Math 443 Homework 7

Xander Naumenko

22/03/23

Question 1. Let G be a r-regular graph that is not Eularian. By Euler's theorem, it must be that r is odd, since otherwise it would be Eularian. The only way this is true is if

$$||G|| = \frac{1}{2}\sum_{v\in G} d(v) = \frac{1}{2}|G|r.$$

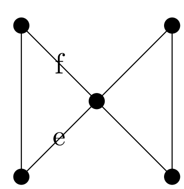
r is odd and the left side is a whole number, so $|G|$ must be even. Since G is regular $\overline{G}$ is also regular. Each of the vertices $v \in V(\overline{G})$ are connected to $(|G|-1)-r$. If this is zero then $\overline{G}$ is disconnected, otherwise $(|G|-1)-r$ is even and each vertex in $\overline{G}$ has this degree, so by Euler's theorem $\overline{G}$ is Eularian. □

Question 2. Since G_1 and $\bar{G_1}$ are Eularian, using the same argument as in question 1, $|G|$ is odd and $|G|$ is even regular (i.e. it is r-regular with r even). Since G_2 is non-Eularian, it must be that G_2 is odd regular, which again using the same argument as question 1 means that $|G_2|$ is even. This same argument applies identically for G_3. The total number of edges in G will be the number of edges in each of G_1, G_2, G_3 plus the number of new edges added. This is:

$$||G|| = ||G_1|| + ||G_2|| + ||G_3|| + (|G_1|)(|G_2|) + (|G_2|)(|G_3|) + (|G_1|)(|G_3|).$$

All the terms above are even except for the $||G_2||$ and $||G_3||$ terms, so the sum is even. By Euler's theorem any graph with an even number of edges is Eularian, so G is Eularian. □

Question 3. The statement is false. Consider the following graph:



Clearly the graph is Eularian by traversing both triangles consecutively, starting from the middle vertex. To show that a closed trail can't use e and f consecutively, by way of contradiction assume there was a closed walk that contains e and f consecutively. Then part of the walk started and ended on the leftmost two vertices without visiting either of the rightmost vertices. However once this happens such a trail will never be able to get back to the rightmost vertices to make the trail Eularian, so such a trail can't exist and the above graph is a counterexample. □

Question 5. The statement is true. Let $S \subset V(G)$ be a set of vertices formed by taking one vertex from each component of H. Since they are taken from separate components of H, each element of S is not adjacent in G, so S is an independent set, which means $\alpha(G) \geq |S|$. We also have that $k(H) = |S|$, since we took one vertex from each component. Putting these together we get $k(H) \leq \alpha(G)$ as required. $\square$

Question 6. As stated in the question the Petersen graph G is non-Hamiltonian, so all we must prove is that for all $S \subset V(G)$, $k(G-S) \leq |S|$. Since G has vertex connectivity 3, $|S| = 1$ and $|S| = 2$ hold since in either case $k(G-S) = 1 \leq |S|$. Also using question 5, we know that $k(G-S) \leq \alpha((G) = 4$, so the $|S| \geq 5$ cases are also handled. Thus either $|S| = 3$ or $|S| = 4$.

Consider removing the edges between the inside sections of the graph, i.e. separating G into two copies of C_5, call this G'. We will prove that this subgraph of G fulfills the property for $|S| \in \{3, 4\}$, and since all we've done is remove edges it must also hold for G.

Case 1 ($|S| = 3$): If all three vertices in S are in one of the cycles, then there are at most two components after the removal from that cycle, for a total $3 \leq |S|$. If they are split two on one cycle and 1 from the other, then again the cycle with two vertices removed forms at most 2 components and the cycle with one vertex removed is still connected, for a total of $3 \leq |S|$ components. By symmetry these are the only ways to split the vertices of S, so we're done.

Case 2 ($|S| = 4$): If all 4 vertices of S are in one cycle then clearly there are only $2 \leq |S|$ components in $G' - S$. If it is split 3-1, there are up to two components resulting from the cycle with 3 vertices removed and the other remains connected, for a total of $3 \leq |S|$ components in $G' - S$. Finally if they are split 2-2 then each cycle has at most 2 components, for a total of $4 \leq |S|$ components. Again by symmetry these are the only ways to split the vertices, so this case is finished.

Since $G' - S$ has fewer than $|S|$ components, this also must hold for G (since $k(G-S) \leq k(G'-S)$). We've covered all possible values of $|S|$, so the Petersen graph is a counterexample to the converse of Theorem 6.5. $\square$

Question 7. Let P_1 be a Hamiltonian cycle in G, and let $G' = G - E(P_1)$. Since each vertex in G had exactly two edges in P_1, we have that $\delta(G') = \delta(G) - 2 = \frac{|G|+4}{2} - 2 = \frac{|G|}{2}$. In class we proved that for any graph G' with $\delta(G') \geq \frac{|G'|}{2}$, G' is Hamiltonian. Let P_2 be a Hamiltonian path on G'. P_1 and P_2 are both Hamiltonian paths on G and they are edge-disjoint by construction as required. $\square$

Question 8. Consider a longest cycle C in G. If $G = C$ then clearly G is Hamiltonian by just removing any edge and taking the remaining path, so assume that $G \neq C$ as the only remaining case to consider. Then there exists a vertex $v \in V(C), u \in V(G)$ such that $uv \in E(G)$. Now consider the set of vertices containing v, u and both neighbors of v in C, call them x and y. It can't be that both $ux \in E(G)$ and $uy \in E(G)$, since then $yCxuy$ would be a longer cycle, contradicting our assumption that C is longest. Thus the subgraph induced by $\{x, y, u, v\}$ is isomorphic to either $K_{1,3}$ or $K_{1,3} + e$. However this was disbarred by the definition of G, so this case must have been impossible and G is Hamiltonian. $\square$

Question 9. Yes, the graph is Hamiltonian. See figure 1 for a Hamiltonian cycle in G.

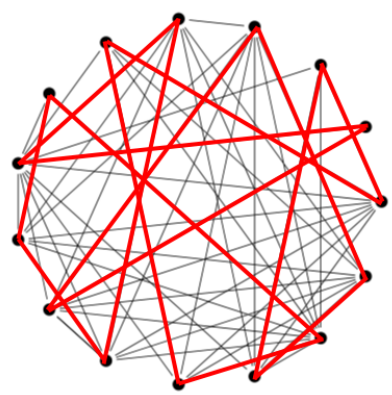


Figure 1: Hamiltonian cycle for question 9