Math 437 Homework 4

Xander Naumenko

05/12/23

Question 1. Without loss of generality assume that $a \geq b$. If $a = b$ then the problem reduces to $n^2 = 2^{a+1} \implies n = 2^{\frac{a+1}{2}} \implies n = 2^m$ works for all $m \in \mathbb{N}$ by choosing $a = b = 2m - 1$. Otherwise for $a > b$, rearranging gives $n^2 = 2^b \left(2^{a-b} + 1\right)$. Since the rightmost term is odd, it must be that $\exp_2 n^2 = b \implies 2|b$, define $k^2 = \frac{n^2}{2^b} = \left(\frac{n}{2^{b/2}}\right)^2 \in \mathbb{N}$. Then we have $k^2 = 2^{a-b} + 1 \implies 2^{a-b} = (k-1)(k+1)$. The only k for which $k+1$ and $k-1$ are powers of 2 is $k = 3$, so for $a > b$, $n = 3 \cdot 2^m, m \in \mathbb{N}$ works with $a = 2m+3, b = 2m$ witnessing the desired equality. Thus the general solution is that any $n \in \mathbb{N}$ in the form $n = 2^m$ or $n = 3 \cdot 2^m, m \in \mathbb{N}$ works. $\square$

Question 2. There are no solutions, by contradiction suppose that there were. If x is even then $y^2 \equiv -1 \mod 8$ which is impossible since 7 isn't a perfect square mod 8, so x is odd and y is even. Moreover since x is odd, $x^3 \equiv x \mod 4$, so $x^3 \equiv x \equiv y^2 + 9 \equiv 1 \mod 4$. Consider the rearrangement of the equation $x^3 - 8 = (x-2)(x^2 + 2x + 4) = y^2 + 1$. Taking the $(x-2)$ factor mod 4 gives $x - 2 \equiv 1 - 2 \equiv 3 \mod 4$, and since $x = \sqrt[3]{y^2 + 9} \geq \sqrt[3]{9} > 2$, we have $x - 2 > 0$. Thus there must exist a prime p in the form $p = 4k + 3$ such that $p|x - 2$ (if all the factors of $x - 2$ were in the form $4k + 1$ then $x - 2$ would also be in that form but we just saw it isn't), and so $p|y^2 + 1$ also. But then $y^2 = -1 \mod p$ which contradicts proposition 12.1[1], so in fact no such x and y exist. $\square$

Question 3. For their fractional parts to be equal, it must be that $\{\sqrt[3]{y}\} = \{\sqrt{x}\} \implies \sqrt[3]{y} - [\sqrt[3]{y}] = \sqrt{x} - [\sqrt{x}] \implies \sqrt[3]{y} = \sqrt{x} + c$, where $c = [y] - [x] \in \mathbb{Z}$. Raising the last equality to the third power gives

$$y = \left(\sqrt{x} + c\right)^3 = x^{\frac{3}{2}} + 3cx + 3c^2 x^{\frac{1}{2}} + c^3 = (3cx + c^3) + \sqrt{x}\left(x + 3c^2\right) \implies \sqrt{x} = \frac{y - 3cx + c^3}{x + 3c^2}.$$

Note that the final division is valid since $x > 0 \implies x + 3c^2 \neq 0$. Thus $\sqrt{x} \in \mathbb{Q}$, so by proposition 24.1, $\sqrt{x} \in \mathbb{N}$. Also $\sqrt[3]{y} = \sqrt{x} + c \in \mathbb{N}$, so x is a perfect square and y is a perfect cube. $\square$

[1] There's a minor typo in the notes in proposition 12.1, it should say $x^2 \equiv -1 \mod p$ is unsolvable instead of $x^2 \equiv -1 \mod 4$.