MAT 128A - Assignment 5

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October 25, 2018

Problem 1

Let $f(x)$ be a polynomial of x of degree N, and

$$f(x) = \sum_{n=0}^{N} a_n T_n(x). \tag{1}$$

The Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

We define a finite sequence of polynomials $\{b_0(x), b_1(x), \cdots, b_N(x), b_{N+1}(x), b_{N+2}(x)\}$ via the formulas:

$$b_{N+1} = b_{N+2} = 0$$
$$b_n(x) = a_n + 2xb_{n+1}(x) - b_{n+2}(x)$$

Show that $f(x) = b_0(x)$.

Hint: first show that if q_{N-1} is defined by

$$q_{N-1}(y) := \sum_{n=0}^{N-1} 2b_{n+1}(x)T_n(y)$$

then

$$(y - x)q_{N-1}(y) + b_0(x) = f(y) \tag{2}$$

and then let $y = x$ in above

Ans: This question looks tricky at first glance but direct substitution of all given formulas will lead us to the desired statement. First, substitute (2):

$$
\begin{aligned}
L.H.S. &= (y-x)\left(\sum_{n=0}^{N-1} 2b_{n+1}(x)T_n(y)\right) + b_0(x) \\
&= \sum_{n=0}^{N-1} b_{n+1}(x) \underbrace{2yT_n(y)}_{T_{n+1}(y)+T_{n-1}(y)} - \sum_{n=0}^{N-1} \underbrace{2xb_{n+1}(x)}_{b_n(x)-a_n+b_{n+2}(x)} T_n(y) + b_0(x) \\
&= \sum_{n=0}^{N-1} b_{n+1}(x)\left(T_{n+1}(y)+T_{n-1}(y)\right) - \sum_{n=0}^{N-1}\left(b_n(x)+b_{n+2}(x)\right)T_n(y) + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= \underbrace{\left(\sum_{n=0}^{N-1} b_{n+1}(x)T_{n+1}(y) - \sum_{n=0}^{N-1} b_n(x)T_n(y)\right)}_{b_N(x)T_N(y)-b_0(x)T_0(y)} + \underbrace{\left(\sum_{n=0}^{N-1} b_{n+1}(x)T_{n-1}(y) - \sum_{n=0}^{N-1} b_{n+2}(x)T_n(y)\right)}_{b_1(x)T_{-1}(y)-b_{N+1}(x)T_{N-1}(y)} \\
&\quad + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= \underbrace{b_N(x)}_{=a_N} T_N(y) - b_0(x)\underbrace{T_0(y)}_{\equiv 1} + b_1(x)\underbrace{T_{-1}(y)}_{=0} - \underbrace{b_{N+1}(x)}_{=0} T_{N-1}(y) + \sum_{n=0}^{N-1} a_nT_n(y) + b_0(x) \\
&= a_NT_N(y) + \sum_{n=0}^{N-1} a_nT_n(y) = f(y) \qquad = R.H.S.
\end{aligned}
$$

It is obvious that by taking $y = x$, we obtain $b_0(x) = f(x)$.

Remark 1. As said in class, the formulation above is an application of the **Clenshaw's recurrence formula**. It can be applied to any classes of function that are defined by a three-term recurrence relation.

Problem 2

The last problem suggests a method for computing the sum (1). How many arithmetic operations does it take to compute $f(x) = b_0(x)$ using this method?

Suppose that instead we sum (1) most directly by first using the recurrence relations $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ to compute the values of $T_0(x), T_1(x), \cdots, T_N(x)$. We then form the values

$$a_0T_0(x), a_1T_1(x), \cdots, a_NT_N(x)$$

and then sum them to form $f(x)$. How many arithmetic operations does this more direct procedure take?

Ans: The problem setting is that given the (N+1) Chebyshev coefficients $\{a_n\}_{n=0}^N$ for a polynomial f of degree N, we want to evalute $f(x)$ for a specified value of x.

First, we count the arithmetic operations by using the algorithm in problem (1), i.e.

- [1st - step:] $\quad b_N(x) = a_N + 2xb_{N+1}(x) - b_{N+2}(x)$
- [2nd - step:] $\quad b_{N-1}(x) = a_{N-1} + 2xb_N(x) - b_{N+1}(x)$
- $\vdots$
- [Nth - step] $\quad b_1(x) = a_1 + 2xb_2(x) - b_3(x)$
- [(N+1)th - step:] $\quad b_0(x) = a_0 + 2xb_1(x) - b_2(x)$

In each step, we have 2 multiplications and 2 additions (subtraction counted as addition), so there are 4 arithmetic operations. In total, we have $4(N+1)$ arithemetic operations after $N+1$ steps.

More precisely, since $b_{N+1} = b_{N+2} = 0$, the first two steps can be simplified to $b_N(x) = a_N$ and $b_{N-1}(x) = a_{N-1} + 2xb_N(x)$. Indeed we only have $4(N-1) + 3 = 4N - 1$ step.

Second, we use the recurrence relation to evaluate the Chebyshev polynomials $\{T_n\}_{n=0}^N$ at x and then compute the sum $\sum a_n T_n(x)$. Given that $T_0(x) = 1$ and $T_1(x) = x$

- [1st - step:] $\quad T_2 = 2xT_1 - T_0$
- [2nd - step:] $\quad T_3 = 2xT_2 - T_1$
- $\vdots$
- [(N-1)th - step] $\quad T_N = 2xT_{N-1} - T_{N-2}$
- [(N+1)th - step:] $\quad f(x) = a_0T_0(x) + a_1T_1(x) + \cdots + a_NT_N(x)$

For all the first (N-1) steps, each steps take 1 addition and 2 multiplication, so there are 3 arithmetic operations. The last step takes N addition and $(N+1)$ multiplication, so there are $(2N+1)$ arithemtic operations. In total, we have $3(N-1) + 2N + 1 = 5N - 2$ operations.

Therefore, **the derivation in problem (1) further accelerate the evaluation of Chebyshev expansion!**

Problem 3

Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$f(x) = \sum_{n=0}^{N}{}' a_n T_n(x)$$

of the functions $f(x) = \sqrt{1-x^2}$ for $-1 \leqslant x \leqslant 1$.

Note that **the dash summation notation indicates that the first term in the series is halved**.

Recall that the coefficients are defined via the formula

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}$$

So computing a_n is equivalent to determine the value of the quantity $a_n = \frac{2}{\pi}\int_{-1}^{1} T_n(x)\,dx$

Ans: First, recall the definition of $T_n(x) = cos(n{\cdot}cos^{-1}(x))$, so $a_n = \frac{2}{\pi}\int_{-1}^{1} cos(n{\cdot}cos^{-1}(x))\,dx$. Second, using the change of variable $x = cost$ ($\Rightarrow\ dx = -sint\,dt$), we have

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_{\pi}^{0} cos(n\cdot t) - sint\,dt \quad \text{note that } 0 < t < \pi \\
&= \frac{2}{\pi}\int_{0}^{\pi} cos(n\cdot t)sint\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} sin(n+1)t - sin(n-1)t\,dt \\
&\overset{(*)}{=} \frac{1}{\pi}\left[-\frac{cos(n+1)t}{n+1} + \frac{cos(n-1)t}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}
\end{aligned}$$

$(*)$ holds due to the sum of angle formula $cos\alpha sin\beta = \frac{1}{2}\left(sin(\alpha+\beta) - sin(\alpha-\beta)\right)$.

Note that for n is odd, $a_n = \frac{1}{\pi}\left\{\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = 0$.

For $n \neq 0$ is even, $a_n = \frac{1}{\pi}\left\{\left[\frac{1}{n+1} - \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = -\frac{4}{(n^2-1)\pi}$.

For $n = 0$, a_0 **is halved**, so $a_0 = -\frac{2}{(0^2-1)\pi} = \frac{2}{\pi}$.

Writing even number $n = 2k$, for $k = 1, 2, 3, \cdots$, the Chebyshev expansion of $f(x) = \sqrt{1-x^2}$ is

$$\sqrt{1-x^2} = \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{(4k^2-1)\pi}T_{2k}(x) = \frac{2}{\pi} - \frac{4}{3\pi}(2x^2-1) - \frac{4}{15\pi}(8x^4-8x^2+1) - \cdots$$

Bonus I: Indeed, we can know that all odd coefficients $a_{2k+1} = 0$ without going through the exact calculation. In order to see that, we need the following proposition

Proposition 1. $T_n(-x) = (-1)^n T_n(x)$

Proof. Notice that $cos^{-1}(x) = \theta \;\Leftrightarrow x = cos\theta$.
So since $cos(\pi - \theta) = -x$, we have $cos^{-1}(-x) = \pi - cos^{-1}(x)$.

$$\begin{aligned} T_n(-x) &= cos(n \cdot cos^{-1}(-x)) = cos(n \cdot (\pi - cos^{-1}(x))) = cos(n\pi - ncos^{-1}(x)) \\ &= \underbrace{cos(n\pi)}_{(-1)^n} cos(ncos^{-1}(x))) + \underbrace{sin(n\pi)}_{=0} sin(ncos^{-1}(x)) = (-1)^n T_n(x) \end{aligned}$$

The above proposition leads to

Corollary 2. $T_n(x)$ is an even polynomial for even n. $T_n(x)$ is an odd polynomial for odd n.

Proof. $T_{2k+1}(-x) = -T_{2k+1}(x), \qquad T_{2k}(-x) = T_{2k}(x). \qquad k = 0, 1, 2, \cdots$

Now look at back the Chebyshev coefficient of $f(x) = \sqrt{1-x^2}$, $a_n = \frac{2}{\pi}\int_{-1}^{1} T_n(x)\,dx$.
For n being odd, the integral is clearly zero from -1 to 1. For n being even, $a_n = \frac{4}{\pi}\int_0^1 T_n(x)\,dx$.

Bonus II: Since $f(x) = \sqrt{1-x^2}$ is an even function, its Chebyshev expansion consist of even polynomials $T_{2k}(x)$ ONLY.
$f(x)$ is not continuously differentiable since its derivative $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ has singularities at $x = \pm 1$. As said in class, its Chebyshev coefficients decay at rate $a_n = \mathcal{O}\left(\frac{1}{n^2}\right)$.

Bonus III: For those of you who want to do numerical experiments on `MATLAB`, I recommend you to download the package `Chebfun` at http://www.chebfun.org/
I use it in `MATLAB` to compute the first 10 Chebyshev's coefficients:

```
x = chebfun('x');
f = sqrt(1-x^2);
p = chebfun(f,'trunc',11);
a = chebcoeffs(p)


a =


    0.636619772367623
    0.000000000000000
```

-0.424413181578414

0.000000000000000

-0.084882636315682

0.000000000000000

-0.036378272706720

0.000000000000000

-0.020210151503733

0.000000000000000

-0.012861005502375

All the odd coefficients vanish as predicted in our calculation!

Problem 4

Find the coefficients $\{a_n\}$ in the Chebyshev expansion

$$g(x) = \sum_{n=0}^{N}{}' a_n T_n(x)$$

of the function

$$g(x) = sgn(x) = \begin{cases} 1, & 1 \geqslant x > 0 \\ 0, & x = 0 \\ -1, & -1 \leqslant x < 0 \end{cases}$$

Note that I restrict the domain of the function to $[-1, 1]$.

Ans: Similar to the previous problem, we use the change of variable $x = cost$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{-1}^{1} sgn(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_{0}^{1} T_n(x)\frac{dx}{\sqrt{1-x^2}} - \frac{2}{\pi}\int_{-1}^{0} T_n(x)\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2}{\pi}\left\{\int_{\frac{\pi}{2}}^{0} -cos(nt)\,dt - \int_{\pi}^{\frac{\pi}{2}} -cos(nt)\,dt\right\} \\ &= \frac{2}{\pi}\left\{\int_{0}^{\frac{\pi}{2}} cos(nt)\,dt - \int_{\frac{\pi}{2}}^{\pi} cos(nt)\,dt\right\} \\ &= \frac{2}{\pi}\left\{\left[\frac{sin(nt)}{n}\right]_0^{\frac{\pi}{2}} - \left[\frac{sin(nt)}{n}\right]_{\frac{\pi}{2}}^{\pi}\right\} = \frac{4}{n\pi}sin(n\frac{\pi}{2}) \\ &= \begin{cases} 0, & n = 2k \text{ even} \\ \frac{4}{(2k+1)\pi}\underbrace{sin((2k+1)\frac{\pi}{2})}_{(-1)^k}, & n = 2k+1 \text{ odd} \end{cases} \end{aligned}$$

for $k = 0, 1, 2, \cdots$

To conclude, we have

$$g(x) = sgn(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{T_{2k+1}(x)}{2k+1}$$

Bonus I: In contrast to the previous function $f(x) = \sqrt{1-x^2}$, $g(x) = sgn(x)$ is not even continuous on $[-1, 1]$, let alone being differentiable. So it is not surprising that the decay rate of coefficient for $g(x)$ is $a_N = \mathcal{O}\left(\frac{1}{N}\right)$, decaying more slowly to zero than that of $f(x)$.

Bonus II: I use the package `chebfun` in `MATLAB` to compute the first 10 Chebyshev's coefficients.

```
>> x = chebfun('x');
g = sign(x);
```

```
p = chebfun(g,'trunc',11);
a = chebcoeffs(p)


a =


  -0.000000000000000
   1.273239544735162
  -0.000000000000000
  -0.424413181578387
  -0.000000000000000
   0.254647908947033
  -0.000000000000000
  -0.181891363533594
  -0.000000000000000
   0.141471060526130
  -0.000000000000000
```

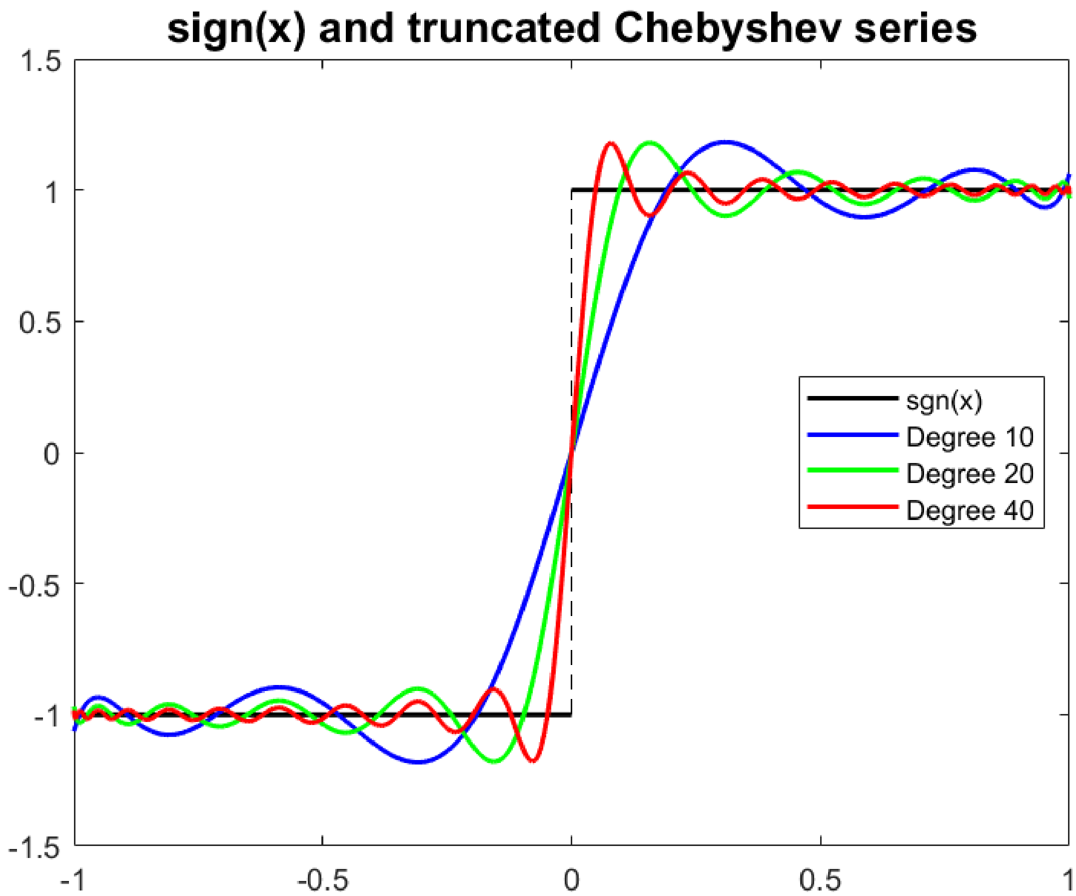
All the even coefficients vanish as predicted in our calculation!

Next, I plot the Chebyshev approximation using the first 10 terms (blue), 20 terms (green), and 40 terms (red). The function $g(x) = sgn(x)$ is plotted in black.

```
p0 = chebfun(g,'trunc',11);
p1 = chebfun(g,'trunc',21);
p2 = chebfun(g,'trunc',41);


FS = 'fontsize'; LW = 'LineWidth'; JL = 'jumpline';
figure
plot(g,'k',LW,1.5,JL,'--'), ylim([-1.5 1.5])
title('sign(x)',FS,14)
hold on
plot(p0,'b',LW,1.5)
plot(p1,'g',LW,1.5)
```

```
plot(p2,'r',LW,1.5)
title('sign(x) and truncated Chebyshev series',FS,14)
lgd = legend('sgn(x)','Degree 10', 'Degree 20', 'Degree 40');
lgd.Location = 'east';
```



Bonus III: Notice that using Chebyshev polynomials to approximate a function with interior discontinuity still leads to the "Gibbs phenomenon" in Fourier series approximation (See my solution for Problem 5 in Homework 2 - Bonus I). In this case $g(x) = sgn(x)$, we observe the Gibbs phenomenon at $x = 0$. It is not surprising since Chebyshev series can be obtained from the Fourier cosine series by the change of variable $x = cost$.