MAT128A SECTION 2
PRACTICE FINAL
December 12, 2018

1. **WRITE YOUR ANSWERS ON THIS EXAM**.
2. **PUT YOUR NAME ON THE COVER OF THIS EXAM NOW**.
3. If you need it, we will be happy to provide scratch paper for your use, but only answers recorded on the exam paper will be graded.
4. You are allowed the use of a page of notes for this exam. No other notes or books are allowed.
5. In addition, no calculators, cellphones or other electronic devices may be used during this exam.
6. If you have a cell phone with you, put it away now. The time remaining to finish the exam will be periodically announced.
7. The problems are equally weighted.
8. The exam will be "curved" and information about the distribution of grades will be provided to you as soon as the exams are graded.
9. The exams should be graded and available on Gradescope by Dec. 15, 2018

Good luck!

Problem 1. Indicate which of the following statements are true and which are false. You do not need to justify your answers.

1. If $f : [-1,1] \to \mathbb{R}$ is a C^k functions and $\{a_n\}$ are the Chebyshev coefficients of f, then $|a_n| = \mathcal{O}\left(\frac{1}{n^k}\right)$.

2. The condition number of evaluation of the function $f(x) = \frac{1}{x}$ goes to ∞ as $x \to 0^+$.

3. The condition number of evaluation of the function $f(x) = \cos(x)$ goes to ∞ as $x \to \pi/2$.

4. The quadrature rule
$$\int_{-\pi}^{\pi} f(t)\ dt \approx \frac{2\pi}{n+1} \sum_{j=0}^{n} f\left(-\pi + \frac{2\pi}{n+1} j\right),$$
is exact for the collection of functions
$$\exp(-ikt), \quad k = -n, -n+1 - n+2, \ldots, -1, 0, 1, 2, \ldots, n-1, n.$$

5. If p is a monic polynomial of degree n, then
$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \geqslant 2^{-n+1}.$$

Problem 2. Find the unique polynomial p of degree less than or equal to 3 such that

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ p(x_1) &= f(x_1) \\ p'(x_1) &= f'(x_1), \end{aligned}$$

where

$$f(x) = \sin(x)$$

and

$$x_0 = 0, \quad \text{and} \quad x_1 = \frac{\pi}{2}.$$

Problem 3. Let $f(x) = \cos(x)$ and, for each positive integer N, let p_N be the polynomial of degree less than or equal to N which interpolates f at the nodes

$$x_j = -1 + \frac{2j}{N}, \quad j = 0, \dots, N.$$

Show that

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \to 0 \quad \text{as} \quad N \to \infty.$$

Problem 4. Find a quadrature rule of the form

$$\int_{-1}^{1} f(x)\ |x|\ dx \approx f(-1)w_0 + f(0)w_1 + f(1)w_2$$

which is exact whenever f is a polynomial of degree less than or equal to 2.

Problem 5. Compute the Chebyshev coefficients of the function

$$f(x) = \sqrt{1 - x^2}.$$

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is smooth, and that $h > 0$. Find coefficients a, b and c such that

$$af(-h) + bf(h) + cf(2h) = f'(0) + \mathcal{O}\left(h^2\right).$$

Problem 7. Let $f(x) = \cos(x)$ and, for each positive integer N, let p_N be the polynomial of degree less than or equal to N which interpolates f at the nodes

$$x_j = \cos\left(\frac{j+\frac{1}{2}}{N+1}\right), \quad j = 0, \ldots, N.$$

Show that

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - p_N(x)| \leqslant \frac{2^{-N}}{(N+1)!}.$$

Problem 8. Show that the Legendre Polynomial P_n of degree n satisfies the differential equation

$$(1 - x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0.$$

Bonus problem 1: Find a quadrature rule of the form

$$\int_{-1}^{1} f(x)\ |x|\ dx \approx f(x_0)w_0 + f(x_1)w_1 + f(x_2)w_2$$

which is exact whenever f is a polynomial of degree less than or equal to 5.

Bonus problem 2: Find the nodes t_0, t_1, t_2, t_3 and weights w_0, w_1, w_2, w_3 of a quadrature rule

$$\int_{-\pi}^{\pi} f(t)\ dt \approx \sum_{j=0}^{3} f(t_j) w_j$$

which is exact for the functions

$$\{\exp(int) : n = -3, -2, -1, 0, 1, 2, 3\}.$$