MAT 168, Fall 2017

Homework 4

(due before class on Wednesday, November 8)

A. Chapter 5: Exercises 5.6, 5.15, and 5.16

B. Chapter 7: Exercise 7.1

C. Let x and x' be two different optimal solutions of a linear program in standard form. Prove that for any number $t \in \mathbb{R}$ with $0 \leq t \leq 1$,

$$x_t := t\,x + (1-t)\,x'$$

is also an optimal solution of the same linear program.

D. For an infeasible linear program, show that if its dual linear program is feasible, then the dual linear program must be unbounded.

E. The dictionary

$$D^* = \begin{bmatrix} b^* & A^* \\ -\zeta^* & \left(c^*\right)^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 1 & -2 \\ 1 & 1 & 3 & 0 & -1 & 0 & 1 \\ -10 & 0 & -1 & 0 & -2 & -3 & -1 \end{bmatrix} \tag{1}$$

is optimal for the linear program

$$\begin{array}{lrl} \max & 7x_1 + 14x_2 + 3x_3 - 6x_4 & \\ \text{s.t.} & 2x_1 + \ 4x_2 + \ x_3 - \ x_4 & \leq 3 \\ & x_1 + \ 3x_2 \qquad\quad - \ x_4 & \leq 1 \\ & x_1,\, x_2,\, x_3,\, x_4 & \geq 0. \end{array} \tag{2}$$

(a) For the dictionary D^*, determine the set B of basic indices and the set N of nonbasic indices. Write down an optimal solution of (2).

(b) Determine the matrix F in the relation

$$D^* = F D_0$$

that connects the optimal dictionary D^* with the initial dictionary D_0 associated with (2).

(c) Determine the optimal solution $y^\star$ of the dual linear program of (2).

F. Consider the linear program (2). Let D^* be the dictionary (1), which is optimal for (2).

(a) The right-hand side b of the inequality constraints of (2) is modified as follows:

$$b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \longrightarrow \quad b(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{where} \quad t \in \mathbb{R}.$$

Determine the range of all $t \in \mathbb{R}$ for which a suitable modification of D^* is optimal for the modified version of (2). What is the corresponding optimal solution?

(b) The first and last coefficients, c_1 and c_4, of the objective function of (2) are modified as follows:

$$c_1 = 7 \longrightarrow c_1(t) = 7 + t \quad \text{and} \quad c_4 = -6 \longrightarrow c_4(t) = -6 + t, \quad \text{where} \quad t \in \mathbb{R}.$$

Determine the range of all $t \in \mathbb{R}$ for which a suitable modification of D^* is optimal for the modified version of (2). What is the corresponding optimal solution?

(c) An additional decision variable $x_0 \geq 0$ is added to (2). Its contribution to the left-hand side of the inequality constraints $Ax \leq b$ of (2) is as follows:

$$Ax \leq b \quad \longrightarrow \quad a_0 x_0 + Ax \leq b, \quad \text{where} \quad a_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The coefficient c_0 of x_0 in the objective function of the modified version of (2) is a real parameter. Determine the range of all $c_0 \in \mathbb{R}$ for which a suitable modification of D^* is optimal for the modified version of (2). What is the corresponding optimal solution?