Problem 6.1:

$$\sum_{i=1}^{n} (y_i - \mu)^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

$$LHS = \sum_{i=1}^{n} (y_i - \mu)^2 = \sum_{i=1}^{n} (y_i - \bar{y} + \bar{y} - \mu)^2$$

$$= \sum (y_i - \bar{y})^2 + \sum \underset{\wedge \text{ constant}}{(\bar{y} - \mu)^2} + 2\sum (y_i - \bar{y})(\bar{y} - \mu)$$

$$= \sum (y_i - \bar{y})^2 + n \cdot (\bar{y} - \mu)^2 + \left(2 \cdot \sum (y_i \bar{y} - \mu y_i - \bar{y}^2 + \bar{y}\mu)\right)$$

$$= 2\left(\bar{y} \sum^{n} y_i - \mu \sum y_i - n\bar{y}^2 + n\bar{y}\mu\right)$$

$$= \cancel{2n\bar{y}^2} - \cancel{2\mu n\bar{y}} - \cancel{2n\bar{y}^2} + \cancel{2n\bar{y}\mu} = 0$$

$$= \sum (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 = RHS \quad \square$$

Problem 6.2: (1) When mean is known, $\underline{\underline{\sigma^2 \sim IG}}$.

(2) $\sigma^2 \sim IG$, $\begin{cases} \text{mean} = \dfrac{\beta}{\alpha - 1} = 12 \\ \text{var} = \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} = 4 \end{cases} \Rightarrow \begin{cases} \beta = 12(\alpha - 1) \\ \dfrac{144(\alpha-1)^2}{(\alpha-1)^2(\alpha-2)} = 4 \end{cases} \Rightarrow \begin{cases} \alpha = 38 \\ \beta = 444 \end{cases}$

$\Rightarrow \sigma^2 \sim \underline{\underline{IG(38, 444)}}$

(3) $L(y|\sigma^2) = \dfrac{1}{(\sigma^2)^{\frac{n}{2}}} \cdot e^{-\frac{\sum (y_i - \mu)^2}{2\sigma^2}}$

$$\Rightarrow p(\sigma^2|y) \propto \text{Prior} \cdot L = \frac{1}{(\sigma^2)^{\alpha + \frac{n}{2} + 1}} \cdot e^{-\frac{\beta + \frac{\sum (y_i - \mu)^2}{2}}{2\sigma^2}}$$

$$\sim \underline{\underline{IG\left(\alpha + \frac{n}{2}, \beta + \overset{?}{\sum (y_i - \mu)^2 / 2}\right)}}$$

$\sum (y_i - \mu)^2 = (46 - 51)^2 + (58 - 51)^2 + (40 - 51)^2 + \dots + (50 - 51)^2 = 368$

$\Rightarrow p(\sigma^2|y) \sim IG(38 + 10, 444 + 368/2) = \underline{\underline{IG(48, 628)}}$

(4) $\mu_{post} = \dfrac{\beta^*}{\alpha^* - 1} = \dfrac{628}{48 - 1} = \underline{\underline{13.362}}$

$\text{Var}_{post} = \dfrac{(\beta^*)^2}{(\alpha^* - 1)^2(\alpha^* - 2)} = \dfrac{628^2}{47^2 \, 46} = \underline{\underline{3.8812}}$

(5) We assume knowing the true pop. mean of candy-making process. It's unreasonable because the properties of machines are very different. Therefore, it's **not applicable** to this case if we want it to be accurate.

(Extra Credit) Problem 6.3:

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f(x)\left(\frac{1}{y}\right)\left|\frac{d}{dy}\left(\frac{1}{y}\right)\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \left(\frac{1}{y}\right)^{\alpha-1} e^{-\beta\left(\frac{1}{y}\right)} \cdot \left|-\frac{1}{y^2}\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1+2} \cdot e^{-\frac{\beta}{y}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{y^{\alpha+1}} \cdot e^{-\frac{\beta}{y}} \quad \sim IG(\alpha, \beta) \quad \square$$