ORF 307: Lecture 6

Linear Programming: Chapter 5
Duality

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Resource Allocation

Given a cache of raw materials and a factory for turning these raw materials into a variety of finished products, how many of each product type should we make so as to maximimze profit?

This is a resource allocation problem:

$$
\begin{array}{lrcrcrcl}
\text{maximize} & c_1 x_1 & + & c_2 x_2 & + \cdots + & c_n x_n & & \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + \cdots + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + \cdots + & a_{2n} x_n & \leq & b_2 \\
 & & \vdots & & \vdots & & \vdots & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + \cdots + & a_{mn} x_n & \leq & b_m \\
 & & & & x_1, \; x_2, \; \ldots, & x_n & \geq & 0,
\end{array}
$$

where

$$
\begin{aligned}
c_j &= \text{profit per unit of product } j \text{ produced} \\
b_i &= \text{units of raw material } i \text{ on hand} \\
a_{ij} &= \text{units raw material } i \text{ required to produce one unit of prod } j.
\end{aligned}
$$

Closing Up Shop

If we produce one unit less of product j, then we *free up* a_{ij} units of raw material i.

Selling these unused raw materials for $y_1,\ y_2,\ \ldots\ y_m$ dollars/unit yields

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \text{ dollars.}$$

Only interested if this exceeds lost profit on each product j:

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \ \geq \ c_j, \qquad j = 1, 2, \ldots, n.$$

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

$$\begin{array}{lrcrcrcl}
\text{minimize} & b_1 y_1 & + & b_2 y_2 & + \cdots + & b_m y_m & & \\
\text{subject to} & a_{11} y_1 & + & a_{21} y_2 & + \cdots + & a_{m1} y_m & \geq & c_1 \\
& a_{12} y_1 & + & a_{22} y_2 & + \cdots + & a_{m2} y_m & \geq & c_2 \\
& & \vdots & & \vdots & & \vdots & \\
& a_{1n} y_1 & + & a_{2n} y_2 & + \cdots + & a_{mn} y_m & \geq & c_n \\
& & & & & y_1,\ y_2,\ \ldots,\ y_m & \geq & 0,
\end{array}$$

Duality

Every Problem:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\
& x_j \geq 0 & j = 1, 2, \ldots, n,
\end{array}
$$

Has a Dual:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\
\text{subject to} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\
& y_i \geq 0 & i = 1, 2, \ldots, m.
\end{array}
$$

Dual of Dual

Primal Problem:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\
& x_j \geq 0 & j = 1, \ldots, n
\end{array}
$$

Original problem is called the *primal problem*.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual in "Standard" Form:

$$
\begin{array}{lll}
-\text{maximize} & \displaystyle\sum_{i=1}^{m} -b_i y_i & \\
\text{subject to} & \displaystyle\sum_{i=1}^{m} -a_{ij} y_i \leq -c_j & j = 1, \ldots, n \\
& y_i \geq 0 & i = 1, \ldots, m
\end{array}
$$

Dual is "negative transpose" of primal.

Theorem *Dual of dual is primal.*

Weak Duality Theorem

If $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual, then*

$$\sum_j c_j x_j \le \sum_i b_i y_i.$$

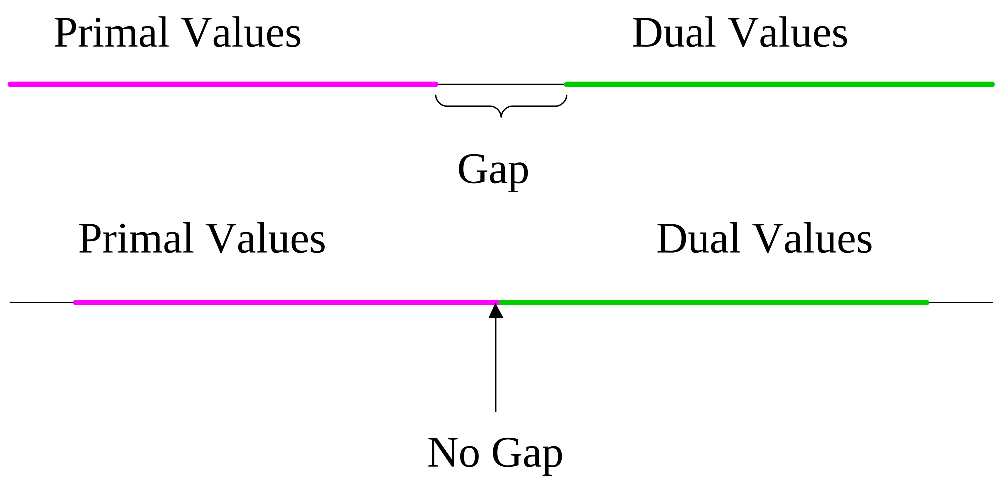
Proof.

$$\begin{aligned} \sum_j c_j x_j &\le \sum_j \left(\sum_i y_i a_{ij} \right) x_j \\ &= \sum_{ij} y_i a_{ij} x_j \\ &= \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\ &\le \sum_i b_i y_i. \end{aligned}$$

Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?



Answer is provided by the Strong Duality Theorem (coming later).

Simplex Method and Duality

A Primal Problem:

maximize ζ	=	0	+	-2	x_1	+	1	x_2	+	1	x_3	+	4	x_4
w_1	=	4	-	0	x_1	-	1	x_2	-	1	x_3	-	2	x_4
w_2	=	3	-	0	x_1	-	0	x_2	-	2	x_3	-	2	x_4
w_3	=	5	-	6	x_1	-	2	x_2	-	-2	x_3	-	5	x_4

Its Dual:

maximize $-\xi$	=	0	+	-4	y_1	+	-3	y_2	+	-5	y_3
z_1	=	2	-	0	y_1	-	0	y_2	-	-6	y_3
z_2	=	-1	-	-1	y_1	-	0	y_2	-	-2	y_3
z_3	=	-1	-	-1	y_1	-	-2	y_2	-	2	y_3
z_4	=	-4	-	-2	y_1	-	-2	y_2	-	-5	y_3

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: x_4 enters, w_3 leaves.

Make analogous pivot in dual: z_4 leaves, y_3 enters.

Seed = 3, GenRand five times

Second Iteration

After First Pivot:

Primal (feasible):

maximize ζ	=	4	+	-34/5	x_1	+	-3/5	x_2	+	13/5	x_3	+	-4/5	w_3
w_1	=	2	-	-12/5	x_1	-	1/5	x_2	-	9/5	x_3	-	-2/5	w_3
w_2	=	1	-	-12/5	x_1	-	-4/5	x_2	-	14/5	x_3	-	-2/5	w_3
x_4	=	1	-	6/5	x_1	-	2/5	x_2	-	-2/5	x_3	-	1/5	w_3

Dual (still not feasible):

maximize $-\xi$	=	-4	+	-2	y_1	+	-1	y_2	+	-1	z_4
z_1	=	34/5	-	12/5	y_1	-	12/5	y_2	-	-6/5	z_4
z_2	=	3/5	-	-1/5	y_1	-	4/5	y_2	-	-2/5	z_4
z_3	=	-13/5	-	-9/5	y_1	-	-14/5	y_2	-	2/5	z_4
y_3	=	4/5	-	2/5	y_1	-	2/5	y_2	-	-1/5	z_4

Note: *negative transpose property intact.*

Again, use primal to pick pivot: x_3 enters, w_2 leaves.

Make analogous pivot in dual: z_3 leaves, y_2 enters.

Third Iteration

After Second Pivot:

Primal (feasible):

$$
\begin{array}{lcrcrlcrlcrlcrl}
\text{maximize } \zeta & = & 69/14 & + & -32/7 & x_1 & + & 1/7 & x_2 & + & -13/14 & w_2 & + & -3/7 & w_3 \\
w_1 & = & 19/14 & - & -6/7 & x_1 & - & 5/7 & x_2 & - & -9/14 & w_2 & - & -1/7 & w_3 \\
x_3 & = & 5/14 & - & -6/7 & x_1 & - & -2/7 & x_2 & - & 5/14 & w_2 & - & -1/7 & w_3 \\
x_4 & = & 8/7 & - & 6/7 & x_1 & - & 2/7 & x_2 & - & 1/7 & w_2 & - & 1/7 & w_3
\end{array}
$$

Dual (still not feasible):

$$
\begin{array}{lcrcrlcrlcrl}
\text{maximize } -\xi & = & -69/14 & + & -19/14 & y_1 & + & -5/14 & z_3 & + & -8/7 & z_4 \\
z_1 & = & 32/7 & - & 6/7 & y_1 & - & 6/7 & z_3 & - & -6/7 & z_4 \\
z_2 & = & -1/7 & - & -5/7 & y_1 & - & 2/7 & z_3 & - & -2/7 & z_4 \\
y_2 & = & 13/14 & - & 9/14 & y_1 & - & -5/14 & z_3 & - & -1/7 & z_4 \\
y_3 & = & 3/7 & - & 1/7 & y_1 & - & 1/7 & z_3 & - & -1/7 & z_4
\end{array}
$$

Note: *negative transpose property intact.*

Again, use primal to pick pivot: x_2 enters, w_1 leaves.

Make analogous pivot in dual: z_2 leaves, y_1 enters.

After Third Iteration

Primal:

$$
\begin{array}{lcrcrlcrlcrlcrl}
\text{maximize } \zeta & = & 26/5 & + & -22/5 & x_1 & + & -1/5 & w_1 & + & -4/5 & w_2 & + & -2/5 & w_3 \\
x_2 & = & 19/10 & - & -6/5 & x_1 & - & 7/5 & w_1 & - & -9/10 & w_2 & - & -1/5 & w_3 \\
x_3 & = & 9/10 & - & -6/5 & x_1 & - & 2/5 & w_1 & - & 1/10 & w_2 & - & -1/5 & w_3 \\
x_4 & = & 3/5 & - & 6/5 & x_1 & - & -2/5 & w_1 & - & 2/5 & w_2 & - & 1/5 & w_3
\end{array}
$$

- Is *optimal*.

Dual:

$$
\begin{array}{lcrcrlcrlcrl}
\text{maximize } -\xi & = & -26/5 & + & -19/10 & z_2 & + & -9/10 & z_3 & + & -3/5 & z_4 \\
z_1 & = & 22/5 & - & 6/5 & z_2 & - & 6/5 & z_3 & - & -6/5 & z_4 \\
y_1 & = & 1/5 & - & -7/5 & z_2 & - & -2/5 & z_3 & - & 2/5 & z_4 \\
y_2 & = & 4/5 & - & 9/10 & z_2 & - & -1/10 & z_3 & - & -2/5 & z_4 \\
y_3 & = & 2/5 & - & 1/5 & z_2 & - & 1/5 & z_3 & - & -1/5 & z_4
\end{array}
$$

- Negative transpose property remains intact.
- Is *optimal*.

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.

Algebra of a Pivot

A primal pivot:

$$
\begin{array}{|c|c|c|}
\hline
d & c & \\
\hline
b & a & \\
\hline
 & & \\
\hline
\end{array}
\xrightarrow{\text{pivot}}
\begin{array}{|c|c|c|}
\hline
d - \dfrac{bc}{a} & c/a & \\
\hline
-b/a & 1/a & \\
\hline
 & & \\
\hline
\end{array}
$$

The corresponding dual pivot:

$$
\begin{array}{|c|c|c|}
\hline
-d & -b & \\
\hline
-c & -a & \\
\hline
 & & \\
\hline
\end{array}
\xrightarrow{\text{pivot}}
\begin{array}{|c|c|c|}
\hline
-d + \dfrac{bc}{a} & b/a & \\
\hline
-c/a & -1/a & \\
\hline
 & & \\
\hline
\end{array}
$$

Strong Duality Theorem

Conclusion on previous slide is the essence of the *strong duality theorem* which we now state:

Theorem. *If the primal problem has an optimal solution,*

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Paraphrase:

If primal has an optimal solution, then there is no duality gap.

Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

Example of *infinite gap*:

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 \\
\text{subject to} & x_1 - x_2 \leq 1 \\
& -x_1 + x_2 \leq -2 \\
& x_1,\ x_2 \geq 0.
\end{array}
$$