Math 443 Homework 5

Xander Naumenko

01/03/23

Question 1. Let k be a positive integer, and let A_1, A_2 be two copies of K_{k+1} (the example also works for K_k, but it's easier to show minimality this way). Let G_k be created by taking a new vertex, v, and connecting it to k vertices in each of A_1, A_2. Clearly $\kappa(G_k) = 1$ since if v is removed G_k gets separated into A_1, A_2 as components. To see that $\lambda(G_k) \leq k$, note that by removing all edges between v and A_1 results in a disconnect graph, which is an edge set of size k.

To see why this is a minimum edge cut, let $E \subset E(G_k)$ s.t. $|E| < k$ (since E is an edge set the $||$ syntax denotes size of set, not number of vertices). Then E couldn't have removed all the edges between A_1 and v, since there are k edges between them. By symmetry the same applies for A_2 and v. $A_1 - E$ and $A_2 - E$ are still connected since they are both copies of K_{k+1}. Since A_1 and A_2 are internally connected and v is still connected to both after the removal of E, the whole graph is still connected and $\lambda(G_k) \geq k$. Thus $\lambda(G_k) = k$ and $\kappa(G_k) = 1$ as required.

Question 2a. The statement is true. Let E be a separating edge set of G, and let A be a smallest resulting component of $G - E$. Clearly $|A| \leq \frac{n}{2}$ since it the smaller of at least two components whose total vertices is n. Note that $||A|| \leq K_{|A|} = |A|(|A| - 1)$. Also note that the total number of edges of the vertices of A in G is $|A| \cdot \delta(G)$. The difference between these numbers is at least the number of vertices taken away by E, i.e. $|E| \geq |A|\delta(G) - |A|(|A| - 1) = |A| \left(\delta(G) - |A| + 1\right)$. This is a downward parabola in $|A|$, so from calculus its minimum must lie on one of the two endpoints, i.e. $|A| = 1$ or $|A| = \frac{n}{2}$. These two values are:

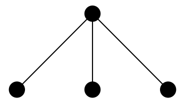
$$|E| \geq \delta(G) - 1 + 1 = \delta(G).$$

$$|E| \geq \frac{n}{2}(\delta(G) - \frac{n}{2} + 1) \geq \delta(G).$$

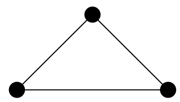
We conclude that $\lambda(G) \geq \delta(G)$. In class we also proved that $\lambda(G) \leq \delta(G)$, so these two inequalities together tell us that $\lambda(G) = \delta(G)$ as required. $\square$

Question 2b. The statement is false. As a counterexample let $k \geq 3$ and consider two copies of K_k, A_1, A_2. Form G by adding two vertices v_1, v_2 connected to all vertices in A_1, A_2. Each vertex in A_1, A_2 has $k - 1$ neighbors from the complete graph as well as v_1, v_2 for a total of $k - 1 + 2 = k + 1$. v_1, v_2 each have $2k$ neighbors since their connected to each vertex in A_1, A_2. The total vertices is $2k + 2$, so $\delta(G) = k + 1 \geq \frac{|G|}{2}$. We can disconnect the graph by removing v_1, v_2, so $\kappa = 2$ (it clearly can't be less than that). Let E be an edge set with $|E| \leq 2$. A_1 remain connected in $G - E$ since it is $k \geq 3$ complete, and there is at least one edge from A_1 to v_1 and v_2 in $G - E$, since there were at least 3 edges before and E removed at most 2. By symmetry the same is true for A_2, so all 4 of A_1, A_2, v_1, v_2 are internally connected and connected together. Thus E is not an edge cut, so $\lambda(G) > 2$ and $\lambda(G) \neq \kappa(G)$.

Question 3. The statement is not true. Consider the following graph G:



The three blocks are three copies of K_2, and they are each connected to each other since they each share a vertex, so this is $\mathcal{G}$:



This obviously isn't a tree since it contains a cycle, so we're done and the statement is false.

Question 4a. The statement is true. Let $v \in G[\mathcal{E}]$. Let $x, y \in G[\mathcal{E}] - v$. Let C be a cycle in $G[\mathcal{E}]$ containing x, y (this exists since you can choose any two edges attached to x, y and they're guaranteed to share a cycle). Let P be a path along C that doesn't include v, since there are two options (each way around C) this will always exist. Then x, y are connected in $G[\mathcal{E}] - v$ through P. This works for any v, so there are no cut vertices in $G[\mathcal{E}]$ so it is nonseparable. $\square$

Question 4b. The statement is true. By was of contradiction suppose it wasn't true, i.e. suppose that $\exists e = xy \in E(G)$ s.t. $x, y \in V(G[\mathcal{E}])$ and $e \notin \mathcal{E}$. Let e_x, e_y be two edges in $\mathcal{E}$ with one endpoint on x, y respectively. Since they are related by R there exists a cycle C in $G[\mathcal{E}]$ that contains both of them. If $e \in C$ then it would be in $\mathcal{E}$, a contradiction, so assume it isn't. C passes through e_x and e_y, so split it into two component paths, both starting at x and ending at y and merge one of them with $e = xy$. The resulting cycle contains e and other edges of $\mathcal{E}$, so $e \in \mathcal{E}$. However this contradicts our assumption it wasn't, so $G[\mathcal{E}]$ must have been an induced subgraph of G. $\square$

Question 4c. The statement is true. We will use proof by contradiction. From part a, $G[\mathcal{E}]$ is nonseparable, so the only way $G[\mathcal{E}]$ could not be a block is if it isn't maximal. By way of contradiction suppose that $\exists B \subset G, v \in G, v \notin G[\mathcal{E}]$ s.t. B is nonseparable and $V(G[\mathcal{E}]) \subset V(B), v \in B,$. It is asserted that there exists paths P_1, P_2 from v to $G[\mathcal{E}]$ in B with the endpoints of $u_1, u_2 \in G[\mathcal{E}], u_1 \neq u_2$. At least one path P_1 must exist for v and $G[\mathcal{E}]$ to be connected, and since removing u_1 shouldn't disconnect $G[\mathcal{E}] + v$ a second path must exist with a different endpoint in $G[\mathcal{E}]$. Choose P_1, P_2 in such a way that u_1, u_2 are their only member in $G[\mathcal{E}]$. Consider the first common ancestor between P_1 and P_2 starting from u_1, u_2, call it x. $G[\mathcal{E}]$ is connected so there exists a path in it between u_1 and u_2, call it P_3. Then $u_1 P_3 u_2 P_1 x P_2 u_1$ is a cycle in G containing an edges not in $G[\mathcal{E}]$, namely the edges of P_1, P_2 before x. This is impossible since we assumed $G[\mathcal{E}]$ was formed by the entire equivalence class, so it must be that $G[\mathcal{E}]$ is a block of G. $\square$

Question 4d. The statement is true. Suppose by contradiction that it wasn't, i.e. suppose there exists a block B of G and $e_1 = xy, e_2 = uv \in E(G)$ s.t. e_1 and e_2 share no common cycle. Additionally choose e_1, e_2 such that the distance between x, u is minimal. It can't be that e_1 and e_2 are adjacent, since otherwise you could delete their shared vertex, find a path between the remaining two (guaranteed by B's nonseparability) and you combine this path with $e_1 e_2$ to make a cycle. Therefore there exists a vertex z such that $xz \in E(B)$ and xz is in the minimal path between x and u, $z \neq x, y, u, v$. z is strictly closer to u than x, so by the minimal choice of e_1, e_2, xz and $e_2 = uv$ share a common cycle, call it C. Define P_1, P_2 to be the two ways of going from x to v on C (one of them contain both uv and xz).

Next by the nonseparability of B, there exists a path P from y to v s.t. $x \notin P$. If P doesn't intersect C, then assume $u \in P_1$ and consider the path $xyPvP_1$, this forms a cycle with xy and uv since $uv \in P_1$ which contradicts our assumption that e_1, e_2 don't share any.

If instead P intersects C, WLOG assume it intersects P_1 (in this specific case we're not assuming $v \in P_1$ anymore, so P_1, P_2 are arbitrary). Define the intersection vertex w to be the shared vertex between P and P_1 closest along P to y, and define $P_1' \subset P_1$ to be the path from w to v and $P' \subset P$ to be the path from y to w. Note that $||P_1|| \geq 1$ since $w \neq v$ (although it might be u), so if $u \in P_1$ then $u \in P_1'$ since u is adjacent to v in C. Then the we can form a cycle $xyP'wP_1'vP_2x$. Since one of P_1', P_2 contain u by C's definition, this cycle we've constructed has both xy and uv as edges which contradicts our assumption that there was no such common cycle. Thus are assumption must have been wrong and $E(B)$ is an equivalence class of G. □

Question 5. Let v be a cut vertex of G. Since it is a cut vertex $d(v) \geq 2$. Let two of its neighbors be x and y, and by v being a cut vertex, choose x, y such that $G - v$ separates x and y into difference components. Therefore there exist no $x - y$ paths that don't contain v. It is claimed that vx and vy are part of separated groups, so v is part of both.

To see why, suppose by contradiction that they weren't. Then vx and vy are part of one group, call it B. $B - v \subset G - v$ contains no xy paths, so it must be disconnected. However this means that B is separable which contradicts our assumption that B was a block, so it must be that vx and vy are in separate blocks. Therefore v is part of both of these blocks as required. □

Question 6.

($\Rightarrow$) Assume G is nonseparable. This means that there is one block of G, $B = G$. By question 4d, $E(G)$ is an equivalence class of R, where R is defined as in question 4. Thus for all $e, f \in E(G$, e and f are on a common cycle, which is a stronger statement than only adjacent edges lie on a common cycle as is required.

($\Leftarrow$) We will prove the contrapositive of the backwards direction, so suppose that G is separable and we will show that there are adjacent edges of G that don't share a common cycle. Let v be a cut vertex of G, and let $x, y \in V(G)$ be two vertices adjacent to v such that x and y are in different components of $G - v$. Since v is a cut vertex, there are no $x - y$ paths in $G - v$. Thus there can't exist a cycle containing vx and vy, since there's no path that doesn't involve v to connect back together x and y. We've proved the contrapositive, so we can conclude that if any two adjacent edges of G lie on a common cycle, G is nonseparable. □