MAT 128A - Practice Final Exam

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Problem 1 (True or False)

(I) If $f\colon [-1,1] \to \mathbb{R}$ is a C^k function and $\{a_n\}$ are the Chebyshev coefficients of f, then $|a_n| = \mathcal{O}(\frac{1}{n^k})$.

(II) The condition number of evaluation of the function $f(x) = \frac{1}{x}$ goes to ∞ as $x \to 0^+$.

(III) The condition number of evaluation of the function $f(x) = cos(x)$ goes to ∞ as $x \to \frac{\pi}{2}$.

(IV) The quadrature rule

$$\int_{-\pi}^{\pi} f(t)\, dt \approx \frac{2\pi}{n+1} \sum_{j=0}^{n} f(-\pi + \frac{2\pi}{n+1} j)$$

is exact for the collection of functions $exp(-ikt)$, for $k = -n, -n+1, \cdots, -1, 0, 1, \cdots n-1, n$.

(V) If p is a monic polynomial of degree n, then

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \geqslant 2^{-n+1}$$

Ans: (I) TRUE

(*Not graded - My reason*): This question is the same as the "True or False" (II) statement. Given f a C^k function, its Chebyshev coefficients $|a_n| = o(\frac{1}{n^k})$ which in turn implies that, $|a_n| = \mathcal{O}(\frac{1}{n^k})$

Remark 1. In general, given two real-valued functions f, g, it is clear that $f = o(g) \Rightarrow f = \mathcal{O}(g)$ since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0 < +\infty$

(II) FALSE

(*Not graded - My reason*): Direct calculation. By definition $\kappa_f = \left|x\frac{f'(x)}{f(x)}\right| = \left|x\frac{d}{dx}ln(f(x))\right|$, substituting $f(x) = \frac{1}{x}$, we have $\kappa_f = \left|x\frac{d}{dx}(-lnx)\right| = |-1| = 1$. So the condition does not go to ∞ as $x \to 0^+$.

Remark 2. This is not surprising. Although $f(x)$ goes to ∞ around $x = 0$, the floating point get denser and denser as $x \to 0$. So these effects offset each other, resulting the condition number equal to a constant.

(III) TRUE

(*Not graded - My reason*): Direction calculation again. $\kappa_f = \left|x\frac{d}{dx}ln(cosx)\right| = |xtanx|$, therefore $\kappa_f(x)$ goes to ∞ as $x \to \frac{\pi}{2}$.

(IV) TRUE

(*Not graded - My reason*): This follows directly from the accruacy of the periodic trapezoidal rule. See in lecture 7 "the n-point periodic trapezoidal rule".

(V) TRUE

(*Not graded - My reason*): This follows from the fact that the scaled Chebyshev polynomial $\frac{1}{2^{n-1}}T_n(x)$ is one of the polynomials whose maximal absolute value on the interval $[-1, 1]$ is minimal, i.e. $\frac{1}{2^{n-1}}T_n(x)$ is one of the polynomials which attain the minimal value with respect to the $\| \|_\infty$. See the slide "Good Interpolation Nodes" in lecture 15.

Therefore,

$$\max_{-1\leqslant x\leqslant 1}|p(x)| \geqslant \min_{\substack{\text{monic } p(x)\\ deg(p(x))=n}} \left(\max_{-1\leqslant x\leqslant 1}|p(x)|\right) = \max_{-1\leqslant x\leqslant 1}\left|\frac{1}{2^{n-1}}T_n(x)\right| = \frac{1}{2^{n-1}}\max_{-1\leqslant x\leqslant 1}|T_n(x)| = 2^{-n+1}$$

where we use the fact $\max\limits_{-1\leqslant x\leqslant 1}|T_n(x)| = 1$ since $T_n(x)$ is a cosine function.

Problem 2 (Hermite Interpolation)

Find the unique polynomial p of degree less than or equal to 3 such that

$$p(x_0) = f(x_0), \quad p'(x_0) = f'(x_0), \quad p(x_1) = f(x_1), \quad p'(x_1) = f'(x_1)$$

where $f(x) = sin(x)$ and $x_0 = 0$, and $x_1 = \frac{\pi}{2}$.

Ans: This question resembles question 1 in homework assignment 7 for the week of November 12.

Substituting $f(x) = sin(x)$, we can rewrite the given condition in

$$p(0) = 0, \quad p'(0) = 1, \quad p(\frac{\pi}{2}) = 1, \quad p'(\frac{\pi}{2}) = 0$$

Let the polynomial $p(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are all unknowns. Substituting into the above condition, we immediately have

$$\begin{cases} d & = 0 \\ c & = 1 \\ a\frac{\pi^3}{8} + b\frac{\pi^2}{4} + \frac{\pi}{2}c + d & = 1 \\ 3a\frac{\pi^2}{4} + 2b\frac{\pi}{2} + c & = 0 \end{cases}$$

We have immediately $c = 1, \; d = 0$. Rearranging the third and fourth equations, we have

$$a\frac{\pi}{2} + b = \frac{4}{\pi^2} - \frac{2}{\pi}, \quad \frac{3\pi^2}{4}a + b\pi = -1$$

Substituting $b = \frac{4}{\pi^2} - \frac{2}{\pi} - \frac{\pi}{2}a$ in the fourth equation, we have

$$\frac{3\pi^2}{4}a + \pi\left(\frac{4}{\pi^2} - \frac{2}{\pi} - \frac{\pi}{2}a\right) = -1$$

After simplying it, we have $a = \frac{4}{\pi^2} - \frac{16}{\pi^3}$ and $b = \frac{12}{\pi^2} - \frac{4}{\pi}$. To conclude, the unique polynomial is $p(x) = \left(\frac{4}{\pi^2} - \frac{16}{\pi^3}\right)x^3 + \left(\frac{12}{\pi^2} - \frac{4}{\pi}\right)x^2 + x$.

Problem 3 (Polynomial Interpolation - equally spaced nodes)

Let $f(x) = cos(x)$ and, for each positive integer N, let p_N be the polynomial of degree less than or equal to N which interpolates f at the nodes $x_j = -1 + \frac{2j}{N}$, $j = 0, 1, \cdots, N$. Show that

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \to 0 \text{ as } N \to \infty$$

Ans: The key idea is to apply the error estimate formula for the Lagrange Formula, in our case, for each x, there exists $\xi_x \in (-1, 1)$ such that

$$f(x) - p(x) = \frac{f^{(N+1)}(\xi_x)}{(N+1)!}(x - x_0)(x - x_1) \cdots (x - x_N) \tag{1}$$

Since $f(x) = cos(x)$ and its derivative can only be $sin(x)$ or $cos(x)$ up to a sign, hence $|f^{(N+1)}(x)| \leqslant 1$ for any natural integer N and for any $x \in (-1, 1)$. Taking absolute value on both sides of (1) and then the maximum over $[-1, 1]$, we have

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \leqslant \frac{1}{(N+1)!} \max_{-1 \leqslant x \leqslant 1} |(x - x_0) \cdots (x - x_N)|$$

Now notice that $|x - x_j| \leqslant 1 - (-1) = 2$ which is the length of the interval $[-1, 1]$, this implies

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \leqslant \frac{2^{N+1}}{(N+1)!}$$

(*I am not saying that this bound is tight! It is certainly not.*)

Since factorial grows faster than exponential function, $(N+1)! > 2^{N+1}$ for sufficiently large N, hence $\lim_{N \to \infty} \frac{2^{N+1}}{(N+1)!} = 0$.

Therefore taking limit on both sides of the above inequality,

$$\lim_{N \to \infty} \left(\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \right) = 0$$

Remark 3. As said in the Bonus part in problem 4 of HW assignmet 7 (Nov 12, 18), we can obtain a bound for equally spaced nodes $x_j = -1 + \frac{2j}{N}$, $j = 0, 1, \cdots, N$:

$$\max_{x \in [-1,1]} |f(x) - p_N(x)| \leqslant \left(\frac{b-a}{n} \right)^{n+1} \frac{1}{4(n+1)} \max_{x \in [-1,1]} |f^{(N+1)}(x)|$$

A more detailed analysis can show that the interpolation error is minimized if the point of interest x is chosen to be as close as possible to the midpoint of $[-1, 1]$.

Problem 4 (Quadrature Rule)

Find a quadrature rule of the form

$$\int_{-1}^{1} f(x)|x|\,dx \approx f(-1)w_0 + f(0)w_1 + f(1)w_2 \tag{2}$$

which is exact whenever f is a polynomial of degree less than or equal to 2.

Ans: Similar to problems $1, 2$ in HW assignment 8 (Nov 26, 18), we substitute

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$$

into (2). We obtain

$$\begin{cases} 1 = \int_{-1}^{1} 1 \cdot |x|\,dx & = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1 \\ 0 = \int_{-1}^{1} x|x|\,dx & = w_0 \cdot (-1) + w_1 \cdot 0 + w_2 \cdot 1 \\ \frac{1}{2} = \int_{-1}^{1} x^2|x|\,dx & = w_0 \cdot (1) + w_1 \cdot 0 + w_2 \cdot (1) \end{cases} \quad \Rightarrow \quad w_0 = \frac{1}{4} = w_2, \ w_1 = \frac{1}{2}$$

(You can also tell that $\int_{-1}^{1} x|x|\,dx = 0$ since $f(x) = x|x|$ is an odd function. Otherwise direct computation.)

Therefore, the quadrature rule (2) becomes

$$\int_{-1}^{1} f(x)|x|\,dx \approx \frac{1}{4}f(-1) + \frac{1}{2}f(0) + \frac{1}{4}f(1) \tag{3}$$

Remark 4. Since

$$0 = \int_{-1}^{1} x^3|x|\,dx = \frac{1}{4} \cdot (-1) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot (1)$$
$$\frac{1}{3} = \int_{-1}^{1} x^4|x|\,dx \neq \frac{1}{4} \cdot (1) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot (1) = \frac{1}{2}$$

Indeed the quadrature rule (3) is exact whenever f is a polynomial of degree less than or equal to 3. **The extra degree of exactness is due to the fact that the points $x = -1, 0, 1$ are symmetrically distributed around $x = 0$**

Problem 5 (Chebyshev coefficients)

Compute the Chebyshev coefficients of the function

$$f(x) = \sqrt{1-x^2}$$

Ans: This problem is exactly the same as problem 3 in Homework 5 (Oct 22,18)

Notice that the function $f(x)$ is ONLY defined on the interval $[-1,1]$. Recall the formula for Chebyshev coefficients:

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}$$

Using the change of variable $x = cost$ ($\Rightarrow\ dx = -sint\,dt$), we have

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_{\pi}^{0} cos(nt) - sint\,dt \quad \text{note that } 0 < t < \pi \\
&= \frac{2}{\pi}\int_{0}^{\pi} cos(nt)sint\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} sin(n+1)t - sin(n-1)t\,dt \\
&\overset{(*)}{=} \frac{1}{\pi}\left[-\frac{cos(n+1)t}{n+1} + \frac{cos(n-1)t}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}
\end{aligned}$$

$(*)$ holds due to the sum of angle formula $cos\alpha sin\beta = \frac{1}{2}\left(sin(\alpha+\beta) - sin(\alpha-\beta)\right)$.

Note that for n is odd, $a_n = \frac{1}{\pi}\left\{\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = 0$.

For $n \neq 0$ is even, $a_n = \frac{1}{\pi}\left\{\left[\frac{1}{n+1} - \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} = -\frac{4}{(n^2-1)\pi}$.

For $n = 0$, a_0 **is halved**, so $a_0 = -\frac{2}{(0^2-1)\pi} = \frac{2}{\pi}$.

Writing even number $n = 2k$, for $k = 1, 2, 3, \cdots$, the Chebyshev expansion of $f(x) = \sqrt{1-x^2}$ is

$$\sqrt{1-x^2} = \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{(4k^2-1)\pi}T_{2k}(x) = \frac{2}{\pi} - \frac{4}{3\pi}(2x^2-1) - \frac{4}{15\pi}(8x^4 - 8x^2 + 1) - \cdots$$

Problem 6 (Numerical Differentiation)

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is smooth, and that $h > 0$. Find the coefficients a, b, and c such that

$$af(-h) + bf(h) + cf(2h) = f'(0) + \mathcal{O}(h^2) \tag{4}$$

Ans: Indeed we want to solve for

$$f'(x) = af(x-h) + bf(x+h) + cf(x+2h) + \mathcal{O}(h^2)$$

The key idea here is Taylor's expansion: for $\xi_1 \in (x-h, x), \xi_2 \in (x, x+h), \xi_3 \in (x+h, x+2h)$

$$\begin{cases} f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(\xi_1) \\ f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(\xi_2) \\ f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{h^3}{3!}f'''(\xi_3) \end{cases}$$

$$\Rightarrow \quad af(x-h) + bf(x+h) + cf(x+2h)$$
$$= (a+b+c)f(x) + (-a+b+2c)hf'(x) + (\frac{a}{2} + \frac{b}{2} + 2c)h^2 f''(x) + \mathcal{O}(h^3)$$

Setting the above equation equal to $f'(x)$, we have

$$\begin{cases} a+b+c &= 0 \\ -a+b+2c &= \frac{1}{h} \\ \frac{a}{2} + \frac{b}{2} + 2c &= 0 \end{cases} \quad \Rightarrow \quad a = -\frac{1}{2h},\ b = \frac{1}{2h},\ c = 0$$

Therefore, we have $f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$. Setting $x = 0$, we get exactly (4).

Remark 5. This is exactly **Three-Point Midpoint Formula** introduced in class. With a bit more work, one can show

$$f'(0) = \frac{f(h) - f(-h)}{2h} - \frac{h^2}{6}f^{(3)}(\xi_1)$$

for some $\xi_1 \in (-h, h)$.

Problem 7 (Chebyshev Interpolation - root grid)

Let $f(x) = cos(x)$ and, for each positive integer N, let p_N be the polynomial of degree less than or equal to N which interpolates f at the nodes

$$x_j = cos\left(\frac{j+\frac{1}{2}}{N+1}\right), \quad j = 0, \cdots, N \tag{5}$$

Show that

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \leqslant \frac{2^{-N}}{(N+1)!}$$

Ans: The solution's idea to this problem is exactly the same as that for Problems 2 to 4 in homework assignment 7 (Nov 12, 18).

First, we notice that the nodes x_j defined in (5) are exactly the roots for the Chebyshev polynomial $T_{N+1}(x)$ of degree $N+1$. Therefore, we have

$$\prod_{j=0}^{N}(x - x_j)(x - x_0) \cdots (x - x_N) \overset{(*)}{=} \frac{1}{2^N} T_{N+1}(x)$$

$(*)$ holds since the product above is a monic polynomials and the leading coefficient for $T_N(x)$ is $\frac{1}{2^{N-1}}$. See the slide "Good Interpolation Nodes" in lecture 15.

Now recall the following theorem for interpolation error from lecture 15.

Theorem 1. Given $f\colon [a,b] \to \mathbb{R}$ is (N+1)-times continuously differentiable, $x_0 < x_1 < \cdots < x_N$ are partition of $[a,b]$, and p_N is the unique polynomial of degree N which interpolates f at nodes $x_0, x_1, \cdots, x_N$. Then for $x \in [a,b]$, there exists a point $\xi_x \in (a,b)$ such that

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N}(x - x_j)$$

Since $f(x) = cos(x)$ and its derivative can only be $sin(x)$ or $cos(x)$ up to a sign, hence $|f^{(N+1)}(x)| \leqslant 1$ for any natural integer N and for any $x \in (-1,1)$.

Also, $|T_n(x)| \leqslant 1$ for any natural integer n and for any $x \in [-1,1]$ since it is a cosine function.

Taking absolute value on both sides of (3) and then the maximum over $[-1,1]$, we have

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \leqslant \frac{1}{(N+1)!} \left| \prod_{j=0}^{N}(x - x_j) \right| = \frac{1}{(N+1)!} \frac{1}{2^N} \underbrace{|T_{N+1}(x)|}_{\leqslant 1} = \frac{2^{-N}}{(N+1)!}$$

Problem 8 (Legendre Polynomial)

Show that the Legendre Polynomial P_n of degree n satisfies the differential equation

$$(1-x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0 \tag{6}$$

Ans: First recall that Legendre Polynomials P_n are a collection of orthogonal polynomials over $[-1,1]$, i.e.

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

The first of them are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

They satisfy the three-term recurrence relations $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$. Similar to the Chebyshev polynomial $T_n(x) = cos(ncos^{-1}(x))$, the Legendre polynomial is defined by a closed formula, for $n \geqslant 1$ and $x \in [-1,1]$

$$P_n(x) := \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{7}$$

This is sometimes called the **Rodrigues' formula** for the Legendre polynomials.
You can easily verify that for $n=1:\ P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$,
$n=2: P_2(x) = \frac{1}{2^2\cdot 2}\frac{d^2(}{dx^2}{}^2-1)^2 = \frac{1}{2}\frac{d}{dx}[x(x^2-1)] = \frac{1}{2}(3x^2-1)$ and so on.

Now we want to show that our formula (7) satisfy the differential equation (6). Direct substitution is hard, since it involes the n^{th} derivative. So we start by observing that

$$p(x) = (x^2-1)^n \quad \Rightarrow \quad p'(x) = 2xn(x^2-1)^{n-1} = 0 \quad \Rightarrow \quad (1-x^2)p'(x) + 2nxp(x) = 0$$

Now we want apply k times derivative to the last equality. Recall the general Leibniz rule in calculus for taking n times derivative on a product function $f(x)g(x)$:

$$\frac{d^n}{dx^n}(fg) = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdot + fg^{(n)} = \sum_{i=0}^{n}\binom{n}{i}f^{(n-i)}g^{(i)}$$

Apply Leibniz rule to $(1-x^2)p'(x)$, notice that $\frac{d^n}{dx^n}(1-x^2) = 0$ for $n \geqslant 3$:

$$\frac{d^k}{dx^k}\left((1-x^2)p'(x)\right) = (1-x^2)p^{(k+1)}(x) + k(-2x)p^{(k)}(x) + \frac{k(k-1)}{2}(-2)p^{(k-1)}(x)$$

Likewise apply Leibniz rule to $2nxp(x)$:

$$\frac{d^k}{dx^k}(2nxp(x)) = 2nxp^{(k)}(x) + 2nkp^{(k-1)}(x)$$

Substituting the two k^{th} derivatives above into $(1-x^2)p'(x) + 2nxp(x) = 0$:

$$(1-x^2)p^{(k+1)}(x) + k(-2x)p^{(k)}(x) + \frac{k(k-1)}{2}(-2)p^{(k-1)}(x) + 2nxp^{(k)}(x) + 2nkp^{(k-1)}(x) = 0$$

Now if we stare at the above equation long enough, we realize that by setting $k = n+1$:

$$\begin{aligned}(1-x^2)p^{(n+2)}(x) - 2(n+1)xp^{(n+1)}(x) - (n+1)np^{(n)}(x) + 2nxp^{(n+1)}(x) + 2n(n+1)p^{(n)}(x) = 0\\ \Rightarrow \quad (1-x^2)p^{(n+2)}(x) - 2xp^{(n+1)}(x) + n(n+1)p^{(n)}(x) =0\\ \Rightarrow \quad (1-x^2)\frac{d^2}{dx^2}\left(\frac{d^n}{dx^n}p(x)\right) - 2x\frac{d}{dx}\left(\frac{d^n}{dx^n}p(x)\right) + n(n+1)\left(\frac{d^n}{dx^n}p(x)\right) = 0\end{aligned}$$

Therefore, the polynomials $\frac{d^n}{dx^n}p(x) = \frac{d^n}{dx^n}(x^2-1)^n$ satisfies the differential equation (6). Moreover, the Legendre polynomials $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ also satisfies (6) since it is multiplied only by a scalar.

Remark 6. Most of the special polynomials satistfies certain differential equations. For Chebyshev polynomials $T_n(x)$, it is the solution to

$$(1-x^2)y'' - xy' + n^2y = 0$$

Bonus: The *Jacobi polynomials* $P_n^{(\alpha,\beta)}$ (`https://en.wikipedia.org/wiki/Jacobi_polynomials`) were briefly mentioned in this class. Indeed both Legendre polynomials and Chebyshev polynomials are special cases of the Jacobi polynomials. And the Jacobi polynomials satisfy

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0$$

Setting $\alpha = 0 = \beta$, we obtain the exact same differential equation (6) as stated in our problem! Setting $\alpha = -\frac{1}{2} = \beta$, we obtain the differential equation for the Chebyshev polynomials stated above.

Problem I (Bonus)

Find a quadrature rule of the form

$$\int_{-1}^{1} f(x)|x|\,dx \approx f(x_0)w_0 + f(x_1)w_1 + f(x_2)w_2 \tag{8}$$

which is exact whenever f is polynomial of degree less than or equal to 5.

Ans: Similar to problem 4 above, we need to solve for a larger nonlinear system now. We substitute

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2, \quad f_4(x) = x^3, \quad f_5(x) = x^4, \quad f_6(x) = x^5$$

into (2). We obtain

$$\begin{cases} 1 = \int_{-1}^{1} 1 \cdot |x|\,dx & = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1 \\ 0 = \int_{-1}^{1} x|x|\,dx & = w_0 \cdot x_0 + w_1 \cdot x_1 + w_2 \cdot x_2 \\ \frac{1}{2} = \int_{-1}^{1} x^2|x|\,dx & = w_0 \cdot x_0^2 + w_1 \cdot x_1^2 + w_2 \cdot x_2^2 \\ 0 = \int_{-1}^{1} x^3|x|\,dx & = w_0 \cdot x_0^3 + w_1 \cdot x_1^3 + w_2 \cdot x_2^3 \\ \frac{1}{3} = \int_{-1}^{1} x^4|x|\,dx & = w_0 \cdot x_0^4 + w_1 \cdot x_1^4 + w_2 \cdot x_2^4 \\ 0 = \int_{-1}^{1} x^5|x|\,dx & = w_0 \cdot x_0^5 + w_1 \cdot x_1^5 + w_2 \cdot x_2^5 \end{cases}$$

$$\Rightarrow \quad w_0 = \frac{3}{8},\ w_1 = \frac{1}{4},\ w_2 = \frac{3}{8},\ x_0 = -\sqrt{\frac{2}{3}},\ x_1 = 0,\ x_2 = \sqrt{\frac{2}{3}}$$

Since

$$\frac{1}{4} = \int_{-1}^{1} x^6|x|\,dx \neq \frac{3}{8}\left(-\sqrt{\frac{2}{3}}\right)^6 + \frac{1}{4}(0) + \frac{3}{8}\left(\sqrt{\frac{2}{3}}\right)^6 = \frac{2}{9}$$

Therefore, the quadrature rule

$$\int_{-1}^{1} f(x)|x|\,dx \approx \frac{3}{8}f\left(-\sqrt{\frac{2}{3}}\right) + \frac{1}{4}f(0) + \frac{3}{8}f\left(\sqrt{\frac{2}{3}}\right)$$

is polynomial of degree less than or equal to 5.

Problem II (Bonus)

Find the nodes t_0, t_1, t_2, t_3 and weights w_0, w_1, w_2, w_3 of a quadrature rule

$$\int_{-\pi}^{\pi} f(t)\, dt \approx \sum_{j=0}^{3} f(t_j) w_j$$

which is exact for the functions

$$\{exp(int) \colon n = -3, -2, -1, 0, 1, 2, 3\}$$

If we look carefully at the set of functions $\{exp(int) \colon n = -3, -2, -1, 0, 1, 2, 3\}$, we realize that it has the exact same form as the functions which can be *exactly evaluated using the* $4-$-*point periodic trapezoidal rule.*

Look at probem 1 (IV) with True/False!

Therefore, the quadrature weights are all the same and they equal to :

$$w_0 = w_1 = w_2 = w_3 = \frac{2\pi}{3+1} \left(= \frac{\pi}{2} \right)$$

The quadrature nodes are exactly $t_j = -\pi + \frac{2\pi}{3+1} j$ for $j = 0, 1, 2, 3$:

$$t_0 = -\pi, \quad t_1 = -\frac{\pi}{2}, \quad t_2 = 0, \quad t_3 = \frac{\pi}{2}$$