MAT 168, Köppe, Spring 2015

Midterm Exam #1

Please fill out the following.

Name: Solution key **Student Id:** 1e308

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Problem	1	2	3	4	5
Points					
of	30	30	30	30	30

Total:

No electronic devices. No books, notes, . . .

Please write your answers in the provided space, and **show your work**.

Please turn over this page only after the start signal.

Problem 1. (30 points.) 5 min.

Consider the following three dictionaries of maximization problems.

$$\begin{array}{l} \zeta = 0 - 4x_1 + 0x_2 - 4x_3 \\ \hline x_4 = 5 + 7x_1 + 0x_2 + 1x_3 \\ x_5 = \frac{1}{2} + 4x_1 + 5x_2 + 6x_3 \end{array} \tag{1}$$

$$\begin{array}{rl} \zeta = & 2015 \; + 4x_1 + 24x_2 + 0x_3 \\ \hline x_4 = & -2 - 17x_1 \; + 0x_2 + 0x_3 \\ x_5 = & 3 \; + 4x_1 \; + 2x_2 - 1x_3 \end{array} \tag{2}$$

$$\begin{array}{rl} \zeta = & -13 + 4x_1 - 5x_2 \; + 2x_3 \\ \hline x_4 = & 0 - 1x_1 - 8x_2 - 11x_3 \\ x_5 = & 7 - 3x_1 - 4x_2 \; - 8x_3 \end{array} \tag{3}$$

(a) Write down the basic solution of dictionary (1).

$x_1 = x_2 = x_3 = 0, \quad x_4 = 5, \quad x_5 = \frac{1}{2}.$

(b) Determine which of the following properties hold for each of the dictionaries. Write your answers ("Yes", "No") in the table. Don't write explanations here; but see (c).)

	Dict (1)	Dict (2)	Dict (3)
Is its basic solution feasible?	Yes	No	Yes
Does it satisfy the optimality criterion?	Yes	No	No
Is its basic solution an optimal solution?	Yes	No	Yes !!

(c) Explain your answers to the last question: "Is its basic solution an optimal solution?"

(1) is feasible & satisfies the opt. criterion, so optimal.

(2) is not feasible, so can't be optimal.

(3) is feasible, but does not satisfy the opt. criterion; however, that is only a sufficient condition. So work harder: Pivot in x_1, then ratio test gives 0, x_4 out. Rewrite objective: $\zeta + 4x_4 = -13 - 37x_2 - 42x_3$, so $\zeta = -13 - 37x_2 - 42x_3 - 4x_4$, which satisfies opt. criterion and is the same basic solution.

Problem 2. (30 points.) 3 min.

Let $A = (a_{ij})_{i,j} \in \mathbf{R}^{m\times n}$ be a matrix of full row rank.

Let $b \in \mathbf{R}^m$ be a column vector.

(a) Find the rank of A.

m.

(b) Is the set

$$F_0 = \{x \in \mathbf{R}^n : Ax = 0\}$$

a linear subspace of $\mathbf{R}^n$? (Don't explain.)

Yes! [it's the kernel, a.k.a. null space of A]

(c) If the answer to (b) is yes, then find its dimension; if it is no, find a linear subspace of $\mathbf{R}^n$ of smallest dimension that contains F_0.

$\dim F_0 = n - m.$

(d) Define "halfspace" of $\mathbf{R}^n$.

a set of the form $\{x \in \mathbb{R}^n : a^T x \leq \alpha\}$, where a is a nonzero vector and $\alpha \in \mathbb{R}$.

(e) Define "(convex) polyhedron" of $\mathbf{R}^n$.

an intersection of finitely many halfspaces.

so this time we don't need to discuss generalized halfspaces

(7 min.)

Problem 3. (30 points.) Same notation as in the last problem.

(a) Consider the set

$$F_b^{\geq} = \{x \in \mathbf{R}^n : Ax = b,\, x \geq 0\}.$$

Prove or disprove: $F_b^{\geq}$ is a (convex) polyhedron. Because A has full row rank, it has no zero rows.

Let a^i be the rows of A, b_i the entries of b. Then $F_b^{\geq}$ is the intersection of the halfspaces $\{a^{i\,T} x \leq b_i\}$, $\{-a^{i\,T} x \leq -b_i\}$, $\{-e^{i\,T} x \leq 0\}$, so it's a convex polyhedron.

(b) Prove or disprove: Every (convex) polyhedron can be written in the form

$$F_b^{\geq} = \{x \in \mathbf{R}^n : Ax = b,\, x \geq 0\}.$$

That's false b/c $\mathbb{R}^1$ itself is a polyhedron [it's the intersection of the empty family], and $-1 \in \mathbb{R}^1$, but $-1 \notin F_b^{\geq}$ for any A, b.

(c) Consider the following definition: A subset $S \subseteq \mathbf{R}^n$ is said to be *convex* if for all $x, y \in S$ and for all $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in S$.

Prove that every convex polyhedron is convex.

It suffices to show this for halfspaces; then it will clearly hold for their intersection as well.

So take $S = \{x : a^T x \leq \alpha\}$ and $x, y \in S$; then with $\lambda \in [0,1]$, have

$$a^T(\lambda x + (1-\lambda)y) = \underbrace{\lambda}_{\geq 0} \underbrace{a^T x}_{\leq \alpha} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{a^T y}_{\leq \alpha}$$

$$\underset{4}{\leq} \lambda \alpha + (1-\lambda)\alpha = \alpha,$$

so $\lambda x + (1-\lambda) y \in S$.

Problem 4. (30 points) Model the following problem. Smin.

4-9 Wobbly Office Equipment (WOE) makes two models of tables for libraries and other university facilities. Both models use the same tabletops, but model A has 4 short (18-inch) legs and model B has 4 longer ones (30-inch). It takes 0.10 labor hour to cut and shape a short leg from stock, 0.15 labor hour to do the same for a long leg, and 0.50 labor hour to produce a tabletop. An additional 0.30 labor hour is needed to attach the set of legs for either model after all parts are available. Estimated profit is \$30 for each model A sold and \$45 for each model B. Plenty of top material is on hand, but WOE wants to decide how to use the available 500 feet of leg stock and 80 labor hours to maximize profit, assuming that everything made can be sold.

Don't forget to write down what your variables and constraints mean. At your option, either use mathematical notation or ZIMPL notation.

Let x_A, x_B be the number of tables of type A, B made.

Let x_S, x_L be the number of short, long legs made.

Model:

$$\max \quad 30x_A + 45x_B$$

s.t.

$$4x_A \le x_S$$
$$4x_B \le x_L$$
$$0.10x_S + 0.15x_L + \underbrace{0.80}_{(=0.50+0.30)}(x_A + x_B) \le 80$$
$$18x_S + 30x_L \le 500 \cdot 12$$
$$x_A, x_B, x_S, x_L \in \mathbb{Z}_+ .$$

[Can rework using just 2 variables.]

4 min.

Problem 5. (30 points) Demand for electricity over the next month is projected to be constant and exactly 424.00000 MW. There are 7 power plants that, when turned on for the month, can be continuously regulated to produce between l_i and u_i MW, $i = 1, \ldots, 7$. There is a fixed cost of s_i \$ for every power plant that is turned on, and a variable cost of c_i \$/MW.

(a) Write a mixed-integer linear optimization model to find a minimum-cost plan to operate the power plants that guarantees that the demand is satisfied. At your option, either use mathematical notation or ZIMPL notation.

Define x_i = power produced by plant i (in MW)

y_i = 1 if plant i is turned on; 0 o/w.

$$\min \sum s_i y_i + \sum c_i x_i$$

$$\text{s.t.} \quad l_i y_i \le x_i \le u_i y_i$$

$$\sum x_i = 424.00000$$

$$x_i \ge 0, \quad i = 1, \ldots, 7$$

$$y_i \in \{0,1\}, \quad i = 1, \ldots, 7$$

(b) Add a constraint: At least 3 plants need to be turned on.

$$\sum y_i \ge 3.$$

(c) Add a constraint: Plant 3 and 7 cannot be both turned on.

$$y_3 + y_7 \le 1$$

(d) Add a constraint: If plants 4 and 5 are both on, then also plant 6 must be on.

$$y_6 \ge y_4 + y_5 - 1.$$

End of the exam.