Delta\sigma=-\epsilon_0n_s$. Since n_S is strictly decreasing with temperature (since $\beta=\frac{1}{k_BT}$), this means that increasing temperature causes the surfactant to mix oil and water less effectively.