

PHYS 403 Homework 4

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Question I. As the hint suggests, first compute $\langle N_{\vec{p}} \rangle$. Since the polarization only changes things by a constant factor 2 as we saw in class, this factor is multiplied at the beginning and the polarization is ignored.

$$\langle N_{\vec{p}} \rangle = 2 \frac{\sum_{N=0}^{\infty} N e^{-\beta c p N}}{\sum_{N=0}^{\infty} e^{-\beta c p N}} = -2 \frac{\partial}{\partial(\beta c p)} \log \left(\frac{1}{1 - e^{-\beta c p}} \right) = \frac{2}{e^{\beta c p} - 1}.$$

We can convert the sum given in the hint to an integral similarly to how we did in class, by adding a factor of $(2\pi\hbar)^3$ and integrating over continuous momentum:

$$\langle N \rangle = 2V \int \frac{dp^3}{(2\pi\hbar)^3} \frac{1}{e^{\beta c p} - 1} = T^3 \frac{2V k_B^3}{(2\pi\hbar)^3} \int \frac{dx^3}{e^x - 1}.$$

The value of the integral doesn't matter, it's clearly bounded so it's just a constant factor. Thus the T^3 dependence in $\langle N \rangle$ is clear.

Question III1. From the geometry of the planets, the sun absorbs $\frac{\pi(6.371 \cdot 10^6)^2}{4\pi(1.5 \cdot 10^{11})^2} = 4.51 \cdot 10^{-10}$ of the sun's radiation.

$$\begin{aligned} \frac{dE_{earth}}{dt} &= P_{in} - P_{out} = \sigma \cdot (4\pi \cdot (7 \cdot 10^8)^2) T_s^4 \cdot 4.51 \cdot 10^{-10} - \sigma \cdot (4\pi \cdot (6.371 \cdot 10^6)^2) T_e^4. \\ &= 1.5696 \cdot 10^2 T_s^4 - 2.892 \cdot 10^7 T_e^4. \end{aligned}$$

Question II2. The steady state corresponds to when the left side the above equation is 0, so plugging in $T_s = 6000\text{K}$ and solving for T_e gives $T_e = 289.6\text{K}$.

Question III1. Recall from in class, the expression for energy in three dimensions is (similarly to question 1):

$$\langle E \rangle = 2V \int \frac{dp^3}{(2\pi\hbar)^3} \frac{cp}{e^{\beta c p} - 1} = T^4 A \int \frac{dx^3}{e^x - 1}.$$

Here A is just an arbitrary constant since we just care about functional dependence. Note that 3 of the powers of T came from the change of variables from $x = \beta c p$, and the last one is from the additional factor of β needed to make the numerator cp into $\beta c p$. Thus for d dimensions this expression is identical except 3 replaced with d , so $\langle E \rangle \propto T^{d+1}$.

Question III2. The key to converting the sums to integrals like we did in question 1 and in the derivation of $\langle E \rangle$ was to send $V \rightarrow \infty$ so that the spacing of the momenta were small. However for very small dimensions this isn't valid, so for such dimensions there is no contribution to the integral.