PHYS 408 Homework 3

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Question 1. From Snell's law, we have $\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i$. For $\theta_i \geq \theta_c = \arcsin\left(\frac{n_t}{n_i}\right)$, we then have that $\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i > \frac{n_i n_t}{n_t n_i} = 1$. This is clearly impossible for $\theta_t \in \mathbb{R}$. To reconcile, this, recall that in the derivation of Snell's law we used that $\hat{u}_n \times \vec{E}_i + \hat{u}_n \times \vec{E}_r = \hat{u}_n \times \vec{E}_t$ to then derive that $\vec{k}_r\vec{r} = \vec{k}_t\vec{r}$. Given that the second statement results in a contradiction for $\theta_i > \theta_c$, it must instead be that $\vec{E}_t = 0$. In that case the boundary conditions for the S and P polarization turn into $E_{0i} = -E_{0r}$ and $E_{0i}\cos\theta_i = E_{0r}\cos\theta_r$ respectively. Given that $\theta_i = \theta_r$, this clearly results in $|r_\perp|^2 = |r_\parallel|^2 = 1$.

Question 2. Calculating the reflection coefficients:

$$n_i \sin\theta_i = n_t \sin\theta_t \implies \theta_t = 19.47°.$$

$$r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} = -0.24.$$

$$r_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t} = 0.159.$$

We can decomposed unpolarized light into the S and P basis and intensity is proportional to E^2, so the degree of polarization will be the difference in intensity of the two components over the total intensiy. Thus we have:

$$\text{Degree of Polarization } = \frac{|r_\perp|^2 - |r_\parallel|^2}{|r_\perp|^2 + |r_\parallel|^2} = 0.39.$$

Question 3a. The KDP crystal is effectively a wave plate with an adjustable phase, for now just refer to it as $\phi(V)$. Then the Jones matrix for the whole system for the transverse axes being rotated by θ is:

$$T = T_p^{(x)} R_\theta^{-1} T_{KDP} R_\theta T_p^{(y)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \cos(\theta)\sin(\theta)(1 - e^{i\phi}) & 0 \end{pmatrix}.$$

For an amplitude modulator we want maximum transmission when it is "open," which occurs when $\frac{d}{d\theta}(\cos\theta\sin\theta) = \sin^2\theta - \cos^2\theta = 0 \implies \theta = \frac{\pi}{2} + n\frac{\pi}{2}, n \in \mathbb{Z}$. Thus the x_1 and x_2 axis would be aligned $-45°$ and $45°$ offset respectively from the x axis.

Question 3b. From the above matrix, we see that the intensity will be $I_{out} = I_{in} \left|\frac{1}{2}\left(1 - e^{i\phi}\right)\right|^2$ (squared because the Jones matrix gives electric field, not intensity). From the textbook we have that $n_1(E) = n_0 - \frac{1}{2}n_0^3 r_{63} E$ and $n_2(E) = n_0 + \frac{1}{2}n_0^3 r_{63} E$, so $\phi = kd\Delta n = kdn_0^3 r_{63}\frac{V}{d}$. Then we have:

$$I_{out} = I_{in}\frac{1}{4}\left|1 - e^{ikn_0^3 r_{63} V}\right|^2 = I_{in}\sin^2\left(\frac{1}{2}kn_0^3 r_{63} V\right).$$

Question 3c. Maximum transmission occurs when the sin in the above expression is 1, which for example occurs when

$$\frac{\pi}{2} = \frac{1}{2}kn_0^3 r_{63} V \implies V = 11\text{kV}.$$

Question 4. For a perfect mirror, we have $r_\perp = -1$ and $r_\parallel = 1$ (even without doing the calculations explicitly, by definition a mirror has no transmission so by conservation of energy $|r_\perp| = |r_\parallel| = 1$, and in class we derived that S light flips polarization but P doesn't). Circularly polarized light can be represented by a Jones vector of $E_0\begin{pmatrix}1\\ \pm e^{i\frac{\pi}{2}}\end{pmatrix}$, without loss of generality choose coordinates so that x is parallel and y is perpendicular to the plan of incidence. Then the Jones vector of the reflected light is $E_0\begin{pmatrix}1\cdot r_\perp\\ \pm e^{i\frac{\pi}{2}} r_\parallel\end{pmatrix} = E_0\begin{pmatrix}1\\ \mp e^{i\frac{\pi}{2}}\end{pmatrix}$, which is exactly the opposite handedness of light.

Question 5a. Forward:

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Backwards:

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Question 5b. Forward:

$$\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\ 1-i\end{pmatrix}.$$

When going backwards after a mirror reflection, the polarization flips but also the quarter wave plate rotates in the opposite direction.

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1+i\\ -1+i\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

However, for example for linear light, this system doesn't block it completely:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1-i\\ 0\end{pmatrix}.$$