Math 437 Homework 1

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Question 1. Let $m \in \mathbb{N}$, and let $s_i = \frac{i}{\sqrt{i+1}}$. s_i is clearly unbounded since $\frac{i}{\sqrt{i+1}} = \frac{i+1}{\sqrt{i+1}} - \frac{1}{\sqrt{i+1}} \geq \sqrt{i+1} - 1$ is unbounded. Also note that $\frac{s_{i+1}}{s_i} = \frac{(i+1)\sqrt{i+1}}{i\sqrt{i+2}} \geq \frac{i+1}{i} > 1$, so s_i is strictly increasing. Thus there exists a smallest number $n \in \mathbb{N}$ with $m \leq s_n$.

I claim that $m = \left[\frac{n}{\sqrt{n+1}}\right]$. To see why suppose by contradiction not, since $m \leq \frac{n}{\sqrt{n+1}}$ this would mean $\frac{n}{\sqrt{n+1}} \geq m+1$. But then $s_{n-1} = \frac{n-1}{\sqrt{n}} \geq \frac{n-\sqrt{n+1}}{\sqrt{n+1}} \geq m$, which contradicts the assumption that n was the smallest possible. Thus it must be that $m = \left[\frac{n}{\sqrt{n+1}}\right]$.

Question 2. Let A be the union of finitely many arithmetic progressions with coefficients $(a_1, b_1), \ldots (a_N, b_N)$ respectively. Let $m = \operatorname{lcm}(a_1, a_2, \ldots, a_N)$ and $B = \max(b_1, b_2, \ldots b_N)$. Then for each $k \in \{B, B+1, \ldots B+m-1\}$, if $k \in S$ then define an arithmetic progression with $a'_k = m, b'_k = k$. Let S' be the union of all arithmetic progressions generated this way, along with the finite set $\{n \in S : n < B\}$.

I claim $S = S'$. Let $c \in S$, if $c < B$ then by construction $c \in S'$. Otherwise, by the division algorithm let $c - B = mn + r, n \geq 0, 0 \leq r < m$. $B + r \in S$, since if it wasn't then the progression (a_i, b_i) with $B + r = a_i n' + b_i$ would also include c since $c = (B+r) + mn$ and $a_i | m$. Thus $S \subseteq S'$.

For the other direction, let $d \in S'$. Again if $d < B$ then by construction $d \in S$ automatically. Otherwise, let a_i, b_i be two coefficients used to generate S originally, and since $d \in S'$ it can be written as $d = mn + B + r$ for some $r \in \{0, 1, \ldots, m-1\}$. Since there exists no $n \geq 0$ with $a_i n + b_i = B + r$ by the definition of S' and $a_i | m$, there also clearly can't be a solution to $d = a_i n + b_i$. Thus $d \in S$, so $S' \subseteq S$. Since both sets contain one another, $S = S'$ as required.

Question 3.