Homework 3, Math 443

Due Wednesday, February 8, 2023

Show full work and justifications for all questions, unless instructed otherwise. You are expected to come up with answers without looking them up on the internet or elsewhere.

1. (6 points) Let G be a graph with $\delta(G) + \Delta(G) \geq |G| - 1$. Show that G is connected, with $diam(G) \leq 4$.

2. (6 points) Prove, or provide a counterexample: Every nontrivial tree with maximum degree k has at least k leaves.

3. (5 points) Find all graphs G such that G and $\overline{G}$ are both forests. Remember to give a complete justification.

4. (5 points) In Theorem 4.9, we showed that for any tree T on n vertices, every graph G with $\delta(G) \geq n-1$ contains T as a subgraph. Show that the bound $n - 1$ is best possible by providing, for all natural n, a tree on n vertices and a graph with minimum degree $n-2$ that does not contain the tree as a subgraph.

5. (6 points) Let H be a graph with $\Delta(H) = k$. Let G be a graph on at least $|H|$ vertices with the property that, for every collection X of k vertices in $V(G)$, $\left|\bigcap_{x \in X} N(x)\right| \geq |H| - 1$.

   Prove, or provide a counterexample: G has a subgraph isomorphic to H.

6. (a) (2 points) Draw a connected weighted graph where every MST formed with Kruskal's algorithm *includes* the unique edge of highest weight, and *excludes* an edge of lowest weight.

Graphs like the one you found in part (a) can feel maddening: including a high-weight edge, while excluding a low-weight edge.

(b) (2 points) Give an example of a graph G with weight function $w \colon E(G) \to \mathbb{Z}$, and a spanning connected subgraph H of G, such that $w(H) < w(T)$ for **every** spanning tree T. Or, show that none exists.

(c) (2 points) Give an example of a graph G with weight function $w \colon E(G) \to \mathbb{N}$, and a spanning connected subgraph H of G, such that $w(H) < w(T)$ for **every** spanning tree T. Or, show that none exists.

7. (6 points) Let G be a connected nontrivial graph with edge-weight function w, and let T be any minimum spanning tree of G. Using an idea similar to our proof of Kruskal's algorithm, prove the following:

   For every $x \in V(G)$, there exists an edge e incident to x in T s.t. $w(e) \leq w(f)$ for every f incident to x in G.

Question:	1	2	3	4	5	6	7	Total
Points:	6	6	5	5	6	6	6	40