Math 318 Homework 10

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Question 1a. In the stationary distribution π, we have that

$$\pi_i = \frac{\pi_{i-1}}{i+1} + \frac{i\pi_{i+1}}{i+1} \implies \pi_i - \pi_{i-1} = i(\pi_{i+1} - \pi_i).$$

This implies we should look for solutions of the form $\pi_i = \frac{i+1}{i!}$, except normalized. The sum is:

$$\sum_{i=0}^{\infty} \frac{i+1}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} + \sum_{i=0}^{\infty} \frac{1}{i!} = 2e$$

$$\implies \pi_i = \frac{1}{2e \cdot i!}.$$

Question 1b. Doing the same manipulations as part a we get

$$\pi_i = \frac{\pi_{i+1}}{i+1} + \frac{i\pi_{i-1}}{i+1} \implies \pi_{i+1} - \pi_i = i(\pi_i - \pi_{i-1}).$$

However this implies that $\pi_i > \pi_{i-1} \forall i$ which is means normalization clearly isn't possible, so there can't be a stationary distribution.

Question 1c. In the stationary distribution π outside the edges, we have

$$\pi_i = \frac{\pi_{i-1}}{2} + \frac{\pi_{i+1}}{2} \implies \pi_i - \pi_{i-1} = \pi_{i+1} - \pi_i.$$

This means that the solution is linear. At the boundary, we have that $\pi_0 = \frac{\pi_1}{2}$ and $\pi_N = \frac{\pi_{N-1}}{2}$. Similarly for the second last edge we have that $\pi_1 = \pi_0 + \frac{1}{2}\pi_2$ and similarly for π_{N-1}. By symmetry it the distribution can't be linear in either direction, so the middle section must be uniform. Thus we get that $\pi_0 = \frac{1}{2N}$, $\pi_N = \frac{1}{2N}$ and $\pi_i = \frac{1}{N}, i \in \{1, 2, \ldots, N-1\}$.

Question 2a. We have a $\frac{n}{n+1}$ chance of going forward and $\frac{1}{n+1}$ chance of going backwards, so

$$q_i = \frac{iq_{i+1}}{i+1} + \frac{q_{i-1}}{i+1}.$$

Question 2b. Plugging the proposed solution into the equation we found:

$$\frac{n}{n+1} \sum_{i<n+1} \frac{1}{i!} + \frac{1}{n+1} \sum_{i<n-1} \frac{1}{i!} = \frac{n}{n+1} \left(\sum_{i<n} \frac{1}{i!} + \frac{1}{n!} \right) + \frac{1}{n+1} \left(\sum_{i<n} \frac{1}{i!} - \frac{1}{(n-1)!} \right)$$

$$= \sum_{i<n} \frac{1}{i} = q_n.$$

Question 2c. a_n fulfills the required equation above, so all we have to do is normalize it. Given $\lim q_n = 1$ this implies that $\sum_i A\frac{1}{i!} = 1$ which means that $q_n = \frac{1}{e}\sum_{i<n}\frac{1}{i!}$.

Question 2d. The equation given is just a random walk and we saw in class that for any $p \neq \frac{1}{2}$ this walk is not recurrent. Since here the probably of going forward is strictly greater than backwards this walk also isn't recurrent, so $\lim q_n = 1$.

Question 3. We can use the fact that reversible markov chain paths are path independent. Let a be the first question mark and b be the second. Then we have:

$$P_{0,1}P_{1,2}P_{2,0} = P_{0,2}P_{2,1}P_{1,0} \implies \frac{1}{3}\frac{2}{3}a = \frac{1}{6}b\frac{1}{3} \implies b = 4a.$$

Since $a + b = 1$, we have $b = \frac{4}{5}$ and $a = \frac{1}{5}$.

Question 4a. We can think of Carol's coin flips independently of those not involving her, let $f(x)$ be the probability that Carol wins given she currently has \$$x$. Clearly $f(0) = 0$ and $f(5) = 1$. To calculate the intermediate values we can use the probability of her winning each flip:

$$f(1) = \frac{1}{2}(0) + \frac{1}{2}f(2) = \frac{1}{2}f(2)$$

$$f(2) = \frac{1}{2}f(1) + \frac{1}{2}f(3) \implies f(2) = \frac{2}{3}f(3)$$

$$f(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) \implies f(3) = \frac{3}{4}f(4)$$

$$f(4) = \frac{1}{2}f(3) + \frac{1}{2}f(5) = \frac{3}{8}f(4) + \frac{1}{2} \implies f(4) = \frac{4}{5}$$

$$\implies f(1) = \frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{4}{5} = \frac{1}{5}.$$

Question 4b. Let $f(a, b, c)$ be the probability of carol going out given that Alice, Bob and Carol start with a, b, c dollars respectively. We're trying to find $f(2, 2, 1)$, and note that because f tracks only the probability of Carol going out first, $f(a, b, c) = f(b, a, c)$. Then summing over the conditional probabilities of Carol being chosen:

$$f(2,2,1) = \frac{2}{3}\left(\frac{1}{2} + \frac{1}{2}f(1,2,2)\right) + \frac{1}{3}f(3,1,1).$$

To calculate $f(3, 1, 1)$ we can look at the possible matchings and calculate conditional probabilities for each:

$$f(3,1,1) = \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}f(1,2,2)\right) + \frac{1}{3}\left(\frac{1}{2}(0) + \frac{1}{2}f(2,2,1)\right).$$

To solve this, note that $f(2, 2, 1) + 2f(1, 2, 2) = 1$. This is because $f(1, 2, 2)$ represents the odds that someone who starts from \$2 goes out first, and the sum of probabilities that Alice, Bob and Carol go out first must be 1. Thus $f(1, 2, 2) = \frac{1}{2} - \frac{1}{2}f(2, 2, 1)$ and we get:

$$f(2,2,1) = \frac{1}{3} + \frac{1}{6} - \frac{1}{6}f(2,2,1) + \frac{1}{3}\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{12} - \frac{1}{12}f(2,2,1) + \frac{1}{6}f(2,2,1)\right).$$

$$\implies \frac{41}{36}f(2,2,1) = \frac{23}{36} \implies f(2,2,1) = \frac{23}{41}.$$

Just to make sure this can be simulated in python, this confirms that the probability is around 0.56 with the following code:

```
import random

n = 10000

c_wins = 0

for _ in range(n):
    coins = [2, 2, 1]
    while True:
        i = random.randint(0,2)

        indices = [0,1,2]
        indices.remove(i)
        if random.random() > 0.5:
            coins[indices[0]]+=1
            coins[indices[1]]-=1
        else:
            coins[indices[0]]-=1
            coins[indices[1]]+=1
        if coins.count(0) != 0:
            print(coins)
            if coins[2] == 0:
                c_wins += 1
            break

print(c_wins / n)
```

Question 5. Simulating this scenario, the average over 10 tries was around 1,000,000 (specifically 983610). The following code was used:

```
import random

n = 10
N = 1000
times = []
for _ in range(n):
    opinions = [i for i in range(N)]
    t = 0
    while len(set(opinions)) > 1:
        t += 1
        x = random.randint(0,N-1)
        y = random.randint(0,N-1)
        while x == y: y = random.randint(0,N-1)
        opinions[x] = opinions[y]
    times.append(t)
et = sum(times) / n
print(times)
print(et)
```