MATH (160)

Mathematics for Data Analytics and Decision Making

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UC Davis, Mathematics

Monday, March 28, 2011

Syllabus, Class Schedule, Office Hours, Textbook, etc.

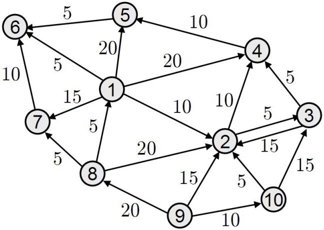
See course website.

`https://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/`

Also I will use your UC Davis email for announcements and quizzes!

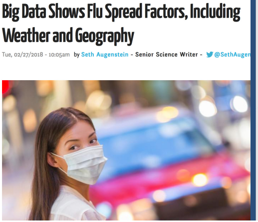
MATHEMATICAL MODELS!!!
That use data to make intelligent decisions!!

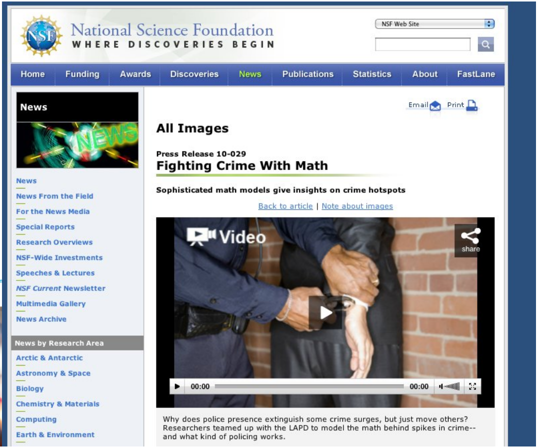
- How does Google's search engine and GPS routing work?



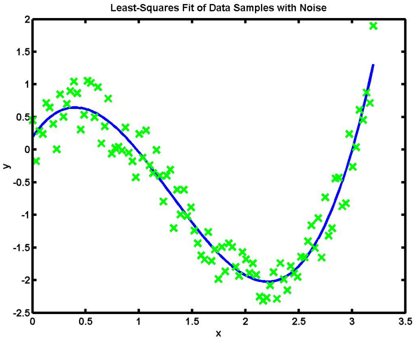
- The engine behind the engines is all MATHEMATICS!!!

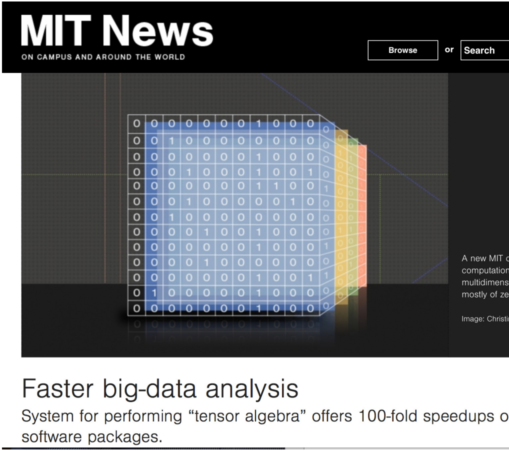
Today the world has abundant data of all sorts, we need to go FROM DATA TO DECISIONS!





and MATH is the only way we can analyze it to extract relevant information.





By the end of this course, you should make BETTER (quantitative) DECISIONS!!

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MULTITRANS is a fictional bus company that is run by a student association in an unnamed college town with partial funding from the city.

(It started out in 1972 with a fleet of historic triple-deck buses from Moscow, but has added more modern buses since.)

During most of the day, though, all scheduled buses are completely empty.

A scientific study was conducted in 2010 to find out why this is the case. The result of the study was:

> *[...] 85% of the representative sample of 1256 potential users of MULTITRANS replied that "I would use it regularly instead of using my bike or car, but the MULTITRANS schedule [is suboptimal]." [...]*

MULTITRANS hires YOU as a consultant to help them improve the schedule.

What are the next steps you need to take to solve the challenge?

Part I of the course:

LINEAR ALGEBRA MODELS

Least-squares applications to modeling.

What is the best team? What is the best candidate? etc: The science of ranking

-Say we are given data about teams with game scores, how could we predict the best team of the season?
-Say we are given voters information about candidates, how can we rank them? (arrange them in order of importance).
-Say we are given a list of movies with user preferences (prefer to star wars v.s. star trek?)

Goal

We will see five ways to make predictions of WHO IS NUMBER ONE using data of comparisons and solving linear equations.

The models depend on how much information we have on the situation. We hope that using these five methods allow users to determine who is the most viable candidate and show the user that different methods may provide different results.

Two old “COUNTING” methods

Plurality

Candidate rated number one is the one with the most wins or the most votes.

The plurality method was originally used in determining a winner in voting systems. It soon became one of the most commonly used methods for selecting a winner.
But what if we knew an ordered list of candidates, one for each voter? We could use more sophisticated methods!! Borda Count is a method that dates back to 1770

Borda Count

Count the number of times a candidate appears in *i*th position and apply the following set of weights as follows (here n is the total number of candidates)
$(n-1)$ points are given to first place, $(n-k)$ to k-th place, and 0 is given to last place. The tallies are summed and the winner is the candidate with the highest sum.

Voter	HC	BS	JK	TC	DT
Voter 1	5	4	3	2	1
Voter 2	5	4	3	2	1
Voter 3	4	5	3	1	2
Voter 4	2	5	1	3	4
Voter 5	3	5	2	1	4
Voter 6	4	5	2	1	3
Voter 7	1	5	3	4	2
Voter 8	5	4	3	1	2
Voter 9	4	5	2	3	1
Voter 10	4	5	2	1	3

Table: Data set (only 10 voters).

1st Place votes	HC	BS	JK	TC	DT
Votes	3	7	0	0	0

Table: Plurality Scores.

	HC	BS	JK	TC	DT
Average Rating	3.7	4.7	2.4	1.9	2.3

Table: Average Scores.

Ranking	HC	BS	JK	TC	DT
1st Place	3	7	0	0	0
2nd Place	4	3	0	1	1
3rd Place	1	0	5	2	2
4th Place	1	0	4	2	4
5th Place	1	0	1	5	3

Table: Borda Rankings.

Candidate	Score
Hillary Clinton	27
Bernie Sanders	37
John Kasich	14
Ted Cruz	9
Donald Trump	11

Table: Borda Scores.

Massey Ranking

- The fundamental idea: Find the rankings or score values of team r_i by noting $r_i - r_j = y_{ij}$ where y_{ij} is the margin of victory of team i over j .
- For each game played there is one such equation, and then there is a system of linear equations. The matrix X is $m \times n$ with $m =$number of games played and $n =$ number of teams.

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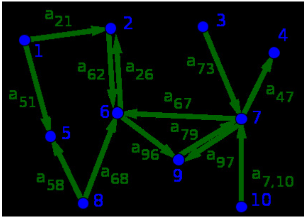
- Each row of the matrix X is nearly all zeros, except a 1 and a -1.
- It is the transpose of a **node-arc incidence matrix of a network** Typically the number of games is very large, larger than the number of teams.



- Note: y_{ij} can be interpreted in many ways: can mean traffic levels, to rank websites, y_{ij} can be number of in-links or outlinks.

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- Recall, solving the least squares problem $Xr = y$ same as solving the system of equations: $X^T Xr = Mr = X^T y = d$.

$$\begin{pmatrix} t_1 & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & t_2 & \cdots & -m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & -m_{n2} & \cdots & t_n \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

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- Outputs the ratings of candidates by approximately solving the system of linear equalities (or least-squares!). We are trying to find the ratings r_i.

	HC	BS	JK	TC	DT
HC	40	-10	-10	-10	-10
BS	-10	40	-10	-10	-10
Jk	-10	-10	40	-10	-10
TC	-10	-10	-10	40	-10
DT	-10	-10	-10	-10	40

Table: Massey Matrix.

	HC	BS	JK	TC	DT
HC	40	-10	-10	-10	-10
BS	-10	40	-10	-10	-10
Jk	-10	-10	40	-10	-10
TC	-10	-10	-10	40	-10
DT	-10	-10	-10	-10	40

Table: Massey Matrix.

Games	HC	BS	JK	TC	DT
1 - 4 (V_1)	10	5	0	-5	-10
5 - 8 (V_2)	10	5	0	-5	-10
9 - 12 (V_3)	5	10	0	-10	-5
13 - 16 (V_4)	-5	10	-10	0	5
17 - 20 (V_5)	0	10	-5	-10	5
21 - 24 (V_6)	5	10	-5	-10	0
25 - 28 (V_7)	-10	10	0	5	-5
29 - 32 (V_8)	10	5	0	-10	-5
33 - 36 (V_9)	5	10	-5	0	-10
37 - 40 (V_{10})	5	10	-5	-10	0
Total	35	85	-30	-55	-35

Table: Candidates' Point Differentials: how many times candidate won or lost to another

	HC	BS	JK	TC	DT
HC	40	-10	-10	-10	-10
BS	-10	40	-10	-10	-10
Jk	-10	-10	40	-10	-10
TC	-10	-10	-10	40	-10
DT	-10	-10	-10	-10	40

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13 - 16 (V_4)	-5	10	-10	0	5
17 - 20 (V_5)	0	10	-5	-10	5
21 - 24 (V_6)	5	10	-5	-10	0
25 - 28 (V_7)	-10	10	0	5	-5
29 - 32 (V_8)	10	5	0	-10	-5
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$$\begin{bmatrix} 40 & -10 & -10 & -10 & -10 \\ -10 & 40 & -10 & -10 & -10 \\ -10 & -10 & 40 & -10 & -10 \\ -10 & -10 & -10 & 40 & -10 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \begin{bmatrix} 35 \\ 85 \\ -30 \\ -55 \\ 0 \end{bmatrix}$$

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$\vec{r}$ =
$$\begin{bmatrix} 0.700 \\ 1.700 \\ -0.600 \\ -1.100 \\ -0.700 \end{bmatrix}$$
The largest rating of 1.700 corresponds to Bernie Sanders. Therefore, Bernie Sanders is the winner for Massey's Method.

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Rank	Candidate
1	Bernie Sanders
2	Hillary Clinton
3	John Kasich
4	Donald Trump
5	Ted Cruz

Table: Ranked Candidates.

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- Intuitively all teams start with score $1/2$ at the beginning and things change. Denote l_i number of games lost by i.

$$w_i = \frac{w_i - l_i}{2} + \frac{w_i + l_i}{2} = \frac{w_i - l_i}{2} + \frac{t_i}{2} = \frac{w_i - l_i}{2} + \sum_{j=1}^{t_i} 1/2$$

Thus

$$\sum_{j=1}^{t_i} 1/2 \sim \sum_{j \text{ opponent of } i} r_j \quad \text{and} \quad w_i \sim \frac{w_i - l_i}{2}$$

Colley's method continued

- We get

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- Rewriting in terms of linear equations and solving for the ratings r_k's, we get a matrix C

$$C = \begin{pmatrix} t_1 + 2 & -n_{12} & \cdots & -n_{1n} \\ -n_{21} & t_2 + 2 & \cdots & -n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -n_{n1} & -n_{n2} & \cdots & t_n + 2 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

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- Where we are trying to find the rating r_i and t_i is the total number of games, $-n_{ij}$ is the number of games team i played against j and $b_i = 1 + \frac{1}{2}(w_i - l_i)$.

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- The matrix C is a real symmetric matrix and $Cr = b$ is the system we need to solve.
- In the example of candidates: Hillary Clinton played forty games in total: ten games against Bernie Sanders, ten games against John Kasich, ten games against Ted Cruz, and ten games against Donald Trump. Each voter rated five candidates which is equivalent to four games since each candidate was indirectly being compared to the other four candidates.

	HC	BS	JK	TC	DT
HC	42	-10	-10	-10	-10
BS	-10	42	-10	-10	-10
Jk	-10	-10	42	-10	-10
TC	-10	-10	-10	42	-10
DT	-10	-10	-10	-10	42

Table: C Matrix.

Games	HC	BS	JK	TC	DT
Won	27	37	14	9	13
Lost	13	3	26	31	27
Total	40	40	40	40	40

Table: Candidate's Scoreboard.

$$\begin{bmatrix} 42 & -10 & -10 & -10 & -10 \\ -10 & 42 & -10 & -10 & -10 \\ -10 & -10 & 42 & -10 & -10 \\ -10 & -10 & -10 & 42 & -10 \\ -10 & -10 & -10 & -10 & 42 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \\ -5 \\ -10 \\ -6 \end{bmatrix}$$

After inputting all the matrices into MATLAB we use $\vec{r}$ = linsolve(C, $\vec{b}$) to get:
$\vec{r}$ =

$$\begin{bmatrix} 0.6346 \\ 0.8269 \\ 0.3846 \\ 0.2885 \\ 0.3654 \end{bmatrix}$$

Is the solution unique? Why?

The largest rating of 0.8269 corresponds to Bernie Sanders. Therefore, Bernie Sanders is the winner for Colley's method.

A classic application: Regression Analysis

- We have data points $x_1, x_2, \ldots, x_m$ each has n input variables x_{ij}, also called **features**.
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- When y can take on discrete values (e.g., want to predict if a dwelling is a house or an apartment), we call it a **classification problem**.
  To perform supervised learning, we must decide how to represent functions/hypotheses h in a computer!

Basic Regression analysis

- LINEAR CHOICE: Approximate y as a linear function of x:

$$h(x) = \sum_{k}^{n} w_i x_{ik} = w^T x_i$$

Basic Regression analysis

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- Formally we wish to find a function that measures, for each value of the w, how close the $h(x_i)$'s are to the corresponding y_i's.
  We define the cost function that minimizes the error.

$$\min \sum_{i}^{m} (h(x_i) - y_i)^2 = \min_{w \in \mathbf{R}^m} ||X^T w - \hat{y}||_2$$

- We use the notation X to denote the $m \times n$ matrix whose rows are the data points x_i. Let $\hat{y}$ the row vector with y_i's. This yields the traditional LEAST SQUARES PROBLEM:
- **Proposition** The optimal solution is the solution of a linear system of equations

$$X^T X w = X^T \hat{y}$$

Polynomial Regression

- We may add nonlinear relations on the features by considering a polynomial of degree M over n variables as the learning function:

$$h(x) = \sum_{a_1+a_2+\cdots+a_n \leq M} c_{i_1,i_2,\ldots,i_n} x_{i1}^{a_1} x_{i2}^{a_2} \cdots x_{in}^{a_n}$$

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- It might seem that the more features we add, the better. However, there is a danger in adding too many features! We have to be careful with **underfitting** or **overfitting**.

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- **Theorem** An $n \times n$ matrix A has n eigenvalues (with multiplicity).

Two key results

Theorem

Let λ_i *for* $i = 1 \dots k$ *the distinct eigenvalues of a matrix* A*. Let* Φ_i *be the* **eigenspace** *of* λ_i*. Then a set of eigenvectors, one for each eigenspace is a set of linearly independent vectors.*

Theorem

Let A *have* n *distinct eigenvalues* λ_i *for* $i = 1 \dots k$*. Let* μ_i *the* **algebraic multiplicity** *of* λ_i*. Moreover, let* Φ_i *the eigenspace of* λ_i *and* $v_i = dim(\Phi_i)$*. Let* $U^{(i)}$ *be the matrix containing as columns a basis of* Φ_i *and* $U = [U^{(1)} U^{(2)} \cdots U^{(k)}]$*. Then*

- $v_i \leq \mu_i$.
- *if* $v_i = \mu_i$ *then* A *is* **diagonalizable***:*

$$A = U \Lambda U^{-1}$$

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- The next stage of the evolution of the process will be given by a **Transition matrix** T.
- Then $u^{(k+1)} = Tu^{(k)}$, with $u^{(0)}$ being the initial state at time zero (MARKOV CHAIN)

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- The equation $u^{(k+1)} = Tu^{(k)}$ represents a chain of probability vectors.
- T having non-zero entries means there is a positive probability to from A to B in the system (matrix is regular!)
- **Theorem (Perron-Frobenius)** If T is a regular transition matrix, then it admits a unique probability eigenvector u^* with eigenvalue $\lambda = 1$. Moreover $Tu^{(k)} \longrightarrow u^*$ as k increases.

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- So eigenvalues can be used to predict dynamic behavior.

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- Intuitively irreducible means you cannot get stuck while walking the network.

What if A is not irreducible? Modify the matrix to be

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Determine the final ratings by solving for the dominant eigenvector $\vec{r}$ (associated with eigenvalue 1) in the following irreducible matrix,

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NOTE: For games of basketball or voting A is a stochastic matrix with entries composed of 0's and $\frac{1}{j_i}$'s where j_i represents that number of loses candidate i obtained (pointing an arrow to someone means you lost to him/her).
If candidate i is undefeated replace the $\mathbf{0}^T$ with $\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$ where n is the number of candidates.

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$A^k(r) \to t$ when $k \to \infty$ converges to the 1 eigenvalue we need. WHY??

- Given an $m \times n$ matrix A, its **singular values** are the positive square roots of the eigenvalues of the matrix $A^T A$ (the **Gram matrix**). The corresponding eigenvectors are called the singular vectors of A.
- By Cholesky's decomposition $A^T A$ is positive semidefinite its eigenvalues are always non-negative and real.
- **Example** Say A is a the square matrix $\begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$ Thus $A^T A = \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix}$ This matrix has eigenvalues 40 and 10. and the corresponding eigenvectors are $(1,1)^T$ and $(1,-1)^T$. Thus the singular values are $\sigma_1 = \sqrt{40} = 6.3246$ and $\sigma_2 = \sqrt{10} = 3.1623$. Note singular values are NOT the same as the eigenvalues of A.

- There are a number of properties we will need to understand the importance of the singular values
- **Theorem** Every real symmetric matrix A can be diagonalized by an orthogonal matrix Q, i.e., $A = Q\Lambda Q^T$ where Q is orthogonal and Λ is a diagonal matrix.
- **Proposition** If $A^T = A$ then its singular values are the absolute values of its non-zero eigenvalues and its singular vectors coincide with the associated non-null eigenvectors.

- **Theorem** Any non-zero real $(m \times n)$ matrix A of rank $r > 0$ can be factored

$$A = P \Sigma Q^T$$

  - P is an $m \times r$ matrix with orthonormal columns ($P^T P = I$).
  - Σ is an $r \times r$ diagonal matrix with the singular values of A in the diagonal.
  - Q^T is an $r \times n$ matrix with orthonormal rows ($Q^T Q = I$).
- **Proof** Same as proving $AQ = P\Sigma$. So find orthonormal vectors $q_1, \ldots, q_r$ for Q and $p_1, \ldots, p_r$!!!
- Let $q_1, \ldots, q_r$ be the orthonormal eigenvectors of Gram matrix $A^T A$ corresponding to the **non-zero eigenvalues**.

$$A^T A q_i = \lambda_i q_i = \sigma_i^2 q_i$$

- **Claim 1:** $w_i = Aq_i$ are orthogonal.
- **Claim 2:** $\|w_i\| = \sqrt{w_i^T w_i} = \sigma$
- **Claim 3:** $p_i = \frac{w_i}{\sigma_i} = \frac{Aq_i}{\sigma}$.

- **Example (continued)** For $A = \begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$, the orthogonal eigenvector basis of A^TA is $q_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T, q_2 = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$

- Thus, following the method of the algorithm, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$

- Thus $A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

- **Theorem** Any non-zero real $m \times n$ A can be factored

$$A = P\Sigma Q^T$$

  where
  - P is an $m \times m$ matrix with orthonormal columns ($P^T P = I$).
  - Σ is an $m \times n$ diagonal matrix with the singular values of A in the diagonal.
  - Q is an $n \times n$ matrix with orthonormal rows ($Q^T Q = I$).
- WHAT IS THE GEOMETRIC MEANING?
- Let us look at what happens to a unit sphere?

Theory Consequences of SVD decomposition:

- **Theorem** Let $A = P\Sigma Q^T$ be the SVD of A.
  Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n$ be the singular values (first r non-zero!) Then
  - The (left) singular vectors $p_1, p_2, \ldots, p_r$ for an orthonormal basis in the range of A and $rank(A) = r$
  - The (right) singular vectors $q_{r+1}, q_{r+2}, \ldots, q_n$ are an orthonormal basis in $N(A)$ and $dim(N(A)) = n - r$.
  - The (right) singular vectors $q_1, q_2, \ldots, q_r$ are an orthonormal basis in the range of A^T.
  - The (left) singular vectors $p_{r+1}, p_{r+2}, \ldots, p_m$ are an orthonormal basis for $N(A^T)$.
  - Moreover, he columns of P are the eigenvectors of AA^T and the columns of Q are the eigenvectors of $A^T A$.
  - The r singular values on the diagonal of σ are the square roots of the non-zero eigenvalues of both $A^T A$ and AA^T.
- **Proposition:** The largest singular value is equal to the 2-norm of A.
- The Frobenius norm of a matrix A, (denoted $\|A\|_F$ is $\sqrt{\sum_{i,j} a_{i,j}^2}$ or $\sqrt{Tr(A^T A)}$. We have:
- The ratio $\sigma_{\mathbf{max}} / \sigma_{\mathbf{min}}$ is the **Condition number**.

WHY CARE? Applications: Numerics, Image Compression, Data Analysis

- The ratio $\sigma_{\mathbf{max}}/\sigma_{\mathbf{min}}$ is the **Condition number**. A matrix with very large condition number is **ill-conditioned**. This is trouble for calculations when the condition number is larger than the reciprocal of the machine precision.
- The singular vectors indicate the **principal components**. One key idea is to discard singular values and vectors that are too small
- The columns of the matrix A represent data vectors (normalized to have mean 0). The matrix A^TA can be thought of as the **covariance matrix**. Its eigenvectors indicate directions of correlation and clustering in the data.
- For Images, instead of sending a 1000×1000 pixels, compute SVD, $A = P\Sigma Q^T = p_1\sigma_1 q_1^T + p_2\sigma_2 q_2^T + \cdots + p_r\sigma_r q_r^T$, we see A is a sum of rank 1 matrices!
  we only keep those pieces with σ_i of "good" size, throw away rest.

Low Rank Approximations

- Given two vectors u, v, What kind of matrix is (uv^T)?
- It is a **rank one** matrix!

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} = \begin{pmatrix} u_1 v_1 & \dots & u_1 v_n \\ \vdots & & \vdots \\ u_n v_1 & \dots & u_n v_n \end{pmatrix}$$

- **Corollary:** A matrix A of rank r can be written as a sum of r, rank one matrices using the SVD decomposition: $A = \sum_{i=1}^{n} \sigma_i (u_i v_i^T)$.
- **Theorem:** For the $m \times n$ matrix A call the truncated sum of matrices $A_k = \sum_{i=1}^{k} \sigma_i (u_i v_i^T)$. Then the optimization problem:

$$\text{Minimize}_{B\ m\times n, rank(B) \leq k} \|A - B\|_2 = \sigma_{k+1}$$

It is achieved by $B = A_k$. Same holds for the Frobenius norm.

Applications: Image Compression

- **key idea for image compression:** discard singular values and vectors that are too small
- For images, instead of sending a 1000×1000 pixels over the internet.
- compute SVD,

$$A = P\Sigma Q^T = p_1\sigma_1 q_1^T + p_2\sigma_2 q_2^T + \cdots + p_r\sigma_r q_r^T$$

We see A is a sum of rank 1 matrices, keep those pieces with σ_i of "good" size, throw away rest.

Principal Component Analysis of Data

- **Goal:** Discover the most important *directions* where data varies the most!!
- **Answer:** The (left) singular vectors are principal components. Why?
  NOTE: If not mean zero, simply translate to the average $\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$. Assume data is centered!!
- **WISH** Find z with $||z||_2 = 1$ such that the variance of the projection of the data points onto the line defined by z
- Components of the data along z are given by $a_i = x_i^T z$.
- The **mean square variation of data** equals:

$$\frac{1}{m}\sum_{i=1}^{m} a_i^2 = \sum_{i=1} mz^T x_i x^T z = z^T AA^T z$$

- Desired direction is z that maximizes:

$$\max_{z \in R^n} z^T AA^T z, \quad ||z||_2 = 1$$

  But by SVD equals

$$\max_{z \in R^n} z^T P_r \Sigma^2 P_r^T z, \quad ||z||_2 = 1$$

- Optimal value $z = p_1$ first column of P and largest variation is σ_1^2