Math 320 Homework 4

Xander Naumenko

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Question 1i. False, let $x_n = n+(-1)^n$. Then clearly $x_n \to \infty$ (for any M choose $N = M+1$, then for $n > N$ we have $x_n > n-1 = M$). However for any n that is even we have $x_n = n+1 > n = x_{n+1}$.

Question 1ii. The statement is true. By contradiction assume $x_n \to \infty$ with no increasing subsequence. Since no increasing subsequence exists, every increasing subset of x_n is of finite length, and choose $n_1, n_2, \ldots, n_K$ be a longest such increasing subsequence. Let $N = x_{n_K}$, then since $x_n \to \infty$ there exists N such that $(n > N) \implies (x_n > x_{n_K})$. Then let $n_{K+1} = \max(n_K, N) + 1$. Then $x_{n_{K+1}} > x_{n_K}$ with $n_{K+1} > n_K$, but this contradicts our assumption that the $n_1, \ldots, n_K$ were chosen to be maximal since adding $x_{n_{K+1}}$ would make a longer increasing subsequence. Thus an increasing subsequence of infinite length must exist.

Question 2a. The sequence converges. Note that we have:

$$a_n = n\frac{1+\frac{1}{n}-1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

I claim that $a_n \to \frac{1}{2}$. To see this let $\epsilon > 0$, and choose $N = \max\left(10, \frac{1}{\left(\frac{1}{\epsilon+1/2}-1\right)^2-1}\right)$. Then for $n > N$,

$$|a_n - \frac{1}{2}| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| < \epsilon.$$

Question 2b. The sequence does not converge. Let $L \in \mathbb{R}, \epsilon = \frac{1}{2}$, and $N > 0$. Choose n to be an arbitrary even integer greater than N if $L < 0$ and an odd integer greater than $\max(N, 3)$ otherwise. Then:

$$|b_n - L| = \left|\frac{(-1)^n n}{n+1} - L\right| = \left|\frac{n}{n+1}\right| + |L| > \frac{1}{2} + |L| \geq \frac{1}{2} = \epsilon.$$

Question 3a. I claim that $\Sigma(A)$ being defined and finite implies that there is a maximum element of A. To see why suppose not, i.e. suppose that $\forall a \in A, \exists b \in A$ s.t. $b > a$. Then let $F \subset A$ be a subset with $\Sigma(F) \geq \Sigma(A)/2$, by hypothesis there exists $b \in A$ s.t. $b > \max(F)$. However then $\Sigma(F \cup \{b\}) > \Sigma(A)/2 + x\Sigma(A)/2 = \Sigma(A)$, which contradicts the definition of Σ. Thus A has a maximum element.

However this implies that A is countable. To see why, consider letting $x_1 = \max(A)$ and $A' = A \setminus \{x_1\}$. Note that since $\Sigma(A')$ is also well defined since we just removed a single element, so it also has a maximum. Then we can let $x_2 = \max(A \setminus x_1)$ and so forth to enumerate all the (potentially infinite) elements of A. Since we've just created an onto map from $\mathbb{N} \to A$ either A is countable or finite.

Question 3b. Let $L = \lim_{n\to\infty} \sum_{n=1}^{N} a_n$. For any $F \subset A$, clearly we have that

$$L = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \geq \sum_{f \in F} f,$$

implying that $\Sigma(A) \leq L$. Let $R < L$, and let $\epsilon = L - R$. Then there exists $N \in \mathbb{N}$ s.t. $\forall n > N$,

$$\left| \sum_{i=1}^{n} a_i - L \right| < \epsilon.$$

Let $F' = \{a_n : n \leq N+1\}$. Then $\Sigma(F') = \sum_{n=1}^{N+1} a_n > L - \epsilon = R$. Since this is true independent of our choice of R, it must be that $\Sigma(A) = L$.

Question 4a. Let $(x, y) \in \mathbb{R}^2$. By their definition note that $W_2(y) \leq f(x, y)$ and $M_1(x) \geq f(x, y)$, since (x, y) is contained in both sets that M_1 and W_2 are taking the supremum and infimum respectively. Putting those two statements together we that $\forall (x, y) \in \mathbb{R}^2, W_2(y) \leq M_1(x)$. Since this holds over all of R^2 taking the supremum and infimum over the left and right sides respectively does nothing to change this inequality, so arrive as required to

$$\sup\{W_2(y) : y \in Y\} \leq \inf\{M_1(x) : x \in X\}.$$

Question 4b. Define

$$f(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

Then we have that

$$\sup\{W_2(y) : y \in Y\} = \sup\{0\} = 0 \leq 1 = \inf\{1\} = \inf\{M_1(x) : x \in X\}.$$

Question 5. Let $x \in [0, \inf(S))$. Then $\forall s \in S$ we have $x < s$, so $x \in [0, s)$. Thus $x \in \bigcap_{S \in S}[0, s)$. Next let $y \in (\inf(S), \infty)$. Since $y > \inf(S)$, there exists $s \in S$ with $s < y$, so $y \notin [0, s) \implies y \notin \bigcap_{S \in S}[0, s)$. These facts about x and y together imply that $\bigcap_{s \in S}[0, s) = [0, \inf(S)$ or $[0, \inf(S)]$.

There are two cases: $\inf(S) \in S$ or $\inf(S) \notin S$. If it's the former, then $\inf(S) \notin [0, \inf(S)) \implies \bigcap_{S \in S}[0, s) = [0, \inf(S))$. If $\inf(S) \notin S$ then $\forall s \in S, \inf(S) \in [0, s)$, so $\bigcap_{S \in S}[0, s) = [0, \inf(S)]$.

Question 6a. Let $\epsilon > 0$, since x_n and y_n are Cauchy there exists N_x, N_y s.t. $\forall n > N_x, p > 0, |x_{n+p} - x_n| < \frac{\epsilon}{2}$ and $\forall n > N_y, p > 0, |y_{n+p} - y_n| < \frac{\epsilon}{2}$. Let $N = \max(N_x, N_y)$. Then $\forall n > N, p > 0$, we have

$$|x_{n+p} + y_{n+p} - x_n - x_p| \leq |x_{n+p} - x_n| + |y_{n+p} - y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $x_n + y_n$ is also Cauchy.

Question 6b. Since x_n and y_n are Cauchy then they are bounded, let X, Y be such that $x_n > X, y_n > Y \forall n$. Define ϵ, N_x, N_y the same as for the previous part, except for $|x_{n+p} - x_n| < \frac{\epsilon}{2M}$ and $|y_{n+p} - y_n| < \frac{\epsilon}{2N}$. Then for $n > \max(N_x, N_y), p > 0$, we have

$$|x_{n+p}y_{n+p} - x_n y_n| \leq |x_{n+p}(y_{n+p} - y_n)| + |y_n\,(x_{n+p} - x_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 6c. Let $\epsilon > 0$, since x_n is Cauchy there exists N_x such that for any $n > N_x, p > 0, |x_{n+p} - x_n| < \frac{\epsilon}{2}$. Also since $(y_n - x_n) \to 0$, $\exists N$ s.t. $\forall n > N, p > 0, |x_{n+p} - y_{n+p} - x_n + y_n| < \frac{\epsilon}{2}$. Choose $N_y =$ and let $p > 0$. Then we have that

$$|y_{n+p} - y_n| \leq |x_{n+p} - y_{n+p} - x_n + y_n| + |x_{n+p} - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 7a and b. I will prove both a and b at once despite the suggestion. We will first solve the recurrence relation. Assume a solution is of the form $a^n = x^n$ for some $x \in \mathbb{R}$. For this to be true, we'd need

$$x^n = (1-\lambda)x^{n-1} + \lambda x^{n-2} \implies x^2 - (1-\lambda)x - \lambda = 0 \implies x = \frac{1}{2}\left(1 - \lambda \pm \sqrt{(1-\lambda)^2 + 4\lambda}\right)$$

$$\implies x = \frac{1}{2}(1 - \lambda \pm \lambda + 1) = -\lambda \text{ or } 1.$$

Since this is a second order recurrence relation, all solutions are a linear combination of these two possibilities, i.e. $a_n = c_1(-\lambda)^n + c_2$ for some $c_1, c_2 \in \mathbb{R}$ determined by the initial conditions. Plugging in the initial conditions, specifically they are determined by the following two linear equations:

$$\begin{cases} a_0 = c_1 + c_2 \\ a_1 = -c_1\lambda + c_2 \end{cases} \implies \begin{cases} c_1 = \frac{a_0 - a_1}{\lambda+1} \\ c_2 = \frac{\lambda a_0 + a_1}{\lambda+1} \end{cases}.$$

I claim that $(-\lambda)^n \to 0$. Let $\epsilon > 0$, and choose $N = \log_\lambda(\epsilon)$. Since $0 < \lambda < N$ note that $(-\lambda)^n$ is a strictly decreasing sequence. Then for $n > N$, we have

$$|(-\lambda)^n - 0| < |\lambda|^{\log_\lambda(\epsilon)} = \epsilon.$$

Thus $(-\lambda)^n \to 0$. Thus we can conclude that:

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_0 - a_1}{\lambda+1}(-\lambda)^n + \frac{\lambda a_0 + a_1}{\lambda+1} = \frac{\lambda a_0 + a_1}{\lambda+1}.$$

Since we've found what α is explicitly, clearly we've also found that the sequence converges.

Question 8a. Assume that $\inf(S) = 0$, and let $R > 0$. Since $\inf(S) = 0, \exists x \in S$ with $x < \frac{1}{R} \implies \frac{1}{x} > R$. Since $\frac{1}{x} \in S^{-1}$ and R was arbitrary, we have that $\sup(S) = +\infty$.

Question 8b. Both directions will be proven:

($\Rightarrow$) Let $N = \frac{2}{\inf(S)}$. If there existed $x \in S^{-1}$ with $x > N$ the that would imply that $\frac{1}{x} < \frac{\inf(S)}{2} < \inf(S)$ which would be a contradiction since $\frac{1}{x} \in S$, so $N < \infty$ is an upper bound of S^{-1}.

($\Leftarrow$) Let $\epsilon = \frac{1}{2\sup(S^{-1})}$. If there existed $x \in S$ with $x < \epsilon$ then that would imply that $\frac{1}{x} > \frac{1}{\epsilon} = 2\sup(S^{-1}) > \sup(S^{-1})$, which would be a contradiction since $\frac{1}{x} \in S^{-1}$, so $\epsilon > 0$ is a lower bound of S^{-1}.

Since both directions hold, the statement is true. When these are true, it remains to be shown that $\sup(S^{-1}) = (\inf(S))^{-1}$. Proof by contradiction, assume that $0 < \inf(S) < +\infty$ and $\sup(S^{-1}) \neq (\inf(S))^{-1}$. Without loss of generality assume that $\sup(S^{-1}) > (\inf(S))^{-1}$ (if it's the other way the whole argument just works in reverse as seen just above, I'm getting tired of rewriting things). Then let $x \in ((\inf(S))^{-1}, \sup(S^{-1})]$ with $x \in S^{-1}$, the existence of such an x is guarantee by the definition of a supremum. Then $\frac{1}{x} < \inf(S) \implies x \notin S$ but by the definition of S^{-1} it should be that $\frac{1}{x} \in S$. This is a contradiction, so equality must hold.

Question 8c. Let $y_n = \sup(\{x_m^{-1} : m > n\})$. There are three cases: $y_n \to 0, y_n \to L \in \mathbb{R}$ or $y_n \to +\infty \in \mathbb{R}$. Note that by part a and b, $y_n = (\inf(\{x_m : m > n\}))^{-1}$, which can be used to convert between the lim inf and lim sup as follows. If $y_n \to 0$, then $\limsup_{n\to\infty}(x_n^{-1}) = 0 = \frac{1}{+\infty} = \left(\liminf_{n\to\infty} x_n\right)^{-1}$. If $y_n \to L$, then it is simply $\limsup_{n\to\infty}(x_n^{-1}) = L = \frac{1}{1/L} = \left(\liminf_{n\to\infty} x_n\right)^{-1}$. Finally if $y_n \to +\infty$, we get $\limsup_{n\to\infty}(x_n^{-1}) = +\infty = \frac{1}{0^+} = \left(\liminf_{n\to\infty} x_n\right)^{-1}$.