Physics 401. Set 1

(Each problem has equal weight.)

Due date: January 16. 2023

4th. Edition. Problems: 1.16, 1.21, 1.22, 1.39, 1.49, 1.64, 8.1, 8.2, 8.6

A. Practice with complex numbers:

Every complex number z can be written in the form $z = x + iy$ where x and y are real; we call x the *real part* of z, written $x = \mathrm{Re}\, z$, and likewise y is the *imaginary part* of z, $y = \mathrm{Im}\, z$. We further define the *complex conjugate* of z as $z^* \equiv x - iy$.

a) Prove the following relations that hold for any complex numbers z, z_1 and z_2:

$$\mathrm{Re}\, z = \frac{1}{2}\left(z + z^*\right) , \tag{1}$$

$$\mathrm{Im}\, z = \frac{1}{2i}\left(z - z^*\right) , \tag{2}$$

$$\mathrm{Re}\,(z_1 z_2) = (\mathrm{Re}\, z_1)(\mathrm{Re}\, z_2) - (\mathrm{Im}\, z_1)(\mathrm{Im}\, z_2)\,, \tag{3}$$

$$\mathrm{Im}\,(z_1 z_2) = (\mathrm{Re}\, z_1)(\mathrm{Im}\, z_2) + (\mathrm{Im}\, z_1)(\mathrm{Re}\, z_2)\,. \tag{4}$$

b) The modulus-squared of z is defined as $|z|^2 \equiv z^* z$. What is $\mathrm{Im}\, |z|^2$, and what is $\mathrm{Im}\, z^2$? In doing quantum mechanics confusing z^2 and $|z|^2$ is very common; be careful!

c) Any complex number can also be written in the form $z = Ae^{i\theta}$, where A and θ are real and θ is usually taken to be in the range $[0, 2\pi)$; A and θ are called the *modulus* and the *phase* of z, respectively. Prove Euler's relation (use a Taylor expansion),

$$e^{ix} = \cos x + i \sin x\,. \tag{5}$$

d) Use (5) to find $\mathrm{Re}\, z$, $\mathrm{Im}\, z$, z^* and $|z|$ in terms of A and θ.

e) Use the above relations on $e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$ to derive trigonometric identities for $\sin(\alpha+\beta)$ and $\cos(\alpha + \beta)$.