Coursework 2: Sturm-Liouville problems and Bessel functions

Hand in solutions to the questions on page 1 only; later pages contain helpful information and supplementary "warm-up" problems to practice on for your own enjoyment. Be as explicit as you can in providing your answers.

(1). Using the method of separation of variables, solve the wave equation inside the semi-circle, $r \leq 1$ and $0 \leq \theta \leq \pi$, applying the boundary conditions, $u(1, \theta, t) = u(r, 0, t) = u(r, \pi, t) = 0$, and initial condition,

$$u(r, \theta, 0) = f(r) \qquad \& \qquad u_t(r, \theta, 0) = 0,$$

expressing your result in terms of Bessel functions and their integrals.

(2). Consider the PDE,

$$u_t + u_r = (ru_r)_r + \frac{1}{r}u_{\theta\theta}, \quad 0 \leq r \leq 1, \quad u(0, \theta, t) = u(1, \theta, t) = 0,$$

subject to the condition that $u(r, \theta, t)$ is $2\pi-$periodic in θ.

(a) Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, establishing the form of the functions $p(r)$, $q(r)$ and $\sigma(r)$ in the ODE, and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of a Bessel function. Given $u(r, \theta, 0) = f(r, \theta)$, express the solution to the PDE in terms of Bessel functions and their integrals.

(b) As shown in figure 1, the numerical solution to the axisymmetric problem, with $u = u(r, t)$ and $f(r) = e^r\sqrt{r}$, eventually decays exponentially at each radial position, with a rate 3.67. Explain this observation.

(c) If $f(r, \theta) = e^r\sqrt{r}\sin\theta$, write down a reduced version of your separation of variables solution. Compute the coefficients for the first five terms of the series, then compare your results with the numerical solution to the problem provided by the MATLAB code, pde23b.m, at the times and positions plotted in the lowest two panels (see figure 2). MATLAB's inbuilt functions, BESSELJ and TRAPZ should help with the computation of the coefficients. Does the truncated analytical solution always provide a good approximation?

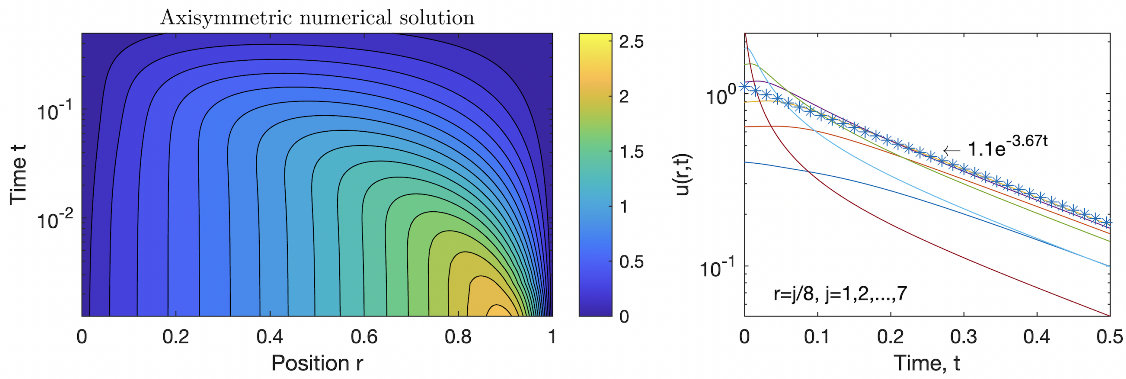

Figure 1: Axisymmetric numerical solution

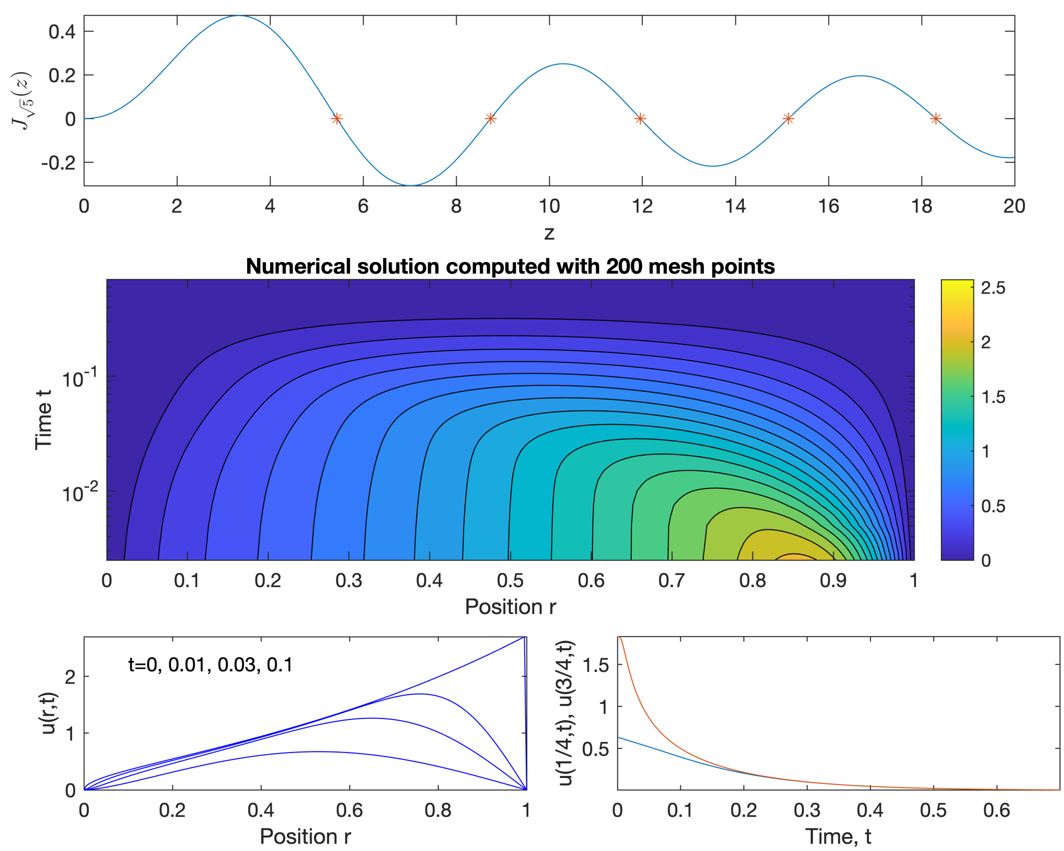

Figure 2: Output from pde23b.m.

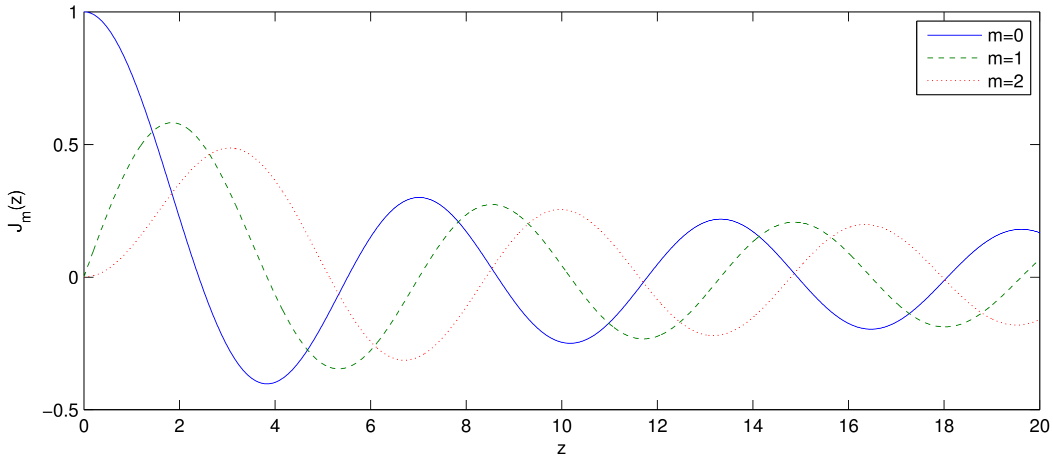

Figure 3: Bessel functions

Helpful notes: Bessel's equation is

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0,$$

and has the two solutions, $y = J_\nu(kx)$ and $Y_\nu(kx)$, of which only the former is regular at $z = 0$. For $z \to 0$, $J_\nu(z) \propto z^\nu$.

The more general ODE,

$$x^2y'' + (1 - 2\alpha)xy' + (\omega^2\beta^2x^{2\beta} + \alpha^2 - \nu^2\beta^2)y = 0,$$

has solutions $y = x^\alpha \mathcal{C}_\nu(\omega x^\beta)$ where $\mathcal{C}_\nu(z)$ is a Bessel function.

If ν is equal to an integer m, the Bessel functions satisfy the recurrence relation,

$$J_{m-1}(z) - J_{m+1}(z) = 2J_m'(z), \qquad J_0'(z) = -J_1(z).$$

For your enjoyment, above is a picture of $J_m(z)$ for $m = 0$, 1 and 2. Remember, Bessel functions are our friends.

Warm-up problems

(1). The equation of motion of a hanging, heavy chain is

$$u_{tt} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right),$$

where $u(x,t)$ is the horizontal deflection at height x and time t (the tension in the chain varies with height due to the weight underneath). The end at $x = 0$ is free, whereas the end at $x = l$ is fixed, so that u is regular for $x = 0$ and $u(l,t) = 0$. Using separation of variables reduce the PDE to two equivalent ODEs. Show that the spatial dependence of the solution is given by the Bessel function, $J_0(z)$. *Hint: the transformation $x = cz^2$ may prove helpful, for some constant c.*

Given that the zeros of $J_0(z)$ are $z = z_1$, z_2, ..., z_n, ..., write down a general solution of the PDE in terms of a sum over Bessel functions with unspecified coefficients. If $u(x,0) = 0$ and $u_t(x,0) = f(x)$, express those coefficients in terms of integrals of $J_0(z)$.

Separation of variables: $u = X(x)T(t)$, with

$$xX'' + X' + \lambda_n^2 X = 0, \qquad T = a_n \cos\lambda_n t + b_n \sin\lambda_n t.$$

Making the suggested change of variable and choosing $c = 1/(4\lambda_n^2)$, leads to

$$X_{zz} + \frac{1}{z}X_z + X = 0 \quad\longrightarrow\quad X = J_0(z) = J_0(2\lambda_n\sqrt{x}),$$

on using the regularity of $J_0(z)$ at $z = 0$. The other boundary condition implies that $\lambda_n = z_n/2\sqrt{l}$, where z_n is the n^{th} zero of $J_0(z)$. Thus,

$$u = \sum_{n=1}^{\infty}(a_n\cos\lambda_n t + b_n\sin\lambda_n t)J_0(2\lambda_n\sqrt{x}),$$

With the given initial condition, $a_n = 0$ and b_n must be computed from a suitable expansion in Bessel functions. Given that the equation for $X(x)$ is a Sturm-Liouville problem with weight $\sigma(x) = 1$, the J_0's form an orthogonal basis set, and we arrive at

$$b_n = \frac{2\sqrt{l}}{z_n}\frac{\int_0^l f(x)J_0(z_n\sqrt{x/l})dx}{\int_0^l J_0^2(z_n\sqrt{x/l})dx}.$$

(2). Using the method of separation of variables, solve Laplace's equation inside the cylinder, $0 \le r \le R$, $0 \le \theta \le 2\pi$, $0 \le z \le L$, in cylindrical polar coordinates (r,θ,z), applying the boundary condition, $u(R,\theta,z) = 0$, $u(r,\theta,0) = 0$ and

$$u(r,\theta,L) = F(r,\theta) = \frac{1}{2}F_0(r) + \sum_{m=1}^{\infty} F_m(r)\cos m\theta$$

expressing your result in terms of Bessel functions (including any constants of integration).

The PDE to solve is

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0.$$

We put $u = X(r)Y(\theta)Z(z)$ and rewrite the PDE as

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2 Y}Y_{\theta\theta} = -\frac{Z_{zz}}{Z}.$$

The right-hand side is a function of z alone, whereas the left-hand side is a function of r and θ, so both must equal a separation constant, $-k^2$. Hence

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ, the left is a function of r; we put both equal the separation constant m^2. Consequently,

$$Z_{zz} = k^2 Z, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2 Y,$$

Thus,

$$Z = \sinh kz, \qquad Y = \cos m\theta \text{ or } \sin m\theta \text{ with } m = 1, 2, ..., \text{ or } \textit{constant} \text{ with } m = 0$$

(since $Z(0) = 0$ and $Y(\theta)$ must be $2\pi-$periodic). Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(R) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue k^2, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each m, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, ...$ Likewise, there are similar solutions for $m = 0$. The general solution of the PDE is therefore

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}X_{0,n}(r)\sinh(k_{0,n}z) + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)X_{m,n}(r)\sinh(k_{m,n}z)\right].$$

Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = \frac{z_{m,n}}{R}$$

where $z_{m,n}$ is the n^{th} zero of $J_m(z)$, and the set of functions can be extended to include $m = 0$. Given also the boundary condition at $z = L$ (a cosine series in θ), we have $b_{m,n} = 0$. Finally,

$$F_m(r) = \sum_{n=1}^{\infty} a_{m,n}J_m(k_{m,n}r)\sinh(k_{m,n}L),$$

with $m = 0$, 1, 2, ..., and so

$$a_{m,n} = \frac{\int_0^R F_m(r)J_m(k_{m,n}r)rdr}{\sinh(k_{m,n}L)\int_0^R [J_m(k_{m,n}r)]^2 rdr}.$$

(3). Using the method of separation of variables, solve the heat equation inside the unit disk, $r \le 1$, applying the boundary condition, $u(1, \theta, t) = 0$, and initial condition,

$$u(r, \theta, 0) = \sum_{m=1}^{\infty} f_m(r)\sin m\theta.$$

expressing your result in terms of Bessel functions and their integrals.

The PDE to solve is (if one includes the diffusivity for completeness but not necessity)

$$\frac{1}{\kappa}u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$

We put $u = X(r)Y(\theta)T(t)$ and rewrite the PDE as

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2 Y}Y_{\theta\theta} = \frac{T_t}{\kappa T}.$$

The right-hand side is a function of t alone, whereas the left-hand side is a function of r and θ, so both equal a separation constant, $-k^2$. Hence

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ, the left is a function of r; we put both equal the separation constant m^2. Consequently,

$$T_t = -\kappa k^2 T, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2 Y,$$

Thus,

$$T = Ce^{-\kappa k^2 t}, \qquad Y = A\cos m\theta \text{ or } B\sin m\theta,\ \ m = 0, 1, 2, ...$$

since $Y(\theta)$ must be $2\pi-$periodic. Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(1) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue k^2, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each m, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, ...$ Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = z_{m,n}$$

where $z_{m,n}$ is the n^{th} zero of $J_m(z)$. The general solution of the PDE is therefore

$$u(r,\theta,z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}J_0(z_{0,n}r)e^{-\kappa z_{0,n}^2 t} + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)J_m(r)e^{-\kappa z_{m,n}^2 t}\right].$$

Finally, we observe that $u(r,\theta,0)$ is a sine series in θ, so $a_{0,n} = a_{m,n} = 0$, and demanding $u(r,\theta,0) = f(r,\theta) = \sum_m f_m(r)\sin m\theta$, implies

$$f_m(r) = \sum_{n=1}^{\infty} b_{m,n}J_m(z_{m,n}r)$$

and so (from the SL expansion formulae)

$$b_{m,n} = \frac{\int_0^1 f_m(r)J_m(z_{m,n}r)rdr}{\int_0^1 [J_m(z_{m,n}r)]^2 rdr}.$$

Two More...

(1). Consider the axisymmetric heat equation,

$$u_t = \frac{1}{r}(ru_r)_r$$

in $r \leq R$, subject to $u(R,t) = 0$ and $u(r,t)$ regular at the origin. Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, $u(r,t) = X(r)T(t)$, establishing the form of the functions $p(r)$, $q(r)$ and $\sigma(r)$ in the ODE and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of $J_0(z)$. Given $u(r,0) = f(r)$, express the solution to the PDE in terms of Bessel functions and their integrals.

(2). Using the method of separation of variables, solve the wave equation inside the unit disk, $r \leq 1$, applying the boundary condition, $u(1,\theta,t) = 0$, and initial conditions,

$$u(r,\theta,0) = \frac{1}{2}f_0(r) + \sum_{m=1}^{\infty} f_m(r)\cos m\theta \qquad \text{and} \qquad u_t(r,\theta,0) = 0,$$

expressing your result in terms of Bessel functions and their integrals.

Solutions:

(1). Separate variables: $u = X(r)T(t)$, giving

$$\frac{1}{rX}(rX_r)_r = \frac{T_t}{T} = -k^2,$$

where $-k^2$ is the separation constant. Hence

$$(rX_r)_r + k^2 rX = 0 \qquad \text{and} \qquad T = e^{-k^2 t}.$$

The first equation is the ODE of a Sturm-Liouville (SL) problem with $p(r) = \sigma(r) = r$, $q(r) = 0$ and eigenvalue k^2. Comparison with Bessel's equation and imposition of $X(R) = 0$ indicates that

$$X(r) = J_0(kr) \qquad \text{and} \qquad J_0(kR) = 0.$$

Denoting z_n as the n^{th} zero of $J_0(z)$, $n = 1, 2, ...$, we find the SL eigenvalues, $k_n = z_n/R$, and eigenfunctions, $X_n(r) = J_0(k_n r)$. Hence,

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} J_0(k_n r).$$

Finally, we apply the initial condition:

$$f(r) = \sum_{n=1}^{\infty} c_n J_0(k_n r) \qquad \longrightarrow \qquad c_n = \frac{\int_0^R f(r)J_0(k_n r)rdr}{\int_0^R [J_0(k_n r)]^2 rdr}.$$

(2). The PDE to solve is

$$u_{tt} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$

We put $u = X(r)Y(\theta)T(t)$ and rewrite the PDE as

$$\frac{1}{rX}(rX_r)_r + \frac{1}{r^2 Y}Y_{\theta\theta} = \frac{T_{tt}}{T}.$$

The right-hand side is a function of t alone, whereas the left-hand side is a function of r and θ, so both equal a separation constant, $-k^2$. Hence

$$\frac{r}{X}(rX_r)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.$$

The right-hand side is now a function of θ, the left is a function of r; we put both equal the separation constant m^2. Consequently,

$$T_{tt} = -k^2 T, \qquad \text{and} \qquad Y_{\theta\theta} = -m^2 Y,$$

Thus,

$$T = \cos kt, \qquad Y = \cos m\theta \text{ or } \sin m\theta, \; m = 0, 1, 2, ...$$

since $u_t(r, \theta, 0) = 0$ (or $T_t(0) = 0$) and $Y(\theta)$ must be $2\pi-$periodic. Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2 X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(1) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue k^2, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each m, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, ...$ The general solution of the PDE is therefore

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}X_{0,n}(r)\cos(k_{0,n}t) + \sum_{m=1}^{\infty}(a_{m,n}\cos m\theta + b_{m,n}\sin m\theta)X_{m,n}(r)\cos(k_{m,n}t)\right].$$

Comparing the ODE of the Sturm-Liouville problem with Bessel's equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \qquad \text{and} \qquad k_{m,n} = z_{m,n}$$

where $z_{m,n}$ is the n^{th} zero of $J_m(z)$.

Finally, we observe that $u(r, \theta, 0)$ is a cosine series in θ, so $b_{m,n} = 0$, and

$$u(r, \theta, z) = \sum_{n=1}^{\infty}\left[\frac{1}{2}a_{0,n}J_0(k_{0,n}r)\cos(k_{0,n}t) + \sum_{m=1}^{\infty}a_{m,n}\cos m\theta\; J_m(k_{m,n}r)\cos(k_{m,n}t)\right].$$

Finally, demanding $u(r, \theta, 0) = f(r, \theta) = \frac{1}{2}f_0(r) + \sum_m f_m(r)\cos m\theta$, implies

$$f_m(r) = \sum_{n=1}^{\infty} a_{m,n}J_m(k_{m,n}r)$$

(for $m = 0, 1, 2...$), and so

$$a_{m,n} = \frac{\int_0^R f_m(r)J_m(k_{m,n}r)rdr}{\int_0^R [J_m(k_{m,n}r)]^2 rdr}.$$

A previous year's assignment:

(1). The temperature $T(r,\theta,t)$ in a heated circular swimming pool satisfies

$$T_t = \frac{1}{r}(rT_r)_r + \frac{1}{r^2}T_{\theta\theta} + \alpha, \qquad T(1,\theta,t) = 0,$$

where the heating rate α is a prescribed constant. First, find the temperature distribution $T(r,\theta,t) = T_{ss}(r)$ if the pool were in steady state. Next, by putting $T(r,\theta,t) = T_{ss}(r) + u(r,\theta,t)$, solve the PDE for $u(r,\theta,t)$ using separation of variables, imposing the initial condition,

$$T(r,\theta,0) = f(r)\sin 2\theta,$$

and expressing your result in terms of Bessel functions and their integrals.

(2). Consider the PDE,

$$u_t = (ru_r)_r + \frac{1}{r}u_{\theta\theta}$$

in $r \le 1$, subject to $u(1,\theta,t) = 0$ and the conditions that $u(r,\theta,t)$ is $2\pi-$periodic in θ and regular at $r = 0$.

(a) Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, establishing the form of the functions $p(r)$, $q(r)$ and $\sigma(r)$ in the ODE, and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of a Bessel function. Given $u(r,\theta,0) = f(r,\theta)$, express the solution to the PDE in terms of Bessel functions and their integrals.

(b) As shown in figure 4, the numerical solution to the axisymmetric problem, with $u = u(r,t)$ and $f(r) = 16r^2(1-r)^2$, eventually decays exponentially at each radial position, with a rate 1.45. Explain this observation.

(c) If $f(r,\theta) = 16r^2(1-r)^2\sin\theta$, write down a reduced version of your separation of variables solution. Compute the coefficients for the first five terms of the series, then compare your results with the numerical solution to the problem provided by the MATLAB code, pde21b.m, at the times and positions plotted in the lowest two panels (see figure 5). MATLAB's inbuilt functions, BESSELJ and TRAPZ should help with the computation of the coefficients.

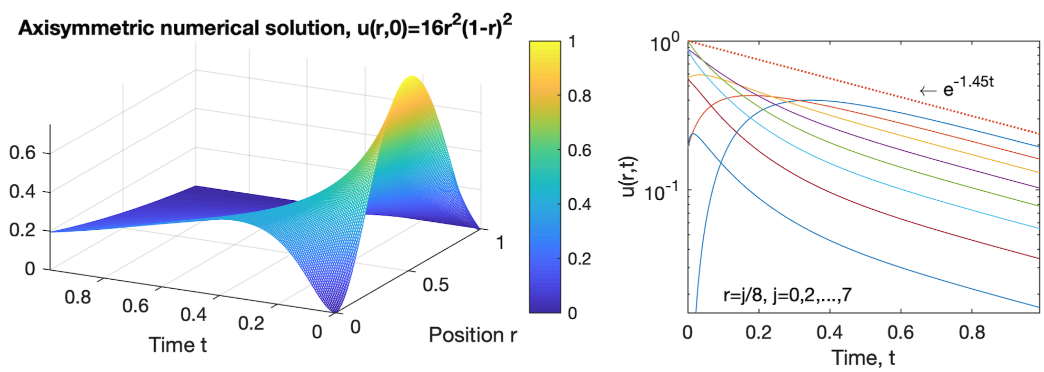

Figure 4: Axisymmetric numerical solution

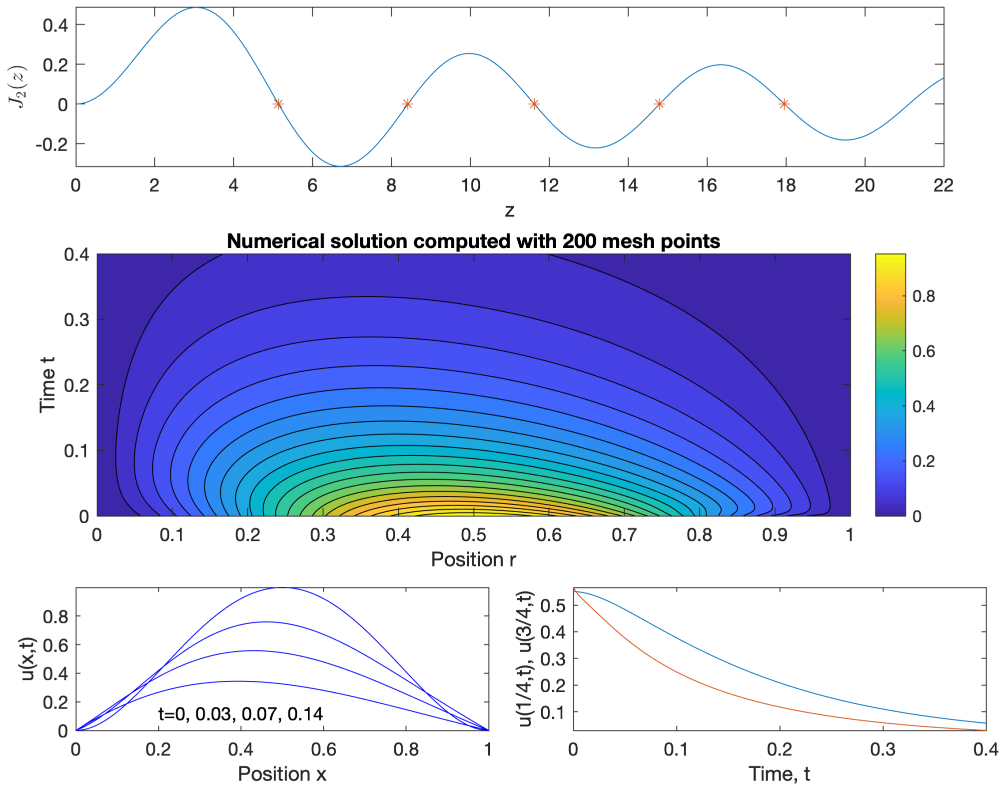

Figure 5: Output from pde21b.m.

Solutions

1. We have

$$(rT'_{ss})' + \alpha r = 0 \quad \rightarrow \quad T'_{ss} + \frac{1}{2}\alpha r = 0 \quad \rightarrow \quad T_{ss} = \frac{1}{4}\alpha(1 - r^2)$$

(avoiding any singularities at $r = 0$ and since $T_{ss}(1) = 0$). If $T(r,t) = T_{ss}(r) + u(r,t)$, then $u(r,t)$ satisfies

$$u_t = \frac{1}{r}(ru_r)_r, \quad u(1,\theta,t) = 0, \quad u(r,\theta,0) = f(r)\sin 2\theta - T_{ss}(r).$$

We separate variables, $u = R(r)\Theta(\theta)T(t)$, giving the ODEs

$$T' + \lambda T = 0, \qquad \Theta'' + m^2\Theta = 0, \qquad (rR')' + \lambda r R - m^2 R = 0,$$

for two separation constants λ and m. We choose $m = 0$ and $\Theta = \frac{1}{2}A_0$, or $m = 1, 2, \ldots$ and $\Theta = B_m \sin m\theta$ or $A_m \cos m\theta$ to guarantee 2π-periodic solutions in θ, in the usual manner of a Fourier series. In fact, the initial condition indicates that we only need the $(m, \Theta) = (0, \frac{1}{2}A_0)$ and $(m, \Theta) = (2, B_2 \sin 2\theta)$ solution pairs. The ODE for $R(r)$ is Bessel's equation, with either $J_0(kr)$ or $J_2(kr)$ as solutions, given that $m = 0$ or 2, with $\lambda = k^2$. But $u(1, \theta, t) = 0$ implies that $R(1) = 0$ and so k must be a zero of the corresponding Bessel function. *i.e.* $\lambda = k_{0,n}^2$ for $m = 0$, or $\lambda = k_{2,n}^2$ for $m = 2$, with $J_m(k_{m,n}) = 0$ and $n = 1, 2, \ldots$ Altogether, we find the general solution,

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \left[a_n J_0(k_{0,n} r) e^{-k_{0,n}^2 t} + b_n J_2(k_{2,n} r) e^{-k_{2,n}^2 t} \sin 2\theta \right],$$

for a suitable set of constants a_n and b_n. Last, in view of the initial condition and the Sturm-Liouville expansion theorem, we see that

$$a_n = -\frac{1}{4}\alpha \int_0^1 (1-r^2) J_0(k_{0,n} r)\ r dr \left[\int_0^1 [J_0(k_{0,n} r)]^2\ r dr\right]^{-1}$$

and

$$b_n = \int_0^1 f(r) J_2(k_{2,n} r)\ r dr \left[\int_0^1 [J_2(k_{2,n} r)]^2\ r dr\right]^{-1}.$$

2(a) Separating variables, we arrive at the ODEs

$$T' + \lambda T = 0, \qquad \Theta'' + m^2\Theta = 0, \qquad (rR')' + \lambda R - \frac{m^2}{r} R = 0,$$

for two separation constants λ and m. We choose $m = 0$ and $\Theta = \mathit{constant}$, or $m = 1, 2, \ldots$ and $\Theta \propto \sin m\theta$ or $\cos m\theta$ to guarantee 2π-periodic solutions in θ. The ODE for the $r-$dependence is a Sturm-Liouville problem with $p \equiv r$, $\sigma \equiv 1$ and $q = -m^2/r$, and type (i) and (ii) boundary conditions ($R(1) = 0$ and we demand regularity at $r = 0$ with $p(0) = 0$). It is also a form of the general ODE that has Bessel functions as solutions with

$$\alpha = 0, \quad \frac{1}{4}\omega^2 = \lambda, \quad \beta = \frac{1}{2}, \quad \frac{1}{4}\nu^2 = m^2.$$

The solutions are therefore $J_{2m}(2\sqrt{\lambda r})$ and $Y_{2m}(2\sqrt{\lambda r})$. However, the latter cannot satisfy the regularity condition at $r = 0$ The other boundary condition therefore implies that $J_{2m}(2\sqrt{\lambda}) = 0$, which demands that $2\sqrt{\lambda}$ is a zero of $J_{2m}(z)$. Denoting the n^{th} such zero by z_{mn}, we have $R \propto J_{2m}(z_{mn}\sqrt{r})$.

A general solution of the PDE is therefore

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \left\{ \frac{1}{2} A_{0n} J_0(z_{0n}\sqrt{r}) e^{-z_{0n}^2 t/4} + \sum_{m=1}^{\infty} (A_{mn}\cos m\theta + B_{mn}\sin m\theta) J_{2m}(z_{mn}\sqrt{r}) e^{-z_{mn}^2 t/4} \right\}.$$

At $t = 0$, and exploiting a Fourier series for the initial condition, we need

$$u(r,\theta,0) = f(r,\theta) = \frac{1}{2}a_0(r) + \sum_{m=1}^{\infty} [a_m(r)\cos m\theta + b_m(r)\sin m\theta].$$

Given the Sturm-Liouville expansion formulae, we may enforce this by setting

$$A_{0n} = \frac{\int_0^1 a_0(r) J_0(z_{0n}\sqrt{r}) dr}{\int_0^1 [J_0(z_{0n}\sqrt{r})]^2 dr}, \qquad [A_{mn}, B_{mn}] = \frac{\int_0^1 [a_m(r), b_m(r)] J_{2m}(z_{mn}\sqrt{r}) dr}{\int_0^1 [J_{2m}(z_{mn}\sqrt{r})]^2 dr}.$$

(b) When the initial condition has no $\theta-$dependence and $u(r,0) = f(r)$, $A_{mn} = B_{mn} = 0$. The long-time behaviour of the solution is then controlled by the smallest value of $z_{0n}^2/4$ (the exponent of the slowest decaying term in the remaining sum). This is given by the first zero of $J_0(z)$, which is $z \approx 2.40$. The long-time decay rate is therefore $(2.40)^2/4 \approx 1.45$, as observed in the numerical solution.

(c) If $f(r,\theta) = 16r^2(1-r)^2\sin\theta$, the entire solution has the factor $\sin\theta$ with only the coefficients B_{1n} non-zero. The revised figure shows a comparison of the numerical solution with the analytical one, truncated to five terms. The updated code pde20bx.m performs the task. Note that, since $u(r,\theta,t) \propto \sin\theta$, one can factor out the $\theta-$dependence and plot the solution as a function of only r and t, as done in the figure (or, equivalently, one could take the nominal value of θ of $\pi/2$, for illustration).

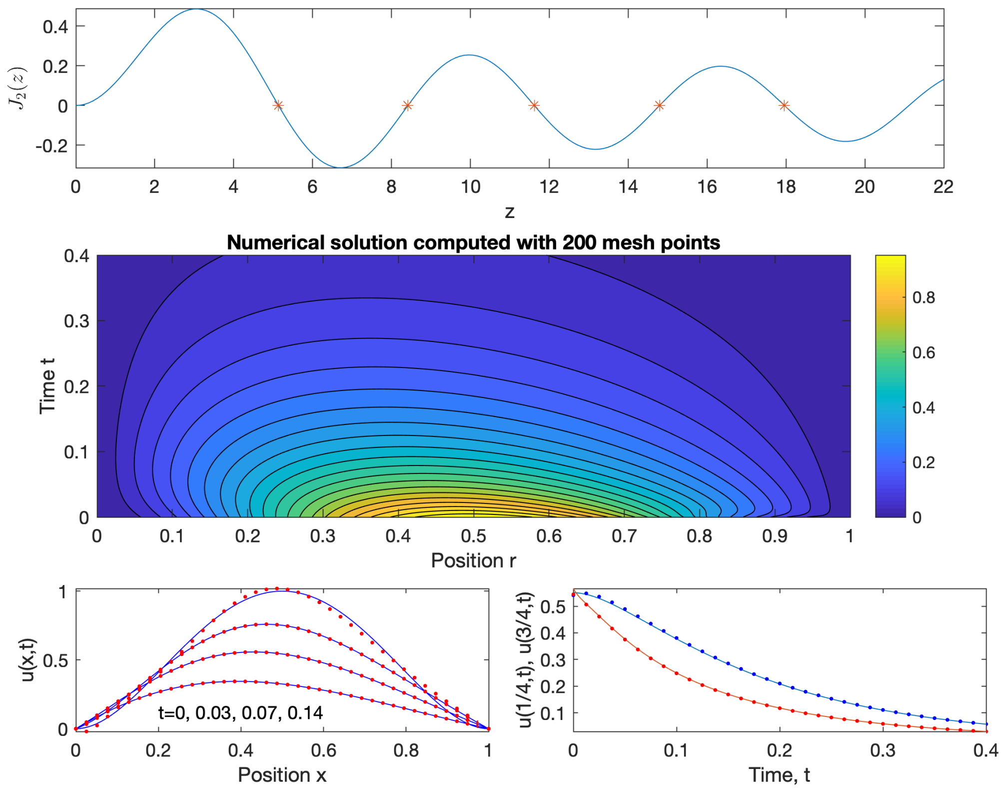

Figure 6: Output from pde21bx.m; the comparison with the truncated analytical solution is included in the lower panels.