PHYS 304 Homework 7

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Question 1. Expanding in terms of energy eigenstates:

$$c_n(t) = \langle n|S(t)\rangle = \langle n| \left(\int |x\rangle \langle x| \right) dx \, |S(t)\rangle = \int \langle n|x\rangle \langle x|S(t)\rangle \, dx = \int \langle x|n\rangle^* \, \Psi(x,t) dx.$$

Since the potential is time independent, we can calculate the energy eigenfunctions:

$$\hat{H} f_n(x) = E_n f_n(x) \implies \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) f_n = E_n f_n \implies f_n(x) = A e^{i \frac{\sqrt{2m(E_n - V)}}{\hbar}} + B e^{-i \frac{\sqrt{2m(E_n - V)}}{\hbar}}.$$

Putting the energy eigenstates into the equation above:

$$c_n(t) = \int f_n(x)^* \Psi(x,t) dx.$$

Question 2. Expanding:

$$\langle n|\hat{x}|S(t)\rangle = \sum_{n'} \sum_{n''} \langle n| \, |n'\rangle \, \langle n'| \, \hat{x} \, |n''\rangle \, \langle n''| \, |S(t)\rangle \, .$$

Applying equation 3.114:

$$= \sum_{n'} \sum_{n''} \delta_{n,n'} \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n''} \delta_{n',n''-1} + \sqrt{n'} \delta_{n'',n'-1} \right) c_{n''}(t).$$

$$= \sum_{n''} \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n''} \delta_{n,n''-1} + \sqrt{n} \delta_{n'',n-1} \right) c_{n''}(t).$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1} c_{n+1}(t) + \sqrt{n} c_{n-1}(t) \right).$$

Question 3. Expanding:

$$\langle S(t)|x|S(t)\rangle = \iint \langle S(t)|p'\rangle \, \langle p'| \, \hat{x} \, |p''\rangle \, \langle p''|S(t)\rangle \, dp' dp''.$$

$$= \iint \Phi(p',t) i\hbar \frac{d}{dp'} \delta(p' - p'') \Phi(p'',t) dp' dp'' = \int \Phi(p',t) i\hbar \frac{d}{dp'} \Phi(p',t) dp'.$$

Question 4. Wavefunction: $\Psi(x,0) = \frac{1}{\sqrt{2}} (\psi_1(x) + \psi_2(x))$. Calculating:

$$\langle H\rangle = \frac{1}{2} (E_1 + E_2) = \frac{5\pi^2 \hbar^2}{4ma^2}.$$

$$\left\langle H^2\right\rangle = \frac{1}{2}\left(E_1^2 + E_2^2\right).$$

$$\sigma_H = \sqrt{\langle H^2\rangle - \langle H\rangle} = \frac{1}{2}\sqrt{E_1^2 + E_2^2 - 2E_1E_2} = \frac{1}{2}(E_2 - E_1).$$

For position:

$$\langle x\rangle = \int \Psi(x,t)^* x \Psi(x,t) dx.$$

From previous homework/tutorials:

$$\langle x\rangle = \frac{a}{2}\left(1 - \frac{32}{9\pi^2}\cos(3\omega t)\right).$$

Calculating:

$$\left\langle x^2\right\rangle = \int \psi(x)^* x^2 \psi(x) dx.$$

$$= \frac{1}{a}\int \left(\sin\left(\frac{\pi}{a}x\right)\psi_1(t) + \sin\left(\frac{2\pi}{a}x\right)\psi_2(t)\right) x^2 \left(\sin\left(\frac{\pi}{a}x\right)\psi_1(t) + \sin\left(\frac{2\pi}{a}x\right)\psi_2(t)\right) dx.$$

$$= -\frac{16a^2}{9\pi^2}\cos\left(\frac{E_2 - E_1}{\hbar}t\right) + \frac{a^2}{3} - \frac{a^2}{2\pi^2} + \frac{a^2}{3} - \frac{a^2}{8\pi^2}.$$

$$= a^2\left(\frac{1}{3} - \frac{5}{16\pi^2} - \frac{16}{9\pi^2}\cos(3\omega t)\right).$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}^2\right)\cos^2(3\omega t)}.$$

By Ehrenfest's theorem:

$$\frac{d\langle x\rangle}{dx} = \frac{8\hbar}{3ma}\sin(3\omega t).$$

Plugging this into the time energy uncertainty principle:

$$\sigma_H^2\sigma_x^2 \geq \frac{\hbar^2}{4}\left(\frac{d\langle x\rangle}{dt}\right)^2.$$

$$\frac{1}{4}\left(E_2 - E_1\right)^2\frac{a^2}{4}\left(\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}^2\right)\cos^2(3\omega t) \geq \frac{\hbar^2}{4}\left(\frac{64\hbar^2}{9m^2a^2}\right)\sin^2(3\omega t).$$

Evaluating both sides numerically we see that the equality holds, as expected.