Math 443 Homework 6

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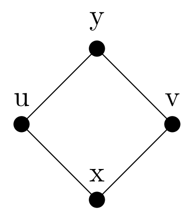
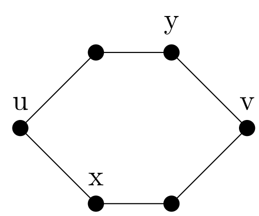
15/03/23

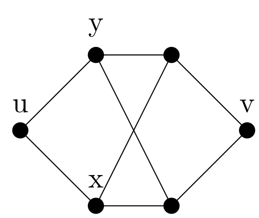
Question 1a. See below, x, y are a minimum separating set and clearly x, y are adjacent to both u and v.



Question 1b. x, y is a minimum separating set with the required properties.



Question 1c. The following graph works. Here I have listed a particular x, y, but once you choose one member of the separating set to be adjacent to u or v the only other option forces the other vertex adjacent to the same choice of u, v.



Question 2. By Menger's Theorem it suffices to provide an upper bound on $\kappa(u, v)$ and a lower bound on $\lambda(u, v)$, as long as they're the same then the two graphs must be equal. In figure 1 I give an example of a degree 5 $u - v$ separating set and in figure 2 I give an example of 5 disjoint paths between u and v, so by Menger's Theorem $5 \geq \kappa(u, v) = \lambda(u, v) \geq 5 \implies \lambda(u, v) = 5$. $\square$

Question 3. Consider a new graph G' that is a copy of G with an additional vertex v that is adjacent to each of the v_i. I claim that G' is k-connected. Let $x, y \in V(G')$. If x and y are both in G then a vertex set of size less than k clearly can't separate them due to the k-connectivity of G,

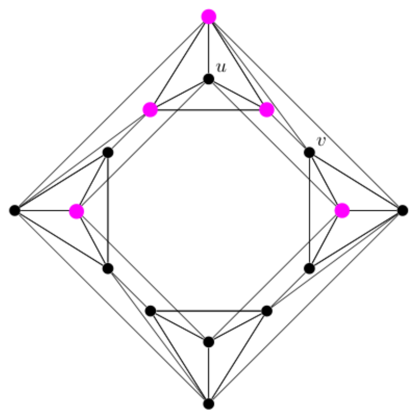


Figure 1: Graph for 2 with a $u - v$ separating set.

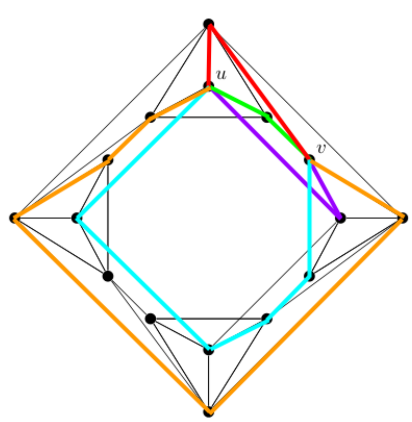


Figure 2: Graph for 2 with disjoint $u - v$ paths.

so assume $y = v$. Let $S \subset V(G)$ be a set of vertices on $G' - x - v$ with $|S| < k$. G is k-connected so there still exists a path from x to v_i in $G - S$ for all v_i that are in $G - S$. $|S| < k$ so at least on such v_i is still in $G' - S$, so there exists a path from x to v_i and an edge from v_i to v, so v and x are connected in $G' - S$. Thus G' is k-connected.

Now apply Menger's Theorem to u and v on G'. G' is k-connected as justified above, so $\kappa(u, v) \geq k \implies \lambda(u, v) \geq k$ (although by our construction we know equality holds, it's not important). Let $P'_1, \ldots, P'_k$ be k disjoint paths from u to v which are guaranteed to exist by our bound of λ. The only neighbors of v for the k paths to go through are each of the v_i of which there are k, so each path must go through exactly one. Letting $P_i = P'_i - v$, these now fill the requirements of the P_i asked for in the question and we're done. $\square$

Question 4. Clearly $\kappa(G_r) \leq \kappa(G) + r$, since you can take a separating set of G and remove all r new vertices to separate G_r. To show that this is also an lower bound, let S be a vertex set with $|S| < \kappa(G) + r$. Let $u, v \in G_r$. If any of the r new vertices are in $G_r - S$, then u and v are both connected to that vertex (or are that vertex), so they are connected. Therefore $G_r - S \subset G$.

It takes r vertices of S to remove all the r new vertices, so S contains less than $\kappa(G)$ vertices of G, so by κ's definition S isn't a separating set. Therefore $\kappa(G_r) \geq \kappa(G) + r$, so equality must hold and $\kappa(G_r) = \kappa(G) + r$. $\square$

Question 5. Both directions will be proven separately.

($\Rightarrow$) Assume G is k-edge-connected, and let $u, v \in V(G)$. By hypothesis a minimum uv separating edge set is of size at least k, so the maximum number of pairwise edge-disjoint uv paths is at least k by Theorem 5.21 which is what we needed to prove.

($\Leftarrow$) Assume that G contains k pairwise edge-disjoint uv-paths for each $u, v \in V(G)$. Let $u, v \in V(G)$. The maximum number of pairwise edge-disjoint uv paths in G is always at least k, so the maximum number of such paths is greater than or equal to k. Thus by Theorem 5.21, a minimum uv separating edge set is of size at least k. Since this is true of each $u, v \in V(G)$, G is k-edge-connected. $\square$

Question 6. As the hint suggests we will use strong induction on k.

Base case (k=2): Let G be 2-connected and let $e_1, e_2 \in E(G)$. G contains no cut vertices so by definition it's a block. From homework 5, question 4, all edges in a block share a cycle. Thus e_1 and e_2 share a cycle in G as required.

Inductive step: Let G be a $(k+1)$-connected graph, and assume the theorem holds for all graphs of connectivity k or less. Let $e_1, e_2 \in E(G)$ and $v_1, \ldots v_{k-1} \in V(G)$. By the inductive hypothesis there exists a cycle C containing e_1, e_2 and $v_i \forall i \in [k-2]$. For the simplicity of the proof, extend C with arbitrary other vertices until is has length at least $k+1$. This is possible since G is $k+1$ connected, so we can take two adjacent vertices $x, y, xy \notin \{e_1, e_2\}$ on C, and replace the edge between them in C with a path between them on $G - (C - x - y) - xy$.

By question 3 of this homework (applied to $k+1$ vertices of C), there exist $k+1$ disjoint paths from v_{k-1} to C with separate endpoints. For each of these paths, consider the shortened version, starting from its first intersection of C, to v_{k-1}, call these paths $P_i, i \in [k+1]$. Since the paths are disjoint their endpoints in C are also still distinct.

Let $S = (v_1, \ldots, e_1, \ldots, e_2, \ldots, v_{k-2})$ be an ordered list of the v_i and e_1, e_2 in the order that they occur in C (the indices on the v_is were arbitrary, so we can rename so that they are in order). Since there are k elements of S and $k+1$ paths ending on C on distinct vertices, by the pigeonhole principle it must be that there are elements of S, s_1 and s_2 that are adjacent such that two paths P_i and P_j have endpoints u_i, u_j between them in C. The awkwardness of the wording comes from the fact that s_1 and s_2 could be either edges or vertices, what I mean by between is that the subpath of C between u_i and u_j does not contain s_1 or s_2 as neither edges nor inner vertices (although if s_1 or s_2 are a vertex it could be that for example $u_i = s_1$). Thus we can extend C by considering the new cycle $C' = u_i P_i v_{k-1} P_j u_j C u_i$, and by our choice of u_i, u_j this cycle contains both e_1, e_2 as well as all the vs. $\square$

Question 7. $\square$ (pretty concise proof, huh)

Question 8. Let c be a circuit in a graph G with ordered vertices $v_1, v_2, \ldots, v_n$ (potentially some repeats). Let v_j be the first repeated vertex (i.e. minimal j), and let i be the index of the vertex that v_j first occurred in. Then $v_i, v_{i+1}, \ldots, v_j$ is a closed walk with no repeated vertices by the minimally of j which is exactly the definition of a cycle so we're done. $\square$