MATH305-201-2021/2022 Homework Assignment 2 (Due Date: Jan. 24, 2022)

10pts each

1. Find all values of the following equation

(a)$z^3 = i - 1$; (b) $z^5 = \frac{2i}{1-\sqrt{3}i}$; (c) $(z-i)^2 = i$; $(d) z^2 + 2iz + 1 = 0$

*2. Let m and n be positive integers that have no common factor and z_0 be a complex number. Let $z_0^{\frac{1}{n}}$ denote the set of all complex numbers such that $z^n = z_0$, i.e., $z_0^{\frac{1}{n}} = \{z \mid z^n = z_0\}$. Prove that the set of numbers $(z_0^{1/n})^m$ is the same as the set of numbers $(z_0^m)^{1/n}$. Use this result to find all values of $(1-i)^{3/2}$. Here $(z_0^{1/n})^m = \{z^m \mid z^n = z_0\}$.

Hint: since m and n have no common factor, for any integer k, we can write it as $k = mk_1 + nk_2$ where k_1, k_2 are two integers.

*: An extra 10points will be awarded to Problem 2 if your answer is correct.

3. Write the following functions in the form $w = u(x,y) + iv(x,y)$.

(a) $f(z) = \frac{z+i}{z+1}$; (b) $f(z) = \frac{e^z}{z}$; (c) $f(z) = \frac{z^2+3}{|z-1|^2}$

4. Describe the image of the following sets under the following maps

(a) $f(z) = (1-i)z + 5$ for $S = \{Re(z) > 0\}$; (b) $f(z) = \frac{z-i}{z+i}$ for $S = \{|z| < 3\}$; (c) $f(z) = -2z^5$ for $S = \{|z| < 1, 0 < Arg z < \frac{\pi}{2}\}$

5. Describe the image of the following sets under the given map

(a) $S = \{Re(z) = 1\}$, $w = e^z$; (b) $S = \{0 \le Im(z) \le \frac{\pi}{4}\}$, $w = e^z$; (c) $S = \{0 \le Re(z) \le 1, Im(z) = 1\}$, $w = z^2$

6. The Joukowski map is defined by

$$w = f(z) = \frac{1}{2}(z + \frac{1}{z})$$

Show that J maps the circle $S = \{|z| = r_0\}$ $(r_0 > 0, r_0 \neq 1)$ onto an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the unit circle $S = \{|z| = 1\}$ onto the real interval $[-1, 1]$.

Hint: use polar form of z.

7. Prove that $|e^{-z^4}| \le 1$ for all z with $-\frac{\pi}{8} \le Arg(z) \le \frac{\pi}{8}$.

8. Show that the function $f(z) = \bar{z}$ is continuous everywhere but not differentiable anywhere.

9. Discuss the differentiability and analyticity of the following functions

(a) $(x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$; (b) $|z|^2 + 2z$

10. Let

$$f(z) = \begin{cases} (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2+y^2), \text{if } z \neq 0; \\ 0 \text{ if } z = 0 \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$ but f is not differentiable at $z = 0$.

Hint: consider the limit with $\Delta z = (1+i)h, h \to 0$.