Math 320 Homework 4

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Question 1i. False, let $x_n = n+(-1)^n$. Then clearly $x_n \to \infty$ (for any M choose $N = M+1$, then for $n > N$ we have $x_n > n-1 = M$). However for any n that is even we have $x_n = n > n-1 = x_{n+1}$.

Question 1ii. The statement is true. By contradiction assume $x_n \to \infty$ with no increasing subsequence. Since no increasing subsequence exists, every increasing subset of x_n is of finite length, and choose $n_1, n_2, \ldots, n_K$ be a longest such increasing subsequence. Let $N = x_{n_K}$, then since $x_n \to \infty$ there exists N such that $(n > N) \implies (x_n > x_{n_K})$. Then let $n_{K+1} = \max(n_K, N) + 1$. Then $x_{n_{K+1}} > x_{n_K}$ with $n_{K+1} > n_K$, but this contradicts our assumption that the $n_1, \ldots, n_K$ were chosen to be maximal since adding $x_{n_{K+1}}$ would make a longer increasing subsequence. Thus an increasing subsequence of infinite length must exist.

Question 2a. The sequence converges. Note that we have:

$$a_n = n\frac{1+\frac{1}{n}-1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

I claim that $a_n \to \frac{1}{2}$. To see this let $\epsilon > 0$, and choose $N = \max\left(10, \frac{1}{\left(\frac{1}{\epsilon+1/2}-1\right)^2-1}\right)$. Then for $n > N$,

$$|a_n - \frac{1}{2}| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| < \epsilon.$$

Question 2b. The sequence does not converge. Let $L \in \mathbb{R}, \epsilon = \frac{1}{2}$, and $N > 0$. Choose n to be an arbitrary even integer greater than N if $L < 0$ and an odd integer greater than $\max(N, 3)$ otherwise. Then:

$$|b_n - L| = \left|\frac{(-1)^n n}{n+1} - L\right| = \left|\frac{n}{n+1}\right| + |L| > \frac{1}{2} + |L| \geq \frac{1}{2} = \epsilon.$$

Question 3a.

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