MAT 128A - Assignment 4 (Revised solutions)

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Problem 1

Suppose that n is a nonnegative integer. Given that the function $y(t) = cos(nt)$ satisfies the second order differential equation

$$\ddot{y}(t) + n^2 y(t) = 0 \quad \text{for all } -\pi < t < \pi$$

Use this observation to show that the function $T_n(x) = cos(ncos^{-1}(x))$, where $cos^{-1}(x) = arccos(x)$, is a solution of the equation

$$(1 - x^2)y''(x) - xy'(x) + n^2 y(x) = 0 \quad \text{for all } -1 < x < 1$$

Here we use $y' := \frac{dy}{dx}$ and $\dot{y} = \frac{dy}{dt}$.

Hint: Use the chain rule to compute $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Ans: The key observation here is the change of variable $t = cos^{-1}x$, so

$$\frac{dx}{dt} = -\sqrt{1 - x^2}, \quad \frac{d^2x}{dt^2} = -\frac{d\sqrt{1 - x^2}}{dt} = \frac{x}{\sqrt{1 - x^2}}\frac{dx}{dt} = -x$$

Using chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\sqrt{1 - x^2}\frac{dy}{dt}, \quad \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} = (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$$

Also, $-\pi < t < \pi \quad \Rightarrow \quad -1 = cos(-\pi) < x < cos(\pi) = 1$. Therefore,

$$0 = \ddot{y}(t) + n^2 y(t) = (1 - x^2)y''(x) - xy'(x) + n^2 y(x) \quad \text{for all } -1 < x < 1$$

Problem 2

Show that

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

Ans: The answer is already given in the class. The key observation is that

$$\int_0^{\pi} cos(nt)cos(mt)dt = \begin{cases} \pi & \text{if } n = m = 0 \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

Proof. For $m = n = 0$, $\int_0^{\pi} 1dt = \pi$.

For $m = n \neq 0$, $\int_0^{\pi} cos^2(mt)dt = \frac{1}{2}\int_0^{\pi} cos(2mt) + 1dt = \frac{\pi}{2}$.

For $m \neq n$, $\int_0^{\pi} cos(nt)cos(mt)dt = \frac{1}{2}\int_0^{\pi} cos[(n+m)t] + cos[(n-m)t]dt = 0$

Use the change of variable $t = cos^{-1}x \ \Rightarrow \ dt = -\frac{1}{\sqrt{1-x^2}}dx$, therefore

$$\int_0^{\pi} cos(nt)cos(mt)dt = -\int_1^{-1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

Problem 3

(a) Using the trigonometric identity

$$cos(nt) = cos((n-1)t)cos(t) - sin((n-1)t)sin(t)$$

Show that

$$T_n(x) = xT_{n-1}(x) - U_{n-1}(x)\sqrt{1-x^2} \tag{1}$$

where U_n is defined via

$$U_n(x) = sin(ncos^{-1}(x)).$$

Ans: Again, substitute $t = cos^{-1}x$ ($\Leftrightarrow x = cost$) into the given trigonometric identity, $L.H.S. = T_n(x)$ follows immediately.

$$R.H.S. = \underbrace{cos\left((n-1)cos^{-1}x\right)}_{T_{n-1}} x - \underbrace{cos\left((n-1)sin^{-1}x\right)}_{U_{n-1}} sin(t) \overset{(*)}{=} xT_{n-1} - \sqrt{1-x^2}U_{n-1}$$

$(*)$ holds since $sin(t) = \sqrt{1-cos^2(t)} = \sqrt{1-x^2}$

Problem 3

(b) Use the trigonometric identity

$$sin(nt) = sin((n-1)t)cos(t) + cos((n-1)t)sin(t)$$

to show that

$$U_n(x) = T_{n-1}(x)\sqrt{1-x^2} + U_{n-1}(x)x \tag{2}$$

Ans: Again, directly substitute $t = cos^{-1}x$.

$L.H.S = U_n(x)$ follows immediately.

$$R.H.S. = \underbrace{cos((n-1)cos^{-1}x)}_{T_n(x)} \underbrace{sin(t)}_{\sqrt{1-x^2}} + \underbrace{sin((n-1)cos^{-1}x)}_{U_{n-1}} cos(cos^{-1}x) = T_n(x)\sqrt{1-x^2} + U_{n-1}x.$$

Problem 3

(c) Combine (1) and (2) to show that

$$U_n(x)\sqrt{1-x^2} = T_{n-1}(x) - xT_n(x) \tag{3}$$

Ans: We multiply (1) by x, multiply (2) by $\sqrt{1-x^2}$, and then add both of them together in order to get rid of the $U_{n-1}(x)$ term, we have

$$xT_n(x) + \sqrt{1-x^2}U_n(x) = x^2T_{n-1}(x) + (1-x^2)T_{n-1}(x) \quad \Rightarrow \quad U_n(x)\sqrt{1-x^2} = T_{n-1}(x) - xT_n(x)$$

Problem 3

(d) Use (3) and (1) — replace n with $n+1$ in (1) — to obtain the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Ans: Following the instruction by replacing n with $n+1$ in (), we have

$$T_{n+1}(x) = xT_n(x) - U_n(x)\sqrt{1-x^2},$$

Substitute (3) into it, we have $T_{n+1}(x) = xT_n(x) - (T_{n-1}(x) - xT_n(x)) = 2xT_n(x) - T_{n-1}(x)$.

Alternatively, one can obtain the desired three-terms recurrence relation without going through parts (a) to (c). Using another trigonometric identity, namely

$$cos(nt) + cos((n-2)t) = 2cos(t)cos((n-1)t) \quad \Rightarrow \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Bonus I: Here is another relation I can think out of, can you prove it by using another trigonometric identity?

$$U_n - U_{n-2} = 2\sqrt{1-x^2}T_n \tag{4}$$

Bonus II: Using the trigonometric identity $2cosntcosmt = cos((n+m)t) + cos((n-m)t)$, we have the following product relation of the Chebyshev polynomials:

$$T_nT_m = \frac{1}{2}\left(T_{n+m} + T_{n-m}\right) \quad \text{assuming that } m \leqslant n$$

Bonus III: Indeed U_n is called the **"Chebyshev polynomials of the second kind"** whereas T_n is called the **"Chebyshev polynomials of the second kind"**. They are closely related. They satisfies other relations apart from those listed here.

Problem 4

Suppose that n is a nonnegative integer. Show that

$$(1-x^2)T_n'(x) = nT_{n-1}(x) - nxT_n(x)$$

for all $-1 < x < 1$.

Ans: Notice that the derivative $T_n'(x) = sin(ncos^{-1}(x))\frac{n}{\sqrt{1-x^2}} = U_n\frac{n}{\sqrt{1-x^2}}$.
Using the formula (3) in problem 3c)., we have

$$(1-x^2)T_n'(x) = n\sqrt{1-x^2}U_n = n\left(T_{n-1}(x) - xT_n(x)\right) = nT_{n-1} - nxT_n(x)$$

Remark 1. Using the recurrence relation in 3d)., we also have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \Rightarrow \quad T_{n+1}(x) - xT_n(x) = xT_n(x) - T_{n-1}(x)$$

Therefore, we can also write $(1-x^2)T_n'(x) = nT_{n-1}(x) - nxT_n(x) = nxT_n(x) - nT_{n+1}(x)$.

Bonus IV: There is one more interesting recurrence relation, namely

$$\begin{cases} T_0(x) &= T_1'(x), \\ T_1(x) &= \frac{1}{4}T_2'(x), \\ T_n(x) &= \frac{1}{2}\left(\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1}\right), \quad n \geqslant 2 \end{cases}$$

You should be able to prove this using results in problems 3 and 4.