PHYS 350 Homework 3

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Question 1a. Taking the Euler Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)=\frac{d}{dt}\left(\alpha q^2\dot{q}\right)=2\alpha q\dot{q}^2+\alpha q^2\ddot{q}=\frac{\partial \mathcal{L}}{\partial q}=\alpha q\dot{q}^2-2\beta q$$

$$\implies \ddot{q}=-\frac{\dot{q}^2}{q}-\frac{2\beta}{q}.$$

Question 1b. There is no time dependence, so energy is conserved. There is q dependence in the lagrangian, so there is no momentum conserved. Computing the energy:

$$E=\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}-\mathcal{L}=\alpha q^2\dot{q}^2-\frac{\alpha}{2}q^2\dot{q}^2+\beta q^2=\frac{\alpha}{2}q^2\dot{q}^2+\beta q^2.$$

Question 1c. From the initial conditions we have the energy is $E=\beta q_0^2$. Rearranging the energy equation we then have that:

$$\beta q_0^2=\frac{\alpha}{2}q^2\dot{q}^2+\beta q^2 \implies \dot{q}=\sqrt{\frac{2\beta\left(q_0^2-q^2\right)}{\alpha q^2}}=\frac{dq}{dt}$$

$$\implies \int_0^{T_c}dt=T_c=\int_{q_0}^{0}\sqrt{\frac{\alpha q^2}{2\beta\left(q_0^2-q^2\right)}}dq.$$

Let $p=q_0^2-q^2 \implies dp=-2qdq$. Also note that the sign could be either positive or negative when we took the square root, so to get a positive time we choose it appropriately. Then the integral becomes:

$$T_c=\sqrt{\frac{\alpha}{2\beta}}\int_0^{q_0^2}\frac{1}{2\sqrt{p}}dp=\sqrt{\frac{\alpha}{2\beta}}\sqrt{p}\Big|_0^{q_0^2}=\sqrt{\frac{\alpha}{2\beta}}q_0.$$

Note that we could have taken the negative sign from the square root to get $-T_c$ as a time when $q=0$. Since the effective potential energy of this Lagrangian behaves like q^2, this time then represents a fourth of the period of oscillation. Thus we have that the period of oscillation is $\sqrt{\frac{8\alpha}{\beta}}q_0$.

Question 2a. Based on the constraints of the system $s=1$. Let ϕ be the angle of m_2 around the z axis be the degree of freedom, while θ is the angle between the z axis and m_2. Then the kinetic terms of the lagrangian for m_2 are simple $\frac{m_2}{2}a^2\dot{\theta}^2\sin^2\phi+\frac{m_2}{2}a^2\dot{\phi}^2$, while the kinetic term for m_1 is $\frac{m_1}{2}\left(4a^2\dot{\phi}^2\sin^2\phi\right)$. The potential terms combined are then $U=-m_2ga\cos\phi-m_1ga\cos\phi$. Also note that $\dot{\theta}=\Omega$. This gives us the Lagrangian:

$$\mathcal{L}=\frac{m_2}{2}a^2\Omega^2\sin^2\phi+\frac{m_2}{2}a^2\dot{\phi}^2+2m_1\left(a^2\dot{\phi}^2\sin^2\phi\right)+m_2ga\cos\phi+m_1ga\cos\phi.$$

Question 2b. $\frac{\partial \mathcal{L}}{\partial \phi} \neq 0$ so there is no momentum conservation. However the time derivative is zero so energy is conserved, and is equal to:

$$E = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = m_2 a^2 \dot{\phi}^2 + 4 m_1 a^2 \dot{\phi}^2 \sin^2 \phi - \mathcal{L}$$

$$\implies E = \frac{m_2}{2} a^2 \dot{\phi}^2 + 2 m_1 \left(a^2 \dot{\phi}^2 \sin^2 \phi \right) - m_2 g a \cos \phi - m_1 g a \cos \phi - \frac{m_2}{2} a^2 \Omega^2 \sin^2 \phi.$$

Note that in this case $E \neq T + U$, which need not always be the case. This makes sense, as does the fact that there is no conserved momentum (since there's an external source driving the system).
Question 2c. If the motion is a circle then the angle ϕ does not change, i.e. $\dot{\phi} = 0$. Next compute the Euler Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{d}{dt} F(\phi, \dot{\phi}) = \frac{\partial \mathcal{L}}{\partial \phi} = m_2 a^2 \Omega^2 \sin \phi \cos \phi - m_2 g a \sin \phi - m_1 g a \sin \phi.$$

Note that every term of F has a $\dot{q}$ factor which goes to zero, so we get

$$m_2 a^2 \Omega^2 \sin \phi \cos \phi = m_2 g a \sin \phi + m_1 g a \sin \phi.$$