PRINCETON UNIVERSITY

ORF 307: Lecture 1

Linear Programming Chapter 1 Introduction

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ORFE Operations Research & Financial Engineering

http://www.princeton.edu/∼rvdb

Course Info

Prereqs:	Three semesters of Calculus
Co-reqs:	Linear Algebra (MAT 202 or MAT 204)
Textbook:	*Linear Programming: Foundations and Extensions, 4th Edition*

Grading:

Homework:	25%
Midterm 1:	25%
Midterm 2:	25%
Final Project:	25%

Homework:

- Will be due every week at noon on Friday.
- All homework must be submitted via Blackboard.
- The lowest homework grade will be dropped.

Midterms: Midterms will be in-class on Thursday of the 6th and 11th weeks.

Lectures: Reading assignments will be posted in advance of each lecture. You should read the reading material before lecture.

Slides: The slides will be posted online. But, they are not a replacement for the lecture. They are just my notes to remind me what to say. You must go to lecture to hear what I have to say.

Webpage: http://orfe.princeton.edu/~rvdb/307/lectures.html

Optimization = Engineering

Engineering is the process of taking the discoveries from science...
implementing them as practical devices, and then ...

making them better, ...

and better, ...

and better.

This is optimization.

In this class, we will take a more mathematical approach.

We will also use computational tools to solve numerically the practical problems we encounter.

Optimization via (Freshman) Calculus

Express an *objective function* to be *minimized* or *maximized* in terms of one independent variable.

Differentiate with respect to this variable.

Set derivative equal to zero.

Solve for the independent variable.

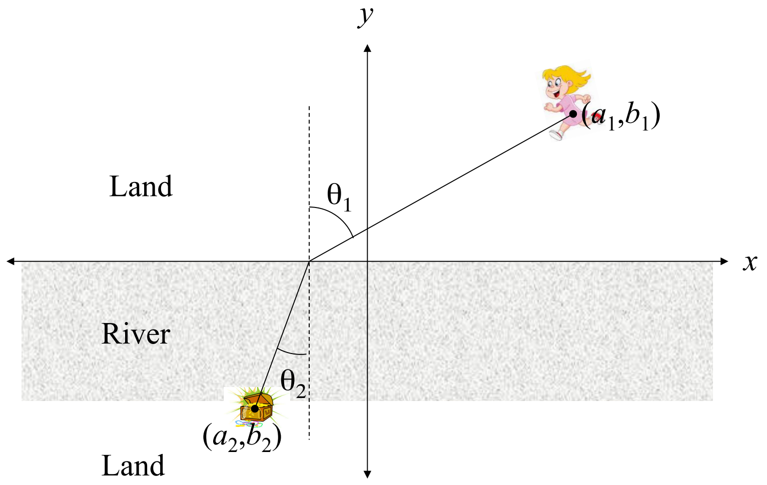
If in doubt as to whether it's a max, min, or saddle point, take second derivative and look at its sign.

If the independent variable is restricted to lying in an interval of the real line, check the endpoints—the optimal solution could be there.

River Crossing

An Example: River Crossing

Suppose you are on one side of a river (at coordinates (a_1, b_1) in the figure) and there is a treasure on the shore at the other side (at coordinates (a_2, b_2)). Worried that someone else might get the treasure before you, you'd like to get there as fast as possible—in *minimum* time.



Assuming that your running speed is v_1 and and your swimming speed is v_2 and that you choose to reach the river at $(x, 0)$, the time is given by:

$$T(x) = \frac{\sqrt{(a_1 - x)^2 + b_1^2}}{v_1} + \frac{\sqrt{(x - a_2)^2 + b_2^2}}{v_2}$$

Derivative is:

$$\frac{dT}{dx} = -\frac{1}{v_1}\frac{a_1 - x}{\sqrt{(a_1 - x)^2 + b_1^2}} + \frac{1}{v_2}\frac{x - a_2}{\sqrt{(x - a_2)^2 + b_2^2}} = 0.$$

At this point, the problem is purely algebra; ugly but doable.

Solving it on the Computer (using AMPL)

```
# Getting to the treasure fast!
param info symbolic, := "File name: river_crossing.txt; Author: R.J. Vanderbei";
display info;

param a1; param b1;
param a2; param b2;
param v1; param v2;

var x;

minimize time: sqrt((a1-x)^2 + b1^2)/v1 + sqrt((x-a2)^2 + b2^2)/v2;

data;

param a1 := 60;
param b1 := 40;
param a2 := -40;
param b2 := -50;
param v1 := 10;
param v2 := 1.5;

solve;
display x;
```

AMPL Intro

Comments: The hashtag symbol (#) starts a "comment".

Statements: Every "statement" ends with a semicolon (;).

Model Section: The first part of the code defines the problem without necessarily providing any specific data values.

Data: Data/parameters are introduced with the `param` command.

Variables: Variables are introduced with the `var` command.

Objective: The objective is to `maximize` or `minimize` a function. The function must be given a name followed by a colon (:) followed by a formula that defines the function.

Constraints: Constraints are introduced with the "`subject to`" command. Each constraint must be given a name followed by a colon (:) followed by the equality or inequality that defines the constraint.

Data Section: The data section starts with the `data` command. In this section parameter definitions are repeated and values are given.

Solve: The `solve` command invokes the solver to solve the problem.

Display: The `display` command is used to see the results.

AMPL Info

- The language is called *AMPL*, which stands for *A Mathematical Programming Language*.

  Note: "Modern" optimization dates back to the 1940's where it was a useful/important tool helping the military prepare their program of activities. Hence, it was called *Mathematical Programming* and if the problem was linear it was called *Linear Programming*. This terminology predates the field of *Computer Programming*. The modern trend (finally!) is to replace the word "programming" with "optimization".

- The book describing the language is called "AMPL" by Fourer, Gay, and Kernighan.

  It is available for free at `http://www.ampl.com/BOOK/download.html`.

  I've also made it available here `http://orfe.princeton.edu/~rvdb/307/textbook/AMPLbook.pdf`.

  There are links to these AMPL websites on the course webpage:

  `http://orfe.princeton.edu/~rvdb/307/lectures.html`.

- There are also online tutorials:
  - Google "AMPL tutorial" for examples.

AMPL

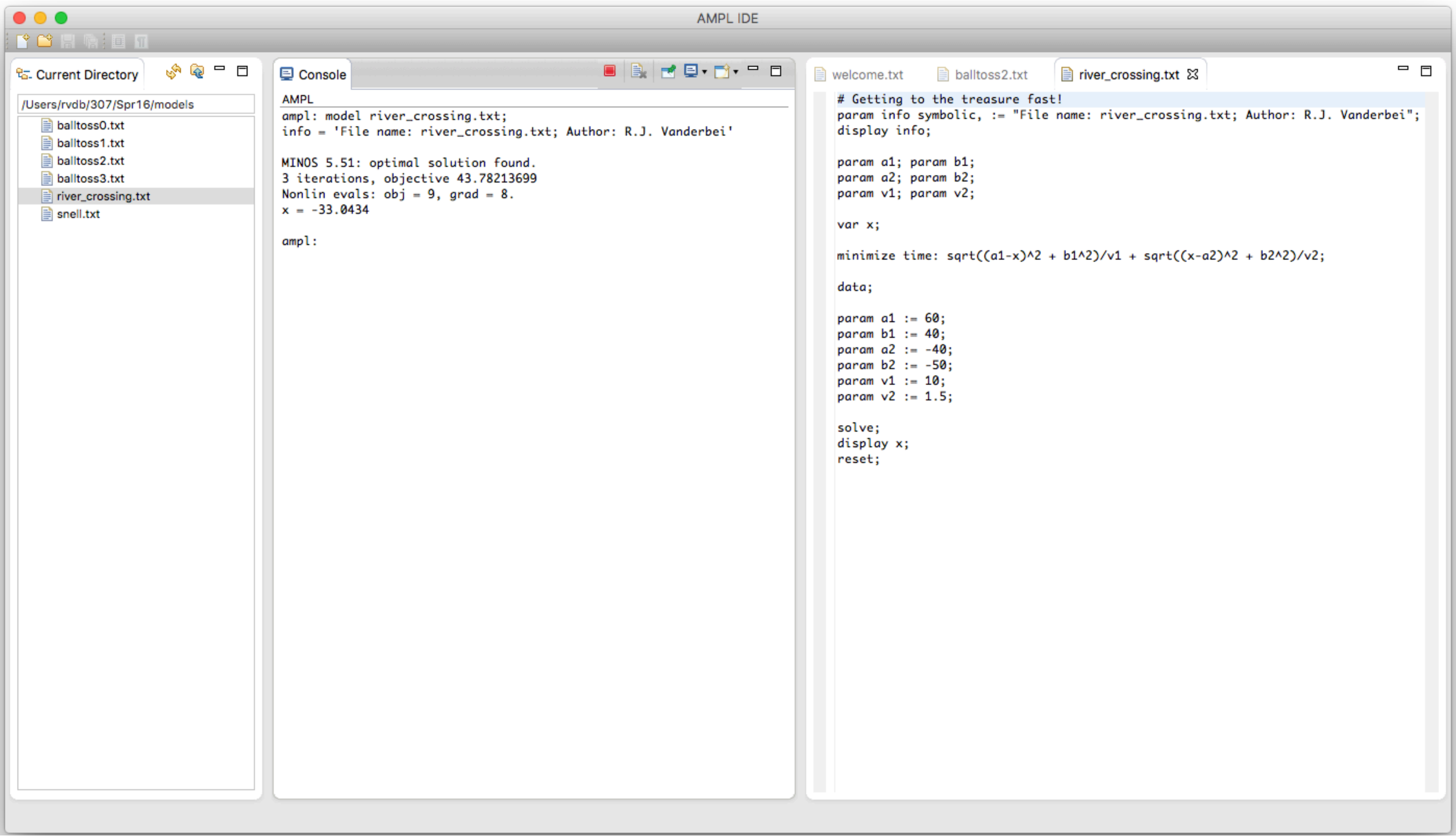
There are three ways to access AMPL:

Online:	The *Network Enabled Optimization Server (NEOS)*.
Download Student Version:	Download from AMPL website (http://ampl.com/try-ampl/download-a-demo-version/) to your own computer. Never expires. Limited number of variables/constraints (500×500).
Download Course Version:	Download from course-specific link to your own computer. Expires at the end of the semester. Unlimited number of variables/constraints. *Preferred method.*

Details about these three methods are available here:

`http://ampl.com/products/ampl/ampl-for-students/`

AMPL IDE



```
/Users/rvdb/307/Spr16/models
balltoss0.txt
balltoss1.txt
balltoss2.txt
balltoss3.txt
river_crossing.txt
snell.txt
```

```
AMPL
ampl: model river_crossing.txt;
info = 'File name: river_crossing.txt; Author: R.J. Vanderbei'

MINOS 5.51: optimal solution found.
3 iterations, objective 43.78213699
Nonlin evals: obj = 9, grad = 8.
x = -33.0434

ampl:
```

```
# Getting to the treasure fast!
param info symbolic, := "File name: river_crossing.txt; Author: R.J. Vanderbei";
display info;

param a1; param b1;
param a2; param b2;
param v1; param v2;

var x;

minimize time: sqrt((a1-x)^2 + b1^2)/v1 + sqrt((x-a2)^2 + b2^2)/v2;

data;

param a1 := 60;
param b1 := 40;
param a2 := -40;
param b2 := -50;
param v1 := 10;
param v2 := 1.5;

solve;
display x;
reset;
```

NEOS Info

NEOS is the *Network Enabled Optimization Server* supported by our federal government and located at the *University of Wisconsin*.

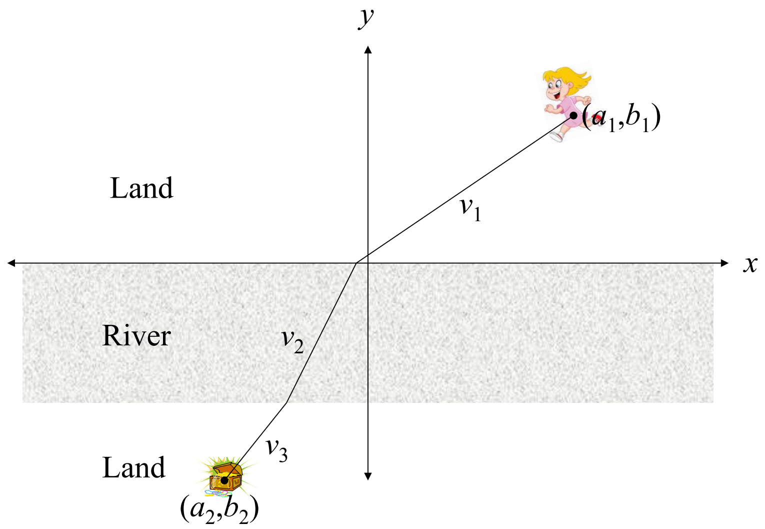
To submit an AMPL model to NEOS...

visit:	`http://www.neos-server.org/neos/`,
click:	on the *Submit a job to NEOS*,
scroll:	to the *Nonlinearly Constrained Optimization* list,
click:	on *LOQO [AMPL input]*,
scroll:	to *Commands File:*,
click:	on *Choose File*,
select:	a file from your computer that contains an AMPL model,
scroll:	to *e-mail address:*,
type:	your email address, and
click:	*Submit to NEOS*.

Piece of cake!

First Problem of First Assignment

Suppose the treasure is not exactly at the shore but rather is a certain distance away from the river. As shown, we are assuming the north shore of the river runs along the x-axis of our coordinate system. Assume that the river is $w = 30$ meters wide. For this problem, we need to figure out two things: (i) where you should enter the river and (ii) where you should exit it.



Write an AMPL model for this problem. Solve the problem using

$$\begin{aligned} (a_1, b_1) &= (60, 40) \\ (a_2, b_2) &= (-50, -50) \\ v_1 &= 10 \\ v_2 &= 1.5 \\ v_3 &= 7 \end{aligned}$$

Report the x-coordinate of the location at which you should enter the river and the x-coordinate of the location at which you should exit the river.

Freshman Calculus

- One variable
- Nonlinear objective function
- Sometimes variable constrained to an interval

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- Thousands of variables
- Linear objective function
- Linear (equality and inequality) constraints

There are multiple objectives for this course.

- Gain experience in formulating real-world problems as optimization problems.
- Learn how to distinguish good formulations from not-so-good ones.
- Learn how to solve real-world problems using `AMPL` software.
- Learn/understand the algorithms one uses to solve the problems.

Diet Problem

The McDonald's Diet Problem

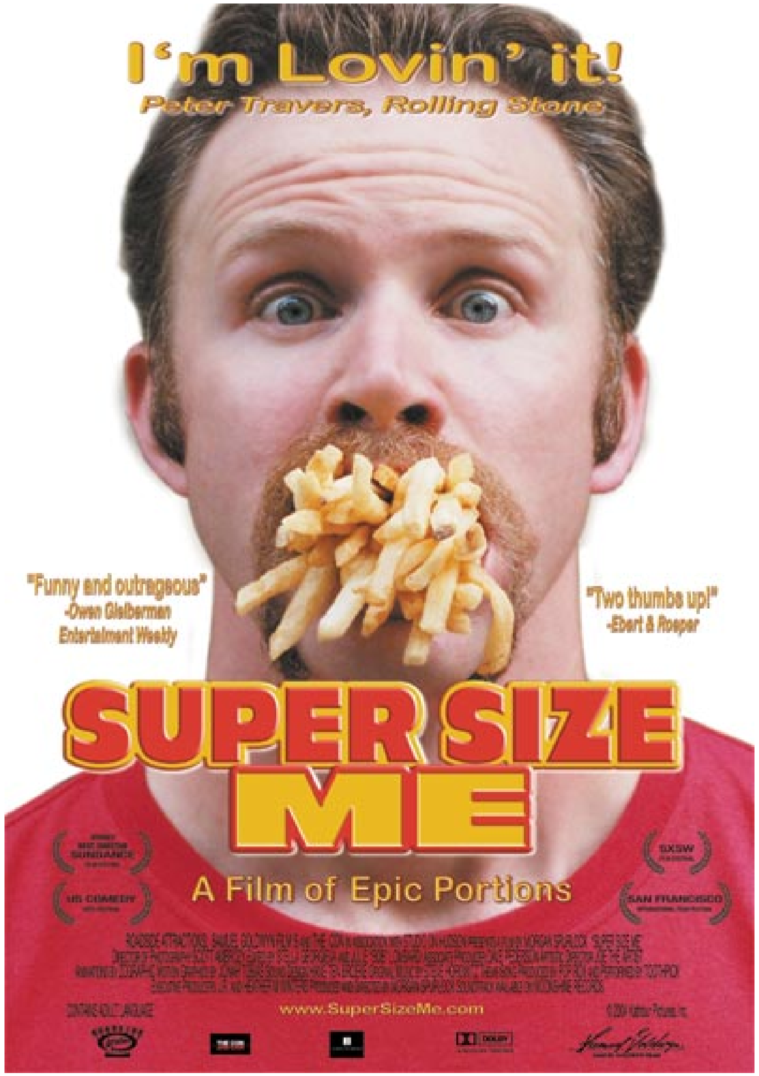
In words:

Minimize:

Calories

Subject to:

Total amounts of nutrients fall between certain minimum and maximum values.


I'm Lovin' it!
Peter Travers, Rolling Stone
"Funny and outrageous" -Owen Gleiberman Entertainment Weekly
"Two thumbs up!" -Ebert & Roeper
SUPER SIZE ME
A Film of Epic Portions
www.SuperSizeMe.com

An AMPL Model

```
# --- Declare the data sets and parameters ---------------

set NUTR;
set FOOD;

param f_min {FOOD} >= 0, default 0;
param f_max {j in FOOD} >= f_min[j], default 200;

param nutr_ideal {NUTR} >= 0;

param amt {NUTR,FOOD} >= 0;

# --- Declare the variables ------------------------------

var Buy {j in FOOD} integer >= f_min[j], <= f_max[j];

# --- State the objective --------------------------------

minimize Calories: sum {j in FOOD} amt["Cal",j] * Buy[j];

# --- State the constraints ------------------------------

subject to Dietary_bounds {i in NUTR}:
   0.8*nutr_ideal[i] <= sum {j in FOOD} amt[i,j] * Buy[j] <= 1.2*nutr_ideal[i];
```

The Data

```
param:  NUTR:   nutr_ideal  :=
        Cal       2500
        CalFat     600
        Fat         65
        SatFat      20
        Chol       300
        Sodium    2400
        Carbo      300
        Protein     50
        VitA       100
        VitC       100
        Calcium    100
        Iron       100
;

set  FOOD :=
"Bacon_Buffalo_Ranch_McChicken 5.6_oz_(159_g)"
"Big_Breakfast_(Large_Size_Biscuit) 10_oz_(283_g)"
"Big_Mac 7.6_oz_(215_g)"
"Chicken_McNuggets_(10_piece) 5.7_oz_(162_g)"
"Coca-Cola_Classic_(Medium) 21_fl_oz_cup"
"Diet_Coke_(Medium) 21_fl_oz_cup"
"Double_Quarter_Pounder_with_Cheese++ 10_oz_(283_g)"
"Egg_McMuffin 4.8_oz_(135_g)"
"Frappe_Caramel_(Medium) 16_fl_oz_cup"
"Hamburger 3.5_oz_(100_g)"
"Hash_Brown 2_oz_(56_g)"
"Large_French_Fries 5.4_oz_(154_g)"
"Mac_Snack_Wrap 4.4_oz_(125_g)"
"McFlurry_with_M&M'S_Candies_(12_fl_oz_cup) 10.9_oz_(310_g)"
"McRib_ 7.3_oz_(208_g)"
"Medium_French_Fries 4.1_oz_(117_g)"
"Mighty_Wings_(10_piece) 11.1_oz_(314_g)"
             # etc
  ;
```

```
param amt (tr):
                                                                        Cal   CalFat   Fat   SatFat   Chol  Sodium :=
  "Bacon_Buffalo_Ranch_McChicken 5.6_oz_(159_g)"                        420    180     20      4       50     1250
  "Big_Breakfast_(Large_Size_Biscuit) 10_oz_(283_g)"                    800    470     52      18      555    1680
  "Big_Mac 7.6_oz_(215_g)"                                              550    260     29      10      75     970
  "Chicken_McNuggets_(10_piece) 5.7_oz_(162_g)"                         470    270     30      5       65     900
  "Coca-Cola_Classic_(Medium) 21_fl_oz_cup"                             200    0       0       0       0      5
  "Diet_Coke_(Medium) 21_fl_oz_cup"                                     0      0       0       0       0      20
  "Double_Quarter_Pounder_with_Cheese++ 10_oz_(283_g)"                  750    380     43      19      160    1280
  "Egg_McMuffin 4.8_oz_(135_g)"                                         290    110     12      5       260    740
  "Frappe_Caramel_(Medium) 16_fl_oz_cup"                                550    200     23      15      80     160
  "Hamburger 3.5_oz_(100_g)"                                            250    80      9       3       25     480
  "Hash_Brown 2_oz_(56_g)"                                              150    80      9       1       0      310
  "Large_French_Fries 5.4_oz_(154_g)"                                   500    220     25      3       0      350
  "Mac_Snack_Wrap 4.4_oz_(125_g)"                                       330    170     19      7       45     670
  "McFlurry_with_M&M'S_Candies_(12_fl_oz_cup) 10.9_oz_(310_g)"          650    210     23      14      50     180
  "McRib_ 7.3_oz_(208_g)"                                               500    240     26      10      70     980
  "Medium_French_Fries 4.1_oz_(117_g)"                                  380    170     19      2       0      270
  "Mighty_Wings_(10_piece) 11.1_oz_(314_g)"                             960    570     63      13      295    2900
            # etc

:                                                                     Carbo  Protein  VitA    VitC  Calcium  Iron :=
  "Bacon_Buffalo_Ranch_McChicken 5.6_oz_(159_g)"                        41     20      2       10      15     15
  "Big_Breakfast_(Large_Size_Biscuit) 10_oz_(283_g)"                    56     28      15      2       15     30
  "Big_Mac 7.6_oz_(215_g)"                                              46     25      4       2       25     25
  "Chicken_McNuggets_(10_piece) 5.7_oz_(162_g)"                         30     22      0       4       2      6
  "Coca-Cola_Classic_(Medium) 21_fl_oz_cup"                             55     0       0       0       0      0
  "Diet_Coke_(Medium) 21_fl_oz_cup"                                     0      0       0       0       0      0
  "Double_Quarter_Pounder_with_Cheese++ 10_oz_(283_g)"                  42     48      10      2       30     35
  "Egg_McMuffin 4.8_oz_(135_g)"                                         31     17      10      0       25     15
  "Frappe_Caramel_(Medium) 16_fl_oz_cup"                                79     9       20      0       30     2
  "Hamburger 3.5_oz_(100_g)"                                            31     12      2       2       10     15
  "Hash_Brown 2_oz_(56_g)"                                              15     1       0       2       0      2
  "Large_French_Fries 5.4_oz_(154_g)"                                   63     6       0       20      2      8
  "Mac_Snack_Wrap 4.4_oz_(125_g)"                                       26     14      2       0       8      15
  "McFlurry_with_M&M'S_Candies_(12_fl_oz_cup) 10.9_oz_(310_g)"          96     13      15      0       45     8
  "McRib_ 7.3_oz_(208_g)"                                               44     22      2       2       15     20
  "Medium_French_Fries 4.1_oz_(117_g)"                                  48     4       0       15      2      6
  "Mighty_Wings_(10_piece) 11.1_oz_(314_g)"                             40     60      4       6       6      15
            # etc
  ;
```

Invoking the Solver and Displaying Results

```
solve;

printf {f in FOOD: Buy[f] > 0.3}: "%-60s %6.2f %4d \n", f, Buy[f], amt["Cal",f];
printf {i in NUTR}: "%-60s %7.1f (%4d)\n", i, sum {j in FOOD} amt[i,j] * Buy[j], nutr_ideal[i];
```

Complete AMPL model can be found here:

http://orfe.princeton.edu/∼rvdb/307/models/mcdonaldsdiet/idealDiet2014b.txt

Complete table of nutritional data can be found at:

http://nutrition.mcdonalds.com/getnutrition/nutritionfacts.pdf

First Run

```
rvdb@stars $ ampl idealDiet2014b.mod
LOQO 7.00: verbose=0
ignoring integrality of 296 variables
LOQO 7.00: optimal solution (14 QP iterations, 14 evaluations)
primal objective 2000
  dual objective 1999.999986

Chocolate_Chip_Cookie 1_cookie_(33_g)                             1.63   160
Diet_Coke_(Child) 12_fl_oz_cup                                    0.38     0
Diet_Coke_(Medium) 21_fl_oz_cup                                   0.32     0
Diet_Coke_(Small) 16_fl_oz_cup                                    0.56     0
EQUAL_0_Calorie_Sweetener 1_pkg_(1.0_g)                          21.32     0
Hotcakes 5.3_oz_(151_g)                                           1.83   350
Iced_Tea_(Child) 12_fl_oz_cup                                     1.45     0
Iced_Tea_(Large) 30_fl_oz_cup                                     0.38     0
Iced_Tea_(Medium) 21_fl_oz                                        0.56     0
Iced_Tea_(Small) 16_fl_oz_cup                                     0.56     0
SPLENDA_No_Calorie_Sweetener 1_pkg_(1.0_g)                       21.32     0
Side_Salad 3.1_oz_(87_g)                                          1.97    20


Cal                                                              2000.0 (2500)
CalFat                                                            609.6 ( 600)
Fat                                                                68.2 (  65)
SatFat                                                             23.9 (  20)
Chol                                                              242.3 ( 300)
Sodium                                                           2870.7 (2400)
Carbo                                                             333.7 ( 300)
Protein                                                            59.9 (  50)
VitA                                                              118.8 ( 100)
VitC                                                              109.2 ( 100)
Calcium                                                            92.8 ( 100)
Iron                                                               80.1 ( 100)
```

Second Run: Limit Amount of Artificial Sweetener

```
subject to NotTooMuchSweetener:
     Buy["EQUAL_0_Calorie_Sweetener 1_pkg_(1.0_g)"]
    +Buy["SPLENDA_No_Calorie_Sweetener 1_pkg_(1.0_g)"]
    <= 2* (
         Buy["Iced_Tea_(Child) 12_fl_oz_cup"]
        +Buy["Iced_Tea_(Large) 30_fl_oz_cup"]
        +Buy["Iced_Tea_(Medium) 21_fl_oz"]
        +Buy["Iced_Tea_(Small) 16_fl_oz_cup"]
 );
```

Output:

```
Chocolate_Chip_Cookie 1_cookie_(33_g)                          2.19  160
Diet_Coke_(Child) 12_fl_oz_cup                                 0.63    0
Diet_Coke_(Medium) 21_fl_oz_cup                                0.44    0
Diet_Coke_(Small) 16_fl_oz_cup                                 0.94    0
EQUAL_0_Calorie_Sweetener 1_pkg_(1.0_g)                       10.47    0
Fat_Free_Chocolate_Milk_Jug 1_carton_(236_ml)                  0.68  130
Hotcakes 5.3_oz_(151_g)                                        1.53  350
Iced_Tea_(Child) 12_fl_oz_cup                                  7.24    0
Iced_Tea_(Large) 30_fl_oz_cup                                  1.02    0
Iced_Tea_(Medium) 21_fl_oz                                     1.98    0
Iced_Tea_(Small) 16_fl_oz_cup                                  1.98    0
SPLENDA_No_Calorie_Sweetener 1_pkg_(1.0_g)                    10.47    0
Side_Salad 3.1_oz_(87_g)                                       1.83   20

Cal                                                          2000.0 (2500)
CalFat                                                        601.9 ( 600)
Fat                                                            67.4 (  65)
SatFat                                                         23.9 (  20)
Chol                                                          243.5 ( 300)
Sodium                                                       2742.7 (2400)
Carbo                                                         313.7 ( 300)
Protein                                                        59.9 (  50)
VitA                                                          118.4 ( 100)
VitC                                                          110.0 ( 100)
Calcium                                                       101.9 ( 100)
Iron                                                           80.1 ( 100)
```

Third Run: Enforce Integrality Constraints

To do that we need a “Mixed Integer Linear Programming” solver such as Gurobi.

Output:

```
Chocolate_Chip_Cookie 1_cookie_(33_g)                      6.00  160
Egg_McMuffin 4.8_oz_(135_g)                                1.00  290
Fat_Free_Chocolate_Milk_Jug 1_carton_(236_ml)              2.00  130
Ketchup_Packet 1_pkg_(10_g)                                5.00   10
Side_Salad 3.1_oz_(87_g)                                   1.00   20
Strawberry_Preserves 0.5_oz_(14_g)                        12.00   35

Cal                                                       2000.0 (2500)
CalFat                                                     530.0 ( 600)
Fat                                                         60.0 (  65)
SatFat                                                      23.0 (  20)
Chol                                                       330.0 ( 300)
Sodium                                                    2060.0 (2400)
Carbo                                                      330.0 ( 300)
Protein                                                     48.0 (  50)
VitA                                                        97.0 ( 100)
VitC                                                        83.0 ( 100)
Calcium                                                     99.0 ( 100)
Iron                                                        83.0 ( 100)
```

Linear Programming

Standard Form.

$$\begin{array}{llcrcrcrcl}
\text{maximize} & c_1x_1 & + & c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & \le & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & \le & b_2 \\
 & & & & \vdots & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \le & b_m \\
 & & & & x_1,\ x_2,\ \ldots & x_n & \ge & 0.
\end{array}$$

Why it's hard:

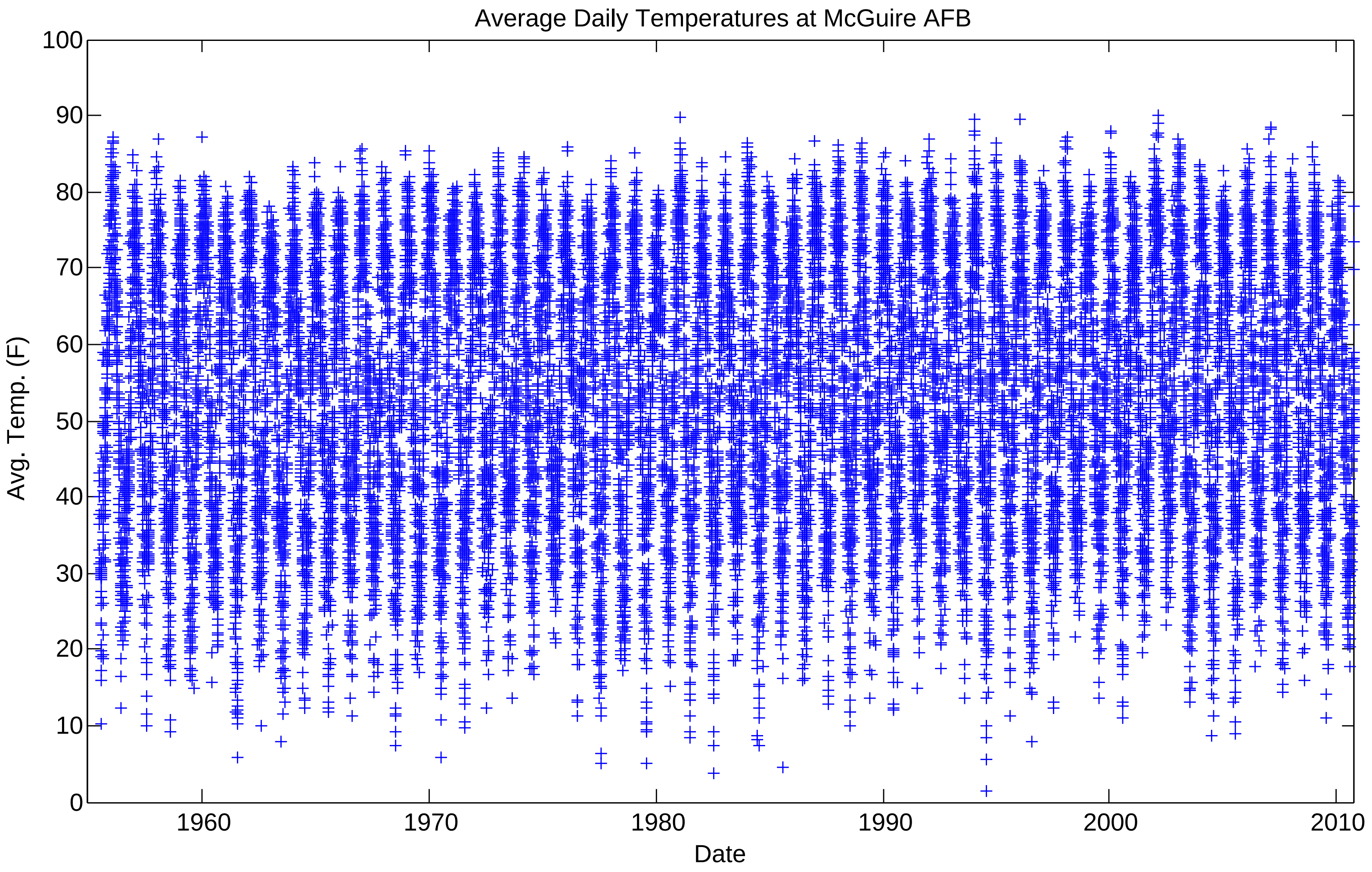
- Lots of variables (n of 'em).
- Lots of "boundaries" to check (the inequalities).

Why it's not impossible:

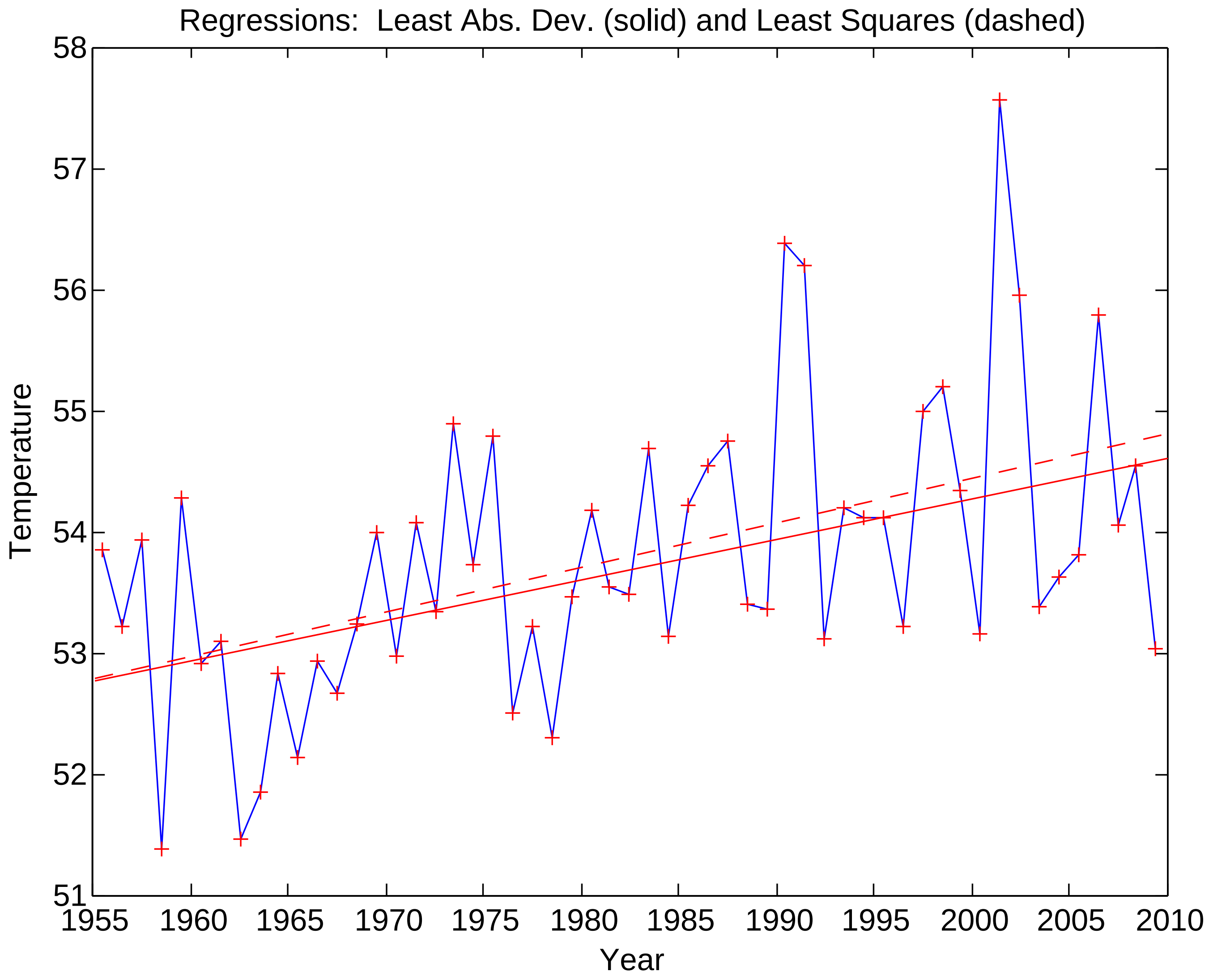
- All expressions are linear.

Climate Change

McGuire AFB Data from NOAA 55+ years



Yearly Averages



Regression Models

Let T_d denote the average temperature in degrees Fahrenheit on year $d \in D$ where D is the set of years from 1955 to 2010.

$$\begin{aligned} T_d = \;& x_0 + x_1 d && \textit{linear trend} \\ & + \varepsilon_d . && \textit{``error'' term} \end{aligned}$$

The parameters x_0 and x_1 are unknown regression coefficients.

Either

$$\min \sum_{d \in D} |\varepsilon_d| \qquad \textit{Least Absolute Deviations (LAD)}$$

or

$$\min \sum_{d \in D} \varepsilon_d^2 \qquad \textit{Least Squares}$$

What You Should Expect To Get From This Course

Learn how to formulate optimization problems.

Learn to distinguish easy problems from hard ones from impossible ones.

Learn some of the theory of Linear Programming (Duality Theory!).

Learn how to express optimization problems in AMPL and solve them.