Math 320 Homework 8

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Question 1. Since $\sum a_n$ converges, $a_n \to 0$. Thus there exists $N \in \mathbb{N}$ s.t. $n \geq N \implies a_n < 1$. Then we have that

$$\sum_{n=N}^{\infty} |a_n| |b_n| < \sum_{n=N}^{\infty} |b_n|$$

which converges. Since the sum converges absolutely, the sum $\sum a_n b_n$ also converges.

The statement is no longer true if the "absolutely" condition is removed. Consider $a_n = b_n = \frac{(-1)^{n+1}}{\lceil n/2 \rceil}$. Then for both series the partial sums are $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots$ and clearly converge to 0. However $\sum a_n b_n = \sum \frac{1}{\lceil n/2 \rceil} = 2 \sum \frac{1}{n} = \infty$ and doesn't converge.

Question 2a. Clearly For $x = 1$ all the terms are zero so it trivially converges, so from now on assume $x \neq 1$. If $c = 1$, then the sum reduces to $\sum_{n=1}^{\infty}(x-1)^n \implies x \in (0,2)$. If $c < 1$ then for $N = \max(1, \lceil \log_c \frac{1}{2(x-1)} \rceil)$, we have $\sum_{n=N+1}^{\infty}(c^n(x-1))^n < \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \infty$. Finally for $c > 1$ we have that for $N = \max(1, \left\lceil \log_c \frac{2}{|x-1|} \right\rceil)$, $|(c^n(x-1))^n| > 2^n$, so by the crude divergence test the sum can't converge. In summary for $x = 1$ the series converges if and only if $[x = 1]$, $[c = 1$ and $x \in (0,2)]$, or $[c < 1]$.

Question 2b. For $x \in (-1,1)$, we have $\sum \left| \frac{x^n - x^{2n}}{n} \right| < \sum 2|x^n| < \infty$, so since it converges absolutely it must converge also. For $|x| > 1$ then we have that $\left| \frac{x^n(1-x^n)}{n} \right| > |x^n - 1| \to \infty$, so by the crude convergence test it diverges for $|x| > 1$. Finally if $x = 1$ then every term is zero so it trivially converges and $x = -1$ corresponds to $\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} < -\sum_{n=1}^{\infty} \frac{1}{n} = -\infty$ so it diverges. Therefore the set of convergence for the series is $(-1, 1]$.

Question 2c. For $x = 0$ then the series turns into $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, so it diverges. For $x \neq 0$ by the ratio test we get

$$\frac{a_{n+1}}{a_n} = \frac{x+1}{2x+1} \left(\frac{\sqrt{n}}{\sqrt{n+1}} \right) \to \frac{x+1}{2x+1}.$$

Solving $\left| \frac{x+1}{2x+1} \right| < 1 \implies x \in (-\frac{2}{3}, 1)$ implies convergence and $x \in (-\infty, -\frac{2}{3}) \cup (1, \infty)$ makes the series diverge. We already checked $x = 1$, and for $x = -\frac{2}{3}$ the series turns into $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the alternating series test. Thus the set of convergence is $(-\frac{2}{3}, 0)$.

Question 2d. For $x = e$ the series trivially converges. Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+2)n}{(n+1)^3}(x-e) \to 4(x-e).$$

Solving $|4(x-e)| < 1 \implies x \in (e - \frac{1}{4}, e + \frac{1}{4})$ implies convergence and $x \in (-\infty, e - \frac{1}{4}) \cup (e + \frac{1}{4}, \infty)$ implies divergence. The only remaining values to check are $x = e - \frac{1}{4}$ and $x = e + \frac{1}{4}$. I claim both converge, note that $x = e + \frac{1}{4}$ converging implies $x = e - \frac{1}{4}$ converges due to the former representing absolute convergence of the latter. Applying Raabe's test:

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{4(n+1)^3}{(2n+1)(2n+2)n} - 1\right) = \frac{3n+2}{2n+1} \to \frac{3}{2} > 1.$$

Thus the region of convergence is $[e - \frac{1}{4}, e + \frac{1}{4}]$

Question 3. I interpret "discuss" to mean state whether each of the series converges or not. For a_n, apply the alternating series test:

$$(-1)^n a_n = \frac{n^n}{(n+1)^{n+1}} < \frac{(n+1)^n}{(n+1)^{n+1}} = \frac{1}{(n+1)} \to 0.$$

Thus by the alternating series test the series converges (but not absolutely as b_n shows). For b_n, using Cauchy's condensation test we get

$$\sum_{n=1}^{\infty} b_n = \sum_{k=0}^{\infty} \frac{2^k (2^k)^{2^k}}{(2^k+1)^{2^k+1}} = \sum_{k=0}^{\infty} \frac{2^{k(2^k+1)}}{(2^k+1)^{2^k+1}} > \sum_{k=0}^{\infty} \frac{1}{2} = \infty.$$

Thus the b_n series diverges. For c_n, the crude convergence test is sufficient since $\frac{(n+1)^n}{n^n} > 1 \not\to 0$, so the c_n series doesn't converge. Finally, since $d_n < b_n \forall n$ by the comparison test d_n also diverges.

Question 4. Using homework 3 question 8a and the fact that $x_n \to 0$, we have that $a_n = \frac{x_1+x_2+...+x_n}{n} \to 0$. Then then the series we're computing is

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) + \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}a_n.$$

Thus by the alternating series test the series converges.

Question 5a. Proof by contrapositive, assume that $\lim_{n\to\infty} na_n \neq 0$. Then $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n > N$ with $|na_n| \geq \epsilon$. Let $S = \{n \in \mathbb{N} : |na_n| \geq \epsilon\}$, because for every N there is an $n > N$ with $|na_n| \geq \epsilon$, $|S| = \infty$. Then we have

$$\sum_{n=1}^{\infty} a_n > \sum_{n \in S} a_n \geq \epsilon |S| = \infty.$$

Thus since the contrapositive holds the original statement is also true.

Question 5b. Note that although part a specifies a decreasing sequence, only the fact that it was positive was used in the proof. Thus we can apply part a to $\frac{b_n^2}{n}$ (which is positive but not necessarily decreasing) to get that $b_n^2 \to 0 \implies b_n \to 0$. Then applying question 8a from homework 3 gives that $s_n = \frac{1}{n}\sum_{m=1}^{n} b_m \to 0$ also.

Question 6a. As the hint suggests, consider the geometric sum formula applied to $e^{2i\theta}$

$$e^{i\theta}\sum_{m=0}^{n-1}\left(e^{2i\theta}\right)^m = e^{i\theta}\frac{1-e^{2ni\theta}}{1-e^{2i\theta}} = \frac{1-e^{2ni\theta}}{e^{-i\theta}-e^{i\theta}} = \frac{\sin(2n\theta)+i(1-\cos(2n\theta)}{2\sin\theta}.$$

Taking the real and imaginary parts of both sides gives the required identities.

Question 6b. Let $\theta \in \mathbb{R}$. If $\sin(\theta) = 0$ then clearly the series converges, so assume that it doesn't. Apply Dirichlet's theorem to this problem, with $a_k = \frac{1}{2k-1}$ and $b_k = \sin((2k-1)\theta)$. The partial sums of the b_n are bounded since $b_1 + b_2 + \ldots + b_n = \sin(\theta) + \sin(3\theta) + \ldots + \sin((2n-1)\theta = \frac{1-\cos(2n\theta)}{2\sin\theta} < \frac{2}{2\sin\theta} < \infty$ and the a_n are decreasing and have limit 0, so Dirichlet's theorem says that $f(\theta)$ converges.

Question 6c. Let $\theta_n = \frac{\pi}{4n}$. Clearly $\theta_n \to 0$, and $S_n(\theta_n) = \frac{1-\cos(\frac{\pi}{2})}{2\sin\theta_n} = \frac{1}{2\sin(\theta_n)}$. Since $\lim_{x\to 0}\frac{1}{\sin(x)} = \infty$ and $\theta_n \to 0$, $S_n(\theta_n) \to \infty$ (I'm being a bit non-rigorous with this given we haven't defined limits on reals yet, but given we haven't rigorously defined sin or imaginary numbers yet either I assume this is fine). This doesn't contradict part b because the order of what is bounded has been changed. When we used Dirichlet's theorem we had already fixed a θ and shown that $S_n(\theta)$ was bounded. Here we've shown that $S_n(\theta)$ is not bounded across all combinations of n and θ, but that's a completely different statement.