Homework 7, Math 443

Due Wednesday, March 22, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (3 points) Let G be a connected nontrivial regular graph that is not Eulerian. Prove that if $\overline{G}$ is connected, then $\overline{G}$ is Eulerian.

2. (5 points) Let G_1, G_2, and G_3 be pairwise-disjoint connected regular nontrivial graphs and let G be a graph constructed from the disjoint union of G_1, G_2, and G_3 by adding every edge uv where u and v are in different subgraphs G_i and G_j. (This is called the *join* of G_1, G_2, and G_3. Your book denotes this $G_1 + G_2 + G_3$; another common notation is $G_1 \vee G_2 \vee G_3$.)

   Prove that if G_1 and $\overline{G_1}$ are Eulerian but G_2 and G_3 are not Eulerian, then G is Eulerian.

3. (4 points) Prove, or provide a counterexample:

   > If e and f are distinct incident edges in a graph G, and G has an Eulerian circuit, then e and f appear consecutively in some Eulerian circuit of G.

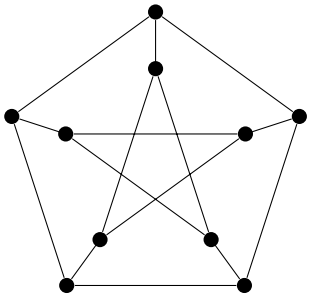
4. (6 points) Characterize the complete multipartite graphs that are Hamiltonian.

5. (2 points) Let $k(H)$ be the number of components in a graph H. Prove, or provide a counterexample: For any graph G and any induced subgraph $H \leq G$, $k(H) \leq \alpha(G)$.

6. (6 points) Theorem 6.5 says:

   > If G is a Hamiltonian graph, then for every nonempty proper set S of vertices of G,
   >
   > $$k(G - S) \leq |S|$$
   >
   > where $k(G - S)$ is the number of components in G_S.

   The Petersen graph is shown below. You may assume without proof that it is non-Hamiltonian, its independence number is 4, and its vertex connectivity is 3.



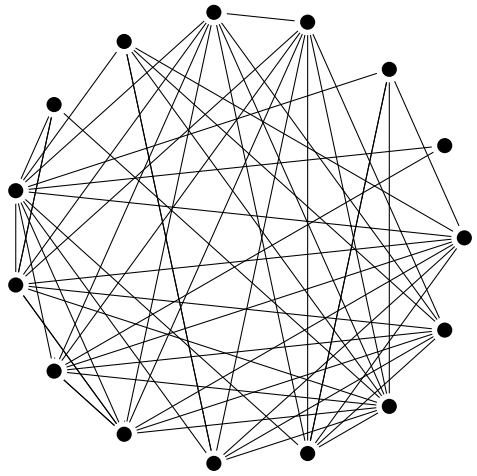
   Use the Petersen graph to show that the converse of Theorem 6.5 is not true.

7. (4 points) Prove the following as a corollary to a result from class: If a graph G has $\delta(G) \geq \frac{|G|+4}{2}$, then G contains two edge-disjoint Hamiltonian cycles.

8. (6 points) Let G be a 2-connected graph. Prove that if G contains no induced subgraph isomorphic to $K_{1,3}$ or $K_{1,3} + e$ (claw plus any edge), then G is Hamiltonian.

   Hint: consider a longest cycle.

9. (4 points) The graph G is shown below. Is G Hamiltonian?



Question:	1	2	3	4	5	6	7	8	9	Total
Points:	3	5	4	6	2	6	4	6	4	40