MATH 406, HWK 6, Due 24 th November

1. Consider determining the eigenvalues for Laplace's equation:

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u \tag{1}$$

on the region Ω subject to Dirichlet boundary condition:

$$u\rfloor_{\partial\Omega} = 0 \tag{2}$$

Starting with the weighted residual statement of this boundary value problem determine the appropriate weak statement of the problem. Consider a triangular tesselation of the domain Ω into triangles T^k such that $\Omega = \cup T^k$. Now use the piecewise linear basis functions $\psi^k_\alpha(x, y)$ defined on the triangles T^k, which we developed in class, to arrive at a Galerkin approximation to this eigenvalue problem.

Now adapt the code supplied in `FEM_Pack.zip` by altering the function `fempoiD.m` so that it calculates the eigenvalues of the Dirichlet problem on a unit circle. Determine the first 6 nonzero eigenvalues for the unit circle $\Omega = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq 2\pi\}$ using `n=32` elements and the meshing parameter `hmax=2`π`/n`. Use `tplot.m` to plot the eigenfunctions associated with $\lambda_{0,1}, \lambda_{1,1}, \lambda_{2,1}$ and $\lambda_{0,2}$. Provide your code for the FEM formulation of the eigenvalue problem.

Note: Recall that the eigenvalues and eigenfunctions for the Dirichlet problem for a circle of radius a are given by:

$$\begin{aligned}
\lambda_{m,n} &= (j_{m,n}/a)^2, \; m = 0, 1, 2, \ldots, n = 1, 2, \ldots \\
\phi_{m,n}(r, \theta) &= J_m(j_{m,n}/a)\left\{\begin{array}{c} 1 \text{ for } m = 0 \\ \cos m\theta \\ \sin m\theta \end{array}\right.
\end{aligned}$$

where $j_{m,n}$ is the n-th positive zero of the $m-$th Bessel J_m. Thus

$$\lambda_{m,n} = \left[\begin{array}{c}
\lambda_{0,1} = (j_{0,1}/1)^2 = (2.4048255576957728)^2 = 5.78318596 \\
\lambda_{1,1} = (j_{1,1}/1)^2 = (3.8317059702075123)^2 = 14.6819706 \\
\lambda_{2,1} = (j_{2,1}/1)^2 = (5.1356223018406826)^2 = 26.3746164 \\
\lambda_{0,2} = (j_{0,2}/1)^2 = (5.5200781102863106)^2 = 30.4712623
\end{array}\right]$$

Using `n=32` nodes on the boundary in your FEM code for the Dirichlet problem complete the first column in following table of Eigenvalues. Now use a mesh with `n=64` nodes on the boundary and complete the second column. Now, assuming that $\lambda_{FEM} = \lambda_{exact} + C \cdot N^{-2}$ use Richardson Extrapolation to obtain improved estimates of the eigenvalues.

$\lambda_{m,n} =$

Exact	FEM (32)	FEM (64)	Richardson Extrap.
$\lambda_{0,1} = 5.78318596$			
$\lambda_{1,1} = 14.6819706$			
$\lambda_{2,1} = 26.3746164$			
$\lambda_{0,2} = 30.4712623$			