MAT128A: Numerical Analysis, Section 2
Homework for the week of October 15

1. Suppose that $n$ is a nonnegative integer. You know that the function $y(t) = \cos(nt)$ satisfies the second order differential equation

$$\ddot{y}(t) + n^2 y(t) = 0 \quad \text{for all} \quad -\pi < t < \pi.$$

Use this observation to show that the function

$$T_n(x) = \cos(n \arccos(x))$$

is a solution of the equation

$$(1 - x^2) y''(x) - x y'(x) + n^2 y(x) = 0 \quad \text{for all} \quad -1 < x < 1.$$

Here, I am using $y'$ to denote the derivative of $y$ with respect to $x$ and $\dot{y}$ to denote the derivative of $y$ with respect to $t$.

Hint: use the chain rule to compute

$$\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2}$$

in terms of

$$\frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2}.$$

2. Show that

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \neq 0. \end{cases}$$

3. (a) Using the trigonometric identity

$$\cos(nt) = \cos((n-1)t)\cos(t) - \sin((n-1)t)\sin(t),$$

show that

$$T_n(x) = x T_{n-1}(x) - U_{n-1}(x) \sqrt{1-x^2}, \tag{1}$$

where $U_n$ is defined via

$$U_n(x) = \sin(n \arccos(x)).$$

(b) Use the trigonometric identity

$$\sin(nt) = \sin((n-1)t)\cos(t) + \cos((n-1)t)\sin(t),$$

to show that

$$U_n(x) = T_{n-1}(x) \sqrt{1-x^2} - U_{n-1}(x) x. \tag{2}$$

(c) Combine (1) and (2) to show that

$$U_n(x)\sqrt{1-x^2} = T_{n-1}(x) + xT_n(x). \tag{3}$$

(d) Use (3) and (1) — replace $n$ with $n+1$ in (1) — to obtain the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

4. Suppose that $n$ is a nonnegative integer. Show that

$$(1-x^2)T_n'(x) = nT_{n-1}(x) - nxT_n(x)$$

for all $-1 < x < 1$.