Math 322 Homework 1

Xander Naumenko

12/09/23

Question 1. To show that $\sim$ is an equivalence relation, we must show that it is reflexive, symmetric and transitive. It is reflexive, as there always exists a line between an arbitrary point and the origin (the meaning of "the line" is a bit ambiguous when $p = q$, so I interpret it to mean there exists a line passing through p, q, and the origin). To show symmetry, note that the line between a and b is the same as the line between b and a, so $a \sim b \implies b \sim a$.

For transitivity, let $p, q, r \in X$ with $p \sim q$ and $q \sim r$. Let the origin be denoted by the point o, and denote the line on the plane defined by two points a, b as ab. Then $o \in pq \implies p \in oq$, and similarly $o \in qr \implies r \in oq$. Thus the line oq is a line passing through p, r that contains the origin which proves transitivity, and thus $\sim$ is an equivalence relation. Qualitatively, $X/\sim$ is a partition of X where every point in radial lines away from the origin are grouped together.

Question 2. "Describe" is a bit vague here, so I'm interpreting this as a qualitative explanation of what $\mathbb{C}^*/\sim$ represents. Since it relates numbers on the complex plane together if they're the same when normalized together, $\sim$ relates elements of $\mathbb{C}^*$ when their angles match. This is similar, but not identical to the quotient set from problem 1 under the bijection $f : \mathbb{C}^* \to X$ defined as $f(c) = (\text{Re}\{c\}, \text{Im}\{c\})$. They both relate elements together based on their angular components, but unlike question 1 this relation differentiates between π radian rotations. For example, under question 1, $(1, 1) \sim (-1, -1)$, but here $1 + i$ does not relate to $-1 - i$.

Question 3. Each element has n other possible pairings and there are n such elements, so the number of total possible pairings is n^2. A relation is defined as a particular choice of a subset of these pairings, and consider the number of ways of choosing from these n^2 pairings:

$$\binom{n^2}{0} + \binom{n^2}{1} + \ldots + \binom{n^2}{n^2} = (1+1)^{n^2} = 2^{n^2}.$$

An equivalence relation is effectively a partition of n elements, so the question is equivalent to enumerating all the partitions of an n element set. We will do so recursively, let B_n be the number of ways of partitioning n elements, e.g. $B_0 = 1$, $B_1 = 1$ and $B_2 = 2$ (these are the Bell numbers). For n elements, we can construct each possible partition by removing each possible subset of elements and counting the number of unique partitions remaining. To remove a subset of size k there are $\binom{n}{k}$ choices to do so, so in formula this is:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Question 4. Let p be prime, and a, b be integers with $p|ab$. If $p|a$ then we're done, so assume it doesn't. Then $\gcd(p, a) = 1 \implies \exists x, y \in \mathbb{Z}$ s.t. $px + ay = 1 \implies pbx + aby = b$. By hypothesis

p divides ab so the left side is divisible by p, which means the right side also is, i.e. $p|b$ and we're done.

Question 5. Proof by contradiction, let n, k be positive integers with n not a perfect k-th power but $n^{1/k} \in \mathbb{Q}$. Then $\exists a, b \in \mathbb{N}$ with a, b sharing no common factors such that $n^{1/k} = \frac{a}{b} \implies nb^k = a^k$. Since a and b share no common factors, $\exists x \in \mathbb{N}$ s.t. $a^k|n \implies n = xa^k \implies xb^k = 1 \implies b = 1 \implies n = a^k$, which contradicts our assumption that n wasn't a k-th power, so $n^{1/k}$ is irrational.

Question 6. Let $s_1 \in (\beta\alpha)^{-1}(U_1)$. By definition, $\beta(\alpha(s_1)) \in U_1 \implies \alpha(s_1) \in \beta^{-1}(U_1) \implies s_1 \in \alpha^{-1}(\beta^{-1}(U_1)$. Similarly, let $s_2 \in \alpha^{-1}(\beta^{-1}(U_1) \implies \alpha(s_2) \in \beta^{-1}(U_1) \implies \beta(\alpha(s_2)) \in U_1 \implies s_2 \in (\beta\alpha)^{-1}(U_1)$. This implies that both sets contain the other, which is only possible if they are equal.