Math 320 Homework 11

Xander Naumenko

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Question 1. Let (x_n) be a Cauchy sequence in Y. For each n, if $x_n \in S$ then define $y_n = x_n$. Otherwise, let $z \in S$ with $d(z, x_n) < \frac{1}{n}$ and let $y_n = z$, this is guaranteed to be possible since S is dense in Y. Let $\epsilon > 0$, and since (x_n) is Cauchy choose N_1 with $n, m > N_1 \implies d(x_n, x_m) < \frac{\epsilon}{2}$. By Archimedes choose N_2 s.t. $\frac{1}{N_2} < \frac{\epsilon}{4}$. Then we have that for $n > \max(N_1, N_2)$,

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.$$

Thus y_n is a Cauchy sequence with values in S, so it converges to some $L \in Y$. I claim $x_n \to L$ also. Let $\epsilon > 0$, since $y_n \to L$ there is some M_1 s.t. $n > M_1 \implies d(x_n, L) < \frac{\epsilon}{2}$. Also by Archimedes choose M_2 s.t. $\frac{1}{M_2} < \frac{\epsilon}{2}$. Then we have

$$d(x_n, L) \leq d(x_n, y_n) + d(y_n, L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 2a. Proof by contradiction, suppose that there exists some (X, d), K and p with $d_K(p) \neq d_{\overline{K}}(p)$, since $K \subseteq \overline{K}$ we then have $d_K(p) > d_{\overline{K}}(p)$. For this to be true there must exist $k' \in \overline{K}$ with $d_K(p) < d(k', p) \leq d_{\overline{K}}(p)$. Since k' can't be in K it must be in K', so there exists some $k \in K$ with $d(k, k') < d_K(p) - d(k', p)$. But then we have

$$d(k, p) \leq d(k, k') + d(k', p) < d_K(p) - d(k', p) + d(k', p) = d_K(p).$$

This is impossible though because $k \in K$ and $d_K(p)$ was chosen to be the minimum possible using the infimium, so in fact $d_K(p) = d_{\overline{K}}(p)$ for all $p \in X$.

Question 2b. Without loss of generality assumethat $d_K(p) > d_K(q)$. Let $\epsilon > 0$, by the construction of $d_K(q)$ there exists $k \in K$ with $d(k, q) < d_K(q) + \epsilon$. Also by the minimality of $d_K(p)$, we get

$$|d_K(p) - d_K(q)| = d_K(p) - d_K(q) \leq |d(p, k) - d(k, q)| + 2\epsilon \leq d(p, q) + 2\epsilon.$$

Since ϵ is arbitrarily small, sending it to 0 gives the desired result.

Question 2c. Let F_n be the set of all $k \in K$ with $d(p, k) < d_K(p) + \frac{1}{n}$. By the definition of $d_K(p)$ each of these sets is nonempty, and clearly $F_1 \supseteq F_2 \supseteq \ldots$. Thus they have the finite intersection property, and since K is compact $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, let $\hat{x}$ be an element of this infinite intersection. Then since $\hat{x} \in F_n \forall n$, we have $d(\hat{x}, p) < d_K(p) + \frac{1}{n} \forall n \implies d(\hat{x}, p) = d_K(p)$.

Question 2d. Consider $K^* = K \cap \mathbb{B}[p; d_K(p) + 1)$. K^* is closed (since K is closed) and bounded (since $K^* \in \mathbb{B}[p, d_K(p) + 1))$, so by the Heine-Borel theorem it is compact. Applying part c gives that there exists $\hat{x} \in K^*$ with $d_K(p) = d(p, \hat{x})$. Since $K^* \subseteq K$ also $\hat{x} \in K$, so we're done.

Question 3a. Both directions:

($\Longrightarrow$) Assume $\overline{A} = \mathbb{R}$, therefore we have $(a, b) = \overline{A} \cap (a, b) = \overline{A \cap (a, b)}$. Clearly $\overline{\emptyset} \neq (a, b)$, so $A \cap (a, b) \neq \emptyset$.

($\Longleftarrow$) Assume that for all nonempty $(a, b) \subseteq \mathbb{R}, A \cap (a, b) \neq \emptyset$. Let $x \in \mathbb{R}$ and choose $\epsilon > 0$. Then $\mathbb{B}[x, \epsilon) \cap A = (x - \epsilon, x + \epsilon) \cap A \neq \emptyset \implies x \in A'$. Therefore $\overline{A} = A \cup A' \subseteq A' = \mathbb{R} \implies \overline{A} = \mathbb{R}$.

Question 3b. Let $(a, b) \subseteq \mathbb{R}$ be nonempty. Since G_1 is dense and open, $F_1 = G_1 \cap (a, b)$ is open and nonempty. Recursively define $F_n = G_n \cap F_{n-1}$, since G_n is dense and open each F_n is a nonempty open set. Clearly $F_1 \supseteq F_2 \supseteq \ldots$, and by Cantor's Intersection theorem therefore there exists a unique point $x \in \bigcap_{n=1}^{\infty} F_n \subseteq S$, and by construction $x \in (a, b)$. Thus by every interval (a, b) contains an element of S, so by the result of part a, S is dense.

Question 3c. By contradiction assume that it was possible, i.e. $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} A_n$ for open sets A_n. Since $\mathbb{Q}$ is dense and $A_n \supset \mathbb{Q}$, each of the A_n are also dense. Next, note that the irrationals also admit a representation as the intersection of countably many open, dense subsets: $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}$. But then $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ is the intersection of countably many open, dense sets and so by the result of part b, it is also dense. $\emptyset$ obviously can't be dense though so the assumption that $\mathbb{Q}$ admitted such a representation must be incorrect.

Question 4. As the hint suggests I will prove the contrapositive, i.e. $E \cap E' = \emptyset \implies E$ is countable. Assume $E \cap E' = \emptyset$, and consider $I = E \cap [0, 1)$. If I is countable then by symmetry every other interval of the form $[a, a + 1)$ is also countable, so E is the union of countably many countable sets so would also be countable. Thus it suffices to show that I is countable.

For each $x \in I$, define $B(x) = \bigcap_{y \in I} \mathbb{B}[x, \frac{|y-x|}{2})$, and since $I \cap I' = \emptyset$ each $B(x)$ is nonempty. Also let $r(x) = \inf\{\frac{|y-x|}{2} : y \in I\}$, i.e. the radius of the smallest ball to fit in $B(x)$. By construction each of the $B(x)$ are disjoint, so for any finite subset $X \subseteq I$, we have $\sum_{x \in X} r(x) \leq 1$. Let $A_n = \{x \in I : r(x) > \frac{1}{n}\}$. Each A_n is finite and has less than n elements, since if it wasn't then $\sum_{i=1}^{n} r(a_n) > \frac{1}{n} n = 1$ for $a_1, \ldots, a_n \in A_n$ which we said was impossible. Also since $r(x) > 0 \forall x$ since $I \cap I' = \emptyset$, every $x \in X$ is in A_n for some n. Then I is the union of countably many finite sets, so it is countable. Since this applies to every interval of length 1, by extension E is countable as well.

Question 5. Let $k_2 \in K_2$. For each $k_1 \in K_1$, use the Hausdorff property of $(X, \mathcal{T})$ to find $U_1^{k_1}, U_2^{k_1}$ with $k_1 \in U_1^{k_1}$, $k_2 \in U_2^{k_1}$ and $U_1^{k_1} \cap U_2^{k_1} = \emptyset$. These $U_1^{k_1}$ cover K_1, so since K_1 is compact there are $k_{1,1}, \ldots, k_{1,n} \in K_1$ such that $K_1 \subseteq \bigcup_{i=1}^n U_1^{k_{1,i}}$, call this union $V_1^{k_2}$. Since n is finite, $V_2^{k_2} = \bigcap_{i=1}^n U_2^{k_{1,i}}$ is an open set containing k_2, and by its construction it both contains k_2 and has null intersection with the union of $U_1^{k_2}$. Thus so far we can distinguish between a point and a compact set using the topology.

To extend this to distinguish between compact sets, now let k_2 vary. Since $K_2 \in \bigcup_{k_2 \in K_2} V_2^{k_2}$, there are finitely many $k_{2,1}, \ldots, k_{2,m} \in K_2$ such that $K_2 \subseteq \bigcup_{i=1}^m V_2^{k_{2,i}}$, call this union U_2. Similar to before, since m is finite, $U_1 = \bigcap_{i=1}^m U_1^{k_{2,m}}$ is open, contains K_1 and doesn't intersect with U_2. Thus we've constructed U_1, U_2 such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$ as desired.

Question 6. Clearly f is increasing, since increasing x only increases the number of terms being summed. Thus it is sufficient to show that for any $p, x \in \mathbb{R}$ with $f(x) > p$, there exists $y \in \mathbb{R}$ with $a < x$ such that $f(a) > p$. Let p, x be such that $f(x) > p$, and let K be such that $\sum_{i=K}^{\infty} \frac{1}{2^i} = 2^{1-K} < f(x) - p$. Let $a < x$ such that $(a, x) \cap \{q_1, q_2, \ldots, q_{K-1}\} = \emptyset$, since there are only finitely many elements in the set on the right this is always possible. Then by our construction of a, we have

$$f(a) = \sum \left\{ \frac{1}{2^k} : q_k < x \right\} \geq f(x) - \sum_{i=K}^{\infty} \frac{1}{2^i} > f(x) - (f(x) - p) = p.$$

Thus $a < x$ with $f(a) > p$, so since f is increasing this implies that $((x - (x - a), x + (x - a)) \in f^{-1}((p, +\infty))$ for every x, so it is open. Since this is true for all $p \in \mathbb{R}$, f is lower semicontinuous.