UBC Mathematics 320(101)—Assignment 11
Due by PDF upload to Canvas at 23:00, Sunday 26 Nov 2023

References: Loewen, lecture notes on Point-Set Topology and CCC (2023-11-12 or newer—see Canvas); Rudin pp. 30b–40; Thomson-Bruckner-Bruckner, Sections 13.1–13.5, 13.8, 13.12

Presentation: To qualify for full credit, submissions must satisfy the detailed specifications provided on Canvas.

1. In a metric space (Y, d), suppose $S \subseteq Y$ is dense, i.e., $\overline{S} = Y$. Suppose that every Cauchy sequence (x_n) for which each $x_n \in S$ converges in Y. Prove that Y is complete.

2. (Nearest Points) Let (X, d) be a metric space, and let K be a nonempty subset of X. Define the function $d_K \colon X \to \mathbb{R}$ representing "distance from K" by

$$d_K(p) = \inf \{d(p, x) \,:\, x \in K\}, \qquad p \in X.$$

(a) Prove that $d_K(p) = d_{\overline{K}}(p)$ for all p in X.
(b) Prove that $|d_K(p) - d_K(q)| \leq d(p, q)$ for any $p, q \in X$.
(c) Suppose K is compact. Prove: $\forall p \in X,\ \exists \widehat{x} \in K \,:\, \ d_K(p) = d(p, \widehat{x})$.
(d) Prove that in the special metric space $\mathbb{R}^k$, the result in part (c) remains valid for every closed set K. (This is interesting because some closed sets are not compact.)

3. Equip $\mathbb{R}$ with its usual topology. Recall that a set $A \subseteq \mathbb{R}$ is called **dense** iff $\overline{A} = \mathbb{R}$.
(a) Prove: A set $A \subseteq \mathbb{R}$ is dense if and only if for every nonempty open interval (a, b), $A \cap (a, b) \neq \emptyset$.
(b) A sequence of sets $G_1, G_2, G_3, \ldots$ in $\mathbb{R}$ is given. Each G_k is *open* and *dense*. Prove that $S = \displaystyle\bigcap_{k \in \mathbb{N}} G_k$ is dense.

[Suggestion: Use the characterization from part (a). Construct a suitable Cauchy sequence.]

(c) Prove that the subset $\mathbb{Q}$ of $\mathbb{R}$ cannot be expressed as a countable intersection of open sets.

[Hint: Such a representation is incompatible with the result in (b).]

4. Prove: If $E \subseteq \mathbb{R}$ is uncountable, then $E' \cap E \neq \emptyset$. Hint: Try the contrapositive.

5. Let $(X, \mathcal{T})$ be a HTS and suppose K_1 and K_2 are nonempty compact sets in X with $K_1 \cap K_2 = \emptyset$. Prove that there are open sets U_1 and U_2 in X such that

$$K_1 \subseteq U_1, \qquad K_2 \subseteq U_2, \qquad U_1 \cap U_2 = \emptyset.$$

6. Given an enumeration of $\mathbb{Q}$ as $(q_1, q_2, q_3, \ldots)$, define $f \colon \mathbb{R} \to (0, 1)$ by

$$f(x) \stackrel{\text{def}}{=} \sum \left\{ \frac{1}{2^k} \,:\, q_k < x \right\}.$$

Prove that f is "lower semicontinuous", i.e., that the following set is open for every $p \in \mathbb{R}$:

$$f^{-1}\left((p, +\infty)\right) = \{x \in \mathbb{R} \,:\, f(x) > p\}.$$

Practice Problems—Not for Credit

These are not to be handed in. Solutions will be provided.

7. Let (x_n) be a sequence in $\mathbb{R}$ and let $M \in \mathbb{R} \cup \{\pm\infty\}$. Prove that the following are equivalent:

(a) $M = \limsup_{n\to\infty} x_n$.

(b) One has both (i) $\forall R > M,\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n < R$,
and (ii) $\forall r < M,\ \forall N \in \mathbb{N},\ \exists n \geq N : x_n > r$.

[Note that for any statement $P(x)$ concerning x, the construction "$\forall x \in \emptyset,\ P(x)$" is always considered *true.* Some case-by-case analysis for different types of M may be required.]

8. Construct a compact subset K of $\mathbb{R}$ for which the set K' is countable. (Use Rudin's terminology, in which a set must be infinite to qualify for the designation of "countable".)