MATH 400 Homework 2

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Question 1. Let $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$. Then the ODE becomes

$$\nabla^2 u = u_{tt} \implies \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = u_{tt}$$

$$\implies \frac{\Theta T}{r}(rR_r)_r + \frac{RT}{r^2}\Theta_{\theta\theta} = R\Theta T_{tt}$$

$$\implies \frac{T_{tt}}{T} = \frac{R_r + rR_{rr}}{rR} + \frac{1}{r^2}\frac{\Theta_{\theta\theta}}{\Theta} = -\lambda.$$

Since both sides are dependent on separate terms $\lambda = \omega^2$ is a constant. Thus we arrive at two more differential equations:

$$T_{tt} = -\lambda T \implies T = A\sin(\omega t) + B\cos(\omega t) \text{ or } T = Cx + D \text{ if } \lambda = 0,$$

$$\frac{r(rR')'}{R} + \lambda r^2 = -\frac{\Theta_{\theta\theta}}{\Theta} = m^2.$$

Again since the two sides have different dependence there is a separation constant m^2. Then we get:

$$\Theta_{\theta\theta} = -m^2\Theta \implies \Theta = A\sin(m\theta), m \in \mathbb{N}.$$

Note we applied boundary conditions to get that only sin terms are possible here. Finally, this leaves us with the radial equation:

$$\left(rR'\right)' + \lambda rR - \frac{m^2}{r}R = 0.$$

This is a Sturm-Liouville equation with $p(r) = r$ and $\sigma(r) = r$ which are both greater than or equal to 0. λ is the eigenvalue here, so it forms an increasing sequence, call them λ_n. Expanding the equation and multiplying by r:

$$r^2R'' + rR' + (\lambda r^2 - m^2)R = 0 \implies r^2R''\left(\frac{r}{\sqrt{\lambda}}\right) + rR'\left(\frac{r}{\sqrt{\lambda}}\right) + \left(r^2 - m^2\right)R\left(\frac{r}{\sqrt{\lambda}}\right) = 0.$$

This is Bessel's equation, so the solutions are $R(r) = J_m(\sqrt{\lambda}r)$ and $R(r) = Y_m(\sqrt{\lambda}r)$. Since only the J solutions are regular the Y solutions are impossible. Also to satisfy the boundary conditions $z_n^m = \omega_n^m = \sqrt{\lambda}$ where z_n^m are the zeros of J_m. Thus the general solution before applying initial conditions:r

$$u(r, \theta, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n^m J_m(\sqrt{\lambda_n^m}r)\sin(m\theta)\left(B_n\sin(\omega t) + C_n\cos(\omega_n^m t)\right).$$

Applying the initial conditions, then $B_n = 0$, so combine C_n and A_n^m (i.e. assume $C_n = 1$). We still must find A_n^m in terms of $f(r)$, but so far we have

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n^m J_m(\omega_n^m r)\sin(m\theta)\cos(\omega_n^m t).$$

To find A_n^m, first expand $f(r,\theta) = f(r)$ from $[0,\pi]$ with its odd extension to $[-\pi, 0]$. Next expand it to be 2π periodic over $\mathbb{R}$. Multiply by $\sin(m\theta)$ and integrate over θ:

$$\frac{2}{\pi}\int_0^{\pi} \sin(m\theta) f(r) d\theta = -2\frac{(-1)^m - 1}{m\pi} f(r) = \sum_{n=1}^{\infty} A_n^m J_m\left(\omega_n^m r\right).$$

Using Sturm-Liouville theory to solve for the remaining:

$$A_n^m = \left(\int_0^1 J_m\left(\omega_n^m\right)\frac{1-(-1)^m}{m\pi} 2f(r) r dr\right)\left(\int_0^1 J_m(\omega_n^m r)^2 r dr\right)^{-1}.$$

Question 2a. Let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$. Then separating:

$$(ru_r)_r + \frac{1}{r}u_{\theta\theta} = u_t + u_r$$

$$\implies \frac{T'}{T} = \frac{1}{r}\frac{\Theta''}{\Theta} + \frac{(rR')'}{R} - \frac{R'}{R} = -\lambda.$$

Since the two sides are dependent on different variables λ is constant. Therefore:

$$T' = -\lambda T \implies T = e^{-\lambda t},$$

$$-\frac{\Theta''}{\Theta} = r\frac{(rR')'}{R} - \frac{rR'}{R} + r\lambda = m^2.$$

Again both are dependent on different variables, so m is constant. Thus:

$$\Theta'' = -m^2\Theta \implies \Theta = \begin{cases} A\sin m\theta + B\cos m\theta & \text{if } m \neq 0 \\ C & \text{if } m = 0 \end{cases}.$$

By the periodic boundary terms there can't be a linear θ dependence. Finally the radial term:

$$r\frac{(rR')'}{R} - \frac{rR'}{R} + \lambda r = m^2$$

$$\implies \left(rR'\right)' - R' + \left(\lambda - \frac{m^2}{r}\right)R = 0$$

$$\implies r^2R'' + \left(\lambda r - m^2\right)R = 0.$$

This is the Sturm-Liouville form of the equation with $p = 1$, $\sigma = \frac{1}{r}, q = -\frac{m^2}{r^2}$. This is also a general Bessel equation with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\omega = 2\sqrt{\lambda}$ and $\nu = \sqrt{4m^2-1}$. Therefore we can order the eigenvalues λ_n in increasing order and the solutions are $R(r) = \sqrt{r}J_{\sqrt{4m^2+1}}\left(2\sqrt{\lambda r}\right)$ (there's also a Y_m term but it's not regular so we can discard it). To satisfy the boundary conditions we must have that $2\sqrt{\lambda} = z_n^m \implies \lambda = \left(\frac{z_n^m}{2}\right)^2$ where z_n^m is the nth zero of $J_{\sqrt{4m^2+1}}$ (NOTE: I'm defining this differently then they're defined in the book to make my indexing easier). Thus the general solution before invoking initial conditions is:

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\left[\frac{1}{2}B_n^0\sqrt{r}J_1\left(z_n^0\sqrt{r}\right)e^{-(z_n^0)^2t/4} + \sum_{m=1}^{\infty}\left(\sqrt{r}J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)e^{-(z_n^m)^2t/4}\left(A_n^m\sin m\theta + B_n^m\cos m\theta\right)\right)\right].$$

To invoke the initial conditions, first note that because f is 2π periodic in θ, we can write it as a combination of sins/coss. Set $t=0$, multiply both sides by $\sin m\theta$ or $\cos m\theta$ and integrate over θ:

$$\frac{1}{\pi}\int_0^{2\pi}\sin m\theta f(r,\theta)d\theta = \sum_{n=1}^{\infty}A_n^m\sqrt{r}J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)$$

$$\frac{1}{\pi}\int_0^{2\pi}\cos m\theta f(r,\theta)d\theta = \sum_{n=1}^{\infty}B_n^m\sqrt{r}J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right).$$

Applying Sturm-Liouville theory to get the final coefficients, we get:

$$A_n^m = \frac{1}{\pi}\left(\int_0^1 J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)^2dr\right)^{-1}\int_0^{2\pi}\int_0^1 f(r,\theta)\sin(m\theta)J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)r^{-1/2}drd\theta.$$

$$B_n^m = \frac{1}{\pi}\left(\int_0^1 J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)^2dr\right)^{-1}\int_0^{2\pi}\int_0^1 f(r,\theta)\cos(m\theta)J_{\sqrt{4m^2+1}}\left(z_n^m\sqrt{r}\right)r^{-1/2}drd\theta$$

Question 2b. When there is no angular dependence, $A_n^m = B_n^m = 0$, so the decay of u over time is controlled by z_n^0, i.e. the smallest zero of $J_1(r)$ (recall I defined z_n^m to be the nth zero of $J_{\sqrt{4m^2+1}}$). Checking the table this is 3.832, so the coefficient in the exponent is $(z_n^m)^2/4 = 3.67$, which is why the solutions appear to decay with this rate. Since all the other terms are killed by the lack of angular dependence, this term dominates.

Question 2c. The only term that corresponds to $\sin\theta$ is $m=1$, so the only term that doesn't vanish is A_n^1. Thus the series becomes

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sqrt{r}J_{\sqrt{5}}\left(z_n^1\sqrt{r}\right)e^{-\left(z_n^1\right)^2t/4}A_n^1\sin\theta.$$

The coefficients can be numerically found by the following:

$$A_n^1 = \left(\int_0^1 J_{\sqrt{5}}\left(z_n^1\sqrt{r}\right)^2dr\right)^{-1}\int_0^1 e^rJ_{\sqrt{5}}\left(z_n^1\sqrt{r}\right)dr.$$

The graphs of the results can be seen in figures 1 and 2. The (Python) code used to produce the graph is here:

```
import numpy as np
from scipy.special import jv, jn_zeros
import matplotlib.pyplot as plt

zn = np.array([5.4336, 8.7388, 11.9533, 15.1365, 18.3053])
r = np.linspace(0, 1, 1000)
Jr = np.array([jv(np.sqrt(5), z*np.sqrt(r)) for z in zn])
```

```
An = np.trapz(np.exp(r)*Jr, r) / np.trapz(Jr**2, r)


for t in [0, 0.01, 0.03, 0.1]:
    u = sum([Jr[i] * An[i] * np.sqrt(r) * np.exp(-zn[i]**2 * t / 4) for i in range(

    plt.plot(r, u, label=f"t={t}")
plt.legend()
plt.xlabel("r")
plt.ylabel("u(r,t)")
plt.show()

for r in [1/4, 3/4]:
    t = np.linspace(0, 0.7, 1000)
    Jr = np.array([jv(np.sqrt(5), z*np.sqrt(r)) for z in zn])
    u = sum([Jr[i] * An[i] * np.sqrt(r) * np.exp(-zn[i]**2 * t / 4) for i in range(

    plt.plot(t, u, label=f"r={r}")

plt.legend()
plt.xlabel("t")
plt.ylabel("u(r,t)")
plt.show()
```

In comparison to the numerical solutions given in the problem statement, the graphs exhibit more oscillatory behavior, especially for $t = 0$. This is because each of the individual solutions fulfills the boundary condition of $u = 0$ at $r = 1$, so to fulfill the initial condition of $u \neq 0$ a large number of terms in the sum are required to converge. Since we truncate after the first 5 our graphs aren't as smooth as the true one. For later $t > 0$ the solutions start to appear more similar.

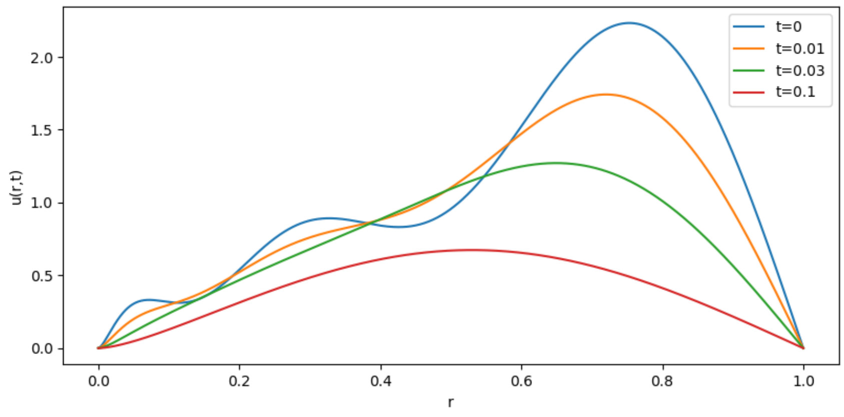

Figure 1: Graph of first 5 terms over position

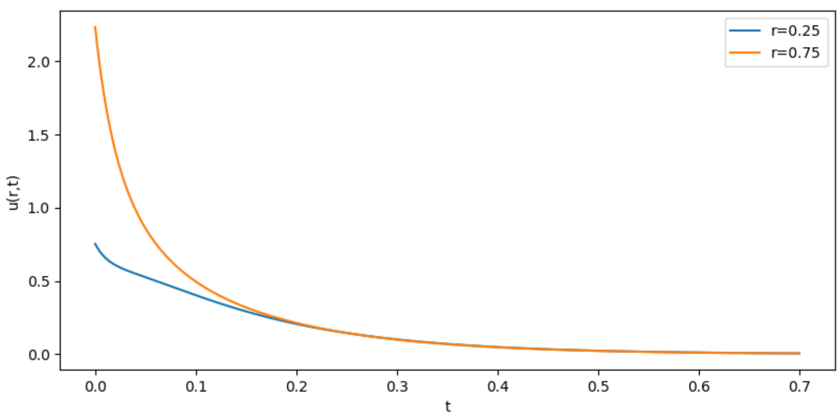

Figure 2: Graph of first 5 terms over time