PHYS 304 Homework 6

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Question 1a. From the definition of operators:

$$\int \psi^* x\psi dx = \int (x\psi)^*\psi dx \implies x^\dagger = x.$$

$$\int \psi^* i\psi dx = \int (-i\psi)^*\psi dx \implies i^\dagger = -i.$$

$$\int \psi^* \frac{d}{dx}\psi dx = \psi\psi^*\Big|_{-\infty}^{\infty} - \int (\frac{d}{dx}\psi)^*\psi dx = \int (-\frac{d}{dx}\psi)^*\psi dx \implies \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}.$$

Question 1b. Expanding:

$$\left\langle f \mid \left(\hat{Q}\hat{R}\right)^\dagger g\right\rangle = \left\langle \hat{Q}\hat{R}f \mid g\right\rangle = \left\langle \hat{R}f \mid \hat{Q}^\dagger g\right\rangle = \left\langle f \mid \hat{R}^\dagger\hat{Q}^\dagger g\right\rangle \implies \left(\hat{Q}\hat{R}\right)^\dagger = \hat{R}^\dagger\hat{Q}^\dagger.$$

$$\left\langle f \mid \left(\hat{Q}+\hat{R}\right)^\dagger g\right\rangle = \left\langle \hat{Q}f \mid g\right\rangle + \left\langle \hat{R}f \mid g\right\rangle = \left\langle f \mid \left(\hat{Q}^\dagger+\hat{R}^\dagger\right) g\right\rangle \implies \left(\hat{Q}+\hat{R}\right)^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger.$$

$$\left\langle f \mid \left(c\hat{Q}\right)^\dagger g\right\rangle = \left\langle c\hat{Q}f \mid g\right\rangle = \left\langle f \mid c^*\hat{Q}g\right\rangle \implies \left(c\hat{Q}\right)^\dagger = c^*\hat{Q}^\dagger.$$

Question 2. Note that from what we derived in the textbook, the space dependent wave function for the ground state of the harmonic oscillator is

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{m\omega}{2\hbar}x^2} e^{-i\omega t/2}.$$

Plugging this into equation 3.54:

$$\Psi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\omega t/2} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{m\omega}{2\hbar}x^2}.$$

Using an integral calculator we find the final equation to be (we could also have completed the square then integrated):

$$\Phi(p,t) = \frac{\exp\left(-\frac{p^2}{2m\omega\hbar} - \frac{i\omega t}{2}\right)}{(\pi\omega m\hbar)^{1/4}}.$$

Question 3. Expanding out the definition of the momentum space wave function:

$$\int \Phi^* \left(i\hbar\frac{\partial}{\partial p}\right)\Phi dp = \frac{1}{\sqrt{2\pi\hbar}} \int \left(\int e^{ipx/\hbar}\Psi^*(x,t)dx\right)\left(\int -i\hbar\frac{ix}{\hbar}e^{-ipx\hbar}\Psi(x,t)dx\right) dp.$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int \left(\int e^{ipx/\hbar} \Psi^*(x,t) dx \right) \left(\int x e^{-ipx\hbar} \Psi(x,t) dx \right) dp.$$

This can be simplified as the hint suggests by combining the integrals and using the definition of delta functions, giving us:

$$= \int \Psi^*(x,t) x \Psi(x,t) dx = \langle x \rangle .$$

Question 4. Because there are two basis elements, there are two eigenstates/eigenvalues. To find them we must solve:

$$\hat{H} |h\rangle = h |h\rangle , |h\rangle = a |1\rangle + b |2\rangle .$$

This is equivalent to the following set of linear equations:

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = h \begin{pmatrix} a \\ b \end{pmatrix} .$$

$$\implies \begin{cases} h = -\sqrt{2}\epsilon \implies |h\rangle = \left(1 - \sqrt{2}\right) |1\rangle + |2\rangle \\ h = \sqrt{2}\epsilon \implies |h\rangle = \left(1 + \sqrt{2}\right) |1\rangle + |2\rangle \end{cases} .$$

As already indicated the matrix representing this set of equations is:

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Question 5a. The measurement collapses the state of the system, so the state of the system is then ψ_1.

Question 5b. The two possible results are ψ_1 or ψ_2 with eigenvalues b_1 and b_2 respectively. The state is currently $\psi_1 = (3\psi_1 + 4\psi_2)/5$, so the probability of ψ_1 is $0.6^2 = 0.36$ and the probability of ψ_2 is $0.8^2 = 0.64$.

Question 5c. The chances of measuring ϕ_1 and then ψ_1 again are $0.36 \cdot 0.36 = 0.1296$ while the probability of measuring ϕ_2 and then ψ_2 are $0.64 \cdot 0.64 = 0.4096$, so the total probability is 0.5392.