Homework 4, Math 443

Due Wednesday, February 15, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

We'll define a directed rooted tree (DRT) as a digraph T with the following properties:

- the underlying graph of T is a tree,
- if uv is a directed edge of T, then uv is not, and
- there exists a vertex r of T, called the root, such that for every vertex v, the $r-v$ path in the underlying tree is a directed path from r to v.

1. (7 points) Prove that in a DRT the root has in-degree 0, and all other vertices have in-degree 1.

2. (7 points) Prove the following:

   Let T_1 and T_2 be disjoint DRTs.
   Let e be any directed edge with one endpoint in T_1 and the other in T_2. The directed graph $(T_1 \cup T_2) + e$ is a DRT if and only if the the second vertex of e is the root of T_1 or T_2.

3. (6 points) Prove, or provide a counterexample:

   If G is a nontrivial connected graph, and P is a longest path in G, then the endpoints of P are not cut-vertices of G.

4. (4 points) In class, we stated Corollary 5.6:

   Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

   Our proof went something like this:

   Proof. Let u and v be vertices of a nontrivial connected graph G such that $d(u,v) = diam(G)$. Since each of u and v is farthest from the other, by Theorem 5.5 both of them are not cut-vertices of G. $\square$

   Consider the similar proof below:

   Proof. Let u and v be endpoints of a longest path in G. Since G is nontrivial and connected, $u \neq v$. Since each of u and v is farthest from the other, by Theorem 5.5 both of them are not cut-vertices of G. $\square$

   What is the fundamental flaw with the second proof?

5. (4 points) True or false (with justification): All components of a graph are 1-connected.

6. (6 points) Prove the following:

Let G be a k-connected graph. Form a graph G' by adding one vertex to G and making it adjacent to at least k vertices. Then G' is also k-connected.

Hint: consider a separating set of G', and whether or not it includes the new vertex.

7. (6 points) Let X be a collection of edges from $E(K_{3n})$, where n is some positive integer. Suppose $K_{3n} - X$ has at least three nonempty components. What is the minimum possible value of $|X|$?

   Remember to give a complete justification. Your justification may include methods from calculus.

Question:	1	2	3	4	5	6	7	Total
Points:	7	7	6	4	4	6	6	40