MATH305-201-2021/2022 Homework Assignment 3 (Due Date: Jan.31, 2022)

1. For the following statements, state if it is true or false. If it is false give a counterexample
   (1) If f is differentiable at $z = z_0$, then f is analytic at $z = z_0$.
   (2) If f is differentiable at $z = z_0$, then f is continuous at $z = z_0$.
   (3) If f is analytic in an open and connected domain D and $Re(f(z)) = Constant$, then f is constant.
   (4) If f is analytic in an open and connected domain D and $|f(z)| = Constant$, then f is constant.

2. Use Cauchy-Riemann equation to find out the harmonic conjugate of the following functions
   (a) $xy - x + y$; (b) $u = \log(x^2 + y^2)$ for $Re(z) > 0$; (c) $u = \sin x \cosh(y)$

3. Let $f(z)$ be an analytic function in D and $Im(f(z)) \neq 0$. Show that $\log|f(z)|$ and $Arg(f(z))$ is harmonic.

4. (a) Let u be a harmonic function in D. Show that if v is a harmonic conjugate of u in a domain D, then both $u^2 - v^2$ and $u^3 - 3uv^2$ are harmonic in D.
   (b)Suppose that functions u and v are harmonic in D. Are the following functions harmonic?
   (1) $u^2 - v^2$; (2) uv; (3) $u - 100v$; (4) $u_{xy} + \Delta v$
   (Assume that harmonic functions are smooth functions with all derivatives.)

5. Find a harmonic function $\phi(x, y)$ in the infinite strip

$$\{z : -2 \leq 2Re(z) - 3Im(z) \leq 3\}$$

such that $\phi = 0$ on the left edge $\{2Re(z) - 3Im(z) = -2\}$ and $\phi = 4$ on the right edge $\{2Re(z) - 3Im(z) = 3\}$. Hint: consider linear functions.

6. Find a harmonic function $\phi(x, y)$ satisfying

$$\Delta\phi = 0, y > 0, -\infty < x < +\infty$$

$$\phi(x, 0) = -1, x < -5; \phi(x, 0) = 0, -5 < x < -1; \phi(x, 0) = 2, -1 < x < 2; \phi(x, 0) = 0, x > 2$$

Write your solution in terms of $\tan^{-1}$ or Arg.

7. Find a harmonic function $\phi(x, y)$ in the annulus $\{z : 1 \leq |z| \leq 2\}$ such that $\phi = 1$ on $\{|z| = 1\}$ and $\phi = 2$ on $\{|z| = 2\}$.

8. Find a harmonic function $\phi(x, y)$ such that

$$\Delta\phi = 0, \text{ in } D = \{(x, y)|y > 0, x^2 + y^2 > 9\}$$

$$\phi(x, 0) = -1, x < -3; \phi(x, y) = 0 \text{ for}, x^2 + y^2 = 9, -3 < x < 3; \phi(x, 0) = 2, x > 3$$

9. Find the image of the $S = \{z : -1 \leq Re(z) \leq 1, -\frac{\pi}{2} \leq Im(z) \leq \pi\}$ under the map $f(z) = e^z$

10. Find all numbers z such that
   (a) $(z + 1)^3 = (1 + i)z^3$; (b) $e^z = -1 - \sqrt{3}i$; (c) $\sin(z) = 4i$; (d) $\sin(z^6) = 0$