MAT 128A - Assignment 1

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Sauer's book, Chapter 0.3 - 3,4,11; Chapter 0.4 - 1,3; Chapter 0.5 - 1a, 2a, 4a

Chapter 0.3 Problem 3

For which positive integer k can the number $5 + 2^{-k}$ be represented exactly (with no rounding error) in double precision floating point arithmetic?

Ans: Since $5 = (101)_2 = 1.01 \times 2^2$,

$$\begin{aligned} 5 + 2^{-k} &= 1.01 \times 2^2 + 1 \times 2^{-k} = 1.01 \times 2^2 + 1 \times 2^2 \times 2^{-k-2} \\ &= (1.01 + 1 \times 2^{-k-2}) \times 2^2 \end{aligned}$$

To make an exact representation possible, i.e. $fl(5 + 2^{-k}) = 5 + 2^{-k}$, we need in the above expression $2^{-k-2} \geqslant \epsilon_{\text{machine}} := 2^{-52}$.

Therefore, $k + 2 \leqslant 52 \quad \Rightarrow \quad k \leqslant 50$. Since k needs to be a positive integer, the final answer is that k can be any integers between 1 and 50.

Chapter 0.3 Problem 4

Find the largest integer k for which $fl(19 + 2^{-k}) > fl(19)$ in double precision floating point arithmetic.

Ans: Similar to the question above,

$$\begin{aligned} 19 + 2^{-k} &= 1.0011 \times 2^4 + 1 \times 2^{-k} \\ &= (1.0011 + 2^{-k-4}) \times 2^4 \end{aligned}$$

For $fl(19+2^{-k}) > fl(19)$, we need $2^{-k-4} \geqslant \epsilon_{\text{machine}}$, so $k \leqslant 48$. Since k can be any integers (even negative), the largest integer is 48.

Chapter 0.3 Problem 11

Does the associative law hold for IEEE computer addition?

Ans: No, $a+(b+c) = (a+b)+c$ is not always true in floating point arithmetic. For example, try to perform the following computation in MATLAB:

```
>> x = (0.1 + 0.2) + 0.3;
>> y = 0.1 + (0.2 + 0.3);
>> x - y

ans =

     1.110223024625157e-16
```

Notice that $1.110223024625157e-16 = 1 \times 2^{-53}$. Can you explain why?

Hint: The logics is similar to the example in section 0.3.3 in Sauer's book.

For those of you who are interested, the floating point multiplication is not distributive over addition. Try to compare in MATLAB $10 \times (0.1+0.2)$ and $10 \times 0.1 + 10 \times 0.2$. Are they the same?

Chapter 0.4 Problem 1

Identify for which values of x there is subtraction of nearly equal numbers, and find an alternate form that avoids the problem.

$$(a). \frac{1-secx}{tan^2x} \qquad (b). \frac{1-(1-x)^3}{x} \qquad (c). \frac{1}{1+x} - \frac{1}{1-x}$$

1a - *Ans:* The loss of significance occurs when $1 - secx \approx 0$. Setting $1 - secx = 0$, we have $x = 2k\pi$ where k is an arbitrary integer. That is the answer for the values of x.

The reformulation makes use of the trigonometric identity $sec^2x = tan^2x + 1$, i.e.

$$\frac{1-secx}{tan^2x}\cdot\frac{1+secx}{1+secx} = \frac{1-sec^2x}{tan^2x(1+secx)} = \frac{-1}{1+secx}$$

Now there will not be any subtraction of nearly equal numbers.

Extra Info.: Using MATLAB, you will also find discrepancy in evaluation between the original form and the alternate form at $x = (2k+1)\pi$. This is again related to floating point arithmetrics.

1b - *Ans:* Similarly, the loss of significance occurs when $1-(1-x)^3 \approx 0$, i.e. when x is near 0. Expanding the term $(1-x)^3 = 1-3x+3x^2-x^3$, we can simplify the expression to x^2-3x^2+3 in which subtraction of nearly equal numbers will not occur.

1c - *Ans:* The loss of significance occurs when $\frac{1}{1+x} \approx \frac{1}{1-x}$, i.e. when x is near 0. The book suggests that we should reformulate it by reducing the given form into a single fraction, i.e.

$$\frac{1}{1+x} - \frac{1}{1-x} = \frac{1-x-(1+x)}{1-x^2} = \frac{-2x}{1-x^2}$$

But we can do better than this! The alternate form above will have no subtraction of nearly equal numbers at $x=0$ but at $x=1$. The better solution here is to **apply Taylor expansion on both fractions**, i.e.

$$\frac{1}{1+x} - \frac{1}{1-x} = (1-x+x^2-x^3\cdots) - (1+x+x^2+x^3+\cdots) = -2x-2x^3-\cdots$$

For example, by using $\frac{1}{1+x} \approx \frac{1}{1-x} \approx -2x-2x^3-2x^5$, we have error of order x^7 with no problem of subtracting nearly equal numbers at any x.

Chapter 0.4 Problem 3

Explain how to most accurately compute the two roots of the equation $x^2 + bx10^{12} = 0$, where $b > 100$

Ans: Similar to examples presented in class, using the quadratice formula, we have

$$x_1 = \frac{-b-\sqrt{b^2+4\times10^{-12}}}{2}, x_2 = \frac{-b+\sqrt{b^2+4\times10^{-12}}}{2}$$

When b is large enough, there is a danger of subtracting two nearly equal numbers in computing x_2. So multiplying both numerator and denominator with $-b-\sqrt{b^2+4\times10^{-12}}$, it becomes

$$x_2 = \frac{2\times10^{-12}}{b+\sqrt{b^2+4\times10^{-12}}}$$

which is numerically stable.

Chapter 0.5 Problem 1a

Use the Intermediate Value Theorem (IVT) to prove that $f(c) = 0$ for some $0 < c < 1$.

$$(a).f(x) = x^3 - 4x + 1$$

Ans: First, $f(x)$ is clearly continuous because it is a polynomial in x.

Second, $f(0) = 1$ and $f(1) = 1 - 4 + 1 = -2 < 0$. So by the IVT, there exists at least one root $c \in (0, 1)$ such that $f(c) = 0$.

Chapter 0.5 Problem 3a

Find c satisfying the Mean Value Theorem for Integrals (MVTI) with $f(x), g(x)$ in the interval $[0, 1]$.

$$(a).\ f(x) = x,\ \ g(x) = x \text{ in } [0, 1]$$

Ans: First, we compute the integral on the R.H.S. of MVTI, $\int_0^1 f(x)g(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$.

Second, we compute the integral on the L.H.S. of MVTI, $\int_0^1 g(x)\,dx = \frac{1}{2}$.

Hence,

$$c = \frac{\int_0^1 f(x)g(x)\,dx}{\int_0^1 g(x)\,dx} = \frac{2}{3}$$

Chapter 0.5 Problem 5a

Find the Taylor polynomial of degree 5 about the point $x = 0$ for:

$$(a).\ f(x) = e^{x^2}$$

The standard way of solving this problem is to calculate all the derivatives of the given function up to degree 5 and evaluate them at $x = 0$, i.e. $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$, and $f^{(5)}(0)$. So the Taylor's expansion is $f(x) \approx 1 + x^2 + \frac{x^4}{2}$ around $x = 0$

A faster way is to make use of the Taylor's expansion of $g(x) = e^x$ around $x = 0$:

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replacing x with x^2, we have

$$f(x) = g(x^2) = 1 + x^2 + \frac{(x^2)^2}{2} + \cdots \quad \Rightarrow \quad f(x) \approx 1 + x^2 + \frac{x^4}{2}$$

And the result follows.

MAT 128A - Assignment 2 (Revised solutions)

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Problem 1

What is the condition number of evaluation of the function

$$f(x) = e^{cos(x)}$$

at the point x?

Ans: Both exponential and cosine functions are smooth (*infinitely differentiable*), their composition is also smooth. So using the formula for condition number

$$\kappa_f(x) = \left| x\frac{f'(x)}{f(x)} \right|,$$

with $f'(x) = -sin(x)e^{cos(x)}$, we have $\kappa_f(x) = |xsin(x)|$.

This computation is straightforward but the implication is more important. Con sider $x_1 = \frac{\pi}{2}$ and $x_2 = 100\pi + \frac{\pi}{2}$. Clearly, both $f(x_1) = 0$ and $f(x_2) = 0$. But the condition number $\kappa_f(x_2) = 100\pi + \frac{\pi}{2}$ are much larger than that of $\kappa_f(x_1) = \frac{\pi}{2}$. Why?

Again, as said in class, it is due the distribution of the double precision numbers! Since the double precision numbers are densest around 0, therefore the condition number of $\kappa_f(x_1)$ is much closer to 1.

Problem 2

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x. Find an expression for the condition number of evaluation of the function $h(x) = f(g(x))$ at x in terms of $\kappa_f(g(x))$ and $g'(x)$.

Ans: Applying the formula for condition number of a function and the chain rule in differentiation, we have

$$\kappa_h(x) = \left| x\frac{h'(x)}{h(x)} \right| = \left| x\frac{f'(g(x))g'(x)}{f(g(x))} \right| = \kappa_f(g(x)) \cdot |g'(x)|$$

Problem 3

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x, and let $\kappa_g(x)$ denote the condition number of evaluation of the function g at x. Find an expression for the condition number of evaluation of the function $h(x) = f(x) \cdot g(x)$ at x in terms of $\kappa_f(x)$ and $\kappa_g(x)$.

Ans: Similar to the previous question, applying the formula for condition number of a function and the product rule in differentiation, we have

$$\kappa_h(x) = \left| x\frac{f'(x)g(x) + f(x)g'(x)}{f(x) \cdot g(x)} \right| = \left| x\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) \right| \leqslant \left| x\frac{f'(x)}{f(x)} \right| + \left| x\frac{g'(x)}{g(x)} \right| = \kappa_f(x) + \kappa_g(x)$$

The upper bound above for $\kappa_h(x)$ shows that the condition number of a product function can be controlled by its component.

Naturally, the above argument can be generalized to $h(x) := f_1(x)f_2(x)\cdots f_n(x)$, i.e.

$$\kappa_h(x) \leqslant \sum_{i=1}^{n} \kappa_{f_i}(x)$$

Problem 4

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x, and let $\kappa_g(x)$ denote the condition number of evaluation of the function g at x. Find an expression for the condition number of evaluation of the function $h(x) = f(x)/g(x)$ at x in terms of $\kappa_f(x)$ and $\kappa_g(x)$.

Ans: Similar to the previous question, applying the formula for condition number of a function and the quotient rule in differentiation, we have

$$\kappa_h(x) = \left| x\frac{h'(x)}{h(x)} \right| = |x| \left| \frac{\frac{g \cdot f' - g' \cdot f}{g^2}}{\frac{f}{g}} \right| = |x| \frac{g \cdot f' - g' \cdot f}{fg} = |x\frac{f'}{f} - x\frac{g'}{g}| \overset{(*)}{\geqslant} \left| \kappa_f(x) - \kappa_g(x) \right|$$

The $(*)$ above refers to the reverse triangle inequality $|a - b| \geqslant \big||a| - |b|\big|$.

The inequality above gives a lower bound to the condition number. As for the upper bound, same as before we have $\kappa_h(x) \leqslant \left|x\frac{f'}{f}\right| + \left|x\frac{g'}{g}\right| \leqslant \kappa_f(x) + \kappa_g(x)$.

To summarize, if either f or g is ill-conditioned, then the quotient function h is guaranteed to be ill-conditioned. On the other hand, if both f and g are well-conditioned, then h is guanranteed to be well-conditioned.

Problem 5

What is the Fourier series of the function $f(x) = x$?

Hint: You can easily find an antiderivative of $xexp(inx)$ using integration by parts.

Ans: $f(x) = x$ is clearly integrable, so for $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, we can use the formula to compute the coefficients

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\, dx$$

Following the given hint to use integration by part, for $n \neq 0$ we have

$$\begin{aligned} a_n &= -\frac{1}{(2\pi)(in)}\int_{-\pi}^{\pi} xd(e^{-inx}) = -\frac{1}{(2\pi)(in)}\left(\left[xe^{-inx}\right]_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} e^{-inx}\, dx}_{0}\right) \\ &= -\frac{1}{(2\pi)(in)}\left(\underbrace{\pi e^{-in\pi} + \pi e^{in\pi}}_{2\pi(-1)^n}\right) = -\frac{1}{in}(-1)^n = \frac{i}{n}(-1)^n \end{aligned}$$

For $n = 0$, we have $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, dx = 0$ since x is a odd function.

Combining everything together, we have

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}(-1)^n e^{-inx}$$

We can further simplify the above expression with the observation that for any non-zero integer n,

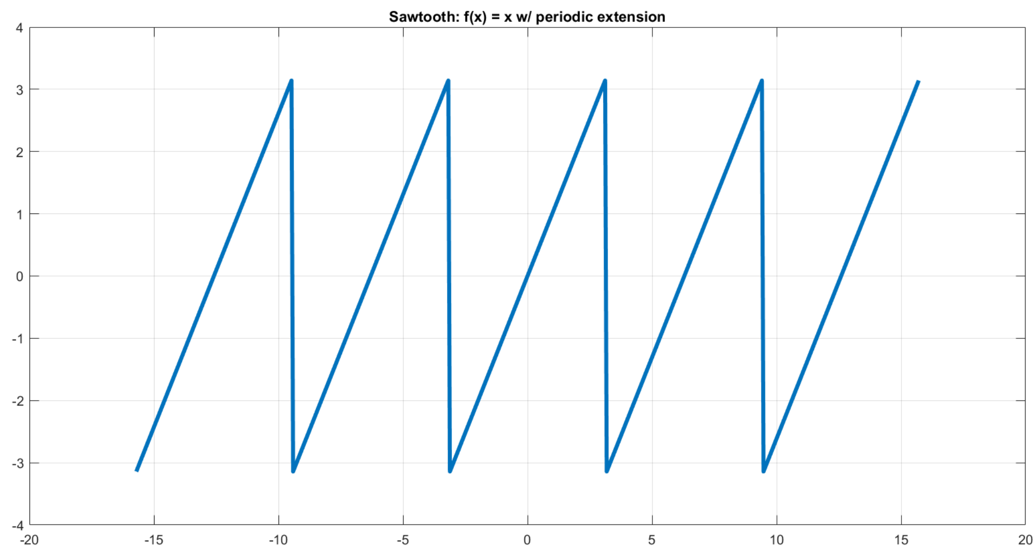
$$\frac{i}{n}(-1)^n e^{-inx} + \frac{i}{-n}(-1)^n e^{inx} = i(-1)^n(2\frac{i}{n}sin(nx)) = -2\frac{(-1)^n}{n}sin(nx)$$

Therefore, in summary

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin(nx) \tag{1}$$

The answer is not surprising since $f(x) = x$ is an odd function. Therefore the Fourier series consists only of $sin(x)$ which are also odd whereas all $cos(x)$ can be cancelled out.

Bonus (I): If we extend the function $f(x) = x$ over $[-\pi, \pi]$ to the entire real line, the extended function is periodic but not continuously differentiable (it is not continuous at the end points). The discontinuities are represented by the vertical lines in the graph below:



As said in class, for such f with discontinuities, the Fourier series does NOT converge uniformly to $f(x) = x$ for all x. Indeed with more advanced analysis, we can show that the Fourier series of this function DOES converge uniformly at all x except at countably many jump points. **More precisely, the above equality (1) holds in the L^2-sense, but NOT in a pointwise sense!** This is beyond the scope of this course...

Also, there are "overshoots" around the discontinuities known as "Gibbs phenomenon": (The dark blue curve is $f(x) = x$, the light blue curve is the approximation by sine series)

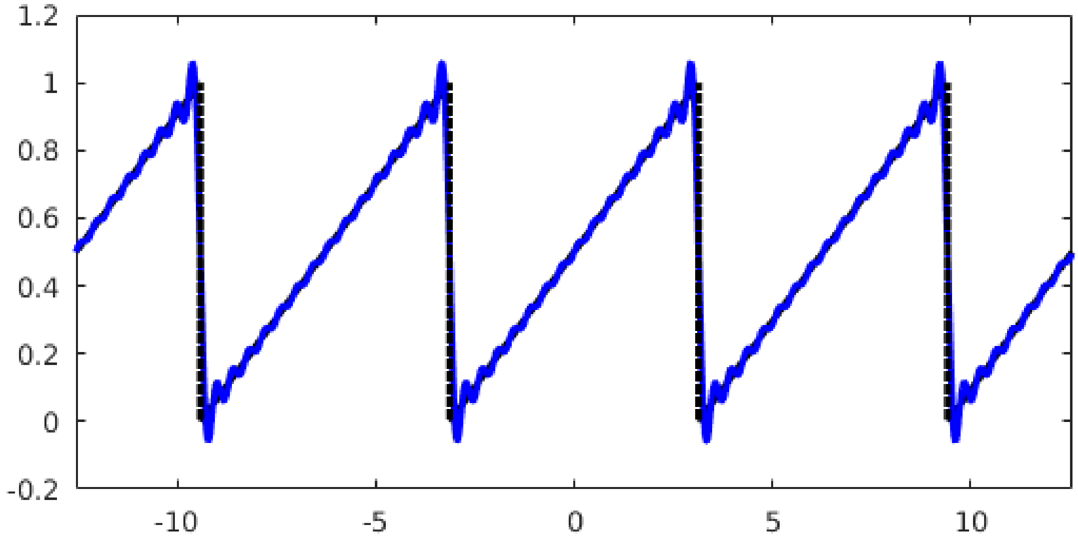


Figure 1: Source: http://www.chebfun.org/

Bonus (II): For those of you who are eager to learn more math, Fourier series are often one way to generate beautiful infinite series. If we substitute $x = \frac{\pi}{2}$ into (1), we have

$$\frac{\pi}{2} = f(\frac{\pi}{2}) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin(n\frac{\pi}{2})$$

Since $sin(n\frac{\pi}{2}) = 0$ when n is even, therefore

$$\frac{\pi}{2} = -2\sum_{k=0}^{\infty}\frac{(-1)^{2k+1}}{2k+1}sin(\frac{2k+1}{2}\pi) \quad \Rightarrow \quad \frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

We have rediscovered the "Leibniz Series"!

Bonus (III): If you are interested in learning more advanced mathematics, using the above Fourier series expansion of $f(x) = x$ together with the "Parseval identity" shown in class

$$\sum_{n=-\infty}^{\infty}|a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$$

Equating the L.H.S. and R.H.S., we have

$$R.H.S. = \sum_{n=-\infty}^{\infty}|a_n|^2 = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}, \qquad L.H.S. = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

This is a indeed very famous problem in mathematics known as the "Basel Problem"!

Problem 6

What is the Fourier series of the function $f(x) = |x|$?

Ans: Similar to the previous question, compute

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}\,dx = \frac{1}{2\pi}\left(\underbrace{\int_0^{\pi} xe^{-inx}\,dx}_{(I)} + \underbrace{\int_{-\pi}^{0} -xe^{-inx}\,dx}_{(II)}\right)$$

For $n \neq 0$, we compute the two integrals using integrating by parts:

$$(I) = \frac{1}{(-in)}\left[\pi(-1)^n + \left(\frac{(-1)^n}{in} - \frac{1}{in}\right)\right]$$
$$(II) = \frac{1}{in}\left[\pi(-1)^n + \left(\frac{1}{in} - \frac{(-1)^n}{in}\right)\right]$$

Combining (I) and (II), the $\pi(-1)^n$ terms will be cancelled out, we have

$$a_n = \frac{1}{2\pi}\left(\frac{2}{in}(\frac{1}{in} - \frac{(-1)^n}{in})\right) = \begin{cases} 0 & n \text{ is even, } n \neq 0 \\ -\frac{2}{\pi n^2} & n \text{ is odd} \end{cases}$$

$$\Rightarrow a_n = \begin{cases} 0, & n = 2k,\ n \neq 0 \\ -\frac{2}{\pi(2k+1)^2}, & n = 2k+1 \end{cases} \quad \text{for } k = 0, 1, 2, 3$$

I skipped many computation steps above. You should make sure that you get the same answer...

Also, for $n = 0$, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{\pi}{2}$.

Combining everything together, we have

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} e^{-i(2k+1)\pi}$$

Using the observation $e^{-inx} + e^{inx} = 2cos(nx)$, we have

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{cos\left((2k+1)x\right)}{(2k+1)^2}$$

Indeed with more advanced analysis, we can show that this Fourier series of the given $f(x) = |x|$, which is continuous over $[-\pi, \pi]$, periodic, and its derivative being piecewise continuous, converges to $f(x)$ uniformly. But this is beyond the scope of this course...

MAT 128A - Assignment 3

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Problem 1

Show that when the n-point periodic trapezoidal rule is used to evaluate the integral $\int_{-\pi}^{\pi} exp(ikt)\, dt$, the result is

$$\begin{cases} (-1)^{|k|}2\pi & \text{if } k = m \cdot n \text{ for some nonzero integer } m \\ 2\pi & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Ans: The desired statment can be shown by applying the n-point periodic trapezoidal rule directly (Clearly the function $f(x) = e^{ikt}$ is 2π-periodic.). As shown in class,

$$\begin{aligned} T(k) &:= \frac{2\pi}{n}\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right) \quad \text{where } x_j = -\pi + \frac{2\pi}{n}j \\ &= \frac{2\pi}{n}\left(e^{-ik\pi} + e^{ik(-\pi+\frac{2\pi}{n})} + \cdots + e^{ik(-\pi+\frac{2\pi}{n}(n-1))}\right) \\ &= \frac{2\pi}{n}e^{-ik\pi}\left(1 + e^{i\frac{2\pi}{n}k} + e^{i\frac{2\pi}{n}k\cdot 2} + \cdots + e^{i\frac{2\pi}{n}k\cdot(n-1)}\right) \\ &= \frac{2\pi}{n}e^{-ik\pi}(1 + r + r^2 + \cdots + r^{n-1}) \quad \text{where } r = e^{i\frac{2\pi}{n}k} \end{aligned}$$

It is tempting to apply the summation formula for geometric series in the last step above right away, BUT let us take a closer look at this sum and consider two special cases:

(i) For $k = 0$, $r = e^0 = 1$, so $T(0) = \frac{2\pi}{n}(1)\underbrace{(1 + 1 + \cdots + 1)}_{n\ '1's} = 2\pi$.

(ii) For $k = m \cdot n$, $r = e^{i\frac{2\pi}{n}(m\cdot n)} = e^{i(2\pi m)} = 1$ and $e^{-ik\pi} = (-1)^{|k|}$, so

$$T(k) = \frac{2\pi}{n}(-1)^{|k|}\underbrace{(1 + 1 + \cdots + 1)}_{n\ '1's} = (-1)^{|k|}2\pi$$

For any other k, applying the summation formula for geometric series, we have

$$T(k) = \frac{2\pi}{n} e^{-ik\pi} \frac{1-r^n}{1-r} = \frac{2\pi}{n} e^{-ik\pi} \frac{1-e^{i2\pi k}}{1-e^{i\frac{2\pi}{n}k}} = \frac{2\pi}{n} e^{-ik\pi} \frac{1-1}{1-e^{i\frac{2\pi}{n}k}} = 0$$

The above calculations explains why **the n-point periodic trapezoidal rule is exact for functions** $e^{ikt},\ k = -n+1, \cdots, n-1$

Problem 2

Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is even — meaning that $f(-x) = f(x)$ for all $0 < x \leqslant \pi$ — then f can be represented via a convergent series of the form

$$f(x) = \sum_{n=0}^{\infty} b_n cos(nx)$$

Ans: Assume that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $f(-x) = f(x)$ implies that $a_{-n} = a_n$ for all integer n (WHY?). Then

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} = a_0 + \sum_{n=1}^{\infty} a_n \left(e^{inx} + e^{-inx}\right) \overset{(*)}{=} a_0 + 2\sum_{n=1}^{\infty} a_n cos(nx)$$

$(*)$ uses the observation $e^{-inx} + e^{inx} = 2cos(nx)$. Finally we set $b_n = \begin{cases} a_0 & \text{for } n = 0 \\ 2a_n & \text{for } n \neq 0 \end{cases}$, then we have the desired statement, $f(x) = \sum_{n=0}^{\infty} b_n cos(nx)$.

As mentioned in the lecture, the Fourier series of a continuously differentiable (i.e. C^1), 2π-periodic function converges uniformly and absolutely to f on $[-\pi, \pi]$. So we don't need to prove the convergence of the above series.

Remark 1. Please look at HW 2 Question 6, the Fourier Series of $f(x) = |x|$ (which is an even function) can be represented by a series of $cos(nx)$.

Bonus: The series above containing only $cos(nx)$ functions is called the "Fourier cosine series". With a bit of extra efforts, one can show that $b_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)cos(nx)\,dx$ for $n = 1, 2, 3, \cdots$

Problem 3

Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is odd — meaning that $f(-x) = -f(x)$ for all $0 < x \leqslant \pi$ — then f can be represented via a convergent series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n sin(nx)$$

Ans: Similar to the last problem, assume that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $f(-x) = -f(x)$ implies that $a_{-n} = -a_n$ for all integer n (WHY?).

Notice that for $n = 0$, $a_0 = -a_0 \Rightarrow a_0 = 0$. Therefore

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} = \underbrace{a_0}_{=0} + \sum_{n=1}^{\infty} a_n \left(e^{inx} - e^{-inx}\right) \overset{(*)}{=} 2i \sum_{n=1}^{\infty} a_n sin(nx)$$

$(*)$ uses the observation $e^{-inx} - e^{inx} = 2isin(nx)$. Finally we set $c_n = 2ia_n$ for $n \neq 0$, then we have the desired statement, $f(x) = \sum_{n=1}^{\infty} c_n sin(nx)$.

Remark 2. Please look at HW 2 Question 5, the Fourier Series of $f(x) = x$ (which is an odd function) can be represented by a series of $sin(nx)$.

Bonus: The series above containing only $sin(nx)$ functions is called the "Fourier sine series". With a bit of extra efforts, one can show that $c_n = \frac{2}{\pi}\int_0^{\pi} f(x)sin(nx)\,dx$ for $n = 1, 2, 3, \cdots$

Problem 4

Suppose that $f(x) = cos(2x) + cos(4x) + \cdots + cos(20x)$. What is the exact value of $\int_{-\pi}^{\pi} f(x)\,dx$?

i.e. How long is the periodic trapezoidal rule of minimum length which evaluates the above integral exactly? That is, what is the least positive integer n such that

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{2\pi}{n} \sum_{j=0}^{n-1} f(-\pi + \frac{2\pi}{n} j) \ ?$$

Here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

Ans: The key idea is the statement shown in class that

The n-point periodic trapezoidal rule is exact for the functions e^{-ikt}, where

$$k = -n+1, -n+2, \cdots, -1, 0, 1, \cdots n-1$$

Using $e^{-inx} + e^{inx} = 2cos(nx)$, rewrite

$$f(x) = \frac{1}{2}\left[(e^{-i2x} + e^{i2x}) + (e^{-i4x} + e^{i4x}) + \cdots + ((e^{-i20x} + e^{i20x}))\right]$$

So we want $20 = k = n - 1 \Rightarrow n = 21$. Clearly using more than 21 points will make the integral evaluation exact.

Therefore, the answer is 21 points on $[-\pi, \pi]$.

Problem 5

Let $f(x) = \sum_{n=1}^{\infty} a_n e^{inx}$ with $|a_n| \leqslant \frac{1}{n^2}$.

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

is bounded above by $\frac{\pi^2}{6}\frac{1}{N^2}$. Again here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

(Hint: look at the solutions from the previous homework assignment to find the value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$).

Ans: The key idea here is use the following corollary shown in class:

Corollary 1. Given $g(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, then $\int_{-\pi}^{\pi} g(x)\, dx = 2\pi a_0$, where the approximation of the integral obtained via the $m-$point periodic trapezoidal rule is

$$2\pi a_0 + 2\pi \sum_{k=1}^{\infty} (-1)^{km}(a_{km} + a_{-km})$$

Observe that for the given function in our problem, $a_n = 0$ for all integer $n \leqslant 0$. Therefore the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$. Therefore, in order to calculate error:

$$\begin{aligned}
\text{Error} &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx - \frac{1}{2\pi}\left[2\pi a_0 + 2\pi\sum_{k=1}^{\infty}(-1)^{kN}(a_{kN} + a_{-kN})\right]\right| \\
&= \left|\sum_{k=1}^{\infty}(-1)^{kN} a_{kN}\right| \leqslant \sum_{k=1}^{\infty}|a_{kN}| \leqslant \sum_{k=1}^{\infty}\frac{1}{k^2N^2} = \frac{\pi^2}{6}\cdot\frac{1}{N^2}
\end{aligned}$$

Remark 3. One should ask right at the beginning whether the given function $f(x) = \sum_{n=1}^{\infty} a_n e^{inx}$ is well-defined, i.e. does this series of function converge? The answer is NO for arbitrary a_n.

But thanks to the condition $|a_n| \leqslant \frac{1}{n^2}$, we can invoke the "Weierstrass M-test" which states that *for a sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$, $f_n\colon A \to \mathbb{C}$, *and suppose for every* $n \in \mathbb{N}$, *there exists constants* $M_n > 0$ *such that* $|f_n(x)| < M_n$ *for all* $x \in A$ *and* $\sum_{n\in\mathbb{N}} M_n < \infty$. *Then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly.* In our case, $M_n = \frac{1}{n^2}$. Therefore, $f(x)$ is well-defined in our case.

Problem 6

Let $f(x) = \sum_{n=0}^{\infty} a_n e^{inx}$ with $|a_n| \leqslant \frac{1}{2^n}$.

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

is bounded above by $\frac{1}{2^N-1}$. Again here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

Ans: Similar to the above problem, here we have $a_n = 0$ for $n < 0$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$. We apply the same corollary to calculate the error:

$$\begin{aligned}
\text{Error } &= \left| a_0 - \frac{1}{2\pi}\left[2\pi a_0 + 2\pi\sum_{k=1}^{\infty}(-1)^{kN}(a_{kN} + a_{-kN})\right]\right| \\
&= \left|\sum_{k=1}^{\infty}(-1)^{kN} a_{kN}\right| \leqslant \sum_{k=1}^{\infty}|a_{kN}| \leqslant \sum_{k=1}^{\infty}\frac{1}{2^{kN}} \overset{(*)}{=} \frac{1}{2^N-1}
\end{aligned}$$

$(*)$ holds since $\sum_{k=1}^{\infty}\frac{1}{2^{kN}} = \frac{1}{2^N}\left(1 + \frac{1}{2^N} + \frac{1}{2^{2N}} + \cdots\right) = \frac{1}{2^N}\frac{1}{1-\frac{1}{2^N}} = \frac{1}{2^N-1}$.

Problem 7

Find the Fourier series for the function

$$f(t) = \frac{2}{e^{it}-2}$$

by using the identity $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$, which holds for all z with $|z| < 1$.

Ans: Rearranging the given function

$$f(t) = -\frac{1}{1-\frac{e^it}{2}} \overset{(*)}{=} -\sum_{k=0}^{\infty}\left(\frac{e^{it}}{2}\right)^k = \sum_{k=0}^{\infty}\left(-\frac{1}{2^k}\right)e^{ikt}$$

This is an example for **exponential decay** of Fourier coefficients due to the fact that the function $g(z) = \frac{2}{z-2}$ is analytic in a strip over the real-axis on the complex plane.

MAT 128A - Assignment 4 (Revised solutions)

Karry Wong

October 18, 2018

Problem 1

Suppose that n is a nonnegative integer. Given that the function $y(t) = cos(nt)$ satisfies the second order differential equation

$$\ddot{y}(t) + n^2 y(t) = 0 \quad \text{for all } -\pi < t < \pi$$

Use this observation to show that the function $T_n(x) = cos(ncos^{-1}(x))$, where $cos^{-1}(x) = arccos(x)$, is a solution of the equation

$$(1 - x^2)y''(x) - xy'(x) + n^2 y(x) = 0 \quad \text{for all } -1 < x < 1$$

Here we use $y' := \frac{dy}{dx}$ and $\dot{y} = \frac{dy}{dt}$.

Hint: Use the chain rule to compute $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Ans: The key observation here is the change of variable $t = cos^{-1}x$, so

$$\frac{dx}{dt} = -\sqrt{1-x^2}, \quad \frac{d^2x}{dt^2} = -\frac{d\sqrt{1-x^2}}{dt} = \frac{x}{\sqrt{1-x^2}}\frac{dx}{dt} = -x$$

Using chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\sqrt{1-x^2}\frac{dy}{dt}, \quad \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$$

Also, $-\pi < t < \pi \quad \Rightarrow \quad -1 = cos(-\pi) < x < cos(\pi) = 1$. Therefore,

$$0 = \ddot{y}(t) + n^2 y(t) = (1-x^2)y''(x) - xy'(x) + n^2 y(x) \quad \text{for all } -1 < x < 1$$

Problem 2

Show that

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

Ans: The answer is already given in the class. The key observation is that

$$\int_{0}^{\pi} cos(nt)cos(mt)dt = \begin{cases} \pi & \text{if } n = m = 0 \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

Proof. For $m = n = 0$, $\int_0^\pi 1dt = \pi$.

For $m = n \neq 0$, $\int_0^\pi cos^2(mt)dt = \frac{1}{2}\int_0^\pi cos(2mt) + 1dt = \frac{\pi}{2}$.

For $m \neq n$, $\int_0^\pi cos(nt)cos(mt)dt = \frac{1}{2}\int_0^\pi cos[(n+m)t] + cos[(n-m)t]dt = 0$

Use the change of variable $t = cos^{-1}x \Rightarrow dt = -\frac{1}{\sqrt{1-x^2}}dx$, therefore

$$\int_{0}^{\pi} cos(nt)cos(mt)dt = -\int_{1}^{-1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

Problem 3

(a) Using the trigonometric identity

$$cos(nt) = cos((n-1)t)cos(t) - sin((n-1)t)sin(t)$$

Show that

$$T_n(x) = xT_{n-1}(x) - U_{n-1}(x)\sqrt{1-x^2} \tag{1}$$

where U_n is defined via

$$U_n(x) = sin(ncos^{-1}(x)).$$

Ans: Again, substitute $t = cos^{-1}x$ ($\Leftrightarrow x = cost$) into the given trigonometric identity, $L.H.S. = T_n(x)$ follows immediately.

$R.H.S. = \underbrace{cos\left((n-1)cos^{-1}x\right)}_{T_{n-1}} x - \underbrace{cos\left((n-1)sin^{-1}x\right)}_{U_{n-1}} sin(t) \overset{(*)}{=} xT_{n-1} - \sqrt{1-x^2}U_{n-1}$

$(*)$ holds since $sin(t) = \sqrt{1-cos^2(t)} = \sqrt{1-x^2}$

Problem 3

(b) Use the trigonometric identity

$$sin(nt) = sin((n-1)t)cos(t) + cos((n-1)t)sin(t)$$

to show that

$$U_n(x) = T_{n-1}(x)\sqrt{1-x^2} + U_{n-1}(x)x \tag{2}$$

Ans: Again, directly substitute $t = cos^{-1}x$.

$L.H.S = U_n(x)$ follows immediately.

$R.H.S. = \underbrace{cos((n-1)cos^{-1}x)}_{T_n(x)} \underbrace{sin(t)}_{\sqrt{1-x^2}} + \underbrace{sin((n-1)cos^{-1}x)}_{U_{n-1}} cos(cos^{-1}x) = T_n(x)\sqrt{1-x^2} + U_{n-1}x.$

Problem 3

(c) Combine (1) and (2) to show that

$$U_n(x)\sqrt{1-x^2} = T_{n-1}(x) - xT_n(x) \tag{3}$$

Ans: We multiply (1) by x, multiply (2) by $\sqrt{1-x^2}$, and then add both of them together in order to get rid of the $U_{n-1}(x)$ term, we have

$$xT_n(x) + \sqrt{1-x^2}U_n(x) = x^2T_{n-1}(x) + (1-x^2)T_{n-1}(x) \quad \Rightarrow \quad U_n(x)\sqrt{1-x^2} = T_{n-1}(x) - xT_n(x)$$

Problem 3

(d) Use (3) and (1) — replace n with $n+1$ in (1) — to obtain the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Ans: Following the instruction by replacing n with $n+1$ in (), we have

$$T_{n+1}(x) = xT_n(x) - U_n(x)\sqrt{1-x^2},$$

Substitute (3) into it, we have $T_{n+1}(x) = xT_n(x) - (T_{n-1}(x) - xT_n(x)) = 2xT_n(x) - T_{n-1}(x)$.

Alternatively, one can obtain the desired three-terms recurrence relation without going through parts (a) to (c). Using another trigonometric identity, namely

$$cos(nt) + cos((n-2)t) = 2cos(t)cos((n-1)t) \quad \Rightarrow \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Bonus I: Here is another relation I can think out of, can you prove it by using another trigonometric identity?

$$U_n - U_{n-2} = 2\sqrt{1-x^2}T_n \tag{4}$$

Bonus II: Using the trigonometric identity $2cosntcosmt = cos((n+m)t) + cos((n-m)t)$, we have the following product relation of the Chebyshev polynomials:

$$T_nT_m = \frac{1}{2}\left(T_{n+m} + T_{n-m}\right) \quad \text{assuming that } m \leqslant n$$

Bonus III: Indeed U_n is called the **"Chebyshev polynomials of the second kind"** whereas T_n is called the **"Chebyshev polynomials of the second kind"**. They are closely related. They satisfies other relations apart from those listed here.

Problem 4

Suppose that n is a nonnegative integer. Show that

$$(1-x^2)T_n'(x) = nT_{n-1}(x) - nxT_n(x)$$

for all $-1 < x < 1$.

Ans: Notice that the derivative $T_n'(x) = sin(ncos^{-1}(x))\frac{n}{\sqrt{1-x^2}} = U_n\frac{n}{\sqrt{1-x^2}}$.
Using the formula (3) in problem 3c)., we have

$$(1-x^2)T_n'(x) = n\sqrt{1-x^2}U_n = n\left(T_{n-1}(x) - xT_n(x)\right) = nT_{n-1} - nxT_n(x)$$

Remark 1. Using the recurrence relation in 3d)., we also have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \Rightarrow \quad T_{n+1}(x) - xT_n(x) = xT_n(x) - T_{n-1}(x)$$

Therefore, we can also write $(1-x^2)T_n'(x) = nT_{n-1}(x) - nxT_n(x) = nxT_n(x) - nT_{n+1}(x)$.

Bonus IV: There is one more interesting recurrence relation, namely

$$\begin{cases} T_0(x) &= T_1'(x), \\ T_1(x) &= \frac{1}{4}T_2'(x), \\ T_n(x) &= \frac{1}{2}\left(\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1}\right), \quad n \geqslant 2 \end{cases}$$

You should be able to prove this using results in problems 3 and 4.

MAT 128A - Assignment 5

Karry Wong

October 25, 2018

Problem 1

Let $f(x)$ be a polynomial of x of degree N, and

$$f(x) = \sum_{n=0}^{N} a_n T_n(x). \tag{1}$$

The Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

We define a finite sequence of polynomials $\{b_0(x), b_1(x), \cdots, b_N(x), b_{N+1}(x), b_{N+2}(x)\}$ via the formulas:

$$b_{N+1} = b_{N+2} = 0$$

$$b_n(x) = a_n + 2xb_{n+1}(x) - b_{n+2}(x)$$

Show that $f(x) = b_0(x)$.

Hint: first show that if q_{N-1} is defined by

$$q_{N-1}(y) := \sum_{n=0}^{N-1} 2b_{n+1}(x)T_n(y)$$

then

$$(y - x)q_{N-1}(y) + b_0(x) = f(y) \tag{2}$$

and then let $y = x$ in above

Ans: This question looks tricky at first glance but direct substitution of all given formulas will lead us to the desired statement. First, substitute (2):

$$
\begin{aligned}
L.H.S. &= (y-x)\left(\sum_{n=0}^{N-1} 2b_{n+1}(x)T_n(y)\right) + b_0(x) \\
&= \sum_{n=0}^{N-1} b_{n+1}(x) \underbrace{2yT_n(y)}_{T_{n+1}(y)+T_{n-1}(y)} - \sum_{n=0}^{N-1} \underbrace{2xb_{n+1}(x)}_{b_n(x)-a_n+b_{n+2}(x)} T_n(y) + b_0(x) \\
&= \sum_{n=0}^{N-1} b_{n+1}(x)\left(T_{n+1}(y)+T_{n-1}(y)\right) - \sum_{n=0}^{N-1}\left(b_n(x)+b_{n+2}(x)\right)T_n(y) + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= \underbrace{\left(\sum_{n=0}^{N-1} b_{n+1}(x)T_{n+1}(y) - \sum_{n=0}^{N-1} b_n(x)T_n(y)\right)}_{b_N(x)T_N(y)-b_0(x)T_0(y)} + \underbrace{\left(\sum_{n=0}^{N-1} b_{n+1}(x)T_{n-1}(y) - \sum_{n=0}^{N-1} b_{n+2}(x)T_n(y)\right)}_{b_1(x)T_{-1}(y)-b_{N+1}(x)T_{N-1}(y)} \\
&\quad + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= \underbrace{b_N(x)}_{=a_N} T_N(y) - b_0(x)\underbrace{T_0(y)}_{\equiv 1} + b_1(x)\underbrace{T_{-1}(y)}_{=0} - \underbrace{b_{N+1}(x)}_{=0} T_{N-1}(y) + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= a_NT_N(y) + \sum_{n=0}^{N-1} a_nT_n(y) = f(y) \qquad = R.H.S.
\end{aligned}
$$

It is obvious that by taking $y = x$, we obtain $b_0(x) = f(x)$.

Remark 1. As said in class, the formulation above is an application of the **Clenshaw's recurrence formula**. It can be applied to any classes of function that are defined by a three-term recurrence relation.

Problem 2

The last problem suggests a method for computing the sum (1). How many arithmetic operations does it take to compute $f(x) = b_0(x)$ using this method?

Suppose that instead we sum (1) most directly by first using the recurrence relations $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ to compute the values of $T_0(x), T_1(x), \cdots, T_N(x)$. We then form the values

$$a_0T_0(x), a_1T_1(x), \cdots, a_NT_N(x)$$

and then sum them to form $f(x)$. How many arithmetic operations does this more direct procedure take?

Ans: The problem setting is that given the (N+1) Chebyshev coefficients $\{a_n\}_{n=0}^{N}$ for a polynomial f of degree N, we want to evalute $f(x)$ for a specified value of x.

First, we count the arithmetic operations by using the algorithm in problem (1), i.e.

- [1st - step:] $b_N(x) = a_N + 2xb_{N+1}(x) - b_{N+2}(x)$
- [2nd - step:] $b_{N-1}(x) = a_{N-1} + 2xb_N(x) - b_{N+1}(x)$
- $\vdots$
- [Nth - step] $b_1(x) = a_1 + 2xb_2(x) - b_3(x)$
- [(N+1)th - step:] $b_0(x) = a_0 + 2xb_1(x) - b_2(x)$

In each step, we have 2 multiplications and 2 additions (subtraction counted as addition), so there are 4 arithmetic operations. In total, we have $4(N+1)$ arithemetic operations after $N+1$ steps.

More precisely, since $b_{N+1} = b_{N+2} = 0$, the first two steps can be simplified to $b_N(x) = a_N$ and $b_{N-1}(x) = a_{N-1} + 2xb_N(x)$. Indeed we only have $4(N-1) + 3 = 4N - 1$ step.

Second, we use the recurrence relation to evaluate the Chebyshev polynomials $\{T_n\}_{n=0}^{N}$ at x and then compute the sum $\sum a_n T_n(x)$. Given that $T_0(x) = 1$ and $T_1(x) = x$

- [1st - step:] $T_2 = 2xT_1 - T_0$
- [2nd - step:] $T_3 = 2xT_2 - T_1$
- $\vdots$
- [(N-1)th - step] $T_N = 2xT_{N-1} - T_{N-2}$
- [(N+1)th - step:] $f(x) = a_0T_0(x) + a_1T_1(x) + \cdots + a_NT_N(x)$

For all the first (N-1) steps, each steps take 1 addition and 2 multiplication, so there are 3 arithmetic operations. The last step takes N addition and $(N+1)$ multiplication, so there are $(2N+1)$ arithemtic operations. In total, we have $3(N-1) + 2N + 1 = 5N - 2$ operations.

Therefore, **the derivation in problem (1) further accelerate the evaluation of Chebyshev expansion!**

Problem 3

Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$f(x) = \sum_{n=0}^{N}{}' a_n T_n(x)$$

of the functions $f(x) = \sqrt{1-x^2}$ for $-1 \leqslant x \leqslant 1$.

Note that **the dash summation notation indicates that the first term in the series is halved**.

Recall that the coefficients are defined via the formula

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}$$

So computing a_n is equivalent to determine the value of the quantity $a_n = \frac{2}{\pi}\int_{-1}^{1} T_n(x)\,dx$

Ans: First, recall the definition of $T_n(x) = cos(n{\cdot}cos^{-1}(x))$, so $a_n = \frac{2}{\pi}\int_{-1}^{1} cos(n{\cdot}cos^{-1}(x))\,dx$. Second, using the change of variable $x = cost$ ($\Rightarrow\ dx = -sint\,dt$), we have

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_{\pi}^{0} cos(n\cdot t) - sint\,dt \quad \text{note that } 0 < t < \pi \\
&= \frac{2}{\pi}\int_{0}^{\pi} cos(n\cdot t)sint\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} sin(n+1)t - sin(n-1)t\,dt \\
&\overset{(*)}{=} \frac{1}{\pi}\left[-\frac{cos(n+1)t}{n+1} + \frac{cos(n-1)t}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}
\end{aligned}$$

$(*)$ holds due to the sum of angle formula $cos\alpha sin\beta = \frac{1}{2}\left(sin(\alpha+\beta) - sin(\alpha-\beta)\right)$.

Note that for n is odd, $a_n = \frac{1}{\pi}\left\{\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = 0$.

For $n \neq 0$ is even, $a_n = \frac{1}{\pi}\left\{\left[\frac{1}{n+1} - \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = -\frac{4}{(n^2-1)\pi}$.

For $n = 0$, a_0 **is halved**, so $a_0 = -\frac{2}{(0^2-1)\pi} = \frac{2}{\pi}$.

Writing even number $n = 2k$, for $k = 1, 2, 3, \cdots$, the Chebyshev expansion of $f(x) = \sqrt{1-x^2}$ is

$$\sqrt{1-x^2} = \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{(4k^2-1)\pi}T_{2k}(x) = \frac{2}{\pi} - \frac{4}{3\pi}(2x^2-1) - \frac{4}{15\pi}(8x^4-8x^2+1) - \cdots$$

Bonus I: Indeed, we can know that all odd coefficients $a_{2k+1} = 0$ without going through the exact calculation. In order to see that, we need the following proposition

Proposition 1. $T_n(-x) = (-1)^n T_n(x)$

Proof. Notice that $cos^{-1}(x) = \theta \Leftrightarrow x = cos\theta$.
So since $cos(\pi - \theta) = -x$, we have $cos^{-1}(-x) = \pi - cos^{-1}(x)$.

$$
\begin{aligned}
T_n(-x) &= cos(n \cdot cos^{-1}(-x)) = cos(n \cdot (\pi - cos^{-1}(x))) = cos(n\pi - ncos^{-1}(x)) \\
&= \underbrace{cos(n\pi)}_{(-1)^n} cos(ncos^{-1}(x))) + \underbrace{sin(n\pi)}_{=0} sin(ncos^{-1}(x)) = (-1)^n T_n(x)
\end{aligned}
$$

The above proposition leads to

Corollary 2. $T_n(x)$ is an even polynomial for even n. $T_n(x)$ is an odd polynomial for odd n.

Proof. $T_{2k+1}(-x) = -T_{2k+1}(x), \qquad T_{2k}(-x) = T_{2k}(x). \qquad k = 0, 1, 2, \cdots$

Now look at back the Chebyshev coefficient of $f(x) = \sqrt{1 - x^2}$, $a_n = \frac{2}{\pi} \int_{-1}^{1} T_n(x)\, dx$.
For n being odd, the integral is clearly zero from -1 to 1. For n being even, $a_n = \frac{4}{\pi} \int_0^1 T_n(x)\, dx$.

Bonus II: Since $f(x) = \sqrt{1 - x^2}$ is an even function, its Chebyshev expansion consist of even polynomials $T_{2k}(x)$ ONLY.
$f(x)$ is not continuously differentiable since its derivative $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ has singularities at $x = \pm 1$. As said in class, its Chebyshev coefficients decay at rate $a_n = \mathcal{O}\left(\frac{1}{n^2}\right)$.

Bonus III: For those of you who want to do numerical experiments on `MATLAB`, I recommend you to download the package `Chebfun` at http://www.chebfun.org/
I use it in `MATLAB` to compute the first 10 Chebyshev's coefficients:

```
x = chebfun('x');
f = sqrt(1-x^2);
p = chebfun(f,'trunc',11);
a = chebcoeffs(p)


a =


   0.636619772367623
   0.000000000000000
```

-0.424413181578414

0.000000000000000

-0.084882636315682

0.000000000000000

-0.036378272706720

0.000000000000000

-0.020210151503733

0.000000000000000

-0.012861005502375

All the odd coefficients vanish as predicted in our calculation!

Problem 4

Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$g(x) = {\sum_{n=0}^{N}}' a_n T_n(x)$$

of the function

$$g(x) = sgn(x) = \begin{cases} 1, & 1 \geqslant x > 0 \\ 0, & x = 0 \\ -1, & -1 \leqslant x < 0 \end{cases}$$

Note that I restrict the domain of the function to $[-1, 1]$.

Ans: Similar to the previous problem, we use the change of variable $x = cost$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{-1}^{1} sgn(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_{0}^{1} T_n(x)\frac{dx}{\sqrt{1-x^2}} - \frac{2}{\pi}\int_{-1}^{0} T_n(x)\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2}{\pi}\left\{\int_{\frac{\pi}{2}}^{0} -cos(nt)\,dt - \int_{\pi}^{\frac{\pi}{2}} -cos(nt)\,dt\right\} \\ &= \frac{2}{\pi}\left\{\int_{0}^{\frac{\pi}{2}} cos(nt)\,dt - \int_{\frac{\pi}{2}}^{\pi} cos(nt)\,dt\right\} \\ &= \frac{2}{\pi}\left\{\left[\frac{sin(nt)}{n}\right]_0^{\frac{\pi}{2}} - \left[\frac{sin(nt)}{n}\right]_{\frac{\pi}{2}}^{\pi}\right\} = \frac{4}{n\pi}sin(n\frac{\pi}{2}) \\ &= \begin{cases} 0, & n = 2k \text{ even} \\ \frac{4}{(2k+1)\pi}\underbrace{sin((2k+1)\frac{\pi}{2})}_{(-1)^k}, & n = 2k+1 \text{ odd} \end{cases} \end{aligned}$$

for $k = 0, 1, 2, \cdots$

To conclude, we have

$$g(x) = sgn(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{T_{2k+1}(x)}{2k+1}$$

Bonus I: In contrast to the previous function $f(x) = \sqrt{1-x^2}$, $g(x) = sgn(x)$ is not even continuous on $[-1, 1]$, let alone being differentiable. So it is not surprising that the decay rate of coefficient for $g(x)$ is $a_N = \mathcal{O}\left(\frac{1}{N}\right)$, decaying more slowly to zero than that of $f(x)$.

Bonus II: I use the package `chebfun` in `MATLAB` to compute the first 10 Chebyshev's coefficients.

```
>> x = chebfun('x');
g = sign(x);
```

```
p = chebfun(g,'trunc',11);
a = chebcoeffs(p)


a =


  -0.000000000000000
   1.273239544735162
  -0.000000000000000
  -0.424413181578387
  -0.000000000000000
   0.254647908947033
  -0.000000000000000
  -0.181891363533594
  -0.000000000000000
   0.141471060526130
  -0.000000000000000
```

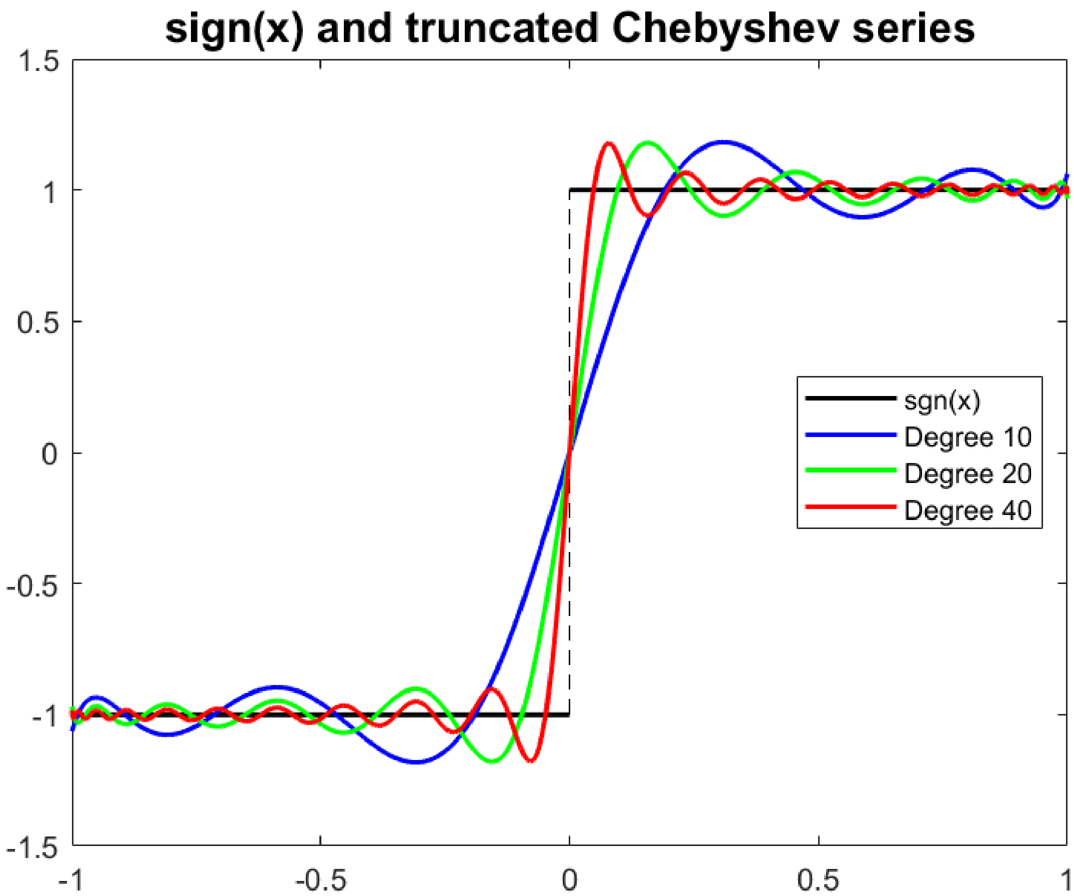
All the even coefficients vanish as predicted in our calculation!

Next, I plot the Chebyshev approximation using the first 10 terms (blue), 20 terms (green), and 40 terms (red). The function $g(x) = sgn(x)$ is plotted in black.

```
p0 = chebfun(g,'trunc',11);
p1 = chebfun(g,'trunc',21);
p2 = chebfun(g,'trunc',41);


FS = 'fontsize'; LW = 'LineWidth'; JL = 'jumpline';
figure
plot(g,'k',LW,1.5,JL,'--'), ylim([-1.5 1.5])
title('sign(x)',FS,14)
hold on
plot(p0,'b',LW,1.5)
plot(p1,'g',LW,1.5)
```

```
plot(p2,'r',LW,1.5)
title('sign(x) and truncated Chebyshev series',FS,14)
lgd = legend('sgn(x)','Degree 10', 'Degree 20', 'Degree 40');
lgd.Location = 'east';
```



Bonus III: Notice that using Chebyshev polynomials to approximate a function with interior discontinuity still leads to the "Gibbs phenomenon" in Fourier series approximation (See my solution for Problem 5 in Homework 2 - Bonus I). In this case $g(x) = sgn(x)$, we observe the Gibbs phenomenon at $x = 0$. It is not surprising since Chebyshev series can be obtained from the Fourier cosine series by the change of variable $x = cost$.

MAT 128A - Assignment 6 (Revised Solution)

Karry Wong

November 14, 2018

Problem 1

Go over the midterm problems and the provided solutions!

You should really do it for your own good! ☺

Problem 2

Show that for all nonnegative integers n, $T_n(1) = 1$ and $T_n(-1) = (-1)^n$.

Ans: The desired statement follows easily by using the definition of the Chebyshev polynomials, i.e. $T_n(x) = cos(ncos^{-1}x)$. Since

$$cos^{-1}(1) = 2k\pi \quad \text{for } k \in \mathbb{Z}, \qquad cos^{-1}(-1) = (2k+1)\pi \quad \text{for } k \in \mathbb{Z}$$

$$\Rightarrow \quad cos(n \cdot cos^{-1}(1)) = cos(2nk \cdot \pi) = 1, \qquad cos^{-1}((-1)) = cos(n(2k+1) \cdot \pi) = \begin{cases} 1, & \text{if } n \text{ even} \\ -1, & \text{if } n \text{ odd} \end{cases}$$

$$\Rightarrow T_n(1) = 1, \qquad T_n(-1) = (-1)^n$$

Remark 1. The symbol $k \in \mathbb{Z}$ means that k is an integer (can be positive or negative).

Problem 3

Show that for all integers $n \geq 2$ and all $-1 < t \leq -1$,

$$\int_{-1}^{t} T_n(x)\,dx = \frac{1}{2}\left(\frac{T_{n+1}(t)}{n+1} - \frac{T_{n-1}(t)}{n-1}\right) - \frac{(-1)^n}{n^2-1}$$

Ans: **This problem is like a generalization of the problem 4 (first part) in our midterm exam!** Can you tell how this problem is related to the midterm problem?

Again, we start with the definition and use the change of variable $x = cosu$, $\Rightarrow$ $dx = -sinu\,du$, also the interval of integration $(-1,t) \mapsto (\pi, cos^{-1}t)$ we have

$$\int_{-1}^{t} T_n(x)\,dx = \int_{\pi}^{cos^{-1}t} cos(n\cdot cos^{-1}x)\,dx = \int_{\pi}^{cos^{-1}t} cos(n\cdot u)(-sinu)\,du = \int_{cos^{-1}t}^{\pi} cos(n\cdot u) sinu\,du$$

Recall the trigonometric identity

$$cos(\alpha u)sin(\beta u) = \frac{1}{2}sin((\alpha+\beta)u) - sin((\alpha-\beta)u),$$

applying this

$$\int_{-1}^{t} T_n(x)\,dx = \frac{1}{2}\int_{cos^{-1}t}^{\pi} sin((n+1)u) - sin((n-1)u)\,du = \frac{1}{2}\left[-\frac{cos((n+1)u)}{n+1} + \frac{cos((n-1)u)}{n-1}\right]_{cos^{-1}t}^{\pi}$$

$$= \frac{1}{2}\left\{\left[-\frac{cos((n+1)\pi)}{n+1} + \frac{cos((n-1)\pi)}{n-1}\right] - \left[-\frac{cos((n+1)cos^{-1}t)}{n+1} + \frac{cos((n-1)cos^{-1}t)}{n-1}\right]\right\}$$

$$= \frac{1}{2}\left\{\underbrace{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]}_{\frac{2\cdot(-1)^{n+1}}{n^2-1}} - \left[-\frac{T_{n+1}(t)}{n+1} + \frac{T_{n-1}(t)}{n-1}\right]\right\}$$

$$= -\frac{(-1)^n}{n^2-1} + \frac{1}{2}\left(-\frac{T_{n+1}(t)}{n+1} + \frac{T_{n-1}(t)}{n-1}\right)$$

Therefore, we have the desired statement.

Remark 2. The above formula clearly does not hold for $n = 0$ or $n = 1$. For $n = 0$, the term $T_{-1}(x)$ is not well-defined. For $n = 1$, the denominator $n-1$ blows up to $+\infty$.

Remark 3. For $t = -1$, the formula is still valid. But both sides of the equation will be equal to zero.

Problem 4

Let $x_0, x_1, \cdots x_N, w_0, w_1, \cdots, w_N$ denote the nodes and weights of the $(N+1)$-point Gauss-Legendre quadrature rule. Suppose that $f\colon[-1,1] \to \mathbb{R}$ is continuously differentiable, and that $c_0, c_1, \cdots c_N$ are defined by the formula

$$c_n = \frac{2n+1}{2}\sum_{j=0}^{N} f(x_j)P_n(x_j)w_j$$

Show that the polynomial

$$p_N(x) = \sum_{n=0}^{N} c_n P_n(x)$$

interpolates f at the points $x_0, x_1, \cdots x_N$.

This problem is tricky. Indeed I spent an entire afternoon to figure it out but it is a very cool problem!

If you have a better solution, please share it with me!

A direct substitution of $c_n = \frac{2n+1}{2} \sum_{j=0}^{N} f(x_j) P_n(x_j) w_j$ into $p_N(x)$ is not very helpful for me. So instead we look more carefully at $p_N(x)$, it is important to notice that for an arbitrary integer m where $0 \le m \le N$,

$$\sum_{i=0}^{N} p_N(x_i) P_m(x_i) w_i = \sum_{i=0}^{N} \left(\sum_{n=0}^{N} c_n P_n(x_i) \right) P_m(x_i) w_i = \sum_{n=0}^{N} c_n \left(\sum_{i=0}^{N} P_n(x_i) P_m(x_i) w_i \right)$$

Now notice that $P_n(x)$ and $P_m(x)$ are both polynomials of degree less than or equal to N, the product polynomial $P_n P_m$ is polynomial of degree less than or equal to $2N$. This implies that the sum inside the bracket in the last step is the same as the integral,

$$\sum_{i=0}^{N} P_n(x_i) P_m(x_i) w_i = \int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} \frac{2}{2m+1} & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

since the Gauss-Legendre quadrature formula is exact for any polynomials of degree less than or equal to $(2N+1)$. This implies that

$$\sum_{i=0}^{N} p_N(x_i) P_m(x_i) w_i = \sum_{n=0}^{N} c_n \left(\sum_{i=0}^{N} P_n(x_i) P_m(x_i) w_i \right) = \frac{2}{2m+1} c_m$$

Now we if substitute the formula for c_m, i.e.

$$\sum_{i=0}^{N} p_N(x_i) P_m(x_i) w_i = \frac{2}{2m+1} c_m = \frac{2}{2m+1} \cdot \frac{2m+1}{2} \sum_{j=0}^{N} f(x_j) P_m(x_j) w_j$$

Comparing both sides gives us $p_N(x_i) = f(x_i)$ for all $1 \le i \le N$.

Illustrative example: This problem might seem abstract to some of you. Let us do a simple example for $N = 2$ and let $f\colon [-1,1] \to \mathbb{R}$ be any "smooth" function.

The normalized Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$. So it has root $x_0 = -\frac{1}{\sqrt{3}}$, $x_1 = \frac{1}{\sqrt{3}}$. We use the formula $w_j = \frac{2}{[1-(x_j)^2]\left(P_2'(x_j)\right)^2}$ with $P_2'(x) = 3x$

$$w_0 = \frac{2}{\left(1 - \frac{1}{3}\right) 3} = 1, \qquad w_1 = \frac{2}{\left(1 - \frac{1}{3}\right) 3} = 1$$

So using the given formula for c_n and $P_0(x) \equiv 1, P_1(x) = x$

$$\begin{aligned}
c_0 &= \frac{1}{2}\left(f(x_0)P_0(x_0)w_0 + f(x_1)P_0(x_1)w_1\right) = \frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}}) \\
c_1 &= \frac{3}{2}\left(f(x_0)P_1(x_0)w_0 + f(x_1)P_1(x_1)w_1\right) = \frac{3}{2}(-\frac{1}{\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})\frac{1}{\sqrt{3}}) \\
\Rightarrow \quad p_2(x) &= \left(\frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}})\right)\cdot 1 + \left(-\frac{3}{2\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + \frac{3}{2\sqrt{3}}f(\frac{1}{\sqrt{3}})\right)\cdot x
\end{aligned}$$

Notice that $p_2(x)$ is a polynomial of degree 1. According to the result we obtained from the previous problem, $p_2(x)$ should interpolate $f(x)$ at two points $x = \pm\frac{1}{\sqrt{3}}$, checking:

$$p_2(x_0) = p_2(-\frac{1}{\sqrt{3}}) = \left(\frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}})\right) + \left(-\frac{3}{2\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + \frac{3}{2\sqrt{3}}f(\frac{1}{\sqrt{3}})\right)\cdot\left(-\frac{1}{\sqrt{3}}\right) = f(-\frac{1}{\sqrt{3}})$$

Similarly, we can verify that $p_2(x_1) = p_2(\frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}})$

MAT 128A - Assignment 7

Karry Wong

November 14, 2018

Problem 1

Find a polynomial p of degree 3 such that

$$p(0) = 0, \quad p(1) = 1, \quad p(2) = 1, \quad \text{and } p'(0) = 1$$

Ans: Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and we solve for a_3, a_2, a_1, a_0.
Since $p(0) = 0$, $a_0 = 0$. Also, since $p'(0) = 1$, $a_1 = 1$. Using $p(1) = 1$ and $p(2) = 1$, we have

$$\begin{cases} a_3 + a_2 + 1 &= 1 \\ 8a_3 + 4a_2 + 2 &= 1 \end{cases} \Rightarrow \begin{cases} a_2 &= \frac{1}{4} \\ a_3 &= -\frac{1}{4} \end{cases}$$

Therefore, $p(x) = -\frac{1}{4}x^3 + \frac{1}{4}x^2 + x$.

Remark 1. A more general question related to the above question would be given n points and all their derivatives up to order m, i.e.

$$\begin{array}{c} (x_0, y_0), \cdots, (x_{n-1}, y_{n-1}) \\ (x_0, y_0'), \cdots, (x_{n-1}, y_{n-1}') \\ \vdots, \quad \ddots, \quad \vdots \\ (x_0, y_0^{(m)}), \cdots, (x_{n-1}, y_{n-1}^{(m)}) \end{array}$$

can we find a polynomial function $p(x)$ to interpolate all n points and satisfies all the derivatives at these n points? If yes, what is the maximal degree of this polynomial?
The answer is yes, and the degree of this polynomial is at most $n(m+1) - 1$. Read more under "**Hermite interpolation**" at https://en.wikipedia.org/wiki/Hermite_interpolation
Notice that if no derivatives are given, then $m = 0$, there exists a polynomial of degree $n - 1$

interpolating the n points given (this polynomial is also unique using the Lagrange interpolation formula).

Problem 2

(a). Show that the roots of

$$p_N(x) = T_{N+1}(x) - T_{N-1}(x)$$

are $x_j = cos\left(\frac{\pi}{N}j\right)$ for $j = 0, 1, \cdots, N$.

(b). Use (a) to prove that

$$\prod_{j=0}^{N}(x - x_j) = (x - x_0)\cdots(x - x_N) = 2^{-N}p_N(x)$$

(c). Show that

$$\left|\prod_{j=0}^{N}(x - x_j)\right| = |(x - x_0)\cdots(x - x_N)| \le 2^{-N+1}$$

for all $x \in [-1, 1]$.

(a). For the sake of simplicity, we let $t = cos^{-1}x$. Using the trigonometric identity, we have

$$\begin{aligned} T_{N+1}(x) - T_{N-1}(x) &= cos((N+1)t) - cos((N-1)t) \\ &= cosNtcost - sinNtsint - (cosNtcost + sinNtsint) \\ &= -2sinNtsint \end{aligned}$$

So for

$$\begin{aligned} -2sinNtsint = 0 \;\Rightarrow\; & sinNt = 0 \quad \text{or} \quad sint = 0 \\ \Rightarrow\; & Nt = j\pi \quad \text{or} \quad t = j\pi \end{aligned}$$

Given that $x \in [-1, 1]$, this implies $t \in [-\pi, \pi]$, therefore $x_j = cos(\frac{\pi}{N}j)$ for $j = 0, 1, \cdots, N$.

(b). Since $x_0, x_1, \cdots x_N$ are roots of $p(x)$, clearly

$$p(x) = c(x - x_0)(x - x_1)\cdots(x - x_N) \tag{1}$$

where c is a constant.

Now, first recall that $T_N(x)$ is a polynomial of degree N with the leading coefficient 2^{N-1}

(You can prove this statement by mathematical induction. See **Bonus**). So $p_N(x) = T_{N+1}(x) - T_{N-1}(x)$ is a polynomial of degree $N+1$ with leading coefficient 2^N.

Degree $N+1$ is confirmed since $p(x)$ is a product of $(N+1)$-times x terms in (1). Also we get $c = 2^N$. Therefore dividing both sides of (1) by 2^N, we have

$$(x - x_0)(x - x_1)\cdots(x - x_N) = 2^{-N}p(x)$$

(c). First, let us recall that $|T_N(x)| \leq 1$ for all integers N since $T_N(x) = cos(ncos^{-1}(x))$ is a cosine function.

Taking aboslute value on both sides and applying the triangular inequality, we have

$$\left|\prod_{j=0}^{N}(x - x_j)\right| = 2^{-N}|p(x)| \leq 2^{-N}\left(|T_{N+1}(x)| + |T_{N-1}(x)|\right) \leq 2^{-N}(1+1) = 2^{-N+1}$$

Bonus: Claim : $T_n(x)$ *is a polynomial of degree* n *with the leading coefficient* 2^{n-1}.

Proof. For $n = 0, 1$, $T_0(x) \equiv 1, T_1(x) = x$ which satisfies the statement.

Assume that the statement holds for arbitrary integer k and $k-1$. For $n = k+1$, recall the recurrence relation $T_{k+1} = 2xT_k(x) - T_{k-1}(x)$. Since $2xT_k(x)$ contains the highest order term which is of degree $k+1$ and with leading coefficient $2 \cdot 2^{k-1} = 2^k$.

So by mathematical induction, the statement holds for all natural numbers n.

Problem 3

Suppose that $f\colon[a,b] \to \mathbb{R}$ is (N+1)-times continuously differentiable, and that $x_0, \cdots x_N$ are the $(N+1)$ nodes of the Chebyshev extrema grid on the interval $[a,b]$ so that

$$x_j = \frac{b-a}{2}cos\left(\frac{j}{N}\pi\right) + \frac{b+a}{2} \tag{2}$$

for all $j = 0, 1, \cdots N$.

Also, let p_N be the polynomial of degree N which interpolates f at the nodes $x_0, x_1, \cdots, x_N$.

Show that there exists $\xi \in (a,b)$ such that

$$|f(x) - p_N(x)| \leq 2^{-N+1}\left(\frac{b-a}{2}\right)^{N+1}\frac{|f^{(N+1)}(\xi)|}{(N+1)!}$$

Hint: Let $g(x) = f(\frac{b-a}{2}x + \frac{b+a}{2})$ and use 2(c) to develop an error bound for g.

Ans: First, nodes $cos\left(\frac{j}{N}\pi\right)$, $j = 0, 1, \cdots, N$ are clearly the extrema nodes for Chebyshev polynomial $T_N(x)$ on $[-1, 1]$.

Using linear transformation $h: [-1, 1] \mapsto [a, b]$ defined by

$$h(x) = \frac{b-a}{2}x + \frac{b+a}{2},$$

the x_j definied in formula (2) are extrema nodes for $T_N(h^{-1}(x))$ on $[a, b]$.

Second, recall the following theorem for interpolation error from lecture 15.

Theorem 1. Given $f: [a, b] \to \mathbb{R}$ is (N+1)-times continuously differentiable, $x_0 < x_1 < \cdots < x_N$ are partition of $[a, b]$, and p_N is the unique polynomial of degree N which interpolates f at nodes $x_0, x_1, \cdots, x_N$. Then for $x \in [a, b]$, there exists a point $\xi_x \in (a, b)$ such that

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N} (x - x_j) \tag{3}$$

We apply the above formula (3) to the function $g(x)$ in the given hint. Note that the function g is defined on $[-1, 1]$, i.e. $g(x) = (f \circ h)(x)$ with $h(x)$ being the linear transformation as defined above. Also, let polynomial $\tilde{p}_N(x) = (p_N \circ h)(x)$.

Then $\tilde{p}_N$ is the polynomial of degree N which interpolates $g(x)$ at the nodes $cos\left(\frac{j}{N}\pi\right)$, $j = 0, 1, \cdots, N$.

From (3) applied on $g(x)$ and $\tilde{p}_N(x)$ with $x \in [-1, 1]$, there exists $\tilde{\xi} \in [-1, 1]$

$$f(h(x)) - p_N(h(x)) = g(x) - \tilde{p}_N(x) = \frac{g^{(N+1)}(\tilde{\xi})}{(N+1)!} \prod_{j=0}^{N} \left(x - cos\left(\frac{j}{N}\pi\right)\right) \tag{4}$$

From part (2c), we know that $\prod_{j=0}^{N} \left(x - cos\left(\frac{j}{N}\pi\right)\right) \le 2^{-N+1}$.

Also, since $g(x) = f(\frac{b-a}{2}x + \frac{b+a}{2})$, therefore by chain rule

$$g^{(N+1)}(\tilde{\xi}) = f^{(N+1)}(\underbrace{\frac{b-a}{2}\tilde{\xi} + \frac{b+a}{2}}_{=\xi \in [a,b]}) \cdot \left(\frac{d}{dx}(\frac{b-a}{2}x + \frac{b+a}{2})\right)^{N+1} = f^{(N+1)}(\xi)\left(\frac{b-a}{2}\right)^{N+1}$$

To conclude, applying absolute value on both sides of 4, we have that for any $x \in [a, b]$,

$$|f(x) - p_N(x)| \le \frac{|f^{(N+1)}(\xi)|}{(N+1)!} \left(\frac{b-a}{2}\right)^{N+1} 2^{-N+1}$$

Remark 2. My argument above might look tedious but the concept is very simple. You apply formula (3) and part 2c. Finally the factor $\frac{b-a}{2}$ comes from the linear transformation.

Problem 4

Suppose that $f(x) = cos(x)$, that N is a positive integer, and that $x_0, \cdots x_N$ are the nodes of the Chebyshev extrema grid on the interval $[0, 1]$. Also, let p_N denote the polynomial of degree N which interpolates f at the nodes $x_0, \cdots x_N$. Show that

$$|f(x) - p_N(x)| \le \frac{2^{-2N}}{(N+1)!}$$

for all $-1 \le x \le 1$.

Ans: We apply the result from problem 3 with $a = 0, b = 1$. Also, note that the function $f(x) = cosx$ is infinitely differentiable, i.e. $f \in C^\infty([0, 1])$. Furthermore, $f^{(N)}(x)$ is either a sine or cosine function up to a sign for all natural numbers N. Therefore $|f^{(N)}(x)| \le 1$, so

$$|f(x) - p_N(x)| \le 2^{-N+1} \cdot \frac{1}{2^{N+1}} \cdot \frac{1}{(N+1)!} = \frac{2^{-2N}}{(N+1)!}$$

Bonus: *What is the advantage of using Chebyshev node?* Think about how we would obtain such a bound **without** knowing Chebyshev polynomial. The most natural thing is to approximate $f(x) = cosx$ with its (Lagrange) interpolation polynomial **with equally spaced points**. Then we can obtain another error bound for $|f(x) - p_N(x)|$ by applying (3). It turns out that for a $(N+1)$-times continuously differentiable function $f : [a, b] \mapsto \mathbb{R}$ and a polynomial of degree N interpolating f at equally spaced points

$$x_i = a + \frac{i}{n}(b - a), \quad i = 0, 1, \cdots n$$

the error bound for any $x \in [a, b]$ becomes

$$|f(x) - p_N(x)| \le \left(\frac{b-a}{n}\right)^{n+1} \frac{1}{4(n+1)} |f^{N+1}(\xi)|$$

for some $\xi \in [a, b]$.

The proof for the inequality above is not obvious. **See pp 7 -8 at** `https://www.math.uh.edu/~jingqiu/math4364/interp_error.pdf`

In practice, as pointed out in class, using the equally spaced point for polynomial interpolation often leads to very large oscillation as x approaches the endpoints a or b of the interval. You can also find more info. in the above link.

MAT 128A - Assignment 8

Karry Wong

December 3, 2018

Problem 1

Let $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$. Find weights w_0, w_1, and w_2 such that the formula

$$\int_0^1 f(x)\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) \tag{1}$$

holds whenever f is a polynomial of degree less than or equal to 2.

Ans: Same as Simpson's rule (the closed Newton-Cotes formula for three points) shown in lecture 18, we substitute

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$$

into (1). We obtain

$$\begin{cases} 1 = \int_0^1 1\,dx & = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1 \\ \frac{1}{2} = \int_0^1 x\,dx & = w_0 \cdot 0 + w_1 \cdot \frac{1}{2} + w_2 \cdot 1 \\ \frac{1}{3} = \int_0^1 x^2\,dx & = w_0 \cdot 0 + w_1 \cdot \left(\frac{1}{2}\right)^2 + w_2 \cdot 1^2 \end{cases} \quad \Rightarrow \quad w_0 = \frac{1}{6},\ w_1 = \frac{2}{3},\ w_2 = \frac{1}{6}$$

As shown in class, for arbitrary interval $[a, b]$, the *weights* for Simpson's rule are $w_0 = \frac{b-a}{6}$, $w_1 = \frac{2(b-a)}{3}$, $w_2 = \frac{b-a}{6}$.

Remark 1. Assume that the function f is at least four times continuously differentiable on $[a, b]$ and c is the midpoint of interval $[a, b]$, i.e. $c = \frac{a+b}{2}$. Let $h = c - a = b - c$. Simpson's rule states that there exists a point $\xi \in (a, b)$ such that

$$\int_a^b f(x)dx = \frac{h}{3}[f(a) + 4f(c) + f(b)] - \underbrace{\frac{h^5}{90} f^{(4)}(\xi)}_{\text{error term}}$$

Since the error term involves the fourth derivative of f, Simpson rule is indeed exact for polynomial of degree equal to or less than three!

Bonus I: For those who are interested in learning more, let me first show you how to "cheat" in obtaining the right coefficient for the error term in Simpson's rule:

Let f be a function that is at least four times continuously differentiable on an interval $[a,b]$. Let midpoint of the interval be $c = \frac{a+b}{2}$ and the spacing be $h = c - a = b - c$. Assume that we know the error term is $\mathcal{O}(h^5)$ instead of $\mathcal{O}(h^4)$ involving the fourth derivative of $f(x)$ (**that is a big assumption to make!**), then we have for $\xi \in (a,b)$,

$$\int_a^b f(x)\,dx = \frac{h}{3}[f(a) + 4f(c) + f(b)] + kf^{(4)}(\xi)$$

where k is a constant. In order to solve for k, the key idea is to apply the above formula on $f(x) = x^4$. Together with $c = \frac{a+b}{2}$, $h = \frac{b-a}{2}$, we have

$$\begin{aligned} \int_a^b x^4\,dx &= \frac{b-a}{6}(a^4 + 4c^4 + b^4) + k(24) \quad \text{since } f^{(4)}(x) \equiv 24 \\ \Rightarrow \quad \frac{b^5 - a^5}{5} &= \frac{b-a}{6}\left(a^4 + \frac{1}{4}(a+b)^4 + b^4\right) + k(24) \\ \Rightarrow \quad 24k &= \frac{b^5 - a^5}{5} - \frac{b-a}{24}\left(4a^4 + (a+b)^4 + 4b^4\right) \end{aligned}$$

If you have lot of time in expanding the RHS of the above equation and calculate carefully (or simply use `Mathematica`), you will arrive at

$$24k = \frac{1}{120}(a-b)^5 \quad \Rightarrow \quad k = -\frac{1}{2880}(b-a)^5 \quad \Rightarrow \quad k = -\frac{h^5}{90}$$

Bonus II: At the end of the lecture 18, Prof. Bremer mentioned the error estimate for Simpson rule, i.e. how to derive the term $-\frac{h^5}{90}f^{(4)}(\xi)$. The main idea is to integrate the error term obtained from the Lagrange interpolation formula. **Unfortunately, unlike the error estimate in the trapezoidal rule, we cannot apply the weighted mean value theorem directly due to change of sign in the cubic polynomial inside the integrand.** Let me show you here how to get around it ☺

Let f be a function that is at least four times continuously differentiable on an interval $[a,b]$. Let midpoint of the interval be $c = \frac{a+b}{2}$ and the spacing be $h = c - a = b - c$. The error term is the integral of the Lagrange interpolation error, i.e.

$$Err \coloneqq \int_a^b \frac{f^{(3)}(\xi(x))}{3!}(x-a)(x-c)(x-b)\,dx, \qquad \text{where } \xi(x) \text{is a function of x!}$$

For those of you who know about *divided difference*, indeed $\frac{f^{(3)}(\xi(x))}{3!} = f[a,b,c,x]$.

the key idea here is to use integration by parts, first define $w(x) \coloneqq \int_a^x (t-a)(t-c)(t-b)\,dt$.

Notice that $w'(x) = (x-a)(x-c)(x-b)$ follows immediately. Also, clearly $w(a) = 0$.

Since the cubic polynomial $g(t) \coloneqq (t-a)(t-c)(t-b)$ is "rotational symmetric" around $t = c$, i.e. $g(-t+2c) = -g(t)$, and a, c, b are equally spaced, so $w(b) = 0$. (Draw a picture or compute the integral explicitly!).

Lastly, $w'(x) = (x-a)(x-c)(x-b) > 0$ for $a < x < c$ and $w'(x) = (x-a)(x-c)(x-b) < 0$ for $c < x < b$, so $w'(x)$ attains local maximum at $x = c$. Combining all the above information, we know that $w(x) > 0$ for all $x \in (a, b)$, i.e. $w(x)$ **does not change sign**.

$$Err = \int_a^b f[a,b,c,x]w'(x)\,dx = \underbrace{\Big[f[a,b,c,x]w(x)\Big]_a^b}_{=0-0=0} - \int_a^b \frac{d}{dx}\left(f[a,b,c,x]\right)w(x)\,dx$$
$$\overset{(\diamond)}{=} -\int_a^b f[a,b,c,x,x]w(x)\,dx \overset{(*)}{=} -f[a,b,c,\eta,\eta]\int_a^b w(x)\,dx$$

for some $\eta \in (a, b)$.

$(\diamond)$ holds since divided difference is invariant under permutation, we have

$$\frac{d}{dx}\left(f[a,b,c,x]\right) = \lim_{h\to 0}\frac{f[a,b,c,x+h]-f[a,b,c,x]}{h} = \lim_{h\to 0}\frac{f[a,b,c,x+h]-f[x,a,b,c]}{h}$$
$$= \lim_{h\to 0} f[x,a,b,c,x+h] = f[x,a,b,c,x] = f[a,b,c,x,x]$$

Important: At $(*)$, the weighted mean value theorem can be applied since $w(x)$ does not change sign.

Now there is a corresponding Mean Value Theorem for divided difference which states that for any $(n+1)$ distinct numbers $x_0, \cdots, x_n$ in $[a, b]$, we have

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\bar\eta)}{n!} \quad \text{for some } \bar\eta \in \left(\min_i\{x_i\}, \max_i\{x_i\}\right)$$

Therefore, $Err = -\frac{f^{(4)}(\bar\eta)}{4!}\int_a^b w(x)\,dx$. for some $\bar\eta \in (a, b)$

Finally, we compute $\int_a^b w(x)\,dx$,

$$\int_a^b w(x)\,dx = \int_a^b \int_a^x (t-a)(t-c)(t-b)\,dt\,dx = \cdots = \frac{4}{15}h^5$$

It took me some time to evaluate this integral. (Again, using `Mathematica` is always an option!)

Therefore, we have

$$Err = \int_a^b \frac{f^{(3)}(\xi(x))}{3!}(x-a)(x-c)(x-b)\,dx = -\frac{f^{(4)}(\bar\eta)}{24}\cdot\frac{4}{15}h^5 = -\frac{h^5}{90}f^{(4)}(\bar\eta)$$

Alternatively, here is a sketch of another idea from Prof. Bremer: Let $p(x)$ be the degree 2 polynomial that interpolates $f(x)$ at a, c, b. We replace the term $\frac{f^{(3)}(\xi(x))}{3!}$ above with

$$r(x) \coloneqq \frac{f(x) - p(x)}{w'(x)} \;\Rightarrow\; f(x) = p(x) + r(x)w'(x)$$

Similarly as the steps above, we can derive

$$\begin{aligned} Err &= \int_a^b f(x) - p(x)\,dx = \int_a^b r(x)w'(x)\,dx = [r(x)w(x)]_a^b - \int_a^b r'(x)w(x)\,dx && (2) \\ &= -r'(\xi)\int_a^b w(x)\,dx && (3) \end{aligned}$$

for some $\xi \in (a, b)$, where we applied the weighted Mean Value Theorem in the last step.

We can use Rolle's theorem to show that for each ξ there exists a $\eta \in (a, b)$ such that $r'(\xi) = \frac{f^{(4)}(\eta)}{4!}$. Then we can obtain the same answer. I will leave the details to you. ☺

Problem 2

Let $x_0 = -\sqrt{\frac{3}{5}}, x_1 = 0$, and $x_2 = \sqrt{\frac{3}{5}}$. Find weights w_0, w_1, and w_2 such that

$$\int_{-1}^{1} f(x)\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) \tag{4}$$

holds whenever f is a polynomial of degree less than or equal to 2. Show that the formula in fact holds when f is a polynomial of degree less than or equal to 5.

Ans: The three points x_0, x_1, x_2 given above are actually Gauss-Legendre quadrature rule with three points! Please refer back to lecture 19 and more on `https://en.wikipedia.org/wiki/Gaussian_quadrature`.

Same as the previous problem, we substitute $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$ into (4). We have

$$\begin{cases} 2 = \int_{-1}^{1} 1\,dx & = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1 \\ 0 = \int_{-1}^{1} x\,dx & = w_0 \cdot \left(-\sqrt{\frac{3}{5}}\right) + w_1 \cdot 0 + w_2 \cdot \left(\sqrt{\frac{3}{5}}\right) \\ \frac{2}{3} = \int_{-1}^{1} x^2\,dx & = w_0 \cdot \left(-\sqrt{\frac{3}{5}}\right)^2 + w_1 \cdot 0 + w_2 \cdot \left(\sqrt{\frac{3}{5}}\right)^2 \end{cases} \Rightarrow \quad w_0 = \frac{5}{9},\ w_1 = \frac{8}{9},\ w_2 = \frac{5}{9}$$

Alternatively, as shown in class, the Gauss-Legendre quadrature weights can be obtained via the Lagranage interpolation formula. Here I demonstrate how to calculate w_0 and you should be able to compute w_1, w_2 on your own:

$$\begin{aligned} w_0 &= \int_{-1}^{1} L_0(x)\,dx = \int_{-1}^{1} \prod_{\substack{0 \le i \le 2 \\ i \ne 0}} \frac{x - x_i}{x - x_0}\,dx = \int_{-1}^{1} \frac{x(x - \sqrt{\frac{3}{5}})}{\left(-\sqrt{\frac{3}{5}}\right)\left(-2\sqrt{\frac{3}{5}}\right)}\,dx \\ &= \frac{5}{6} \int_{-1}^{1} x\left(x - \sqrt{\frac{3}{5}}\right)dx = \frac{5}{6} \cdot \frac{2}{3} = \frac{5}{9} \end{aligned}$$

Now we want show that the quadrature rule (4) above is exact for any polynomials of degree up to (including equal to) 5.

As said, the given points x_0, x_1, x_2 given are Gauss-Legendre points of degree with $N = 2$. They are the roots of the Legendre polynomial of degree 3, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$. Therefore, it is exact for polynomial of degree less than or equal to $2(2) + 1 = 5$ in our case.

Remark 2. For those of you who have difficulty in understanding the proof in class, I repeat the same proof for $N = 2$.

Assume we only know that the Gauss-Legendre quadrature (4) holds exactly for degree less than or equal to 2.

Suppose that p is a polynomial of degree 5. Then by the polynomial division, we can write $p(x) = q(x)P_3(x) + r(x)$, where $P_3(x) = \frac{3}{2}x^2 - \frac{1}{2}$ is the Legendre polynomial of degree 3, $q(x)$ a polynomial of degree at most 2 and $r(x)$ is a polynomial of degree at most 2.

Since the Legendre polynomials form an orthogonal basis for functions on interval $[-1, 1]$ with respect to the L^2 inner product by construction, i.e.

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} \frac{2}{2n+1}, & m = n \\ 0, & m \neq n \end{cases}$$

The Legendre polynomial P_3 is orthogonal to any polynomials of degree up to (including equal to) 2, i.e. $\int_{-1}^{1} q(x)P_3(x)\,dx = 0$. So

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} q(x)P_3(x) + r(x)\,dx = \int_{-1}^{1} r(x)\,dx$$

Since Gauss-Legendre quadrature holds exactly for degree less than or equal to 2 and x_0, x_1, x_2 are roots of $P_3(x)$, we have

$$\sum_{k=1}^{n} p(x_k)w_k = \sum_{k=1}^{n} \left(q(x_k)\underbrace{P_3(x_k)}_{=0} + r_(x_k) \right) w_k = \sum_{k=1}^{n} \underbrace{r(x_k)}_{deg(r)\leq 2} w_k = \int_{-1}^{1} r(x)\,dx = \int_{-1}^{1} p(x)\,dx$$

So (4) holds exactly for degree less than or equal to 5. The above proof cannot work for polynomials of degree higher than or equal to 6, why?

Problem 3

Let $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$. Find weights w_0, w_1, and w_2 such that the formula

$$\int_0^1 f(x)\sqrt{x}\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) \tag{5}$$

when f is a polynomial of degree less than or equal to 2. Use this quadrature rule to approximate

$$\int_0^1 cos(x)\sqrt{x}\,dx$$

How accurate is your approximation?

Ans: We substitute $f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2$ into (5).

$$\begin{cases} 1 = \int_0^1 1\,dx & = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1 \\ \frac{2}{5} = \int_0^1 x\sqrt{x}\,dx & = w_0 \cdot 0 + w_1 \cdot \frac{1}{2} + w_2 \cdot 1 \\ \frac{2}{7} = \int_0^1 x^2\,dx & = w_0 \cdot 0 + w_1 \cdot \left(\frac{1}{2}\right)^2 + w_2 \cdot 1^2 \end{cases} \Rightarrow \quad w_0 = \frac{13}{35},\ w_1 = \frac{16}{35},\ w_2 = \frac{6}{35}$$

Set $f(x) = cos(x)$,

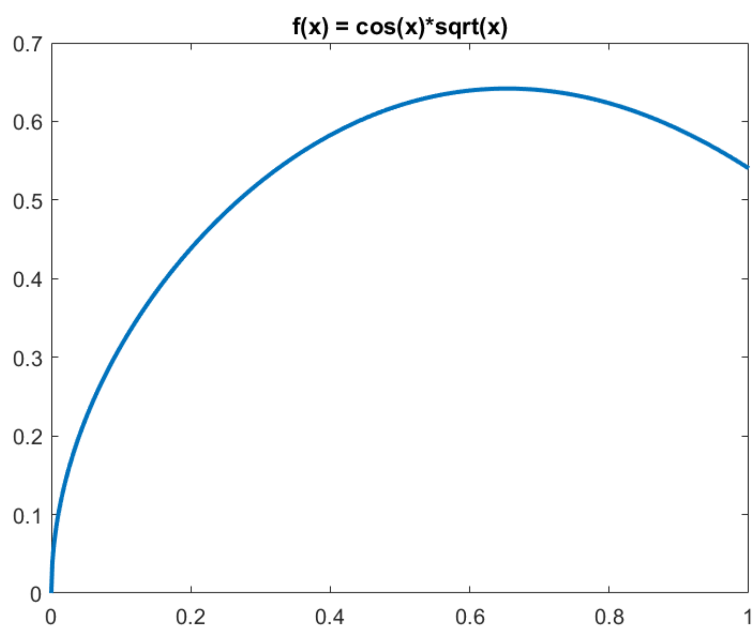
$$\int_0^1 f(x)\sqrt{x}\,dx = \frac{13}{35}f(0) + \frac{16}{35}f(\frac{1}{2}) + \frac{6}{35}f(1)$$
$$\Rightarrow \quad \int_0^1 cos(x)\sqrt{x}\,dx = \frac{13}{35} \cdot 1 + \frac{16}{35} \cdot cos\left(\frac{1}{2}\right) + \frac{6}{35} \cdot cos(1)$$

On one hand, using `MATLAB`, we compute the above sum to 15 decimal places ≈ 0.865232423584423. On the other hand, using `Mathematica` , the exact value for this definite integral should be ≈ 0.531202683084515.

The approximation is actually bad since it is wrong already in the first decimal place. The reason is the abrupt increase of the function $g(x) = cos(x)\sqrt{x}$ near $x = 0$ (*cos(x) is NOT a polynomial of any degrees!*), compute the derivative

$$g'(x) = -sin(x)\sqrt{x} + \frac{cos(x)}{2\sqrt{x}} \quad \Rightarrow \quad \lim_{x\to 0^+} g'(x) = +\infty$$

Below is a plot of $g(x)$ over $[0, 1]$, observe that the tangent line of $g(x)$ near $x = 0$ behaves like a vertical line.



Also, the error for Simpson rule should be bound by the term $\frac{h^5}{90}g^{(4)}(\xi)$. One can also show that the fourth derivative of $g(x)$ is unbounded as $x \to 0^+$. This also explains why the approximation is bad in this case.

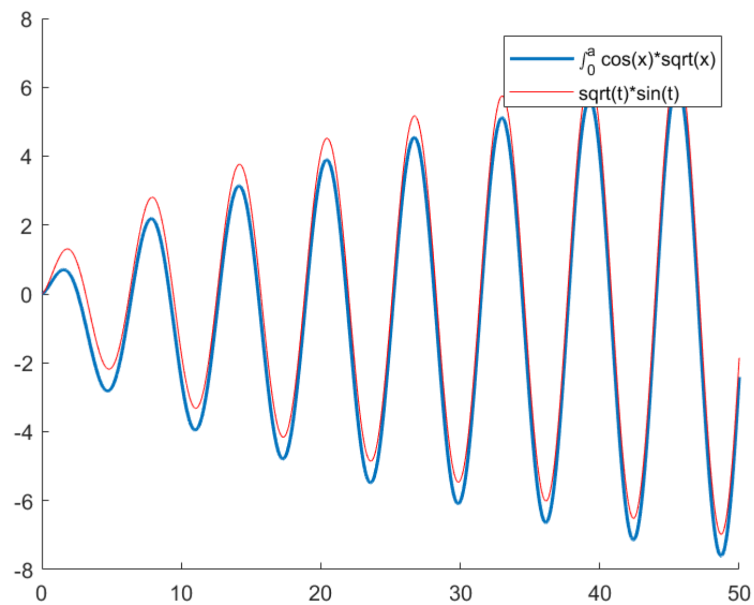
Bonus: For those who are interested in special integral, here is more to learn from this problem: If we look at our definite integral $F(a) \coloneqq \int_0^a cos(x)\sqrt{x}\,dx$ for real number $a \geq 0$, it seems hard to find a closed form (algebraic expressions) for this integral at first glance. But integrate by part and then use change of variable $x = t^2$, we have:

$$F(a) = \int_0^a cos(x)\sqrt{x}\,dx = \left[\sqrt{x}sin(x)\right]_0^a - \frac{1}{2}\int_0^a \frac{sin(x)}{\sqrt{x}}\,dx = \sqrt{a}sin(a) - \frac{1}{2}\int_0^{\sqrt{a}} \frac{sin(t^2)}{t}\,(2tdt)$$

$$\Rightarrow\ F(a) = \sqrt{a}sin(a) - \underbrace{\int_0^{\sqrt{a}} sin(t^2)\,dt}_{\coloneqq S(\sqrt{a})}$$

where $S(x) \coloneqq \int_0^x sin(t^2)\,dt$ is called the **Fresnel Integral** which has many interesting properties and is closely related to special curves. Please refer to https://en.wikipedia.org/wiki/Fresnel_integral for more info.

Lastly, our definite integral $F(x)$ has oscillatory behavior due to the dominant term $\sqrt{x}sin(x)$. Below is a plot showing a blue curve as $F(x)$ over the interval $[0, 50]$ and a red curve $g(x) = \sqrt{x}sin(x)$. The discrepancy between the two is due to the correction term, the Fresnel integral, $S(\sqrt{x})$.



Problem 4

Suppose that

$$f(x) = 2T_0(x) + 4T_1(x) - 6T_2(x) + 12T_3(x) - 14T_4(x)$$

Find

$$\int_{-1}^{1} f(x)\,dx$$

Ans: This should very straight-forward if you remember the integral formula for the n-th Chebyshev polynomial:

$$\int_{-1}^{1} T_n(x)\,dx = \begin{cases} 0, & n \text{ odd} \\ \frac{2}{1-n^2}, & n \text{ even} \end{cases}$$

The above formula holds for all $n = 0, 1, 2, \cdots$. This was the problem 4 in the midterm exam. Can you recall how to derive this formula? Make sure you can! ☺

So we have

$$\int_{-1}^{1} f(x)\,dx = 2\int_{-1}^{1} T_0(x)\,dx + 4\cdot 0 - 6\int_{-1}^{1} T_2(x)\,dx + 12\cdot 0 - 14\int_{-1}^{1} T_4(x)\,dx$$
$$= 2\frac{2}{1-0^2} - 6\frac{2}{1-2^2} - 14\frac{2}{1-4^2} = 4 + 4 + \frac{28}{15} = \frac{148}{15}$$