Math 220 Homework 1

September 20, 2021

1. Let $n \in \mathbb{Z}$. Prove that if $3|n+1$ then $3 \not| n^2+5n+5$.

Because $3|n+1$, by definition $\exists m \in \mathbb{Z}$ s.t. $n+1 = 3m$, i.e. $n = 3m-1$. Using this identity we get that

$$n^2+5n+5 = (3m-1)^2 + 5(3m-1) + 5 = 9m^2 + 9m + 1 = 3(3m^2+3m) + 1$$

By axiom, because $m \in \mathbb{Z}$ we know that $3m^2+3m \in \mathbb{Z}$. To show that the expression above is not divisible by 3, we will use proof by contradiction, so suppose that it was. Then we would have that for some $m' \in \mathbb{Z}$,

$$n^2+5n+5 = 3(3m^2+3m)+1 = 3m' \Rightarrow m' - 3m^2 - 3m = \frac{1}{3}$$

The right side of this expression is clearly not an integer and the left side has to be an integer by axiom, so our assumption must be incorrect and $3 \not| n^2+5n+5$ as desired. $\square$

2. Let $a \in \mathbb{Z}$. Prove that if $5a+11$ is odd then $9a+13$ is odd.

By definition, if $5a+11$ is odd then $\exists m \in \mathbb{Z}$ s.t. $5a+11 = 2m+1$. Rearranging, we get

$$5a+11 = 2m+1 \Rightarrow 5a = 2m-10$$

Using this we get that

$$9a+13 = 5a+4a+13 = 2m-10+4a+13 = 2(m+2a)+3 = 2(m+2a+1)+1$$

Since m and a are both integers by axiom $m+2a+1 \in \mathbb{Z}$, which means that $9a+13$ matches the definition of being odd. $\square$

3. If $-1 < x < 2$, then $x^2 - x - 2 < 0$.

First note that because $x > -1$, we have that

$$x+1 > -1+1 = 0$$

Next note that because $x < 2$, we have that

$$x-2 < 2-2 < 0$$

Thus we have that $x+1$ is always positive and $x-2$ is always negative. Therefore their product is negative, i.e. $(x-2)(x+1) = x^2-x-2 < 0$. $\square$

4. Let a, b, c, d be integers. Suppose that $a, c, b+d$ are all odd numbers. Prove that $ab+cd$ is odd.

By definition of being odd, we have that $\exists m, n, o$ s.t. $a = 2m+1, b = 2n+1, b+d = 2o+1$. Using this we get that

$$ab + cd = (2m+1)b + (2n+1)d = 2(mb+nd) + b + d = 2(mb+nd) + 2o + 1 = 2(mb+nd+o) + 1$$

Since $mb + nd + o$ is an integer by axiom, we have that $ab + cd$ matches the definition for being odd. $\square$

5. Let x and y be real numbers. Show that

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

First, note that $z^2 \geq 0 \forall z \in \mathbb{R}$ (this was stated in class). Thus we have

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

Rearranging the inequality, we arrive at

$$2xy \geq x^2 + y^2 \Rightarrow \geq \frac{1}{2}(x^2 + y^2)$$

This is what was desired, so we are done. $\square$

6. Let x and y be real numbers. Suppose that $x < y$ and $y^2 < x^2$. Show that $x + y < 0$.

Starting from the second inequality given, we rearrange to get

$$y^2 < x^2 \Rightarrow 0 > y^2 - x^2 = (y+x)(y-x)$$

Since $x < y$, $y - x > 0$ and $x \neq 0$. Therefore we can divide the above inequality on both sides by $y - x$ without switching the inequality or dividing by zero. This leaves us with

$$y + x < 0$$

as required. $\square$

7. Since $5|(n+7)$, by definition $\exists m$ s.t. $n + 7 = 5m \Rightarrow n = 5m - 7$. Using this, we get that

$$n^2 + 1 = (5m+7)^2 + 1 = 25m^2 + 70m + 49 + 1 = 5(5m^2 + 14m + 10)$$

Since $5m^2 + 14m + 10$ is an integer by axiom, we have that $5|n^2 + 1$ as required. $\square$

8. Let $n, a, b, x, y \in \mathbb{Z}$. If $n|a$ and $n|b$, then $n|(ax + by)$.

By definition of divisibility $\exists c, d$ s.t. $a = cn$ and $b = dn$. Using this we have that

$$ax + by = cnx + dny = n(cx + dy)$$

$cx + dy$ is an integer by axiom, which means that $ax + by$ matches the definition required for $n|(ax + by)$. $\square$

9. If a and b are integer roots, prove that so is ab.

By the given definition of integer roots, we know that $\exists k_1, k_2 \in \mathbb{N}, m_1, m_2 \in \mathbb{Z}$ s.t. $a^{k_1} = m_1$ and $b^{k_2} = m_2$. Using this we get

$$(ab)^{k_1 k_2} = (a^{k_1})^{k_2} (b^{k_2})^{k_1} = m_1^{k_2} \cdot m_2^{k_1}$$

Let $k' = k_1 k_2$ and $m' = (m_1)^{k_2}(m_2)^{k_1}$. By the axioms given in class both k' and m' are integers since $k_1, k_2 \in \mathbb{N}, m_1, m_2 \in \mathbb{Z}$. Thus we have that

$$(ab)^{k'} = m'$$

which matches the definition for an integer roots given, so we're done. $\square$