MATH 305 Homework 8

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1. Use Rouche's theorem to count the number of zeroes for $p(z) = 4z^5 + z^2 + 2z - 1$ in $|z| \leq 1$.
   Let $g(z) = 4z^5$. Then for $|z| = 1$, we have

$$|p(z) - g(z)| = \left|z^2 + 2z - 1\right| = \left|z + 2 - z^{-1}\right| = |2 + 2i\sin\theta| \leq 2\sqrt{2} < 4 \leq |4z^5|.$$

Thus by Rouché's theorem there are 5 zeros inside $|z| \leq 1$.

2. Use Rouche's theorem to count the number of zeroes for $p(z) = z^5 + 7z^2 + 2$ in $1 \leq |z| \leq 2$.
   Let $g(z) = z^5$. Then for $|z| = 2$, we have

$$|p(z) - g(z)| = |7z^2 + 2| \leq 26 < 32 = |z^5|.$$

Let $h(z) = 7z^2$. Then for $|z| = 1$, we have

$$|p(z) - h(z)| = |z^5 + 2| \leq 3 < 7 = |7z^2|.$$

Thus by Rouché's theorem, there are 5 zeros within $|z| \leq 2$ and 2 in $|z| \leq 1$, which means that there are 3 zeros with $1 \leq |z| \leq 2$.

3. Use Rouche's theorem to count the number of zeroes for $f(z) = z^2 - 4 + 3e^{-z}$ on the right half plane $\{Re(z) > 0\}$.
   Let $g(z) = z^2 - 4$, and consider the right half circle with radius R and origin $z = 0$ as $R \to \infty$. Then we have for $Re(z) \geq 0$ and $z = R$:

$$|f(z) - g(z)| = \left|3e^{-z}\right| = 3e^{-Re(z)} \leq 3 < \left|z^2 - 4\right| < R^2 \text{ as } R \to \infty.$$

Instead for $0 \leq |z| \leq R, Re(z) = 0$, we have:

$$|f(z) - g(z)| = \left|3e^{-z}\right| = 3 < 4 \leq \left|z^2 - 4\right|.$$

Since the equality holds true on the contour as $R \to \infty$, we have that the number of zeros of g is the same as the number of zeros of f. In this case g has 1 zero on the right half plane, so f has one zero on the right half plane.

4. Use Nyquist criterion to find the number of zeroes of $p(z) = z^3 + 2z^2 + 4$ in the right half plane $\{Re(z) > 0\}$.
   Using the Nyquist criterion:

$$N = \frac{1}{2\pi}\left(3\pi + 2[argp]_{\Gamma_{I_+}}\right).$$

Note that at $p(iy) = -iy^3 - 2y^2 + 4 = (4 - 2y^2) - iy^3 = p_r(y) + ip_i(y)$

Γ	p_r	p_i
∞	< 0	< 0
2	0	< 0
0	4	0

Therefore

$$2[argp]_{\Gamma_{I_+}} = \frac{\pi}{2}.$$

Therefore by the Nyquist criteria listed above we have $N = 2$

5. Use Nyquist criterion to find the number of zeroes of $p(z) = z^3 + 2z^2 + 4z + 2$ in the right half plane $\{Re(z) > 0\}$.

Expanding for $z = iy$: $p(iy) = -iy^3 - 2y^2 + 4iy + 2 = 2 - 2y^2 + i(-y^3 + 4y) = p_r(y) + p_i(y)$. To calculate the argument for the imaginary line construct a table:

Γ	p_r	p_i
∞	< 0	< 0
2	< 0	0
1	0	> 0
0	2	0

Thus we have:

$$2[argp]_{\Gamma_{I_+}} = \frac{-3\pi}{2}.$$

And by Nyquist's criterion:

$$N = \frac{1}{2\pi}\left(3\pi + 2[argp]_{\Gamma_{I_+}}\right) = 0.$$

6. (20pts) (a) Use Nyquist criterion to show that there are no zeroes of $p(z) = z^3 + z^2 + 4z + 1$ in $\{Re(z) \geq 0\}$.

Expanding for $z = iy$: $p(iy) = -iy^3 - y^2 + 4iy + 1 = 1 - y^2 + i(-y^3 + 4y) = p_r(y) + p_i(y)$. To calculate the argument for the imaginary line construct a table:

Γ	p_r	p_i
∞	< 0	< 0
2	< 0	0
1	0	> 0
0	2	0

Thus we have:

$$2[argp]_{\Gamma_{I_+}} = \frac{-3\pi}{2}.$$

And by Nyquist's criterion:

$$N = \frac{1}{2\pi}\left(3\pi + 2[argp]_{\Gamma_{I_+}}\right) = 0.$$

(b). Show that all the solutions $y = y(t)$ to

$$y^{'''} + y^{''} + 4y^{'} + y = 0$$

must approach to zero as $t \to +\infty$.

Taking the laplace transform of the differential equation:

$$\mathcal{L}(y''' + y'' + 4y' + y) = Y(s)\left(s^3 + s^2 + 4s + 1\right) = Q(s) \implies Y(s) = \frac{Q(s)}{s^3 + s^2 + 4s + 1}.$$

We just proved in part a that all zeros of the denominator lie in the right half plane, which means all solutions trend to 0 as $t \to 0$.

7. (20pts) Find the Laurent series for the function $\frac{z}{(z+1)(z-2)}$ in each of the following domains

(a) $|z| < 1$

Partial fraction decomposition:

$$\frac{z}{(z+1)(z-2)} = \frac{1}{3(z+1)} + \frac{2}{3(z-2)} = \frac{1}{3}\left(\frac{1}{1+z} - \frac{1}{1-\frac{z}{2}}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(z^n(-1)^n - \frac{z^n}{2^n}\right).$$

(b) $|z| > 2$

Expanding:

$$\frac{z}{(z+1)(z-2)} = \frac{1}{3(z+1)} + \frac{2}{3(z-2)} = \frac{1}{3z}\left(\frac{1}{1+\frac{1}{z}} + \frac{2}{1-\frac{2}{z}}\right) = \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{z^n} + \frac{2^{n+1}}{z^n}\right).$$

$$= \frac{1}{3}\sum_{n=1}^{\infty}\left(-\frac{(-1)^n}{z^n} + \frac{2^n}{z^n}\right).$$

8. Find the first three terms of Laurent series for $\frac{z}{\mathrm{Log}(z)}$ in $|z-1| < 1$, where $Log(z)$ is the principal branch.

First note that we have that the Taylor series of $\log(z)$ is:

$$\log(1 - z + 1) = \sum_{n=1}^{\infty}\frac{(z-1)^n(-1)^n}{n}.$$

Thus we have:

$$\frac{z}{\log(z)} = \frac{(z-1)+1}{z-1}\frac{1}{1 - ((z-1)/2 - (z-1)^2/3 - \dots)}.$$

$$= \left(1 + \frac{1}{z-1}\right)\left(1 + ((z-1)/2 - (z-1)^2/3 - \dots) + ((z-1)/2 - (z-1)^2/3 - \dots)^2 + \dots\right).$$

$$= \frac{1}{z-1} + \frac{3}{2} + \frac{5(z-1)}{12}.$$