Math 318 Homework 9

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Question 1a. Any states that involve more than one book to the front can never happen, and otherwise the probability is just $\frac{1}{6}$ if the new front is B/C and $\frac{2}{3}$ if A is (I wrote the matrix backwards originally so note the transpose):

$$P = \begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}^T.$$

Question 1b. In the stationary distribution we have that $P\pi = \pi \implies (P - I)\pi = 0$, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}^T \begin{pmatrix} -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & -\frac{5}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{5}{6} \end{pmatrix}^T = 0$$

$$\begin{pmatrix} -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & -\frac{5}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{5}{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0$$

$$\implies \pi = \begin{pmatrix} 10 \\ 10 \\ 4 \\ 1 \\ 4 \\ 1 \end{pmatrix} \frac{1}{30}.$$

Question 1c. The state BCA is $i = 4$. As we proved in class then the expect number of steps to return is $\frac{1}{\pi_i} = 30$.

Question 2a. If it doesn't rain then $X_{n+1} = 4 - X_n$. If it does rain, then if $n \neq 1$ then $X_{n+1} = 5 - X_n$. Thus the transition matrix, with the indices being $0, 1, 2, 3, 4$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 1-p & p & 0 \\ 0 & 1-p & p & 0 & 0 \\ 1-p & p & 0 & 0 & 0 \end{pmatrix}.$$

Question 2b. Finding the stationary distribution, we first find an eigen vector by setting $\pi_4 = 1$ and solving the equation $\pi(P - I) = 0$ then scaling to be a proper probability vector:

$$\implies v = \begin{pmatrix} 1-p \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{5-p}.$$

We can now verify that $\pi_i P_{ij} = \pi_j P_{ji}$. Checking this for each $i, j \in [4]$, this confirms that the chain is reversible.

Question 2c. The only way that he gets wet is if both he is in state π_0 and the p chance that it rains occurs. In the stationary distribution being in π_0 is $\frac{1-p}{5-p}$, so the probability that he's wet on a given trip is $\frac{p(1-p)}{5-p}$.

Question 2d. To maximize this probability you can take the derivative and set it to 0:

$$\frac{d}{dp}\frac{p(1-p)}{5-p} = 0 \implies p^2 - 10p + 5 = 0 \implies p = 5 - 2\sqrt{5}.$$

This is greater than the value at the boundary ($p = 0, p = 1$), so it is the maximum.

Question 2e. The transition probability would be the same as part a in the same pattern (i.e. diagonal going upwards with p then $1 - p$), just extended to a larger N. Therefore the stationary state would be

$$\pi = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \frac{1}{1-p} \end{pmatrix} \frac{1}{N+1-p}.$$

The probability that he gets wets on a given day is then $\frac{p(1-p)}{N+1-p}$, and to find the p that maximizes this probability would be the solution to the problem $\frac{d}{dp}\left(\frac{p(1-p)}{N+1-p}\right)$ (given the question is quite vague I assume I don't have to actually solve this).

Question 3. Same as question 2.

Question 4. As the hint suggests, consider q to be the probability of ever getting to $n-1$ starting on n. This is independent on n, as the probability transitions are also independent of n. From a given n, there are two ways to go to $n-1$: either directly there with probability $1-p$, or go up once with probability p and eventually make it back to n then $n-1$, which happens with probability q^2. Thus

$$q = 1 - p + pq^2 \implies pq^2 - q + 1 - p = 0 \implies q = \frac{1}{p} - 1 \text{ or } .$$

Note that if $p < \frac{1}{2}$ then the formulation above isn't valid since $q = 1$ (since the random walk is then recurrent). Thus if $p > \frac{1}{2}$ then $q = \frac{1}{p} - 1$, and $q = 1$ otherwise.

Now for the actual problem, note that if the first state that is taken is downward then we are guaranteed to return to 0 since $p > \frac{1}{2}$, and otherwise the probability to return is q. Thus we have

$$P = 1 - p + pq = 1 - p + p(\frac{1}{p} - 1) = 2 - 2p.$$

Question 5. Consider this random walk as a walk on graph with 52! nodes, each node denoting a particular permutation of the cards. Two nodes are connected if they are identical except one node flipped. Each node has $52 \cdot 51$ neighbors (number of choices of 2 distinct neighbors). For the uniform distribution π over all possible states, for each state i we have $\pi_i = \frac{1}{52!}$. The next state is given by

$$\pi_i P_i = 52 \cdot 51 \frac{1}{52! 52 \cdot 51} = \frac{1}{52!}.$$

Since $\pi_i P_i = \pi_i$ for all i, the uniform distribution π_i is the stationary distribution.

Question 6a. See figure 1 for the graph. The code is as follows:

```
import random
import matplotlib.pyplot as plt

F_n = []
n = 1000
for s in range(n):
    cards = list(range(52))
    F_n.append([52])
    for _ in range(200):
        i = random.randint(0, 51);
        j = random.randint(0, 51);
        temp = cards[i]
        cards[i] = cards[j]
        cards[j] = temp

        f_n = F_n[-1][-1]
        if cards[i] == j:
            f_n-=1
        if cards[j] == i:
            f_n-=1
        if cards[i] == i:
            f_n+=1
        if cards[j] == j:
            f_n+=1
        F_n[-1].append(f_n)
```

```
F_n_avg = [sum([f[i] for f in F_n]) / n for i in range(len(F_n[0]))]
print(F_n_avg)

plt.plot(F_n_avg);

plt.show()
```

Question 6b. As confirmed with the code, the expected value of F in the stationary state is $F = 1$. Given that this is a simulation question I don't think I have to justify my answer, but intuitively at least it's because the stationary state is randomly shuffled, and in this state each card has a $\frac{1}{52}$ chance of being in its correct place (although properly proving that you can sum these is more complicated since they're not independent).

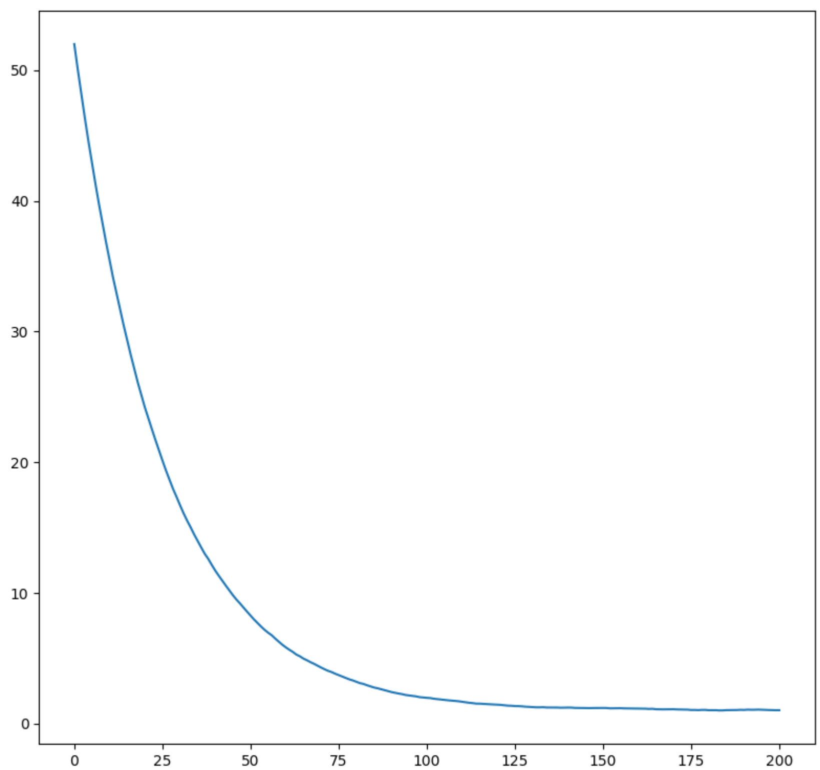

Figure 1: Graph for question 6