Math 322 Homework 9

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Question 1. Clearly $N(P) \subseteq N(N(P))$, since a group normalizes itself by closure. For the other direction, let $g \in N(N(G))$. Then consider g acting on P: $H = gPg^{-1}$. H is a group and $|H| = |P| = p^r$. By the lemma in section 1.13, we then have that $H \subseteq P$, and since they have the same cardinality $H = P$. Since $gPg^{-1} = P$, g normalizes P and so $g \in N(P)$. Since this is true of all $g \in N(N(G))$, $N(N(G)) \subseteq N(G)$ and we're done.

Question 2. Factoring, we have $148 = 2^2 \cdot 37$. Consider Sylow 37-subgroups, by Sylow II we have that $n_{37} \equiv 1 \mod 37$ and $n_{37}|4$ (where n_p is the number of p-Sylow groups in G). The only solution to these equations is $n_{37} = 1$. Since there is only one Sylow 37-subgroup P and conjugation preserves group cardinality, we have $gPg^{-1} = P \forall g \in G$, i.e. P is normal and G isn't simple.

For $56 = 2^3 \cdot 7$, By Sylow II we have that $n_7 \equiv 1 \mod 7$ and $n_7|8$. Thus either $n_7 = 1$ or $n_7 = 8$. If $n_7 = 1$ then by the same logic as for 148, the unique Sylow 7-subgroup is normal. If $n_7 = 8$, then there are 8 distinct Sylow 7-subgroups. Since each of these are cyclic and unique, they don't intersect other than 1, so there are $6 \cdot 8 = 48$ different elements of order 7. By Sylow I there's at least one subgroup of order 8, which must must be comprised of the remaining 7 elements as well as the identity. But then there is only one subgroup of order 8, so it is normal and G isn't simple.

Question 3. If $p = q$ then the group is of order p^2 which by exercise 5 from the previous homework implies that G is abelian and thus any subgroup (e.g. subgroup of order p) is normal. Without loss of generality assume that $p > q$. Then we have that $n_p \equiv 1 \mod p$ and $n_p|q$, but since $p > q$ this means that $n_p = 1$. But a unique subgroup of a given order must be normal, so the group is simple.

Question 4. Let G be a non-abelian group of order 6. Then by Sylow II there is a unique subgroup H of order 3 since $n_3 \equiv 1 \mod 3$ & $n_3|2 \implies n_3 = 1$, so it is normal. Since $|H| = 3$ is prime it is cyclic, call it's elements $H = \{1, \sigma, \sigma^2\}$. Then G/H is a subgroup of order 2, so can be written as $G/H = \{H, \tau H\}$. Thus G is given by $G = \{1, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$.

The only remaining choice in specifying G is the behavior of $\sigma\tau$, with this any combination of σ and τ can be reduced to one of the forms above. Since G isn't abelian, $\sigma\tau \neq \tau\sigma$. Clearly $\tau\sigma \neq 1, \sigma, \sigma^2, \tau$ since σ and τ are invertible. The only remaining choice is $\sigma\tau = \tau\sigma^2$. This is exactly S_3 under the map $\sigma \to (123)$ and $\tau \to (23)$, so G is isomorphic to S_3 using this map.

Question 5. Let G be a group of order 15. Using Sylow's theorems there is a unique subgroup H of order 5 in any group of order 15 (since $n_5 \equiv 1 \mod 5$ and $n_5|3 \implies n_5 = 1$). Then H is cyclic as it is of prime order and normal since it's the only subgroup of order 5, and thus G/H is a cyclic group of order 3. We proved in class that G is abelian, so using these facts we can write every element in G as $a^i b^j$ where a is order 3 and b is order 5. Thus $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, so there is only one possible G up to isomorphisms.

Question 6. Let n be the order of uv. Then I claim that $\langle u, v\rangle$ is isomorphic to D_n. Let $\sigma = uv$ and $\tau = u$. Then $|\langle\sigma\rangle| = n$, $|\langle\tau\rangle| = 2$, and $(\sigma\tau)^2 = (uvu)(uvu) = uvu^2vu = uv^2u = u^2 = 1$. D_n

is generated as $\langle \sigma, \tau \mid \sigma^2 = 1, \tau^n = 1, (\sigma\tau)^2 = 1 \rangle$, so since the multiplication and the cardinalities ($2n$) are preserved $\langle u, v \rangle \cong D_n$.

Question 7. Since u, v are order 2 then $u^{-1} = u$ and $v^{-1} = v$. Then using the fact that $(uv)^{-1} = v^{-1}u^{-1} = vu$ we have:

$$(uv)^n = 1 \implies v = (uv)^{n-1}u = (uv)^{\frac{n-1}{2}}u(uv)^{-\frac{n-1}{2}}$$

Letting $g = (uv)^{\frac{n-1}{2}}$ (n is odd so this is well defined) this fulfills the definition of conjugate.

Question 8. Assume (uv) has order $2n$ (an unfortunate choice of variable name given I was previously using n to be the order of uv). Then we have:

$$uw = u(uv)^n = v(uv)^{n-1} = (vu)^{n-1}v = (vu)^n u^{-1} = (uv)^{-n}u = (uv)^n u = wu.$$

Similarly:

$$vw = v(uv)^n = (vu)^n v = (uv)^{-n}v = (uv)^n v = wv.$$

Thus $\{u, v\} \subseteq C(w)$.

Question 9. As the hint suggests, we will count the number of ordered pairs (x, y) with x conjugate to u_1 and y conjugate to u_2 in two different ways. First look at how many choices for x there are. The orbit of u_1 under G by conjugation by theorem 1.10 is $[G : C(u_1)] = \frac{|G|}{|C(u_1)|} = \frac{|G|}{c_1}$. By symmetry the same is also true of y and each choice is independent, so the number of such combinations of x and y is $\frac{|G|^2}{c_1c_2}$.

Consider x, y with x conjugate to u_1 and y conjugate to u_2. If $o(xy)$ is odd then by question 7 we have that x is conjugate to y which isn't possible since u_1 isn't conjugate to u_2, so $o(xy)$ must be even. But then by question 8 we have that for $n = \frac{o(xy)}{2}$, $(xy)^n$ has order 2, and since G only has two conjugacy classes it must either be conjugate to u_1 or u_2. Then another way of counting the number of possible such x, y is to divide them into two groups: those with $(xy)^n$ conjugate to u_1 and those with $(xy)^n$ conjugate to u_2. For each member g of the conjugacy class of u_i we can consider the set $\{(x, y) : x \text{ conjugate to } u_1, y \text{ conjugate to } u_2, (xy)^n = g\}$. The cardinality of this set is s_i regardless of g and there are $\frac{|G|}{c_i}$ choices for g, so the total possible choices of x, y with $(xy)^n$ conjugate to u_i, x conjugate to u_1 and y conjugate to u_2 is $\frac{|G|s_i}{c_i}$. Summing over $i = 1, 2$ and comparing with our previous computation, we arrive at:

$$\frac{|G|^2}{c_1c_2} = \frac{|G|s_1}{c_1} + \frac{|G|s_2}{c_2} \implies |G| = c_1s_2 + c_2s_1.$$