Math 320 Homework 10

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Question 1. Consider the family of sets $\mathcal{F}$ containing $f^{-1}((-\infty, p])^c = f^{-1}((p, \infty)) = \{x \in X : f(x) > p\}$ for $p > \inf f(X)$, these are all open since f is lower semicontinuous. By contradiction assume $\inf f(X) = -\infty$, then for any $x \in X$ we have $x \in f^{-1}((-\infty, f(x)-1])^c$, so these sets provide an open cover of X. The compactness of X guarantees the existence of $F_1 = f^{-1}((p_1, \infty)), \ldots, F_n = f^{-1}((p_n, \infty)) \in \mathcal{F}$ which make an open cover of X. But then $\min\{p_1, \ldots, p_n\}$ is a lower bound for $f(X)$ which contradicts our assumption that $\inf f(X) = -\infty$, so f must be bounded below.

Next, to show that f attains it's lower bound, consider the set $\mathcal{F}^c = \{F^c : F \in \mathcal{F}\} = \{f^{-1}((-\infty, p]) : p > \inf f(X)\}$. $\mathcal{F}^c$ contains only closed sets since f is lower semicontinuous, and the finite intersection property holds since for any $p_1, \ldots, p_n \in \mathbb{R}$ with $p_i > \inf f(X) \forall i$, we have $\bigcap_{i=1}^n f^{-1}((-\infty, p_i]) = f^{-1}([\inf f(X), \min\{p_1, \ldots, p_i\}]) \neq \emptyset$. Then because X is compact, $\mathcal{F}^c$ has nonempty intersection. Thus we have $\bigcap \mathcal{F}^c = f^{-1}((-\infty, \inf f(X)]) = f^{-1}(\inf f(X)) \neq \emptyset$, so any $z \in f^{-1}(\inf f(X))$ works.

Question 2. Let $K \subseteq X$ be a compact set, and assume that $\mathcal{G}$ is an open cover of K. For each $z \in K$, we can find some $G_z \in \mathcal{G}$ such that $z \in G_z$. Since $\mathcal{G}$ is comprised of open sets, there exists an r_z s.t. $\mathbb{B}[z, 2r_z) \subseteq G_z$. Consider $\mathcal{F} = \{\mathbb{B}[z, r_z) : z \in K\}$. $\mathcal{F}$ is an open covering of K (since $z \in \mathbb{B}[z, r_z) \forall z \in K$), so by the compactness of K there is a finite collection $z_1, z_2, \ldots, z_n$ such that $\bigcap_{i=1}^{n} \mathbb{B}[z_i, r_{z_i}) = K$. Let $r = \min\{r_{z_1}, \ldots, r_{z_n}\}$, I claim that this r fulfills the question's requirements.

Let $x, y \in K$ obeying $d(x, y) < r$. By the construction earlier there is z_k s.t. $x \in \mathbb{B}[z_k, r_{z_k})$. Then we have $d(z_k, y) \leq d(z_k, x) + d(x, y) < r_{z_k} + r \leq 2r_{z_k} \implies y \in \mathbb{B}[z_k, 2r_{z_k})$. In addition, by construction there exists $G_{z_k} \in \mathcal{G}$ with $\mathbb{B}[z_k, 2r_{z_k}) \subseteq G_{z_k}$, so both x and y are contained in G_{z_k} as required.

Question 3a. If S is bounded then $p_1(S)$ is also bounded. To see why, suppose that there exists $D > 0$ such that for any $x, y \in \mathbb{R}^2$, we have $d(x, y) < D$. Then for any $(x_1, x_2), (y_1, y_2) \in S$, note that $D > d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \geq |x_1 - y_1| = d(x_1, y_1) = d(p_1((x_1, x_2)), p_1((y_1, y_2)))$. Thus D is also an bound for $p_1(S)$.

Question 3b. The statement is not true. Consider $S = \{(x, \frac{1}{x}) : x \in \mathbb{R} \setminus \{0\}\}$. For any $(x, y) \in \mathbb{R}^2$ with $\frac{1}{x} \neq y$, let ϵ be the distance between (x, y) and the graph of $y = \frac{1}{x}$, then we have that $\mathbb{B}[(x, y), \epsilon) \cap S = 0$. Thus S^c is open and S is closed. However, $p_1(S) = (-\infty, 0) \cup (0, \infty)$ isn't closed ($p_1(S)^c = \{0\}$ which isn't open).

Question 3c. The statement is true. From the notes, we proved that an equivalent defintion of compactness is that every subsequence has a convergent subsequence. Assume that S is compact, and let (x_n) be a sequence in $p_1(K)$. Then for each x_n there exists y_n such that $(x_n, y_n) \in S$, so construct a new sequence $((x_n, y_n))$ in S this way. Since S is compact $((x_n, y_n))$ has a convergent subsequence $((x_{n_k}, y_{n_k}))$ (where k is now the indexing variable of this sequence rather than n). Then by the definition of convergence in $\mathbb{R}^2$, x_{n_k} is a converging subsequence of x_n and therefore $p_1(S)$ is compact.

Question 4a. Let $V \subseteq \ell^2$ be finite and $x \in \ell^2$. Let $G = \Omega(x; V)$, and choose $y \in G$. By Archimedes let k be a natural number large enough that $\frac{1}{k} < 1 - \max\{|\langle v, y - x\rangle| : v \in V\}$. Let $V' = kV = \{(kv_n) : (v_n) \in V\}$. Consider $z \in \Omega(y; V')$, by the definition of V' we have that $|\langle v, z - y\rangle| < \frac{1}{k}$. Using this, we get

$$|\langle v, z - x\rangle| = \left|\sum_{n=1}^{\infty} v_n(z_n - x_n)\right| = \left|\sum_{n=1}^{\infty} v_n((z_n - y_n) - (x_n - y_n))\right|$$

$$\leq |\langle v, z - y\rangle| + |\langle v, x - y\rangle| < \left(1 - \frac{1}{k}\right) + \frac{1}{k} = 1.$$

Thus $|\langle v, z - x\rangle| < 1$ and so $z \in \Omega(x; V)$. Since z was arbitrary we get $y \in \Omega(y; V') \subseteq \Omega(x, V) = G$ as required, so $G \in \mathcal{T}$.

Question 4b. Each property will be proven separately:

(HTS1) Choosing $V = \emptyset$ and $x \in \ell^2$ gives that $\Omega(x; \emptyset) = \ell^2$, as the condition in the definition of Ω is vacuously true and part a shows that $\Omega(x; \emptyset) \in \mathcal{T}$. Also $\emptyset \in \mathcal{T}$, as every point $x \in \emptyset$ satisfies the condition to be in $\mathcal{T}$ trivially since there aren't any x to choose.

(HTS2) Let $\mathcal{G} \subseteq \mathcal{T}$. Let $x \in \bigcup \mathcal{G}$, then since $x \in G$ for some $G \in \mathcal{G}$ we have that $\Omega(x; V) \subseteq G$ for some finite set $V \subseteq \ell^2$. Then since the union only makes the set bigger we also have $\Omega(x; V) \subseteq \bigcup \mathcal{G} \implies \bigcup \mathcal{G} \in \mathcal{T}$.

(HTS3) Let $U_1, \ldots, U_N \in \mathcal{T}$. Let $G = \bigcap_{i=1}^{N} U_i$ and $x \in G$. Then for each $i = 1, \ldots, N$, we have $x \in \Omega(x; V_i) \subseteq U_i$. Let $V = \bigcup_{i=1}^{N} V_i$. Then $x \in \Omega(x, V) \subseteq U_i \forall i$, so $x \in \Omega(x, V) \subseteq G$ as required.

(HTS4) Let $x, y \in \ell^2$ with $x \neq y$. Let $U = \Omega(x, \{\frac{1}{|x_1|}\hat{e}_1\}) = \Omega(x, \{(\frac{1}{|x_1|}, 0, \ldots)\})$ and $V = \Omega(y, \{-\frac{1}{|y_1|}\hat{e}_1\})$. Then for $z \in U$ we have $|\langle \frac{1}{|x_1|}\hat{e}_1, x - z\rangle| = |1 - z_1| < 1 \implies z_1 > 0$, and for $w \in V$ we have $|\langle -\frac{1}{|y_1|}\hat{e}_1, y - w\rangle| = |-1 - w_1| < 1 \implies w_1 < 0$. Since their first entries differ in sign $z \neq w \implies U \cap V = \emptyset$, but they are each open sets that contain x and y respectively as required.

Question 4c. Let $U \in \mathcal{N}(0)$. Then there exists $V \subseteq \ell^2$ finite such that $\Omega(0; V) \subseteq U$. Let $p \in \mathbb{N}$ be such that $|v_p| < 1 \forall v \in V$. This is possible since $\|v\| < \infty \forall v \in V$. Then $\langle v, \hat{e}_p\rangle = |v_p| < 1 \forall v \in V$, so $\hat{e}_p \in \Omega(0; V) \subseteq U$, i.e. $U \cap S \supseteq \{e_p\} \neq 0$, so $0 \in S'$.

Question 4d. Let $G \in \mathcal{T}$ and $x \in G$. Let $V \subseteq \ell^2$ finite be such that $x \in \Omega(x; V) \subseteq G$. Let $r = \frac{1}{\max\{\|v\| : v \in V\}}$. Then using homework 7 question 3b (it states that $|\langle x, y\rangle| \leq \|x\|\|y\| \forall x, y \in \ell^2$), we get that for $y \in \ell^2$ with $\|y - x\| < r$ and $v \in V$,

$$|\langle v, y - x\rangle| \leq \|v\|\|y - x\| < \frac{\|v\|}{\|v\|} = 1.$$

Thus $y \in \Omega(x; V)$. Since y was chosen arbitrarily as long as $\|y - x\| < r$, we have $\mathbb{B}[x; r) \subseteq G$.

Question 4e. Let $G = \{y \in \ell^2 : \|y\| > 1\}$, proving that $\mathbb{B}[0, 1]$ is closed is equivalent to proving that G is open. Let $x \in G$ and $\epsilon < \frac{\|x\|^2 - 1}{2}$. Since $\|x\| < \infty$, let $M = \max\{x_p : p \in \mathbb{N}\}$ and choose N such that $\sum_{n=N+1}^{\infty} |x_n|^2 < \frac{\epsilon}{2}$. Choose $V = \{\frac{4NM}{\epsilon}\hat{e}_p : 1 \leq p \leq N\}$. I claim that $\Omega(x; V) \in G$.

Let $y \in \Omega(x; V)$. Then $\left|\langle \frac{4NM}{\epsilon}\hat{e}_p, y - x\rangle\right| < 1 \implies |y_p - x_p| < \frac{\epsilon}{4NM} \implies |y_p| > |x_p| - \frac{\epsilon}{4NM} \implies |y_p|^2 > |x_p|^2 - \frac{\epsilon}{2NM}|x_p| + \left(\frac{\epsilon}{4NM}\right)^2 > |x_p|^2 - \frac{\epsilon}{2N}$. Thus we have

$$\|y\|^2 = \sum_{n=1}^{N} |y_n|^2 + \sum_{n=N+1}^{\infty} |y_n|^2 > \sum_{n=1}^{N} \left(|x_n|^2 - \frac{\epsilon}{2N}\right) + 0 \geq \|x\|^2 - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \|x\| - \epsilon > 1.$$

Thus $\|y\| > 1$ so $y \in G$, as required. This implies that $x \in \Omega(x; V) \subseteq G$ so G is open, and thus $\mathbb{B}[0, 1]$ is closed.