Math 160 (De Loera) **Homework (linear algebra methods)**
Due date: Wed April 18, 2018

INSTRUCTIONS

All homeworks will have many problems, both theoretical and practical.

Solutions need to be submitted via CANVAS by uploading files. No homeworks will be accepted in person. Mark your team members clearly.

Write legibly preferably using word processing if your hand-writing is unclear.

Be organized and use the notation appropriately. Show your work on every problem. Correct answers with no support work will not receive full credit.

1. **DO NOT SUBMIT SOLUTION to problem one, only for your review!**: This first three weeks of the course, I assume you know linear algebra. The 3 exercises below should give you a chance to remember.

   (a) For the matrix A below, under what conditions on b does the system of equations has a solution?
   $$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}, \quad b = (b_1, b_2, b_3)^T.$$
   (b) Find a basis for the nullspace of A.

   (c) Find a general solution of $Ax = b$, when solution exists.

   (d) Find a basis for the column space of A.

   (e) What is the rank of A^T?

   (f) Use the Gram-Schmidt procedure to find an orthonormal basis for the row space, column space and the nullspace of the matrix A below.

   (b) If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, without using MATLAB find the value of A^{100}. HINT: You dont need to multiply 100 matrices.

   (c) For what range of numbers a,b are the matrices A,B positive definite?
   $$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

   (d) Let A, B be real $m \times n$ matrices. Show that if the nullspace of B is contained in the nullspace of A implies that the range of B^T contains that of A^T.

   (e) Please decide whether each statement is TRUE or FALSE (no reasoning gives you zero points):

      i. For any matrix A, the nullspace of $A^T A$ equals the nullspace of A.

      ii. For any matrix A, the rank of $A^T A$ equals the rank of A^T.

      iii. If A, B are orthogonal matrices then $A + B$ is orthogonal too.

      iv. A orthogonal implies $\|Ax\| = \|x\|$.

v. A orthogonal if and only if $\|Ax - Ay\| = \|x - y\|$.

vi. Let A orthogonal matrix and $x_1, x_2, \ldots x_n$ be an orthonormal basis for R^n, then $Ax_1, \ldots Ax_n$ is an orthonormal basis for R^n too.

vii. Every permutation matrix P satisfies $P^2 = I$

viii. A matrix that satisfies $P^2 = I$ is a permutation matrix.

ix. Multiplication of permutation matrices is commutative.

x. Let P be a permutation matrix their determinant is always 1.

xi. If the matrix A has eigenvalues $2, 2, 5$ then the matrix is invertible.

xii. If Q is an orthogonal matrix then Q^{-1} is an orthogonal matrix

xiii. If A has eigenvalues $1, 1, 2$ then A is diagonalizable

xiv. If S is a matrix whose columns are linear independent eigenvectors of A then A is invertible

xv. If A is PSD and Q is orthogonal $Q^T AQ$ is PSD

xvi. If A is PSD and Q is orthogonal $Q^T AQ$ is diagonal

(f) The Singular Value Decomposition of a matrix is very important. Here are a few theoretical questions:

- What are the singular values of a $1 \times n$ matrix? Write down its singular value decomposition.
- Prove that if A is a square non-singular matrix then the singular values of the A^{-1} are the reciprocals of the singular values of A.
- Suppose A is an $m \times m$ matrix with SVD $A = U\Sigma V^T$. Use this to find the **singular value** decomposition of the $2m \times 2m$ matrix:

$$\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$$

(g) Find the best straight-line fit (least squares) to the measurements $b = 4$ at $t = -2$, $b = 3$ at $t = -1$, $b = 1$ at $t = 0$ and $b = 0$ at $t = 2$. The find the projection of $b = (4, 3, 1, 0)$ onto the column space of $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

(h) Let $f : R^n \leftrightarrow R^m$ be a linear map. Show how to compute the unique matrix such that $f(x) = Ax$ for every vector in R^n. If we think of the space of polynomials of degree less or equal to 5 as a vector space, its derivative is a linear map. If we think of the corresponding isomorphic R^n's, what is the matrix?

2. PROJECT (**linear algebra for ranking**): You are supposed to use the linear algebra methods to make a decision regarding choice of winners in elections and in rating the value of objects or services.

For the first part of the homework we have collected the ranking of 5 presidential candidates for more than 200 people. Using the ranking data available at

`https://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/rankingcandidates.dat`

write MATLAB code to predict the winner of the presidential election based on the following rank-aggregation (aka voting) methods. You will make a total of 8 predictions:

- Using *Plurality vote method* (voters top choice is counted for each candidate, winner has the most first-place votes.)
- Using *Average rank method* (in this case each of n candidates has been given a position from 1 to n by each of the voters, the integers representing the positions are averaged to create a rank-aggregated list. If ties occur, then pick a ranking list that appears most often as the "tie-breaking list". If i, j are in a tie, but in the list we choose i is ahead of j then that is the order.)
- *Borda count method* (For n candidates each voter awards his or her first choice candidate $n-1$ points, second choice $n-2$ points, and so on with 0 points for person last place. these are borda points. Winner is the candidate with the most total Borda points as awarded by the voters.
- *W-borda count method* (For a given vector $W = (w_1, w_2, \dots, w_n)$ with $w_1 \geq w_2 \geq \dots \geq w_n$, each voter awards his or her first choice candidate w_1 points, second choice w_2 points, and so on with w_n points for person last place. These are W-Borda points. Winner is the candidate with the most total W-borda points as awarded by the voters. Try five different vectors W making 5 predictions of the election. Can you choose them to make any of the five candidates win?
- Finally, use the Pagerank algorithm to rank the candidates.
- Report your predictions and compare the situation. Which is in your opinion the fairest method to count votes?

3. PROJECT **Using SVD's or a network to decide the ranking of difficulty:** An exam with m question is given to n students of MATH 16000. The clever instructor collects all grades in an $n \times m$ matrix G, with G_{ij} the grade obtained by student i on question j. The professor would like to assign a *difficulty score or rating* to each question based on the available data, rather than use the subjective perception of students.

    - From the theory of SVD's we know G can be decomposed as a sum of rank-many rank-one matrices. Suppose that G is approximated by a rank-one matrix sq^T with $s \in R^n$ and $q \in R^m$ with non-negative components. Can you use this fact to give a difficulty score or rating? What is the possible meaning of the vector s? Note one can use the top singular value decomposition to get this score vector!
    - There is another way to rank the difficulty of test questions using Networks: Each student gives scores for each problem, then we construct a network whose nodes/vertices are the problems of the exam. There is an arc from problem A to B for student k that did better in problem B than in problem A, (i.e., if s_A, s_B are the scores of those problems, $s_B > s_A$). The weight of the arc y_k associated to student k equals (approximately) the difference of score the student received in those two problems $s_B - s_A = y_k$. Explain why the Massey least square method we saw in class for rating movies can be use for rating exam problems by difficulty.
    - The data available at
      `https://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/examscores.dat`
      has the scores of 31 students in a 7 question exam (each problem was graded on a scale of 0 to 6). Use the data and give a rating of the difficulty of each question in the test using
        - SVD method.
        - Massey's network method.
        - Colley's method.

4. PROJECT **Using SVD to analyze images** Download the image called mandril.mat (available at `https://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/mandril.mat`) using the following MATLAB command (This loads a matrix X containing a face of a cute mandrill, and a map containing a colormap of the image.) *load mandrill;*

Display this matrix on your screen by:

```
>> image(X); colormap(map)
```

Then, attach it in your HW sheets.

(b) Compute the SVD of this mandrill image and plot the distribution of its singular values on your screen (Note that the MATLAB `svd` function returns three matrices U, S, V for a given input matrix. So, the singular values are nicely plotted by:

```
>> stem(diag(S)); grid
```

Then print this figure and attach it in your HW sheets.

(c) Let σ_j, $\boldsymbol{u}_j$, $\boldsymbol{v}_j$ be the jth singular value, the jth left and right singular vectors of the mandrill image, respectively. In other words, they are `S(j,j)`, `U(:,j)`, `V(:,j)` of the SVD of X in MATLAB. Let us define the rank k approximation of the image X as

$$X_k := \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^\top + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^\top.$$

Then, for $k = 1, 6, 11, 31$, compute X_k of the mandrill, and display the results. Fit these four images in one page by using subplot function in MATLAB (i.e., use `subplot(2,2,1)` to display the first image, `subplot(2,2,2)` to display the second image, etc.)

(d) For $k = 1, 6, 11, 31$, display the residuals, i.e., $X - X_k$, fit them in one page, print them, and attach that page in your HW sheets.

(e) For $k = 1, 6, 11, 31$, compute $\|X - X_k\|_2$ by the `norm` function of MATLAB. Then, compare the results with σ_{k+1}. More precisely, compute the relative error and report the results:

$$\frac{|\sigma_{k+1} - \|X - X_k\|_2\,|}{\sigma_{k+1}}.$$