Hw 5:

STAT: 4520.

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7.1. (1). Non-info prior: $p(\mu, \sigma^2) \propto \frac{1}{\sigma^2}$ $(-\infty < \mu < \infty)$ $(0 < \sigma^2 < \infty)$

$$L(y|\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}$$

$$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}}} e^{-1/2\sigma^2 \cdot \sum_{i=1}^{n}(y_i-\mu)^2}$$

The ~~prior~~ post. : $P(\mu,\sigma^2|y) \propto P(\mu,\sigma^2) \cdot L(y|\mu,\sigma^2)$

$$= \frac{1}{\sigma^2} \times \frac{1}{(\sigma^2)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2\right]}$$

$$= \frac{1}{(\sigma^2)^{\frac{n+1}{2}}} \cdot e^{-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]}$$

Normalize: $P(\mu,\sigma^2|y) = \frac{\left(\frac{(n-1)s^2}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{(\sigma^2)^{\frac{n+1}{2}}} e^{-\frac{(n-1)s^2}{2\sigma^2}} \cdot \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2}}$

$$P(\mu|y) = \int_0^\infty P(\mu,\sigma^2|y)\, d\sigma^2.$$

$(t_{n-1} = t_{19} \Rightarrow)$ $= \dfrac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi} \cdot \frac{s}{\sqrt{n}}\left[1+\frac{1}{n-1}\left(\frac{\mu-\bar{y}}{s/\sqrt{n}}\right)^2\right]^{\frac{n}{2}}}$

scale: $\frac{s^2}{n} = \frac{4.2797^2}{20} = 0.9158$

$$= \frac{\Gamma(10)}{\Gamma(9.5)\sqrt{19\pi} \cdot \frac{4.2797}{\sqrt{20}} \cdot \left[1+\frac{1}{19}+\left(\frac{\mu-50}{4.2797/\sqrt{20}}\right)^2\right]^{10}}$$

post. mean $= \bar{y} = 50$.

$\Rightarrow$ 95% CS for μ: $\boxed{(47.997,\ 52.003)}$

(R: 50 + qt(c(.025, .975), 19) * sqrt(0.9158))

(2). $p(\sigma^2|y) \propto \int_{-\infty}^{\infty} \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{-\frac{1}{2\sigma^2}[(n-1)s^2+n(\bar{y}-\mu)^2]} d\mu$

$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} \cdot e^{-\frac{(n-1)s^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\frac{n(\bar{y}-\mu)^2}{2\sigma^2}} d\mu$

$\propto \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{\frac{(n-1)s^2}{2\sigma^2}} \quad \sim IG\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

$= \boxed{IG(9.5, 17.4)}$

$p(\sigma^2|y) = \frac{\left(\frac{348}{2}\right)^{\frac{19}{2}}}{\Gamma\left(\frac{19}{2}\right)} \cdot \frac{1}{(\sigma^2)^{\frac{19}{2}+1}} \cdot e^{-\frac{348}{2\sigma^2}}$

$= \frac{174^{9.5}}{\Gamma(9.5)} \cdot \frac{1}{\sigma^{21}} \cdot e^{-\frac{174}{\sigma^2}} \quad (0<\sigma^2<\infty)$

$E(\sigma^2|y) = \frac{\beta}{\alpha-1} = \frac{\frac{348}{2}}{\frac{20-1}{2}-1} = 20.4705\underline{9}$

$\text{Sd. Err} = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} = \sqrt{\frac{174^2}{8.5^2 \times 7.5}} = 7.4748$

95% CS for σ^2 : $(10.5929, 39.0725)$

(R: 1/ gamma (c(.975, .025)), 9.5 , 174)

7.2. (1). Conj. Prior: $P(\mu, \sigma^2) = P(\sigma^2) P(\mu|\sigma^2) = IG(\alpha, \beta) \times N(\mu_0, \frac{\sigma^2}{k})$

$$= IG(38, 444) \times N(51, \frac{\sigma^2}{10})$$

$$= \frac{444^{38}}{\Gamma(38)} \frac{1}{(\sigma^2)^{39}} e^{-\frac{444}{y}} \times \frac{\sqrt{10}}{\sqrt{2\pi\sigma^2}} e^{-\frac{10(\mu-51)^2}{2\sigma^2}}$$

$$P(\mu, \sigma^2 | y) \propto \frac{\sqrt{k}}{\sigma^2} e^{-\frac{k(\mu-\mu_0)^2 + n(\mu-\bar{y})^2}{2\sigma^2}}$$

$$\propto \frac{\sqrt{k}}{\sigma} e^{-(k+n)\left(\frac{\mu - \frac{k\mu_0 + n\bar{y}}{k+n}}{2\sigma^2}\right)^2}$$

$$= \frac{\sqrt{10}}{\sigma} e^{-(30)\frac{\left(\mu - \frac{510+1000}{30}\right)^2}{2\sigma^2}}$$

$$\Rightarrow \mu, \sigma^2 | y \sim IG\left(\alpha + \frac{n}{2}, \beta + \frac{(n-1)S^2}{2} + \frac{kn(\bar{y}-\mu_0)^2}{2(k+n)}\right) \times N\left(\frac{k\mu_0 + n\bar{y}}{k+n}, \frac{\sigma^2}{k+n}\right)$$

$$= IG\left(38 + \frac{20}{2}, 444 + \frac{348}{2} + \frac{10 \times 20 \times (50-51)^2}{2(10+20)}\right) \times N\left(\frac{10 \times 51 + 20 \times 50}{10+20}, \frac{\sigma^2}{10+20}\right)$$

$$= IG(48, 621.333) \times N(50.333, \frac{\sigma^2}{30})$$

(2). $p(\mu|y)$ is t density w/

mean: $\frac{k\mu_0 + n\bar{y}}{k+n} = \frac{10 \times 51 + 20 \times 50}{10+20} = 50.333$

scale parameter: $\dfrac{n_0\sigma_0^2 + (n-1)S^2 + \frac{kn(\bar{y}-\mu_0)^2}{k+n}}{(k+n)(n_0+n)}$

$$= \left(19 \times 46.7368 + 348 + \frac{10 \times 20 (50-51)^2}{30}\right) / (30 \times (19+20))$$

$$= 0.4315$$

$df = n + n_0 = 20 + 19 = 39$

95% CS: $\mu \pm t_{39,(0.95)} \sqrt{\text{scale para.}}$

$$= (48.2488, 52.4179)$$

13.7. $p(\sigma^2 | y) \sim IG\left(\frac{n}{2} + \alpha, \frac{\sum (y_i - \mu)^2}{2} + \beta\right)$

$$= IG\left(\frac{20}{2} + 38, \frac{368}{2} + 444\right) = IG(48, 621.333)$$

$$\text{mean} = \frac{\beta'}{\alpha' - 1} = \frac{628}{48 - 1} = 13.3617 = E(\sigma^2 | y)$$

95% CI for σ^2: $(10.04799, 17.74442)$

Extra Credit:

7.3 $$p(\mu, \sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{(\sigma^2)^{\alpha+1}} e^{-\frac{\beta}{\sigma^2}} \cdot \frac{\sqrt{k}}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-k(\mu-\mu_0)^2}{2\sigma^2}}$$

where $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$

$$\therefore p(\mu, \sigma^2) \propto \frac{1}{(\sigma^2)^{\frac{n_0+3}{2}}} e^{-\frac{n_0\sigma_0^2 + k(\mu-\mu_0)^2}{2\sigma^2}}$$

$$\to \frac{1}{\sigma^2} \quad (\text{as } n_0 \to -1, \; k, \sigma_0^2 \to 0)$$

Extra Credit:

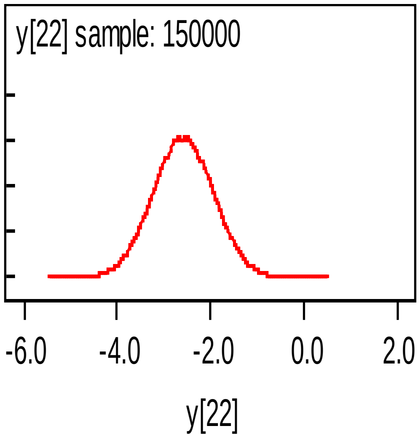
7.4. $$\int_0^\infty \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{\frac{-(n-1)s^2 + n(\mu-\bar{y})^2}{2\sigma^2}} d\sigma^2 = \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}$$

$$= \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}} \cdot \int_0^\infty \frac{\left[\frac{(n-1)s^2 + n(\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \cdot \frac{1}{(\sigma^2)^{\frac{n}{2}+1}} e^{\frac{-(n-1)s^2 + n(\bar{y}-\mu)^2}{2\sigma^2}} \cdot d\sigma^2$$

$$= \sim \cdot \int_0^\infty \underbrace{IG\left(\frac{n}{2}, \frac{(n-1)s^2 + n(\bar{y}-\mu)^2}{2}\right) d\sigma^2}_{=1}$$

$$= \frac{\Gamma(\frac{n}{2})}{\left[\frac{(n-1)s^2 + (\mu-\bar{y})^2}{2}\right]^{\frac{n}{2}}}$$

Problem 8.2:



	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
y[22]	-2.589	0.6481	0.001669	-3.858	-2.59	-1.322	1001	150000

Comparing to the result of 6.2.8.3, the mean is quite similar. We obtained -2.589 here and -2.586 in 6.2.8.3. Standard deviation here is 0.6481 while the sd is 1/60 =0.01667 is 6.2.8.3. The difference indicates that the **Openbugs output is more spread out**. In other words, the distribution is 6.2.8.3 is more concentrated around the center of -2.586.

Problem 8.4:

1. Since the conjugate family of the prior is IG(38,444), the conjugate family prior for normal precision is **Gamma(38,444).** Presidion = 1/variance.

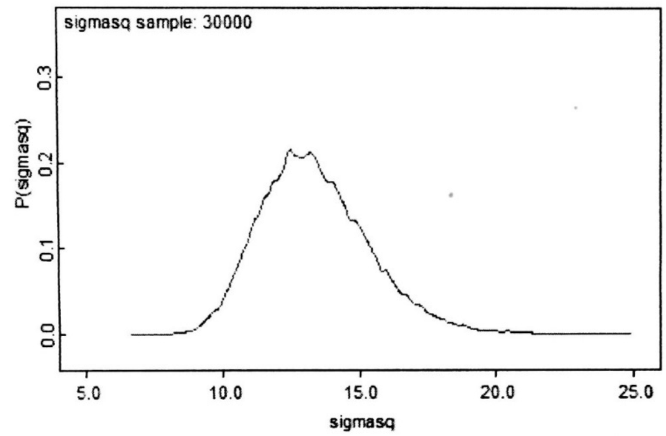
2.

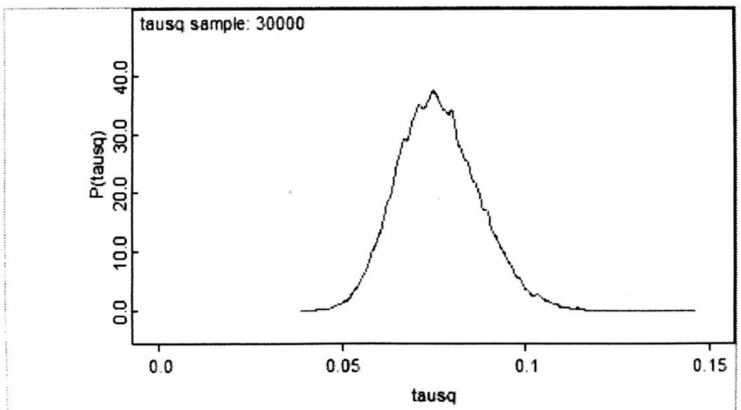
```
model
{

# likelihood
for (i in 1:N) {
y[i] ~ dnorm( mu, tausq )
}
# priors
tausq ~ dgamma( 38, 444)
sigmasq <- 1/tausq
}

#data
list(y=c(46, 58, 40, 47, 47, 53, 43, 48, 50, 55, 49, 50, 52, 56, 49,
54, 51, 50, 52, 50), N=20, mu=51)

#inits for model 1
list(mu = 5)
list(mu = 10)
list(mu = 15)
```





The stat output is:

	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
sigmasq	13.37	1.975	0.01074	10.04	13.19	17.73	1	30000
tausq	0.07641	0.01109	6.012E-5	0.05639	0.07582	0.09961	1	30000

Sigmasq mean in 6.2 is 13.362, which is very close to that of Openbugs but **slightly smaller** than what Openbugs generates. It makes sense because Openbugs drew a large sample size to make the result more preciously. For variance, the estimated posterior variance is $1.975^2 = 3.901$, which is **consistent with** the previous result.