Math 318 Homework 1

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Question 1a. That sample space is listed in table 1. The probabilities can be calculated by calculating the number of permutations of that length.

Table 1: Sample space for question 1

Sequence	Probability
TT	$\frac{1}{4}$
HTT	$\frac{1}{8}$
HHTT	$\frac{1}{16}$
THTT	
HHHHH	$\frac{1}{32}$
HHHHT	
HHHTH	
HHHTT	
HTHTT	
THHTT	
HHTHH	
HHTHT	
HTHHH	
HTHHT	
HTHTH	
THHHH	
THHHT	
THHTH	
THTHH	
THTHT	

Question 1b. Since they are independent events we can sum the probabilities from the previous questions, so we get:

$$i = 2 : \frac{1}{4}.$$

$$i = 3 : \frac{1}{8}.$$

$$i = 4 : \frac{1}{8}.$$

$$i = 5 : \frac{1}{2}.$$

As expected the probabilities sum to 1.

Question 2. There are 40 marked frogs, so the odds of getting exactly 14 frogs are

$$L(n) = \frac{\binom{40}{14}\binom{n-40}{36}}{\binom{n}{50}} = \frac{40!(n-40)!50!(n-50)!}{14!26!36!(n-76)!n!}.$$

As the hint suggests, we can solve $\frac{l(n)}{L(n-1)} = 1$ to find where $L(n)$ is at its maximum.

$$\frac{L(n)}{L(n-1)} = \frac{(n-40)(n-50)}{n(n-76)} = 1$$

$$\implies (n-40)(n-50) = n^2 - 76n \implies n = \frac{1000}{7}.$$

Thus $L(n)$ is increasing for $n < \frac{1000}{7}$ and decreasing for $n > \frac{1000}{7}$. Thus either $n = 142$ or $n = 143$ maximizes $L(n)$, computing them shows that $n_* = 143$ is the most likely.

Question 3a. Plugging in values:

$$P_1 = \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{44}{49} \cdot \frac{40}{48} = 0.42.$$

$$P_2 = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 11 \cdot 4}{\binom{52}{5}} = 0.047.$$

Question 3b. Same as before:

$$P_1 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \binom{5}{2}}{6^5} = 0.46.$$

$$P_2 = \frac{6 \cdot 5 \cdot 4 \cdot \binom{5}{2} \cdot \binom{3}{2}}{6^5} = 0.23.$$

Question 4a. Using number of possibilities:

$$p_n = \frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{n!^2 2^{2n}}.$$

Question 4b.

$$p_{n+1}/p_n = \frac{(2n+2)(2n+1)}{(n+1)^2 2^2} = \frac{(n+1)(n+\frac{1}{2})}{(n+1)^2} \leq 1.$$

The ratio of successive elements being less than one means that the next term is smaller than the previous, so the sequence is decreasing.

Question 4c. Using sterling's approximation:

$$p_n = \frac{\sqrt{4\pi n}\,(2n/e)^{2n}}{2\pi n\,(n/e)^{2n}\,2^{2n}} = \frac{1}{\sqrt{\pi n}} \implies \alpha = \frac{1}{\sqrt{\pi}}.$$

Question 5a. There are n balls and $m-1$ lines to place so the number of configurations is just the number of ways to arrange this which is $\binom{n+m-1}{m-1}$.

Question 5b. Here we are just choosing which of the m urns to put our n balls in which is the definition of the choice function, which is just $\binom{m}{n}$.

Question 6. See the code below.

```
import random
import math
import matplotlib.pyplot as plt

for days in {365, 669}:
    def birthday(n):
        return len({random.randint(1, days) for i in range(n)}) != n

    N = list(range(2, 61))
    X = []
    for n in N:
        matches = 0
        runs = 0
        for i in range(1000):
            matches += birthday(n)
            runs += 1
        X.append(matches / runs)

    Y = [1 - math.factorial(days) / math.factorial(days - n) / days**n for n in N]

    plt.plot(N, X)
    plt.plot(N, Y)
    plt.show()
```