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Question 1a. Let $a, b, c, d \in \mathbb{Z}$ with either $a \neq c$ or $b \neq d$. By contradiction suppose that $f(a, b) = f(c, d) \implies a + b\sqrt{2} = c + d\sqrt{2} \implies a - c = \sqrt{2}\,(d - b) \implies \sqrt{2} = \frac{a-c}{d-b}$ or $d - b = 0$. However $\sqrt{2}$ isn't rational, so in the former case $a - c = 0$. However $a - c = 0 \implies a - d = 0$ and vice versa, but this implies that both $a = c$ and $b = d$ which contradicts the definition of a, b, c, d. Thus $f(a, b) \neq f(c, d)$ and f is one-to-one.

Question 1b. To show this I will prove that for any $M \in \mathbb{N} \setminus \{1\}$, there exists a pair (m, n) with either $m = M$ or $m = M - 1$ such that $(m, n) \in S \cap (0, 1)$. Since f is one-to-one, if $S \cap (0, 1)$ was finite then there would be a maximum M for which this is no longer possible, so proving it is sufficient.

Let $M \in \mathbb{N} \setminus \{1\}$, and consider $m_1 = M, n_1 = -\left[\frac{M}{\sqrt{2}}\right]$, where $[x]$ represents the integer part (or floor) of x. Then $m_1 + n_1\sqrt{2} = m_1 - \left[\frac{m_1}{\sqrt{2}}\right]\sqrt{2} > 0$. Also note that $m_1 - \left[\frac{m_1}{\sqrt{2}}\right]\sqrt{2} < m_1 - \frac{m_1}{\sqrt{2}}\sqrt{2} + \sqrt{2} = \sqrt{2}$. If $m_1 + n_1\sqrt{2}$ is less than 1 already then we're done, since $0 < m_1 + n_1\sqrt{2} < 1$. Otherwise, note that the pair $m_2 = m_1 - 1, n_2 = n_1$ works, since using the fact that $1 < \sqrt{2} < 2$, we get:

$$m_2 + n_2\sqrt{2} = m_1 + n_1\sqrt{2} - 1 > 1 - 1 = 0$$

$$m_2 + n_2\sqrt{2} = m_1 + n_1\sqrt{2} - 1 < \sqrt{2} - 1 < 1.$$

Question 1c. Let $\epsilon > 0$, by the Archimedes principle there exists $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon} \implies \frac{1}{N} < \epsilon$. Consider dividing the interval $(0, 1)$ into N equally spaced intervals. By the result from part b we know that there are infinite members of S to be split between N intervals, so by the pigeonhole principle there exists an interval containing two members of S, without loss of generality call them m_1, n_1, m_2, n_2 with $0 < m_1 + n_1\sqrt{2} - m_2 - n_2\sqrt{2} = (m_1 - m_2) + (n_1 - n_2)\sqrt{2} \leq \frac{1}{N} < \epsilon$. Thus the difference between these elements lies in the interval $(0, \epsilon)$ and is in the required form so we're done.

Question 1d. Let $(a, b) \subset \mathbb{R}$. Apply the result from part b to obtain an element $m + n\sqrt{2} \in S$ with $0 < m + n\sqrt{2} < b - a$. Let N be the smallest integer such that $N > \frac{a}{m+n\sqrt{2}}$. Then clearly by definition $(m + n\sqrt{2})N > a\frac{m+n\sqrt{2}}{m+n\sqrt{2}} = a$. Also, since N is minimal we have

$$(m + n\sqrt{2})(N - 1) < a \implies (m + n\sqrt{2})N < a + b - a = b.$$

Thus $(m + n\sqrt{2}) = mN + nN\sqrt{2}$ is in the required form and is contained in (a, b).

Question 2. I claim that $f(x) = \delta(x)$, where $\delta(x)$ is 1 when $x = 0$ and 0 otherwise. If $x = 0$, then clearly $f(1) = \lim_{n\to\infty} \frac{1}{1} = 1$. Let $x \in \mathbb{R}, x \neq 0, \epsilon > 0$. By Archimedes' principle, there exists $m > \frac{1}{\epsilon x} \implies \frac{1}{mx} < \epsilon$ Choose $N = m$. Then we have that for $n > m$,

$$\left|\frac{1}{1+nx}\right| < \left|\frac{1}{mx}\right| < \epsilon.$$

Question 3. Let $(a_n)_n$ be a real sequence that converges to A. To show that $a_n^3 \to A^3$, set $\epsilon > 0$. $a_n \to A$, so $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$, $|a_n - A| < 1$. Also $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$, $|a_n - A| < \frac{\epsilon}{3|A|^2+3|A|+1}$. Then for $n > \max(N_1, N_2)$, we have

$$\begin{aligned} \left|a_n^3 - A^3\right| = \left|(a_n - A)\left(a_n^2 + a_n A + A^2\right)\right| &< |a_n - A|\left(|A|^2 + 2|A| + 1 + |A|^2 + |A| + |A|^2\right) \\ &= |a_n - A|\left(3|A|^2 + 3|A| + 1\right) < \epsilon. \end{aligned}$$

To show that $a_n^{1/3} \to A^{1/3}$, let $\epsilon > 0$. If $A = 0$, then since $a_n \to A$, $\exists N$ s.t. $\forall n > N$, $|a_n - A| = |a_n| < \epsilon^3$. Then for $n > N$, we have

$$|a_n^{1/3} - A^{1/3}| = |a_n^{1/3}| = |a_n|^{\frac{1}{3}} < \epsilon.$$

This handles the case where $A = 0$, so now let $A \neq 0$. $a_n \to A$, so $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$, $|a_n - A| < \frac{A}{2}$, note that this implies that a_n and A have the same sign for $n > N_1$. Also $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$, $|a_n - A| < \frac{1}{|A|^{2/3}}$. Let $n > \max(N_1, N_2)$. If $A = 0$, then we have:

$$\left|a_n^{1/3} - A^{1/3}\right| = \left|(a_n - A)\left(\frac{1}{a_n^{2/3} + \sqrt[3]{a_n A} + A^{2/3}}\right)\right|$$

All terms in the denominator are positive, so we get:

$$< |a_n - A|\frac{1}{|A|^{2/3}} < \epsilon.$$

Question 4a. By the Archimedes principle, $\exists R \in \mathbb{N}$ s.t. $N > \frac{b}{M-m}$. Then for all $x > R$, we have:

$$x > R > \frac{b}{M-m} \implies Mx > mx + b.$$

Question 4b. Let y_n, M, m, b be defined as in the question. Since $(y_n/n) \to M$, $\exists N_1$ s.t. $\forall n > N_1$, $|y_n/n - M| < b$. Let $N = \max(N_1, \lceil 2b/(M-m) \rceil)$. Note that the second part of the maximum implies that for $n > N$, $n > \frac{2b}{M-m} \implies Mn - b > mn + b \implies Mn - b > mn + b$The first part of that maximum means that for $n > N$, $y_n > Mn - b$. Putting these together we get that for $n > N$,

$$y_n > Mn - b > mn + b$$

as required.

Question 4c. False, let $y_n = n + 1$. Let $\epsilon > 0$, and let n be an integer with $n > \frac{1}{\epsilon}$. Then for $n > N$, we have

$$|y_n/n - 1| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \epsilon.$$

Thus $y_n/n \to 1$. However $|y_n - n| = |n + 1 - n| = 1 \not\to 0$, so the statement isn't true.

Question 5. Without loss of generality assume that $\alpha > \beta$, so we are trying to prove that $\lim_{n\to\infty} (\alpha^n + \beta^n)^{1/n} = \alpha$. Let $\epsilon > 0$, and choose any $N > \frac{1}{\log_2(1+\epsilon/\alpha)}$. Then for $n > N$, using the fact that log is monotonic we have

$$\left|(\alpha^n + \beta^n)^{1/n} - \alpha\right| \leq \left|(\alpha^n + \alpha^n)^{1/n} - \alpha\right| = \left|\alpha 2^{1/n} - \alpha\right| < \alpha \left|1 + \frac{\epsilon}{\alpha} - 1\right| = \epsilon.$$

Question 6a. Let $r' = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$, by assumption we have $r' < 1$. Let $\epsilon = \frac{1-r'}{2}$, then $\exists N$ s.t. $\forall n > N$, $\left|\frac{x_{n+1}}{x_n} - r'\right| < \frac{1-r'}{2}$. Let $r = \frac{1+r'}{2}$ and $C = \max(x_1, x_2, \ldots, x_N)$. Then we get that for all $n > N$:

$$x_n = x_1 \cdot \frac{x_2}{x_1} \cdots \frac{x_n}{x_{n-1}} < C\left(r' + \frac{1-r'}{2}\right)^n = C\left(\frac{1+r'}{2}\right)^n = Cr^n.$$

Let $\epsilon > 0, r \in (0,1)$ and choose $N = \log_r \frac{\epsilon}{C}$. Then we have for $n > N$:

$$|Cr^n - 0| < \left|C\frac{\epsilon}{C}\right| = \epsilon.$$

Thus $Cr^n \to 0$, and since x_n is bounded below by 0 and above by Cr^n, by the squeeze theorem it also goes to 0.

Question 6b. By contradiction, assume that $\frac{1}{x_n} \to M$ for some $M \in \mathbb{R}$. Choose $\epsilon = M$, and let $N \in \mathbb{N}$. By the convergence of x_n, there exists $n > N$ with $|x_n - 0| = |x_n| < \frac{1}{2M}$, but this would mean

$$\left|\frac{1}{x_n} - M\right| > |2M - M| = |M| = \epsilon.$$

Thus our assumption that $\frac{1}{x_n} \to M$ was wrong and $\frac{1}{x_n}$ doesn't converge.

Question 6c. Checking the ratios of the first:

$$\left(\frac{10^{n+1}}{(n+1)!}\right)\left(\frac{10^n}{n!}\right) = \frac{10}{n+1} \to 0.$$

Thus by part a we get that $\left(\frac{10^n}{n!}\right) \to 0$. For the second, note that $\left(\frac{n}{2^n}\right) \to 0$ (the question states that $\epsilon - N$ arguments aren't required so I won't include one), so by part b its reciprocal $\frac{2^n}{n}$ doesn't converge. Finally for the last one, we can take the ratios of the sequence once more:

$$\left(\frac{2^{3(n+1)}}{3^{2(n+1)}}\right)\left(\frac{3^{2n}}{2^{3n}}\right) = \frac{8}{9}.$$

Thus by part a $\left(\frac{2^{3n}}{3^{2n}}\right)$ converges.

Question 7. Proof by contradiction, assume that $x_n \to M$ for $M \neq 0$. Let $L = \lim_{n\to\infty}(x_n y_n)$ Then let $\epsilon = 1, N \in \mathbb{N}$. Since $x_n \to M, \exists N_1$ s.t. $\forall n > N_1$, $|x_n - M| < \frac{M}{2}$. Also since $y_n \to \infty$, $\exists N_2$ s.t. $\forall n > N_2, y_n > 2\frac{1+L}{M}$. Then for $n > \max(N, N_1, N_2)$, we have:

$$|x_n y_n - L| > \left|\frac{M}{2}2\frac{1+L}{M} - L\right| = |L + 1 - L| = 1 = \epsilon.$$

Since this is true of all choices of $N \in \mathbb{N}$, this contradicts our assumption that $(x_n y_n)$ converges, so the assumption that $M \neq 0$ must have been incorrect and it actually was $M = 0$ (technically it's possible that x_n diverges, but clearly since y_n also diverges $x_n y_n$ wouldn't be able to converge).

Question 8a. Let $\epsilon > 0$. Since $a_n \to a$ there exists $N_1 \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \frac{\epsilon}{2}$. Let $M = \max(|a_1 - a|, |a_2 - a|, \ldots, |a_{N_1} - a|)$ and choose any $N > \frac{2N_1 M}{\max(1,\epsilon)}$. The for $n > N$, we have

$$|s_n - a| < \frac{1}{n}\left(|a_1 - a| + \ldots + |a_{N_1} - a| + |a_{N_1+1} - a| + \ldots + |a_N - a|\right)$$
$$< \frac{1}{N}\left(MN_1 + \frac{\epsilon}{2}N\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 8b. False. Let $a_n = -(-1)^n$. Setting $\epsilon = 1$ shows that clearly a_n can't converge, but $s_n = \frac{a_1+a_2+\ldots+a_n}{n}$ is $\frac{1}{n}$ if n is odd and 0 otherwise. Let $\epsilon > 0$, and choose $N = \epsilon$, then $\forall n > N$, $|s_n - 0| < |\frac{1}{n}| = \epsilon$, so $s_n \to 0$.

Question c, part a. Let $R > 0$. Since $a_n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$, $a_n > 2R$. Let $N = 2N_1$. Then for all $n > N$, we have

$$s_n - R = \frac{1}{N}\left(a_1 - R + \ldots + a_{N_1} - R + a_{N_1+1} - R + \ldots a_N - R\right)$$

$$> \frac{1}{N}\left((N - N_1)2R\right) > R$$

Thus $s_n \to \infty$.

Question c, part b. False. Define

$$a_n = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$

Then $s_n = \frac{1+2+\ldots+n/2}{n} = \frac{n(n/2+1)}{4n} = \frac{n}{2} + 1$ for n even and $s_n = \frac{1+2+\ldots+\frac{n-1}{2}}{n} = \frac{n-1}{2} + 1$ for n odd, which clearly grows linearly and diverges. However a_n alternates between 0 and $\frac{n}{2}$ and doesn't go to infinity for the same reason as the example in part b. Thus the statement isn't true.