# Math 305 Homework 3

Xander Naumenko

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1. For the following statements, state if it is true or false. If it is false give a counterexample

(1) If $f$ is differentiable at $z = z_0$, then $f$ is analytic at $z = z_0$.

This statement is false. As a counterexample, consider $f(z) = \overline{z}^2$. Then $\frac{\partial f}{\partial \overline{z}} = 2\overline{z}$, which means $f$ is only differentiable at $z_0 = 0$. Thus it is not differentiable in a small region around $z_0 = 0$ so it is not analytic.

(2) If $f$ is differentiable at $z = z_0$, then $f$ is continuous at $z = z_0$.

The statement is true. A prerequisite for differentiablity was continuity, so if a function has differentiability then it must be continuous.

(3) If $f$ is analytic in an open and connected domain $D$ and $Re(f(z)) = Constant$, then $f$ is constant.

The statement is true. By Cauchy Riemann, $u_x = v_y = 0 \implies v_y = 0$ and $u_x = -v_y = 0 \implies v_x = 0$, so $v = Im(f(z))$ must also be constant.

(4) If $f$ is analytic in an open and connected domain $D$ and $|f(z)| = Constant$, then $f$ is constant.

The statement is true. From Cauchy Riemann, we have that $|f(z)| = u^2 + v^2 = c \implies 2uu_x + 2vv_x = 0$ and $2uu_y + 2vv_y = 0 \implies u_x = u_y = v_x = v_y = 0$.

2. Use Cauchy-Riemann equation to find out the harmonic conjugate of the following functions

(a) $xy - x + y$

$$u = xy - x + y \implies v_y = u_x = y - 1 \implies v = \frac{y^2}{2} - y + F(x)$$

$$v_x = -u_y = -x - 1 \implies v = \frac{y^2}{2} - y - x - \frac{x^2}{2} + C.$$

(b) $u = \log(x^2 + y^2)$ for $Re(z) > 0$

$$u = \log(x^2 + y^2) \implies u_x = \frac{2x}{x^2 + y^2} = v_y \implies v = 2\arctan\left(\frac{y}{x}\right) + F(x)$$

$$v_x = -u_y = -\frac{2y}{x^2 + y^2} \implies v = 2\arctan\left(\frac{y}{x}\right) + C.$$

(c) $u = \sin x \cosh(y)$

$$u = \sin x \cosh y \implies v_y = u_x = \cos x \cosh y \implies v = \cos x \sinh y + F(x)$$

$$v_x = -u_y = -\sin x \sinh y \implies F(x) = 0 \implies v = \cos x \sinh y + C.$$

3. Let $f(z)$ be an analytic function in $D$ and $Im(f(z)) \neq 0$. Show that $\log|f(z)|$ and $Arg(f(z))$ is harmonic.

First of all note that since $Im(f(z)) \neq 0$, both $u = \log|f(z)|$ and $v = Arg(f(z))$ are differentiable. Assume that $f(z) = a(x,y) + ib(x,y)$ which satisfy Cauchy-Riemann. Checking Cauchy-Riemann for $u, v$:

$$u_x = \frac{d}{dx}\log(\sqrt{a^2+b^2}) = \frac{aa_x + bb_x}{a^2+b^2}$$

$$v_y = \frac{d}{dy}\arctan\left(\frac{b}{a}\right) = \frac{1}{1+\left(\frac{b}{a}\right)^2}\left(\frac{b_y}{a} - \frac{ba_y}{a^2}\right) = \frac{ab_y - ba_y}{a^2+b^2} = \frac{aa_x + bb_x}{a^2+b^2}$$

$$u_y = \frac{d}{dy}\log(\sqrt{a^2+b^2}) = \frac{aa_y + bb_y}{a^2+b^2}$$

$$v_x = \frac{d}{dx}\arctan\left(\frac{b}{a}\right) = \frac{1}{1+\left(\frac{b}{a}\right)^2}\left(\frac{b_x}{a} - \frac{ba_x}{a^2}\right) = \frac{ab_x - ba_x}{a^2+b^2} = \frac{-aa_y - bb_y}{a^2+b^2}.$$

Since the Cauchy Riemann equations are fulfilled $u$ and $v$ are harmonic.

4. (a) Let $u$ be a harmonic function in $D$. Show that if $v$ is a harmonic conjugate of $u$ in a domain $D$, then both $u^2 - v^2$ and $u^3 - 3uv^2$ are harmonic in $D$.

Computing the derivatives:

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\left(u^2 - v^2\right) = \frac{d}{dx}(2uu_x - 2vv_x) + \frac{d}{dy}(2uu_y - 2vv_y)$$
$$= 2u_x^2 + 2uu_{xx} - 2v_x^2 - 2vv_{xx} + 2u_y^2 + 2uu_{yy} - 2v_y^2 - 2vv_{yy}$$
$$= 2u_x^2 + 2uy_{xy} - 2v_x^2 + 2vu_{xy} + 2v_x^2 - 2uv_{xy} - 2u_x^2 - 2vu_{xy} = 0.$$

Similarly for the other given function:

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\left(u^3 - 3uv^2\right) = \frac{d}{dx}(3u^2u_x - 3u_xv^2 - 6uvv_x) + \frac{d}{dy}(3u^2u_y - 3u_yv^2 - 6uvv_y)$$

$$= \frac{d}{dx}(3u^2u_x - 3u_xv^2 - 6uvv_x) + \frac{d}{dy}(-3u^2v_x + 3v_xv^2 - 6uvu_y) = 0.$$

Since the laplacian of both functions is zero they are harmonic in $D$.

(b)Suppose that functions $u$ and $v$ are harmonic in $D$. Are the following functions harmonic?

(1) $u^2 - v^2$

Expanding:

$$\Delta(u^2 - v^2) = 2u_x^2 + 2uu_{xx} - 2v_x^2 - 2vv_{xx} + 2u_y^2 + 2uu_{yy} - 2v_y^2 - 2vv_{yy}$$
$$= 2u_x^2 - 2v_x^2 + 2u_y^2 - 2v_y^2 \neq 0.$$

Since the laplacian does not necessarily equal zero, $u^2 - v^2$ is not harmonic.

(2) $uv$

Expanding:

$$\Delta(uv) = u_{xx}v + 2u_xv_x + uv_{xx} + u_{yy}v + 2u_yv_y + uv_{yy} = 2u_xv_x + 2u_yv_y \neq 0.$$

Since the laplacian does not necessarily equal zero, $uv$ is not harmonic.

(3) $u - 100v$

Expanding:

$$\Delta(u - 100v) = \Delta u - 100\Delta v = 0 - 0 = 0.$$

Since the laplacian is zero $u - 100v$ is harmonic.

(4) $u_{xy} + \Delta v$

Expanding:

$$\Delta(u_{xy} + \Delta v) = \Delta(u_{xy} + 0) = u_{xyyy} + v_{xxxy} = (u_{xx} + v_{yy})_{xy} = 0.$$

Since the laplacian is zero $u_{xy} + \Delta v$ is harmonic.

5. Find a harmonic function $\phi(x, y)$ in the infinite strip

$$\{z : -2 \leq 2Re(z) - 3Im(z) \leq 3\}$$

such that $\phi = 0$ on the left edge $\{2Re(z) - 3Im(z) = -2\}$ and $\phi = 4$ on the right edge $\{2Re(z) - 3Im(z) = 3\}$. Hint: consider linear functions.

Consider the linear functions of the form $\phi = A(2x - 3y) + B$. Such functions are always harmonic so that requirement is already satisfied. To make the boundary conditions true, we must solve the system of 2 equations. Note that for the right line $y = \frac{2}{3}x - 1$ and for the left one $y = \frac{2}{3}x + \frac{2}{3}$.

$$\phi_l = A(2x - 3y) + B = B - 2A = 0$$

$$\phi_r = A(2x - 3y) + B = B + 3A = 4 \implies A = \frac{4}{5} \implies B = \frac{8}{5}.$$

Therefore a harmonic function that satisfies the requirements is $\phi = \frac{4}{5}(2x - 3y) + \frac{8}{5}$.

6. Find a harmonic function $\phi(x, y)$ satisfying

$$\Delta\phi = 0, y > 0, -\infty < x < +\infty$$

$$\phi(x, 0) = -1, x < -5; \phi(x, 0) = 0, -5 < x < -1; \phi(x, 0) = 2, -1 < x < 2; \phi(x, 0) = 0, x > 2$$

Write your solution in terms of $\tan^{-1}$ or $Arg$.

In class we found that forms to such boundary conditions take, so assume that $\phi$ is in the form

$$\phi(z) = AArg(z + 5) + BArg(z + 1) + CArg(z - 2) + D.$$

From the bondary conditions, we can generate a system of equations to solve for the constants:

$$-1 = \pi A + \pi B + \pi C + D$$

$$0 = \pi B + \pi C + D$$

$$2 = \pi C + D$$

$$0 = D$$

$$\implies D = 0 \implies C = \frac{2}{\pi} \implies B = -\frac{2}{\pi} \implies A = -\frac{1}{\pi}.$$

Thus the final function that satisfies the requirements is

$$\phi(z) = -\frac{1}{\pi}Arg(z + 5) - \frac{2}{\pi}Arg(z + 1) + \frac{2}{\pi}Arg(z - 2).$$

7. Find a harmonic function $\phi(x, y)$ in the annulus $\{z : 1 \leq |z| \leq 2\}$ such that $\phi = 1$ on $\{|z| = 1\}$ and $\phi = 2$ on $\{|z| = 2\}$.

Let $\phi = A \log r + B$. We proved $\phi$ is harmonic in class, all that is left is to determine the constants. The boundary conditions give us two equations:

$$1 = A \log 1 + B = B, 2 = A \log 2 + B \implies A = \frac{1}{\log 2}.$$

Thus the function that satisfies the requirements is

$$\phi = A \frac{\log r}{\log 2} + 1.$$

8. Find a harmonic function $\phi(x, y)$ such that

$$\Delta\phi = 0, \text{ in } D = \{(x, y)|y > 0, x^2 + y^2 > 9\}$$

$$\phi(x, 0) = -1, x < -3; \phi(x, y) = 0 \text{ for}, x^2 + y^2 = 9, -3 < x < 3; \phi(x, 0) = 2, x > 3$$

Consider the map $w = \frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) = u + iv$. This transforms the given domain into the much simpler upper half plane, where we know the solution is in the form:

$$\phi = AArg(u + iv + 1) + BArg(u + iv - 1) + C.$$

With the given initial conditions this results in the following constant:

$$-1 = \pi A + \pi B + C$$

$$0 = \pi B + C$$

$$2 = C \implies B = -\frac{2}{\pi} \implies A = -\frac{5}{\pi}.$$

The final form of the solution is then

$$\phi = -\frac{5}{\pi} Arg(u + iv + 1) - \frac{2}{\pi} Arg(u + iv - 1) + 2$$

where we define

$$u = \frac{1}{2}\left(\frac{x}{3} + \frac{3x}{x^2 + y^2}\right), v = \frac{1}{2}\left(\frac{y}{3} + \frac{3y}{x^2 + y^2}\right).$$

9. Find the image of the $S = \{z : -1 \leq Re(z) \leq 1, -\frac{\pi}{2} \leq Im(z) \leq \pi\}$ under the map $f(z) = e^z$

Let $z = x+iy$. Plugging this into the map we get $f(z) = e^{x+iy} = e^x e^{iy}$. Then since $-\frac{\pi}{2} \leq y \leq \pi$, the argument of $f(z)$ is in the same range. Similarly since $-1 \leq x \leq 1$, the magnitude of $f(z)$ is constrained between $e^{-1} \leq |f(z)| \leq e$. Putting this together, we get that the image is

$$f(S) = \{w \mid e^{-1} \leq |w| \leq e, -\frac{\pi}{2} \leq Arg(w) \leq \pi\}.$$

Graphically this forms an annulus with a segment cut out of it in the bottom left quadrant.

10. Find all numbers $z$ such that

(a) $(z + 1)^3 = (1 + i)z^3$

Taking the log of both sides:

$$3\log(z+1) = \log(\sqrt{2}e^{\frac{i\pi}{4}}) + 3\log z + 2\pi ki \implies 3\log(1+\frac{1}{z}) = \frac{1}{2}\log 2 + \frac{\pi}{4}i + 2\pi ki$$

$$\implies 1+\frac{1}{z} = 2^{\frac{1}{6}}e^{i\left(\frac{\pi}{12}+\frac{2}{3}\pi k\right)} \implies z = \left(2^{\frac{1}{6}}e^{i\left(\frac{\pi}{12}+\frac{2}{3}\pi k\right)} - 1\right)^{-1}, k \in \mathbb{Z}.$$

(b) $e^z = -1 - \sqrt{3}i$

$$z = \log\left(2e^{-i\frac{2\pi}{3}}\right) = \log 2 - i\frac{2\pi}{3} + 2\pi ki, k \in \mathbb{Z}.$$

(c) $\sin(z) = 4i$

$$\sin z = \cosh y \sin x + i \sinh y \cos x \implies \sin x = 0 \implies \pm \sinh y = 4$$

$$\implies z = 2\pi k + i\sinh^{-1}(4) \text{ or } z = 2\pi k - i\sinh^{-1}(4), k \in \mathbb{Z}.$$

(d) $\sin(z^6) = 0$
Let $w = z^6 = u + iv$.

$$\sin(z^6) = \sin w = \cosh v \sin u + i \sinh v \cos u = 0 \implies \sin u = 0 \implies \sinh v = 0$$

$$\implies w = 2\pi k, k \in \mathbb{Z}.$$

Now solving for $z$, we get

$$z = \sqrt[6]{2\pi k}e^{2n\frac{\pi}{6}}, k \in \mathbb{Z}, n = 0, 1, \ldots, 5.$$