MAT128A: Numerical Analysis, Section 2
Homework for the week of October 8

1. Show that when the n-point periodic trapezoidal rule is used to evaluate the integral

$$\int_{-\pi}^{\pi} \exp(ikt)\ dt,$$

the result is

$$\begin{cases} (-1)^{|k|}\ 2\pi & \text{if } k = m \cdot n \text{ for some nonzero integer } m \\ 2\pi & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is even — meaning that $f(-x) = f(x)$ for all $0 < x \leq \pi$ — then f can be represented via a convergent series of the form

$$f(t) = \sum_{n=0}^{\infty} b_n \cos(nt).$$

3. Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is odd — meaning that $f(-x) = -f(x)$ for all $0 < x \leq \pi$ — then f can be represented via a convergent series of the form

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(nt).$$

4. Suppose that

$$f(t) = \cos(2t) + \cos(4t) + \cos(6t) + \ldots + \cos(20t).$$

What is the exact value of

$$\int_{-\pi}^{\pi} f(t)\ dt? \tag{1}$$

How long is the periodic trapezoidal rule of minimum length which evaluates (1) exactly? That is, what is the least positive integer n such that

$$\int_{-\pi}^{\pi} f(t)\ dt = \frac{2\pi}{n} \sum_{j=0}^{n-1} f\left(-\pi + \frac{2\pi}{n} j\right)? \tag{2}$$

Here, we are assuming that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

5. Let

$$f(t) = \sum_{n=1}^{\infty} a_n \exp(int)$$

with

$$|a_n| \leq \frac{1}{n^2}.$$

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt$$

is bounded above by

$$\frac{\pi^2}{6} \frac{1}{N^2}.$$

Again, we are assuming that exact arithmetic is used to perform the calculations so that we needn't worry about roundoff error.

(Hint: look at the solutions from the previous homework assignment to find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

6. Let

$$f(t) = \sum_{n=0}^{\infty} a_n \exp(int)$$

with

$$|a_n| \leq \frac{1}{2^n}.$$

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt$$

is bounded above by

$$\frac{1}{2^N - 1}.$$

Once again, we are assuming that exact arithmetic is used to perform the calculations so that we needn't worry about roundoff error.

7. Find the Fourier series for the function

$$f(t) = \frac{2}{\exp(it) - 2}$$

by using the identity

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j,$$

which holds for all complex-valued z such that $|z| < 1$.