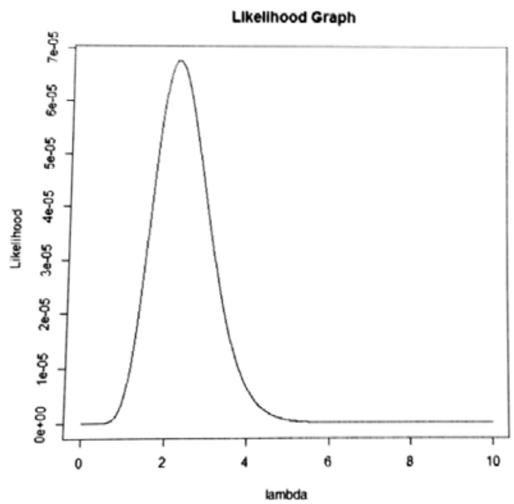
Problem 5.4:

1. $fy(y1 \ldots yn|\lambda) = \prod_{i=0}^{n} \frac{e^{-\lambda}\lambda^{yi}}{yi!} = \boldsymbol{\frac{e^{-n\lambda}\lambda^{n\bar{y}}}{\prod_{i=1}^{n}(yi!)}}$

2. $\bar{y} = \frac{1}{5}(2+5+1+0+3) = 2.2$

$$L(\lambda) = \prod_{i=0}^{5} \frac{e^{-\lambda}\lambda^{yi}}{yi!} = \frac{e^{-5\lambda}\lambda^{5*2.2}}{2!\,5!\,1!\,0!\,3!} = \boldsymbol{\frac{e^{-5\lambda}\lambda^{11}}{1440}}$$



3. $\frac{\partial L(*)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{e^{-5\lambda}\lambda^{11}}{1440}\right) = \frac{-5e^{-5\lambda}\lambda^{10}(-5\lambda+11)}{1440} = 0$

$$\frac{\partial^2}{\partial x^2}\left(\frac{e^{-5\lambda}\lambda^{11}}{1440}\right) < \mathbf{0}$$

Therefore, $\hat{\lambda} = \frac{11}{5} = 2.2 = \bar{y}$ is **the mle of $\boldsymbol{\lambda}$**.

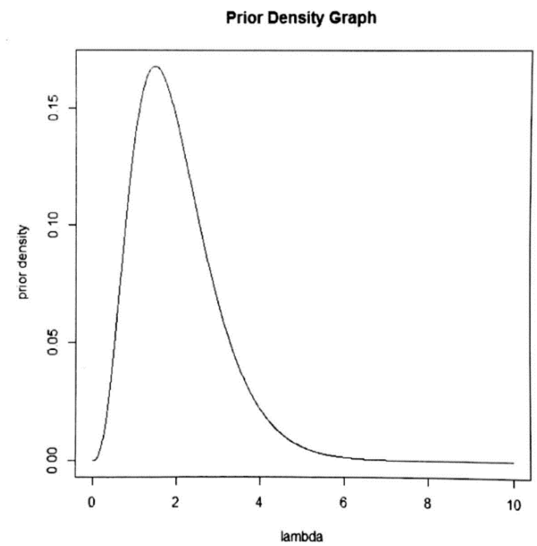
```
> poisson.test(11,5,alternative = c("two.sided","less","greater"),
+ conf.level=0.95)

        Exact Poisson test

data:  11 time base: 5
number of events = 11, time base = 5, p-value = 0.02043
alternative hypothesis: true event rate is not equal to 1
95 percent confidence interval:
 1.098232 3.936408
sample estimates:
event rate
       2.2
```

The R output has a same **mle of 2.2**. The 95% CI for λ is **(1.0982, 3.9364)**

Problem 5.5:



1. The prior is a member of **Gamma (4,2).**

2. The plot indicates that the president's prior belief about rate parameter is maximized around somewhere between 1.6 to 2.5. The majority of expected number of breakdowns are distributed around 0 to 5.

3. $p(\lambda|y) = p(\lambda)L(\lambda) \propto \lambda^3 e^{-2\lambda}\left(\frac{e^{-5\lambda}\lambda^{11}}{1440}\right) = \frac{\lambda^{14}e^{-7\lambda}}{1440} = \boldsymbol{Gamma(15,7)}$

4. Mean is 15/7 = **2.143**

5. **Yes**, it was. It's a Poisson distribution because they have the same likelihood as the one in problem 5.4.

Problem 5.6:

(1). $L(\lambda|y) = \frac{e^{-\lambda}\lambda^{y}}{y!}$ $\quad l(\lambda|y) = -\lambda + y\log\lambda - \log(y!)$

$\frac{\partial l}{\partial \lambda} = -1 + \frac{y}{\lambda}$; $\frac{\partial^2 l}{\partial \lambda^2} = -\frac{y}{\lambda^2}$

$\Rightarrow I(\lambda|y) = -E\left(\frac{\partial^2 l(\lambda|y)}{\partial \lambda^2}\right) = E\left(\frac{y}{\lambda^2}\right) = \frac{1}{\lambda}$

$I(\lambda|y^*) = n \times I(\lambda|y) = \frac{n}{\lambda}$

$P(\lambda) \propto \sqrt{I(\lambda|y)} = \sqrt{\frac{n}{\lambda}} \propto \sqrt{\frac{1}{\lambda}}$

(2) $p(\lambda|y) = p(\lambda)p(y|\lambda) \propto \sqrt{\frac{1}{\lambda}}\,\frac{e^{-5\lambda}\lambda^{11}}{1440} = \frac{e^{-5\lambda}\lambda^{10.5}}{1440} = Gamma(11.5, 5)$

(3) mean $= \frac{11.5}{5} = 2.3$

The 95% credible set is **(1.1689, 3.8076).**

```
> qgamma(c(0.025,0.975),11.5,rate=5)
[1] 1.168855 3.807563
```

4. The Bayesian point estimate is **larger** than the classical estimate (2.3 > 2.2). In addition, the interval estimate is **wider** than the classical interval estimate.

5. The employee's estimate is **larger** than that of the president since the employee has larger λ.

6. Employee:

```
> pgamma(2,11.5,rate=5,lower.tail=FALSE)
[1] 0.6419118
```

7. President:

```
> pgamma(2,15,rate=7,lower.tail=FALSE)
[1] 0.5704367
```

(problems in Section 6 are on next page)

Problem 6.1:

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2$$

$$LHS = \sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n}(y_i-\bar{y}+\bar{y}-\mu)^2$$

$$= \sum(y_i-\bar{y})^2 + \sum(\bar{y}-\mu)^2 + 2\sum(y_i-\bar{y})(\bar{y}-\mu)$$

^ constant

$$= \sum(y_i-\bar{y})^2 + n\cdot(\bar{y}-\mu)^2 + \left(2\cdot\sum(y_i\bar{y} - \mu y_i - \bar{y}^2 + \bar{y}\mu)\right)$$

$$= 2\left(\bar{y}\sum^{n} y_i - \mu\sum y_i - n\bar{y}^2 + n\bar{y}\mu\right)$$

$$= \cancel{2n\bar{y}^2} - \cancel{2\mu n\bar{y}} - \cancel{2n\bar{y}^2} + \cancel{2n\bar{y}\mu} = 0$$

$$= \sum(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2 = RHS \quad \square$$

Problem 6.2:

(1) When mean is known, $\sigma^2 \sim IG$.

(2) $\sigma^2 \sim IG$, $\begin{cases} \text{mean} = \dfrac{\beta}{\alpha-1} = 12 \\ \text{var} = \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} = 4 \end{cases} \Rightarrow \begin{cases} \beta = 12(\alpha-1) \\ \dfrac{144(\alpha-1)^2}{(\alpha-1)^2(\alpha-2)} = 4 \end{cases} \Rightarrow \begin{cases} \alpha = 38 \\ \beta = 444 \end{cases}$

$\Rightarrow \sigma^2 \sim IG(38, 444)$

(3) $L(y|\sigma^2) = \dfrac{1}{(\sigma^2)^{\frac{n}{2}}} \cdot e^{-\frac{\sum(y_i-\mu)^2}{2\sigma^2}}$

$$\Rightarrow p(\sigma^2|y) \propto \text{Prior}\cdot L = \frac{1}{(\sigma^2)^{\alpha+\frac{n}{2}+1}} \cdot e^{-\frac{\beta + \frac{\sum(y_i-\mu)^2}{2}}{2\sigma^2}}$$

?

$$\sim IG\left(\alpha + \frac{n}{2},\ \beta + \sum(y_i-\mu)^2/2\right)$$

$\sum(y_i-\mu)^2 = (46-51)^2 + (58-51)^2 + (40-51)^2 + \dots + (50-51)^2 = 368$

$\Rightarrow p(\sigma^2|y) \sim IG(38+10,\ 444 + 368/2) = IG(48, 628)$

(4) $\mu_{post} = \dfrac{\beta^*}{\alpha^*-1} = \dfrac{628}{48-1} = 13.362$

$Var_{post} = \dfrac{(\beta^*)^2}{(\alpha^*-1)^2(\alpha^*-2)} = \dfrac{628^2}{47^2\,46} = 3.8812$

(5) We assume knowing the true pop. mean of candy-making process. It's unreasonable because the properties of machines are very different. Therefore, it's <u>not applicable</u> to this case if we want it to be accurate.

(Extra Credit) Problem 6.3:

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f(x)\left(\tfrac{1}{y}\right)\left|\frac{d}{dy}\left(\tfrac{1}{y}\right)\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \left(\frac{1}{y}\right)^{\alpha-1} e^{-\beta\left(\frac{1}{y}\right)} \cdot \left|-\frac{1}{y^2}\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1+2} \cdot e^{-\frac{\beta}{y}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{y^{\alpha+1}} \cdot e^{-\frac{\beta}{y}} \sim IG(\alpha, \beta) \quad \square$$