Math 406 Homework 6

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Question 1. Using the method of variations, we get that for every small perturbation v,

$$\int_\Omega v\left(\Delta u + \lambda u\right) dv = 0.$$

Using the fact that $\nabla(v\nabla u) = \nabla v \nabla u + v\nabla^2 u$, this is equivalent to:

$$\int_\Omega \nabla(v\nabla u) dv - \int_\Omega \nabla v \nabla u dv + \lambda \int_\Omega uv dv = 0$$

$$\implies \int_\Omega \nabla u \nabla v dv = \lambda \int_\Omega uv dv.$$

Thus the weak form of the PDE is to find $u \in H_0^1 = \{u : \int_\Omega |\nabla u|^2 dv < \infty, u\big|_{\partial\Omega} = 0\}$ such that the above equation holds for all $v \in H_0^1$. Let $u(x,y) = \sum_{n=1}^N u_n \psi(x,y)$ and $v(x,y) = \sum_{m=1}^N v_m \psi_m(x,y)$. The plugging this into the weak form above and rearranging the sum to bring v to the outside, we get

$$\sum_{m=1}^N v_m \left(\sum_{n=1} u_n \int_\Omega \nabla\psi_m \nabla\psi_n dv - \lambda \sum_{n=1}^N u_n \int_\Omega \psi_m \psi_n dv \right) = 0$$

$$\implies Ku = \lambda M u.$$

Similarly to the 1d case, K is the stiffness matrix with entries coming from $K_{mn} = \int_\Omega \nabla\psi_m \nabla\psi_n dv$ and M is the mass matrix coming from $M_{mn} = \int_\Omega \psi_m \psi_n dv$. These entries were derived in class specifically for the linear basis functions, where for an individual triangle T they were found to be

$$M_{mn}^e = \frac{A(T)}{12}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, K_{mn}^e = \frac{2A(t)}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

where $A(T)$ is the area of T.