VII. CCC

A. Compactness

Definition. Given a HTS $(X, \mathcal{T})$, let $K \subseteq X$. To call K **compact** means that **for every** collection of open sets $\mathcal{G}$ such that

$$K \subseteq \bigcup \mathcal{G} = \bigcup_{G \in \mathcal{G}} G, \tag{1}$$

there exist $N \in \mathbb{N}$ and sets $G_1, \ldots, G_N$ in $\mathcal{G}$ such that

$$K \subseteq G_1 \cup \cdots \cup G_N.$$

A collection of open sets $\mathcal{G}$ obeying (1) is "an open cover" of K; calling K compact means, "**every** open cover has a finite subcover."

Remarks.

1. Compactness is the next best thing to finiteness. It's so valuable that when it is absent, we sometimes switch to a new topology in which compactness is present. This is why some abstract understanding of topologies is useful.
2. The quantifier **every** concerning open covers is essential. When compactness is present, we profit by choosing an open cover $\mathcal{G}$ whose open sets all share some desirable property, which is inherited by all the sets in the finite subcover. See Example (c) below, and the proofs to follow.

Lemma. *In any HTS, every finite set is compact.*

Proof. If $K = \{x_1, \ldots, x_N\}$ and $\mathcal{G}$ is an open cover for K, then each x_i in K must belong to some set G_i in $\mathcal{G}$. Thus $\{G_1, \ldots, G_N\} \subseteq \mathcal{G}$ is a finite subcover of K. ////

$\boxed{\text{08 Nov 2023}}$

Failure modes. Compactness is a special property, so non-compact sets should be more ordinary. A set S will fail to be compact exactly when there exists an open cover of S that fails to have a finite subcover.

Lemma. *In* $(\mathbb{R}, |\cdot|)$, *the set* $\mathbb{Z}$ *is not compact.*

Proof. Let $\mathcal{G} = \{(n-1, n+1) : n \in \mathbb{Z}\}$. Each element of $\mathcal{G}$ is an open set, and it's clear that

$$\bigcup \mathcal{G} = \bigcup_{n \in \mathbb{Z}} (n-1, n+1) \supseteq \mathbb{Z}.$$

But for any finite subset of $\mathcal{G}$, say $G_1, \ldots, G_N$, there is only one integer in each interval G_k, so

$$G_1 \cup G_2 \cup \cdots \cup G_N \not\supseteq \mathbb{Z}$$

because the left side only covers N points in the infinite set $\mathbb{Z}$. ////

Definition. In a metric space (X, d), a set $S \subseteq X$ is called bounded if there exist $x \in X$ and $R > 0$ such that $S \subseteq \mathbb{B}[x; R)$.

Prop. *Every compact subset of a metric space must be bounded.*

Proof. Let (X,d) be a metric space, with $K \subseteq X$ a compact set. Pick any x in X and define

$$\mathcal{G} = \{\mathbb{B}[x;n) \,:\, n \in \mathbb{N}\}\,.$$

Clearly $\bigcup \mathcal{G} = X \supseteq K$, so $\mathcal{G}$ is an open cover for K. Extract a finite subcover, $\mathbb{B}[x;n_1),\ldots,\mathbb{B}[x;n_N)$. Then let $R = \max\{n_1,n_2,\ldots,n_N\}$: this will show that K is bounded, because

$$K \subseteq \bigcup_{k=1}^{N} \mathbb{B}[x;n_k) = \mathbb{B}[x;R).$$

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Lemma. *In* $(\mathbb{R},|\cdot|)$, *the set* $S = \{1/n \,:\, n \in \mathbb{N}\}$ *is not compact, but the set* $\overline{S} = S \cup \{0\}$ *is compact.*

Proof. For each n, the ball $\mathbb{B}[1/n;1/(n+1)^2)$ is an open interval that contains exactly one element of the set S. The set S is infinite, so this infinite collection of intervals makes up an open cover. No finite subcover exists.

The story is different for $\overline{S}$. If $\mathcal{G}$ is an open cover for $\overline{S}$, then some open $G_0 \in \mathcal{G}$ contains 0. Since G_0 is open, there exists $\varepsilon > 0$ such that $\mathbb{B}[0;\varepsilon) \subseteq G_0$. Pick any integer $N > 1/\varepsilon$. Then every $n > N$ has $1/n < \varepsilon$, so $1/n \in G_0$. Each remaining $n = 1,2,\ldots,N$ must belong to some open set $G_n \in \mathcal{G}$. (That's what it means for $\mathcal{G}$ to be a cover of $\overline{S}$.) It follows that all points of $\overline{S}$ lie in the finite union

$$G_0 \cup G_1 \cup G_2 \cup \cdots \cup G_N.$$

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Remark. The proof above can be adapted to show that for any convergent sequence in any metric space, the closure of the range is a compact set. The idea that every compact set should contain all its limit points turns out to be correct in a rather general interpretation, as we show next.

Prop. *Every compact set in a HTS is closed.*

Proof. Let $(X,\mathcal{T})$ be a HTS, and let $K \subseteq X$ be compact. We will show that K^c is open. So, pick any $y \in K^c$. For each x in K, use (HTS4) to find open sets U_x, V_x such that

$$x \in U_x, \quad y \in V_x, \quad U_x \cap V_x = \emptyset.$$

Note that the third property here is equivalent to $V_x \subseteq U_x^c$. Now clearly

$$K \subseteq \bigcup_{x\in K} \{x\} \subseteq \bigcup_{x\in K} U_x.$$

So $\mathcal{G} = \{U_x : x \in K\}$ is an open cover of K. By compactness, $\mathcal{G}$ must admit a finite subcover. That is, there must be some $N \in \mathbb{N}$ and some $x_1, \ldots, x_N$ such that $K \subseteq \displaystyle\bigcup_{k=1}^{N} U_{x_k}$. Hence

$$K^c \supseteq \left[\bigcup_{k=1}^{N} U_{x_k}\right]^c \supseteq \bigcap_{k=1}^{N} U_{x_k}^c \supseteq \bigcap_{k=1}^{N} V_{x_k}.$$

Here $V \stackrel{\text{def}}{=} \bigcap_{k=1}^{N} V_{x_k}$ is open (it's a finite intersection of open sets) and $y \in V$ (since $y \in V_x\ \forall x \in K$), so $y \in V \subseteq K^c$. Thus $y \in \operatorname{int}(K^c)$. Since $y \in K^c$ is arbitrary, this shows that K^c is open, and K is closed. ////

Prop. *If F is closed, and $F \subseteq K$ for some compact K, then F is compact.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be an open cover of F. Since F is closed, F^c is open, so an open cover of the compact K is provided by

$$\{G_\alpha : \alpha \in A\} \cup \{F^c\}.$$

Extract a finite subcover. It will consist of $G_{\alpha_1}, \ldots, G_{\alpha_N}$ for some N and possibly F^c:

$$K \subseteq (F^c) \cup \bigcup_{k=1}^{N} G_{\alpha_k}.$$

Now since $F \subseteq K$ and $F \cap F^c = \emptyset$, the sets $G_{\alpha_1}, \ldots, G_{\alpha_N}$ are enough to cover F. These sets provide a finite subcover consisting of sets from the original $\mathcal{G}$. ////

$\boxed{\text{10 Nov 2023}}$

Corollary 1. *In any HTS, if K is compact and F is closed, then $F \cap K$ is compact.*

Proof. Since K is compact, it must be closed. Thus $F \cap K$ is closed. This makes $F \cap K$ a closed subset of a compact set (namely K), so $F \cap K$ is compact. ////

Corollary 2. *Let K be a compact set in some HTS. If $A \subseteq K$ is an infinite set, then $A' \neq \emptyset$.*

Proof. Let's do the contrapositive. That is, suppose $S \subseteq K$ is a set for which $S' = \emptyset$. From a known property of closed sets,

$$\overline{S} = S \cup S' = S.$$

Thus S is closed, and the result above implies that S is compact. For each x in S, use the property of not being a limit point to find an open set U_x for which $U_x \cap S = \{x\}$. Now $\mathcal{G} = \{U_x : x \in S\}$ is an open cover for S, so (by compactness) it must have a finite subcover. That is, there must be some finite set $\{x_1, x_2, \ldots, x_N\}$ such that

$$S \subseteq U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_N}.$$

But each open set on the right contains exactly one element of S, so this shows

$$S \subseteq \{x_1, x_2, \ldots, x_N\}.$$

That is, S is a finite set. ////

A Complementary Perspective. As we noted in class, the whole theory of topology could be built by starting with closed sets instead of open ones as the fundamental ingredients. Here's what the compactness setup would look like from that point of view.

Definition. A family of sets $\mathcal{F}$ has the **finite intersection property** if whenever $N \in \mathbb{N}$ and $F_1, \ldots, F_N$ are sets in $\mathcal{F}$, one has $\displaystyle\bigcap_{n=1}^{N} F_n \neq \emptyset$.

Theorem. *Given a HTS $(X, \mathcal{T})$ and a closed set $K \subseteq X$, TFAE:*

(a) *K is compact.*

(b) *Every collection of closed subsets of K with the finite intersection property has nonempty intersection.*

Proof. (Home Practice.) The result is equivalent to $\neg$(a) $\Leftrightarrow$ $\neg$(b), where

$\neg$(a) K is not compact, i.e., there exists an open cover of K that admits no finite subcover.

$\neg$(b) There exists some collection of closed subsets of K with the finite intersection property that has empty intersection.

$\neg$(b)$\Rightarrow$ $\neg$(a) Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a collection of closed subsets of K that has the FIP but $\bigcap \mathcal{F} = \emptyset$. For each $\alpha \in A$, let $G_\alpha = X \setminus F_\alpha$: each G_α is open, and

$$\bigcup_{\alpha \in A} G_\alpha = \left[\bigcap_{\alpha \in A} F_\alpha\right]^c = \emptyset^c = X.$$

Therefore $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ is an open cover for K. However, for any finite subset $G_{\alpha_1}, \ldots, G_{\alpha_N}$, we have

$$K \setminus \bigcup_{j=1}^{N} G_{\alpha_j} = K \cap \left[\bigcup_{j=1}^{N} F_{\alpha_j}^c\right]^c = K \cap \left[\bigcap_{j=1}^{N} F_{\alpha_j}\right].$$

Since $\mathcal{F}$ has the FIP, the right side is nonempty. So the finite subset just mentioned cannot cover K. But it's an arbitrary finite subset, so $\mathcal{G}$ admits no finite subcover.

$\neg$(a) $\Rightarrow$ $\neg$(b): Suppose K has an open cover $\mathcal{G} = \{G_\alpha : \alpha \in A\}$, admitting no finite subcover. We may assume $G_\alpha \cap K \neq \emptyset$ for each $\alpha \in A$. (Removing from $\mathcal{G}$ all sets containing no elements of K does not dilute the covering property, and produces a family which still has no finite subcover.) For each $\alpha \in A$, define $F_\alpha = K \setminus G_\alpha$; let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$. Then $\mathcal{F}$ is a family of closed subsets of K, and any finite collection indexed by $\alpha_1, \ldots, \alpha_N$ obeys

$$F_{\alpha_1} \cap \cdots \cap F_{\alpha_N} = \bigcap_{n=1}^{N} K \cap G_{\alpha_n}^c = K \cap \bigcap_{n=1}^{N} G_{\alpha_n}^c = K \cap \left[\bigcup_{n=1}^{N} G_{\alpha_n}\right]^c.$$

Since K has no finite subcover in $\mathcal{G}$, this set is nonempty. Thus $\mathcal{F}$ has the FIP. However, the intersection of all sets in $\mathcal{F}$ is

$$\bigcap_{\alpha\in A} F_\alpha = \bigcap_{\alpha\in A} K\cap G_\alpha^c = K\cap \bigcap_{\alpha\in A} G_\alpha^c = K\cap\left[\bigcup_{\alpha\in A} G_\alpha\right]^c = \emptyset,$$

since $\mathcal{G}$ covers K. This establishes ($\neg$b). ////

B. Convergence

In a metric space, compactness has a sequential characterization.

Theorem. *In a metric space (X, d), with $K \subseteq X$, TFAE:*

(a) *The set K is compact.*

(b) *Every sequence (x_n) in K has convergent subsequence, whose limit lies in K.*

Proof. (a$\Rightarrow$b) Let K be compact, and let (x_n) be any sequence of elements in K. Define $A = \{x_n : n \in \mathbb{N}\}$. If A is a finite set, then (x_n) has a constant subsequence, and (b) follows. If A is infinite, then we know $A' \neq \emptyset$, $A' \subseteq K$ from general HTS theory. So pick $z \in A'$. Since every open set containing z (e.g., $\mathbb{B}[z, 1/n)$) contains infinitely many points of A, it's easy to build a subsequence x_{n_k} such that $x_{n_k} \in \mathbb{B}[z, 1/k)$ for each k. This does the job.

(b$\Rightarrow$a) Let K be a set with the stated sequential property, and let $\mathcal{G}$ be an open cover for K. We'll select a finite subcover. For each $x \in K$, define

$$R(x) = \begin{cases} \sup\left\{\varepsilon > 0 \,:\, \mathbb{B}[x,\varepsilon) \subseteq G\ \exists G \in \mathcal{G}\right\}, & \text{if this set is bounded,}\\ 1, & \text{otherwise.}\end{cases}$$

Note: $R(x) > 0\ \forall x \in K$. Let $r(x) = \frac{1}{2}R(x)$. Then for any $x \in K$, $r(x) < R(x)$, so the definition of "sup" implies that some set $G \in \mathcal{G}$ contains $\mathbb{B}[x, r(x))$. Now proceed by induction:

Pick any x_1 in K; write $r_1 = r(x_1)$.
Pick any x_2 in $K \setminus \mathbb{B}[x_1, r_1)$; write $r_2 = r(x_2)$.
Pick any x_3 in $K \setminus [\mathbb{B}[x_1, r_1) \cup \mathbb{B}[x_2, r_2)]$; write $r_3 = r(x_3)$.
Continue.

Note: If $q > p$, then $x_q \notin \mathbb{B}[x_p; r_p)$, so $d(x_q, x_p) \geq r_p$.

Claim. The construction above cannot go on forever.

Pf. Seeking contradiction, suppose the process never stops, producing a sequence (x_n) in K.

By hypothesis (x_n) has a subsequence $(x_{n_k})_k$ converging to some point $\widehat{x}$ in K. Along this subsequence, $\boxed{r_{n_k} \to 0,}$ because (by the note above)

$$r_{n_k} \leq d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}}, \widehat{x}) + d(\widehat{x}, x_{n_k}) \to 0 \text{ as } k \to \infty. \qquad (*)$$

Now since $\widehat{x} \in K$, $\widehat{r} \stackrel{\text{def}}{=} r(\widehat{x}) > 0$; recall that some $\widehat{G} \in \mathcal{G}$ obeys $\mathbb{B}[\widehat{x}; \widehat{r}) \subseteq \widehat{G}$. Choosing $\varepsilon = \frac{1}{2}\widehat{r}$ in the definition of $x_{n_k} \to \widehat{x}$, we find that for all k sufficiently large,

$$d(x_{n_k}, \widehat{x}) < \frac{\widehat{r}}{2}, \quad \text{which implies} \quad \mathbb{B}\left[x_{n_k}, \frac{\widehat{r}}{2}\right) \subseteq \mathbb{B}[\widehat{x}, \widehat{r}) \subseteq \widehat{G}.$$

It follows that $R(x_{n_k}) \geq \widehat{r}/2$, so

$$\boxed{r_{n_k} \geq \widehat{r}/4 \text{ for all } k \text{ sufficiently large.}}$$

This contradicts "$r_{n_k} \to 0$", so the claim must hold.

Conclusion. The construction above can only break because, at some stage M, there are no points left to choose for x_{M+1}. That is, some M in $\mathbb{N}$ must satisfy

$$K \setminus [\mathbb{B}[x_1, r_1) \cup \cdots \cup \mathbb{B}[x_M, r_M)] = \emptyset. \tag{$**$}$$

For each j obeying $1 \leq j \leq M$, some set G_j from $\mathcal{G}$ obeys $G_j \supseteq \mathbb{B}[x_j; r_j)$. Therefore $(**)$ implies the required finite-subcover relation:

$$K \subseteq \bigcup_{j=1}^{M} G_j.$$

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Corollary (Heine-Borel Theorem). *In $\mathbb{R}^k$ (with the usual topology), a set is compact if and only if it is both closed and bounded.*

Proof. $(\Rightarrow)$ Shown earlier.

$(\Leftarrow)$ Let C be closed and bounded, and let (x_n) be a sequence in C. The scalar sequence $\widehat{\mathbf{e}}_1 \bullet x_n$ is bounded, so it has a convergent subsequence. Along that subsequence, consider $\widehat{\mathbf{e}}_2 \bullet x_n$: it's bounded, so it has a convergent subsequence. Repeat. After k steps of subsequence extraction, we have a subsequence in which every component converges. Hence the full vectors x_n converge (along that subsequence) in the metric topology of $\mathbb{R}^k$. Since C is closed, the limit must be a point in C. ////

Convergence with Caution. In the general context of a HTS $(X, \mathcal{T})$, there is a natural definition for convergence of a sequence: if $\widehat{x}$ and all x_n lie in X, we say $x_n \to \widehat{x}$ exactly when

$$\forall U \in \mathcal{N}(\widehat{x}),\ \exists N \in \mathbb{N} :\ \ \forall n > N,\ x_n \in U.$$

This is a faithful extension of the definition for convergence in a metric space, but it must be used with great care. Examples largely beyond the scope of MATH 320 will show that there are interesting HTS's in which some sets with property (b) are not compact. In ultimate generality, property (b) defines the phrase, "K is *sequentially compact.*" The distinction between sequential compactness and [ordinary topological] compactness is an advanced topic.

C. Completeness

17 Nov 2023

Definition. Let (X, d) be a metric space.

(a) A sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \,:\, \forall m, n > N,\ d(x_m, x_n) < \varepsilon.$$

(b) The space (X, d) is called **complete** if every Cauchy sequence in X converges [in X].

Lemma. *Every Cauchy sequence is bounded. That is, if (x_n) is Cauchy in (X, d), then there exists $R > 0$ and $z \in X$ such that*

$$\{x_n \,:\, n \in \mathbb{N}\} \subseteq \mathbb{B}[z; R].$$

Proof. (Similar to the story in $\mathbb{R}^k$ discussed earlier this term.) Challenge def'n of Cauchy with $\varepsilon = 1$, get $N \in \mathbb{N}$ such that $d(x_m, x_N) < 1$ whenever $m > N$. Then use $z = x_1$ and

$$R = 1 + \max\left\{d(x_j, x_1) \,:\, 1 \le j \le N + 1\right\}.$$

The inclusion $x_j \in \mathbb{B}[z; R]$ is obvious by construction if $j \le N + 1$, and follows from the triangle inequality if $j > N$:

$$d(x_j, x_1) \le d(x_j, x_{N+1}) + d(x_{N+1}, 1) < 1 + d(x_{N+1}, 1).$$

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Example. (a) $(\mathbb{R}^k, |\cdot|)$ is complete. $(\mathbb{Q}, |\cdot|)$ is not.

[Proof for $\mathbb{R}^k$ given above; any sequence of rational numbers converging in $\mathbb{R}$ to an irrational limit will be Cauchy, but not convergent, relative to $\mathbb{Q}$.]

(b) Every closed subset of a complete metric space is itself a complete metric space.

[Let X be the space and S the subset: any Cauchy sequence in S is also Cauchy in X, so it converges in X, to some $\widehat{x}$. But since S is closed, $\widehat{x} \in S$.]

Proposition. *On the set ℓ^∞ of all bounded real sequences $x = (x_1, x_2, \ldots)$, the following metric is complete:*

$$d\left(x, y\right) = \sup_n |y_n - x_n|.$$

Call each $x \in \ell^\infty$ a "vector".

Proof. To see that the function d defines a metric on ℓ^∞, read through the solution for HW05 Q8 with the input set $[0, 1]$ replace by the domain $\mathbb{N}$. Everything just works.

If $\left(x^{(n)}\right)_n$ is a Cauchy sequence of vectors in ℓ^∞, then

$$\forall \varepsilon' > 0,\ \exists N \in \mathbb{N} : \forall m, n \geq N,\ \sup_j \left| x_j^{(m)} - x_j^{(n)} \right| < \varepsilon'. \qquad (*)$$

This implies that for each $j \in \mathbb{N}$, the component sequence $\left(x_j^{(m)}\right)_m$ is Cauchy in $\mathbb{R}$, and hence converges to some real number $\widehat{x}_j$. We define $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \ldots)$ and propose to show $x^{(n)} \to \widehat{x}$ in ℓ^∞.

Claim. $\widehat{x} \in \ell^\infty$.

Proof. Consider $\mathbf{0} = (0, 0, 0, \ldots)$. Since every Cauchy sequence is bounded,

$$\exists M > 0 : \quad d(\mathbf{0}, x^{(n)}) \left(= \sup_j \left| x_j^{(n)} \right| \right) \leq M \qquad \forall n \in \mathbb{N}.$$

This implies that $\left| x_j^{(n)} \right| \leq M$ for all n and all j. It follows that, for each $j \in \mathbb{N}$,

$$|\widehat{x}_j| = \lim_{n \to \infty} \left| x_j^{(n)} \right| \leq M.$$

Hence $\sup_j |\widehat{x}_j| \leq M$, as required.

Claim. $x^{(n)} \to \widehat{x}$ in (ℓ^∞, d).

Proof. We seek to prove that

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n > N,\ \sup_j \left| x_j^{(n)} - \widehat{x}_j \right| < \varepsilon. \qquad (**)$$

So let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ so large that $(*)$ holds with $\varepsilon' = \varepsilon/2$. Then look at a specific component $j \in \mathbb{N}$. For any $m, n > N$, we have

$$\left| x_j^{(n)} - \widehat{x}_j \right| \leq \left| x_j^{(n)} - x_j^{(m)} \right| + \left| x_j^{(m)} - \widehat{x}_j \right| \leq \frac{\varepsilon}{2} + \left| x_j^{(m)} - \widehat{x}_j \right|.$$

While holding n fixed, send $m \to \infty$ on both sides: the result is

$$\left| x_j^{(n)} - \widehat{x}_j \right| \leq \frac{\varepsilon}{2} + 0 \qquad \forall n \geq N.$$

(Note that N depends only on ε: in particular N is independent of m.) Since this works for any component j, we have

$$d(x^{(n)}, \widehat{x}) = \sup_j \left| x_j^{(n)} - \widehat{x}_j \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

And this inequality holds for each $n > N$. This confirms $(**)$. ////

Example. *With the usual notation for unit vectors, the following subset of ℓ^∞ is closed and bounded, but not compact:*

$$S = \{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3, \dots\}.$$

Proof. For each n, let

$$S_n = \{\widehat{\mathbf{e}}_n, \widehat{\mathbf{e}}_{n+1}, \widehat{\mathbf{e}}_{n+2}, \dots\}.$$

Note that $S = S_1$. Clearly each S_n is bounded, since

$$d(\widehat{\mathbf{e}}_k, \mathbf{0}) = 1 \qquad \forall k \in \mathbb{N}.$$

Also, since $d(\widehat{\mathbf{e}}_k \widehat{\mathbf{e}}_j) = 1$ whenever $k \neq j$, we must have $S_n' = \emptyset \subseteq S_n$ for each n. Therefore each set S_n is closed. It's obvious that the family $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ has the finite intersection property, yet $\bigcap \mathcal{S} = \emptyset$. This implies that S_1 is not compact. ////

Theorem (Cantor's Intersection Theorem). *Let (X,d) be a metric space. TFAE:*

(a) *(X,d) is complete, i.e., every Cauchy sequence converges.*

(b) *For every nested sequence of nonempty closed sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ in X with $\operatorname{diam}(F_n) \to 0$, the infinite intersection*

$$F = \bigcap_{n \in \mathbb{N}} F_n$$

is a set containing exactly one point. Here we refer to "the diameter of a set," defined as

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\} \qquad \forall A \subseteq X.$$

Proof. (a⇒b) For each n, pick $x_n \in F_n$. Then (x_n) is a Cauchy sequence. Indeed, let $\varepsilon > 0$ be given and choose N so large that $\operatorname{diam}(F_N) < \varepsilon$. This can be done, because $\operatorname{diam}(F_n) \xrightarrow[n]{} 0$. Then whenever $m, n \geq N$, we have $x_m, x_n \in F_N$ and consequently $d(x_m, x_n) \leq \operatorname{diam}(F_N) < \varepsilon$. Since (X,d) is complete, the sequence (x_n) converges to some $\widehat{x}$ in X. Now for each $p \in \mathbb{N}$, we have $x_j \in F_p$ $\forall j \geq p$, so $\widehat{x} \in F_p$: hence $\widehat{x} \in F$. But for any point $y \neq \widehat{x}$, we have $r = d(y, \widehat{x}) > 0$, so $\operatorname{diam}(F_n) < r$ for all n sufficiently large. Since $\widehat{x} \in F_n$, it follows that $y \notin F_n$. In particular, $y \notin F$: this shows that

$$\{\widehat{x}\}^c \subseteq F^c, \quad \text{i.e.,} \quad \{\widehat{x}\} \supseteq F,$$

as required.

(b⇒a) Given a Cauchy sequence in X, define

$$F_n = \overline{\{x_n, x_{n+1}, x_{n+2}, \dots\}}.$$

Each F_n is closed and nonempty, and the nesting property is evident. To prove $\operatorname{diam}(F_n) \to 0$, let $\varepsilon > 0$ be given and use the Cauchy property to find $N \in \mathbb{N}$ so large that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$. This implies that $\operatorname{diam}(F_N) \leq \varepsilon$. (Detail: Any p in F_N can be realized as a subsequential limit of (x_n), so for any $p, q \in F_N$ there are subsequences (x_{n_i}), $\left(x_{n_j}\right)$ converging to p, q, respectively, such that

$$d(p,q) \leq d(p, x_{n_i}) + d(x_{n_i}, x_{n_j}) + d(x_{n_j}, q).$$

In the limit as $i, j \to \infty$ we have $d(p,q) \leq 0 + \varepsilon + 0$ from the Cauchy condition. Since $p, q \in F_N$ are arbitrary, the result follows.)

Let $\widehat{x}$ be the unique point in the intersection F: we will show that $x_n \to \widehat{x}$. Indeed, let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so large that $\operatorname{diam}(F_N) < \varepsilon$, and then consider any n with $n \geq N$. Since $\widehat{x} \in F_n$, we have $d(x_n, \widehat{x}) \leq \operatorname{diam}(F_N) < \varepsilon$, confirming the definition of the limit. (Compare Rudin's proof for Thm. 3.11(b), pp. 53–54.) ////

Discussion. It's interesting to compare and contrast Cantor's Intersection Theorem (a story about completeness) with the Finite Intersection Property (a formulation of compactness). The FIP can handle any collection of sets (not necessarily countable or nested in any simple way). Compactness implies completeness, as we now show.

Corollary (Rudin Thm. 3.11(b)). *Every compact metric space is complete.*

Proof. If X is compact, then every family $\mathcal{F}$ of closed sets for which any finite subfamily has nonempty intersection must have $\bigcap \mathcal{F} \neq \emptyset$. In particular, consider a family $\mathcal{F}$ of nested closed nonempty sets as described in part (b) of the previous theorem: it's clear that every finite subfamily has nonempty intersection, so, by compactness, $\cap \mathcal{F} = \cap_n F_n$ is nonempty. (Clearly this set cannot contain more than one point.) By Cantor's intersection theorem, it follows that X is complete.

[Direct proof: Let (K, d) be a compact metric space. Let $(x_n)_n$ be a Cauchy sequence in K. Then, by compactness, we have $x_{n_j} \xrightarrow[j]{} \widehat{x}$ for some subsequence $\left(x_{n_j}\right)_j$ and limit point $\widehat{x}$. To show that in fact $x_n \to \widehat{x}$ along the full original sequence, let $\varepsilon > 0$ be given. Use Cauchy (C1) to find N so large that $d(x_n, x_p) < \varepsilon/2 \ \forall n, p \geq N$. Then whenever $n \geq N$, choose j so large that both $n_j \geq N$ and $d(x_{n_j}, \widehat{x}) < \varepsilon/2$. This shows

$$d(x_n, \widehat{x}) \leq d(x_n, x_{n_j}) + d(x_{n_j}, \widehat{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall n \geq N,$$

as required.] ////