Math 406 Homework 4

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Question 1a.

Question 2a. Integrating:

$$\int_0^1 vLudx = \int_0^1 v\left(x^2u'' + 3xu' - u\right) = \left[vx^2u' + 3vx\right]_0^1 - \int_0^1 (2x^2v)'u' + (3xv)'u + uvdx$$

$$= \left[2x^2vu' + 3xvu + v'u\right]_0^1 + \int_0^1 u\left((2x^2v)'' - (3xv)' - v\right)dx.$$

Thus the adjoint operator is $L_s^*v = (2s^2v)'' - (3sv)' - v = 2s^2v'' + 5sv' \implies L^* = 2s^2\frac{d^2}{ds^2} + 5s\frac{d}{ds}$. We want to find $v(s,x)$ with $v(0,x) < \infty$ and $v(1,x) = 0$ so we can write $u(x) = v_s(1,x) + \int_0^1 v(s,x)f(s)ds$. We want to find v such that $L^*v(s,x) = \delta(x-s)$. Try $v = s^r$ in the homogeneous equation:

$$L^*v = 2s^2v_{ss} + 5sv_s = 2r(r-1)s^r + 5rs^r = 0 \implies 2r(r-1) + 5r = r(2r+3) = 0 \implies r = 0 \text{ or } -\frac{3}{2}.$$

For non-homogeneous $L^*v = \delta(s-x)$, we can solve to the right and left of $x = s$:

$$v(s,x) = \begin{cases} A_- + B_-s^{-\frac{3}{2}} & 0 < s < x \\ A_+ + B_+s^{-\frac{3}{2}} & x < s < 1 \end{cases}.$$

The regularity condition implies that $B_- = 0$, and the $s = 1$ condition imposes $B_+ = -A_+$.We also need continuity, so $A_- = A_+(1 - x^{-\frac{3}{2}})$. Finally, the jump condition:

$$1 = \int_{x-\epsilon}^{x+\epsilon} 2s^2v_{ss} + 5sv_sds = (2s^2v)_s\big|_{x-\epsilon}^{x+\epsilon} = 2s^2v_s\big|_{x-\epsilon}^{x+\epsilon} = 2x^2\left(\frac{3}{2}A_+x^{-\frac{5}{2}} - 0\right) \implies A_+ = \frac{1}{3}\sqrt{x}.$$

Putting this all together, we get

$$v(s,x) = \begin{cases} \frac{1}{3}\left(x^{\frac{1}{2}} - x^{-1}\right) & 0 < s < x \\ \frac{1}{3}\sqrt{x}\left(1 - s^{-\frac{3}{2}}\right) & x < s < 1 \end{cases}.$$

Finally, plugging this into the solution form given above:

$$u(x) = 2v_s(1,x)u(1) + \int_0^1 v(s,x)f(s)ds = \int_0^x \frac{1}{3}\left(x^{\frac{1}{2}} - x^{-1}\right)f(s)ds + \frac{1}{3}\sqrt{x}\int_x^1 \left(1 - s^{-\frac{3}{2}}\right)f(s)ds.$$

The original question asks for G, but of course here $G(s,x) = v(s,x)$ since they represent the same thing.

Question 2b. From class, the factor to multiply the equation by is:

$$F = e^{\int \frac{a_1}{a_0} dx} \frac{1}{a_0} = e^{\int \frac{3}{2x} dx} \frac{1}{2x^2} = \frac{1}{2\sqrt{x}}.$$

Multiplying this, we get:

$$FLu = x^{\frac{3}{2}} u'' + \frac{3}{2} x^{\frac{1}{2}} u' - \frac{1}{2} x^{-\frac{1}{2}} u = \frac{1}{2} x^{-\frac{1}{2}} f.$$

Call this new self adjoint operator L'. Running through the same process as for part a again, we first find the boundary terms for our expression of u. We know that the new operator is self adjoint though, so we can immediately write (choosing $v(1,x) = 0$ and $v(0,x) < \infty$:

$$\int_0^1 vL'udx = \left[vx^{\frac{3}{2}} u' + \frac{3}{2} x^{\frac{1}{2}} v - \frac{3}{2} x^{\frac{1}{2}} vu - x^{\frac{3}{2}} v'u \right]_0^1 + \int_0^1 uL'vdx.$$

From the boundary terms we get that $v(1) = 0$ and $v(0) < \infty$ gives enough information for all of the terms, so the operator L' is also essentially self adjoint. To solve the homogeneous case try $v = s^r$:

$$L's^r = 0 \implies r(r-1) + \frac{3}{2} r - \frac{1}{2} = 0 \implies r = -1 \text{ or } \frac{1}{2}.$$

Applying the boundary conditions $v(0,x) < \infty$ and $v(1,x) = 0$, we can write the solution to $L'v = \delta(x-s)$ as:

$$v(s,x) = \begin{cases} A_- s^{\frac{1}{2}} & 0 < s < x \\ A_+ \left(s^{-1} - s^{\frac{1}{2}} \right) & x < s < 1 \end{cases}.$$

Continuity gives $A_- x^{\frac{1}{2}} = A_+ \left(x^{-1} - x^{\frac{1}{2}} \right) \implies A_- = A_+ \left(x^{-\frac{3}{2}} - 1 \right)$. Finally, the jump condition results in

$$s^{\frac{3}{2}} v_s \Big|_{x-\epsilon}^{x+\epsilon} = 1 \implies x^{\frac{3}{2}} \left(A_+ \left(-x^{-2} - \frac{1}{2} x^{-\frac{1}{2}} \right) - A_+ \left(x^{-\frac{3}{2}} - 1 \right) \frac{1}{2} x^{-\frac{1}{2}} \right) = 1.$$

$$\implies A_+ = -\frac{2}{3} x^{\frac{1}{2}}.$$

Thus our expression for the Green's function $v(s,x) = G(s,x)$ is

$$v(s,x) = \begin{cases} -\frac{2}{3} \left(x^{-1} - x^{\frac{1}{2}} \right) s^{\frac{1}{2}} & 0 < s < x \\ -\frac{2}{3} x^{\frac{1}{2}} \left(s^{-1} - s^{\frac{1}{2}} \right) & x < s < 1 \end{cases}.$$

Using this to solve for u:

$$u(x) = \frac{1}{3} \int_0^x \left(1 - x^{-\frac{3}{2}} \right) s^{\frac{1}{2}} f(s) ds + \frac{1}{3} \int_x^1 \left(s^{\frac{1}{2}} - s^{-1} \right) f(s) ds.$$

Question 3. Because $a_0' = 0 = a_1$, the operator L is self adjoint. This is a special case of the form $(pu')' + qu = f$ which in class we showed can be expressed as

$$u(x) = \left[vu' - v'u \right]_0^\infty + \int_0^\infty v(s,x) f(s) ds$$

for $Lv = \delta(s,x)$. The required boundary conditions on v to make each term knowable are $v \to 0, v' \to 0$ as $x \to \infty$. First solving the homogeneous equation, we have

$$Lv = v'' + v = 0 \implies v(s,x) = A\sin(s) + B\cos(s).$$

Applying this to the non-homogeneous equation $Lv = \delta(s-x)$, we have

$$v(s,x) = \begin{cases} A_- \sin(s) + B_- \cos(s) & 0 < s < x \\ A_+ \sin(s) + B_+ \cos(s) & s > x \end{cases}.$$

The boundary conditions on v at infinity force $A_+ = B_+ = 0$. Continuity forces $A_- \sin(x) + B_- \cos(x) = 0 \implies B_- = -\tan(x)A_-$. Finally, the jump condition gives:

$$\int_{x-\epsilon}^{x+\epsilon} v_{ss} + v ds = 1 \implies v_s(x^+, x) - v_s(x^-, x) = (0 - A_-(\cos(x) + \tan(x)\sin(x))) = 1 \implies A_- = -\cos(x).$$

Thus our final expression for v is:

$$v(s,x) = \begin{cases} -\cos(x)\sin(s) + \sin(x)\cos(s) & 0 < s < x \\ 0 & s > x \end{cases}.$$

For the boundary terms seen previous, we then have $v(0,x) = -\sin(x)$ and $v_s(0,x) = \cos(x)$. Expressing u in terms of these Green's functions:

$$u(x) = \left[vu' - v'u\right]_0^\infty + \int_0^\infty v(s,x)f(s)ds = \sin(x)v_0 + \cos(x)u_0 - \int_0^x (\cos x \sin s - \sin x \cos s)f(s)ds$$

Question 4a. The question states to solve it assuming that solutions are in the form r^i, but as far as I can tell solutions aren't in this form. Instead, I claim that $G_{ij} = ki + c$:

$$k(i+1) - 2ki + k(i-1) = 0.$$

For the non-homogeneous equation, the boundary conditions and continuity enforce what constants are allowed. Thus solutions are in the form:

$$G_{ij} = \begin{cases} \frac{k}{j} i & 0 \le i < j \\ k & i = j \\ \frac{k}{N-j}(N-i) & j < i \le N \end{cases}.$$

The final condition is that $G_{j+1j} - 2G_{jj} + G_{j-1j} = 1$, so $1 = \frac{k}{j}(j-1) - 2k + \frac{k}{N-j}(N-j-1) \implies k = \frac{j(j-N)}{N}$. Thus the explicit solution to (3) is

$$G_{ij} = \begin{cases} \frac{j-N}{N} i & 0 \le i < j \\ 1 & i = j \\ -\frac{j}{N}(N-i) & j < i \le N \end{cases}.$$