Math 220 Homework 9

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Question 1. f must be bijective. To show this we will use two facts we proved in class: $g \circ h$ injective implies g injective and $g \circ h$ surjective implies h surjective. Since $f \circ f$ is bijective it is both injective and surjective, so applying the first statement gives that f is injective and the second gives f is surjective. Thus f is bijective. $\square$

Question 2. First we will show that $f(f^{-1}(Y)) \subseteq Y$. Let $y \in f(f^{-1}(Y))$. Then $\exists x \in f^{-1}(Y)$ s.t. $f(x) = y \implies y \in Y$. Next we will show that $Y \subseteq f(f^{-1}(Y))$. Let $y \in Y$. Then since f is a surjection $\exists x \in A$ s.t. $f(x) = y \implies x \in f^{-1}(Y) \implies y \in f(f^{-1}(Y))$. Since each set is contained in the other they must be equal. $\square$

Question 3. For the first direction, suppose that f is surjective and we will show that $\forall A \in \mathcal{P}(E), F \setminus f(A) \subseteq f(E \setminus A)$. Let $A \in \mathcal{P}(E)$ and $y \in F \setminus f(A)$. Since $y \in F$ and f is surjective, $\exists x \in E$ s.t. $f(x) = y$. Note that $x \notin A$ since otherwise we would have $f(x) \in f(A) \not\subseteq F \setminus f(A)$. Since $x \notin A$ then we must have that $f(x) = y \in f(E \setminus A)$ and this direction is done.

For the other direction, assume that $\forall A \in \mathcal{P}(E), F \setminus f(A) \subseteq f(E \setminus A)$ and we will show f is surjective. Let $y \in F$ and $A = \{x \in E : f(x) \neq y\} \in \mathcal{P}(E)$. Then $F \setminus f(A) = \{y\} \cup G$ for some arbitrary set G by construction (in fact $G = \emptyset$ since f is surjective, but since we're in the middle of proving that we can leave it as an arbitrary set). Since by hypothesis $F \setminus f(A) = \{y\} \cup G \subseteq f(E \setminus A) \implies y \in f(E \setminus A)$, $\exists x \in E \setminus A$ such that $f(x) = y$. Since y was arbitrary this means f is injective and we're done with the second direction. $\square$

Question 4a. Let $z \in \mathbb{R}$. if $z \geq 0$ then let $x = \sqrt{z}$, otherwise let $y = \sqrt{z}$. Then $g(x, y) = x^2 - y^2 = \sqrt{z}^2 = z$ if $z \geq 0$ and $g(x, y) = x^2 - y^2 = -\sqrt{-z}^2 = z$ if $z < 0$. In either case $\exists x, y \in \mathbb{R}$ s.t. $g(x, y) = z$, so g is surjective. $\square$

Question 4b. $g^{-1}(\{0\})$ is the set of all x, y s.t. $g(x, y) = 0$, i.e. the solutions to the equation $x^2 - y^2 = 0$. This is equivalent to the following set:

$$g^{-1}(\{0\}) = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$$

Question 4c. In a similar vein to the previous part, $h^{-1}(\{c\})$ is the set of x that are a solution to the equation $f(x) = x^4 + 3 = c$. This is the following set:

$$h^{-1}(\{c\}) = \{x \in \mathbb{R} : x^4 = c - 3\} = \begin{cases} \emptyset & \text{if } c < 3 \\ \{\sqrt[4]{3 - c}\} & \text{if } c \geq 3 \end{cases}$$