Homework 5, Math 443

Due Wednesday, March 1, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (4 points) For all positive integers k, describe a graph G_k such that $\lambda(G_k) = k$ and $\kappa(G_k) = 1$.

   (We've seen graphs with $\lambda(G) = \kappa(G)$. Your examples will show that $\kappa(G)$ can be far away from $\lambda(G)$.)

2. Let G be a graph with $|G| = n$.

   (a) (4 points) Prove, or provide a counterexample: if $\delta(G) \geq \frac{n}{2}$, then $\lambda(G) = \delta(G)$.

   (b) (4 points) Prove, or provide a counterexample: if $\delta(G) \geq \frac{n}{2}$, then $\lambda(G) = \kappa(G)$.

   Hint: think about *small* components resulting from removing edges from G.

3. (4 points) Let G be a connected nontrivial graph with blocks $B_1, \ldots, B_k$. Define another graph $\mathcal{G}$ to have $V(\mathcal{G}) = \{B_i : i \in [k]\}$ and $E(\mathcal{G}) = \{B_iB_j : B_i \cap B_j \neq \emptyset\}$. Prove, or provide a counterexample: $\mathcal{G}$ is a tree.

4. The results of this question are used in the proof of Corollary 5.9(c), so you may not use Corollary 5.9(c) in your proofs.

   Given a nonempty collection of edges $\mathcal{E}$ of a graph G, we define a graph $G[\mathcal{E}]$ to be the graph with edge set $\mathcal{E}$ and no isolated vertices.

   Let G be a nontrivial connected graph, and let R be the equivalence relation from Theorem 5.8. That is, R is the relation on the edge set of a nontrivial connected graph G defined as follows: for all $e, f \in E(G)$, eRf iff $e = f$ or e and f lie on a common cycle of G.

   (a) (4 points) Prove, or provide a counterexample: given an equivalence class $\mathcal{E}$ of R, $G[\mathcal{E}]$ is nonseparable.

   (b) (4 points) Prove, or provide a counterexample: given an equivalence class $\mathcal{E}$ of R, $G[\mathcal{E}] \leq G$.

   (c) (4 points) Prove, or provide a counterexample: given an equivalence class $\mathcal{E}$ of R, $G[\mathcal{E}]$ is a block of G.

   (d) (4 points) Prove, or provide a counterexample: for every block B of G, $E(B)$ is an equivalence class of R.

5. (4 points) Prove or disprove: if v is a cut vertex in a connected graph G with order at least three, then v is contained in more than one block of G.

6. (4 points) Prove that a connected graph G with $\|G\| \geq 2$ is nonseparable if and only if any two adjacent edges of G lie on a common cycle of G.

Question:	1	2	3	4	5	6	Total
Points:	4	8	4	16	4	4	40