# Math 406 Homework 4

Xander Naumenko

05/11/23

**Question 1a.** Consider the generalized functions operating on a test function $\phi$:

$$T_{a(x)\delta(x)}(\phi) = \int_{-\infty}^{\infty} a(x)\delta(x)\phi(x)dx = a(0)\phi(0)$$

$$T_{a(0)\delta(x)}(\phi) = \int_{-\infty}^{\infty} a(0)\delta(x)\phi(x)dx = a(0)\phi(0).$$

Since they act identically on all test functions, we have $a(x)\delta(x) = a(0)\delta(0)$.

**Question 1b.** Consider the generalized function operating on a test function $\phi$ (the boundary terms all vanish since $\phi$ vanishes at infinity):

$$T_{x^2\delta^{(3)}(x)}(\phi) = \int_{-\infty}^{\infty} x^2\delta^{(3)}(x)\phi(x)dx = \int_{-\infty}^{\infty} \delta^{(2)}(x)(2x\phi(x) + x^2\phi'(x))dx$$

$$= \int_{-\infty}^{\infty} \delta^{(1)}(x)(2\phi(x) + 4x\phi'(x) + x^2\phi''(x))dx = \int_{-\infty}^{\infty} \delta(x)\left(6\phi'(x) + 6x\phi''(x) + x^2\phi^{(3)}(x)\right)dx$$

$$= \int_{-\infty}^{\infty} 6\delta(x)\phi'(x)dx = T_{6\delta'}(\phi).$$

Thus $x^2\delta^{(3)}(x) = 6\delta'(x)$. For the equation $x^2g(x) = 0$, the solution must be zero for all $x \neq 0$, so assume that the solution is a linear combination of $\delta$ and it's derivatives. The above computation shows that for any $n > 1$, when doing integration by parts there is a non $x$ dependent term in the product $(x^2\delta^{(n)}, \phi)$ so the result is non-zero and can't be solutions. Clearly $x^2\delta(x) = 0$ (since $x\delta(x) = 0$), and for $\delta'(x)$ we have

$$(x^2\delta'(x), \phi) = \int_{-\infty}^{\infty} x^2\delta'(x)\phi(x)dx = \int_{-\infty}^{\infty} (2x\phi(x) + x^2\phi'(x))\delta(x) = 0,$$

so $\delta'(x)$ is also a solution. Thus the general solution to $x^2g(x) = 0$ is $g(x) = A\delta(x) + B\delta'(x)$.

**Question 1c.** Expanding for a test function $\phi$:

$$(\delta(\cos(x)), \phi) = \int_{-\infty}^{\infty} \delta(\cos(x))\phi(x)dx = \sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} \delta(\cos(x))\phi(x)dx = \sum_{k=-\infty}^{\infty} \frac{\phi\left(\left(k+\frac{1}{2}\right)\pi\right)}{\left|\sin\left(\left(k+\frac{1}{2}\right)\pi\right)\right|}$$

$$= \left(\sum_{k=-\infty}^{\infty} \delta\left(\left(k+\frac{1}{2}\right)\pi\right), \phi\right).$$

**Question 1d.** Again with a test function $\phi$:

$$\big(f(x)\delta'(x), \phi(x)\big) = \int_{-\infty}^{\infty} f(x)\delta'(x)\phi(x)dx = -\int_{-\infty}^{\infty} \delta(x)(f'(x)\phi(x) + f(x)\phi'(x))dx$$

$$= (f(x)\delta'(x) - f'(x)\delta(x), \phi).$$

Using part (a) we can replace $f(x)$ with $f(0)$ in last line above to get the required result: $f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x)$.

**Question 2a.** Integrating:

$$\int_0^1 vLudx = \int_0^1 v\left(x^2u'' + 3xu' - u\right) = \left[vx^2u' + 3vx\right]_0^1 - \int_0^1 (2x^2v)'u' + (3xv)'u + uvdx$$

$$= \left[2x^2vu' + 3xvu + v'u\right]_0^1 + \int_0^1 u\left((2x^2v)'' - (3xv)' - v\right)dx.$$

Thus the adjoint operator is $L_s^*v = (2s^2v)'' - (3sv)' - v = 2s^2v'' + 5sv' \implies L^* = 2s^2\frac{d^2}{ds^2} + 5s\frac{d}{ds}$. We want to find $v(s,x)$ with $v(0,x) < \infty$ and $v(1,x) = 0$ so we can write $u(x) = v_s(1,x) + \int_0^1 v(s,x)f(s)ds$. We want to find $v$ such that $L^*v(s,x) = \delta(x-s)$. Try $v = s^r$ in the homogeneous equation:

$$L^*v = 2s^2v_{ss} + 5sv_s = 2r(r-1)s^r + 5rs^r = 0 \implies 2r(r-1) + 5r = r(2r+3) = 0 \implies r = 0 \text{ or } -\frac{3}{2}.$$

For non-homogeneous $L^*v = \delta(s-x)$, we can solve to the right and left of $x = s$:

$$v(s,x) = \begin{cases} A_- + B_-s^{-\frac{3}{2}} & 0 < s < x \\ A_+ + B_+s^{-\frac{3}{2}} & x < s < 1 \end{cases}.$$

The regularity condition implies that $B_- = 0$, and the $s = 1$ condition imposes $B_+ = -A_+$.We also need continuity, so $A_- = A_+(1 - x^{-\frac{3}{2}})$. Finally, the jump condition:

$$1 = \int_{x-\epsilon}^{x+\epsilon} 2s^2v_{ss} + 5sv_sds = (2s^2v)_s\big|_{x-\epsilon}^{x+\epsilon} = 2s^2v_s\big|_{x-\epsilon}^{x+\epsilon} = 2x^2\left(\frac{3}{2}A_+x^{-\frac{5}{2}} - 0\right) \implies A_+ = \frac{1}{3}\sqrt{x}.$$

Putting this all together, we get

$$v(s,x) = \begin{cases} \frac{1}{3}\left(x^{\frac{1}{2}} - x^{-1}\right) & 0 < s < x \\ \frac{1}{3}\sqrt{x}\left(1 - s^{-\frac{3}{2}}\right) & x < s < 1 \end{cases}.$$

Finally, plugging this into the solution form given above:

$$u(x) = 2v_s(1,x)u(1) + \int_0^1 v(s,x)f(s)ds = \int_0^x \frac{1}{3}\left(x^{\frac{1}{2}} - x^{-1}\right)f(s)ds + \frac{1}{3}\sqrt{x}\int_x^1\left(1 - s^{-\frac{3}{2}}\right)f(s)ds.$$

The original question asks for $G$, but of course here $G(s,x) = v(s,x)$ since they represent the same thing.

**Question 2b.** From class, the factor to multiply the equation by is:

$$F = e^{\int \frac{a_1}{a_0}dx}\frac{1}{a_0} = e^{\int \frac{3}{2x}dx}\frac{1}{2x^2} = \frac{1}{2\sqrt{x}}.$$

Multiplying this, we get:

$$FLu = x^{\frac{3}{2}}u'' + \frac{3}{2}x^{\frac{1}{2}}u' - \frac{1}{2}x^{-\frac{1}{2}}u = \frac{1}{2}x^{-\frac{1}{2}}f.$$

Call this new self adjoint operator $L'$. Running through the same process as for part a again, we first find the boundary terms for our expression of $u$. We know that the new operator is self adjoint though, so we can immediately write (choosing $v(1,x) = 0$ and $v(0,x) < \infty$:

$$\int_0^1 vL'udx = \left[vx^{\frac{3}{2}}u' + \frac{3}{2}x^{\frac{1}{2}}v - \frac{3}{2}x^{\frac{1}{2}}vu - x^{\frac{3}{2}}v'u\right]_0^1 + \int_0^1 uL'vdx.$$

From the boundary terms we get that $v(1) = 0$ and $v(0) < \infty$ gives enough information for all of the terms, so the operator $L'$ is also essentially self adjoint. To solve the homogeneous case try $v = s^r$:

$$L's^r = 0 \implies r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0 \implies r = -1 \text{ or } \frac{1}{2}.$$

Applying the boundary conditions $v(0,x) < \infty$ and $v(1,x) = 0$, we can write the solution to $L'v = \delta(x-s)$ as:

$$v(s,x) = \begin{cases} A_- s^{\frac{1}{2}} & 0 < s < x \\ A_+\left(s^{-1} - s^{\frac{1}{2}}\right) & x < s < 1 \end{cases}.$$

Continuity gives $A_- x^{\frac{1}{2}} = A_+\left(x^{-1} - x^{\frac{1}{2}}\right) \implies A_- = A_+\left(x^{-\frac{3}{2}} - 1\right)$. Finally, the jump condition results in

$$s^{\frac{3}{2}}v_s\Big|_{x-\epsilon}^{x+\epsilon} = 1 \implies x^{\frac{3}{2}}\left(A_+\left(-x^{-2} - \frac{1}{2}x^{-\frac{1}{2}}\right) - A_+\left(x^{-\frac{3}{2}} - 1\right)\frac{1}{2}x^{-\frac{1}{2}}\right) = 1.$$

$$\implies A_+ = -\frac{2}{3}x^{\frac{1}{2}}.$$

Thus our expression for the Green's function $v(s,x) = G(s,x)$ is

$$v(s,x) = \begin{cases} -\frac{2}{3}\left(x^{-1} - x^{\frac{1}{2}}\right)s^{\frac{1}{2}} & 0 < s < x \\ -\frac{2}{3}x^{\frac{1}{2}}\left(s^{-1} - s^{\frac{1}{2}}\right) & x < s < 1 \end{cases}.$$

Using this to solve for $u$:

$$u(x) = \frac{1}{3}\int_0^x \left(1 - x^{-\frac{3}{2}}\right)s^{\frac{1}{2}}f(s)ds + \frac{1}{3}\int_x^1 \left(s^{\frac{1}{2}} - s^{-1}\right)f(s)ds.$$

**Question 3.** Because $a_0' = 0 = a_1$, the operator $L$ is self adjoint. This is a special case of the form $(pu')' + qu = f$ which in class we showed can be expressed as

$$u(x) = \left[vu' - v'u\right]_0^\infty + \int_0^\infty v(s,x)f(s)ds$$

for $Lv = \delta(s,x)$. The required boundary conditions on $v$ to make each term knowable are $v \to 0, v' \to 0$ as $x \to \infty$. First solving the homogeneous equation, we have

$$Lv = v'' + v = 0 \implies v(s,x) = A\sin(s) + B\cos(s).$$

Applying this to the non-homogeneous equation $Lv = \delta(s-x)$, we have

$$v(s,x) = \begin{cases} A_- \sin(s) + B_- \cos(s) & 0 < s < x \\ A_+ \sin(s) + B_+ \cos(s) & s > x \end{cases}.$$

The boundary conditions on $v$ at infinity force $A_+ = B_+ = 0$. Continuity forces $A_- \sin(x) + B_- \cos(x) = 0 \implies B_- = -\tan(x)A_-$. Finally, the jump condition gives:

$$\int_{x-\epsilon}^{x+\epsilon} v_{ss} + v ds = 1 \implies v_s(x^+, x) - v_s(x^-, x) = (0 - A_-(\cos(x) + \tan(x)\sin(x))) = 1 \implies A_- = -\cos(x).$$

Thus our final expression for $v$ is:

$$v(s,x) = \begin{cases} -\cos(x)\sin(s) + \sin(x)\cos(s) & 0 < s < x \\ 0 & s > x \end{cases}.$$

For the boundary terms seen previous, we then have $v(0,x) = -\sin(x)$ and $v_s(0,x) = \cos(x)$. Expressing $u$ in terms of these Green's functions:

$$u(x) = \left[vu' - v'u\right]_0^\infty + \int_0^\infty v(s,x)f(s)ds = \sin(x)v_0 + \cos(x)u_0 - \int_0^x (\cos x \sin s - \sin x \cos s)f(s)ds$$

**Question 4a.** For solutions of the form $G_{ij} = r^i$ in the homogeneous equation then we have $G_{i+1j} - 2G_{ij} + G_{i-1j} = r^{i+1} - 2r^i + r^{i-1} = 0 \implies r = 1$ or 0. Since the $r = 1$ root has multiplicity 2 solutions are thus in the form $b1^i + ai1^i = ai + b$, i.e. linear. For the non-homogeneous equation, the boundary conditions and continuity enforce what constants are allowed. Thus solutions are in the form:

$$G_{ij} = \begin{cases} \frac{k}{j} i & 0 \le i < j \\ k & i = j \\ \frac{k}{N-j}(N-i) & j < i \le N \end{cases}.$$

The final condition is that $G_{j+1j} - 2G_{jj} + G_{j-1j} = 1$, so $1 = \frac{k}{j}(j-1) - 2k + \frac{k}{N-j}(N-j-1) \implies k = \frac{j(j-N)}{N}$. Thus the explicit solution to (3) is

$$G_{ij} = \begin{cases} \frac{j-N}{N} i & 0 \le i < j \\ 1 & i = j \\ -\frac{j}{N}(N-i) & j < i \le N \end{cases}.$$

Note that because $G$ is fixed at the endpoints there's an off-by-one comparison with $A_N$. I had trouble signing into Matlab due to the new cwl two-factor authentication so I did the computation in Python, I hope that's fine:

```
import numpy as np

n=5


A5 = np.zeros((n, n))
G = np.zeros((n+2, n+2))
```

```
for i in range(n):
    A5[i][i] = -2
    if i > 0:
        A5[i][i-1] = 1
    if i < n-1:
        A5[i][i+1] = 1

for i in range(0,n+2):
    for j in range(0,n+2):
        k = j*(j-(n+1))/(n+1)
        if i < j:
            G[i][j] = k*i/j
        elif i == j:
            G[i][j] = k
        else:
            G[i][j] = k*((n+1)-i)/((n+1)-j)

print(A5)
inv = np.linalg.inv(A5)
print(inv)
print(G)
```

The output of the program gives the values for $G$ and $A_N^{-1}$ to be:

$$A_5^{-1} \begin{bmatrix} -0.83 & -0.67 & -0.50 & -0.33 & -0.17 \\ -0.67 & -1.33 & -1.00 & -0.67 & -0.33 \\ -0.50 & -1.00 & -1.50 & -1.00 & -0.50 \\ -0.33 & -0.67 & -1.00 & -1.33 & -0.67 \\ -0.17 & -0.33 & -0.50 & -0.67 & -0.83 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.83 & -0.67 & -0.50 & -0.33 & -0.17 & 0.00 \\ 0.00 & -0.67 & -1.33 & -1.00 & -0.67 & -0.33 & 0.00 \\ 0.00 & -0.50 & -1.00 & -1.50 & -1.00 & -0.50 & 0.00 \\ 0.00 & -0.33 & -0.67 & -1.00 & -1.33 & -0.67 & 0.00 \\ 0.00 & -0.17 & -0.33 & -0.50 & -0.67 & -0.83 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}.$$

As expected they're identical except with more entries in G since it's tied down at the endpoints.