ELEC 221 Homework 2

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Question 1a1. $z_1+z_2 = 3+4e^{j\frac{\pi}{4}} = 3+\frac{4}{\sqrt{2}}+j\frac{4}{\sqrt{2}} = (3+2\sqrt{2})+j2\sqrt{2} = 5.8284+2.8284j = 6.48e^{0.451j}$

Question 1a2. $\cos(\pi/4) - j\sin(\pi/4) + \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}} = 0.707 - 0.707j + 0.5 + 0.5 = 1.207 - 0.207j = 1.225e^{-0.17j}$

Question 1b1. $z_1 z_2 = 12e^{j\frac{\pi}{4}}$

Question 1b2. $z_1 z_2 = e^{-j\frac{\pi}{4}}\frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$

Question 1c1. Let $z = a + bi$. Then

$$\frac{(z + z*)}{2} = \frac{a + bj + a - bj}{2} = a = Re(z)$$

Question 1c2. Let $z = a + bi$. Then

$$\frac{(z - z*)}{2j} = \frac{a + bj - a + bj}{2j} = b = \Im(z)$$

Question 1c3. Let $z = a + bi$. Then

$$\sqrt{zz*} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2} = |z|$$

Question 1d. By definition, $z* = re^{-j\theta}$, so $\frac{1}{z*} = \frac{1}{re^{-j\theta}} = \frac{1}{r}e^{j\theta}$ as desired. For the graphs see figure 1.

Question 1e. Computing, we get

$$\mathbf{B}^H = \begin{bmatrix} \cos(\pi/4) & 0 & e^{-j(\frac{1}{3}+\frac{\pi}{2})} \\ e^j & 2-3j & e^2 \end{bmatrix}$$

Similarly computing for $\mathbf{v}$, we get

$$\mathbf{v}^H = \begin{bmatrix} 1 & e^{-j\pi/4} & e^{-j\pi/2} & e^{-j3\pi/4} \end{bmatrix}$$

Question 2a. Writing it out, we get

$$\phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(\pi/2)} & e^{j(\pi)} & e^{j3(\pi/2)} \\ 1 & e^{j(\pi)} & e^{j2\pi} & e^{j3\pi} \\ 1 & e^{j3(\pi/2)} & e^{j3\pi} & e^{j9(\pi/2)} \end{bmatrix}$$

Question 2b. By inspection, we find that

$$x_m = e^{j(2\pi/N)(m-1)}$$

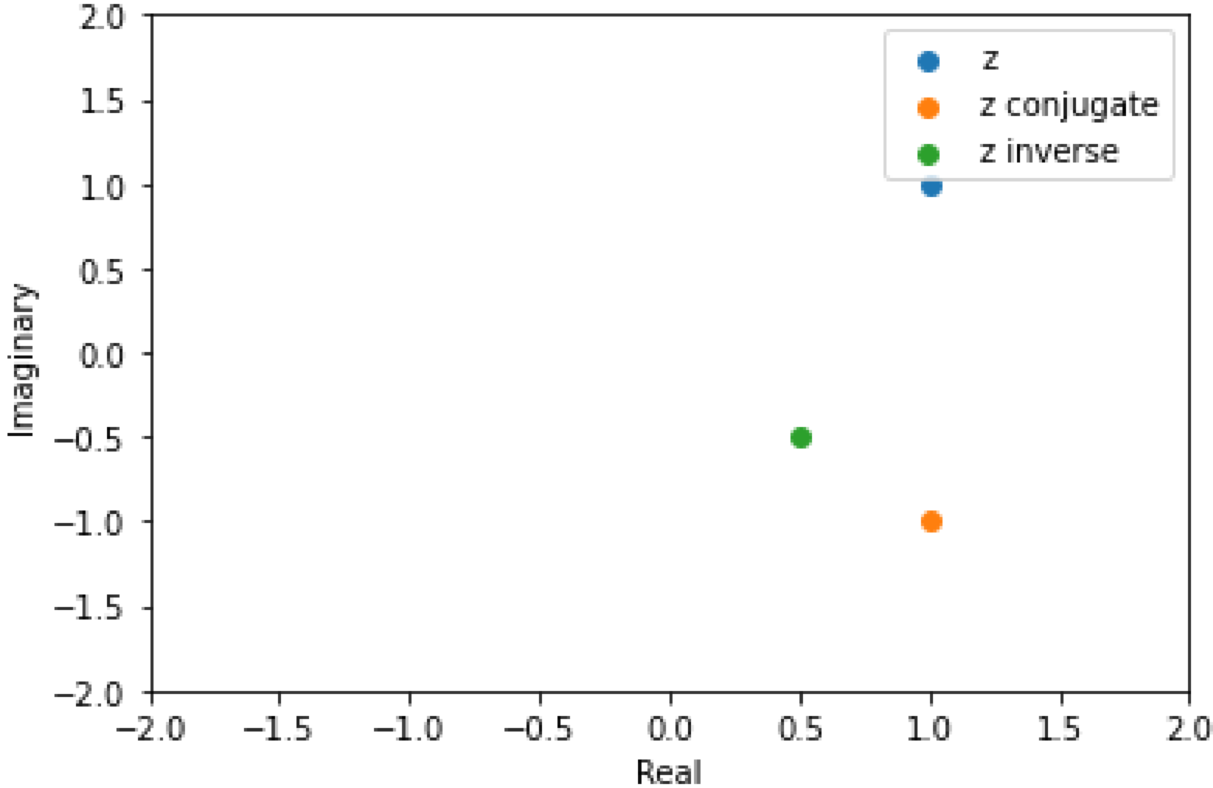

Figure 1: Graph for questio 1d

Question 2c. First note that because the matrix is symmetric about the diagonal, the transpose of the matrix is the matrix itself. Also since each entry has magnitude 1, taking the conjugate of each entry just changes the sign in the exponent. Thus the entry in the ith row and jth column in the final resulting matrix will be the ith row dotted with the conjugates of the jth column. Writing this out in summation form we have

$$[\Phi]^H[\Phi]_{m,l} = \sum_{n=1}^{N} e^{-j(2\pi/N)((m-1)(n-1)-(n-1)(l-1))} = \sum_{n=1}^{N} e^{j(2\pi(n-1)/N)(l-m)}$$

If $l = m$, i.e. along the diagonals we have that the expression in the summation is just 1, so the diagonals have value N. If $l \neq m$, then let $r = e^{j(2\pi(n-1)/N)(l-m)}$ and using the geometric formula we find that

$$[\Phi]^H[\Phi]_{m,l} = \frac{1-r^n}{1-r} = \frac{1-e^{j(2\pi(n-1)/N)(l-m)N}}{1-r} = \frac{1-e^{j(2\pi(n-1))(l-m)}}{1-r} = 0$$

This shows that all non-diagonal entries are 0, and the diagonals are non-zero so the matrix is diagonal.

Question 2d. Computing the dot product, we get

$$\langle \phi_k, \phi_l \rangle = \sum_{n=1}^{N} e^{j(2\pi/N)k(n-1)} e^{-j(2\pi/N)l(n-1)} = \sum_{n=1}^{N} e^{j(2\pi/N)(k-l)(n-1)}$$

If $k = l$, then the expression above becomes N. Otherwise, we again apply the geometric series and we see that

$$\langle \phi_k, \phi_l \rangle = \frac{1-e^{j(2\pi)(k-l)}}{1-e^{j(2\pi/N)(k-l)}} = 0$$

Thus the inner product between each of the vectors is zero and they are all orthogonal to each other.

Question 2e. As we saw in the previous part, if $k = l$ (i.e. the vectors are the same) the result was of magnitude N, so to normalize them they would have to be divided by $\sqrt{N}$

Question 2f. Since we know that $\mathbf{\Phi}^H\mathbf{\Phi}$ is a diagonal matrix, it must be the inverse of Φ up to a normalization. Since we found that the diagonals are N we get that $\mathbf{\Phi}^{-1} = \frac{1}{N}\mathbf{\Phi^H}$.

Question 2g. Applying the inverse to both sides, we have

$$\alpha = \mathbf{\Phi}^H s$$

Question 2h. First note that $\alpha_k = \langle s, \phi_k \rangle$. Thus we can write α as a row vector as

$$\alpha = \begin{bmatrix} \langle s, \phi_0 \rangle & \dots & \langle s, \phi_{N-1} \rangle \end{bmatrix} = \frac{1}{N} s\Phi^H$$

Where s is also represented as a row vector. Taking the inverse of both sides using the previous parts of this question, we get

$$\alpha N \frac{1}{N}\Phi = \frac{N}{N} s\Phi^H\Phi = s$$

Expanding the left side of this equation we get

$$s = \alpha\Phi = \begin{bmatrix} \langle \alpha, \phi_0^* \rangle & \dots & \langle \alpha, \phi_{n-1}^* \rangle \end{bmatrix}$$

Looking at this row vector term by term, we find that

$$s[n] = \langle \alpha, \phi_n^* \rangle = \sum_{k=0}^{N-1} \alpha_k e^{j(2\pi/N)kn}$$

Since this is exactly what was desired in the question, we are done.
Question 3a. The support set is $\{0, 1, 2, 3, 4, \}$. The length of the signal is 5.
Question 3b. The support set of the impulse is $\{0, 1, 2, 3\}$. The order $M = 4$.
Question 3c. The filter is causal because $h[n] = 0 \forall n < 0$.
Question 3d. The length of the output is $M + N - 1 = 8$.
Question 3e. The plot of their convulutions can be seen in figure 2. The computed values associated with the graph are as follows:

$$y = [6, 10, 14, -4, 10, 4, 0, 2, 0]$$

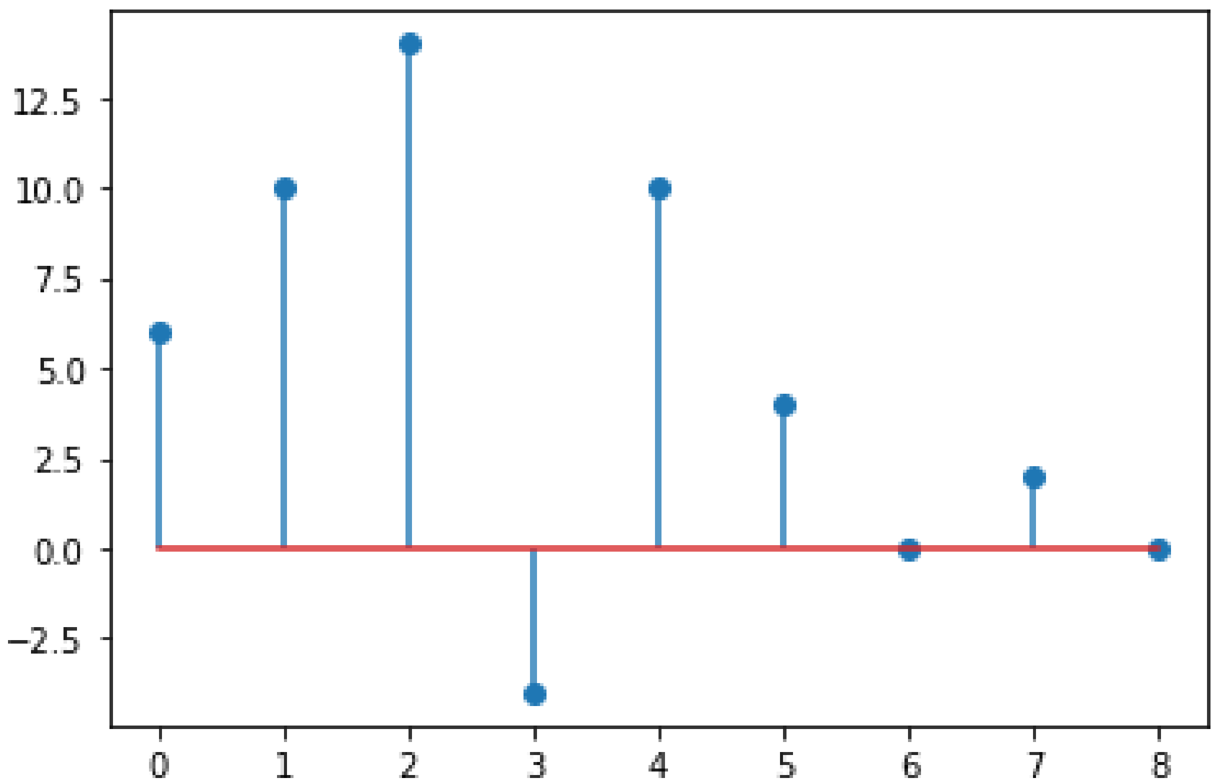

Figure 2: Plot for question 3e.

Question 5a. See figure 3.
Question 5b. Let $n, l \in \mathbb{Z}$. If $n \neq l$, then $s[n]\delta[n-\ell] = s[n](0) = 0 = s[\ell]\delta[n-\ell]$. If $n = l$, then we have $s[n]\delta[n-\ell] = s[n]\delta[0] = s[n] = s[\ell] = s[\ell]\delta[0]$. In either case theyr'e the same so we're done.
Question 5c. Doing the sum, we get

$$\sum_{\ell \in \mathbb{Z}} s[n]\delta[n-\ell] = s[n]\delta[0] + \sum_{\ell \in \mathbb{Z}, \ell \neq 0} s[\ell] \cdot 0 = s[n]$$

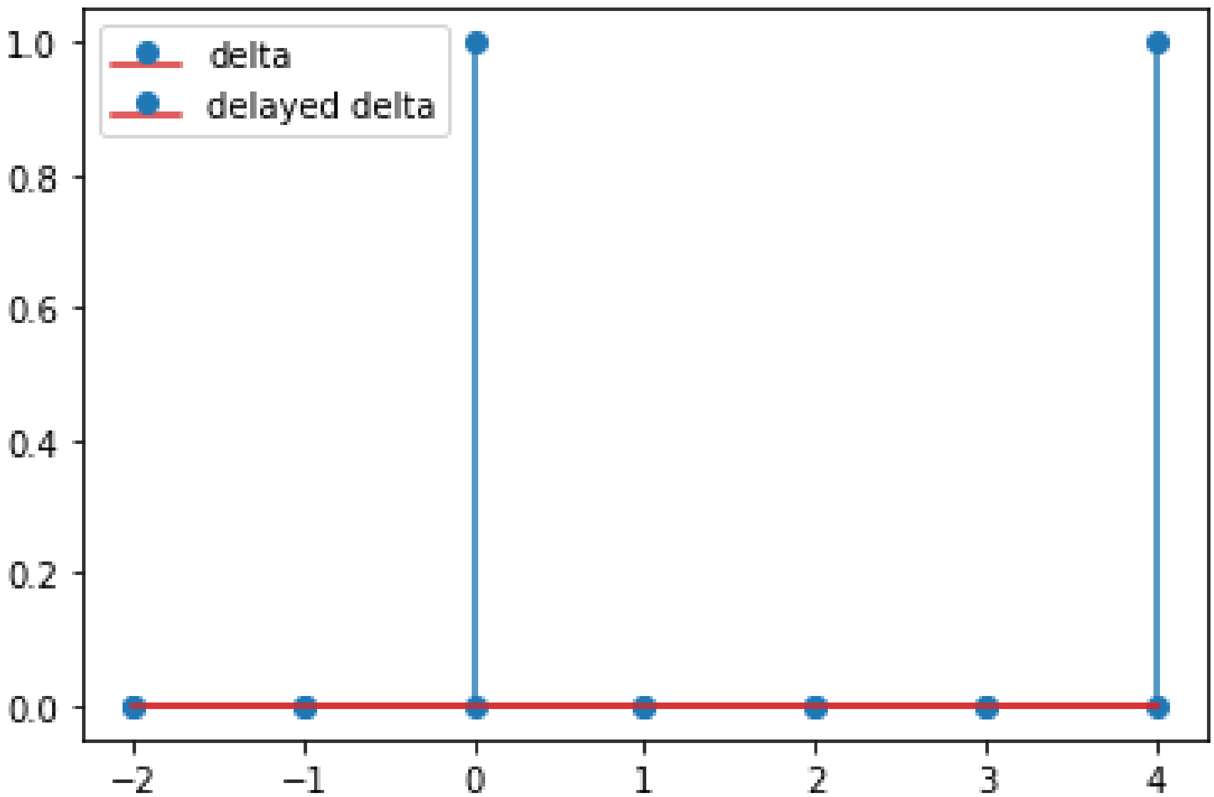

Figure 3: Figure for question 5a

Question 5d. Applying part b to the summand, we get

$$\sum_{\ell\in\mathbb{Z}} s[\ell]\delta[n-\ell] = \sum_{\ell\in\mathbb{Z}} s[n]\delta[n-\ell]$$

By part c this simplifies to $s[n]$ so we're done.

Question 5e. Let the system be T. Using part d, we have that

$$s[n] = \sum_{\ell\in\mathbb{Z}} s[\ell]\delta[n-\ell]$$

$$T(s[n]) = y[n] = \sum_{\ell\in\mathbb{Z}} s[\ell]T(\delta[n-\ell]) = \sum_{\ell\in\mathbb{Z}} s[\ell]h_\ell[n]$$

The last step was because by definition, $T(\delta[n-\ell]) = h_\ell[n]$.

Question 5f. Because the system is time invariant, it's impulse response to $\delta[n-\ell] = \delta[n] \forall \ell \in \mathbb{Z}$. This means that all of the h_ks are the same, i.e. $\exists h$ s.t. $h_k[n] = h[n] \forall k$. Then applying part e, we have that

$$y[n] = \sum_{k\in\mathbb{Z}} s[k]h_k[n-k] \sum_{k\in\mathbb{Z}} s[k]h[n-k]$$

as required.

Question 5g. See figures 4 and 5.

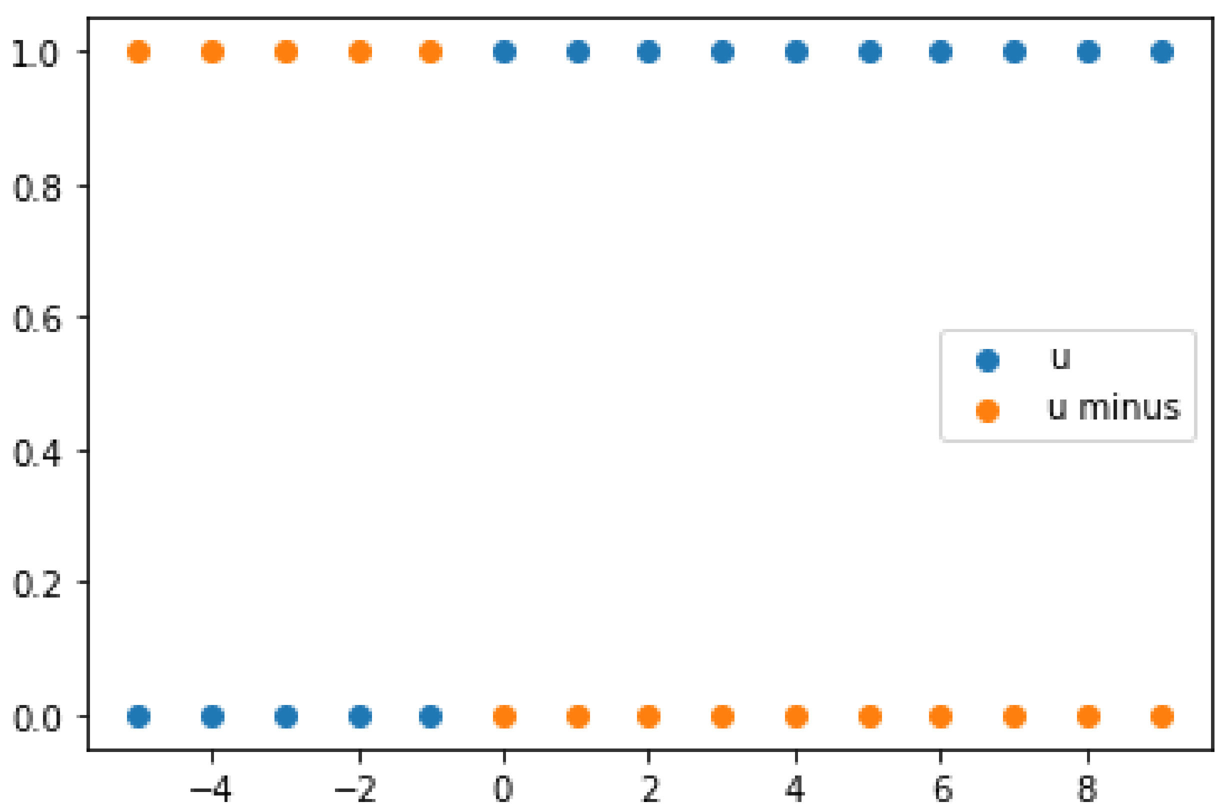

Figure 4: Graph for question 5g

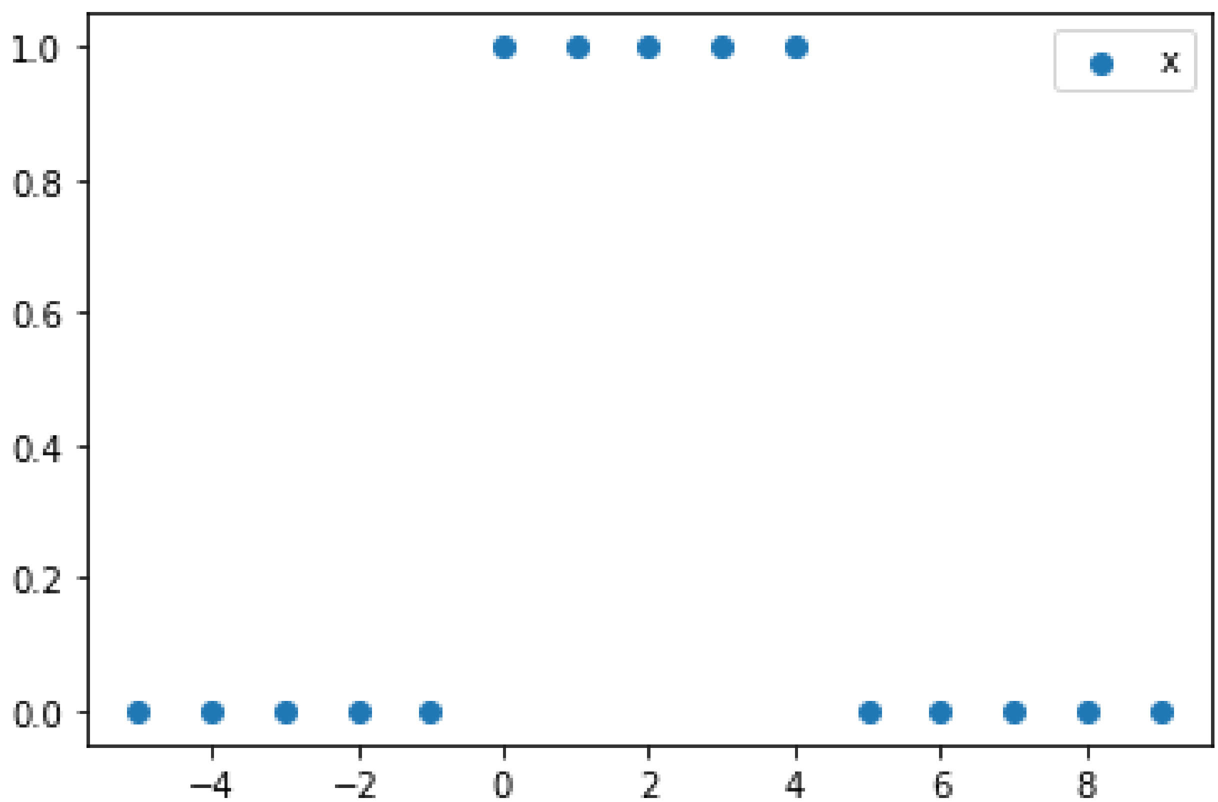

Figure 5: Graph for question 5g