Math 443 Homework 3

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Question 1. Let x, y be arbitrary vertices and v be a maximum degree vertex of G. First it will be show that the distance between x, y and v is at most 2. Since x, y are arbitrary consider x, but the same argument holds for y. If $x = v$ or $v \in N(x)$ then we are done, so assume neither is true. x has $\deg x \geq \delta(G)$ neighbors and v has $\deg v = \Delta(G)$ neighbors. There are $|G| - 2$ vertices other than x and v but $\deg x + \deg v \geq \delta(G) + \Delta(G) \geq |G| - 1$ vertices that are neighbors to either x or v, so by the pigeonhole principle there must be a vertex that is adjacent to both x and y, so there exists a path P_x between x and v with length less than or equal to 2. As mentioned previously by symmetry this argument also works for y so there exists a path P_y between y and v with length less than or equal to 2. Thus the walk xP_xP_yy has length at most 4, and there exists a subpath of smaller or equal length, so the distance between x and y is less than or equal to 4. This holds for all x, y so $diam(G) \leq 4$. $\square$

Question 2. Let T be a nontrivial tree with $\Delta(T) = k$ and v be a vertex with degree k in T. Next consider removing each edge incident to v, and let the resulting forest be F. Since each edge removed from a tree results in two separate tree and we removed k edges, the result is $k+1$ disjoint trees. Let $T_1, \ldots, T_k$ be the trees created this way other than the trivial tree created out of v since we've removed all of it's vertices. We will show that each T_i contributed at least one leaf to T.

Let $i \in [k]$. If $|T_i| = 1$ then let $V(T_i) = \{u\}$, and so uv was the only edge incident to u in T, so u was a leaf in T. If $|T_i| \geq 2$, then we proved in class that it has at least two leaves. However we only deleted one vertex incident to T_i to separate it from T, so only one of these two leaves could have been created by doing deleting the edges incident to v. Thus T_i has at least one vertex that is a leaf and was also a leaf in T. Since this is true for all i and each T_i is disjoint, there are at least k leaves in T.

Question 3. Recall in class that we found that all trees T have $||T|| = |T| - 1$. Also note that the number of edges in a forest is less than or equal to that of a tree, since it is possible to convert a forest into a tree strictly by adding edges. For any G with $G, \overline{G}$ both forests then, $||G|| + ||\overline{G}|| \geq 2|G| - 2$. Note also that $\{V(G), E(G) \cup E(\overline{G})\} = K_{|G|}$ and $E(G) \cap E(\overline{G}) = \emptyset$ by definition, so $||G|| + ||\overline{G}|| = ||K_{|G|}|| = \frac{1}{2}|G|(|G| - 1)$. Putting these two facts together:

$$4|G| - 4 \leq |G|^2 - |G| \implies |G| \leq 4.$$

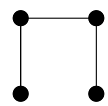
4 is reasonably small, so we can brute force check each graph with degree less that 4.

One vertex:

Two vertices:

Three vertices:

Four vertices:



Question 4. Let $n \in \mathbb{N}$. Let T be a tree created by starting with a central vertex r and adding a new vertex connected only to r, $n-1$ times. The graph created this way is a tree because it is connected (everything is connected to r) and there are no cycles by construction. Let G be any graph with $\delta(G) = n-2$ and $\Delta(G) = n-2$, i.e. a regular graph. Then since $\deg r = n-1$ but each vertex in G has degree $n-2$, clearly T can't be a subgraph of G.

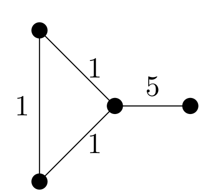
Question 5. We will prove that G has a subgraph isomorphic to H by induction on n, where $n = |H|$.

For the base case $n = 1$, the result clearly holds as H is the trivial graph, so choosing any vertex of G works.

For the inductive step, assume that G has a subgraph F' isomorphic to H', for all $|H'| \leq n$ and G with the required properties. Let H be a graph with $\Delta(H) = k$ and $|H| = n+1$, and assume that the given property of G holds as in the question for H. Let w be an arbitrary vertex in H, and let $H' = H - w$. Let F' be a subgraph of G isomorphic to H', and let X be the set of vertexes in F' that correspond to neighbors of w. Since $\Delta(F') \leq k$, the number of such vertices is less than or equal to k, if it is less than that pad X with arbitrary other vertices until $|X| = k$. Using the property of G given in the question, there are at least $|H| - 1 = n$ vertices in G that adjacent to each element of X. Choose u in $V(G)$ from these vertices, so we have that $\forall v \in V(F'), uv \in E(G)$ and $u \notin F'$. We can then construct the vertex set of F, a subgraph of G isomorphic to H, as $V(F') \cup \{u\}$, and its edge set is generated by including the edges corresponding to the edges attached to w in H as well as the edge set of F'. Since u is connected to every vertex in F' this process will always be able to create F to be isomorphic to H so the inductive step is done.

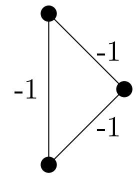
This induction proves that G has a subgraph isomorphic to H. $\square$

Question 6a. Consider the following graph:



Any graph formed by Kruskal's algorithm will include some permutation of 2 edges of weight 1 but not all three, and will include the edge of weight 5 because the tree must be spanning.

Question 6b. Consider the following graph G, and set $H = G$:



In class we showed that for a tree T, $||T|| = |T| - 1$, so any spanning tree T of G must have weight $w(T) \geq w(H) = -3$.

Question 6c. No such graph exists. By way of contradiction suppose that such a graph H did exist. Let T be a minimum spanning tree of G. If H is a tree, clearly $w(T) \leq w(H)$ by minimality of T. Thus H isn't a tree, so it must have an edge that isn't a bridge. Note that removing this edge decreases the weight of H, as $w(e) \geq 0 \forall e \in E(G)$. Repeat this process until this process results in a tree T'. Since each step decreased the total weight we definitely have $w(T') \leq w(H)$. But this contradicts our assumption that H is lighter than all spanning trees, so no such H exists. $\square$

Question 7. Let G, T be as in the question and assume by way of contradiction that $\exists x \in V(G), e \in E(G)$ s.t. $w(e) < w(f) \forall f \in E(T)$ with $x \in f, x \in e$ but $e \notin E(T)$. Let y be the other endpoint of e, i.e. $e = xy$. Since T is connected there exists a unique path from x to y, call it P. Since the path starts at x the second vertex in the path, z, must be in $N(x)$ (it isn't possible that $y = z$ as $xy \notin E(T)$ but xz is). Then consider the graph $T' = T - xz + e$. $w(T') < w(T)$ since by assumption e had smaller weight then all other edges incident to x. T' is still connected, since any walk that used to take xz can now take the walk $zPyx$. To see that T' is a tree, T being a tree means xz was a bridge. Thus $T - xz$ isn't connected but as just described $T - xz + e$ is, so e is a bridge in T'. Therefore T' is a tree with a lower weight than T, but this contradicts the assumption that T has minimum weight so e must have been in T. $\square$