PHYS 301 Tutorial 6

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October 27, 2021

Question 1a. First we use the definition of $\vec{D}$. Consider a Gaussian cylinder of radius r around one of the plates. Then we have

$$\oint \vec{D} \cdot d\vec{A} = Q_{free} = D\pi r^2 = \frac{Q}{LW}\pi r^2 \implies \vec{D} = -\frac{Q}{LW}\hat{z}$$

To calculate the E field we would get the exact same as what we previously calculated except with a term coming from σ_b in the Gaussian cylinder, so we get

$$\vec{E} = -\frac{1}{\epsilon_0}(\frac{Q}{LW} + \sigma_b)\hat{z}$$

Finally to find σ_b we can use the relationship between E and D:

$$D = \epsilon_0 \epsilon_r E$$

$$\epsilon_0 \epsilon_r E = -\epsilon_r(\frac{Q}{LW} + \sigma_b) = -\frac{Q}{LW} \implies \sigma_b = \frac{Q}{\epsilon_r LW} - \frac{Q}{LW} = \frac{Q}{LW}(\frac{1}{\epsilon_r} - 1)$$

Question 1b. To find the voltage we do the line integral of the electric field and set the voltage at one of the plates to be zero:

$$\int \vec{E} \cdot \vec{dl} = \int_0^z \frac{1}{\epsilon_0 \epsilon_r} \frac{Q}{LW} dl = \frac{Qz}{\epsilon_0 \epsilon_r LW}$$

Question 1c. We calculate capacitance with the standard formula:

$$C = \frac{Q}{V} = Q\frac{\epsilon_0 \epsilon_r LW}{Qz} = \frac{\epsilon_0 \epsilon_r LW}{z}$$

Question 1d. Using the formula for capacitor energy we get

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 \epsilon_r LW}{z}\left(\frac{Qz}{\epsilon_0 \epsilon_r LW}\right)^2 = \frac{Q^2 z}{2\epsilon_0 \epsilon_r LW}$$

Question 1e. Inspecting our formulas derived only the capacitance increases as ϵ_r increases.

Question 1f. We assume the charge distribution remains uniform despite the voltage difference on the plate. Then we have that for the left section, we can treat it as a seperate capacitor and the voltage difference is $V_1 = \frac{(Q\frac{x}{L})z}{\epsilon_0 x W} = \frac{Qz}{\epsilon_0 LW}$. The one on the right is the same as what we calculated for part b except with a smaller area, so the voltage difference for that section is $V_2 = \frac{(Q\frac{L-x}{L})z}{\epsilon_0 \epsilon_r (L-x)W} = \frac{Qz}{\epsilon_0 \epsilon_r LW}$. We can then treat the two sections as capacitors in parrallel and

$$C_{total} = C_1 + C_2 = \frac{Q_1}{V_1} + \frac{Q_2}{V_2} = \frac{\epsilon_0 x W}{z} + \frac{\epsilon_0 \epsilon_r (L-x) W}{z} = \frac{\epsilon_0 W}{z}(x + \epsilon_r(L-x))$$

Question 1g. Using the same method as question 1d we get

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{zQ^2}{\epsilon_0 W(x + \epsilon_r(L-x))}$$

Question h. There is a force on the dialectric. The energy is the integral of force, so we also know that $F = -\frac{dU}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$. Expanding we get

$$F = \frac{1}{2}\frac{Q^2}{C^2}\frac{\epsilon_0 W}{z}(1 - \epsilon_r)$$

Question i. We conclude that the get polarized and produce an electric field in opposition to the applied one.

Question 2a. For above the liquid, we get that the voltage difference is

$$\Delta V_1 = \frac{-\sigma r_a}{\epsilon_0}\log(\frac{r_b}{r_a})$$

For below the liquid, we get that the voltage is

$$\Delta V_2 = \frac{-\sigma r_a}{(1+\chi_e)\epsilon_0}\log(\frac{r_b}{r_a})$$

The total capacitance is then

$$\begin{aligned} C = C_1 + C_2 = \frac{Q_1}{V_1} + \frac{Q_2}{V_2} &= \frac{\sigma 2\pi r_a (L-h)\epsilon_0}{\sigma r_a \log(\frac{r_b}{r_a})} + \frac{\sigma 2\pi r_a h \epsilon_0 (1+\chi_e)}{\sigma r_a \log(\frac{r_b}{r_a})} \\ &= \frac{2\pi\epsilon_0}{\log(\frac{r_b}{r_a})}\Big((L-h) + h(1+\chi_e)\Big) \\ &= \frac{2\pi\epsilon_0}{\log(\frac{r_b}{r_a})}\Big(L + h\chi_e\Big) \end{aligned}$$

Question 2b. To calculate energy we use $U = \frac{Q^2}{2C}$:

$$U = \frac{Q^2}{2C} = 4\pi^2 r_a^2 \sigma^2 L^2 \frac{\log(\frac{r_b}{r_a})}{2\pi\epsilon_0(L + h\chi_e)} = \frac{2\pi r_a^2 \sigma^2 L^2 \log(\frac{r_b}{r_a})}{\epsilon_0(L + h\chi_e)}$$

Question 2c. Using the same method as for the previous question, we get

$$\begin{aligned} F = -\frac{dU}{dh} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dh} &= \frac{1}{2}4\pi^2 r_a^2 \sigma^2 L^2 \left(\frac{\log(\frac{r_b}{r_a})}{2\pi\epsilon_0(L + h\chi_e)}\right)^2 \frac{2\pi\epsilon_0\chi_e}{\log(\frac{r_b}{r_a})} \\ &= \frac{\pi r_a^2 \sigma^2 L^2 \log(\frac{r_b}{r_a})\chi_e}{\epsilon_0(L + h\chi_e)} \end{aligned}$$

Question 2d. The force of gravity on the liquid will be $\rho\pi g(r_b^2 - r_a^2)h$. For equilibrium the forces should be equal so

$$\rho\pi g(r_b^2 - r_a^2)h = \frac{\pi r_a^2 \sigma^2 L^2 \log(\frac{r_b}{r_a})\chi_e}{\epsilon_0(L + h\chi_e)}$$

$$\rho\pi g(r_b^2 - r_a^2)h = \frac{\pi r_a^2 \sigma^2 L^2 \log(\frac{r_b}{r_a})\chi_e}{\epsilon_0(L + h\chi_e)}$$

$$\implies \rho\pi g(r_b^2 - r_a^2)\epsilon_0\chi_e h^2 + \rho\pi g(r_b^2 - r_a^2)\epsilon_0 L h - \pi r_a^2 \sigma^2 L^2 \log(\frac{r_b}{r_a})\chi_e = 0$$

This is a quadratic equation which can be solved with the quadratic formula for h.