III. Sequences and Series in $\mathbb{R}$

UBC M320 Lecture Notes by Philip D. Loewen

A. Sequences and Convergence

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Definition. (Sequences) Let X be a set. A **sequence in** X is a function $x\colon \mathbb{N} \to X$. We often write x_n for the value $x(n)$, and denote the sequence itself by $(x_n)_{n=1}^{\infty}$. [Notation resembles "vector with $\aleph_0$ components".] The **range of the sequence** $(x_n)_{n=1}^{\infty}$ is an (unordered) subset of X, namely, $\{x_n : n \in \mathbb{N}\}$.

Remark. The real sequence $x_n = 1\ \forall n$ defines infinitely many *symbols* $x_1, x_2, x_3, \ldots$, but its range is just the one-point set $\{x_n : n \in \mathbb{N}\} = \{1\}$.

Definition. (Limits) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$; and let $\widehat{x} \in \mathbb{R}$. To say that **the sequence** (x_n) **converges to** $\widehat{x}$**,** or that $\widehat{x}$ **is a limit of** (x_n)**,** means this:

$$\forall\varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n > N,\ |x_n - \widehat{x}| < \varepsilon. \tag{$*$}$$

Situation $(*)$ is also expressed as

$$x_n \to \widehat{x} \text{ as } n \to \infty, \qquad \text{or} \qquad x_n \xrightarrow[n\to\infty]{} \widehat{x}.$$

Intuition/Terminology. In $(*)$, the number $\varepsilon > 0$ is an "error tolerance". The final inequality is equivalent to

$$-\varepsilon < x_n - \widehat{x} < \varepsilon, \quad \text{i.e.,} \quad x_n \in (\widehat{x} - \varepsilon, \widehat{x} + \varepsilon).$$

The quantifiers on N and n say that this inclusion holds *for all n sufficiently large,* or *for all but finitely many n.* The quantifier on ε requires the statement above to be valid for every $\varepsilon > 0$.

Remark. The order of the quantifiers ("$\exists, \forall$") in the definition is critical: for every $\varepsilon > 0$ something must happen, but *how* it happens may vary from one ε-value to another. In particular, the choice of N almost always *depends on ε.*

Extensions. The final inequality in line $(*)$ can be phrased geometrically by saying, "sequence element x_n is within distance ε of the point $\widehat{x}$". With this insight, extending the definition above to sequences and limits in $\mathbb{R}^k$ is trivial: just allow each of x_n and $\widehat{x}$ to be a vector, and use $|\cdot|$ to denote the vector length instead of just the absolute value. One could go further: given any set X and a function $d\colon X \times X \to \mathbb{R}$ that measures some form of "distance" between its inputs, requiring $d(x_n, \widehat{x}) < \varepsilon$ instead of $|x_n - \widehat{x}| < \varepsilon$ in $(*)$ opens the way to even deeper generalizations.

Definition. (Convergence/Divergence) To say simply that **the sequence** (x_n) **converges** means

$$\exists\widehat{x} \in \mathbb{R} : (*) \text{ holds.}$$

"Diverges" is the opposite of "converges." That is,

$$\begin{aligned}
(x_n) \text{ diverges} &\Leftrightarrow \neg\,[\,(x_n) \text{ converges}] \\
&\Leftrightarrow \neg\,[\exists\widehat{x} \in \mathbb{R} : \forall\varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N,\ |x_n - \widehat{x}| < \varepsilon] \qquad \text{(DNE)} \\
&\Leftrightarrow \forall\widehat{x} \in \mathbb{R},\ \exists\varepsilon > 0 : \forall N \in \mathbb{N},\ \exists n > N : |x_n - \widehat{x}| \geq \varepsilon.
\end{aligned}$$

(Think about how to express this symbolic assertion in simple words.)

Home Practice. Show that the order of the quantifiers in $(*)$ really matters by proving that The Following Are Equivalent (TFAE):

(i) $\exists N \in \mathbb{N} : \forall \varepsilon > 0,\ \forall n > N,\ |x_n - \widehat{x}| < \varepsilon$. [NOT our definition of a limit.]

(ii) $x_n = \widehat{x}$ for all but at most finitely many $n \in \mathbb{N}$. [NOT our intuitive concept of a limit.]

Example. Simple examples in $(\mathbb{R}, |\cdot|)$:

(a) $x_n = 1/n$ converges to $\widehat{x} = 0$.

Pf: Given $\varepsilon > 0$, choose any integer $N \geq 1/\varepsilon$: then for all $n > N \geq 1/\varepsilon$, one has $|x_n - 0| = 1/n < \varepsilon$.

(b) For any real $M \neq 0$ and $p > 0$, $x_n = M/n^p$ converges to $\widehat{x} = 0$.

Pf: Given $\varepsilon > 0$, choose any integer $N \geq \left(\frac{|M|}{\varepsilon}\right)^{1/p}$: then for all $n > N$, one has

$$n^p > \frac{|M|}{\varepsilon} \implies \frac{1}{n^p} < \frac{\varepsilon}{|M|} \implies |x_n - 0| = \frac{|M|}{n^p} < \varepsilon.$$

(c) $x_n = 1$ converges to $\widehat{x} = 1$.

Pf: Given $\varepsilon > 0$, choose $N = 320$. Then $n > N \Rightarrow |x_n - 1| = 0 < \varepsilon$.

NB: Any $N \in \mathbb{N}$ (not just 320) will work. Choice $N = 1$ would be most "efficient".

Remarks. 1. Convergence statement $(*)$ is self-contained: variable quantities ε and N are internal, not visible from outside. It does not make sense to ask, "What is the value of ε in line $(*)$?" [Analogy: A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *increasing* when "$a < b \Rightarrow f(a) < f(b)$". Given an increasing function, it is meaningless to ask, "Find a to six sig figs."]

2. To confirm $(*)$, imagine finding a **function** $N = N(\varepsilon)$ that is
   (i) Defined for all $\varepsilon > 0$ [both very small and very large],
   (ii) Consistent with $\forall n > N(\varepsilon),\ |x_n - \widehat{x}| < \varepsilon$.
3. The key issue is *existence,* not efficiency. A useful strategy is to invent a simple overestimate of $|x_n - \widehat{x}|$ and choose N to control that instead. ("To make $|x_n - \widehat{x}|$ small, it would be *more than enough* to make this simpler thing small.")
4. Insisting that $N \in \mathbb{N}$ in the definition is not very meaningful: allowing $N \in \mathbb{R}$ gives a concept with exactly the same meaning. [Exercise: Writing the definition to require N to be a prime number instead of a generic positive integer also gives exactly the same concept.] So we're not going to enforce the distinction on homework and tests.

Example. (More Serious Examples)

(a) $x_n = \sin(n)/(n^5 + n^4 + n^3 + n^2 + n + 1)$ converges to $\widehat{x} = 0$.

Pf: For any $n \in \mathbb{N}$,

$$|x_n - \widehat{x}| = \left|\frac{\sin(n)}{n^5+n^4+n^3+n^2+n+1}\right| < \left|\frac{\sin(n)}{0+0+0+0+n+0}\right| \le \frac{1}{n},$$

so any integer $N \ge 1/\varepsilon$ will satisfy definition $(*)$.

NB: Estimating $|x_n - \widehat{x}| \le 1/n^5$ instead would show that in fact any $N > \varepsilon^{-1/5}$ would work; for small $\varepsilon > 0$ (like $\varepsilon = 10^{-5}$), $1/\varepsilon^{1/5}$ is much smaller than $1/\varepsilon$, so the latter choice allows much smaller N. This does not matter! The logic only requires one N that works!

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(b) $x_n = (-1)^n$ diverges (or "has no limit", or "$\lim_n (-1)^n$ DNE").

Pf: We confirm (DNE) directly. Given any $\widehat{x}$ in $\mathbb{R}$, consider $\varepsilon = 1$: fix any $N \in \mathbb{N}$. Then there exist an odd integer $n_o > N$ and an even integer $n_e > N$, for which the triangle inequality gives

$$2 = |x_{n_e} - x_{n_o}| = \left|(x_{n_e} - \widehat{x}) + (\widehat{x} - x_{n_o})\right| \le |x_{n_e} - \widehat{x}| + |x_{n_o} - \widehat{x}|.$$

This implies that either $|x_{n_e} - \widehat{x}| \ge 1 = \varepsilon$ or else $|x_{n_o} - \widehat{x}| \ge 1 = \varepsilon$. ////

(c) The sequence $x_n = \dfrac{n^2 - 320n^{3/2}}{2n^2 - 801}$ converges to $\widehat{x} = \frac{1}{2}$.

Pf: Given an arbitrary $\varepsilon > 0$, choose the smallest integer $N \ge \max\left\{30, \left(\frac{750}{\varepsilon}\right)^2\right\}$. Then every $n > N$ satisfies $n^2 > 30^2 = 900$, so

$$(*) \quad 2n^2 - 801 = n^2 + (n^2 - 801) > n^2 \qquad \text{and} \qquad (**) \quad \sqrt{n} > \frac{750}{\varepsilon},$$

and these inequalities (among others) imply

$$\begin{aligned}
|x_n - \widehat{x}| &= \left|\frac{n^2 - 320n^{3/2}}{2n^2 - 801} - \frac{1}{2}\right| \\
&= \frac{\left|(2n^2 - 640n^{3/2}) - (2n^2 - 801)\right|}{2|2n^2 - 801|} \\
&\le \frac{640n^{3/2} + 801}{2|2n^2 - 801|} && \text{(triangle inequality)} \\
&< \frac{640n^{3/2} + 801n^{3/2}}{2n^2} && \text{(from } (*)\text{, and } 1 < n^{3/2}\text{)} \\
&= \frac{750}{\sqrt{n}} && (640 + 801 < 1500) \\
&< \frac{750}{750/\varepsilon} && \text{(from } (**)\text{)} \\
&= \varepsilon,
\end{aligned}$$

as required. ////

Remarks. Loose overestimates like "$640 + 801 < 1500$" and "$801 \leq 801n^{3/2}$" embedded above are fair play: they are admittedly sloppy, but not wrong. The point is simply to show **existence** of some $N = N(\varepsilon)$ that satisfies the definition, **not** to find the best one. The point at which an inequality becomes too sloppy is when it changes the dominant order of growth of the multinomial where it appears. Think about why the argument that succeeded with the sloppy inequality "$801 \leq 801n^{3/2}$", would break down utterly if we had opted for "$801 \leq 801n^2$" instead.

Famous Limits. Recall the Binomial Theorem, which says that for arbitrary $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for } 0 \leq k \leq n.$$

(Recall $0! = 1$.) This implies that for each $z \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2} z^2 + \frac{n(n-1)(n-2)}{6} z^3 + \cdots + nz^{n-1} + z^n.$$

In particular, when $z = x \geq 0$ is a nonnegative real number, one has both

$$(1+x)^n \geq 1 + nx, \qquad (1+x)^n \geq \frac{n(n-1)}{2} x^2 \qquad \forall n \in \mathbb{N}.$$

Lemma. *One has $n^{1/n} \to 1$ as $n \to \infty$.*

Proof. Following Rudin, we define $x_n = n^{1/n} - 1 > 0$ and apply the estimate above:

$$n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2.$$

When $n \geq 2$, we have $n - 1 \geq n/2$, so the inequality above gives

$$x_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{n/2}} = \frac{2}{\sqrt{n}}.$$

Given any $\varepsilon > 0$, the inequality $x_n < \varepsilon$ can be guaranteed by insisting on both $n \geq 2$ and $2/\sqrt{n} < \varepsilon$, i.e., $n > 4/\varepsilon^2$. So we choose any integer

$$N \geq \max\left\{2, \frac{4}{\varepsilon^2}\right\}.$$

Every $n \geq N$ will give $\left|n^{1/n} - 1\right| < \varepsilon$, as required. ////

B. Limit Laws

$$\boxed{(*):\qquad x_n \to \widehat{x} \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N,\ |x_n - \widehat{x}| < \varepsilon.}$$

Observations. 1. For a sequence (x_n) in $\mathbb{R}^k$, one has

$$x_n \to \widehat{x} \iff (x_n - \widehat{x}) \to 0.$$

(Compare the definitions.)

2. The letters "n", "ε", "N" in the definition are strictly internal (think of a stand-alone procedure in computer programming): that logical T/F statement is equivalent to

$$(*_R): \qquad \forall \rho > 0,\ \exists R \in \mathbb{N} : \forall r > R,\ |x_r - \widehat{x}| < \rho.$$

In particular, the N-vs-ε relationship for one sequence is typically completely unrelated to the R-vs-ρ relationship for another.

Lemma (Uniqueness of Limits). *If a sequence (x_n) in $\mathbb{R}$ obeys both $x_n \to \widehat{x}$ and $x_n \to z$, then $z = \widehat{x}$.*

Proof. Let any $\eta > 0$ be given. Define $\varepsilon = \eta/2 > 0$.

(i) Since "$x_n \to \widehat{x}$", $\exists N_1 \in \mathbb{N}$ so large that $|x_n - \widehat{x}| < \varepsilon\ \forall n > N_1$.

(ii) Since "$x_n \to z$", $\exists N_2 \in \mathbb{N}$ so large that $|x_n - z| < \varepsilon\ \forall n > N_2$.

Pick any $n > \max\{N_1, N_2\}$: then both inequalities hold, so

$$|\widehat{x} - z| \le |\widehat{x} - x_n| + |x_n - z| < \varepsilon + \varepsilon = \eta.$$

This proves, "$\forall \eta > 0,\ |\widehat{x} - z| < \eta$". Hence (Archimedes) $|\widehat{x} - z| = 0$, i.e., $z = \widehat{x}$. ////

It is thanks to the lemma above that we can rely on the notation

$$\widehat{x} = \lim_{n\to\infty} x_n \iff x_n \xrightarrow[n\to\infty]{} \widehat{x}.$$

Proposition (Squeeze Theorem). *Suppose sequences (a_n), (x_n), and (b_n) are given, along with a real number L, and*

(a) *$a_n \to L$ as $n \to \infty$;*

(b) *$b_n \to L$ as $n \to \infty$;*

(c) *$\exists N \in \mathbb{N} : \forall n > N,\ a_n \le x_n \le b_n$.*

Then $x_n \to L$ as $n \to \infty$.

Proof. Let $\varepsilon > 0$ be given. Challenge the definition in (a) to produce some $N_a \in \mathbb{N}$ such that

$$\forall n > N_a,\ L - \varepsilon < a_n < L + \varepsilon.$$

Similarly, apply (b) to get some $N_b \in \mathbb{N}$ such that

$$\forall n > N_b,\ L - \varepsilon < b_n < L + \varepsilon.$$

Finally, choose $N_c \in \mathbb{N}$ with the property described in (c). Let $N = \max\{N_a, N_b, N_c\}$. Then every $n > N$ will obey

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon, \quad \text{so} \quad |x_n - L| < \varepsilon,$$

as required ////

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Example. *For each $r \geq 1$, $r^{1/n} \to 1$ as $n \to \infty$.*

Proof. Fix $r \geq 1$. Then there exists some $N \in \mathbb{N}$ such that $r < N$. Whenever $n > N$, we have $1 \leq r < n$ so $1 \leq r^{1/n} < n^{1/n}$. Send $n \to \infty$ and apply the Squeeze Theorem with $a_n = 1$, $x_n = r^{1/n}$, and $b_n = n^{1/n}$. ////

Lemma. *Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^k$; denote $\widehat{x} = \lim_{n\to\infty} x_n$. Then*

(a) *(x_n) is bounded, i.e.,*

$$\exists M > 0 :\ \forall n \in \mathbb{N},\ |x_n| \leq M.$$

(b) *If $\widehat{x} \neq 0$, then (x_n) is "eventually bounded away from zero", i.e.,*

$$\exists r > 0,\ N \in \mathbb{N} :\ \forall n > N,\ |x_n| > r.$$

Remark. The converse of (a) is false: some bounded sequences, like $(-1)^n$ in $\mathbb{R}$, are not convergent. Contraposing (a) ensures "not bounded implies not convergent".

Proof. (a) Take $\varepsilon = 1$ in definition $(*)$: then $\exists N$ such that

$$|x_n - \widehat{x}| < 1 \quad \forall n > N.$$

Hence for all $n \in \mathbb{N}$,

$$|x_n - \widehat{x}| \leq \max\{|x_1 - \widehat{x}|, |x_2 - \widehat{x}|, \ldots, |x_N - \widehat{x}|, 1\} \stackrel{\text{def}}{=} M_0.$$

Since $|x_n| = |\widehat{x} + (x_n - \widehat{x})| \leq |\widehat{x}| + |x_n - \widehat{x}|$, this gives

$$|x_n| \leq |\widehat{x}| + M_0 \stackrel{\text{def}}{=} M,$$

as required.

(b) The statement is true for any $r > 0$ such that $r < |\widehat{x}|$. Indeed, pick one. Then apply definition $(*)$ to $\varepsilon = |\widehat{x}| - r > 0$. That gives $N \in \mathbb{N}$ s.t.

$$\forall n > N,\ \varepsilon > |x_n - \widehat{x}| = |\widehat{x} - x_n| \geq |\widehat{x}| - |x_n|.$$

From the definition of ε, this gives $-r > -|x_n|$, i.e., $|x_n| > r$. ////

Theorem. *If the sequences (x_n), (y_n) in $\mathbb{R}$ converge, with $\widehat{x} = \lim_n x_n$ and $\widehat{y} = \lim_n y_n$, and if $c \in \mathbb{R}$, then*

(a) $\lim\limits_n (cx_n) = c\widehat{x}$,

(b) $\lim\limits_n (x_n + y_n) = \widehat{x} + \widehat{y}$,

(c) $\lim\limits_n (x_n y_n) = \widehat{x}\,\widehat{y}$,

(d) *if $\widehat{y} \neq 0$ and $y_n \neq 0$ for all n, then* $\lim\limits_n \left(\dfrac{1}{y_n}\right) = \dfrac{1}{\widehat{y}}$.

Proof. See Rudin, Theorems 3.3, 3.4. ////

Corollary. *For any $r > 0$, $r^{1/n} \to 1$ as $n \to \infty$.*

Proof. The case of $r \geq 1$ was dealt with earlier, using the Squeeze Theorem. So let $r \in (0, 1)$. Define $R = 1/r > 1$ and apply (d) with $y_n = R^{1/n}$ and $\widehat{y} = 1$:

$$\lim_{n\to\infty} r^{1/n} = \lim_{n\to\infty} \left(\frac{1}{R}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{R^{1/n}}\right) = \frac{1}{\left(\lim\limits_{n\to\infty} R^{1/n}\right)} = \frac{1}{1}.$$

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The Vector Case. Any sequence $(x_n)_n$ in $\mathbb{R}^k$ has k "component sequences" in $\mathbb{R}$. The first one is $(\mathbf{e}_1 \bullet x_n)_n$; the p-th one is $(\mathbf{e}_p \bullet x_n)_n$. E.g., if $x_n = (n^{-1} \sin n, \dfrac{n+1}{n-1}, n^{1/n})$ in $\mathbb{R}^3$, there are 3 component sequences:

$$\mathbf{e}_1 \bullet x_n = n^{-1} \sin n, \qquad \mathbf{e}_2 \bullet x_n = \frac{n+1}{n-1}, \qquad \mathbf{e}_3 \bullet x_n = n^{1/n}.$$

We have $x_n \to (0, 1, 1)$ by the following result.

Theorem. *A sequence in $\mathbb{R}^k$ converges iff each of its k component sequences converges. In this case, componentwise evaluation of the limit produces the correct result.*

Proof. Invent the notation $\underline{k} = \{1, 2, 3, \ldots, k\}$.

($\Rightarrow$) Let (x_n) be a convergent sequence in $\mathbb{R}^k$; define $\widehat{x} = \lim x_n$. Fix any $p \in \underline{k}$. To prove $(\mathbf{e}_p \bullet x_n)_n$ converges [to $\mathbf{e}_p \bullet \widehat{x}$], let any $\varepsilon > 0$ be given. Use the convergence of (x_n) to find $N \in \mathbb{N}$ so large that $|x_n - \widehat{x}| < \varepsilon$ for all $n > N$. Since

$$|(\mathbf{e}_p \bullet x_n) - (\mathbf{e}_p \bullet \widehat{x})| = |\mathbf{e}_p \bullet (x_n - \widehat{x})\,| \leq 1 \cdot |x_n - \widehat{x}|,$$

this same N will serve in the definition for $(\mathbf{e}_p \bullet x_n)$.

($\Leftarrow$) Suppose that for each $p \in \underline{k}$, the component sequence $(\mathbf{e}_p \bullet x_n)$ converges; define $\widehat{x}_p = \lim(\mathbf{e}_p \bullet x_n)$. Then define $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_k)$. To prove (x_n) converges to $\widehat{x}$, let any

$\varepsilon > 0$ be given. Challenge the k definitions of convergence for component sequences with the same error tolerance, namely $\varepsilon/k > 0$, to find $N_1, N_2, \ldots, N_k$ such that

$$\forall n > N_p,\ |\mathbf{e}_p \bullet x_n - \widehat{x}_p| < \varepsilon/k.$$

Then let $N = \max\{N_1, N_2, \ldots, N_k\}$: for all $n > N$, the triangle inequality gives

$$|x_n - \widehat{x}| = \left|\sum_{p=1}^{k} [\mathbf{e}_p \bullet (x_n - \widehat{x})]\, \mathbf{e}_p\right| \le \sum_{p=1}^{k} |\mathbf{e}_p \bullet (x_n - \widehat{x})| < \sum_{p=1}^{k} \frac{\varepsilon}{k} = \varepsilon.$$

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Three Viewpoints on Convergence. One can approach the definition of "$x_n \to \widehat{x}$" given in line $(*)$ above in three distinct ways:

1. As a prover. If your job is to justify the statement "$x_n \to \widehat{x}$", then you must find a convincing reason why, for arbitrary $\varepsilon > 0$, the favourable outcome described in $(*)$ actually occurs. The first line in such an argument is often, "Let any $\varepsilon > 0$ be given."

2. As a detractor. To show that $x_n \not\to \widehat{x}$, you must prove "$\neg(*)$". Now $\neg(*)$ begins, "$\exists \varepsilon > 0 \ldots$". To prove a statement like this, it suffices to choose a particular $\varepsilon > 0$ and show that the outcome described in $\neg(*)$ follows.

3. As a user. In a context where the statement "$x_n \to \widehat{x}$" is *given,* you can use it to produce other ingredients for use in logical constructions. In particular, if $(*)$ is known to be true, then you can use it by choosing any $\varepsilon > 0$ you like and feeding it into $(*)$ to generate a number $N \in \mathbb{N}$ with certain potentially useful features.

C. Completeness

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The fundamental property that makes $\mathbb{R}$ better than $\mathbb{Q}$ for analysis has three different-looking manifestations. Their contexts are (a) Cauchy sequences, (b) bounded sets, and (c) monotone sequences. Let's look at the setup for each of these.

(a) Cauchy Sequences.

A sequence (x_n) in $\mathbb{R}^k$ is called **Cauchy** iff

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : |x_m - x_n| < \varepsilon \quad \forall m, n \ge N. \tag{C1}$$

An equivalent condition is

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : |x_n - x_{n+p}| < \varepsilon \quad \forall n \ge N,\ p \in \mathbb{N}. \tag{C2}$$

[Home practice: prove that (C1)$\Leftrightarrow$(C2).]

Proposition. *Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.*

Proof. Fix any convergent sequence (x_n); let $\widehat{x} = \lim_{n\to\infty} x_n$. Estimate

$$|x_m - x_n| \le |x_m - \widehat{x}| + |\widehat{x} - x_n|.$$

Given $\varepsilon > 0$, define $\varepsilon' = \varepsilon/2$ and use it in the definition of convergence: this gives some $N' \in \mathbb{N}$ such that all $k > N'$ make $|x_k - \widehat{x}| < \varepsilon'$. Put $N = N' + 1$ and use it once with $k = m$ and again with $k = n$ to extend the above inequality:

$$m, n \ge N \implies |x_m - x_n| \le |x_m - \widehat{x}| + |\widehat{x} - x_n| < \varepsilon' + \varepsilon' = \varepsilon.$$

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Corollary. *Any sequence (x_n) that is not Cauchy must diverge.*

Proof. This is the contrapositive form of the statement above. ////

Example. Let $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. This sequence diverges.

Proof. Use form (C2) above. For any $n, p \in \mathbb{N}$,

$$\begin{aligned} x_{n+p} - x_n &= \left(1 + \frac{1}{2} + \ldots + \frac{1}{n+p}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+p} \\ &> \frac{p}{n+p}. \end{aligned}$$

Consider $\varepsilon = \frac{1}{2}$. For any $N \in \mathbb{N}$, and any $n > N$, take $p = n$ to get

$$x_{n+p} - x_n > \frac{n}{2n} = \tfrac{1}{2} = \varepsilon.$$

This proves $\neg$(C2), i.e.,

$$\exists \varepsilon > 0 \,:\, \forall N \in \mathbb{N}, \exists n \ge N, \exists p \in \mathbb{N} \,:\, |x_{n+p} - x_n| \ge \varepsilon.$$

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Proposition. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let (x_n) be a Cauchy sequence. Take $\varepsilon = 1$ and apply definition (C1) to get $N \in \mathbb{N}$ so large that

$$\forall m, n \ge N, \quad |x_m - x_n| < 1.$$

Now $|x_m| \le |x_m - x_n| + |x_n|$. Taking $n = N$, we deduce that

$$\forall m \in \mathbb{N}, \qquad |x_m| \le \max\left\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|, \right\}.$$

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Theorem (Metric Completeness). *Every Cauchy sequence in $\mathbb{R}$ converges.*

Proof. Later. ////

(b) Bounded Sets.

Theorem (Order Completeness). *Given any nonempty set $S \subseteq \mathbb{R}$, let*

$$\begin{aligned} A &= \{a \in \mathbb{R} \,:\, \forall s \in S,\ a \le s \quad\ \}, \\ B &= \{b \in \mathbb{R} \,:\, \forall s \in S, \qquad s \le b\}. \end{aligned}$$

Then ...

(a) *either $A = \emptyset$ or $A = (-\infty, \alpha]$ for some $\alpha \in \mathbb{R}$, and*

(b) *either $B = \emptyset$ or $B = [\beta, +\infty)$ for some $\beta \in \mathbb{R}$.*

Proof. Later. ////

Discussion. In the statement above, every $b \in B$ has $[b, +\infty) \subseteq B$. This implies the identity

$$B = \bigcup_{b \in B} [b, +\infty).$$

This is not much help in identifying B, since it appears on both sides, but at least it shows that B must be some kind of interval. The big news in the Theorem is that *the left endpoint of this interval must be included* (provided both $S \neq \emptyset$ and $B \neq \emptyset$).

Terminology. With the notation in the Theorem, we say S *is bounded above* when $B \neq \emptyset$; every $b \in B$ is called *an upper bound* for S. Likewise, we call S *bounded below* when $A \neq \emptyset$; every $a \in A$ is a *lower bound* for A. We call S *bounded* when both $A \neq \emptyset$ and $B \neq \emptyset$.

In the Theorem, α is called the greatest lower bound or infimum of S, written $\alpha = \inf(S)$; β is called the least upper bound or supremum of S, written $\beta = \sup(S)$. These are robust alternatives to the more fragile notions of "minimum" and "maximum" respectively. (Discussion.)

An equivalent and more often-seen characterization of $\beta = \sup(S)$ has 2 parts that give structure to typical proofs involving this concept:

(i) $\forall x \in S$, $x \le \beta$
(i.e., β is an upper bound for S); and

(ii) $\forall \gamma < \beta$, $\exists x \in S \,:\, x > \gamma$
(i.e., every γ less than β fails to be an upper bound for S—which makes β the *least* upper bound).

Example. $S = (0, 1)$ has $\inf(S) = 0$, $\sup(S) = 1$, whereas $\max(S)$ and $\min(S)$ do not exist. (Of course, if $\max(S)$ exists then it must equal $\sup(S)$, etc.)

(c) Monotonic Sequences.

Theorem (Monotone Convergence Property). *For any sequence* (x_n) *with*

$$x_1 \le x_2 \le x_3 \le \cdots,$$

exactly one of the following holds:

(a) $x_n \to +\infty$ *as* $n \to \infty$*;*

(b) x_n *converges, i.e.,* $\exists \widehat{x} \in \mathbb{R}$ *with* $x_n \to \widehat{x}$ *as* $n \to \infty$*.*

Theme. The main point in each of the big theorems stated above is the existence of a certain real number: in (a), it's the limit of a Cauchy sequence; in (b), it's the supremum of a nonempty set with an upper bound; in (c), it's the limit of a bounded nondecreasing sequence. All of these existence properties can fail in $\mathbb{Q}$, but they hold in $\mathbb{R}$, and they provide the foundation for all sorts of theorems that assert the existence of a real number with certain properties. (Examples include the Intermediate Value Theorem, Rolle's Theorem, the Mean Value Theorem, etc.)

Linkages. We continue to defer the hard work of constructing the real number system. But at this stage it's instructive to show that the information-content of the three important principles above is comparable.

Theorem. *Metric Completeness implies Order Completeness.*

Proof. Suppose we know that every Cauchy sequence converges. Let S be a nonempty set, and define $B = \{b \in \mathbb{R} : \forall x \in S,\ x \le b\}$, the set of upper bounds for S. Assuming $B \ne \emptyset$, we must show that $B = [\beta, +\infty)$ for some real number β.

Let

$$b_n = \min\left(B \cap \left\{\frac{k}{2^n} : k \in \mathbb{Z}\right\}\right), \qquad n \in \mathbb{N}.$$

(This makes sense.) Here are 3 key observations, valid for each n:

(i) By definition of the minimum, $b_n - \frac{1}{2^n} \notin B$, so there must be some $s_n \in S$ with $s_n > b_n - \frac{1}{2^n}$;

(ii) Defining b_{n+1} involves fractions $\frac{k}{2^{n+1}}$ that include all the ones competing for the minimum defining b_n (obtained when k is even). Thus it's certain that $b_n \ge b_{n+1}$.

(iii) Of course $b_{n+1} \in B$, so $b_{n+1} \ge s_n$ for the element mentioned in (i). With (ii), this gives

$$b_n \ge b_{n+1} \ge b_n - \frac{1}{2^n}, \quad \text{so} \quad 0 \le b_n - b_{n+1} \le \frac{1}{2^n}.$$

Claim: *The sequence* (b_n) *is Cauchy.*

Proof. Given $\varepsilon > 0$, choose N so large that $\frac{2}{2^N} < \varepsilon$. Then for any $n \geq N$ and $p \in \mathbb{N}$,

$$\begin{aligned}
|b_{n+p} - b_n| &= b_n - b_{n+p} && \text{(by (iii))} \\
&= (b_n - b_{n+1}) + (b_{n+1} - b_{n+2}) + \ldots + (b_{n+p-1} - b_{n+p}) \\
& && \text{(add-subtract trick)} \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+p}} \\
&\leq \frac{1}{2^n}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^p}\right] \\
&< \frac{2}{2^n} < \varepsilon.
\end{aligned}$$

Here's the big moment: by assumption, we can assert that $b_n \to \beta$ for some real β. It remains only to show that $B = [\beta, +\infty)$. This takes two steps, which correspond closely with the two-part characterization of $\sup(S)$ discussed in the text above.

Claim: $B \supseteq [\beta, +\infty)$.

Proof. It's enough to show $\beta \in B$. (Why? Oh.) Suppose not. If some $s \in S$ obeys $\beta < s$, then let $\varepsilon = s - \beta > 0$. Infinitely many b_n must have $b_n < \beta + \varepsilon = s$, but that's impossible: each b_n is an upper bound for S.

Claim: $B \subseteq [\beta, +\infty)$.

Proof. Let's prove this equivalent statement: $B^c \supseteq (-\infty, \beta)$. Pick any $\gamma < \beta$. Then $\beta - \gamma > 0$, so we can pick N so large that $\frac{1}{2^N} < \beta - \gamma$. For this N, recall point (i) above: some $s_N \in S$ obeys

$$s_N > b_N - \frac{1}{2^N} \geq \beta - \frac{1}{2^N} > \gamma.$$

So indeed $\gamma \notin B$. ////

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Theorem. *Order Completeness implies the Monotone Sequence Property.*

Proof. Let (x_n) be a nondecreasing sequence, i.e., assume

$$x_1 \leq x_2 \leq x_3 \leq \cdots.$$

If the set $\mathcal{R} = \{x_k : k \in \mathbb{N}\}$ (the $\mathcal{R}$ange of the sequence) is not bounded, then $x_n \to +\infty$ as $n \to \infty$. [Prove at home for practice.] We must show that the only alternative is that the sequence converges. So suppose the set $\mathcal{R}$ is bounded, and let

$$\beta = \sup(\mathcal{R})$$

We claim $x_n \to \beta$ as $n \to \infty$.

To prove this, let any $\varepsilon > 0$ be given. Then $\gamma = \beta - \varepsilon < \beta$, so there must be some element of $\mathcal{R}$, say x_N, with $x_N > \gamma$. This N works in the definition of the limit. Indeed, $x_n \leq \beta$ for every $n \in \mathbb{N}$, so (thanks to monotonicity) every $n > N$ satisfies

$$|x_n - \beta| = \beta - x_n \leq \beta - x_N < \beta - \gamma = \varepsilon.$$

////

Theorem. *The Monotone Sequence Property implies Metric Completeness.*

Proof. This is a nontrivial project. Here is one 3-step approach.

Step 1. *Every given sequence contains either a nondecreasing subsequence or a nonincreasing subsequence.*

Proof. Let (x_n) be given. Call an integer p a "peak point" for this sequence if $x_p \geq x_{p+k}$ for each $k \in \mathbb{N}$. If there are infinitely many peak points, list them in order:

$$p_1 < p_2 < \cdots .$$

Evidently $x_{p_1} \geq x_{p_2} \geq x_{p_3} \geq \ldots$, so (x_n) has a nonincreasing subsequence. If the number of peak points is finite, choose some n_1 larger than all of them. It's not a peak point, so there must be some $n_2 > n_1$ satisfying $x_{n_2} > x_{n_1}$. But then n_2 is not a peak point, so there must be some $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. This process can be continued forever, producing a subsequence with the property

$$x_{n_1} < x_{n_2} < x_{n_3} < \cdots .$$

////

Step 2. *Every Cauchy sequence is bounded.*

Proof. [Home practice.] ////

Now for the big moment: Let an arbitrary Cauchy sequence (x_n) be given. Use Step 1 to get a monotonic subsequence. Since the original Cauchy sequence is bounded (by Step 2), the subsequence is too. Therefore, by the Monotone Sequence Property (which we are assuming here), the subsequence converges. To be specific, denote the subsequence by $(x_{n_k})_{k\in\mathbb{N}}$ and let $\widehat{x}$ denote its limit.

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Step 3. *The original sequence $x_n \to \widehat{x}$.*

Proof. The key estimate, valid for any and all choices of n and k, looks like this:

$$|x_n - \widehat{x}| \leq |x_n - x_{n_k}| + |x_{n_k} - \widehat{x}|. \qquad (*)$$

To exploit this, let any $\varepsilon > 0$ be given. Invent $\varepsilon' = \frac{1}{2}\varepsilon > 0$ and use the Cauchy property of the original sequence to find some N so large that

$$\forall m, n \geq N,\ |x_m - x_n| < \varepsilon' = \tfrac{1}{2}\varepsilon.$$

This N works. To see why, fix any $n > N$ and use the convergence of the subsequence to find some K so large that

$$\forall k > K,\ |x_{n_k} - \widehat{x}| < \varepsilon' = \tfrac{1}{2}\varepsilon.$$

Use this line to get one specific $k^* > K$ such that $n_{k^*} \geq N$. Put that into line $(*)$ above (taking $m = n_{k^*}$):

$$|x_n - \widehat{x}| \leq |x_n - x_{n_{k^*}}| + |x_{n_{k^*}} - \widehat{x}| < \varepsilon' + \varepsilon' = \varepsilon.$$

////

Eventually we will prove one of the three forms of completeness discussed here, and immediately harvest this important consequence.

Theorem (Bolzano-Weierstrass). *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Recall Step 1 in the proof above. Any bounded sequence in $\mathbb{R}$ will have a monotone subsequence. That subsequence will inherit boundedness from the original sequence, and therefore the subsequence will converge by the "monotone sequence" version of the completeness property. ////

Hunting License. Students are now authorized to use the completeness property of $\mathbb{R}$ in any of the forms shown above to solve homework and test problems. Here's a textbook problem to show how that might go.

Example (Rudin exercise 3.3): Let $s_1 = \sqrt{2}$ and define

$$s_{n+1} = \sqrt{2+\sqrt{s_n}}, \qquad n = 1, 2, 3, \ldots.$$

(a) Prove that $s_n < 2$ for all $n \in \mathbb{N}$, and that (s_n) converges.

(b) Prove that the number $\widehat{s} \overset{\text{def}}{=} \lim_n s_n$ is algebraic.

Solution. (a) Use induction. For each n in $\mathbb{N}$, consider this statement about n:

$$P(n): \qquad s_n < 2 \quad \text{and} \quad s_n \leq s_{n+1}.$$

Then $P(1)$ says $s_1 < 2$ and $s_1 \leq s_2$. This is true because $s_1 = \sqrt{2} < 2$ and $s_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2+0} = s_1$. Next, assume that $n \in \mathbb{N}$ is an integer for which $P(n)$ is true. Then $s_n < 2$, so

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < \sqrt{2+2} = 2, \tag{$\dagger$}$$

and $s_{n+1} \geq s_n$, so

$$s_{n+2} = \sqrt{2+\sqrt{s_{n+1}}} \geq \sqrt{2+\sqrt{s_n}} = s_{n+1}. \tag{$\ddagger$}$$

By proving (†) and (‡), we have established $P(n+1)$. By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

We have shown that (s_n) is a nondecreasing sequence that is bounded above. Such a sequence must converge. Hence $s_n \to \widehat{s}$ for some number $\widehat{s} \in \left[\sqrt{2}, 2\right]$.

(b) Manipulating the definition leads to an identity valid for all n:

$$s_{n+1}^2 = 2 + \sqrt{s_n}, \qquad \text{i.e.,} \qquad \left(s_{n+1}^2 - 2\right)^2 = s_n.$$

Sending $n \to \infty$ here, and applying the limit laws, shows that $p(\widehat{s}) = 0$ for the polynomial

$$p(s) = \left(s^2 - 2\right)^2 - s = s^4 - 4s^2 - s + 4.$$

Since the coefficients in p are integers, $\widehat{s}$ is an algebraic number.

(Observation (not required for credit): the unique root of p in $(1, 2]$ can be found exactly by factoring $p(s) = (s-1)(s^3+s^2-3s-4)$ and applying the cubic formula, but it's messy. A good decimal approximation is 1.831177207. The polynomial p has three other roots: one at 1 and two with nonzero imaginary parts.)

D. Practice with Supremum and Infimum

Calculating with Infinity. Given a set $S \subseteq \mathbb{R}$, we considered the set of upper bounds,

$$B = \{b \in \mathbb{R} : \forall x \in S,\ x \leq b\}.$$

There are only 3 possibilities for the "shape" of B:

- $B = \emptyset$ when the given set S has no upper bound;
- $B = [\beta, +\infty)$ when the given set S is nonempty and bounded above; and
- $B = (-\infty, +\infty)$ when $S = \emptyset$.

In the middle scenario, we have defined $\beta = \sup(S)$. A common-sense extension to cover the other cases would be to say

$\sup S = +\infty \Leftrightarrow$ the set S has no upper bound,

$\sup S = -\infty \Leftrightarrow S = \emptyset$.

Of course we would also say

$\inf S = +\infty \Leftrightarrow S = \emptyset$,

$\inf S = -\infty \Leftrightarrow$ the set S has no lower bound,

This scheme gives a value to the supremum and infimum for any set $S \subseteq \mathbb{R}$. [Note that "$\inf S = -\sup(-S)$" holds in this extended interpretation, for any set S at all.] Let's do this, and also use the obvious extensions when S includes one or both of the "extended real numbers" $\pm\infty$. (See Rudin 1.23, p. 11.)

Upper and Lower Limits. Now let (x_n) be a real sequence. We define the **limit superior** (or **upper limit**) and **limit inferior** (or **lower limit**) as

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \left[\sup_{k\geq n} x_k\right], \qquad \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \left[\inf_{k\geq n} x_k\right].$$

These are well-defined, with values in $\mathbb{R} \cup \{\pm\infty\}$, for absolutely any given sequence.

Example. (a) $x_n = 1/n$, (b) $x_n = (-1)^n + 1/n$, (c) $x_n = n$.

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Proposition. *Let (x_n) be a real sequence.*

(a) $\liminf_n x_n \leq \limsup_n x_n$.

(b) *One has $x_n \to L$ if and only if* $\limsup_{n\to\infty} x_n = L = \liminf_{n\to\infty} x_n$.

Proof. For each $n \in \mathbb{N}$, define the tail set

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\},$$

and let

$$i_n = \inf T_n = \inf_{k\geq n} x_k, \quad s_n = \sup T_n = \sup_{k\geq n} x_k.$$

Observe that for each n,

$$i_n \leq s_n, \qquad i_n \leq i_{n+1}, \qquad s_n \geq s_{n+1}.$$

(a) For any pair $m, n \in \mathbb{N}$, pick any integer $N > \max\{m, n\}$ and combine the three observations above:

$$i_m \leq i_{m+1} \leq \cdots \leq i_N \leq s_N \leq \cdots \leq s_{n+1} \leq s_n.$$

This establishes the inequality $i_m \leq s_n$ for all $m, n \in \mathbb{N}$. For fixed n, this shows that s_n is *an* upper bound for the set $\{i_m : m \in \mathbb{N}\}$, so s_n must dominate the set's *least* upper bound (which we recognize):

$$s_n \geq \sup_m i_m = \liminf_m x_m.$$

Since this holds for each n, the value on the right is a lower bound for the set $\{s_n : n \in \mathbb{N}\}$. Therefore it cannot exceed that set's *greatest* lower bound (which we recognize):

$$\liminf_m x_m \leq \inf_n s_n = \limsup_n x_n.$$

(b) ($\Rightarrow$) Suppose $x_n \to L$, with $L \in \mathbb{R} \cup \{+\infty\}$. For any real $\gamma < L$, the definition provides some $N \in \mathbb{N}$ such that

$$\gamma < x_n, \qquad \forall n > N.$$

That is, γ is a lower bound for the set T_{N+1}, and this forces

$$\gamma \leq i_{N+1} \leq \sup_n i_n = \liminf_{n\to\infty} x_n.$$

Since this holds for all $\gamma < L$, we have $L \leq \liminf_{n\to\infty} x_n$; in conjunction with part (a), we have

$$L \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n, \qquad \text{for } L \in \mathbb{R} \cup \{+\infty\}. \tag{$\dagger$}$$

Similarly, suppose $x_n \to L$ with $L \in \mathbb{R} \cup \{-\infty\}$. For any real $\lambda > L$, the definition provides some $N \in \mathbb{N}$ such that

$$x_n < \lambda, \qquad \forall n > N.$$

This implies that λ is an upper bound for the set T_{N+1}, and therefore

$$\lambda \geq s_{N+1} \geq \inf_n s_n = \limsup_{n\to\infty} x_n.$$

Since this holds for all $\lambda > L$, we must have $\limsup_{n\to\infty} x_n \leq L$. Recalling part (a), we have

$$L \geq \limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n, \qquad \text{for } L \in \mathbb{R} \cup \{-\infty\}. \tag{‡}$$

Now consider the possibilities: if $L \in \mathbb{R}$, then both (†) and (‡) apply and the equations in (b) follow. If $L = -\infty$, then line (‡) is enough to confirm (b), and if $L = +\infty$, then (b) follows from (†).

($\Leftarrow$) Consider first the case where $L \in \mathbb{R} \cup \{+\infty\}$. Since $L = \sup_n i_n$, for any $\gamma < L$ there must exist some N_1 such that $i_{N_1} > \gamma$. In short

$$L > -\infty \implies \forall \gamma < L,\ \exists N_1 \in \mathbb{N} : \forall n \geq N_1, \quad x_n > \gamma. \tag{$*$}$$

On the other hand, suppose $L \in \mathbb{R} \cup \{-\infty\}$. Since $L = \inf_n s_n$, for any $\lambda > L$ there must exist some N_2 such that $s_{N_2} < \lambda$. Put succinctly,

$$L < +\infty \implies \forall \lambda > L,\ \exists N_2 \in \mathbb{N} : \forall n \geq N_2, \quad x_n < \lambda. \tag{$**$}$$

Now consider the possibilities: If $L = +\infty$, then line ($*$) shows that $x_n \to +\infty$, and if $L = -\infty$ then line ($**$) shows that $x_n \to -\infty$. If $L \in \mathbb{R}$, then both lines apply. Given any $\varepsilon > 0$, choose $\gamma = L - \varepsilon$ in ($*$) to get an integer N_1 and choose $\lambda = L + \varepsilon$ in ($**$) to get an integer N_2, and chain together the guaranteed inequalities:

$$\forall n > N \stackrel{\text{def}}{=} \max\{N_1, N_2\}, \qquad L - \varepsilon < x_n < L + \varepsilon.$$

This confirms the definition of $x_n \to L$. ////

Marginal Notes: Here are some words about important topics that students encountered through the homework or a guest lecture. Some are already used implicitly above.

Infinite Limits. For a sequence (x_n) in $\mathbb{R}$, to say $x_n \to +\infty$ or $\lim_n x_n = +\infty$ means $\liminf_n x_n = +\infty$. In elementary terms, this means

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n > M.$$

We say, "(x_n) diverges to $+\infty$". Similarly, (x_n) diverges to $-\infty$ (written as $x_n \to -\infty$ or as $\lim_n x_n = -\infty$) iff $-x_n \to \infty$, i.e.,

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n < M.$$

Note: the original phrase "(x_n) converges" *continues to mean "has a limit in $\mathbb{R}$".*

Subsequences. . . . allow another description of lim sup and lim inf. Please read Rudin's presentation in items 3.5, 3.15–3.20.

Definition. (Subsequence) Let $x\colon \mathbb{N} \to X$ be a sequence in some set X. A sequence $y\colon \mathbb{N} \to X$ is a **subsequence of** (x_n) if $y(k) = x(n(k))$ for some increasing function (sequence) $n\colon \mathbb{N} \to \mathbb{N}$, i.e.,

$$y_k = x_{n_k}, \quad \text{where} \quad 1 \le n_1 < n_2 < n_3 < \dots.$$

Example. Subsequences of $x_n = n$ include

$$(1, 3, 5, 7, \dots),\ (2, 4, 6, 8, \dots),\ (2, 3, 5, 8, 13, 21, \dots),\ \text{even}\ (x_n)\ \text{itself},$$

but **not** $(3, 1, 4, 1, 5, 9, \dots)$ [order permuted and elements re-used].

Proposition. *Let $(x_n)_n$ be a sequence in $\mathbb{R}$. TFAE:*

(a) *(x_n) converges.*

(b) *Every subsequence of (x_n) converges.*

Proof. (a⇒b) Obvious.

(b⇒a) [Contraposition] If (x_n) does not converge, let $n_k = k$: then the subsequence $(x_{n_k})_k = (x_k)_k$ also fails to converge. ////

Proposition. *Let (x_n) be a real sequence; define $\mu = \liminf_{n\to\infty} x_n$, $M = \limsup_{n\to\infty} x_n$.*

(a) *If $\ell = \lim_k x_{n_k}$ for some subsequence $(x_{n_k})_k$, then $\mu \le \ell \le M$.*

(b) *There exist subsequences $\left(x_{n_j}\right)_j$ and $(x_{n_k})_k$ obeying $x_{n_j} \to \mu$ and $x_{n_k} \to M$.*

(*Note*: *values $\pm\infty$ allowed for μ, ℓ, M throughout.*)

Proof. For each $n \in \mathbb{N}$, define

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}, \quad i_n = \inf T_n = \inf_{k\ge n} x_k, \quad s_n = \sup T_n = \sup_{k \ge n} x_k.$$

(a) For each $k \in \mathbb{N}$, $x_{n_k} \ge \inf\{x_j : j \ge n_k\} = i_{n_k}$. As $k \to \infty$, we have $x_{n_k} \to \ell$ and $i_{n_k} \to \mu$: hence $\ell \ge \mu$. Similarly, $x_{n_k} \le s_{n_k}$ $\forall k$, so $\ell \le M$.

(b) Let's build $(x_{n_k})_k$ with $x_{n_k} \xrightarrow[k]{} M$. (Similar arguments work for μ.)

Case 1: $M = +\infty$. Here $\inf_n s_n = +\infty$, so $s_n = +\infty$ for all n. Pick $n_1 = 1$. Now $s_1 = \infty$ means that $\{x_k : k \in \mathbb{N}\}$ has no upper bound. So pick $n_2 \in \mathbb{N}$ such that $x_{n_2} > 2$. Take care to choose $n_2 > n_1$. Similarly, for each $k \ge 2$, there must be some $n_k > \max\{n_1, \dots, n_{k-1}\}$ such that $x_{n_k} > k$. This produces the desired subsequence.

Case 2: $M \in \mathbb{R}$. Pick $n_1 = 1$. For each $k \ge 2$, define $r_k = M - 1/k$ and $R_k = M + 1/k$ and apply this two-step reasoning:

(i) Since $R_k > M = \inf_n s_n$, some N_k must obey $R_k > s_{N_k}$. In particular, $x_n < R_k$ for all $n \geq N_k$. Define $\widehat{N}_k = 1 + \max\{N_k, n_1, \ldots, n_{k-1}\}$.

(ii) Since $r_k < M = \inf_n s_n$, we certainly have $r_k < s_{1+\widehat{N}_k}$. Hence there exists $n_k > \widehat{N}_k$ such that $r_k < x_{n_k}$. Since $n_k > N_k$ also, we have

$$M - \frac{1}{k} = r_k < x_{n_k} < R_k = M + \frac{1}{k}.$$

This construction works for all k: it guarantees both $n_k < n_{k+1}$ for all k and $x_{n_k} \to M$.

Case 3: $M = -\infty$. [Exercise: Adapt Case 2 using $R_k = -k$, omitting part (ii).] ////

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There's a lot more to know about upper and lower limits, particularly about how they work with combinations of given sequences. Here is a particularly simple case, showing that upper and lower limits preserve elementwise inequalities; please read the textbook (including the exercises) for more.

Lemma. *Let (x_n) and (y_n) be real sequences. If the element-by-element inequality $x_n \leq y_n$ holds for all n sufficiently large, then also*

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \quad \text{and} \quad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

Proof. Extend the notation of earlier proofs to define

$$\begin{aligned}
T_n(x) &= \{x_n, x_{n+1}, x_{n+2}, \ldots\}, \quad i_n(x) = \inf T_n(x), \quad s_n(x) = \sup T_n(x),\\
T_n(y) &= \{y_n, y_{n+1}, y_{n+2}, \ldots\}, \quad i_n(y) = \inf T_n(y), \quad s_n(y) = \sup T_n(y).
\end{aligned}$$

The pointwise inequality implies that $i_n(x) \leq i_n(y)$ and $s_n(x) \leq s_n(y)$ for all n sufficiently large. Therefore, for all such n,

$$\begin{aligned}
i_n(x) \leq i_n(y) &\leq \sup_m i_m(y) = \liminf_{m\to\infty} y_m,\\
s_n(y) \geq s_n(x) &\geq \inf_m s_m(x) = \limsup_{m\to\infty} x_m.
\end{aligned}$$

The extended-valued sequences on the left are, respectively, nondecreasing and nonincreasing. So knowing these inequalities only for all n sufficiently large is enough to write them in reverse and reason a little further:

$$\begin{aligned}
\liminf_{m\to\infty} y_m &\geq \sup_n i_n(x) = \liminf_{n\to\infty} x_n,\\
\limsup_{m\to\infty} x_m &\leq \inf_n s_n(y) = \limsup_{n\to\infty} y_n.
\end{aligned}$$

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This gives a fresh pathway to a key result we have mentioned earlier.

Corollary (Bolzano-Weierstrass). *Every bounded real sequence has a convergent subsequence.*

Proof. If (x_n) is a bounded real sequence, there must be real numbers a and b for which $a \leq x_n \leq b$ for each n. Define constant sequences $a_n = a$ and $b_n = b$ and use the Lemma above to infer that $M = \limsup_{n\to\infty} x_n$ lies in the interval $[a, b]$. Then the previous proposition provides a subsequence of (x_n) that tends to M. Since M is not $-\infty$ or $+\infty$, that subsequence converges. ////

E. Helpful Hints for Homework

The Squeeze Theorem is a powerful tool for proving convergence results. The upper and lower limits from Section D give the same underlying idea immense power.

Many homework problems request a proof that $s_n \to L$ for some given sequence (s_n) and real number L. We know that $s_n \to L$ if and only if

$$L \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L. \tag{$*$}$$

(The middle inequality always holds.)

The chain of inequalities in $(*)$ is equivalent to a quantified logical statement in which the inequalities are weaker, hence easier to prove:

$$\forall \varepsilon > 0, \qquad L - \varepsilon \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L + \varepsilon. \tag{$\dagger$}$$

To establish this, it's helpful to remember that the upper and lower limits are always defined, and they preserve non-strict inequalities. Therefore one can prove $(\dagger)$ by inventing sequences (a_n) and (b_n) for which $a_n \to L$ and $b_n \to L$ and showing this:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \geq N, \qquad a_n - \varepsilon \leq s_n \leq b_n + \varepsilon. \tag{$\ddagger$}$$

Starting with $(\ddagger)$, the standard prose would go something like this:

> For fixed $\varepsilon > 0$, taking upper and lower limits in $(\ddagger)$ gives $(\dagger)$. Since this holds for every $\varepsilon > 0$, we must have $(*)$. This completes the proof.