Problem 4.1:

- **My prior beta (41, 81):**

The posterior is beta (41+5, 81+25) = beta (46, 106).

(a) The posterior mean is $\frac{\alpha}{\alpha+\beta} = \frac{46}{46+106} = \mathbf{0.3026}$. The mode is $\frac{\alpha-1}{\alpha+\beta-2} = \frac{46-1}{46+106-2} = \mathbf{0.3}$.

(b) The 95% posterior interval is **(0.2324, 0.3778)**.

```
> qbeta(c(.025,.975),46,106)
[1] 0.2324309 0.3777516
```

(c) $Pr(\theta > 0.25|y) = 1 - \Pr(\theta \le 0.25|y) = \mathbf{0.9252}$

```
> 1-pbeta(0.25,46,106)
[1] 0.9252302
```

- **Mother's prior beta (40, 80):**

The posterior is beta (40+5, 80+25) = beta (45, 105).

(a) The posterior mean is $\frac{\alpha}{\alpha+\beta} = \frac{45}{45+105} = \mathbf{0.3}$. The mode is $\frac{\alpha-1}{\alpha+\beta-2} = \frac{45-1}{45+105-2} = \mathbf{0.2973}$.

(b) The 95% posterior interval is **(0.2296, 0.3755)**.

```
> qbeta(c(.025,.975),45,105)
[1] 0.2295700 0.3754814
```

(c) $Pr(\theta > 0.25|y) = 1 - \Pr(\theta \le 0.25|y) = \mathbf{0.9130}$

```
> 1-pbeta(0.25,45,105)
[1] 0.9130111
```

- **Uniform prior beta (1, 1):**

The posterior is beta (1+5, 1+25) = beta (6, 26).

(a) The posterior mean is $\frac{\alpha}{\alpha+\beta} = \frac{6}{6+26} = \mathbf{0.1875}$. The mode is $\frac{\alpha-1}{\alpha+\beta-2} = \frac{6-1}{6+26-2} = \mathbf{0.1667}$.

(b) The 95% posterior interval is **(0.0745, 0.3373)**.

```
> qbeta(c(.025,.975),6,26)
[1] 0.0745199 0.3372716
```

(c) $Pr(\theta > 0.25|y) = 1 - \Pr(\theta \le 0.25|y) = \mathbf{0.1764}$

```
> 1-pbeta(0.25,6,26)
[1] 0.176416
```

Problem 4.2:

1. Since **p-value = 0.0002** < 0.05, we conclude there is **sufficient evidence** to say less than half of the voters support the tax.

```
> binom.test(131,327,p=0.5,alternative="less")

        Exact binomial test

data:  131 and 327
number of successes = 131, number of trials = 327, p-value = 0.0001928
alternative hypothesis: true probability of success is less than 0.5
95 percent confidence interval:
 0.0000000 0.4472553
sample estimates:
probability of success
             0.4006116
```

2. Uniform prior beta (1, 1), binomial likelihood is $L(\pi|y) = \binom{327}{131}\pi^{131}(1-\pi)^{327-131}$.

The posterior is $\propto \pi^{131}(1-\pi)^{196}$, which is beta (132, 197).

For null hypothesis $\pi < 0.5$, **the probability is 0.0001591, which is very small**. Therefore, we conclude there is **sufficient evidence** that less than half of the voter support the tax.

```
> 1-pbeta(0.5,132,197)
[1] 0.0001590998
```

Problem 4.3:

1. The posterior predictive probability of y* = 8 of n = 20 is

$$\Pr(y^*|\pi) = \int_0^1 \binom{20}{8}\pi^8(1-\pi)^{12} * \pi^{131}(1-\pi)^{196}d\pi = \cdots \pi^{139}(1-\pi)^{208}d\pi$$
$$= E\big(beta(140,209)\big) = \mathbf{\frac{140}{349}}$$

2. By trying several sets, the moment when the total probability exceeds 0.95 is the y* set from 4 to 12. Therefore, we find the posterior predictive interval for y* to be **(4, 5, 6, ..., 11, 12).**

Problem 5.1:

1. The likelihood function is $\binom{30}{5}\pi^5(1-\pi)^{25}$

The log likelihood is $L(\pi|y) = 5log\pi + 25(1-\pi) + c.\ L' = \frac{5}{\pi} + \frac{25}{1-\pi}.\ L'' = \frac{5}{\pi^2} + \frac{25}{(1-\pi)^2}$

$I(\pi|y) = -E(L'') = \frac{30}{\pi(1-\pi)}$, we get $\Pr(\pi) \propto \pi^{-\frac{1}{2}}(1-\pi)^{-\frac{1}{2}} \sim Beta\left(\frac{1}{2},\frac{1}{2}\right)$

The posterior is Beta (0.5+5, 0.5+25) = **Beta (5.5, 25.5).**

(a) The posterior mean is 5.5/(5.5+25.5) = **0.1774**. The mode is **0.1552**.

(b) The 95% posterior interval is **(0.0666, 0.3274)**.

```
> qbeta(c(.025,.975),5.5,25.5)
[1] 0.06657395 0.32742775
```

(c) $Pr(\theta > 0.25|y) = 1 - \Pr(\theta \le 0.25|y) = \mathbf{0.1445}$

```
> 1-pbeta(0.25,5.5,25.5)
[1] 0.1445038
```

2. For improper Beta (0, 0), the posterior is Beta (0+5, 0+25) = **Beta (5, 25).**

(a) The posterior mean is 5 /(5+25) = **0.1667**. The mode is **0.1429**.

(b) The 95% posterior interval is **(0.0585, 0.3166)**.

```
> qbeta(c(.025,.975),5,25)
[1] 0.05845608 0.31664061
```

(c) $Pr(\theta > 0.25|y) = 1 - \Pr(\theta \le 0.25|y) = \mathbf{0.1153}$

```
> 1-pbeta(0.25,5,25)
[1] 0.1153243
```

The two priors **are both prior** because their integral is finite.

Problem 5.2:

Name:	**Prior Density:**	$\boldsymbol{Pr(\theta > 0.25|y)}$	**95% equal tail credible set**
My prior	Beta (41, 81)	0.9252	(0.2324, 0.3778)
Mother's prior	Beta (40, 80)	0.9130	(0.2296, 0.3755)
Uniform	Beta (1, 1)	0.1764	(0.0745, 0.3373)
Jeffery Prior	Beta (0.5, 0.5)	0.1445	(0.0666, 0.3274)
Improper (0,0)	Beta (0, 0)	0.1153	(0.0585, 0.3166)

The main purpose of our test is comparing whether the record of player getting hits at bat is similar as our expectations. We might designate our primary test of hypothesis of $\pi > 0.25$. In this case, the analysis is **sensitive and not robust** to different priors since there's about 92.52% of chance getting $\pi > 0.25$.

If we decide to obtain a point estimate and 95% credible set for π, we still conclude the analysis is **sensitive and moderately robust** to different priors. It's because their posteriors are quite different.