MATH 305

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1. Use Fundamental Theorem of Calculus to evaluate
   (a) $\int_C e^z dz$, C : arc $e^{it}, -\frac{\pi}{2} \leq t \leq \pi$
   Let $F(z) = e^z$. Then $\frac{d}{dz}F = e^z$, so by FTC:

$$\int_C e^z dz = e^{e^{i\pi}} - e^{e^{-\frac{\pi}{2}i}} = \frac{1}{e} - e^{-i}.$$

   (b) $\int_C \frac{1}{z}dz$, C : part of the ellipse $\frac{x^2}{4} + y^2 = 1, x \geq 0$
   Let $F = Logz$. Then by FTC taking the contour to be slightly less than all the way around the circle:

$$\int_C \frac{1}{z}dz = \frac{1}{2} \cdot 2\pi i - 0 = \pi i.$$

   (c) $\int_C \frac{1}{z^2}dz$, C : part of the ellipse $\frac{x^2}{4} + y^2 = 1, y \geq 0$.
   Let $F = -\frac{1}{2z}$. Then by FTC:

$$\int_C \frac{1}{z^2}dz = \frac{1}{2} + \frac{1}{2} = 1.$$

2. (15pts) Use the inequality $|\int_\Gamma f(z)dz| \leq \max_{z\in\Gamma} |f(z)| \times$ length of(Γ) to prove
   (a) $|\int_C \frac{dz}{z^2-i}| \leq \frac{3\pi}{4}$, C: circle $|z| = 3$ traversed once
   The maximum of $\frac{1}{z^2-i}$ over the circle is when $z = \pm 3e^{i\frac{\pi}{4}}$, where $|f(z)| = \frac{1}{8}$ (this is obvious geometrically). Then

$$\left|\int_C \frac{1}{z^2-i}dz\right| = \frac{1}{8} \cdot 6\pi = \frac{3\pi}{4}.$$

   (b) $|\int_C Log(z)dz| \leq \frac{\pi^2}{\mathbf{2}}$, C : arc $e^{it}, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
   The maximum is attained when $z = e^{\pm i\frac{\pi}{2}}$ where $|Log(z)| = \frac{\pi}{2}$, so we have:

$$\left|\int_C Log(z)dz\right| \leq \frac{\pi}{2} \cdot \pi = \frac{\pi^2}{2}.$$

   (c) $|\int_C \frac{e^{3z}}{e^z+1}dz| \leq \frac{2\pi e^{3R}}{e^R-1}$, C is the vertical line segment from $z = R(> 0)$ to $z = R + 2\pi i$.
   For the numerator the imaginary component doesn't matter for the magnitude, and for the denominator the magnitude is minimized when $Im(z) = \pi$. Thus by FTC:

$$\left|\int_C \frac{e^{3z}}{e^z+1}dz\right| \leq 2\pi \cdot \frac{e^{3R}}{e^R-1}.$$

3. (15pts) Show that

(a) $\int_{C_\epsilon} \frac{Log(z)}{1+z^2} dz \to 0$ as $\epsilon \to 0$, where C_ϵ is the contour ϵe^{it}, $-\pi + \epsilon \le t \le \pi - \epsilon$
Bounding the limit:

$$\left| \frac{Log(z)}{1+z^2} \right| \le \frac{|\ln \epsilon + i\pi|}{|z|^2 - 1}.$$

Then we can bound the integral as:

$$\left| \int_{C_\epsilon} \frac{Log(z)}{1+z^2} dz \right| \le \frac{|\ln \epsilon + i\pi|}{||z|^2 - 1|} 2(\pi - \epsilon)\epsilon.$$

$$\lim_{\epsilon \to 0} \frac{\sqrt{(\ln \epsilon)^2 + \pi^2}}{1 - \epsilon^2} 2(\pi - \epsilon)\epsilon = 2\pi \lim_{\epsilon \to 0} \epsilon \ln \epsilon = 2\pi \lim_{\epsilon \to 0} \frac{\ln \epsilon}{1/\epsilon} = 2\pi \lim_{\epsilon \to 0} \epsilon = 0.$$

The last step was by L'Hopital's rule, and we're done.
(b) $\int_{C_R} \frac{Log(z)}{1+z^2} dz \to 0$ as $R \to +\infty$, where C_R is the contour Re^{it}, $-\pi + \frac{1}{R} \le t \le \pi - \frac{1}{R}$;
Using the exact same bounds for the integral as last time except with $\epsilon = \frac{1}{R}$, we get:

$$\left| \int_{C_R} \frac{Log(z)}{1+z^2} dz \right| \le \frac{|-\ln R + i\pi|}{||z|^2 - 1|} 2(\pi - \frac{1}{R})R.$$

$$\lim_{R \to \infty} \frac{|-\ln R + i\pi|}{|R^2 - 1|} 2(\pi - \frac{1}{R})R = 2\pi \lim_{R \to \infty} \frac{\ln R}{R} = 2\pi \lim_{R \to \infty} \frac{1}{R} = 0.$$

4. Use Fundamental Theorem of Calculus to compute
(a) $\int_\Gamma z^{\frac{1}{2}} dz$ for the principal branch of $z^{\frac{1}{2}}$, where Γ is $r = 2\cos\frac{\theta}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
Let $F = \frac{2}{3} z z^{\frac{1}{2}}$. Then $F'(x) = \frac{2}{3} z^{\frac{1}{2}} + \frac{1}{3} z^{\frac{1}{2}} = z^{\frac{1}{2}}$. Then using FTC we get:

$$\int_\Gamma z^{\frac{1}{2}} dz = \frac{2}{3} z z^{\frac{1}{2}} \Big|_{\sqrt{2}e^{-i\frac{\pi}{2}}}^{\sqrt{2}e^{i\frac{\pi}{2}}} = \frac{2}{3}\left(2^{3/4} e^{i\frac{3\pi}{4}} - 2^{3/4} e^{-i\frac{3\pi}{4}}\right) = \frac{2^{11/4}}{3} i \sin(\frac{3\pi}{4}) = \frac{2^{9/4}}{3} i.$$

(b) $\int_\Gamma (Log(z))^2 dz$, where Γ is $r = 2\cos\frac{\theta}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
Let $F(z) = zLog(z))^2$. Then as the hint suggests note that $(z(Log(z))^2)' = (Log(z))^2 + 2Log(z)$, so we have that

$$\int_\Gamma (Log(z))^2 dz = z(Log(z))^2 \Big|_{\sqrt{2}e^{-i\frac{\pi}{2}}}^{\sqrt{2}e^{i\frac{\pi}{2}}} - 2\int_\Gamma Log(z) dz$$

$$= \sqrt{2}i\left(\frac{1}{2}\ln 2 + i\frac{\pi}{2}\right)^2 + \sqrt{2}i\left(\frac{1}{2}\ln 2 - i\frac{\pi}{2}\right)^2 - 2\left(zLog(z) - z\right)\Big|_{-\sqrt{2}i}^{\sqrt{2}i}.$$

$$= \sqrt{2}i\left(\frac{1}{2}\ln 2 + i\frac{\pi}{2}\right)^2 + \sqrt{2}i\left(\frac{1}{2}\ln 2 - i\frac{\pi}{2}\right)^2 - 2\left(\sqrt{2}i\left(\frac{1}{2}\ln 2 + i\frac{\pi}{2} - 1\right) + \sqrt{2}i\left(\frac{1}{2}\ln 2 - i\frac{\pi}{2} - 1\right)\right).$$

5. Let C be the contour of ellipse $\frac{x^2}{4} + y^2 = 1$ traversed once. Compute
(a) $\int_C \frac{1}{(z-1)^2} dz$
Applying Cauchy's integral formula with $f = 1$, this gives:

$$\int_C \frac{1}{(z-1)^2} dz = 2\pi i f'(1) = 0.$$

(b) $\int_C \frac{e^z}{z(z-1)} dz$

$$\int_c \frac{e^z}{z(z-1)}dz = 2\pi i e^z \frac{1}{z}\Big|_{z=1} + 2\pi i e^z \frac{1}{z-1}\Big|_{z=0} = 2\pi i e - 2\pi i.$$

(c) $\int_C \frac{1}{z(z^2-1)}dz$

$$\int_c \frac{1}{z(z^2-1)}dz = 2\pi i \frac{1}{z(z-1)}\Big|_{z=-1} + 2\pi i \frac{1}{z(z+1)}\Big|_{z=1} + 2\pi i \frac{1}{(z+1)(z-1)}\Big|_{z=0}$$
$$= \pi i + \pi i - 2\pi i = 0.$$

(d) $\int_C \frac{1}{2z^2+1}dz$

$$\int_C \frac{1}{2z^2+1}dz = \frac{2\pi i}{(\sqrt{2}z+i)}\Big|_{i/\sqrt{2}} + \frac{2\pi i}{(\sqrt{2}z-i)}\Big|_{-i/\sqrt{2}} = \frac{2\pi i}{2i} - \frac{2\pi i}{2i} = 0.$$

6. Determine the domain of the analyticity of the following function and explain why

$$\int_{|z|=2} f(z)dz = 0$$

(a) $f(z) = \frac{\cos z}{z^2+6z+10}$

The domain of analyticity is $\mathbb{C}\setminus\{z = 3\pm i\}$. The integral is zero because the function is analytic in the domain and so the Cauchy integral formula tells us the integral is zero.

(b) $f(z) = \text{Log}(2z+5)$

The domain of analyticity is when $Re(2z+5) \geq 0, Im(2z+5) \neq 0$, i.e. $\mathbb{C}\setminus\{z \in \mathbb{C} : Re(z) \leq -\frac{5}{2}, Im(z) = 0\}$. The integral is zero because again, the function is function is analytic within the circle and the Cauchy integral formula says it must be zero.

(c) $f(z) = \sin^{-1}(\frac{z}{3})$

Expanding:

$$\sin^{-1}(z) = -iLog\left(\frac{iz}{3} + \sqrt{(1-\frac{z}{3}^2)}\right).$$

This gives us that the domain of analyticity is $\mathbb{C}\setminus(-\infty,-3]\cup[3,\infty)$. The integral is zero since the function is analytic in the circle of radius two and Cauchy's integral formula.

(d) $f(z) = \tan(\frac{z}{2})$

This function is analytic when $\cos(z) \neq 0$, i.e. $\mathbb{C}\setminus\{z \in \mathbb{Z} \mid z = 2\pi n + \pi, n \in \mathbb{N}, Im(z) = 0\}$. The integral is zero since the function is analytic in the domain.

7. Evaluate the contour integral $\int_C \frac{z}{(z^2+1)(z-1)}dz$ along the following contours

(a) $C : |z-i| = 1$, counter-clockwise

$$\int_C \frac{z}{(z^2+1)(z-1)}dz = 2\pi i \frac{z}{(z+i)(z-1)}\Big|_{z=i} = \frac{\pi i}{(i-1)}.$$

(b) $C = C_1 \cup C_2$, $C_1 : |z-i| = 1$,counter-clockwise; $C_2 = |z+i| = 1$,clockwise

$$\int_C \frac{z}{(z^2+1)(z-1)}dz = 2\pi i \frac{z}{(z+i)(z-1)}\Big|_{z=i} - 2\pi i \frac{z}{(z-i)(z-1)}\Big|_{z=-i} = \frac{\pi i}{(i-1)} - \frac{\pi i}{-i-1}.$$

(c) $C = C_1 \cup C_2$, $C_1 : |z-1| = 1$, counter-clockwise; $C_2 : |z+1| = 1$, clockwise.

$$\int_C \frac{z}{(z^2+1)(z-1)}dz = -2\pi i \frac{z}{(z-i)(z+i)}\bigg|_{z=1} = -\frac{2\pi i}{(i+1)(i-1)} = \pi i.$$

8. Evaluate the contour integral $\int_C \frac{2z^2-z+1}{(z-1)(z+1)^2}dz$ along the contour $C = C_1 \cup C_2$, where C_1 : $|z-1| = 1$, counter-clockwise; $C_2 : |z+1| = 1$, clockwise.

Hint: you can do partial fractions first.

First note that

$$\frac{2z^2-z+1}{(z-1)(z+1)^2} = \frac{A}{z-1} + \frac{Bz+C}{(z+1)^2} = \frac{1/2}{z-1} + \frac{3z/2-1/2}{(z+1)^2}.$$

Applying the Cauchy integral formula this gives us:

$$\int_C \frac{2z^2-z+1}{(z-1)(z+1)^2}dz = \int_C \frac{1/2}{z-1} + \frac{3z/2-1/2}{(z+1)^2}dz.$$

$$= \frac{1}{2}2\pi i - \frac{3}{2}2\pi i = -2\pi i.$$

9. Evaluate

(a) $\int_{|z|=2} \frac{1}{z^2+2z+2}dz$

The roots of the polynomial are $z = -1 \pm i$, so the integral is:

$$\int_{|z|=2} \frac{1}{z^2+2z+2}dz = \frac{2\pi i}{z+1+i}\bigg|_{z=-1+i} + \frac{2\pi i}{z+1-i}\bigg|_{z=-1-i}.$$

$$= \pi - \pi = 0.$$

(b) $\int_{|z|=2} \frac{1}{z^2-2z-3}dz$

The roots are $z = 3, z = -1$, so:

$$\int_{|z|=2} \frac{1}{z^2-2z-3}dz = \frac{2\pi i}{z-3}\bigg|_{z=-1} = -\pi\frac{i}{2}.$$