MATH 305 Homework 9

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1. (20) Compute the Laurent series for

(a) $\frac{1}{z(z+2)}, 1 < |z-1| < 3$

Partial fractions:

$$\frac{1}{z(z+2)} = \frac{1}{2z} - \frac{1}{2(z+2)} = \frac{1}{2}\left(\frac{1}{(z-1)}\frac{1}{1+1/(z-1)} - \frac{1}{3}\frac{1}{1+(z-1)/3}\right).$$

$$= \frac{1}{2(z-1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n} - \frac{1}{3}\sum_{n=0}^{\infty}\left((-1)\frac{z-1}{3}\right)^n = -\frac{1}{6} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2(z-1)^n} - \frac{(z-1)^n}{2(-3)^{n+1}}.$$

(b) $\frac{1}{z^2+4}, |z-2i| > 4$

$$\frac{1}{z^2+4} = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{(z-2i)\left(1+4i/(z-2i)\right)}\right).$$

$$= \frac{1}{4i(z-2i)} + \frac{1}{4i}\sum_{n=0}^{\infty}\frac{(-4i)^n}{(z-2i)^{n+1}} = (\frac{1}{4i}-1)\frac{1}{z-2i} + \sum_{n=1}^{\infty}\frac{(-1)^n(4i)^{n-1}}{(z-2i)^{n+1}}.$$

2. (20) Determine the types of all the isolated singularities of the following functions and compute the residue at each isolated singularity

(a) $\frac{z}{\tan z}$

$$\frac{z}{\tan z} = \frac{z\cos z}{\sin z}.$$

This function has singularities for $z = n\pi$. For $z = 0$, this is a removable singularity since $\lim_{z\to 0}\frac{z\cos z}{\sin z} = 1$, so $Res[\frac{z}{\tan z}; 0] = 0$. For $z = n\pi, n \neq 0$ the function has simple poles, which gives:

$$Res[\frac{z}{\tan z}; n\pi] = \frac{n\pi}{\sec^2 nz} = n\pi.$$

(b) $\frac{\cos z}{z^3}$

The only singularity is $z = 0$, which is a pole of order 3 (since $\cos(z) \neq 0$, which gives that

$$Res[\frac{\cos z}{z^3}; 0] = \frac{1}{2}\frac{d^2}{dz^2}\cos z = -\frac{1}{2}.$$

(c) $\frac{\text{Log}(z)}{(z^2+1)^2}$

The two isolated singularities are at $z = \pm i$, which are simple poles of order 2. Since $\text{Log}(\pm i) \neq 0$, we get that the residue is

$$Res[\frac{\text{Log}(z)}{(z^2+1)^2}; \pm i] = \frac{d}{dz}\left(\frac{\text{Log}(z)}{(z\pm i)^2}\right) = \left.\frac{\frac{1}{z}(z\pm i)^2 - 2(z\pm i)\text{Log}(z)}{(z\pm i)^4}\right|_{z=\pm i}.$$

$$= \frac{\mp i(-2i)^2 \pm 4i(\mp \frac{\pi}{2} i)}{16} = \pm \frac{1}{4} i + \frac{\pi}{8}.$$

(d) $\frac{e^z}{1-\sqrt{z}}$

Since we're using the principle branch the only pole is that $z = 1$. Consider the function as follows:

$$\frac{e^z}{1-\sqrt{z}} = \frac{e^z(1+\sqrt{z})}{1-z}.$$

Then $z = 1$ is clearly a simple pole, so the residue is

$$Res[\frac{e^z(1+\sqrt{z})}{1-z}; 1] = -2e.$$

3. (20) Evaluate the following integrals by Cauchy residue Theorem

(a) $\int_{|z|=3} \frac{e^z}{(z-1)^2 z^3}$

Calculating residue:

$$Res[f(z); 1] = \frac{d}{dx}\left(\frac{e^z}{z^3}\right) = \left.\frac{z^3 e^z - 3z^2 e^z}{z^6}\right|_{z=1} = -2e.$$

$$Res[f(z); 0] = 2\frac{d^2}{dx^2}\left(\frac{e^z}{(z-1)^2}\right) = \frac{1}{2}\frac{d}{dz}\left(\frac{(z-1)^2 e^z - 2(z-1)e^z}{(z-1)^4}\right).$$

$$= \frac{1}{2}\left(\frac{\left(2(z-1)e^z + (z-1)^2 e^z - 2e^z - 2(z-1)e^z\right)(z-1)^4 - \left((z-1)^2 e^z - 2(z-1)e^z\right)4(z-1)^3}{(z-1)^8}\right)\Bigg|_{z=0}.$$

$$= \frac{1}{2}\left((-2+1-2+2) + 4(1+2)\right) = \frac{11}{2}.$$

$$\implies \int_{|z|=3} \frac{e^z}{(z-1)^2 z^3} = (\frac{11}{2} - 2e)2\pi i.$$

(b) $\int_{|z|=1} \frac{1}{z^2 \sin z} dz$

The one singularity is $z = 0$.

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3}\frac{1}{1 - z^2/6 + O(z^4)} = \frac{1}{z^3}\left(1 + \left(\frac{z^2}{6} - O(z^4)\right) + \dots\right) \implies Res[\frac{1}{z^2 \sin z}; 0] = \frac{1}{6}.$$

$$\implies \int_{|z|=1} \frac{1}{z^2 \sin z} dz = \frac{\pi i}{3}.$$

(c) $\int_{|z|=1} e^{\frac{1}{z}} \cos(z) dz$

Calculating residue:

$$e^{\frac{1}{z}} \cos(z) = \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots\right)\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right) \implies Res[e^{\frac{1}{z}} \cos(z); 0] = \sum_{n=1} \frac{1}{n((n-1)!)^2}.$$

$$\implies \int_{|z|=1} e^{\frac{1}{z}} \cos(z) dz = 2\pi i \sum_{n=1} \frac{(-1)^{n+1}}{n((n-1)!)^2}.$$

(d) $\int_{|z|=1} \frac{e^z}{\sin^3 z} dz$

Computing residue:

$$\frac{e^z}{\sin^3 z} = \frac{1+z+z/2}{\left(z-z^3/6+\ldots\right)^3} = \frac{1}{z^3}\left(1+z+z^2/2\right)\left(1-3\left(1-z^2/6+\ldots\right)+\ldots\right).$$

$$\Longrightarrow Res[\frac{e^z}{\sin^3 z};0] = \frac{1}{2}+\frac{3}{3!} = 1 \Longrightarrow \int_{|z|=1} \frac{e^z}{\sin^3 z}dz = 2\pi i.$$

4. Computing the following integrals
   (a) $\int_0^\pi \frac{1}{1+\sin^2\theta}d\theta$
   Let $z = e^{i\theta} \Longrightarrow dz = ie^{i\theta}d\theta$.

$$\int_{|z|=1} \frac{iz}{1+\left(z-z^{-1}\right)^2/4}dz = \int_{|z|=1} \frac{4iz}{2+z^2+z^{-2}}dz.$$

   (b) $\int_0^{2\pi} \frac{\sin^2\theta}{3+\cos\theta}d\theta$

5. (30) Using contour integrals to compute the following integrals
   (a) $\int_0^\infty \frac{x^2}{(x^2+4)^2}dx$, (b) $\int_0^\infty \frac{1}{x^4+x^2+1}dx$, (c) $\int_0^\infty \frac{1}{x^3+1}dx$.
   (d) $\int_0^\infty \frac{\cos x}{x^4+1}dx$, (e) $\int_{-\infty}^\infty \frac{\sin x}{x^2+2x+2}dx$