MATH 443 Homework 4

Xander Naumenko

13/02/23

Question 1. Suppose by way of contradiction that the root of a DRT T had in-degree nonzero. Let r be the root of T and v be a vertex such that $vr \in E(G)$. Then vr is a path from v to r in the underlying tree of T. Since T is a tree it is the only path between r and v, so there doesn't exist a directed path from r to v, which is a contradiction of our assumption that T is a DRT. Therefore r has in-degree zero since no vertices can lead into it.

To see why each other vertex $v \neq r$ must have in-degree 1, first note that they clearly can't have in-degree 0 since there exists a directed path from r to v, and the last edge in this path will contribute 1 to the in-degree of that vertex. To see why the number of in vertices can't be more than one, supposed by contradiction that it was, i.e. suppose there exist $u_1, u_2 \in V(T)$ s.t. $u_1v \in E(T), u_2v \in E(T)$. Since $u_1, u_2 \in V(T)$ there exists ordered paths P_1, P_2 such that the first vertex of both is r and the last is u_1 and u_2 respectively. Let w be the last vertex of P_1 that is also in P_2. Let Q_1, Q_2 be the sub paths of P_1, P_2 that go from w to v. $Q_1 \cap Q_2 = \emptyset$ since w was chosen to be the last shared vertex. However then wQ_1vQ_2w is a cycle which is impossible in a tree. Therefore the in-degree number can't be 0 and can't be 2 or greater, so it must be 1. $\square$

Question 2. Let T_1, T_2 be disjoint DRTs and let e be a directed edge with one endpoint in T_1 and the other in T_2.

($\Rightarrow$) Assume $(T_1 \cup T_2) + e$ is a DRT, we will prove that the second vertex of e is the root of T_1 or T_2, call them r_1, r_2. To see why suppose by contradiction that that the second vertex of $e = uv$ is not r_1, r_2, and WLOG assume $u \in E(T_1), v \in E(T_2), v \neq r_2$. Let P_1 be a directed path from r_1 to u and let P_2 be a directed path from r_2 to v. Then $P_1 + e$ is a directed path from r_1 to v in the new graph. However this implies that in the new graph v has an in-degree of at least 2 (since the second last vertex of P_1 is in T_1 and the second last vertex of P_2 is in T_2). However problem 1 in this homework proved that no vertex on a DRT has in-degree 2 or greater, so it must be that e was the root of T_1 or T_2.

($\Leftarrow$) Assume the endpoint of e is the root of T_1 or T_2. WLOG assume $e = ur_2$ where r_2 is the root of T_2, and let r_1 be the root of T_1. Let $T = (T_1 \cup T_2) + e$ and let $x \in V(T)$. If $x \in T_1$ then since T_1 is a DRT there exists a directed path from r_1 to x. If $x \in T_2$ then let P_1 be a directed path from r_1 to u and P_2 be a directed path from r_2 to x. Then $r_1P_1eP_2x$ is a directed path from r_1 to x, so r_1 fulfills all the root requirements for T. Also note that since T_1, T_2 were disconnected before adding e, e is a bridge. T_1, T_2 were both trees and we added a bridge to connect them so T is a tree. Finally since T_1, T_2 were both DRTs and we added an edge $e = ur_2$ such that $r_2u \notin E(T)$, the second requirement for a DRT is also satisfied. We've shown that T fulfills all the requirements for a DRT, so it is one and we're done. $\square$

Question 3. The statement is true. Let P be a longest path of a connected graph G, and let u, v be its two endpoints. The statement will be shown for u, although since u, v were arbitrary it also holds for v. By way of contradiction let $x, y \in G - u$ such that x, y are disconnected in $G - u$.

G is connected so there exists a path $Q \subset G$ from x to y. Given that x, y became disconnected after removing u it must be that $u \in V(Q)$. $x \neq u, y \neq u$, so there exist paths Q_1 between x and u_1 and Q_2 between u_2 and y, where $u_1, u_2 \in N(u)$. We proved in class that $N(u) \in V(P)$, since otherwise you could extend P by including a vertex of $N(u)$ not already in P. Let $P_1 \subset P$ be the portion of P between u_1, u_2. Note that $u \notin P_1$ since u is an endpoint of P, so it's not a midpoint of any subpath. But then $xQ_1u_1P_1u_2Q_2y$ is a path of $G - u$ between x and y which contradicts our assumption that x, y disconnected. Therefore u couldn't have been a cut vertex, and by the symmetry of the argument v couldn't have been either. $\square$

Question 4. The flaw is that the distance between the endpoints of a longest path of a graph G are not necessarily the farthest from each other. To see why consider C_4. Then the longest path includes all 4 vertices, but the distance between two endpoints of such a path is just 1 whereas the actual farthest vertices are distance 2. Therefore it's not valid to assume that the endpoints of a longest path are the vertices that are the farthest from one another.

Question 5. The statement is true. Let C be a component of G. Clearly C is connected, since otherwise it wouldn't be a component. Let V be a set of vertices of size smaller than 1. The only possibility is that $|V| = 0 \implies V = \emptyset$. However $C - V = C - \emptyset = C$ is connected, so C is 1-connected. $\square$

Question 6. Let v be the vertex added to G and V be a minimal separating set of G'. We will consider two cases: $v \in V$ and $v \notin V$.
Case 1 ($v \in V$): Note that $V - v$ must be a separating set for G (since $V - v \subset G$ and V separates G'). Therefore $|V - v| \geq k \implies |G| \geq k + 1 > k$ as required.
Case 2 ($v \notin V$): If v is a trivial component in $G' - V$, $N(v) \subset V$ (in fact $N(v) = V$ by V's minimality but this isn't required) and $|V| \geq |N(v)| \geq k$ which is what we're trying to show. If v isn't its own component, it is part of a component $C \subset G', |C| \geq 2$ and $G' - V$ is disconnected. Then $C - v \neq \emptyset$ so $G - V = G' - V - v$ separates G, so $|V| \geq k$. Either way we've shown that $|V| \geq k$, which is the definition of being k-connected. $\square$

Question 7. Consider the number of edges that must be removed to generate three components. Let x be the number of vertices of the first component G_1, and y be the number of vertices of the second component G_2. Then the last vertex has $3n - x - y$ vertices, call it G_3. The number of edges to disconnect G_1 is $x(3n - x)$, since each of the x vertices is attached to $3n - x$ vertices. G_2 has y vertices attached to $3n - y$ vertices, but x of the edges attached to the second of those vertices was already removed in the first step. Therefore you must remove $y(3n - x - y)$ vertices to separate the second component. After removing both of these the last will be disconnected, since neither of the G_1, G_2 are connected to it by design. Thus the total number of edges required to disconnect is:

$$f(x, y) = x(3n - x) + y(3n - x - y) = 3n(x + y) - (x^2 + xy + y^2).$$

Fix x and consider optimizing y. The optimum occurs either on the edges ($y = 1, y = 3n - 2$) or where the derivative is zero. Note however that this is a inverted parabola with its vertex at $y = \frac{-b}{2a} = \frac{3n-x}{2}$ which is its maximum point, so if we were to take the derivative and set it to zero we would find a maximum, not am minimum. Thus $y = 1$ or $y = 3n - 2$.

If $y = 3n - 2$, then the only option for x would be $x = 1$ and the total number of edges is $f(1, 3n - 2) = 6n - 3$. If instead $y = 1$, then using the same logic as we used to optimize y, $x = 3n - 2$ or $x = 1$. Since the remaining component will be the opposite of whatever choice of x we use (and switching them doesn't change f), WLOG assume $x = 1$ as well. Then the total number of removed edges is still $f(1, 1) = 6n - 3$. We've covered all possible cases, so the minimum number of edges required is $|X| = 6n - 3$.