UBC Mathematics 320(101)—Assignment 1
Due by PDF upload to Canvas at 18:00, Saturday 16 Sep 2023

Readings: Loewen, lecture notes for Week 1; Rudin, pages 24–30.

1. Prove or disprove: For each $n \in \mathbb{N}$, $n^2 - n + 41$ is prime.

2. If A, B, and C are sets, prove that
(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(b) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$,
(c) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

3. Let $f\colon A \to B$. Let C, C_1, and C_2 be subsets of A, and let D be a subset of B. Prove:
(a) If f is one-to-one, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.
(b) If f is 1-1, then $f^{-1}(f(C)) = C$.
(c) If f is onto, then $f\left(f^{-1}(D)\right) = D$.
In each part, find an inclusion relation (either "$\subseteq$" or "$\supseteq$") that can be used to replace the symbol "=" and produce a true statement even without the given hypothesis.
[Recall that $f^{-1}(y) = \{x \in A : f(x) = y\}$ is, in general, a set-valued operation. It is not safe to infer that f is invertible just because the symbol f^{-1} appears.]

4. For Question 3(a), construct a specific example in which the indicated equation fails. (Of course the given hypothesis will have to be false too.)
Repeat for parts 3(b) and 3(c).

5. Prove that there is no (a, b) in $\mathbb{Z} \times \mathbb{Z}$ for which $a^2 = 4b + 3$.
(*Hint*: Every integer a must be either even or odd.)

6. Let $f\colon A \to B$ and $g\colon B \to C$ be given functions. Use the symbol $g \circ f$ to denote the function from A to C defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$. Prove:
(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
(b) If $g \circ f$ is one-to-one, then f is one-to-one.
(c) If f is onto and $g \circ f$ is one-to-one, then g is one-to-one.
(d) It can happen that $g \circ f$ is one-to-one, but g is not. (To "prove" this, simply provide a specific example with the indicated properties.)

7. (a) Suppose $f\colon X \to X$ is a function, and define $g = f \circ f$. Prove: If $g(x) = x$ for all $x \in X$, then f is one-to-one and onto.
(b) Extend the result in (a) to the function $g = f \circ f \circ \cdots \circ f$ defined by composing f with itself n times. Show that the result is valid for each $n \in \mathbb{N}$.

8. Let $f\colon A \to B$ and let $C \subseteq A$.
(a) Proof or counterexample: $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
(b) Proof or counterexample: $f(A \setminus C) \supseteq f(A) \setminus f(C)$.
(c) What condition on f will guarantee $f(A \setminus C) = f(A) \setminus f(C)$?
(Choose between "f is 1-1" and "f is onto"; prove that your answer is correct.)