# MA5206
# HOMEWORK 2

BY

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**Exercise 1** (8.6)**.** *Show that the closed convex hull $\breve{M}$ of $M$ is the closure of the convex hull of $M$.*

*Proof.* Note that the convex hull $\hat{M}$ of $M$ is the smallest convex set containing $M$, so $\hat{M} \subset \breve{M}$ since $\breve{M}$ is convex. Also, the closure of $\hat{M}$ is the smallest closed set containing $\hat{M}$, so $\mathrm{cl}(\hat{M}) \subset \breve{M}$ since $\breve{M}$ is closed. Then because $\breve{M}$ is the intersection of all closed convex sets containing $M$, showing $\mathrm{cl}(\hat{M})$ is convex will imply $\breve{M} \subset \mathrm{cl}(\hat{M})$.

Let $x \in \mathrm{cl}(\hat{M})$ and $y \in \mathrm{cl}(\hat{M})$. Then there exist sequences $(x_n)$ and $(y_n)$ in $\hat{M}$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then for all $\lambda \in [0,1]$, $\lambda x_n + (1-\lambda)y_n \in \hat{M}$ for all $n \in \mathbb{N}$ since $\hat{M}$ is convex. Then

$$\begin{aligned}\|\lambda x_n + (1-\lambda)y_n - (\lambda x + (1-\lambda)y)\| &= \|\lambda(x_n - x) + (1-\lambda)(y - y_n)\| \\ &\le \|\lambda(x_n - x)\| + \|(1-\lambda)(y-y_n)\| \\ &= \lambda\|x - x_n\| + (1-\lambda)\|y - y_n\| \to 0\end{aligned}$$

as $n \to \infty$. This shows the sequence $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$ as $n \to \infty$ for each $\lambda \in [0,1]$, and since $\mathrm{cl}(\hat{M})$ is closed, $\lambda x + (1-\lambda)y \in \mathrm{cl}(\hat{M})$ for all $\lambda \in [0,1]$. Hence $\mathrm{cl}(\hat{M})$ is convex.
Therefore since $\breve{M} \subset \mathrm{cl}(\hat{M}) \subset \breve{M}$, $\breve{M} = \mathrm{cl}(\hat{M})$. □

**Exercise 2** (8.7)**.** *For any bounded subset $M$ of a normed linear space $X$ over $\mathbb{R}$, prove that the support function $S_M : X' \to \mathbb{R}$ defined by*

$$S_M(\ell) = \sup_{y\in M} \ell(y)$$

*has the following properties:*

*(i) Subadditivity: for all $\ell, m \in X'$, $S_M(\ell + m) \le S_M(\ell) + S_M(m)$.*
*(ii) $S_M(0) = 0$.*
*(iii) Positive homogeneity: $S_M(a\ell) = aS_M(\ell)$ for $a > 0$.*
*(iv) Monotonicity: for $M \subset N$, $S_M(\ell) \le S_N(\ell)$.*
*(v) Additivity: $S_{M+N} = S_M + S_N$.*
*(vi) $S_{-M}(\ell) = S_M(-\ell)$.*
*(vii) $S_{\overline{M}} = S_M$.*
*(viii) $S_{\hat{M}} = S_M$.*

*Proof.* Let $\ell \in X'$.

(i) Let $m \in X'$, and let $(y_n)$ be a sequence in $M$ such that $\ell(y_n) + m(y_n) \to S_M(\ell + m)$. By definition, $\ell(y_n) \le S_M(\ell)$ and $m(y_n) \le S_M(m)$ for all $n \in \mathbb{N}$. Then $\ell(y_n) + m(y_n) \le S_M(\ell) + S_M(m)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $S_M(\ell + m) \le S_M(\ell) + S_M(m)$.

(ii) $0(y) = 0$ for all $y \in M$. Thus $S_M(0) = \sup_{y\in M} 0 = 0$.

(iii) By linearity, $\ell(ay) = a\ell(y)$ for all $a \in \mathbb{R}$. Then for $a > 0$,

$$S_M(a\ell) = \sup_{y\in M} \ell(ay) = \sup_{y\in M} a\ell(y) = a \sup_{y\in M} \ell(y) = aS_M(\ell).$$

(iv) Let $M \subset N$. Any sequence $(y_n)$ in $M$ for which $\ell(y_n) \to S_M(\ell)$ as $n \to \infty$ is also a sequence in $N$. Hence

$$S_M(\ell) = \sup_{y\in M} \ell(y) = \lim_{n\to\infty} \ell(y_n) \le \sup_{y\in N} \ell(y) = S_N(\ell).$$

(v) Note that

$$\ell(y) = \ell(y_1) + \ell(y_2) \le \sup_{z_1\in M} \ell(z_1) + \sup_{z_2\in N} \ell(z_2)$$

for all $y = y_1 + y_2 \in M + N$, where $y_1 \in M$ and $y_2 \in N$. On the other hand,

$$\ell(y_1) + \ell(y_2) = \ell(y_1 + y_2) \le \sup_{y\in M+N} \ell(y)$$

for all $y_1 \in M$ and $y_2 \in N$. Hence

$$S_{M+N}(\ell) = \sup_{y \in M+N} \ell(y) = \sup_{y \in M} \ell(y) + \sup_{y \in N} \ell(y) = S_M(\ell) + S_N(\ell)$$

for all $\ell \in X'$. Therefore $S_{M+N} = S_M + S_N$.

(vi) Note that by linearity, $-\ell(y) = \ell(-y)$ for all $y \in M$. Then since $y \in M$ if and only if $-y \in -M$, we have

$$S_M(-\ell) = \sup_{y \in M} -\ell(y) = \sup_{y \in M} \ell(-y) = \sup_{z \in -M} \ell(z) = S_{-M}(\ell).$$

(vii) Since $M \subseteq \overline{M}$, from (iv) we have $S_M(\ell) \leq S_{\overline{M}}(\ell)$. Conversly, let $y \in \overline{M}$. Then there exists $(y_n)$ in $M$ suct that $y_n \to y$ as $n \to \infty$. Since $\ell$ is continuous, we have

$$\ell(y) = \lim_{n \to \infty} \ell(y_n) \leq \sup_{z \in M} \ell(z) = S_M(\ell).$$

Therefore, $S_{\overline{M}}(\ell) \leq S_M(\ell)$. Thus, $S_{\overline{M}}(\ell) = S_M(\ell)$.

(viii) Since $M \subseteq \hat{M}$, from (iv) we have $S_M(\ell) \leq S_{\hat{M}}(\ell)$. Conversly, let $y \in \hat{M}$. Then, there exist $y_1, y_2, \ldots, y_n \in M$ and $a_1, a_2, \ldots, a_n \in [0,1]$ such that $\sum_{i=1}^{n} a_i = 1$ and $y = \sum_{i=1}^{n} a_i y_i$. Since $\ell$ is linear, we have

$$\ell(y) = \sum_{i=1}^{n} a_i \ell(y_i) \leq \sum_{i=1}^{n} a_i \sup_{z \in M} \ell(z) = \sup_{z \in M} \ell(z) = S_M(\ell).$$

Therefore, $S_{\hat{M}}(\ell) \leq S_M(\ell)$. Thus, $S_{\hat{M}}(\ell) = S_M(\ell)$.

This concludes the proof.

*Remark*: From (vii) and (viii), we also have $S_{\check{M}} = S_{\overline{\hat{M}}} = S_{\hat{M}} = S_M$. □

**Exercise 3** (10.5)**.** *Prove that every weakly sequentially compact set is bounded.*

*Proof.* Let $C$ be a weakly sequentially compact subset of a Banach space $X$. Suppose $C$ is not bounded. Then there exists a sequence $(x_n)$ in $C$ such that for every $N \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $n > m$ implies $\|x_n\| > N$. Note that this property holds for every subsequence of $(x_n)$ as well.

Since $C$ is weakly sequentially compact, there exists a subsequence $(y_n)$ of $(x_n)$ that is weakly convergent. Recall that we have shown every weakly convergent sequence in a normed linear space is uniformly bounded in the norm. Then there exists $c > 0$ such that $\|y_n\| < c$ for all $n \in \mathbb{N}$. On the other hand, there exists $m \in \mathbb{N}$ such that $n > m$ implies $\|y_n\| > c$. Thus $n > m$ implies $c < \|y_n\| < c$, a contradiction. Therefore every weakly sequentially compact set is bounded. □

**Exercise 4** (10.6)**.** *Prove that if the sequence $\{u_n\}$ is weak* convergent to $u$, then*

$$\|u\| \leq \liminf \|u_n\|.$$

*Proof.* We make a simple calculation:

$$\|u\| = \sup_{\|x\|=1} |u(x)| = \sup_{\|x\|=1} \lim_{n \to \infty} |u_n(x)| \leq \sup_{\|x\|=1} \liminf \|u_n\| \|x\| = \liminf \|u_n\|.$$

Therefore if $\{u_n\}$ is weak* convergent to $u$, then $\|u\| \leq \liminf \|u_n\|$. □

**Exercise 5** (Ex 1)**.** *Let $f$ be a continuous function on $[a,b] \times (-\infty, \infty)$, $y_0 \in \mathbb{R}$. Consider the following operator on $C[a,b]$:*

$$Tu(x) = y_0 + \int_a^x f(t, u(t))dt, \quad \text{for } x \in [a,b].$$

*Show that $T$ is continuous and maps bounded sets in $C[a,b]$ to precompact sets in $C[a,b]$.*

*Proof.* Note that $C[a,b]$ is a norm space with supremum norm $(\|.\|_\infty)$. First, we show that $T$ is continuous on $C[a,b]$. Let $u \in C[a,b]$ and $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $C[a,b]$ that converges to $u$. Then, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $C[a,b]$. Futhermore, there exists $M > 0$ such that $\|u\|_\infty \leq M$ and $\|u_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Note that $f$ is continuous on $[a,b] \times [-M, M]$, which is closed and bounded in $\mathbb{R}^2$ and hence compact by the Heine-Borel theorem. Thus we may apply the Heine-Cantor theorem to conclude $f$ is uniformly continuous on $[a,b] \times [-M, M]$.

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$$

for all $(x_1, y_1), (x_2, y_2) \in [a, b] \times [-M, M]$ with $\|(x_1, y_1) - (x_2, y_2)\|_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \delta$. Since $u_n \to u$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\|u_n - u\|_\infty < \delta$ for all $n \geq n_0$. Then, for all $n \geq n_0$ and $t \in [a, b]$, we have

$$\|(t, u_n(t)) - (t, u(t))\|_2 = |u_n(t) - u(t)| \leq \|u_n - u\|_\infty < \delta,$$

and therefore

$$\begin{aligned} |Tu_n(x) - Tu(x)| &= \left| \int_a^x f(t, u_n(t))dt - \int_a^x f(t, u(t))dt \right| \\ &= \left| \int_a^x [f(t, u_n(t)) - f(t, u(t))]dt \right| \\ &\leq \int_a^x |f(t, u_n(t)) - f(t, u(t))| \, dt \\ &< \epsilon(x - a) \leq \epsilon(b - a) \end{aligned}$$

for all $x \in [a, b]$. Hence, $\|Tu_n - Tu\|_\infty \leq \epsilon(b - a)$ for all $n \geq n_0$. That means $Tu_n \to Tu$ as $n \to \infty$. Thus, $T$ is continuous on $C[a, b]$.

Next, we show that $T$ maps bounded set in $C[a, b]$ to precompact set in $C[a, b]$. It suffices to show that for every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $C[a, b]$, the sequence $\{Tu_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $C[a, b]$. Then, there exists $M_0 > 0$ such that $\|u_n\|_\infty \leq M_0$ for all $n \in \mathbb{N}$. Since, $[a, b] \times [-M_0, M_0]$ is compact and $f$ is continuous on $[a, b] \times [-M_0, M_0]$, $f$ is uniformly continuous and bounded on $[a, b] \times [-M_0, M_0]$. That means there exists $M_1 > 0$ such that $f$ is bounded by $M_1$ on $[a, b] \times [-M_0, M_0]$. Then we have

$$|Tu_n(x)| \leq |y_0| + \int_a^x |f(t, u_n(t))|dt \leq |y_0| + M_1(b - a)$$

for all $n \in \mathbb{N}$ and $x \in [a, b]$. Futhermore, for any $\epsilon > 0$,

$$|Tu_n(x) - Tu_n(y)| \leq \left| \int_a^x f(t, u_n(t))dt - \int_a^y f(t, u_n(t))dt \right| \leq M_1|x - y| < \epsilon$$

for all $x, y \in [a, b]$ with $|x - y| < \dfrac{\epsilon}{M_1}$ and $n \in \mathbb{N}$. Hence, $\{Tu_n\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous on $[a, b]$. By Arzela-Ascoli Theorem, $\{Tu_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $C[a, b]$. Therefore the continuous map $T$ maps bounded sets in $C[a, b]$ to precompact sets in $C[a, b]$. □

**Exercise 6** (Ex 2)**.** *Let $\Omega$ be a bounded open connected set in $\mathbb{R}^3$. Let $A = [a_{ij}]$ be a symmetric and constant matrix that has only positive eigenvalues, $b_i \in L^3(\Omega)$ for $i \in \{1, 2, 3\}$, and let*

$$L(u) = \sum_{i=1,j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(b_i(x)u).$$

*Show that $Lu = f$ on $\Omega$, $u = 0$ on $\partial\Omega$ has a (weak) solution in $W_0^{1,2}(\Omega)$ for any $f \in L^2(\Omega)$.*

*Proof.* Define a function $T : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ by

$$T(u, v) = \int_\Omega A\nabla u \nabla v dx + \int_\Omega \sum_{i=1}^{3} b_i(x)u \frac{\partial v}{\partial x_i} dx.$$

Integrating by parts shows in fact

$$T(u, v) = -\int_\Omega v \sum_{i=1,j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} dx - \int_\Omega \sum_{i=1}^{3} v \frac{\partial}{\partial x_i}(b_i(x)u)dx.$$

Certainly by linearity of the integral and derivative, $T(u, v)$ is linear in $v$ for fixed $u$ and linear in $u$ for fixed $v$. Define a functional $\ell : W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$\ell(v) = -\int_\Omega fv dx$$

for all $v \in W_0^{1,2}(\Omega)$. It's clear by Hölder's inequality that $\ell$ is a linear and bounded with respect to the norm

$$\|v\|_{W_0^{1,2}(\Omega)} = (\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2)^{\frac{1}{2}}, \text{ where } \|\nabla v\|_{L^2(\Omega)} = \left( \sum_{i=1}^{3} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Note that if $T(u,v)$ is bounded and coercive, then by Lax-Milgram Lemma there exists $u \in W_0^{1,2}(\Omega)$ such that $T(u,v) = \ell(v)$ for all $v \in W_0^{1,2}(\Omega)$, and therefore $Lu = f$, $u = 0$ on $\partial\Omega$ has a weak solution in $W_0^{1,2}(\Omega)$. Thus it suffices to show that $T$ is bounded and coercive.

Fix $u, v \in W_0^{1,2}(\Omega)$, let $\mu = \min\{\lambda : \lambda \text{ is an eigenvalue of } A\}$, and let $\tilde{\mu} = \max\{\lambda : \lambda \text{ is an eigenvalue of } A\}$. Since $A$ has only positive eigenvalues, $\mu > 0$, and we have

$$\mu \|\nabla u\|_{L^2(\Omega)}^2 = \mu \int_\Omega |\nabla u|^2 dx \leq \int_\Omega A\nabla u \nabla u dx. \tag{1}$$

Using the Cauchy-Schwarz inequality, we also have

$$\int_\Omega |A\nabla u \cdot \nabla v dx| \leq \int_\Omega |A\nabla u||\nabla v| dx \leq \tilde{\mu} \int_\Omega |\nabla u||\nabla v| dx. \tag{2}$$

Next, let $B(x) = (b_1(x), b_2(x), b_3(x))$. Note $B : \Omega \to \mathbb{R}^3$ and

$$\|B\|_{L^3(\Omega)}^3 = \sum_{i=1}^3 \|b_i\|_{L^3(\Omega)}^3 < \infty.$$

Also,

$$\left|\sum_{i=1}^3 b_i u \frac{\partial v}{\partial x_i}\right| = |u| \left|\sum_{i=1}^3 b_i \frac{\partial v}{\partial x_i}\right| = |u||B \cdot \nabla v| \leq |u||B||\nabla v| \tag{3}$$

by the Cauchy-Schwarz inequality.

We can now show $T$ is bounded. Recall that a special case of the Sobolev Embedding Theorem says if $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $n > p$ then $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. In particular with $p = 2$ and $n = 3$, we see $W_0^{1,2}(\Omega) \subset L^6(\Omega)$, and hence $u \in L^6(\Omega)$. Furthermore, since $u \in W_0^{1,2}(\Omega)$, by the Sobolev inequality, there exists a constant $C > 0$ such that $\|u\|_{L^6(\Omega)} \leq C\|u\|_{W_0^{1,2}(\Omega)}$. Then we apply Hölder's inequality, (2), and (3) to see:

$$\begin{aligned}
|T(x,y)| &= \left|\int_\Omega A\nabla u \nabla v dx + \int_\Omega \sum_{i=1}^3 b_i u \frac{\partial v}{\partial x_i} dx\right| \\
&\leq \int_\Omega |A\nabla u \nabla v| dx + \int_\Omega \left|\sum_{i=1}^3 b_i u \frac{\partial v}{\partial x_i}\right| dx \\
&\leq \tilde{\mu} \int_\Omega |\nabla u||\nabla v| dx + \int_\Omega |u||B||\nabla v| dx \\
&\leq \tilde{\mu}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^6(\Omega)}\|B\|_{L^3(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\leq \tilde{\mu}\|u\|_{W_0^{1,2}(\Omega)}\|v\|_{W_0^{1,2}(\Omega)} + C\|u\|_{W_0^{1,2}(\Omega)}\|B\|_{L^3(\Omega)}\|v\|_{W_0^{1,2}(\Omega)} \\
&= (\tilde{\mu} + C\|B\|_{L^3(\Omega)})\|u\|_{W_0^{1,2}(\Omega)}\|v\|_{W_0^{1,2}(\Omega)}.
\end{aligned}$$

This shows $T$ is bounded.

Next, we wish to show $T$ is coercive. We can apply the Poincare inequality to find a constant $c \geq 0$ such that

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}.$$

Squaring both sides and adding $\|\nabla u\|_{L^2(\Omega)}^2$ gives

$$\|u\|_{W_0^{1,2}(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq c^2\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 = (c^2+1)\|\nabla u\|_{L^2(\Omega)}^2.$$

Applying this inequality to (1) yields

$$\int_\Omega A\nabla u \nabla u dx \geq \frac{\mu}{c^2+1}\|u\|_{W_0^{1,2}(\Omega)}^2.$$

Note that if we have

$$\int_\Omega \sum_{i=1}^3 b_i u \frac{\partial u}{\partial x_i} dx = \int_\Omega \sum_{i=1}^3 b_i \frac{1}{2} \frac{\partial u^2}{\partial x_i} dx \geq 0, \tag{4}$$

then

$$T(u,u) \geq \frac{\mu}{1+c^2}\|u\|_{W_0^{1,2}(\Omega)}^2,$$

and so $T$ is coercive. Unfortunately (4) is not true in general. For a counterexample, let $\Omega = (-3,0)^3 = (-3,0)\times(-3,0)\times(-3,0)$, $b_1 = b_2 = b_3 = 1$ on $\Omega$, $u(x) = u(x_1,x_2,x_3) = x_1$ for all $x \in (-2,-1)^3$ and zero elsewhere. We have $u \in W_0^{1,2}(\Omega)$, but

$$\int_\Omega \sum_{i=1}^n b_i u \frac{\partial u}{\partial x_i} dx = \int_{-2}^{-1}\int_{-2}^{-1}\int_{-2}^{-1} x_1 dx_1 dx_2 d_3 = -\frac{3}{2}.$$

Now, we give some possibility of additional assumptions for $b_i(x)$ in the problem such that $T$ is coercive.

(i) We assume that $b_i(x)$ satisfy

$$\int_\Omega \sum_{i=1}^3 b_i u \frac{\partial u}{\partial x_i} dx \geq -\frac{\alpha\mu}{c^2+1}\|u\|^2_{W_0^{1,2}(\Omega)} \quad \forall u \in W_0^{1,2}(\Omega)$$

for some $\alpha < 1$. Then we conclude $T(u,u) \geq \frac{(1-\alpha)\mu}{c^2+1}\|u\|^2_{W_0^{1,2}(\Omega)}$ so that $T$ is coercive.

(ii) We assume that $b_i(x)$ satisfy

$$C\|B\|_{L^3(\Omega)} < \frac{\mu}{1+c^2}.$$

Then

$$\left|\int_\Omega \sum_{i=1}^3 b_i u \frac{\partial u}{\partial x_i} dx\right| \leq C\|B\|_{L^3(\Omega)}\|u\|^2_{W_0^{1,2}(\Omega)}$$

and hence

$$|T(u,u)| \geq \left|\int_\Omega A\nabla u \nabla u dx\right| - \left|\int_\Omega \sum_{i=1}^3 b_i u \frac{\partial u}{\partial x_i} dx\right| \geq \left(\frac{\mu}{1+c^2} - C\|B\|_{L^3(\Omega)}\right)\|u\|^2_{W_0^{1,2}(\Omega)}.$$

That means $T$ is coercive.

By using either one of these additional assumptions, we conclude $T$ is bounded and coercive. Therefore, $Lu = f$, $u = 0$ on $\partial\Omega$ has a weak solution in $W_0^{1,2}(\Omega)$ for any $f \in L^2(\Omega)$. □

**Exercise 7** (Ex 3)**.** *Given $w \in C[a,b]$ such that $w > 0$ on $(a,b)$, show that $L^2_w[a,b]$ is a Hilbert space, where*

$$f \in L^2_w[a,b] \text{ if } \int_a^b f(x)^2 w(x) dx < \infty.$$

*Proof.* Imbue $L^2_w[a,b]$ with the inner product

$$\langle f, g\rangle_w = \int_a^b f(x)g(x)w(x)dx.$$

Clearly this satisfies the conditions to be an inner product. In particular, it is guaranteed to be positive definite because $w(x) > 0$ on $(a,b)$. We must show $L^2_w[a,b]$ is complete with respect to the norm induced by this inner product. Let $\{f_n\}$ be a Cauchy sequence of functions in $L^2_w[a,b]$. For every $n \in \mathbb{N}$, define

$$\tilde{f}_n(x) = \begin{cases} f_n(x) & \text{if } x \in \{a,b\} \\ f_n(x)\sqrt{w(x)} & \text{if } x \in (a,b) \end{cases}.$$

Observe that for every $n \in \mathbb{N}$,

$$\|\tilde{f}_n\|^2_{L^2[a,b]} = \int_a^b |f_n(x)\sqrt{w(x)}|^2 dx = \int_a^b |f_n(x)|^2 w(x) dx = \|f_n\|^2_{L^2_w[a,b]},$$

where we do not need the absolute value on $w(x)$ since $w(x) > 0$ on $(a,b)$. Then since $\{f_n\}$ is Cauchy in $L^2_w[a,b]$, $\{\tilde{f}_n\}$ is Cauchy in $L^2[a,b]$. Since $L^2[a,b]$ is complete, $\{\tilde{f}_n\}$ must converge to some limit $\tilde{f} \in L^2[a,b]$. Define

$$f(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \{a,b\} \\ \frac{\tilde{f}(x)}{\sqrt{w(x)}} & \text{if } x \in (a,b) \end{cases}.$$

Note that we may divide by $\sqrt{w(x)}$ since $w(x) > 0$ on $(a,b)$. Observe that $f \in L^2_w[a,b]$ since

$$\|f\|^2_{L^2_w[a,b]} = \int_a^b \left|\frac{\tilde{f}(x)}{\sqrt{w(x)}}\right|^2 w(x)dx = \int_a^b |\tilde{f}(x)|^2 dx = \|\tilde{f}\|^2_{L^2[a,b]} < \infty.$$

We claim $\{f_n\}$ converges to $f$ in $L^2_w[a,b]$. Let $\epsilon > 0$. Since $\{\tilde{f}_n\}$ converges to $\tilde{f}$ in $L^2[a,b]$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\begin{aligned}\epsilon^2 &> \|\tilde{f}_n - \tilde{f}\|^2_{L^2[a,b]} \\ &= \int_a^b |\tilde{f}_n(x) - \tilde{f}(x)|^2 dx \\ &= \int_a^b |f_n(x)\sqrt{w(x)} - f(x)\sqrt{w(x)}|^2 dx \\ &= \int_a^b |f_n(x) - f(x)|^2 w(x) dx \\ &= \|f_n - f\|^2_{L^2_w[a,b]}.\end{aligned}$$

Therefore $\{f_n\}$ converges to $f$, and we conclude $L^2_w[a,b]$ is a Hilbert space. $\square$

*Remark:* The mapping $f \mapsto \tilde{f} := \begin{cases} f(x) & \text{if } x \in \{a,b\} \\ f(x)\sqrt{w(x)} & \text{if } x \in (a,b) \end{cases}$ is a bijection and a linear isometry from $L^2_w[a,b]$ to $L^2[a,b]$.

**Exercise 8** (Ex 4)**.** *Let $\Omega$ be a bounded open connected set in $\mathbb{R}^n$. Show that there exist a countable number of nonoverlapping closed cubes $\{Q_j\}$ that cover $\Omega$ such that for all $j$,*

$$\sqrt{n}l(Q_j) \le d(Q_j, \partial\Omega) = \inf\{|x-y| : x \in Q_j, y \notin \Omega\} \le 4\sqrt{n}l(Q_j)$$

*where $l(Q_j)$ is the length of the cubes $Q_j$. Moreover, if $Q_j \cap Q_i \neq \emptyset$, then*

$$\frac{1}{4} \le \frac{l(Q_i)}{l(Q_j)} \le 4.$$

*Let $W^{1,2}(\Omega)$ be the completion of $C^\infty(\Omega)$ functions in $L^2(\Omega)$ (and also their first derivatives) under the norm*

$$\|f\|_{W^{1,2}(\Omega)} = \left( \int_\Omega |f|^2 + \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}}.$$

*Show that $u \in W^{1,2}(\Omega)$ if and only if $u \in L^2(\Omega)$ and for each $i \in \{1, \ldots, n\}$, there exists $u_i \in L^2(\Omega)$ with*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega u_i \phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega). \tag{$*$}$$

*You may assume there exist $\psi_j \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_{Q_j} \le \psi_j \le \chi_{\lambda Q_j}$, for any $\lambda > 1$.*

### Lecturer comments

(i) The first part regarding the countable nonoverlapping closed cubes is a standard result called Whitney Decomposition that can be found in " Singular integrals and differentiability properties of functions" page 167 by Elias Stein. It is usually used as standard reference regarding this. Indeed, there is also a construction of a partition of unity on page 168-170, that is, given any $0 < \varepsilon < 1/4$, for each $k$, there exists $\psi_k \in C_0^\infty(\mathbb{R}^n)$ with $\chi_{Q_k} \le \psi_k \le \chi_{(1+\varepsilon)Q_k}$. Normalize $\psi_k$ appropriately, we can get $\phi_k \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \phi_k \le \chi_{(1+\varepsilon)Q_k}$ such that $\sum_{k=1}^\infty \phi_k = 1$ on $\Omega$.

(ii) Suppose there exists a sequence $\{g_j\} \subset C^\infty(\Omega)$ such that it is a Cauchy sequence in $W^{1,2}(\Omega)$. Then by similar technique as in $W_0^{1,2}(\Omega)$, we can see that there exists $u, u_i \in L^2(\Omega)$, $i = 1, \cdots, n$ such that $g_j \to u$ in $L^2(\Omega)$ and $\frac{\partial g_j}{\partial x_i} \to u_i$ in $L^2(\Omega)$. It is then easy to check that $(*)$ holds.

(iii) Now suppose $u, u_i \in L^2(\Omega)$ such that $(*)$ holds. We will now show that given any $\alpha > 0$ there exists $g \in C^\infty(\Omega)$ such that $\|u - g\|_{W^{1,2}(\Omega)}$. To this end, we will break it down into 2 steps.

**Step 1** We show that if $u, u_i \in L^2$ has compact support in $\mathbb{R}^n$, then the above is true. For this purpose, fix any nonnegative function $G \in C_0^\infty(\mathbb{R}^n)$ with $\int G dx = 1$ and define $G_\varepsilon(x) = G(x/\varepsilon)\varepsilon^{-n}$. It has been shown in MA5205 that given any $\alpha > 0$, there exists $G_\varepsilon * u \in C_0^\infty(\mathbb{R}^n)$ such that $\|G_\varepsilon * u - u\|_{L^2(\mathbb{R}^n)}, \|G_\varepsilon * u_i - u_i\|_{L^2(\mathbb{R}^n)} < \alpha$ **(this team of students also provide a proof for this fact; see below)**.

On the other hand, let us verify that $\frac{\partial}{\partial x_i} G_\varepsilon * u = G_\varepsilon * u_i$ a.e.. To this end, first, it is easy to verify that for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,
$G_\varepsilon(y)u(x-y)\partial_i\varphi(x), G_\varepsilon(y)u_i(x-y)\varphi(x)$ are both in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Hence by Fubini's theorem, divergence theorem (integration by parts) and $(*)$, we have

$$\begin{aligned}
\int \partial_i(G_\varepsilon * u)(x)\varphi(x)dx &= -\int G_\varepsilon * u(x)\partial\varphi(x)dx \\
&= -\int\int G_\varepsilon(y)u(x-y)dy\partial_i\varphi(x)dx \\
&= -\int G_\varepsilon(y)\int u(x-y)\partial_i\varphi(x)dxdy \\
&= \int G_\varepsilon(y)\int u_i(x-y)\varphi(x)dxdy \quad \text{(also need change of variables)} \\
&= \int\int G_\varepsilon(y)u_i(x-y)\varphi(x)dydx \\
&= \int G_\varepsilon * u_i(x)\varphi(x)dx.
\end{aligned}$$

Since it holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we see that our claim is true.

**Step 2** Now, note that if $u, u_i \in L^2(\Omega)$ such that $(*)$ holds. We can write $u = \sum_{k=1}^\infty \phi_k u$. Clearly $\phi_k u, u\partial_i\phi_k + u_i\phi_k$ has compact support and check that they satisfy $(*)$. Given any $\alpha > 0$, from the above, for each $k$, there exists $\varepsilon_k$ such that

$$\|G_{\varepsilon_k} * (u\phi_k) - u\phi_k\|_{W^{1,2}(\Omega)} < \alpha/2^k.$$

Furthermore, we may assume $\varepsilon_k > 0$ is small enough such that $G_{\varepsilon_k} * (u\phi_k) = f_k$ has support in $\frac{5}{4}Q_k$. As $\{\frac{5}{4}Q_k\}$ has bounded overlaps, it is clear that $\sum f_k$ is well defined and is $C^\infty$. Finally, note that

$$\|\sum_k f_k - \sum_k u\phi_k\|_{W^{1,2}(\Omega)} \le \sum \|f_k - u\phi_k\|_{W^{1,2}(\Omega)} < \alpha.$$

**End of comment**

*Proof.* We will construct the cubes in an iterative way. Begin by inscribing $\Omega$ inside the smallest closed cube $\Gamma$ which will contain $\Omega$. Divide $\Gamma$ into $2^n$ equal nonoverlapping smaller closed cubes each with half the length of $\Gamma$ in the obvious way, so that each resulting cube shares exactly one vertex with $\Gamma$. With each of these smaller cubes, we will proceed in the following way:

If a cube $Q_j$ is disjoint from $\Omega$, we will remove $Q_j$ from the collection of remaining cubes. If $Q_j$ satisfies $\sqrt{n}l(Q_j) \leq d(Q_j, \partial\Omega)$, then we will leave $Q_j$ alone. Finally, if $Q_j$ does not satisfy $\sqrt{n}l(Q_j) \leq d(Q_j, \partial\Omega)$, we will further divide $Q_j$ into $2^n$ equal nonoverlapping closed cubes each with half the side length of $Q_j$. When we divide the cube $Q_j$, we will say $Q_j$ is the parent of the $2^n$ cubes into which it was split.

Iterating in this way will produce a countable collection of cubes $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{N}}$, each of which satisfies $\sqrt{n}l(Q_j) \leq d(Q_j, \partial\Omega)$. The union $\bigcup_{j\in\mathbb{N}} Q_j$ must be contained in $\Omega$, for if not, there would be a cube which is either disjoint from $\Omega$ or intersects the boundary of $\Omega$. However, this cannot be the case since at each step we remove any cubes disjoint from $\Omega$, and any cube which intersects the boundary would be further subdivided in the next step in the iteration. Thus we can guarantee that $\bigcup_{j\in\mathbb{N}} Q_j \subseteq \Omega$.

We claim that $\bigcup_{j\in\mathbb{N}} Q_j$ contains $\Omega$ as well. Let $M = l(\Gamma)$. Let $x \in \Omega$. Since $\Omega$ is open, $d(x, \partial\Omega) > 0$. Then there exists $m \in \mathbb{N}$ such that $\sqrt{n}\frac{M}{2^m} < d(x, \partial\Omega)$. Note that if we divide the cube $\Gamma$ into $2^{mn}$ smaller closed cubes each with the length equals to $\frac{M}{2^m}$, then there exists a cube $Q$ (from those cubes) such that $x \in Q$. Since $\operatorname{diam}(Q) = \sqrt{n}\frac{M}{2^m}$ and $d(x, \partial\Omega) > \sqrt{n}\frac{M}{2^m}$, we have $d(Q, \partial\Omega) > 0$. That means $Q \subseteq \Omega$. If $Q \subseteq Q_j$ for some $j$, then $x \in Q_j$ and we are done. Otherwise, from the iteration, $d(Q, \partial\Omega) < \sqrt{n}\frac{M}{2^m}$. Suppose that $\sqrt{n}\frac{M}{2^{m'}} < d(Q, \partial\Omega)$ for some $m' > m$. If we divide the cube $Q$ into $2^{(m'-m)n}$ smaller closed cubes each with the length equals to $\frac{M}{2^{m'}}$, then there exists a cube $Q'$ (from those cubes) such that $x \in Q' \subseteq Q$. We have

$$\sqrt{n}l(Q') = \sqrt{n}\frac{M}{2^{m'}} < d(Q, \partial\Omega) \leq d(Q', \partial\Omega).$$

Then from the iteration on $Q$, there must be a cube $Q_j \in \mathcal{Q}$ with $l(Q_j) = \frac{M}{2^{m_1}}$ $(m < m_1 \leq m')$ such that $Q'$ is contained in $Q_j$. Hence, $x \in Q_j$ and we are done.

We must argue that the other conditions are satisfied. Each cube is the result of the division its parent cube, which did not satisfy the lower bound, into $2^n$ equal cubes. That is, for any cube $Q_j \in \mathcal{Q}$, its parent cube $Q_p \supset Q_j$ has length $2l(Q_j)$ and satisfies $2\sqrt{n}l(Q_j) > d(Q_p, \partial\Omega)$. Note that

$$\sup\{|x - y| : x \in Q_j, y \in Q_p\} \leq \operatorname{diam}(Q_p) = 2\sqrt{n}l(Q_j).$$

That is, the maximum possible distance between points in $Q_j$ and points in $Q_p$ is no more than $2\sqrt{n}l(Q_j)$, the diameter of $Q_p$. It follows by the triangle inequality that

$$d(Q_j, \partial\Omega) \leq \operatorname{diam}(Q_p) + d(Q_p, \partial\Omega) < 4\sqrt{n}l(Q_j).$$

Now we show the second condition. Let $Q_i, Q_j \in \mathcal{Q}$ with $Q_j \cap Q_i \neq \emptyset$. If these cubes are the same size, then we have nothing to prove, so suppose arbitrarily that $Q_j$ is the larger cube. Then $\frac{1}{4}l(Q_i) \leq l(Q_j)$, and we must show $l(Q_j) \leq 4l(Q_i)$.

Note that $Q_j$ satisfies $\sqrt{n}l(Q_j) \leq d(Q_j, \partial\Omega)$. Let $q_0 \in Q_i \cap Q_j$. Since $q_0 \in Q_j$,

$$\sqrt{n}l(Q_j) \leq d(Q_j, \partial\Omega) \leq d(q_0, \partial\Omega).$$

Denote by $Q_p$ the parent cube of $Q_i$. By the triangle inequality, for all $y \in Q_p$, we can say

$$d(q_0, \partial\Omega) \leq |q_0 - y| + d(y, \partial\Omega).$$

Since $Q_p$ is the parent cube of $Q_i$ recall that it satisfies $d(Q_p, \partial\Omega) < 2\sqrt{n}l(Q_i)$. Also since $q_0 \in Q_p$, note that $\sup\limits_{y\in Q_p} |q_0 - y| \leq 2\sqrt{n}l(Q_i)$. Then taking the infimum over all $y \in Q_p$, we get

$$\begin{aligned}\sqrt{n}l(Q_j) &\leq d(q_0, \partial\Omega)\\ &\leq \inf_{y\in Q_p} (|q_0 - y| + d(y, \partial\Omega))\\ &\leq \sup_{y\in Q_p} (|q_0 - y|) + d(Q_p, \partial\Omega)\end{aligned}$$

$$< 2\sqrt{n}l(Q_i) + 2\sqrt{n}l(Q_i) = 4\sqrt{n}l(Q_i).$$

Dividing both sides by $\sqrt{n}$ gives $l(Q_j) < 4l(Q_i)$. Hence we have shown that $\mathcal{Q}$ covers $\Omega$ and all elements of $\mathcal{Q}$ satisfy both required inequalities.

($\Rightarrow$) Suppose $u \in W^{1,2}(\Omega)$. Then there exists a sequence $(f_m)$ of $C^\infty(\Omega)$ functions in $L^2(\Omega)$ whose first derivatives are also in $L^2(\Omega)$ and a function $u_i \in L^2(\Omega)$ for each $i \in \{1, \dots, n\}$ such that $\|u - f_m\|_{L^2(\Omega)} \to 0$ and $\left\|u_i - \frac{\partial f_m}{\partial x_i}\right\|_{L^2(\Omega)} \to 0$ as $m \to \infty$. Integrating by parts shows for all $m \in \mathbb{N}$,

$$\int_\Omega f_m \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega \frac{\partial f_m}{\partial x_i} \phi dx \tag{1}$$

for all $\phi \in C_0^\infty(\Omega)$. Then applying Holder's inequality, we have

$$\begin{aligned} \left| \int_\Omega u_i \phi dx - \int_\Omega \frac{\partial f_m}{\partial x_i} \phi dx \right| &= \left| \int_\Omega \left( u_i - \frac{\partial f_m}{\partial x_i} \right) \phi dx \right| \\ &\le \int_\Omega \left| \left( u_i - \frac{\partial f_m}{\partial x_i} \right) \phi \right| dx \\ &\le \left\| u_i - \frac{\partial f_m}{\partial x_i} \right\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} \to 0 \end{aligned}$$

as $m \to \infty$ for each $i \in \{1, \dots, n\}$ and $\phi \in C_0^\infty(\Omega)$, and similarly since $\frac{\partial \phi}{\partial x_i}$ also has compact support,

$$\left| \int_\Omega f_m \frac{\partial \phi}{\partial x_i} dx - \int_\Omega u \frac{\partial \phi}{\partial x_i} dx \right| \le \|f_m - u\|_{L^2(\Omega)} \left\| \frac{\partial \phi}{\partial x_i} \right\|_{L^2(\Omega)} \to 0$$

as $m \to \infty$ for each $i \in \{1, \dots, n\}$ and $\phi \in C_0^\infty(\Omega)$. Thus, taking the limit on both sides of equation (1) for each $i \in \{1, \dots, n\}$ yields

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega u_i \phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

($\Leftarrow$) Suppose $u \in L^2(\Omega)$ and there exist $u_i$ such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega u_i \phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Partition $\Omega$ into countably many cubes $Q_j$, $j \in \mathbb{N}$ as before, and define the set

$$\Omega_\epsilon = \bigcup_{j \in \mathbb{N}, d(Q_j, \partial\Omega) > \epsilon} Q_j.$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$\phi(x) = \begin{cases} c e^{\frac{1}{|x|^2 - 1}} & \text{if } |x| < 1, \\ 1 & \text{if } |x| \ge 1, \end{cases}$$

for some $c > 0$, and note that $\text{supp}(\phi) \subset \overline{B_1(0)}$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$. For every $\epsilon > 0$ define $\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi(\frac{x}{\epsilon})$. Note that $\text{supp}(\phi_\epsilon) \subset \overline{B_\epsilon(0)}$ and $\int_{\mathbb{R}^n} \phi_\epsilon(x) dx = \int_{B_\epsilon(0)} \phi_\epsilon(x) dx = 1$. For $x \in \Omega_\epsilon$, define

$$u_\epsilon = (u * \phi_\epsilon)(x) = \int_\Omega \phi_\epsilon(x - y) u(y) dy.$$

Note that $u_\epsilon \in C^\infty(\Omega_\epsilon)$.

We will need a lemma in order to proceed further.

**Lemma 1.** *For any $u \in L^2(\Omega)$ with compact support such that there exist $u_i \in L^2(\Omega)$ for $i \in \{1, \dots, n\}$ which satisfy*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega u_i \phi dx \quad \textit{for all } \phi \in C_0^\infty(\Omega)$$

*and any open set $\Omega' \subset \Omega$ with $d(\Omega', \partial\Omega) > 0$, the function $u_\epsilon$ defined as before is defined on $\Omega'$ for sufficiently small $\epsilon > 0$ and satisfies $\|u_\epsilon - u\|_{W^{1,2}(\Omega')} \to 0$ as $\epsilon \to 0^+$.*

*Proof.* First, we show for $v \in C(\Omega)$ that $v_\epsilon \to v$ as $\epsilon \to 0^+$ on compact subsets of $\Omega$. Let $K \subset \Omega$ be compact. Then $d(K, \partial\Omega) > 0$, and since $\bigcup_{i\in\mathbb{N}} Q_i = \Omega$ and $d(Q_i, \partial\Omega) \leq 4\sqrt{n}l(Q_i) \to 0$ as $i \to \infty$, there exists $\eta > 0$ such that $0 < \epsilon < \eta$ implies $K \subset \Omega_\epsilon$, and since $u_\epsilon$ is defined on $\Omega_\epsilon$, $u_\epsilon$ is defined on $K$.

Observe that because $\Omega_\epsilon$ is a finite union of closed cubes, it is a compact set. Then if $v \in C(\Omega)$, $v$ is uniformly continuous on all $\Omega_\epsilon$. We may define $v_\epsilon = (v * \phi_\epsilon)(x)$ analogously to $u_\epsilon$. Then for all $\alpha > 0$ there exist $\delta > 0$ such that if $|x - y| < \delta$,

$$|v(x) - v(y)| < \frac{\alpha\epsilon^n}{|B_\epsilon(0)|(1 + \|\phi\|_\infty)}.$$

Let $\epsilon < \min\{\delta, \eta\}$. For all $x \in K$, we use the facts that $\text{supp}(\phi_\epsilon) = B_\epsilon(0)$ and $\int_{B_\epsilon(0)} \phi_\epsilon(x)dx = 1$ to see

$$\begin{aligned}
|v_\epsilon(x) - v(x)| &= \left| \int_\Omega \phi_\epsilon(x-y)v(y)dy - v(x) \right| \\
&= \left| \int_{B_\epsilon(x)} \phi_\epsilon(x-y)v(y)dy - v(x)\int_{B_\epsilon(x)} \phi_\epsilon(x-y)dy \right| \\
&= \left| \int_{B_\epsilon(x)} \frac{1}{\epsilon^n}\phi\left(\frac{x-y}{\epsilon}\right)v(y)dy - \int_{B_\epsilon(x)} \frac{1}{\epsilon^n}\phi\left(\frac{x-y}{\epsilon}\right)v(x)dy \right| \\
&= \frac{1}{\epsilon^n}\left| \int_{B_\epsilon(x)} \phi\left(\frac{x-y}{\epsilon}\right)(v(y) - v(x))dy \right| \\
&\leq \frac{1}{\epsilon^n}\|\phi\|_\infty \int_{B_\epsilon(x)} |v(x) - v(y)|dy \\
&< \frac{1}{\epsilon^n}\|\phi\|_\infty \int_{B_\epsilon(x)} \frac{\alpha\epsilon^n}{|B_\epsilon(0)|(1 + \|\phi\|_\infty)}dy \\
&= \frac{1}{\epsilon^n}\|\phi\|_\infty \frac{\alpha\epsilon^n}{|B_\epsilon(0)|(1 + \|\phi\|_\infty)}|B_\epsilon(0)| \\
&\leq \alpha.
\end{aligned}$$

Thus if $v \in C(\Omega)$, $v_\epsilon \to v$ uniformly as $\epsilon \to 0^+$ on each compact set $K \subset \Omega$.

Second, we claim that

$$\|u_\epsilon - v_\epsilon\|_{L^2(\Omega_\epsilon)} \leq \|u - v\|_{L^2(\Omega)}.$$

We apply Holder's Inequality with the fact that $\int_{\mathbb{R}^n} \phi dx = 1$ to see

$$\begin{aligned}
|u_\epsilon(x) - v_\epsilon(x)| &= \left| \int_\Omega \phi_\epsilon(x-y)(u(y) - v(y))dy \right| \\
&= \left| \int_\Omega (\phi_\epsilon(x-y))^{\frac{1}{2}}(\phi_\epsilon(x-y))^{\frac{1}{2}}(u(y) - v(y))dy \right| \\
&\leq \left( \int_\Omega \phi_\epsilon(x-y)dy \right)^{\frac{1}{2}} \left( \int_\Omega \phi_\epsilon(x-y)|u(y) - v(y)|^2 dy \right)^{\frac{1}{2}} \\
&\leq \left( \int_\Omega \phi_\epsilon(x-y)|u(y) - v(y)|^2 dy \right)^{\frac{1}{2}}.
\end{aligned}$$

Then applying Tonelli's theorem and again the fact that $\int_{\mathbb{R}^n} \phi = 1$, we get

$$\begin{aligned}
\int_{\Omega_\epsilon} |u_\epsilon(x) - v_\epsilon(x)|^2 dx &\leq \int_{\Omega_\epsilon}\int_\Omega \phi_\epsilon(x-y)|u(y) - v(y)|^2 dydx \\
&= \int_\Omega |u(y) - v(y)|^2 \int_{\Omega_\epsilon} \phi_\epsilon(x-y)dxdy \\
&\leq \int_\Omega |u(y) - v(y)|^2 dy.
\end{aligned}$$

This proves the claim.

Third, let $\alpha > 0$. Since $L^2(\Omega)$ is the completion of $C(\Omega)$ functions under the $L^2$ norm and $u$ has compact support, we may choose $v \in C_0(\Omega)$ such that

$$\|v-u\|_{L^2(\Omega)} < \frac{\alpha}{3}.$$

Since $K := \operatorname{supp}(v)$ is compact, we know $v_\epsilon \to v$ uniformly as $\epsilon \to 0^+$, so there exists $\delta > 0$ such that $0 < \epsilon < \delta$ implies

$$\|v_\epsilon - v\|_{C(K)} < \frac{\alpha}{3\sqrt{|K|}}.$$

Also, there exists $\mu > 0$ such that $K \subset \Omega_\epsilon$ for $0 < \epsilon < \mu$. Then since $v = v_\epsilon = 0$ on $\Omega_\epsilon \setminus K_\eta$, if $\epsilon < \min\{\delta, \mu\}$,

$$\int_{\Omega_\epsilon} |v_\epsilon - v|^2 dx = \int_K |v_\epsilon - v|^2 dx \le \left(\|v_\epsilon - v\|_{C(K)}\right)^2 |K| < \left(\frac{\alpha}{3}\right)^2.$$

Thus by Minkowski's inequality on $L^2(\Omega)$, for $\epsilon < \min\{\delta, \eta, \mu\}$,

$$\begin{aligned}\|u_\epsilon - u\|_{L^2(\Omega_\epsilon)} &\le \|u_\epsilon - v_\epsilon\|_{L^2(\Omega_\epsilon)} + \|v_\epsilon - v\|_{L^2(\Omega_\epsilon)} + \|v-u\|_{L^2(\Omega_\epsilon)} \\ &\le 2\|u-v\|_{L^2(\Omega)} + \|v_\epsilon - v\|_{L^2(\Omega_\epsilon)} \\ &< 3\frac{\alpha}{3} = \alpha.\end{aligned}$$

Hence, for any open subset $\Omega' \subset \Omega$ with $d(\Omega', \partial\Omega) > 0$, $u_\epsilon \to u$ as $\epsilon \to 0^+$ in $L^2(\Omega')$.

Finally, we use partial derivative of $u_\epsilon$ derived above and the hypothesis to see

$$\begin{aligned}\frac{\partial u_\epsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial \phi_\epsilon}{\partial x_i}(x-y)u(y)dy &= -\int_\Omega \frac{\partial \phi_\epsilon}{\partial y_i}(x-y)u(y)dy \\ &= \int_\Omega \phi_\epsilon(x-y)u_i(y)dy \\ &= (\phi_\epsilon * u_i)(x).\end{aligned}$$

Then replacing $u$ with $u_i$ in the previous argument shows $\lim\limits_{\epsilon\to 0^+} \frac{\partial u_\epsilon}{\partial x_i} = u_i$ in $L^2(\Omega)$ for each $i \in \{1, \ldots, n\}$. Hence, recognizing the fact that $u_i$ is the unique weak derivative of $u$, we can say

$$\|u_\epsilon - u\|^2_{W^{1,2}(\Omega')} = \int_{\Omega'} |u_\epsilon - u|^2 + \sum_{i=1}^n \left|\frac{\partial u_\epsilon}{\partial x_i} - u_i\right|^2 dx = \|u_\epsilon - u\|_{L^2(\Omega')} + \sum_{i=1}^n \left\|\frac{\partial u_\epsilon}{\partial x_i} - u_i\right\|_{L^2(\Omega')} \to 0$$

as $\epsilon \to 0^+$. Therefore $u_\epsilon \to u$ as $\epsilon \to 0^+$ in $W^{1,2}(\Omega')$ on every open subset $\Omega' \subset \Omega$. $\square$

We continue the proof of Exercise 4 ($\Leftarrow$). Since $\sqrt{n}\,l(Q_i) \le d(Q_i, \partial\Omega)$ for all $i \in \mathbb{N}$, for sufficiently small $\lambda > 1$ we have $d(\lambda Q_i, \partial\Omega) > 0$ and $\bigcup\limits_{i\in\mathbb{N}} \lambda Q_i = \Omega$, where $\lambda Q_i$ is the open enlargement of $Q_i$. Note as well that because $\frac{1}{4} \le \frac{l(Q_i)}{l(Q_j)} \le 4$ for $Q_i \cap Q_j \ne \emptyset$ and $d(\lambda Q_i, \partial\Omega) > 0$ for $i \in \mathbb{N}$, each open cube $\lambda Q_i$ intersects only finitely many other cubes in the cover. Since $\{\lambda Q_i\}$ is an open cover of $\Omega$, we may construct a partition of unity $\{\psi_i\}$ subordinate to the open cover $\{\lambda Q_i\}$, so that $\operatorname{supp}(\psi_i) \subset \lambda Q_i$ for all $i \in \mathbb{N}$, and $\sum_{i=1}^\infty \psi_i(x) = 1$ for all $x \in \Omega$. In particular, we may write

$$u(x) = \sum_{i=1}^\infty \psi_i(x)u(x)$$

for all $x \in \Omega$. Let $\alpha > 0$. Since $\operatorname{supp}(\psi_i u) \subset \lambda Q_i$ and $\lambda Q_i$ is an open subset of $\Omega$ with $d(\lambda Q_i, \partial\Omega) > 0$, by Lemma 1, for all $i \in \{1, \ldots, n\}$ we may find $\epsilon_i > 0$ such that

$$\operatorname{supp}((\psi_i u)_{\epsilon_i}) \subset \lambda Q_i \text{ and } \|(\psi_i u)_{\epsilon_i} - \psi_i u\|_{W^{1,2}(\Omega)} < \frac{\alpha}{2^i},$$

where $(\psi_i u)_{\epsilon_i}$ is defined analogously to $u_\epsilon$ above and the norm can be taken over all of $\Omega$ because of the compact support. Define

$$v(x) = \sum_{i=1}^\infty (\psi_i u)_{\epsilon_i}(x).$$

The limit of the sum must exist because pointwise each $x$ is contained in only finitely many cubes $\lambda Q_i$ and we have compact support of the terms of the sum contained in their respective cubes. Moreover,

$v \in C^\infty(\Omega)$ because each $\lambda Q_i$ is covered by finitely many cubes $\lambda Q_j$, which implies on each cube in the open cover of $\Omega$ that $v$ is a finite sum of $C^\infty(\Omega)$ functions. By the triangle inequality,

$$\|\sum_{i=1}^N \phi_i u(x) - \sum_{i=1}^N (\phi_i u)_{\epsilon_i}(x)\|_{W^{1,2}(\Omega)} \le \sum_{i=1}^N \|(\phi_i u)_{\epsilon_i} - \phi_i u\|_{W^{1,2}(\Omega)} < \sum_{i=1}^N \frac{\alpha}{2^i} \le \alpha$$

for all $N \in \mathbb{N}$. Letting $N \to \infty$ gives

$$\lim_{N\to\infty} \|\sum_{i=1}^N \phi_i u(x) - \sum_{i=1}^N (\phi_i u)_{\epsilon_i}(x)\|_{W^{1,2}(\Omega)} \le \alpha.$$

By the continuity of the norm, we may move the limit inside the norm to conclude $\|u - v\|_{W^{1,2}(\Omega)} \le \alpha$. Note that the definition of $v$ does depend on $\alpha$. What we have shown then, is that for any $\alpha > 0$ there exists a $C^\infty(\Omega)$ function $v$ such that $\|u - v\|_{W^{1,2}(\Omega)} \le \alpha$. Thus we can, for example, choose $\alpha = \frac{1}{m}$ for $m \in \mathbb{N}$ to construct a sequence $(v_m) \subset C^\infty(\Omega)$ such that $v_m \to u$ as $m \to \infty$ under the $W^{1,2}(\Omega)$ norm. Therefore $u \in W^{1,2}(\Omega)$. □

*Remark:* The primary reference for the last problem was Giovanni Leoni: A First Course in Sobolev Spaces. See Theorems 10.15, C.19, C.20, and Lemma 10.16.

**Exercise 9** (Ex 5)**.** *Let $X$ be a reflexive Banach space and $Y$ a closed subspace of $X$. For any $z \in X \setminus Y$, show there exists $y_0 \in Y$ such that*

$$\|y_0 - z\| \le \|y - z\| \quad \textit{for all } y \in Y.$$

*Proof.* Let $z \in X \setminus Y$, and let $(y_n)$ be a sequence in $Y$ such that $\|y_n - z\| \to \inf_{y\in Y} \|y - z\|$ as $n \to \infty$. Let $B$ be the intersection of $Y$ and the closed ball of radius $2\inf_{y\in Y} \|y - z\|$ centered at $z$. Recall that every bounded, closed, convex set in a reflexive Banach space is weakly sequentially compact. Then as $B$ is bounded, closed, and convex, we can conclude $B$ is weakly sequentially compact.

By the convergence of $y_n$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $y_n \in B$. Then the sequence $(y_n)_{n>N}$ has a weakly convergent subsequence $(x_n)$ which converges weakly to some point $y_0 \in B \subset Y$. Then the sequence $(x_n - z)$ converges weakly to $y_0 - z$. As we have previously shown for weakly convergent sequences,

$$\|y_0 - z\| \le \liminf \|x_n - z\|.$$

Since $\|x_n - z\| \to \inf_{y\in Y} \|y - z\|$, we can in fact write

$$\|y_0 - z\| \le \inf_{y\in Y} \|y - z\|.$$

Therefore we have found $y_0 \in Y$ such that $\|y_0 - z\| \le \|y - z\|$ for all $y \in Y$. □