Homework 9, Math 443

Due Wednesday, April 7, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (3 points) Prove the following, or provide a counterexample:

   Let G be a graph with degree sequence

   $$2, \ldots, 2, 4, 4, 4, 4$$

   (That is: G has exactly four degree-4 vertices, and all other vertices of G have degree 2.) Then G is planar.

2. (3 points) In class, we showed the following.

   If a triangulation has minimum degree 5, then has an edge xy with $d(x) = 5$ and $d(y) \in \{5, 6\}$.

   Prove the following as a corollary, or provide a counterexample:

   If a planar graph has minimum degree 5, then has an edge xy with $d(x) = 5$ and $d(y) \in \{5, 6\}$.

3. (6 points) Prove the following using discharging.

   Let G be a triangulated planar graph, $\delta(G) \geq 5$, with no two degree-5 vertices adjacent to one another. Then G has at least 60 edges connecting a vertex of degree 5 with a vertex of degree 6.

4. (6 points) Prove the following, using the method of discharging.

   Let G be a triangulation with $\delta(G) = 4$ and no vertices of degree 7. Suppose every degree-5 vertex in v has only neighbours of degree 8 or higher. Then G contains an edge xy such that $d(x) = 4$ and $d(y) \in \{4, 6\}$.

5. (6 points) Let G be a triangulated plane graph with $\delta(G) \geq 4$ and no vertices of degree 6 or 7. Use the method of discharging to show that there exists a subgraph of G isomorphic to $K_{1,3}$ where all leaves have degrees in $\{4, 5\}$ in G.

6. (4 points) Prove the following, or provide a counterexample:

   Let G be a planar graph with at least one edge. Then there exists an edge xy of G with $d(x) + d(y) \leq 13$.

7. (6 points) Given a face f, we denote by $\ell(f)$ the length of f: the number of edges on the shortest closed walk traversing its boundary. Every edge of a plane graph either appears once on the boundary of two faces, or twice on a closed walk traversing one face. Therefore $\sum_{f \in F(G)} \ell(f) = 2\|G\|$, where $F(G)$ is the set of faces of a plane drawing of G.

   In lecture, when we used discharging, we assigned initial charges to vertices. This is not the only method. Three common methods are listed below.

   Vertex charging Assign to every vertex v an initial charge $\mu(v) = 6 - d(v)$, and assign to every face f an initial charge $\mu(f) = 6 - 2\ell(f)$.

   Face charging Assign to every vertex v an initial charge $\mu(v) = 6 - 2d(v)$, and assign to every face f an initial charge $\mu(f) = 6 - \ell(f)$.

   Balanced charging Assign to every vertex v an initial charge $\mu(v) = 4 - d(v)$, and assign to every face f an initial charge $\mu(f) = 4 - \ell(f)$.

   For each of these three cases, simplify the total charges in a planar graph G,

$$\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) \ .$$

   You may use n, m, and r for the number of vertices, edges, and faces (regions) of G.

8. (6 points) Let G be a 2-connected plane graph with girth $g(G) \geq 7$ and $\delta(G) \geq 2$.

   Use balanced charging to show that G has an edge xy such that $d(x) = 2$ and $d(y) \in \{2, 3\}$.

   You may assume without proof that in a 2-connected planar graph on at least 3 vertices, the boundary of every face is a cycle, so a vertex of degree k is incident to precisely k faces.

   Hint 1: Initial charges for all faces are strongly negative. So, faces can accept a fairly large amount of positive charge, without becoming positively charged themselves.

   Hint 2: After you've set up your discharging rules, suppose uv is an edge incident to a face f. If uv is not the type of edge the theorem says must exist, what is the maximum amount of charge that u and v can impart to f?

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	3	6	6	6	4	6	6	40