MAT128A: Numerical Analysis, Section 2
Homework for the week of October 22

1. Suppose that f is a polynomial of degree N, and that

$$f(x) = \sum_{n=0}^{N} a_n T_n(x). \tag{1}$$

The Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Suppose that we define a finite sequence of polynomials $\{b_0(x), b_1(x), \ldots, b_N(x), b_{N+1}(x), b_{N+2}(x)\}$ via the formulas

$$b_{N+1} = b_{N+2} = 0$$

and

$$b_n(x) = a_n + 2xb_{n+1}(x) - b_{n+2}(x).$$

Show that

$$f(x) = b_0(x).$$

Hint: first show that if q_{N-1} is defined by

$$q_{N-1}(y) = \sum_{n=0}^{N-1} 2b_{n+1}(x)T_n(y),$$

then

$$(y - x)q_{N-1}(y) + b_0(x) = f(y) \tag{2}$$

and then let $y = x$ in (2).

2. The last problem suggests a method for computing the sum (1). How many arithmetic operations does it take to compute $f(x) = b_0(x)$ using this method?

Suppose that instead we sum (1) in the most direct way. That is, we first use the recurrence relations

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

to compute the values of $T_0(x), T_1(x), \ldots, T_n(x)$. We then form the values

$$a_0T_0(x), a_1T_1(x), \ldots, a_nT_n(x), \tag{3}$$

and sum them to obtain the value of $f(x)$. How many arithmetic operations does this more direct procedure take?

3. Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$f(x) = {\sum_{n=0}^{N}}' a_n T_n(x)$$

of the function $f(x) = \sqrt{1-x^2}$. Recall that the coefficients are defined via the formula

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}},$$

so computing a_n is equvalent to determing the value of the quantity

$$a_n = \frac{2}{\pi}\int_{-1}^{1} T_n(x)\ dx.$$

4. Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$f(x) = \sum_{n=0}^{N}{}' a_n T_n(x)$$

of the function

$$f(x) = \mathrm{sign}\,(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0. \end{cases}$$

Recall that the coefficients are defined via the formula

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}.$$