# MAT128A SECTION 2
# PRACTICE MIDTERM
# November 5, 2018

1. **WRITE YOUR ANSWERS ON THIS EXAM**.
2. **PUT YOUR NAME ON THE COVER OF THIS EXAM NOW**.
3. If you need it, we will be happy to provide scratch paper for your use, but only answers recorded on the exam paper will be graded.
4. You are allowed the use of a page of notes for this exam. No other notes or books are allowed.
5. In addition, no calculators, cellphones or other electronic devices may be used during this exam.
6. If you have a cell phone with you, put it away now. The time remaining to finish the exam will be periodically announced.
7. The problems are equally weighted.
8. The exam will be "curved" and information about the distribution of grades will be provided to you as soon as the exams are graded.
9. The exams should be graded and returned to you by Friday, November 9, 2018.

**Good luck!**

**Problem 1.** Suppose that $f : [-1,1] \to \mathbb{R}$ is a continuous function, and that $\{a_n\}$ are its Chebyshev coefficients. That is, the sequence $\{a_n\}$ is defined by the formula

$$a_n = \frac{2}{\pi} \int_{-1}^{1} f(x) T_n(x) \frac{dx}{\sqrt{1-x^2}}.$$

Moreover, suppose that for each positive integer $N$, $p_N$ denotes the polynomial

$$p_N(x) = {\sum_{n=0}^{N}}{}' a_n T_n(x).$$

Indicate whether each of the following statements is true or false. You do not need to justify your answers.

1. If $f$ is continously differentiable, then $\|p_N - f\|_\infty \to 0$ as $N \to \infty$.
2. If $f$ is $k$-times continuously differentiable, then
$$|a_n| = \mathcal{O}\left(\frac{1}{n^k}\right).$$
3. If $f$ is infinitely differentiable, then there exists an $r > 0$ such that $|a_n| = \exp(-rn)$.
4. If $|a_n| \leqslant 2^{-n}$ for all nonnegative integers $n$, then
$$|f(x) - P_N(x)| \leqslant 2^{-N}$$
for all positive integers $N$.
5. For each positive integer $N$, $p_N$ is the unique polynomial of degree $N$ which interpolates $f$ at the points
$$\cos\left(\frac{j+\frac{1}{2}}{N+1}\pi\right), \quad j = 0, 1, \ldots, N.$$

**Problem 2.** Compute

$$\int_{-\pi}^{\pi} f(t)\ dt,$$

where $f : [-\pi, \pi] \to \mathbb{C}$ is the function defined by the Fourier series

$$f(t) = \sum_{n=0}^{\infty} 2^{n+1} \exp(int).$$

**Problem 3.** Let $\kappa_f(x)$ denote the condition number of evaluation of the function $f$ at the point $x$. Find a function $f$ which is infinitely differentiable on the interval $(0, 1)$ (but which may have singularities at $x = 0$) such that

$$\lim_{x \to 0^+} \kappa_f(x) = \infty.$$

**Problem 4.** Suppose that $f : [0, 1] \to \mathbb{R}$ is a twice differentiable function such that

$$|f''(x)| \leqslant 1 \quad \text{for all} \quad 0 \leqslant x \leqslant 1.$$

Let $p$ be the unique polynomial of degree 1 which interpolates $f$ at the points 0 and 1. Show that

$$|p(x) - f(x)| \leqslant \frac{1}{8}$$

for all $x \in [0, 1]$.

**Problem 5.** Suppose that $f : [-1, 1] \to \mathbb{R}$ is an infinitely differentiable function, and that $\{a_n\}$ is the sequence of Chebyshev coefficients of $f$ — that is, $\{a_n\}$ is defined via the formula

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) T_n(x) \, \frac{dx}{\sqrt{1-x^2}}.$$

Suppose also that $N$ is a positive integer, and that $p$ is the polynomial

$$p(x) = \sum_{n=0}^{N}{}' a_n T_n(x).$$

Show that if $q$ is any polynomial of degree $N$ then

$$|a_{N+1}| \leqslant \frac{4}{\pi} \sup_{-1 \leqslant x \leqslant 1} |f(x) - q(x)|$$

for all $x \in [-1, 1]$.

Hint: you may use the fact that

$$\int_{-1}^{1} |T_{n+1}(x)| \, \frac{dx}{\sqrt{1-x^2}} = 2.$$

for all $n \geqslant 0$ without proving it.