Homework 1, Math 443

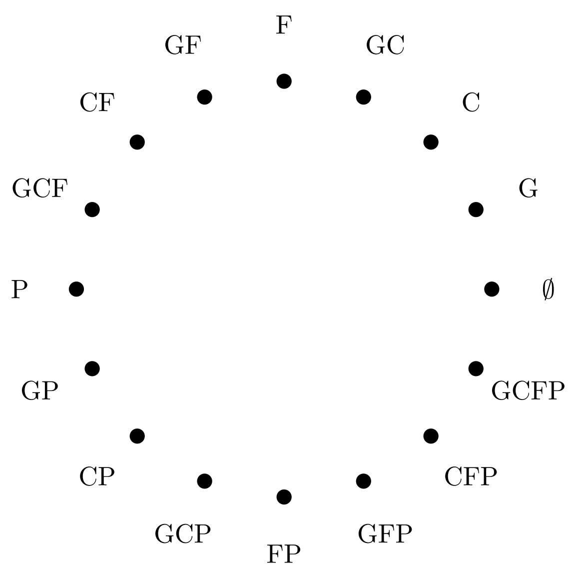
Due Wednesday, January 25, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. The well-known ferryboat problem goes something like this:

   > A person wants to move a fox, a chicken, and a bag of grain from one bank of a river to the other. The person has access to a stand-up paddleboard, which can only carry themselves and one other character (the fox, or the chicken, or the grain). If left unsupervised by the person, the fox will eat the chicken. If left unsupervised by the person, the chicken will eat the grain. The person does not want anything to eat anything else right now. How can they move all three actors from the current shore to the other shore?
   >
   > *The only way for anything to move from one shore to the other is for the person to paddleboard them over; the fox and chicken are semi-domesticated so they may be left unattended (albeit not together) while the person is elsewhere engaged.*

   We want to make a graph G to find all possible solutions to the problem. Each vertex in $V(G)$ is a subset of $\{P, F, C, G\}$, corresponding to the actors populating the first side of the river. So, the initial state is GCFP, and the goal is $\emptyset$.



   Two vertices are adjacent in G iff it is possible to change from one configuration to another with a single paddleboard trip.

   (a) (2 points) Is G bipartite? If not, why not? If so, what are the two parts?

(b) (2 points) Draw the subgraph H induced by the set of vertices corresponding to safe configurations. (For example, do not include any vertex that corresponds to a situation where the chicken and grain are on a different shore from the person.)

(c) (1 point) Is H connected? Just answer "yes" or "no," no need to explain.

(d) (2 points) Describe, **in terms of the graph(s)** G and/or H, what a solution to the ferryboat problem will look like. (Use graph theory terms.)

(e) (2 points) List **all** solutions to the ferryboat problem that do not involve repeating a configuration.

2. We build a graph, with countably infinite vertices, as follows.

   - Start with a vertex labelled v_1, marked as unfinished.
   - $(*)$ For each existing vertex v_i that is marked as unfinished, do the following:
     - Create a vertex labelled v_{2i}, add the edge from v_{2i} to v_i, and mark v_{2i} as unfinished.
     - If $i \equiv 2 \mod 3$ and $i > 2$, create a vertex labelled $v_{\frac{2i-1}{3}}$, add the edge from $v_{\frac{2i-1}{3}}$ to v_i, and mark $v_{\frac{2i-1}{3}}$ as unfinished.
     - Mark v_i as finished.
   - Now there are more unfinished vertices, so repeat $(*)$.

   Call this graph G. (A portion of G is shown on the next page.)

   (a) (1 point) *Prove, or provide a counterexample:* If v_i was created later than v_j, then $i > j$.

   (b) (1 point) *Prove, or provide a counterexample:* If i is even, then v_i is adjacent to $v_{i/2}$, and not adjacent to $v_{\frac{3i+1}{2}}$

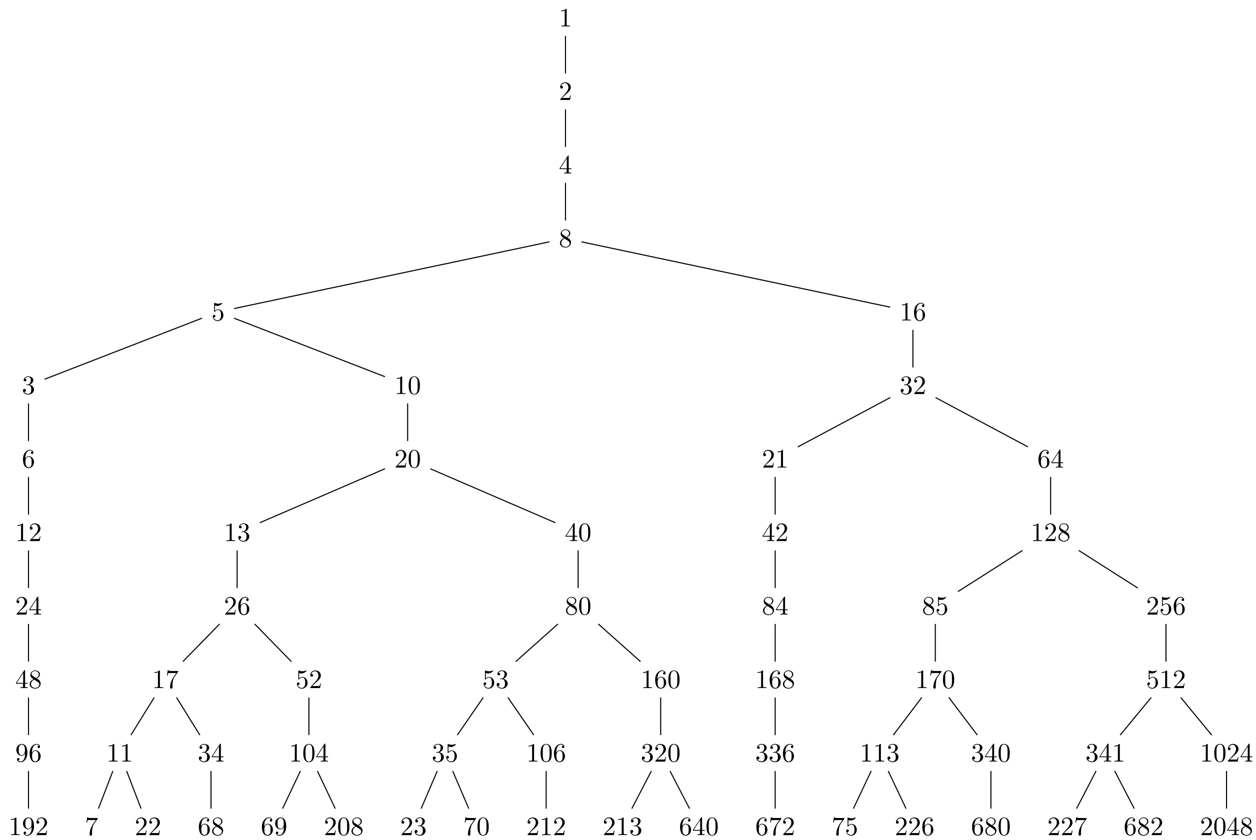
   (c) (3 points) *Prove, or provide a counterexample:* No two vertices have the same labels. That is, we never create a vertex labelled v_i if there already exists a vertex labelled v_i.

   (d) (3 points) *Prove, or provide a counterexample:* The infinite graph G that we generate in this manner it contains no (finite) cycles.

   Hint: consider the youngest vertex in a cycle.

   (e) (1 point) Restate the Collatz conjecture (link) as a conjecture that the graph we created has a particular property.

   A subgraph of G:

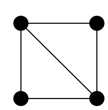


3. (2 points) In class, we defined a subgraph H of G to be **proper** iff $V(H) \neq V(G)$ or $E(H) \neq E(G)$.

   True or false: If we replace "$V(H) \neq V(G)$ or $E(H) \neq E(G)$" with "$|E(H)| < |E(G)|$," the resulting definition is equivalent.

   (Remember you need to justify your answers, including this one.)

4. (4 points) Let G be the (unlabeled) graph shown below.

   List all distinct (i.e. non-isomorphic) subgraphs of G. **Your list should be in some logical order**, to make sure that (a) no two are isomorphic, and (b) you haven't missed any.

5. In class, we proved the following:

   > Given a graph G with $|G| \geq 3$, G is connected iff there exist distinct $u, v \in V(G)$ st $G - u$ and $G - v$ are both connected.

   (a) (2 points) Prove or provide a counterexample: Given a graph G with $|G| \geq 4$, if there exist distinct $u, v, w \in V(G)$ st $G - u$, $G - v$, and $G - w$ are all connected, then G itself is connected.

   (b) (2 points) Prove or provide a counterexample: Given a graph G with $|G| \geq 4$, if G is connected, then there exist distinct $u, v, w \in V(G)$ st $G - u$, $G - v$, and $G - w$ are all connected.

6. (3 points) The **diameter** of a connected graph G, denoted $diam(G)$, is the greatest distance between any two vertices of G.

   Let G be a connected graph that contains at least one cycle. The **girth** of G, denoted $g(G)$, is the length of the smallest cycle in G.

   Prove, or give a counterexample: Every connected graph G containing a cycle satisfies

$$g(G) \leq 2diam(G) + 1.$$

7. (a) (2 points) Let G be a graph with distinct vertices a and b. Suppose P and Q are $a - b$ paths in G such that $P \neq Q$ and $V(P) \cap V(Q) = \{a, b\}$.

      Show that G contains a cycle of length at least $|P|$.

   (b) (4 points) Suppose G is a graph containing subgraphs C and P, where C is a cycle and P is a path on k vertices with both endpoints in $V(C)$. Show that G contains a cycle of length greater than $\sqrt{k}$.

      Hint: use the result from (a).

8. (3 points) A *chorded cycle* is the graph obtained by adding one more edge to a cycle. (For example, the graph from Question 4 is a chorded C_4.)

   Prove the following statement, or give a counterexample: Every graph G with $\delta(G) \geq 3$ contains a chorded cycle.

Question:	1	2	3	4	5	6	7	8	Total
Points:	9	9	2	4	4	3	6	3	40