MATH 305 Homework 7

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1. Let f be analytic inside and on the simple closed loop C and let z_0 lie outside C. What is the value of $\int_C \frac{f(z)}{z-z_0}dz$?

By Cauchy's integral theorem, the integral is 0.

2. Evaluate

(a) $\int_{|z|=3} \frac{e^{iz}}{(z^2+1)^2}dz$.

There are singularities at $\pm i$:

$$\int_{|z|=3} \frac{e^{iz}}{(z-i)^2(z+i)^2}dz = 2\pi i \frac{d}{dx}\left(\frac{e^{iz}}{(z+i)^2}\right)\bigg|_{z=i} + 2\pi i \frac{d}{dx}\left(\frac{e^{iz}}{(z-i)^2}\right)\bigg|_{z=-i}.$$

$$= 2\pi i \frac{ie^{iz}(z+i)^2 - 2e^{iz}(z+i)}{(z+i)^4}\bigg|_{z=i} + 2\pi i \frac{ie^{iz}(z-i)^2 - 2e^{iz}(z-i)}{(z-i)^4}\bigg|_{z=-i}.$$

$$= 2\pi\left(\frac{4e^{-1}+2e^{-1}}{16} + \frac{4e-2e}{16}\right).$$

(b) $\int_{|z|=2} \frac{\cos z}{z^2(z-1)}dz$

Singularities at $0, 1$:

$$\int_{|z|=2} \frac{\cos z}{z^2(z-1)}dz = 2\pi i \frac{\cos z}{z^2}\bigg|_{z=1} + 2\pi i \frac{-\sin z(z-1) - \cos z}{(z-1)^2}\bigg|_{z=0}.$$

$$= 2\pi i\left(\cos 1 - 1\right).$$

3. Evaluate

(a) $\int_{|z|=2} \frac{z^2+1}{(z-1)^3}dz$.

$$\int_{|z|=2} \frac{z^2+1}{(z-1)^3}dz = 2\pi i \frac{d^2}{dx^2}\left(z^2+1)\right)\bigg|_{z=1} = 2\pi i.$$

(b) $\int_{|z|=2} \frac{\sin z}{z^2(z-3)}dz$

$$\int_{|z|=2} \frac{\sin z}{z^2(z-3)}dz = 2\pi i \frac{\cos z(z-3) - \sin z}{(z-3)^2}\bigg|_{z=0} = -\frac{2}{3}\pi i.$$

4. Evaluate

(a) $\int_{|z|=5} \frac{z^2+1}{z^4+z+1}dz$.

Note that for $|z| \geq 5$, we have $\left|z^4+z+1\right| \geq |z^4| - |z| - 1 \geq 5^4$

(b) $\int_{|z|=2} \frac{z}{(z-3)(z^4+z+1)}dz$

Hint: show that $\int_{|z|=R}(...)dz \to 0$ as $R \to +\infty$.

5. Evaluate
(a) $\int_0^{2\pi} \frac{1}{2+\sin\varphi}d\varphi$. (b) $\int_0^{\pi} \frac{1}{2-\cos\varphi}d\varphi$. (c) $\int_0^{2\pi} \sin^{10}\varphi d\varphi$.

6. Suppose that $f(z)$ is entire and $|f(z)| \le 2(1+|z|)^3$. Show that $f(z)$ is a polynomial of degree at most three.

7. Let f be entire and suppose that $Re(f(z)) \le 2Im(f(z))$. Show that $f(z)$ must be a constant function.
Hint: consider $g = e^{\alpha f}$ for some complex number α.

8. Let f be analytic in $D = \{|z| \le 1\}$. Assume that $|f(z)| \le M$ for $|z| = 1$. Show that
(a) $|f''(0)| \le 2M$, (b) $|f^{('')}(\frac{1}{2})| \le 16M$

9. Find the maximum value of $|z^2 + 3z - 1|$ in the disk $|z| \le 1$.

10. Show that $\max_{|z|\le 1} |4z^{100} - 5z| = 9$.