MAT 128A - Assignment 6 (Revised Solution)

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Problem 1

Go over the midterm problems and the provided solutions!

You should really do it for your own good! ☺

Problem 2

Show that for all nonnegative integers n, $T_n(1) = 1$ and $T_n(-1) = (-1)^n$.

Ans: The desired statement follows easily by using the definition of the Chebyshev polynomials, i.e. $T_n(x) = cos(ncos^{-1}x)$. Since

$$cos^{-1}(1) = 2k\pi \quad \text{for } k \in \mathbb{Z}, \qquad cos^{-1}(-1) = (2k+1)\pi \quad \text{for } k \in \mathbb{Z}$$

$$\Rightarrow \quad cos(n \cdot cos^{-1}(1)) = cos(2nk \cdot \pi) = 1, \qquad cos^{-1}((-1)) = cos(n(2k+1) \cdot \pi) = \begin{cases} 1, & \text{if } n \text{ even} \\ -1, & \text{if } n \text{ odd} \end{cases}$$

$$\Rightarrow T_n(1) = 1, \qquad T_n(-1) = (-1)^n$$

Remark 1. The symbol $k \in \mathbb{Z}$ means that k is an integer (can be positive or negative).

Problem 3

Show that for all integers $n \geq 2$ and all $-1 < t \leq -1$,

$$\int_{-1}^{t} T_n(x)\,dx = \frac{1}{2}\left(\frac{T_{n+1}(t)}{n+1} - \frac{T_{n-1}(t)}{n-1}\right) - \frac{(-1)^n}{n^2-1}$$

Ans: **This problem is like a generalization of the problem 4 (first part) in our midterm exam!** Can you tell how this problem is related to the midterm problem?

Again, we start with the definition and use the change of variable $x = cosu$, $\Rightarrow$ $dx = -sinu\,du$, also the interval of integration $(-1,t) \mapsto (\pi, cos^{-1}t)$ we have

$$\int_{-1}^{t} T_n(x)\,dx = \int_{\pi}^{cos^{-1}t} cos(n\cdot cos^{-1}x)\,dx = \int_{\pi}^{cos^{-1}t} cos(n\cdot u)(-sinu)\,du = \int_{cos^{-1}t}^{\pi} cos(n\cdot u)sinu\,du$$

Recall the trigonometric identity

$$cos(\alpha u)sin(\beta u) = \frac{1}{2}sin((\alpha+\beta)u) - sin((\alpha-\beta)u),$$

applying this

$$\int_{-1}^{t} T_n(x)\,dx = \frac{1}{2}\int_{cos^{-1}t}^{\pi} sin((n+1)u) - sin((n-1)u)\,du = \frac{1}{2}\left[-\frac{cos((n+1)u)}{n+1} + \frac{cos((n-1)u)}{n-1}\right]_{cos^{-1}t}^{\pi}$$

$$= \frac{1}{2}\left\{\left[-\frac{cos((n+1)\pi)}{n+1} + \frac{cos((n-1)\pi)}{n-1}\right] - \left[-\frac{cos((n+1)cos^{-1}t)}{n+1} + \frac{cos((n-1)cos^{-1}t)}{n-1}\right]\right\}$$

$$= \frac{1}{2}\left\{\underbrace{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]}_{\frac{2\cdot(-1)^{n+1}}{n^2-1}} - \left[-\frac{T_{n+1}(t)}{n+1} + \frac{T_{n-1}(t)}{n-1}\right]\right\}$$

$$= -\frac{(-1)^n}{n^2-1} + \frac{1}{2}\left(-\frac{T_{n+1}(t)}{n+1} + \frac{T_{n-1}(t)}{n-1}\right)$$

Therefore, we have the desired statement.

Remark 2. The above formula clearly does not hold for $n = 0$ or $n = 1$. For $n = 0$, the term $T_{-1}(x)$ is not well-defined. For $n = 1$, the denominator $n-1$ blows up to $+\infty$.

Remark 3. For $t = -1$, the formula is still valid. But both sides of the equation will be equal to zero.

Problem 4

Let $x_0, x_1, \cdots x_N, w_0, w_1, \cdots, w_N$ denote the nodes and weights of the $(N+1)$-point Gauss-Legendre quadrature rule. Suppose that $f\colon[-1,1] \to \mathbb{R}$ is continuously differentiable, and that $c_0, c_1, \cdots c_N$ are defined by the formula

$$c_n = \frac{2n+1}{2}\sum_{j=0}^{N} f(x_j)P_n(x_j)w_j$$

Show that the polynomial

$$p_N(x) = \sum_{n=0}^{N} c_n P_n(x)$$

interpolates f at the points $x_0, x_1, \cdots x_N$.

This problem is tricky. Indeed I spent an entire afternoon to figure it out but it is a very cool problem!

If you have a better solution, please share it with me!

A direct substitution of $c_n = \frac{2n+1}{2}\sum_{j=0}^{N} f(x_j)P_n(x_j)w_j$ into $p_N(x)$ is not very helpful for me. So instead we look more carefully at $p_N(x)$, it is important to notice that for an arbitrary integer m where $0 \le m \le N$,

$$\sum_{i=0}^{N} p_N(x_i)P_m(x_i)w_i = \sum_{i=0}^{N}\left(\sum_{n=0}^{N} c_n P_n(x_i)\right)P_m(x_i)w_i = \sum_{n=0}^{N} c_n\left(\sum_{i=0}^{N} P_n(x_i)P_m(x_i)w_i\right)$$

Now notice that $P_n(x)$ and $P_m(x)$ are both polynomials of degree less than or equal to N, the product polynomial P_nP_m is polynomial of degree less than or equal to $2N$. This implies that the sum inside the bracket in the last step is the same as the integral,

$$\sum_{i=0}^{N} P_n(x_i)P_m(x_i)w_i = \int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} \frac{2}{2m+1} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

since the Gauss-Legendre quadrature formula is exact for any polynomials of degree less than or equal to $(2N+1)$. This implies that

$$\sum_{i=0}^{N} p_N(x_i)P_m(x_i)w_i = \sum_{n=0}^{N} c_n\left(\sum_{i=0}^{N} P_n(x_i)P_m(x_i)w_i\right) = \frac{2}{2m+1}c_m$$

Now we if substitute the formula for c_m, i.e.

$$\sum_{i=0}^{N} p_N(x_i)P_m(x_i)w_i = \frac{2}{2m+1}c_m = \frac{2}{2m+1}\cdot\frac{2m+1}{2}\sum_{j=0}^{N} f(x_j)P_m(x_j)w_j$$

Comparing both sides gives us $p_N(x_i) = f(x_i)$ for all $1 \le i \le N$.

Illustrative example: This problem might seem abstract to some of you. Let us do a simple example for $N = 2$ and let $f\colon[-1,1] \to \mathbb{R}$ be any "smooth" function.

The normalized Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$. So it has root $x_0 = -\frac{1}{\sqrt{3}}$, $x_1 = \frac{1}{\sqrt{3}}$. We use the formula $w_j = \frac{2}{[1-(x_j)^2](P_2'(x_j))^2}$ with $P_2'(x) = 3x$

$$w_0 = \frac{2}{\left(1 - \frac{1}{3}\right)3} = 1, \qquad w_1 = \frac{2}{\left(1 - \frac{1}{3}\right)3} = 1$$

So using the given formula for c_n and $P_0(x) \equiv 1, P_1(x) = x$

$$\begin{aligned}
c_0 &= \frac{1}{2}\left(f(x_0)P_0(x_0)w_0 + f(x_1)P_0(x_1)w_1\right) = \frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}}) \\
c_1 &= \frac{3}{2}\left(f(x_0)P_1(x_0)w_0 + f(x_1)P_1(x_1)w_1\right) = \frac{3}{2}(-\frac{1}{\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})\frac{1}{\sqrt{3}}) \\
\Rightarrow \quad p_2(x) &= \left(\frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}})\right)\cdot 1 + \left(-\frac{3}{2\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + \frac{3}{2\sqrt{3}}f(\frac{1}{\sqrt{3}})\right)\cdot x
\end{aligned}$$

Notice that $p_2(x)$ is a polynomial of degree 1. According to the result we obtained from the previous problem, $p_2(x)$ should interpolate $f(x)$ at two points $x = \pm\frac{1}{\sqrt{3}}$, checking:

$$p_2(x_0) = p_2(-\frac{1}{\sqrt{3}}) = \left(\frac{1}{2}f(-\frac{1}{\sqrt{3}}) + \frac{1}{2}f(\frac{1}{\sqrt{3}})\right) + \left(-\frac{3}{2\sqrt{3}}f(-\frac{1}{\sqrt{3}}) + \frac{3}{2\sqrt{3}}f(\frac{1}{\sqrt{3}})\right)\cdot\left(-\frac{1}{\sqrt{3}}\right) = f(-\frac{1}{\sqrt{3}})$$

Similarly, we can verify that $p_2(x_1) = p_2(\frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}})$