MAT 128A - Assignment 7

Karry Wong

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Problem 1

Find a polynomial p of degree 3 such that

$$p(0) = 0, \quad p(1) = 1, \quad p(2) = 1, \quad \text{and } p'(0) = 1$$

Ans: Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and we solve for a_3, a_2, a_1, a_0.
Since $p(0) = 0$, $a_0 = 0$. Also, since $p'(0) = 1$, $a_1 = 1$. Using $p(1) = 1$ and $p(2) = 1$, we have

$$\begin{cases} a_3 + a_2 + 1 &= 1 \\ 8a_3 + 4a_2 + 2 &= 1 \end{cases} \Rightarrow \begin{cases} a_2 &= \frac{1}{4} \\ a_3 &= -\frac{1}{4} \end{cases}$$

Therefore, $p(x) = -\frac{1}{4}x^3 + \frac{1}{4}x^2 + x$.

Remark 1. A more general question related to the above question would be given n points and all their derivatives up to order m, i.e.

$$\begin{aligned} &(x_0, y_0), \cdots, (x_{n-1}, y_{n-1}) \\ &(x_0, y_0'), \cdots, (x_{n-1}, y_{n-1}') \\ &\vdots, \quad \ddots, \quad \vdots \\ &(x_0, y_0^{(m)}), \cdots, (x_{n-1}, y_{n-1}^{(m)}) \end{aligned}$$

can we find a polynomial function $p(x)$ to interpolate all n points and satisfies all the derivatives at these n points? If yes, what is the maximal degree of this polynomial?
The answer is yes, and the degree of this polynomial is at most $n(m+1) - 1$. Read more under "**Hermite interpolation**" at https://en.wikipedia.org/wiki/Hermite_interpolation
Notice that if no derivatives are given, then $m = 0$, there exists a polynomial of degree $n - 1$

interpolating the n points given (this polynomial is also unique using the Lagrange interpolation formula).

Problem 2

(a). Show that the roots of

$$p_N(x) = T_{N+1}(x) - T_{N-1}(x)$$

are $x_j = cos\left(\frac{\pi}{N}j\right)$ for $j = 0, 1, \cdots, N$.

(b). Use (a) to prove that

$$\prod_{j=0}^{N}(x - x_j) = (x - x_0)\cdots(x - x_N) = 2^{-N}p_N(x)$$

(c). Show that

$$\left|\prod_{j=0}^{N}(x - x_j)\right| = |(x - x_0)\cdots(x - x_N)| \leq 2^{-N+1}$$

for all $x \in [-1, 1]$.

(a). For the sake of simplicity, we let $t = cos^{-1}x$. Using the trigonometric identity, we have

$$\begin{aligned} T_{N+1}(x) - T_{N-1}(x) &= cos((N+1)t) - cos((N-1)t) \\ &= cosNtcost - sinNtsint - (cosNtcost + sinNtsint) \\ &= -2sinNtsint \end{aligned}$$

So for

$$\begin{aligned} -2sinNtsint = 0 \ \Rightarrow \ & sinNt = 0 \quad \text{or} \quad sint = 0 \\ \Rightarrow \ & \ Nt = j\pi \quad \text{or} \quad t = j\pi \end{aligned}$$

Given that $x \in [-1, 1]$, this implies $t \in [-\pi, \pi]$, therefore $x_j = cos(\frac{\pi}{N}j)$ for $j = 0, 1, \cdots, N$.

(b). Since $x_0, x_1, \cdots x_N$ are roots of $p(x)$, clearly

$$p(x) = c(x - x_0)(x - x_1)\cdots(x - x_N) \tag{1}$$

where c is a constant.

Now, first recall that $T_N(x)$ is a polynomial of degree N with the leading coefficient 2^{N-1}

(You can prove this statement by mathematical induction. See **Bonus**). So $p_N(x) = T_{N+1}(x) - T_{N-1}(x)$ is a polynomial of degree $N+1$ with leading coefficient 2^N.

Degree $N+1$ is confirmed since $p(x)$ is a product of $(N+1)$-times x terms in (1). Also we get $c = 2^N$. Therefore dividing both sides of (1) by 2^N, we have

$$(x - x_0)(x - x_1)\cdots(x - x_N) = 2^{-N}p(x)$$

(c). First, let us recall that $|T_N(x)| \leq 1$ for all integers N since $T_N(x) = cos(ncos^{-1}(x))$ is a cosine function.

Taking aboslute value on both sides and applying the triangular inequality, we have

$$\left|\prod_{j=0}^{N}(x - x_j)\right| = 2^{-N}|p(x)| \leq 2^{-N}\left(|T_{N+1}(x)| + |T_{N-1}(x)|\right) \leq 2^{-N}(1+1) = 2^{-N+1}$$

Bonus: Claim : $T_n(x)$ *is a polynomial of degree* n *with the leading coefficient* 2^{n-1}.

Proof. For $n = 0, 1$, $T_0(x) \equiv 1, T_1(x) = x$ which satisfies the statement.

Assume that the statement holds for arbitrary integer k and $k-1$. For $n = k+1$, recall the recurrence relation $T_{k+1} = 2xT_k(x) - T_{k-1}(x)$. Since $2xT_k(x)$ contains the highest order term which is of degree $k+1$ and with leading coefficient $2 \cdot 2^{k-1} = 2^k$.

So by mathematical induction, the statement holds for all natural numbers n.

Problem 3

Suppose that $f\colon[a,b] \to \mathbb{R}$ is (N+1)-times continuously differentiable, and that $x_0, \cdots x_N$ are the $(N+1)$ nodes of the Chebyshev extrema grid on the interval $[a,b]$ so that

$$x_j = \frac{b-a}{2}cos\left(\frac{j}{N}\pi\right) + \frac{b+a}{2} \tag{2}$$

for all $j = 0, 1, \cdots N$.

Also, let p_N be the polynomial of degree N which interpolates f at the nodes $x_0, x_1, \cdots, x_N$.

Show that there exists $\xi \in (a,b)$ such that

$$|f(x) - p_N(x)| \leq 2^{-N+1}\left(\frac{b-a}{2}\right)^{N+1}\frac{|f^{(N+1)}(\xi)|}{(N+1)!}$$

Hint: Let $g(x) = f(\frac{b-a}{2}x + \frac{b+a}{2})$ and use 2(c) to develop an error bound for g.

Ans: First, nodes $cos\left(\frac{j}{N}\pi\right)$, $j = 0, 1, \cdots, N$ are clearly the extrema nodes for Chebyshev polynomial $T_N(x)$ on $[-1, 1]$.

Using linear transformation $h: [-1, 1] \mapsto [a, b]$ defined by

$$h(x) = \frac{b-a}{2}x + \frac{b+a}{2},$$

the x_j definied in formula (2) are extrema nodes for $T_N(h^{-1}(x))$ on $[a, b]$.

Second, recall the following theorem for interpolation error from lecture 15.

Theorem 1. Given $f: [a, b] \to \mathbb{R}$ is (N+1)-times continuously differentiable, $x_0 < x_1 < \cdots < x_N$ are partition of $[a, b]$, and p_N is the unique polynomial of degree N which interpolates f at nodes $x_0, x_1, \cdots, x_N$. Then for $x \in [a, b]$, there exists a point $\xi_x \in (a, b)$ such that

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N} (x - x_j) \tag{3}$$

We apply the above formula (3) to the function $g(x)$ in the given hint. Note that the function g is defined on $[-1, 1]$, i.e. $g(x) = (f \circ h)(x)$ with $h(x)$ being the linear transformation as defined above. Also, let polynomial $\tilde{p}_N(x) = (p_N \circ h)(x)$.

Then $\tilde{p}_N$ is the polynomial of degree N which interpolates $g(x)$ at the nodes $cos\left(\frac{j}{N}\pi\right)$, $j = 0, 1, \cdots, N$.

From (3) applied on $g(x)$ and $\tilde{p}_N(x)$ with $x \in [-1, 1]$, there exists $\tilde{\xi} \in [-1, 1]$

$$f(h(x)) - p_N(h(x)) = g(x) - \tilde{p}_N(x) = \frac{g^{(N+1)}(\tilde{\xi})}{(N+1)!} \prod_{j=0}^{N} \left(x - cos\left(\frac{j}{N}\pi\right)\right) \tag{4}$$

From part (2c), we know that $\prod_{j=0}^{N} \left(x - cos\left(\frac{j}{N}\pi\right)\right) \le 2^{-N+1}$.

Also, since $g(x) = f(\frac{b-a}{2}x + \frac{b+a}{2})$, therefore by chain rule

$$g^{(N+1)}(\tilde{\xi}) = f^{(N+1)}(\underbrace{\frac{b-a}{2}\tilde{\xi} + \frac{b+a}{2}}_{=\xi \in [a,b]}) \cdot \left(\frac{d}{dx}(\frac{b-a}{2}x + \frac{b+a}{2})\right)^{N+1} = f^{(N+1)}(\xi)\left(\frac{b-a}{2}\right)^{N+1}$$

To conclude, applying absolute value on both sides of 4, we have that for any $x \in [a, b]$,

$$|f(x) - p_N(x)| \le \frac{|f^{(N+1)}(\xi)|}{(N+1)!} \left(\frac{b-a}{2}\right)^{N+1} 2^{-N+1}$$

Remark 2. My argument above might look tedious but the concept is very simple. You apply formula (3) and part 2c. Finally the factor $\frac{b-a}{2}$ comes from the linear transformation.

Problem 4

Suppose that $f(x) = cos(x)$, that N is a positive integer, and that $x_0, \cdots x_N$ are the nodes of the Chebyshev extrema grid on the interval $[0,1]$. Also, let p_N denote the polynomial of degree N which interpolates f at the nodes $x_0, \cdots x_N$. Show that

$$|f(x) - p_N(x)| \leq \frac{2^{-2N}}{(N+1)!}$$

for all $-1 \leq x \leq 1$.

Ans: We apply the result from problem 3 with $a = 0, b = 1$. Also, note that the function $f(x) = cosx$ is infinitely differentiable, i.e. $f \in C^\infty([0,1])$. Furthermore, $f^{(N)}(x)$ is either a sine or cosine function up to a sign for all natural numbers N. Therefore $|f^{(N)}(x)| \leq 1$, so

$$|f(x) - p_N(x)| \leq 2^{-N+1} \cdot \frac{1}{2^{N+1}} \cdot \frac{1}{(N+1)!} = \frac{2^{-2N}}{(N+1)!}$$

Bonus: *What is the advantage of using Chebyshev node?* Think about how we would obtain such a bound **without** knowing Chebyshev polynomial. The most natural thing is to approximate $f(x) = cosx$ with its (Lagrange) interpolation polynomial **with equally spaced points**. Then we can obtain another error bound for $|f(x) - p_N(x)|$ by applying (3). It turns out that for a $(N+1)$-times continuously differentiable function $f : [a,b] \mapsto \mathbb{R}$ and a polynomial of degree N interpolating f at equally spaced points

$$x_i = a + \frac{i}{n}(b-a), \quad i = 0, 1, \cdots n$$

the error bound for any $x \in [a,b]$ becomes

$$|f(x) - p_N(x)| \leq \left(\frac{b-a}{n}\right)^{n+1} \frac{1}{4(n+1)} |f^{N+1}(\xi)|$$

for some $\xi \in [a,b]$.

The proof for the inequality above is not obvious. **See pp 7 -8 at** https://www.math.uh.edu/~jingqiu/math4364/interp_error.pdf

In practice, as pointed out in class, using the equally spaced point for polynomial interpolation often leads to very large oscillation as x approaches the endpoints a or b of the interval. You can also find more info. in the above link.