Math 318 Homework 3

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03/02/23

Question 1a. Clearly $P(X = x) = 0 \forall x < 4$ since it takes at least 4 games for the game to end. For the remaining games, fix the number of games x and consider the games as repeated Bernoulli trials. There are two possible cases: A won or B won (where the event A is A winning and the event B is B winning). Regardless of which one, the winner must have won the last game (otherwise the game would have ended earlier). The remaining games are distributed in a binomial distribution. Therefore:

$$P(X = x \cap A) = \binom{x-1}{3} p^3 (1-p)^{x-3}.$$

$$P(X = x \cap B) = \binom{x-1}{3} p^{x-3} (1-p)^3.$$

Since these are independent events their probability can be summed, so

$$P(X = x) = \binom{x-1}{3} p^3 (1-p)^{x-3} + \binom{x-1}{3} p^{x-3} (1-p)^3.$$

Question 1b. If $X = 4$ and A won then clearly A won all four games which occurs with probability $P = p^4$.

Question 1c. To get to the point that 7 games are played it must have been tied 3-3 previously or else the game would already have been over. The probability that A wins from that point is just the probability that A wins the last game, which is just $P = p$.

Question 2a. Let y be the number of unique results. There are $\binom{6}{y}$ combinations of y numbers that are unique, while there are 6^4 possible combinations of rolls. The number of unique ways of ordering the rolls depend on what y is.

If $y = 1$ then there's only one way of arranging them, and if $y = 4$ then there's 4! orderings of rolls. If $y = 3$ then $\binom{4}{2}3!$ ways of arranging the rolls, since one pair of rolls must result in the same number and there are 3! ways of choosing which number has the pair. Finally if $y = 2$ then either the rolls are distributed with 2 in each repeated number or there are 3 in 1 and 1 in the other. If it's 2-2 then there are $\binom{4}{2}$ ways of arranging the rolls, with 4 choices for the first pair and the remaining in the second. If it's arrange 3-1 there are $2 \cdot \binom{4}{1} = 8$ ways of this occurring. Putting this together:

$$P(Y = 1) = \frac{\binom{6}{1} 1^4}{6^4} = \frac{1}{216}.$$

$$P(Y = 2) = \frac{\binom{6}{2} \left(\binom{4}{2} + 2 \cdot 4\right)}{6^4} = \frac{35}{216}.$$

$$P(Y=3) = \frac{\binom{6}{3}\binom{4}{2}3!}{6^4} = \frac{5}{9}.$$

$$P(Y=4) = \frac{\binom{6}{4}4!}{6^4} = \frac{5}{18}.$$

The expected value is just the sum of the mass function:

$$EY = \sum_{i=1}^{4} iP(Y=i) = \frac{1}{216} + \frac{70}{216} + \frac{15}{9} + \frac{20}{18} \approx 3.1.$$

Question 2b. Consider that the probability that the minimum is z is the probability that all the numbers are z or greater minus the probability that they are all $z+1$ or greater. Thus we have:

$$P(Z=z) = 1 - \frac{(6-z)^4}{6^4} - \left(1 - \frac{(7-z)^4}{6^4}\right) = \frac{(7-z)^4 - (6-z)^4}{6^4}.$$

For the expected value, again we can just sum over the possibilities:

$$EZ = \sum_{i=1}^{4} iP(Z=i) = \sum_{i=1}^{4} \frac{(7-i)^4 - (6-i)^4}{6^4} = \frac{2275}{1296} \approx 1.76.$$

Question 3a. Summing over the possible outcomes:

$$P(Y>m) = \sum_{i=m+1}^{\infty} p(1-p)^{i-i} = p(1-p)^m \sum_{i=0}^{\infty} (1-p)^i = (1-p)^m.$$

Question 3b. Computing the limit:

$$\lim_{\delta \to 0} (1 - \lambda\delta)^{\frac{t}{\delta}}.$$

Let $\delta' = -\delta\lambda$. Then:

$$\lim_{\delta' \to 0} \left(1 + \delta'\right)^{-\lambda \frac{t}{\delta'}}.$$

This is a well known identity for e, so it converges to $e^{-\lambda t}$

Question 3c. From the previous parts, we know that $P(Y>m)$ is approximately an exponential random variable. We also have:

$$P(Y>m) = \int_m^{\infty} P(Y=y)dy \implies \frac{d}{dt}(1-e^{-\lambda t}) = P(Y=y) \implies P(Y=y) = \lambda e^{-\lambda t}.$$

Question 4. Suppose without loss of generality that a 0 is being transmitted (the same argument in reverse works in that case). The probability that the message is received correctly is the probability that the normal distribution is less than $\frac{1}{2}$. Plugging this into a calculator (since the CDF of the normal distribution isn't elementary):

$$P = P(N(0, 0.04) < 0.5) = 0.9938.$$