MAT 128A - Assignment 2 (Revised solutions)

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Problem 1

What is the condition number of evaluation of the function

$$f(x) = e^{cos(x)}$$

at the point x?

Ans: Both exponential and cosine functions are smooth (*infinitely differentiable*), their composition is also smooth. So using the formula for condition number

$$\kappa_f(x) = \left| x \frac{f'(x)}{f(x)} \right|,$$

with $f'(x) = -sin(x)e^{cos(x)}$, we have $\kappa_f(x) = |xsin(x)|$.

This computation is straightforward but the implication is more important. Con sider $x_1 = \frac{\pi}{2}$ and $x_2 = 100\pi + \frac{\pi}{2}$. Clearly, both $f(x_1) = 0$ and $f(x_2) = 0$. But the condition number $\kappa_f(x_2) = 100\pi + \frac{\pi}{2}$ are much larger than that of $\kappa_f(x_1) = \frac{\pi}{2}$. Why?

Again, as said in class, it is due the distribution of the double precision numbers! Since the double precision numbers are densest around 0, therefore the condition number of $\kappa_f(x_1)$ is much closer to 1.

Problem 2

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x. Find an expression for the condition number of evaluation of the function $h(x) = f(g(x))$ at x in terms of $\kappa_f(g(x))$ and $g'(x)$.

Ans: Applying the formula for condition number of a function and the chain rule in differentiation, we have

$$\kappa_h(x) = \left|x\frac{h'(x)}{h(x)}\right| = \left|x\frac{f'(g(x))g'(x)}{f(g(x))}\right| = \kappa_f(g(x)) \cdot |g'(x)|$$

Problem 3

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x, and let $\kappa_g(x)$ denote the condition number of evaluation of the function g at x. Find an expression for the condition number of evaluation of the function $h(x) = f(x) \cdot g(x)$ at x in terms of $\kappa_f(x)$ and $\kappa_g(x)$.

Ans: Similar to the previous question, applying the formula for condition number of a function and the product rule in differentiation, we have

$$\kappa_h(x) = \left|x\frac{f'(x)g(x) + f(x)g'(x)}{f(x) \cdot g(x)}\right| = \left|x\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right)\right| \leqslant \left|x\frac{f'(x)}{f(x)}\right| + \left|x\frac{g'(x)}{g(x)}\right| = \kappa_f(x) + \kappa_g(x)$$

The upper bound above for $\kappa_h(x)$ shows that the condition number of a product function can be controlled by its component.

Naturally, the above argument can be generalized to $h(x) := f_1(x)f_2(x)\cdots f_n(x)$, i.e.

$$\kappa_h(x) \leqslant \sum_{i=1}^{n} \kappa_{f_i}(x)$$

Problem 4

Suppose that f and g are continuously differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $\kappa_f(x)$ denote the condition number of evaluation of the function at x, and let $\kappa_g(x)$ denote the condition number of evaluation of the function g at x. Find an expression for the condition number of evaluation of the function $h(x) = f(x)/g(x)$ at x in terms of $\kappa_f(x)$ and $\kappa_g(x)$.

Ans: Similar to the previous question, applying the formula for condition number of a function and the quotient rule in differentiation, we have

$$\kappa_h(x) = \left|x\frac{h'(x)}{h(x)}\right| = |x|\left|\frac{\frac{g\cdot f' - g'\cdot f}{g^2}}{\frac{f}{g}}\right| = |x|\frac{g\cdot f' - g'\cdot f}{fg} = |x\frac{f'}{f} - x\frac{g'}{g}| \overset{(*)}{\geqslant} \left|\kappa_f(x) - \kappa_g(x)\right|$$

The $(*)$ above refers to the reverse triangle inequality $|a-b| \geqslant \big||a|-|b|\big|$.

The inequality above gives a lower bound to the condition number. As for the upper bound, same as before we have $\kappa_h(x) \leqslant \left|x\frac{f'}{f}\right| + \left|x\frac{g'}{g}\right| \leqslant \kappa_f(x) + \kappa_g(x)$.

To summarize, if either f or g is ill-conditioned, then the quotient function h is guaranteed to be ill-conditioned. On the other hand, if both f and g are well-conditioned, then h is guanranteed to be well-conditioned.

Problem 5

What is the Fourier series of the function $f(x) = x$?

Hint: You can easily find an antiderivative of $xexp(inx)$ using integration by parts.

Ans: $f(x) = x$ is clearly integrable, so for $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, we can use the formula to compute the coefficients

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx$$

Following the given hint to use integration by part, for $n \neq 0$ we have

$$\begin{aligned} a_n &= -\frac{1}{(2\pi)(in)}\int_{-\pi}^{\pi} xd(e^{-inx}) = -\frac{1}{(2\pi)(in)}\left(\left[xe^{-inx}\right]_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} e^{-inx}\,dx}_{0}\right) \\ &= -\frac{1}{(2\pi)(in)}\left(\underbrace{\pi e^{-in\pi} + \pi e^{in\pi}}_{2\pi(-1)^n}\right) = -\frac{1}{in}(-1)^n = \frac{i}{n}(-1)^n \end{aligned}$$

For $n = 0$, we have $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$ since x is a odd function.

Combining everything together, we have

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}(-1)^n e^{-inx}$$

We can further simplify the above expression with the observation that for any non-zero integer n,

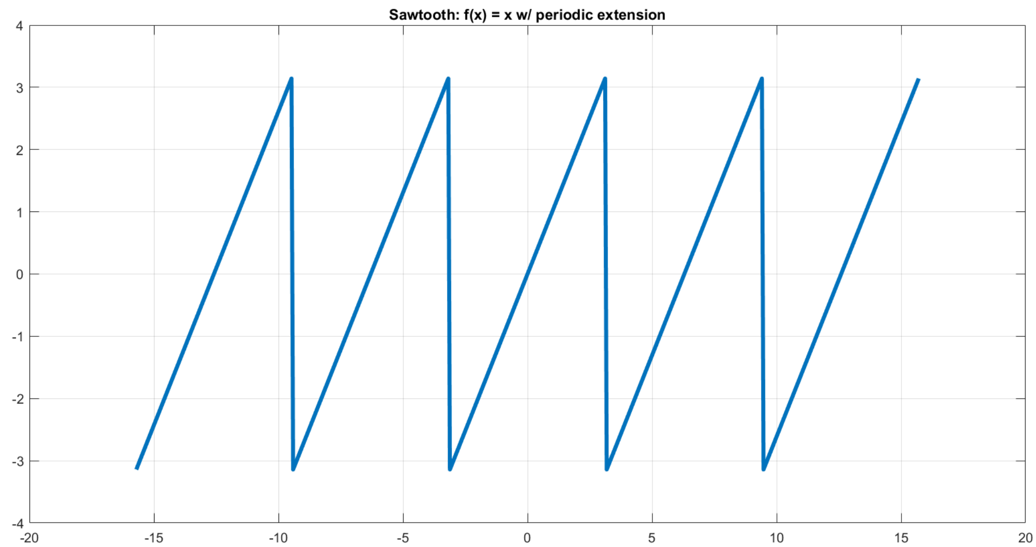
$$\frac{i}{n}(-1)^n e^{-inx} + \frac{i}{-n}(-1)^n e^{inx} = i(-1)^n(2\frac{i}{n}sin(nx)) = -2\frac{(-1)^n}{n}sin(nx)$$

Therefore, in summary

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin(nx) \tag{1}$$

The answer is not surprising since $f(x) = x$ is an odd function. Therefore the Fourier series consists only of $sin(x)$ which are also odd whereas all $cos(x)$ can be cancelled out.

Bonus (I): If we extend the function $f(x) = x$ over $[-\pi, \pi]$ to the entire real line, the extended function is periodic but not continuously differentiable (it is not continuous at the end points). The discontinuities are represented by the vertical lines in the graph below:



As said in class, for such f with discontinuities, the Fourier series does NOT converge uniformly to $f(x) = x$ for all x. Indeed with more advanced analysis, we can show that the Fourier series of this function DOES converge uniformly at all x except at countably many jump points. **More precisely, the above equality (1) holds in the L^2-sense, but NOT in a pointwise sense!** This is beyond the scope of this course...

Also, there are "overshoots" around the discontinuities known as "Gibbs phenomenon": (The dark blue curve is $f(x) = x$, the light blue curve is the approximation by sine series)

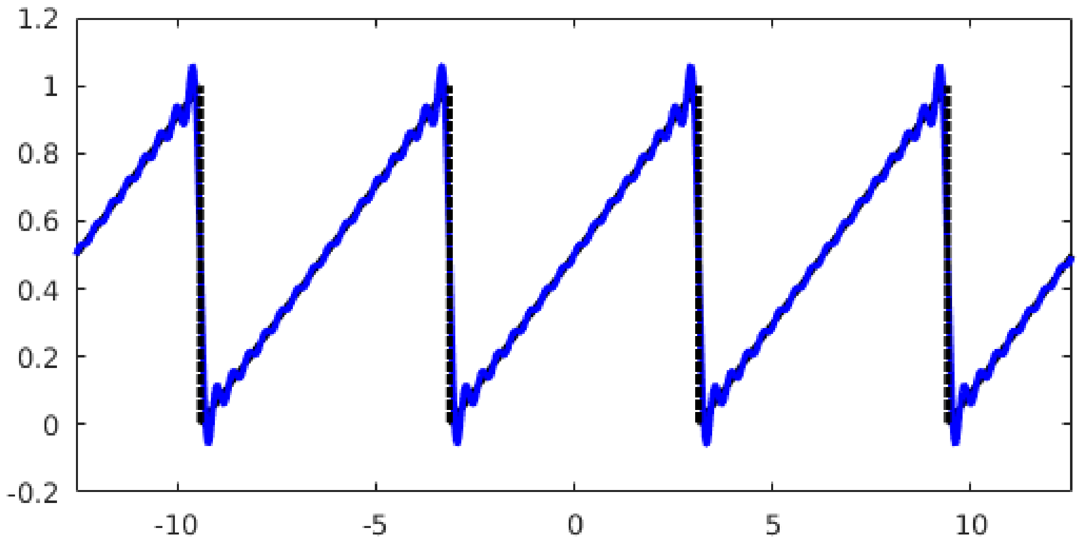


Figure 1: Source: http://www.chebfun.org/

Bonus (II): For those of you who are eager to learn more math, Fourier series are often one way to generate beautiful infinite series. If we substitute $x = \frac{\pi}{2}$ into (1), we have

$$\frac{\pi}{2} = f(\frac{\pi}{2}) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin(n\frac{\pi}{2})$$

Since $sin(n\frac{\pi}{2}) = 0$ when n is even, therefore

$$\frac{\pi}{2} = -2\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} sin(\frac{2k+1}{2}\pi) \quad \Rightarrow \quad \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

We have rediscovered the "Leibniz Series"!

Bonus (III): If you are interested in learning more advanced mathematics, using the above Fourier series expansion of $f(x) = x$ together with the "Parseval identity" shown in class

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Equating the L.H.S. and R.H.S., we have

$$R.H.S. = \sum_{n=-\infty}^{\infty} |a_n|^2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad L.H.S. = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This is a indeed very famous problem in mathematics known as the "Basel Problem"!

Problem 6

What is the Fourier series of the function $f(x) = |x|$?

Ans: Similar to the previous question, compute

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| e^{-inx}\, dx = \frac{1}{2\pi}\left(\underbrace{\int_0^{\pi} xe^{-inx}\, dx}_{(I)} + \underbrace{\int_{-\pi}^{0} -xe^{-inx}\, dx}_{(II)}\right)$$

For $n \neq 0$, we compute the two integrals using integrating by parts:

$$(I) = \frac{1}{(-in)}\left[\pi(-1)^n + \left(\frac{(-1)^n}{in} - \frac{1}{in}\right)\right]$$
$$(II) = \frac{1}{in}\left[\pi(-1)^n + \left(\frac{1}{in} - \frac{(-1)^n}{in}\right)\right]$$

Combining (I) and (II), the $\pi(-1)^n$ terms will be cancelled out, we have

$$a_n = \frac{1}{2\pi}\left(\frac{2}{in}(\frac{1}{in} - \frac{(-1)^n}{in})\right) = \begin{cases} 0 & n \text{ is even, } n \neq 0 \\ -\frac{2}{\pi n^2} & n \text{ is odd} \end{cases}$$

$$\Rightarrow a_n = \begin{cases} 0, & n = 2k,\ n \neq 0 \\ -\frac{2}{\pi(2k+1)^2}, & n = 2k+1 \end{cases} \quad \text{for } k = 0, 1, 2, 3$$

I skipped many computation steps above. You should make sure that you get the same answer...

Also, for $n = 0$, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{\pi}{2}$.

Combining everything together, we have

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} e^{-i(2k+1)\pi}$$

Using the observation $e^{-inx} + e^{inx} = 2cos(nx)$, we have

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{cos\left((2k+1)x\right)}{(2k+1)^2}$$

Indeed with more advanced analysis, we can show that this Fourier series of the given $f(x) = |x|$, which is continuous over $[-\pi, \pi]$, periodic, and its derivative being piecewise continuous, converges to $f(x)$ uniformly. But this is beyond the scope of this course...