MAT 128A - Practice Midterm Exam

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Problem 1 (True or False)

Given that $f\colon [-1,1] \to \mathbb{R}$ is a continuous function, and that $\{a_n\}$ are its Chebyshev coefficients.

Also, for $N \in \mathbb{N}$, $p_N(x) = {\sum_{n=0}^{N}}' a_n T_n(x)$. Determine whether each of the following statements is true or false. No need to justify your answers.

(I) If f is continuously differentiable, then $\|p_N - f\|_\infty \to 0$ as $N \to \infty$.

(II) if f is k-times continuously differentiable, then $|a_n| = \mathcal{O}(\frac{1}{n^k})$.

(III) If f is infinitely differentiable, then there exists an $r > 0$ such that $|a_n| = exp(-rn)$.

(IV) If $|a_n| \leqslant 2^{-n}$ for all nonnegative integers n, then $|f(x) - p_N(x)| \leqslant 2^{-N}$ for all positive integers N and **all** $x \in [-1,1]$.

(V) For each positive integer N, p_N is the unique polynomial of degree N which interpolates f at the points

$$cos\left(\frac{j+\frac{1}{2}}{N+1}\pi\right), \quad j = 0, 1, 2, \cdots, N$$

Ans: (I) TRUE

(*Not graded - My reason*): for continuously differentiable $f\colon [-1,1] \to \mathbb{C}$, its Chebyshev series converges absolutely and uniformly on the interval $[-1,1]$. In particular, uniform convergence means exactly $\lim\limits_{N\to\infty} \|f - p_N\|_\infty = 0$

(II) TRUE

(*Not graded - My reason*): In class, we learned that given f being k-times continuously differentiable, its Chebyshev coefficients $|a_n| = o(\frac{1}{n^k})$, which in turn implies that, $|a_n| = \mathcal{O}(\frac{1}{n^k})$

Remark 1. In general, given two real-valued functions f, g, it is clear that $f = o(g) \Rightarrow f = \mathcal{O}(g)$ since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 < +\infty$

(III) FALSE

(*Not graded - My reason*): exponential decay of Chebyshev coefficients is a consequence when f is analytic over an ellipse. Given that f is infinitely differentiable (C^∞), it does not imply that f is analytic. Indeed the class of analytic functions is a proper subset of C^∞-function. Here is an example of a real-valued function that is C^∞ but not analytic

$$f(x) = \begin{cases} exp(-\frac{1}{x}), & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

We can show that f is infinitely differentiable but its Taylor's expansion at $x = 0$ converges to zero and hence fails to converge to $f(x)$ for $x > 0$.

See https://en.wikipedia.org/wiki/Non-analytic_smooth_function for more details.

Remark 2. For those of you who have taken complex analysis, the picture is fundamentally different for complex-valued functions. Indeed, *any holomorphic (= complex differentiable) function is analytic and vice versa!*

(IV) TRUE

(*Not graded - My reason*): first, note that $|T_n(x)| < 1$ for all n and all $x \in [-1, 1]$ since $T_n(x) = cos(n \cdot cos^{-1}(x))$ is basically a cosine function.

Second,

$$|f(x) - p_n(x)| = \left|\sum_{k=n+1}^{\infty} a_k T_k(x)\right| \leqslant \sum_{k=n+1}^{\infty} |a_k||T_k(x)| \leqslant \sum_{k=n+1}^{\infty} |a_k| \leqslant \sum_{k=n+1}^{\infty} 2^{-n} = \frac{1}{2^n}$$

Replace n with N, we are done.

(V) TRUE

(*Not graded - My reason*): This is a theorem shown in lecture 15. In addition, the Chebyshev coefficient is computed by the quadrature rule (which is exact in this case) $a_n = \frac{2}{N+1}\sum_{j=0}^{N} f(x_j)T_n(x_j)$, where

$$x_j = cos\left(\frac{j + \frac{1}{2}}{N + 1}\pi\right), \quad j = 0, 1, 2, \cdots, N$$

Problem 2 (Fourier Series)

Compute

$$\int_{-\pi}^{\pi} f(t)\, dt$$

where $f\colon [-\pi, \pi] \to \mathbb{C}$ is the function defined by the Fourier series

$$f(t) = \sum_{n=0}^{\infty} 2^{n+1} e^{int}$$

Ans: As far as I can tell, there is only one way to get to the right answer. (*if you have a better solution, please let me know!*)

It is tempting for some of you to interchange the integral and the infinite sum. But **it is wrong**!

(Correct). Since the given form of $f(t)$ is already a Fourier series,

$$f(t) = 2 + 2^2 e^{it} + 2^3 e^{2it} + \cdots = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{where } a_n = \begin{cases} 0, & n < 0 \\ 2^{n+1}, & n \geqslant 0 \end{cases}$$

The integral $\int_{-\pi}^{\pi} f(t)\, dt$ we want to compute is part of the definition of a_0, therefore

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt \quad \Rightarrow \quad \int_{-\pi}^{\pi} f(t)\, dt = 2\pi a_0 = 2\pi(2) = 4\pi$$

(Incorrect). It is tempting to write down the following answer:

$$\int_{-\pi}^{\pi} f(t)\, dt = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} 2^{n+1} e^{int}\, dt \overset{(*)}{=} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} 2^{n+1} e^{int}\, dt = 2 \int_{-\pi}^{\pi} e^0\, dt + \sum_{n=1}^{\infty} 2^{n+1} \underbrace{\int_{-\pi}^{\pi} e^{int}\, dt}_{=0} = 4\pi$$

But the step $(*)$ is wrong! I hope that you learned in your analysis class that **it is NOT always possible to interchange the integral and the infinite sum.** Here is a counterexample:

Let $\{f_n(x) = cos(nx)\}$. Then consider $\sum_{n=0}^{N} cos(nx)$,

Since $\sum_{n=0}^{N} cos(nx) sin(\frac{x}{2}) = \frac{1}{2} \sum_{n=0}^{\infty} \left[sin(\left(n + \frac{1}{2}\right) x) - sin(\left(n - \frac{1}{2}\right) x) \right]$

$$= \frac{1}{2} \left[\left(sin(\frac{x}{2}) - sin(-\frac{x}{2}) \right) + \left(sin(\frac{3x}{2}) - sin(\frac{x}{2}) \right) + \cdots + \left(sin(\frac{(N+1)}{2} x) - sin(\frac{(N-1)}{2} x) \right) \right]$$

$$= \frac{1}{2} \left[sin(\frac{x}{2}) + sin(\frac{(N+1)}{2} x) \right]$$

So we have

$$\sum_{n=0}^{N} cos(nx) = \frac{sin(\frac{x}{2}) + sin(\frac{(N+1)}{2}x)}{2sin(\frac{x}{2})}$$

Now for $\sum_{n=0}^{\infty} cos(nx)$ is divergent (try to let $N \to \infty$ in the above expression.) Therefore, $\int_{-\pi}^{\pi} \sum_{n=0}^{N} cos(nx)\, dx$ is undefined!

But at the same time, for all integers $n \geqslant 1$, $\int_{-\pi}^{\pi} cos(nx)\, dx = 0$ and $\int_{-\pi}^{\pi} cos(0x)\, dx = 2\pi$. Therefore, $\sum_{n=0}^{N} \int_{-\pi}^{\pi} cos(nx)\, dx = 2\pi$. So we see in this example that generally

$$\sum_{n=0}^{\infty} \int_X f_n(x)\, dx \neq \int_X \sum_{n=0}^{\infty} f_n(x)\, dx$$

Remark 3. Some of you might recall that **Fubini's theorem** justifies the interchange of integrals for an integrable function. As a special case, Fubini's theorem also justifies the interchange of the integral and the infinite sum since infinite sum can be identified as the integration with respect to the counting measure:

Theorem 1. For a sequence of measurable functions $\{f_n\}_{n=0}^{\infty}$ with $f_n \colon X \to \mathbb{R}$ and μ the Lebesgue measure on $\mathbb{R}$, if

$$\text{either} \quad \sum_{n=0}^{\infty} \int_X |f_n|\, d\mu < +\infty \quad \text{or} \quad \int_X \sum_{n=0}^{\infty} |f_n|\, d\mu < +\infty,$$

Then we can interchange the integral and infinite sum over $\{f_n\}_{n=0}^{\infty}$, i.e.

$$\sum_{n=0}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=0}^{\infty} f_n d\mu$$

But this theorem is NOT applicable in the above example, since $f_n(x) = 2^{n+1}e^{int}$,

$$\int_X \sum_{n=0}^{\infty} |f_n|\, d\mu = \int_{-\pi}^{\pi} \underbrace{\sum_{n=0}^{\infty} 2^{n+1} \underbrace{|e^{int}|}_{1}}_{+\infty} dx = +\infty$$

Problem 3 (Condition Number)

Let $\kappa_f(x)$ denote the condition number of evaluation of the function f at the point x. Find a function f which is infinitely differentiable on the interval $(0, 1)$ (but which may have singularities at $x = 0$) such that

$$\lim_{x \to 0^+} \kappa_f(x) = \infty$$

Note: Since $\kappa_f \geqslant 0$ by definition, therefore ∞ here is clearly $+\infty$

Ans: $f(x) = exp(\frac{1}{x})$, then

$$\kappa_f = \left| x \frac{f'(x)}{f(x)} \right| = \left| x \frac{-\frac{1}{x^2} exp(\frac{1}{x})}{exp(\frac{1}{x})} \right| = \frac{1}{x}$$

Therefore, $\lim_{x \to 0^+} \kappa_f(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty$.

(**Idea - Not graded:**) First recall the formula for the condition number of a function $f(x)$:

$$\kappa_f = \left| x \frac{f'(x)}{f(x)} \right|$$

If we look at this formula carefully, any $f(x)$ as some power of x will NOT give $\lim_{x \to 0^+} \kappa_f = \infty$, since assume $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R}$:

$$\kappa_f = \left| x\alpha \frac{x^{\alpha-1}}{\alpha} \right| = |\alpha| < +\infty$$

This simply wiil not work.

How about we draw inspiration from example $f(x) = e^x$ presented in class:

$$\kappa_f = \left| x \frac{e^x}{e^x} \right| = |x|$$

That is actually insightful since we learn that **exponential growth will lead to blow-up of the condition number**, i.e.

$$\lim_{x \to +\infty} \kappa_f(x) = +\infty$$

Can we produce exponential growth around $x = 0$**? Yes!** Simplest answer: $f(x) = exp(\frac{1}{x})$.

By the same token, we can produce infinitely many functions as such, i.e. $exp(\frac{1}{x^2}), exp(\frac{1}{x^3}), \cdots$.

Problem 4 (Error of interpolant)

Suppose that $f\colon [0,1] \to \mathbb{R}$ is twice differentiable function such that $|f''(x)| \leqslant 1$ for all $0 \leqslant x \leqslant 1$.

Let p be the unique polynomial of degree 1 which interpolates f at the points 0 and 1.

Show that

$$|p(x) - f(x)| \leqslant \frac{1}{8}$$

for all $x \in [0,1]$).

Ans: we apply the interpolation error formula in this case, i.e. $x_0 = 0,\ x_1 = 1$

$$f(x) - p(x) = \frac{f''(\xi_x)}{2!}(x - x_0)(x - x_1) = \frac{f''(\xi_x)}{2!}x(x-1)$$

where $\xi_x \in (0,1)$.

In order to find an upper bound for $|f(x) - p(x)|$, we look at the function $g(x) := x(x-1)$. Using calculus,

$$g'(x) = 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

$$g''(x) = 2 \quad \Rightarrow \quad g(\frac{1}{2}) > 0$$

So $g(x)$ has an absolute maxium at $x = \frac{1}{2}$ and $g(\frac{1}{2}) = \frac{1}{4}$.

Using $|f''(x)| \leqslant 1$, we have

$$|f(x) - p(x)| \leqslant \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

for all $x \in [0,1]$.

Problem 5 (Chebyshev coefficients)

Suppose that $f\colon [-1,1] \to \mathbb{R}$ is an infinitely differentiable function, and that $\{a_n\}$ is the sequence of Chebyshev coefficients of f — that is, $\{a_n\}$ defined via the formula

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}$$

Suppose also that N is a positive integer, and p is the polynomial, $p = \sum_{n=0}^{N}{}' a_n T_n(x)$ Show that if q is any polynomial of degree N, then

$$|a_{N+1}| \leqslant \frac{4}{\pi}|f(x) - q(x)|$$

for all $x \in [-1,1]$.

Hint: You might want to use the fact that $\int_{-1}^{1} |T_{n+1}(x)| \frac{dx}{\sqrt{1-x^2}} = 2$ *for all* $n \geqslant 1$ *without proving it.*

Ans: **The desired statement as above is NOT correct.** Choosing $f(x) = T_0(x) + T_1(x) = 1 + x$ which is infinitely differentiable. Now choose $q(x)$ to be a polynomial of degree 0, namely a constant, let say $q(x) \equiv 1$. Then $L.H.S = |a_1| = 1$ but $R.H.S. = \frac{4}{\pi}(1 + x - 1) = \frac{4}{\pi}x$. Taking $x = 0$, we have $L.H.S < R.H.S.$.

The desired statement should be

$$|a_{N+1}| \leqslant \frac{4}{\pi}\sup_{-1\leqslant x\leqslant 1}|f(x) - q(x)|$$

where the $N+1$-coefficients provides a lower bound for the "distance" between f and any polynomials of degree N. The proof's idea is as follows:

$$\begin{aligned}
\frac{\pi}{2}|a_{N+1}| &= \left|\int_{-1}^{1} f(x)T_{N+1}(x)\frac{dx}{\sqrt{1-x^2}}\right| \leqslant \left|\int_{-1}^{1} \left[f(x) - q(x) + q(x)\right] T_{N+1}(x)\frac{dx}{\sqrt{1-x^2}}\right| \\
&\leqslant \left|\int_{-1}^{1} (f(x) - q(x))T_{N+1}(x)\frac{dx}{\sqrt{1-x^2}}\right| + \left|\int_{-1}^{1} q(x)T_{N+1}(x)\frac{dx}{\sqrt{1-x^2}}\right| \\
&\leqslant \int_{-1}^{1} |f(x) - q(x)||T_{N+1}(x)|\frac{dx}{\sqrt{1-x^2}} + \underbrace{\langle q(x), T_{N+1}(x)\rangle_{w(x)}}_{=0}
\end{aligned}$$

The reason for $\int_{-1}^{1} q(x)T_{N+1}(x)\frac{dx}{\sqrt{1-x^2}} = 0$ is that since $q(x)$ is a polynomial of degree N, $q(x)$ can be expressed as a sum using $T_0(x), T_1(x), T_2(x), \cdots, T_N(x)$, i.e.

$$q(x) = \sum_{n=0}^{N}{}' a_n T_n(x) = \frac{1}{2}a_0T_0(x) + a_1T_1(x) + \cdots + a_NT_N(x)$$

And $T_{N+1}(x)$ is "orthogonal" to all $T_0(x), T_1(x), \cdots T_N(x)$ with respect to the weighted inner product, i.e.

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0, \qquad n \neq m$$

Therefore,

$$\int_{-1}^{1} q(x) T_{N+1}(x) \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} {\sum_{n=0}^{N}}' a_n T_n(x) T_{N+1}(x) \frac{dx}{\sqrt{1-x^2}} = {\sum_{n=0}^{N}}' \int_{-1}^{1} a_n T_n(x) \frac{dx}{\sqrt{1-x^2}} = 0$$

Now going on with the upper bound for $|a_{N+1}|$, we have

$$\begin{aligned}
\frac{\pi}{2}|a_{N+1}| &\leqslant \sup_{-1\leqslant x\leqslant 1} |f(x)-q(x)| \int_{-1}^{1} |T_{N+1}(x)| \frac{dx}{\sqrt{1-x^2}} \\
&\leqslant 2 \sup_{-1\leqslant x\leqslant 1} |f(x)-q(x)| \\
\Rightarrow &\qquad \boxed{|a_{N+1}| \leqslant \frac{4}{\pi} \sup_{-1\leqslant x\leqslant 1} |f(x)-q(x)|}
\end{aligned}$$