MATH 305 Homework 5

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1. Find the region where $f(z) = Log(1-z^3)$ is analytic.

f is not analytic when $1-z^3$ has only a negative real part, i.e.

$$Im(1-z^3) = Im(1-(x+iy)^3) = -x^2y + y^3 = 0.$$

$$Re(1-z^3) = Re(1-(x+iy)^3) = 1 - x^3 + xy^2 < 0.$$

The first requirement tells us that either $y = 0$ or $y^2 = x^2$. However if $x^2 = y^2$ then the second requirement would need $1 < 0$ which is never true, so it must be that $y = 0$. If $y = 0$ the second requirement gives $1 - x^3 < 0 \implies x^3 > 1 \implies x > 1$. Thus the domain of analycity is $D = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x > 1\}$.

2. Find a branch of each of the following multivalued functions that is analytic in the given domain

(a) $(9+z^2)^{\frac{1}{2}}$ in $C \backslash \{x = 0, -3 \leq y \leq 3\}$

Factoring we get $(9+z^2)^{\frac{1}{2}} = (z+3i)^{\frac{1}{2}}(z-3i)^{\frac{1}{2}} = |z+3i||z-3i|e^{\frac{1}{2}(\phi_1+\phi_2)}$. Let ϕ_1 be the angle around $z = 3i$ with $\frac{\pi}{2} < \phi_1 < \frac{5\pi}{2}$ and ϕ_2 be the angle around $z = -3i$ with $\frac{\pi}{2} < \frac{5\pi}{2}$. Then for $Re(z) = 0, Im(z) > 3$ the difference between the angles on both sides are separated by 2π in the exponent, so the function is analytic on the required domain.

(b) $(z^4-1)^{\frac{1}{2}}$ in $\{|z| > 1\}$.

Factoring:

$$(z^4-1)^{\frac{1}{2}} = (z^2-1)^{\frac{1}{2}}(z^2+1)^{\frac{1}{2}} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}(z-i)^{\frac{1}{2}}(z+i)^{\frac{1}{2}}.$$
$$= |z-1||z+1||z-i||z+i|e^{\frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4)}.$$

Consider ϕ_1, ϕ_2 to be the angles around -1 and 1 respectively, both going between $0 < \phi_{1/2} < 2\pi$. Similarly let ϕ_3, ϕ_4 to be the angles around $-i$ and i respectively, both going between $\frac{\pi}{2} < \phi_{1/2} < \frac{5\pi}{2}$. Then above (the line $x = 0, y > 1$) and below (the line $y = 0, x < -1$), the angles are separated by multiples of 2π (after being divided by 2), so the domain of analycity is $\{z \mid |z| > 1\}$.

3. Find all solutions to

(a) $\sin(z) = -i$

$$\arcsin \sin z = z = \arcsin(-i) = -i \log\left(1 + \left(2^{\frac{1}{2}}\right)\right) = -i \log\left(1 \pm \sqrt{2}\right).$$

$$\implies z = -i\ln(\sqrt{2}+1) + 2\pi k \text{ or } -i\ln(\sqrt{2}-1) + 2\pi k + \pi, k \in \mathbb{Z}.$$

(b) $\sin^{-1}(i)$

$$\sin^{-1}(i) = -iLog\left(-1 + \left(2^{\frac{1}{2}}\right)\right) = -iLog\left(-1+\sqrt{2}\right).$$

$$= -i\ln\left(\sqrt{2}-1\right).$$

(c) $\cos(z) = 2i$

$$\arccos\cos z = z = \arccos(2i) = \frac{1}{2}\pi + i\log\left(-2 \pm \sqrt{5}\right).$$

$$= \frac{1}{2}\pi + 2\pi k + i\ln(\sqrt{5}-2) \text{ or } \frac{3}{2}\pi + 2\pi k + i\ln\left(\sqrt{5}+2\right).$$

(d) $\cos^{-1}(2i)$

$$\cos^{-1}(2i) = \frac{1}{2}\pi + iLog\left(iz + \sqrt{1-z^2}\right) = \frac{1}{2\pi} + iLog\left(-2+\sqrt{5}\right) = \frac{1}{2\pi} + i\ln\left(-2+\sqrt{5}\right).$$

4. Find a solution to the boundary value problem

$$\phi_{xx} + \phi_{yy} = 0, y > 0, -1 < x < 1, y > 0$$

$$\phi(x,y) = 0, \text{on } x = -1, y > 0; 0, \text{on } y = 0, -1 < x < 1; 2, \text{on } x = 1, y > 0.$$

Let $f(z) = \sin(\pi z) = u + iv$ and $\Phi(u,v) = \phi(x,y)$. Then this maps the given domain to the upper half plane with the same boundary conditions. Thus we can write the solution as:

$$\Phi(w) = AArg(w+1) + BArg(w-1) = 2Arg(w-1).$$

Since $w = \sin(\pi z)$, the final solution would be $\phi(z) = 2Arg(\sin(\pi z))$.

5. Find a solution to the boundary value problem

$$\phi_{xx} + \phi_{yy} = 0, \quad x > 0, y > 0$$

$$\phi = 1 \text{ on } x = 0, y > 0; \phi_y = 0 \text{ on } 0 < x < 1, y = 0; \phi = 2 \text{ on } x > 1, y = 0$$

Let $f(z) = \sin^{-1}(z) = u + iv$ and $\Phi(u,v) = \phi(x,y)$. Then under this map the given region becomes a rectangle above the $y = 0$ line between 0 and $\frac{\pi}{2}$. The solution to this new boundary problem is just linear, which using the initial conditions comes to $\Phi(u,v) = \frac{2}{\pi}(u+1)$ Switching variables to the original x, y, we get that $\phi = \frac{2}{\pi}\sin^{-1}\left(\frac{1}{2}\left((x+1)^2+y^2\right)^{\frac{1}{2}} - \left((x-1)^2+y^2\right)^{\frac{1}{2}}\right)$ (the derivation for the u, v parts of $\sin^{-1}$ was done in lecture).

6. Find an inverse function for $sinh(z) = \frac{e^z - e^{-z}}{2}$ such that its value at 0 equals 0.

Starting from the given expression for sinh:

$$2z = e^w - e^{-w} \implies 0 = e^{2w} - 2ze^w - 1 \implies e^w = z \pm \frac{1}{2}(4z^2+4)^{\frac{1}{2}}.$$

$$\implies w = \log\left(z \pm \left(z^2+1\right)^{\frac{1}{2}}\right).$$

Since we want the inverse to have value 0 at 0, we can take the principle branches of both log and $z^{\frac{1}{2}}$. Thus we end up with:

$$w = Log\left(z + \sqrt{z^2+1}\right).$$

7. Show that $|\sin z| < 3$ when $|z| < 1$.

From the definition of sin with the triangle identity, we have

$$|\sin z| = |\sin x \cosh y + i \cos x \sinh y| \leq |\sin x||\cosh y| + |\cos x||\sinh y|.$$

$$\leq |\cosh y| + |\sinh y| < 1.176 + 1.544 < 3.$$

Note the last step uses the fact that $x < 1, y < 1$ to get direct bounds on the hyperbolic trig functions.

8. Compute the integral $\int_C f dz$ using the contour (always counter-clockwise) given

(a) $f = x - 2xyi$; $C = \{y = x^2, 0 \leq x \leq 1\} \cup \{y = 1, -1 \leq x \leq 1\}$

$$\int_C f dz = \int_0^1 (t - 2it^3)(1 + 2it)dt + \int_0^2 (1 - t - 2(1-t)i)(-1)dt.$$

$$= \int_0^1 \left(t + 4t^4 + i(2t^2 - 2t^3)\right) dt + \int_0^2 (t - 1 + 2(1-t)i)\, dt.$$

$$= \frac{1}{2} + \frac{4}{5} + i\left(\frac{2}{3} - \frac{1}{2}\right) + 2 - 2 + i(4 - 4) = \frac{13}{5} + \frac{i}{6}.$$

(b) $f = \bar{z}^2$; C: square with vertices $z = 0, z = 1, z = 1 + i$ and $z = i$

$$\int_C f dz = \int_0^1 t^2 dt + \int_0^1 (1 - it)^2\, i dt + \int_0^1 (t - i)^2 (-1)dt + \int_0^1 (it)^2 (-i)dt.$$

$$= \frac{1}{3} + i + 1 - \frac{1}{3} + i - 1 = 2i.$$

(c) $f = Log(z)$; $C = \{|z| = 1, Re(z) \geq 0\}$

Let $z(t) = e^{it}$.

$$\int_C f dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Log(e^{it}) ie^{it} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -te^{it} dt = -e^{it}(1 - it)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2i.$$

9. Evaluate $\int_C (z^2 + 1)dz$, where C is the following contour from $z = -i$ to $z = 1$:

(a) the simple line segment

Let $z = t + i(t - 1)$.

$$\int_C f dz = \int_0^1 \left((t + i(t-1))^2 + 1\right)(1 + i)\, dt = (1 + i) + (1 + i)\left(\frac{1}{3} - \frac{1}{3} + 1 - 1 + i\left(\frac{2}{3} - 2\right)\right).$$

$$= 1 + i + 2 - \frac{2}{3} + i\left(\frac{2}{3} - 2\right) - 1 + i = \frac{4}{3} + \frac{2i}{3}.$$

(b) two simple line segments, the first from $z = -i$ to $z = 0$ and the second from $z = 0$ to $z = 1$

$$\int_C f dz = \int_{-1}^0 (1 - t^2)\, i dt + \int_0^1 (t^2 + 1)\, dt = 1 - \frac{1}{3} + \frac{1}{3} + 1 = \frac{4}{3} + \frac{2i}{3}.$$

(c) the circular arc $z = e^{it}, -\frac{\pi}{2} \leq t \leq 0$

$$\int_C f dz = \int_{-\frac{\pi}{2}}^0 (e^{2it} + 1)\, ie^{it} dt = i\left(\frac{1}{3} - \frac{1}{3}e^{i\frac{3\pi}{2}} + 1 - e^{i\frac{\pi}{2}}\right) = \frac{4}{3} + \frac{2i}{3}.$$

10. Evaluate $\int_C \bar{z} dz$, where

(a) C is the circle $|z| = 2$ traversed once counterclockwise

Let $z = e^{it}$.

$$\int_C f dz = 4 \int_0^{2\pi} e^{-it} i e^{it} dt = 8\pi i.$$

(b) C is the circle $|z| = 2$ traversed twice counterclockwise

Going around twice counterclockwise is just two times the previous answer, so the integral would be $16\pi i$

(c) C is the circle $|z| = 2$ traversed three times clockwise. Three times clockwise would be negative 3 times the answer for part a, so the integral would be $-24\pi i$