Math 322 Homework 3

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Question 2. Let f be a map from the rotation $\frac{2\pi}{n}$ to $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The subgroup generated by the following is isomorphic to these groups, as a rotation effectively shifts each point on the n-gon or complex unit circle 1 unit in a direction:

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}.$$

Question 4. They are not isomorphic. To see why suppose by contradiction not, i.e. suppose there exists an isomorphism $f : \mathbb{Z} \to \mathbb{Q}$. Let $p = f^{-1}(1)$. Then $f\left(\frac{p}{2}\right)$ is an integer that when added to itself gives $f\left(\frac{p}{2}\right) + f\left(\frac{p}{2}\right) = f\left(\frac{p}{2} + \frac{p}{2}\right) = f(p) = 1$, but obviously no such integer exists so f can't actually exist.

Question 5. They are not isomorphic. Note that $(-1)^{-1} = -1$ in the group of multiplicative group of non-zero rationals, but there is no such element under that additive group of rationals that is it's own inverse. Thus for any bijective map $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q}$, $f(-1) = f((-1)(-1)(-1)) = f(-1) + f(-1) + f(-1) = 3f(-1) \implies 3 \neq 1$ which is clearly ridiculous, so f can't be an isomorphism.

Question 1. Simply applying C to the equation $C(C(A)) \supset A$, we get that $C(C(C(A))) \subset A$. For the other direction, let $A = C(B)$, then $C(C(C(B) \supset C(B)$. Since this holds for any two sets A, B, both sides contain one another and $C(A) = C(C(C(A)))$.

For the second part, let $c \in C(A)$, where $A \subset C(c)$. Then $< A >\subset C(c)$, and $C(A) \subset C(< A >)$ since $< A >$ commutes with everything in $C(A)$.

Finally for the last part, we can use the previous result and the fact that $S \subset C(S)$ (since S commutes) to say $C(S) = C(< S >)) = C(M) \implies S \subset C(M)$. Then $M =< S >\subset C(M) \implies$ M is commutative.

Question 3. Let G be an abelian group, and let $g_1, g_2, \ldots, g_n$ be a finite set of generators in G. Then every element $g \in G$ can be expressed as a combination of $g_1^{a_1} g_2^{a_2} \ldots g_n^{a_n}, 0 \leq a_i < o(g_1)$. The number of total possibilities is just the product of the number of choices on each each generator, i.e. $|G| = \prod_i o(g_i) < \infty$.

Question 4. Let g as described in the question. $(g^k)^{[n,k]/k} = 1$, since $n|[n,k]$ by definition. Thus $o(g^k)|[n,k]/k$. Also $(g^k)^{o(g^k)} = 1$ by definition, so $n|ko(g^k)$. Thus since $ko(g^k)$ both divides and is divided by $[n,k]$, they must be equal and thus $o(g^k) = [n,k]/k = n/(n,k)$.