MAT 128A - Assignment 3

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Problem 1

Show that when the n-point periodic trapezoidal rule is used to evaluate the integral $\int_{-\pi}^{\pi} exp(ikt)\, dt$, the result is

$$\begin{cases} (-1)^{|k|}2\pi & \text{if } k = m \cdot n \text{ for some nonzero integer } m \\ 2\pi & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Ans: The desired statment can be shown by applying the n-point periodic trapezoidal rule directly (Clearly the function $f(x) = e^{ikt}$ is 2π-periodic.). As shown in class,

$$\begin{aligned} T(k) &:= \frac{2\pi}{n}\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right) \quad \text{where } x_j = -\pi + \frac{2\pi}{n}j \\ &= \frac{2\pi}{n}\left(e^{-ik\pi} + e^{ik(-\pi+\frac{2\pi}{n})} + \cdots + e^{ik(-\pi+\frac{2\pi}{n}(n-1))}\right) \\ &= \frac{2\pi}{n}e^{-ik\pi}\left(1 + e^{i\frac{2\pi}{n}k} + e^{i\frac{2\pi}{n}k\cdot 2} + \cdots + e^{i\frac{2\pi}{n}k\cdot(n-1)}\right) \\ &= \frac{2\pi}{n}e^{-ik\pi}(1 + r + r^2 + \cdots + r^{n-1}) \quad \text{where } r = e^{i\frac{2\pi}{n}k} \end{aligned}$$

It is tempting to apply the summation formula for geometric series in the last step above right away, BUT let us take a closer look at this sum and consider two special cases:

(i) For $k = 0$, $r = e^0 = 1$, so $T(0) = \frac{2\pi}{n}(1)\underbrace{(1 + 1 + \cdots + 1)}_{n\ '1's} = 2\pi$.

(ii) For $k = m \cdot n$, $r = e^{i\frac{2\pi}{n}(m\cdot n)} = e^{i(2\pi m)} = 1$ and $e^{-ik\pi} = (-1)^{|k|}$, so

$$T(k) = \frac{2\pi}{n}(-1)^{|k|}\underbrace{(1 + 1 + \cdots + 1)}_{n\ '1's} = (-1)^{|k|}2\pi$$

For any other k, applying the summation formula for geometric series, we have

$$T(k) = \frac{2\pi}{n} e^{-ik\pi} \frac{1-r^n}{1-r} = \frac{2\pi}{n} e^{-ik\pi} \frac{1-e^{i2\pi k}}{1-e^{i\frac{2\pi}{n}k}} = \frac{2\pi}{n} e^{-ik\pi} \frac{1-1}{1-e^{i\frac{2\pi}{n}k}} = 0$$

The above calculations explains why **the n-point periodic trapezoidal rule is exact for functions** e^{ikt}, $k = -n+1, \cdots, n-1$

Problem 2

Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is even — meaning that $f(-x) = f(x)$ for all $0 < x \leqslant \pi$ — then f can be represented via a convergent series of the form

$$f(x) = \sum_{n=0}^{\infty} b_n cos(nx)$$

Ans: Assume that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $f(-x) = f(x)$ implies that $a_{-n} = a_n$ for all integer n (WHY?). Then

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} = a_0 + \sum_{n=1}^{\infty} a_n \left(e^{inx} + e^{-inx}\right) \stackrel{(*)}{=} a_0 + 2\sum_{n=1}^{\infty} a_n cos(nx)$$

$(*)$ uses the observation $e^{-inx} + e^{inx} = 2cos(nx)$. Finally we set $b_n = \begin{cases} a_0 & \text{for } n = 0 \\ 2a_n & \text{for } n \neq 0 \end{cases}$, then we have the desired statement, $f(x) = \sum_{n=0}^{\infty} b_n cos(nx)$.

As mentioned in the lecture, the Fourier series of a continuously differentiable (i.e. C^1), 2π-periodic function converges uniformly and absolutely to f on $[-\pi, \pi]$. So we don't need to prove the convergence of the above series.

Remark 1. Please look at HW 2 Question 6, the Fourier Series of $f(x) = |x|$ (which is an even function) can be represented by a series of $cos(nx)$.

Bonus: The series above containing only $cos(nx)$ functions is called the "Fourier cosine series". With a bit of extra efforts, one can show that $b_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)cos(nx)\,dx$ for $n = 1, 2, 3, \cdots$

Problem 3

Suppose that f is a continuously differentiable 2π-periodic function. Show that if f is odd — meaning that $f(-x) = -f(x)$ for all $0 < x \leqslant \pi$ — then f can be represented via a convergent series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n sin(nx)$$

Ans: Similar to the last problem, assume that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $f(-x) = -f(x)$ implies that $a_{-n} = -a_n$ for all integer n (WHY?).

Notice that for $n = 0$, $a_0 = -a_0 \;\Rightarrow\; a_0 = 0$. Therefore

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} = \underbrace{a_0}_{=0} + \sum_{n=1}^{\infty} a_n \left(e^{inx} - e^{-inx}\right) \overset{(*)}{=} 2i \sum_{n=1}^{\infty} a_n sin(nx)$$

$(*)$ uses the observation $e^{-inx} - e^{inx} = 2isin(nx)$. Finally we set $c_n = 2ia_n$ for $n \neq 0$, then we have the desired statement, $f(x) = \sum_{n=1}^{\infty} c_n sin(nx)$.

Remark 2. Please look at HW 2 Question 5, the Fourier Series of $f(x) = x$ (which is an odd function) can be represented by a series of $sin(nx)$.

Bonus: The series above containing only $sin(nx)$ functions is called the "Fourier sine series". With a bit of extra efforts, one can show that $c_n = \frac{2}{\pi} \int_0^{\pi} f(x) sin(nx)\, dx$ for $n = 1, 2, 3, \cdots$

Problem 4

Suppose that $f(x) = cos(2x) + cos(4x) + \cdots + cos(20x)$. What is the exact value of $\int_{-\pi}^{\pi} f(x)\, dx$?

i.e. How long is the periodic trapezoidal rule of minimum length which evaluates the above integral exactly? That is, what is the least positive integer n such that

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{2\pi}{n} \sum_{j=0}^{n-1} f(-\pi + \frac{2\pi}{n} j) \;?$$

Here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

Ans: The key idea is the statement shown in class that

The n-point periodic trapezoidal rule is exact for the functions e^{-ikt}, where

$$k = -n+1, -n+2, \cdots, -1, 0, 1, \cdots n-1$$

Using $e^{-inx} + e^{inx} = 2cos(nx)$, rewrite

$$f(x) = \frac{1}{2}\left[(e^{-i2x} + e^{i2x}) + (e^{-i4x} + e^{i4x}) + \cdots + ((e^{-i20x} + e^{i20x}))\right]$$

So we want $20 = k = n - 1 \Rightarrow n = 21$. Clearly using more than 21 points will make the integral evaluation exact.

Therefore, the answer is 21 points on $[-\pi, \pi]$.

Problem 5

Let $f(x) = \sum_{n=1}^{\infty} a_n e^{inx}$ with $|a_n| \leqslant \frac{1}{n^2}$.

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

is bounded above by $\frac{\pi^2}{6}\frac{1}{N^2}$. Again here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

(Hint: look at the solutions from the previous homework assignment to find the value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$).

Ans: The key idea here is use the following corollary shown in class:

Corollary 1. Given $g(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, then $\int_{-\pi}^{\pi} g(x)\, dx = 2\pi a_0$, where the approximation of the integral obtained via the $m-$point periodic trapezoidal rule is

$$2\pi a_0 + 2\pi \sum_{k=1}^{\infty}(-1)^{km}(a_{km} + a_{-km})$$

Observe that for the given function in our problem, $a_n = 0$ for all integer $n \leqslant 0$. Therefore the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$. Therefore, in order to calculate error:

$$\begin{aligned} \text{Error } &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx - \frac{1}{2\pi}\left[2\pi a_0 + 2\pi\sum_{k=1}^{\infty}(-1)^{kN}(a_{kN} + a_{-kN})\right]\right| \\ &= \left|\sum_{k=1}^{\infty}(-1)^{kN} a_{kN}\right| \leqslant \sum_{k=1}^{\infty}|a_{kN}| \leqslant \sum_{k=1}^{\infty}\frac{1}{k^2N^2} = \frac{\pi^2}{6}\cdot\frac{1}{N^2} \end{aligned}$$

Remark 3. One should ask right at the beginning whether the given function $f(x) = \sum_{n=1}^{\infty} a_n e^{inx}$ is well-defined, i.e. does this series of function converge? The answer is NO for arbitrary a_n.

But thanks to the condition $|a_n| \leqslant \frac{1}{n^2}$, we can invoke the "Weierstrass M-test" which states that *for a sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$, $f_n\colon A \to \mathbb{C}$, *and suppose for every* $n \in \mathbb{N}$, *there exists constants* $M_n > 0$ *such that* $|f_n(x)| < M_n$ *for all* $x \in A$ *and* $\sum_{n\in\mathbb{N}} M_n < \infty$. *Then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly.* In our case, $M_n = \frac{1}{n^2}$. Therefore, $f(x)$ is well-defined in our case.

Problem 6

Let $f(x) = \sum_{n=0}^{\infty} a_n e^{inx}$ with $|a_n| \leqslant \frac{1}{2^n}$.

Show that the error incurred when the periodic trapezoidal rule of length N is used to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

is bounded above by $\frac{1}{2^N-1}$. Again here we assume that exact arithmetic is used to perform the calculations so that we need not worry about roundoff error.

Ans: Similar to the above problem, here we have $a_n = 0$ for $n < 0$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$. We apply the same corollary to calculate the error:

$$\begin{aligned}\text{Error } &= \left| a_0 - \frac{1}{2\pi}\left[2\pi a_0 + 2\pi\sum_{k=1}^{\infty}(-1)^{kN}(a_{kN} + a_{-kN})\right]\right| \\ &= \left|\sum_{k=1}^{\infty}(-1)^{kN} a_{kN}\right| \leqslant \sum_{k=1}^{\infty}|a_{kN}| \leqslant \sum_{k=1}^{\infty}\frac{1}{2^{kN}} \overset{(*)}{=} \frac{1}{2^N-1}\end{aligned}$$

$(*)$ holds since $\sum_{k=1}^{\infty}\frac{1}{2^{kN}} = \frac{1}{2^N}\left(1 + \frac{1}{2^N} + \frac{1}{2^{2N}} + \cdots\right) = \frac{1}{2^N}\frac{1}{1-\frac{1}{2^N}} = \frac{1}{2^N-1}$.

Problem 7

Find the Fourier series for the function

$$f(t) = \frac{2}{e^{it}-2}$$

by using the identity $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$, which holds for all z with $|z| < 1$.

Ans: Rearranging the given function

$$f(t) = -\frac{1}{1-\frac{e^it}{2}} \overset{(*)}{=} -\sum_{k=0}^{\infty}\left(\frac{e^{it}}{2}\right)^k = \sum_{k=0}^{\infty}\left(-\frac{1}{2^k}\right)e^{ikt}$$

This is an example for **exponential decay** of Fourier coefficients due to the fact that the function $g(z) = \frac{2}{z-2}$ is analytic in a strip over the real-axis on the complex plane.