MAT128A: Numerical Analysis, Section 2
Homework for the week of November 12, 2018

1. Find a polynomial p of degree 3 such that

$$p(0) = 0, \quad p(1) = 1, \quad p(2) = 1, \quad \text{and} \quad p'(0) = 1.$$

2. (a) Show that the roots of

$$p_N(x) = T_{N+1}(x) - T_{N-1}(x)$$

are

$$x_j = \cos\left(\frac{\pi}{N} j\right), \qquad j = 0, 1, \ldots, N.$$

(b) Use (a) to prove that

$$(x - x_0) \cdots (x - x_N) = 2^{-N} p_N(x).$$

(c) Show that

$$|(x - x_0) \cdots (x - x_N)| \leq 2^{-N+1}$$

for all $x \in [-1, 1]$.

3. Suppose that $f : [a, b] \to \mathbb{R}$ is $(N + 1)$-times continuously differentiable, and that $x_0, \ldots, x_N$ are the $(N + 1)$ nodes of the Chebyshev extrema grid on the interval $[a, b]$ so that

$$x_j = \frac{b-a}{2} \cos\left(\frac{j}{N}\pi\right) + \frac{b+a}{2} \quad \text{for all} \quad j = 0, 1 \ldots, N.$$

Also, let p_N be the polynomial of degree N which interpolates f at the nodes $x_0, x_1, \ldots, x_N$. Show that there exists $\xi \in (a, b)$ such that

$$|f(x) - p_k(x)| \leq 2^{-N+1} \left(\frac{b-a}{2}\right)^{N+1} \left|\frac{f^{(N+1)}(\xi)}{(N+1)!}\right|$$

Hint: Let $g(x) = f\left(\frac{b-a}{2}\, x + \frac{b+a}{2}\right)$ and use 2(c) to develop an error bound for g.

4. Suppose that $f(x) = \cos(x)$, that N is a positive integer, and that $x_0, \ldots, x_N$ are the nodes of the Chebyshev extrema grid on the interval $[0, 1]$. Also, let p_N denote the polynomial of degree N which interpolates f at the nodes $x_0, \ldots, x_N$. Show that

$$|f(x) - p_N(x)| \leq \frac{2^{-2N}}{(N+1)!}$$

for all $-1 \leq x \leq 1$.