UBC Mathematics 320(101)—Assignment 5
Due by PDF upload to Canvas at 18:00, Saturday 14 Oct 2023

References: Loewen, lecture notes on Sequences and Series (2023-10-07 or newer—see Canvas); Rudin, pages 11b–12a, 47–58; Thomson-Bruckner-Bruckner, Chapter 2.

1. Consider a real-valued sequence (x_n) and a real number $\widehat{x}$. Prove that the following are equivalent:

(a) $x_n \to \widehat{x}$,

(b) $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall n \geq 23N,\ |x_n - \widehat{x}| < 20\varepsilon$.

2. Extend our collection of equivalent formulations of the completeness property for $\mathbb{R}$ by proving that that the following are equivalent (TFAE). Proceed directly, without relying on the completeness property in one of its other forms. (So, for example, do not assume existence of suprema and infima.)

(i) For any sequence of nonempty closed real intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, …, such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ (such intervals are called "nested"), one has

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

(ii) Every bounded monotonic sequence in $\mathbb{R}$ converges. (Recall Rudin's Definition 3.13.)

(*Note*: The interval notation $[a, b] = \{t \in \mathbb{R}\ :\ a \leq t \leq b\}$ is reserved for the case where both a and b are real numbers. To encode $\{t \in \mathbb{R}\ :\ t \geq 0\}$, for example, we would write $[0, +\infty)$, not $[0, +\infty]$.)

Note: *Questions 3–6 contribute to the major project of constructing $\mathbb{R}$ from $\mathbb{Q}$. Therefore they must be completed entirely in the context of the rational numbers. Present solutions that make no reference at all to the completeness property of $\mathbb{R}$, in any of its equivalent forms.*

3. Introduce the following notation:

$CS(\mathbb{Q})$: the set of all Cauchy sequences with rational elements.

x, y, z: typical symbols for elements of $CS(\mathbb{Q})$. Thus, e.g., $x = (x_1, x_2, \ldots)$.

$R[x]$: the subset of $CS(\mathbb{Q})$ associated with a given $x \in CS(\mathbb{Q})$ as follows:

$$R[x] = \left\{ x' \in CS(\mathbb{Q})\ :\ \lim_{n \to \infty} |x'_n - x_n| = 0 \right\}.$$

Φ: the function that takes each rational number q into the subset of $CS(\mathbb{Q})$ containing the corresponding constant sequence, i.e.,

$$\Phi(q) = R\left[(q, q, \ldots)\right] \qquad \forall q \in \mathbb{Q}.$$

(a) Prove: $R[x] \neq \emptyset$ for every $x \in CS(\mathbb{Q})$.

(b) Prove: For any $x, y \in CS(\mathbb{Q})$, $\qquad R[x] = R[y] \quad \iff \quad R[x] \cap R[y] \neq \emptyset$.

4. Continue with the notation from Question 3. We would like to define a relation denoted "$<$" on $\mathbb{Q}^*$ as follows:

$$R[x] < R[y] \iff \exists r > 0\,(r \in \mathbb{Q}),\ \exists N \in \mathbb{N}\ :\ \forall n > N,\ y_n - x_n > r.$$

This relation looks like one that is familiar for rational numbers, but here it compares two *sets*. Each of the properties we take for granted when manipulating inequalities relating numbers requires careful thinking in this new context. Prove the following.

(a) Whenever $R[x'] = R[x]$ and $R[y'] = R[y]$ for some given $x, x', y, y' \in CS(\mathbb{Q})$, the definition proposed above guarantees that

$$R[x'] < R[y'] \iff R[x] < R[y].$$

(That is, the proposed definition is unambiguous. Or, more conventionally, "the relation $<$ is well-defined".)

(b) If $x, y, z \in CS(\mathbb{Q})$ obey $R[x] < R[y]$ and $R[y] < R[z]$, then $R[x] < R[z]$.

(c) The inequality $R[x] < R[x]$ never happens, for any $x \in CS(\mathbb{Q})$.

(d) For any $p, q \in \mathbb{Q}$, we have $\quad p < q \iff \Phi(p) < \Phi(q)$.

5. Continue with the notation from Questions 3 and 4. Prove the following:

(a) For any $x \in CS(\mathbb{Q})$, exactly one of the following holds:

$$R[x] < \Phi(0), \qquad R[x] = \Phi(0), \qquad \Phi(0) < R[x].$$

(b) For each x in $CS(\mathbb{Q})$, there exist $q, r \in \mathbb{Q}$ such that $\Phi(q) < R[x] < \Phi(r)$.

(c) For any $x, y \in CS(\mathbb{Q})$ with $R[x] < R[y]$, there exists $q \in \mathbb{Q}$ such that $R[x] < \Phi(q) < R[y]$.

6. Continue with the notation from Questions 3 and 4. Prove the following:

If $x \in CS(\mathbb{Q})$ has $R[x] \neq \Phi(0)$, then there exists $z \in CS(\mathbb{Q})$ for which $R[x \cdot z] = \Phi(1)$.

Here $x \cdot z$ denotes the sequence whose nth term is $x_n z_n$.
(Recall from Assignment 4, Question 6, that $x \cdot z \in CS(\mathbb{Q})$ whenever $x, z \in CS(\mathbb{Q})$.)

7. [Rudin problem 3.5] For any two real sequences (a_n) and (b_n), prove that the inequality

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

holds whenever the right side is not of the form $(+\infty) + (-\infty)$. Give a specific example to show that strict inequality may hold.

8. Let X denote the collection of all functions $f\colon [0,1] \to \mathbb{R}$ for which the set of real numbers $f([0,1]) = \{f(x) : x \in [0,1]\}$ is bounded. For each $f \in X$, define

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}.$$

Prove that for all real c and all $f, g, h \in X$,

(a) $\|cf\| = |c|\,\|f\|$,

(b) $\|f + g\| \leq \|f\| + \|g\|$,

(c) $\|f - h\| - \|g - h\| \leq \|f - g\|$.

Give an example where (b) holds as a strict inequality ("$<$").