Hw 5:

STAT : 4520.

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7.1. (1). Non-info prior: $p(\mu, \sigma^2) \propto \frac{1}{\sigma^2}$ $(-\infty < \mu < \infty)$

$(0 < \sigma^2 < \infty)$

$L(y \mid \mu, \sigma^2) = \prod\limits_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(y_i - \mu)^2}{2\sigma^2}}$

$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}}} e^{-1/2\sigma^2 \cdot \sum\limits_{i=1}^{n}(y_i - \mu)^2}$

The prior post. : $P(\mu, \sigma^2 \mid y) \propto P(\mu, \sigma^2) \cdot L(y \mid \mu, \sigma^2)$

$= \frac{1}{\sigma^2} \times \frac{1}{(\sigma^2)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\sigma^2}[\sum\limits_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2]}$

$= \frac{1}{(\sigma^2)^{\frac{n+1}{2}}} \cdot e^{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y} - \mu)^2]}$

Normalize: $P(\mu, \sigma^2 \mid y) = \frac{((n-1)s^2/2)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{1}{(\sigma^2)^{\frac{n+1}{2}}} e^{-\frac{(n-1)s^2}{2\sigma^2}} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2}}$

$P(\mu \mid y) = \int_{0}^{\infty} P(\mu, \sigma^2 \mid y)\, d\sigma^2$.

$(t_{n-1} = t_{19} \Rightarrow) = \dfrac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2}) \sqrt{(n-1)\pi} \cdot \frac{s}{\sqrt{n}} [1 + \frac{1}{n-1}(\frac{\mu - \bar{y}}{s/\sqrt{n}})^2]^{\frac{n}{2}}}$

Scale : $\frac{s^2}{n} = \frac{4.2797^2}{20} = 0.9158$

$= \dfrac{\Gamma(10)}{\Gamma(9.5)\sqrt{19\pi} \cdot \frac{4.2797}{\sqrt{20}} \cdot [1 + \frac{1}{19} + (\frac{\mu - 50}{4.2797/\sqrt{20}})^2]^{10}}$

post. mean $= \bar{y} = 50$.

$\Rightarrow$ 95% CS for μ: $\boxed{(47.997,\ 52.003)}$

(R : 50 + qt(c(.025, .975), 19) * sqrt(0.9158))

(2). $P(\sigma^2 | y) \propto \int_{-\infty}^{\infty} \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y}-\mu)^2]} d\mu$

$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} \cdot e^{-\frac{(n-1)s^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\frac{n(\bar{y}-\mu)^2}{2\sigma^2}} d\mu$

$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{\frac{(n-1)s^2}{2\sigma^2}} \qquad \sim IG\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

$$= \boxed{IG(9.5, 17.4)}$$

$P(\sigma^2 | y) = \frac{(\frac{348}{2})^{\frac{19}{2}}}{\Gamma(\frac{19}{2})} \cdot \frac{1}{(\sigma^2)^{\frac{19}{2}+1}} \cdot e^{-\frac{348}{2\sigma^2}}$

$= \frac{174^{9.5}}{\Gamma(9.5)} \cdot \frac{1}{\sigma^{22}} \cdot e^{-\frac{174}{\sigma^2}} \qquad (0 < \sigma^2 < \infty)$

$E(\sigma^2 | y) = \frac{\beta}{\alpha-1} = \frac{\frac{348}{2}}{\frac{20-1}{2}-1} = \underline{20.4705}$

$Sd. Err = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} = \sqrt{\frac{174^2}{8.5^2 \times 7.5}} = \underline{7.4748}$

95% CS for σ^2 : $\underline{(10.5929, 39.0725)}$

(R : 1/ gamma (C(.975, .025), 9.5, 174)

7.2. (1). Conj. Prior: $P(\mu, \sigma^2) = P(\sigma^2) P(\mu | \sigma^2) = IG(\alpha, \beta) \times N(\mu_0, \frac{\sigma^2}{k})$

$$= IG(38, 444) \times N(51, \frac{\sigma^2}{10})$$

$$= \frac{444^{38}}{\Gamma(38)} \frac{1}{(\sigma^2)^{39}} e^{-\frac{444}{\sigma^2}} \times \frac{\sqrt{10}}{\sqrt{2\pi\sigma^2}} e^{-\frac{10(\mu-51)^2}{2\sigma^2}}$$

$$P(\mu, \sigma^2 | y) \propto \frac{\sqrt{k}}{\sigma^2} e^{-\frac{k(\mu-\mu_0)^2 + n(\mu-\bar{y})^2}{2\sigma^2}}$$

$$\propto \frac{\sqrt{k}}{\sigma} e^{-(k+n)\left(\frac{\mu - \frac{k\mu_0 + n\bar{y}}{k+n}}{2\sigma^2}\right)^2}$$

$$= \frac{\sqrt{10}}{\sigma} e^{-(30)\left(\frac{\mu - \frac{510 + 1000}{30}}{2\sigma^2}\right)^2}$$

$$\Rightarrow \mu, \sigma^2 | y \sim IG\left(\alpha + \frac{n}{2}, \beta + \frac{(n-1)s^2}{2} + \frac{kn(\bar{y}-\mu_0)^2}{2(k+n)}\right) \times N\left(\frac{k\mu_0 + n\bar{y}}{k+n}, \frac{\sigma^2}{k+n}\right)$$

$$= IG\left(38 + \frac{20}{2}, 444 + \frac{348}{2} + \frac{10 \times 20 \times (50-51)^2}{2(10+20)}\right) \times N\left(\frac{10 \times 51 + 20 \times 50}{10 + 20}, \frac{\sigma^2}{10 + 20}\right)$$

$$= IG(48, 621.333) \times N\left(50.333, \frac{\sigma^2}{30}\right)$$

(2). $P(\mu | y)$ is t density w/

mean: $\frac{k\mu_0 + n\bar{y}}{k+n} = \frac{10 \times 51 + 20 \times 50}{10 + 20} = 50.333$

scale parameter: $\frac{n_0\sigma_0^2 + (n-1)s^2 + \frac{kn(\bar{y}-\mu_0)^2}{k+n}}{(k+n)(n_0+n)}$

$$= \left(19 \times 46.7368 + 348 + \frac{10 \times 20 (50-51)^2}{30}\right) / (30 \times 19 + 20)$$

$$= 0.4315$$

$df = n + n_0 = 20 + 19 = 39$

95% CS: $\mu \pm t_{39, (0.95)} \sqrt{\text{scale para.}}$

$$= (48.2488, 52.4179)$$

137. $P(\sigma^2|y) \sim IG\left(\frac{n}{2} + \alpha, \frac{\sum(y_i - \mu)^2}{2} + \beta\right)$

$$= IG\left(\frac{20}{2} + 38, \frac{368}{2} + 444\right) = IG(48, 621.333)$$

$$\text{mean} = \frac{\beta'}{\alpha'-1} = \frac{628}{48-1} = 13.3617 = E(\sigma^2|y)$$

95% CS for σ^2: $(10.0479, 17.74442)$

Extra Credit:

7.3 $p(\mu, \sigma^2) = \frac{\beta^2}{\Gamma(\alpha)} \cdot \frac{1}{(\sigma^2)^{\alpha+1}} e^{-\frac{\beta}{\alpha^2}} \cdot \frac{\sqrt{k}}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{k(\mu-\mu_0)^2}{2\sigma^2}}$

where $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$

$\therefore p(\mu, \sigma^2) \propto \frac{1}{(\sigma^2)^{\frac{n_0+3}{2}}} e^{-\frac{n_0\sigma_0^2 + k(\mu-\mu_0)^2}{2\sigma^2}}$

$\to \frac{1}{\sigma^2}$ (as $n_0 \to -1$, $k\sigma_0^2 \to 0$)

Extra Credit:

7.4 $\int_0^\infty \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{-\frac{(n-1)s^2 + n(\mu-\bar{y})^2}{2\sigma^2}} d\sigma^2 = \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}$

$= \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}} \cdot \int_0^\infty \frac{\left[\frac{(n-1)s^2 + n(\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \cdot \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{-\frac{(n-1)s^2 + n(\bar{y}-\mu)^2}{2\sigma^2}} \cdot d\sigma^2$

$= \sim \cdot \int_0^\infty IG\left(\frac{n}{2}, \frac{(n-1)s^2 + n(\bar{y}-\mu)^2}{2}\right) d\sigma^2$

$\underbrace{\phantom{\int_0^\infty IG\left(\frac{n}{2}, \frac{(n-1)s^2 + n(\bar{y}-\mu)^2}{2}\right)}}_{=1}$

$= \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}$