MATH 406, HWK 5, Due 17 th November

1. Consider the following boundary value problem

$$\mathcal{L}u = u'' + k^2 u(x) = f(x), \quad u(0) = \alpha, \quad u'(1) = \beta \tag{1}$$

   (a) Use integration by parts to obtain the weak statement of the BVP.

   (b) If f is sufficiently differentiable show that the strong and weak formulations of the BVP are equivalent.

   (c) Write down the Galerkin formulation of this BVP.

   (d) Use piecewise linear basis functions and the weak formulation to obtain a Finite Element discretization of the BVP.

   (e) Solve the BVP with $f(x) = x^3$, $k = 10$, $\alpha = 0$, and $\beta = 1$ and $N = 10, 20, 30$ and compare with the exact solution.

   (f) In the case $f(x) = 0$, $\alpha = 0$, and $\beta = 0$ we have an eigenvalue problem. By considering the minimization of the appropriate Rayleigh quotient or the appropriate weak formulation, use Finite Elements to discretize the problem and to reduce it to a corresponding generalized matrix eigenvalue problem of the form $Ax = \lambda Bx$. Use the MATLAB function [V,D] = EIG(A,B) to determine the eigenvalues and corresponding eigenvectors for $N = 10$. Discretize the same problem using finite differences, determine an explicit expression for the approximate eigenvalues of the finite difference equations, and determine the order of the error. Compare the results of the FEM and FD solutions by providing the following plots: k_j vs mode number j, and the first three eigenfunctions.

2. We consider a class of boundary value problems (BVP) for $p(r,t)$ of the form

$$D\Delta p = D\frac{1}{r}(rp_r)_r = f(r,t) \quad \text{where } 0 < r < R(t) \tag{2}$$

$$\text{Left BC:} \quad \text{Specified flux: } \lim_{r\to 0} r\frac{\partial p(r,t)}{\partial r} = -\frac{Q_0}{2\pi D} \tag{3}$$

$$\text{Right BC} \ : \ p(R(t),t) = 0 \tag{4}$$

$$\text{Initial Condition} \ : \ p(r,0) = 0 \ \text{ and } R(0) = 0 \tag{5}$$

Here Δp is the Laplacian, which reduces to an ordinary differential operator since the problem is assumed to be radially symmetric. This boundary value problem represents the pressure distribution within a circular fluid-filled zone occupying the cylindrical region $0 < r < R(t), -\frac{w_0}{2} < z < \frac{w_0}{2}$ between two parallel plates a distance w_0 apart. The flow velocity, according to Poiseuille's law, is given by

$$v = -\frac{w_0^2}{\mu'}\frac{dp}{dr} \tag{6}$$

Here $\mu' = 12\mu$ is the scaled fluid viscosity and the velocity is obtained by integrating the parallel plate solution to the Navier Stokes equations in the z directions across the gap between the plates. Associated with the Poiseuille velocity is the fluid flux within the parallel disks, which is given by

$$q = w_0 v = -\frac{w_0^3}{\mu'}\frac{dp}{dr} = -D\frac{dp}{dr}, \text{ where } D = \frac{w_0^3}{\mu'} \tag{7}$$

The ODE (11) expresses the conservation of mass in which the flux gradient is balanced by the sources or sinks $g(r,t)$ distributed within the expanding domain $0 < r < R(t)$. Though the BVP is relatively simple to solve, the fact that the extent of the domain is unknown complicates the problem considerably. This type of problem is known as a "free boundary problem" or "moving boundary problem". At the moving front Poiseuille's law provides the so-called Stefan condition for the front velocity:

$$\dot{R}(t) = q(R(t))/w_0 = v = -\frac{w_0^2}{\mu'}\left.\frac{dp}{dr}\right]_{r=R} \tag{8}$$

(a) **Analytic solution with no losses:** Assuming no distributed source/sink term (i.e. $f(r,t) = 0$) determine the pressure distribution $p(r)$, the velocity $\dot{R}(t)$ of the moving front, and an expression for the radius $R(t)$ of the moving front dfined by the boundary value problem defined in (2)-(4) along with the Stefan condition (8).

(b) **Analytic solution with a distributed loss using Green's functions:** Assume that fluid is being lost through the parallel plates via a diffusion process that leads to a sink term of the form

$$f(r,t) = \frac{C' H(t - t_0(r))}{\sqrt{t - t_0(r)}}$$

where $H(t)$ is the Heaviside function, C' is a given constant, and $t_0(r)$ is the time at which the fluid font arrives at the ring of radius r. Thus $t_0(r) = R^{-1}(r)$ the inverse function of $R(t)$.

Steps:

i. Determine the Green's function for the boundary value problem

$$\begin{aligned} \Delta p &= \frac{1}{r}(r p_r)_r = g(r,t) \quad \text{where } 0 < r < R \\ \lim_{r\to 0} r\frac{\partial p(r,t)}{\partial r} &= -\frac{Q_0}{2\pi D}, \; p(R) = 0 \end{aligned}$$

Note that the radial part of the integral over the circular domain should be of the form $\int_0^R h(r)\Delta p(r) r dr$. Now use the Green's function to determine an expression for $p(r,t)$ in terms of $R(t)$ and $g(r,t) = f(r,t)/D$.

ii. Use this expression to determine an expression for $p_r(R)$. Now use the Stefan condition (8) to derive an expression for the front velocity $\dot{R}(t)$. In the integral that results use the transformation of variables $\rho = R(\tau)$, $d\rho = \dot{R}(\tau)d\tau$ to arrive at an Abel integral equation for $\phi(R, \dot{R}) = R\dot{R}$ of the form:

$$\phi(R, \dot{R}) = A + B \int_0^t \frac{\phi(R(\tau), \dot{R}(\tau))}{\sqrt{t-\tau}} d\tau \tag{9}$$

iii. Since this integral equation is in the form of a Laplace Transform convolution, take the Laplace transform of (9) to determine Laplace transform of $\phi(t)$. The Laplace transform $\mathcal{L}(\frac{1}{t^{1/2}}) = \left(\frac{\pi}{s}\right)^{1/2}$ may prove useful.

iv. Now invert the Laplace transform of ϕ to determine an expression for $\dot{R}(t)$ and thence an expression for $R(t)$. The inverse Laplace Transform $\mathcal{L}^{-1}(\frac{1}{s+\alpha s^{1/2}}) = e^{\alpha^2 t} \operatorname{erf} c(\alpha t^{1/2})$ and the integral

$$\int_0^t e^{\alpha^2 \tau} \operatorname{erf} c(\alpha \tau^{1/2}) d\tau = \frac{1}{\alpha^2}\left(e^{\alpha^2 t} \operatorname{erf} c(\alpha t^{1/2}) - 1\right) + \frac{2t^{1/2}}{\alpha \pi^{1/2}}$$

may prove useful. Now obtain an expression for $p(r, t)$.

v. a)Assuming $w_0 = \mu' = C' = Q_0 = 1$, plot $R(t)$ for $0 < t < 500$. From part 4 you will have an expression of the form

$$F(R) = \lambda \left(\frac{1}{\alpha^2}\left(e^{\alpha^2 t} \operatorname{erf} c(\alpha t^{1/2}) - 1\right) + \frac{2t^{1/2}}{\alpha \pi^{1/2}} \right) \tag{10}$$

You will need to step through the desired sample times t and use Newton's method applied to (10) to determine the corresponding value of R.

b) Now use `quadgk` to determine $p(r, t)$ and plot $p(r, t)$ at $t = 500$. Your expression for $p(r, t)$ will involve $t_0(r)$. You can use `pchip` to determine $t_0(r)$, which is essentailly the inverse function of $R(t)$, as follows. Store the values of $R(t)$ in a vector for a relatively fine sampling of t values. Now use the MATLAB Piecewise Cubic Hermite Polynomial routine `pchip` to determine an approximation for the inverse function $t_0(r)$ by defining the following function: `t0 = @(s)ppval(pchip(Re,t),s);` where `Re` is the radius vector and `t` is the vector of corresponding sample times. You can now use `quadgk` to determine $p(r, t)$.

3. Consider the boundary value problem that determines $p(r, t)$ for the mold

filling problem without the loss term (i.e. $f = 0$) described in 2 (a)

$$\frac{1}{r}(rp_r)_r = 0 \text{ where } 0 < r < R(t) \tag{11}$$

Left BC: Flux boundary condition: $$\lim_{r\to 0} r\frac{\partial p(r,t)}{\partial r} = -\frac{Q_0}{2\pi D} \tag{12}$$

Right BC : $$p(R(t),t) = 0 \tag{13}$$

(a) Assuming that $R(t)$ is known use integration by parts to obtain the weak statement of the BVP (11)-(13).

(b) Using piecewise linear finite elements write down the Galerkin formulation for the problem.

(c) For $R = 10$ and $D = 1$ and $Q_0 = 1$ determine the solution to this problem using the Galerkin approximation determined in (b). Compare your results to the exact solution determined in part 2(a) using $N = 10$ elements and plot the error.

(d) This type of problem is known as a "free boundary problem" or "moving boundary problem". At the moving front Poiseuille's law provides the so-called Stefan condition for the front velocity:

$$\dot{R}(t) = q/w_0 = v = -\frac{w_0^2}{\mu'}\frac{dp}{dr}\bigg]_{r=R} \tag{14}$$

Using the exact solution along with the Stefan condition evaluated as $r \to R$, determine a simple ODE for R and an expression defining R implicitly in terms of t assuming $R(0) = 0$. Now adapt your code developed in (c) to solve the free boundary problem by marching forward $Ntime$ steps in time and iterating $Nitfront$ times on the location of the front:

`for` $k = 1 : Ntime$

- $t = t + \Delta t$
  $v_{k+1} = v_k$
  $R^o_{k+1} = R_k$
  `for` $itf = 1 : Nitfront$
    $R_{k+1} = R_k + \Delta t v_{k+1}$
    `if` $\left|R^o_{k+1} - R_{k+1}\right| < tol \cdot R_{k+1}$ `break`
    `Set up FEM mesh for` $[r_0, R_{k+1}]$ `and solve for` $p^{k+1} = [p_1\ p_2, \ldots, p_{N-1}, p_N = 0]$
    $v_{k+1} = -\frac{w_0^2}{\mu'}\left(\frac{p_{N+1}-p_N}{\Delta r_N}\right)$
    $R^o_{k+1} = R_{k+1}$
  `end (front iteration loop)`

`end (time step loop)`

Where $\Delta r_N = r_{N+1} - r_N = R - r_N$.

(e) Assume $w_0 = 1$, $\mu' = 1$, $D = 1$, and $Q_0 = 1$ and plot t vs $R(t)$ for $t \in [0, 50]$. Since the objective of the exercise is to determine if the FEM can be used to track the solution, as a starter solution use the exact solution for the moving front that has progressed 5 elements of length Δr for example $R(t) = 5\Delta r$. Also use the exact pressure distribution to determine the starting front velocity $v = -\frac{w_0^2}{\mu'}\left(\frac{p_{N+1}-p_N}{\Delta r_N}\right)$ and initialize the time t to correspond to $R(t)$. Plot the exact and FEM solution $p(r, 50)$ vs r for $r \in (0, R(50)]$.Why do you think the FEM solution overestimates R?