Math 320 Homework 1

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Question 1. The statement is false. Let $n = 41$. Then $41^2 - 41 + 41 = 41^2$ which is clearly divisible by 41.

Question 2a. Let $x \in A \cap (B \cup C)$. Then $x \in A$, and $x \in B$ or $x \in C$ which implies either $x \in (A \cap B)$ or $x \in (A \cap C)$. In either case $x \in (A \cap B) \cup (A \cap C)$. Similarly, let $y \in (A \cap B) \cup (A \cap C)$. y is either in $A \cap B$ or $A \cap C$, in either case $y \in A \cap (B \cup C)$. Since both sets contain the other, they must be equal.

Question 2b. Let $x \in C \setminus (A \cup B)$. x is in C but in neither A nor B, which means that $x \in C \setminus A$ and $x \in C \setminus B$, implying $x \in (C \setminus A) \cap (C \setminus B)$. Let $y \in (C \setminus A) \cap (C \setminus B)$. If x is in either of A or B it would be excluded by the intersection, so $x \in (A \cup B)$. Since both sets contain the other, they must be equal.

Question 2c. Let $x \in C \setminus (A \cap B)$. By it's definition x is in C but not in both A and B. Thus either $x \in C \setminus A$ or $x \in C \setminus B$, which means $x \in (C \setminus A) \cup (C \setminus B)$, so $x \in (C \setminus A) \cup (C \setminus B)$. Let $y \in (C \setminus A) \cup (C \setminus B)$. Either $y \in C \setminus A$ or $y \in C \setminus B$, in either case $y \in C$, and is excluded only if $y \in A \cup B$. Thus $y \in C \setminus (A \cup B)$. Both sets contain the other, so they are equal.

Question 3a. Let $b_1 \in f(C_1 \cap C_2)$. Then $\exists a_1 \in C_1 \cap C_2$ s.t. $f(a_1) = b_1$. Then $a_1 \in C_1$ and $a_1 \in C_2$, so $b_1 \in f(C_1) \cap f(C_2)$. Thus $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$, and since we didn't use the fact f is one-to-one this is always true.

Let $b_2 \in f(C_1) \cap f(C_2)$. Since f is one-to-one there exists exactly on $a_2 \in A$ with $f(a_2) = b_2$. Since $b_2 \in f(C_1) \cap f(C_2)$ and a_2 is unique, $a_2 \in C_1 \cap C_2 \implies b_2 \in f(C_1 \cap C_2)$. Since both sets contain one another, they are the same.

Question 3b. Let $a_1 \in f^{-1}(f(C))$. Since f is one-to-one, there exists $b_1 \in f(C)$ s.t. a_1 is the only member of A with $f(a_1) = b_1$, but since a_1 the only such element then $a_1 \in C$. Since this is true of all a_1, then $f^{-1}(f(C)) \subseteq C$.

Let $a_2 \in C$. Let $b_2 = f(a_2)$, so $b_2 \in f(C)$. By the definition of preimage $a_2 \in f^{-1}(\{b_2\}) \subseteq f^{-1}(f(C))$, so $f^{-1}(f(C)) \supseteq C$, and since we didn't use the fact that f is one-to-one this is always true. Both sets contain each other so they're the same.

Question 3c. Let $b_1 \in f(f^{-1}(D))$. Then $\exists a_1 \in A$ s.t. $f(a_1) = b_1$ and $a_1 \in f^{-1}(D)$. Then by the definition of preimage $b_1 \in D$. Thus $f(f^{-1}(D)) \subseteq D$, and since we didn't use the fact that f is onto this is always true.

Let $b_2 \in D$. Since f is onto, $\exists a_2 \in A$ s.t. $f(a_2) = b_2 \implies a_2 \in f^{-1}(D)$. Thus $b_2 \in f(f^{-1}(D))$, and $f(f^{-1}(D)) \supseteq D$. Both sets contain each other so they're the same.

Question 4. For all parts of this question, let $A = \{1, 2\}, B = \{1, 2\}$, and $f(x) = 1$.

Question 4a. Let $C_1 = \{1\}, C_2 = \{2\}$. Then $f(C_1 \cap C_2) = f(\emptyset) = \emptyset \neq \{1\} = f(C_1) \cap f(C_2)$.

Question 4b. Let $C = \{1\}$. Then $f^{-1}(f(C)) = \{1, 2\} \neq \{1\} = C$.

Question 4c. Let $D = \{2\}$. Then $f(f^{-1}(D)) = f(\emptyset) = \emptyset \neq \{2\} = D$.

Question 5. a is either even or odd. If it is even, $\exists k \in \mathbb{Z}$ s.t. $a = 2k \implies a^2 = 4k^2$, which can never be in the form $4b + 3$ since $4b + 3$ is odd while $4k^2$ is even. If a is odd, then $\exists m \in \mathbb{Z}$ s.t. $a = 2m + 1 \implies a^2 = 4(m^2 + m) + 1$. However then $a^2 = 1 \mod 4 \neq 3 \mod 4b + 3$. Since both cases are impossible, there exist no $a, b \in \mathbb{R}^2$ s.t. $a^2 = 4b + 3$.

Question 6a. Suppose not, i.e. suppose $\exists a, b \in A, a \neq b$ s.t. $(g \circ f)(a) = (g \circ f)(b)$. Since f is one-to-one, $f(a) \neq f(b)$, but this implies that $g(f(a)) = g(f(b))$ for different inputs of g which should be impossible since g is injective. Thus our original assumption was wrong and $g \circ f$ is injective.

Question 6b. Again by contradiction assume not, i.e. assume $\exists a, b \in A, a \neq b$ s.t. $f(a) = f(b)$. However then $g(f(a)) = g(f(b))$ since g is receiving the same input in both case, which contradicts the fact that $g \circ f$ is one-to-one. Thus our assumption was wrong and f is injective.

Question 6c. Again contradiction, assume that $\exists x, y \in B, a \neq b$ s.t. $g(x) = g(y)$. Since f is onto and one-to-one (by part b of this question), $\exists a, b \in A, a \neq b$ with $f(a) = x, f(b) = y$. But then $(g \circ f)(a) = (g \circ f)(b)$, which contradicts our assumption that $g \circ f$ is one-to-one. Thus our assumption was wrong and g is one-to-one.

Question 6d. Let $A = \{x \in \mathbb{N}\}, B = \mathbb{Z}, C = \mathbb{Z}, f(x) = x, g(x) = x^2$. Clearly g isn't one-to-one ($f(-x) = f(x)$), but $g \circ f(x)$ is one-to-one, since for positive inputs x^2 is injective.

Question 7. I'll prove part b straight away, which has part a as an immediate special case. Let $h = f \circ f \circ \cdots \circ f$ composed together $n - 1$ times, so $g = f \circ h = h \circ f = x$. Since the identity is bijective and in specific one-to-one, we can apply the result from part 6b to $h \circ f$ to conclude that f is one-to-one.

To prove that f is onto, let $y \in X$, and let $x = h(y) \in X$. Then $f(x) = f(h(y)) = y$. Since this is true of all $y \in X$, f is onto. Thus f is injective and surjective, so is bijective. Apply this result for $n = 2$, and part a follows immediately.

Question 8a. False: let $A = \mathbb{Z}, C = \mathbb{N}$. Then $f(A \setminus C) = \{x^2 : x < 0, x \in \mathbb{Z}\} \neq \{0\} = f(A) \setminus f(B)$.

Question 8b. True: Let $y \in f(A) \setminus f(C)$. Then $\exists x \in A$ s.t. $f(x) = y$, and by y's definition we know that $x \notin C$. Thus $x \in A \setminus C \implies y \in f(A \setminus C)$. Since this is true of all $y \in f(A) \setminus f(C)$, we have that $f(A \setminus C) \supseteq f(A) \setminus f(C)$.

Question 8c. f being one-to-one is sufficient. To see this, let $y \in f(A \setminus C) \implies \exists x \in A \setminus C$ s.t. $f(x) = y$. Since f is one-to-one then x is unique, specifically there does not exist a $z \in C$ s.t. $f(z) = y$. Thus $y \notin f(C)$, so $y \in f(A) \setminus f(C)$. Thus we have that if f is one-to-one, $f(A \setminus C) = f(A) \setminus f(C)$.