Solutions to ORF 307 HW4

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Exercise 5.1

There are 4 variables and 3 constraints in the primal, so there will be 3 variables and 4 constraints in the dual. Table 5.1 tells that an equality constraint corresponds to a free variables and an inequality constraint corresponds to a nonnegative variable. Therefore y_3 is free and y_1 and y_2 are nonnegative. x_1 and x_4 are free variables so that means the first and the fourth constraints will be equality and the other two will be inequality. Now it is easy to write down the dual as:

$$\min\ 3y_2 + y_3$$

$$\begin{aligned}
-y_1 + 4y_2 - y_3 &= 1 \\
-2y_1 + 3y_2 - y_3 &\geq -2 \\
y_1 + 4y_2 + 2y_3 &\geq 0 \\
-y_1 - 2y_2 + y_3 &= 0 \\
y_1 &\geq 0 \\
y_2 &\geq 0
\end{aligned}$$

Exercise 5.5

(a)

$$\min\ 6y_1 + 1.5y_2 + 4y_3$$

$$
\begin{aligned}
2y_1 - 2y_2 + 3y_3 &\geq 2 \\
3y_1 + 4y_2 + 2y_3 &\geq 8 \\
+ 3y_2 - 2y_3 &\geq -1 \\
6y_1 \qquad - 4y_3 &\geq -2 \\
y_1 &\geq 0 \\
y_2 &\geq 0 \\
y_3 &\geq 0
\end{aligned}
$$

(b) Nonbasics: w_1. x_2, w_3 and x_4. Basics: x_1, w_2 and x_3.

(c) $x_1 = 3$, $w_2 = 0$ $x_3 = 2.5$, $w_1 = 0$. $x_2 = 0$, $w_3 = 0$ and $x_4 = 0$. It is feasible and it is degenerate.

(d)

$$
\begin{aligned}
\xi &= -3.5 - 3z_1 \qquad\qquad - 2.5z_3 \\
y_1 &= 0.25 + 0.5z_1 - 1.25y_2 + 0.75z_3 \\
z_2 &= -6.25 + 1.5z_1 + 3.25y_2 + 1.25z_3 \\
y_3 &= 0.5 \qquad\qquad + 1.5y_2 - 0.5z_3 \\
z_4 &= 1.5 + 3z_1 \quad - 13.5y_2 + 6.5z_3
\end{aligned}
$$

(e) No, it is not dual feasible.

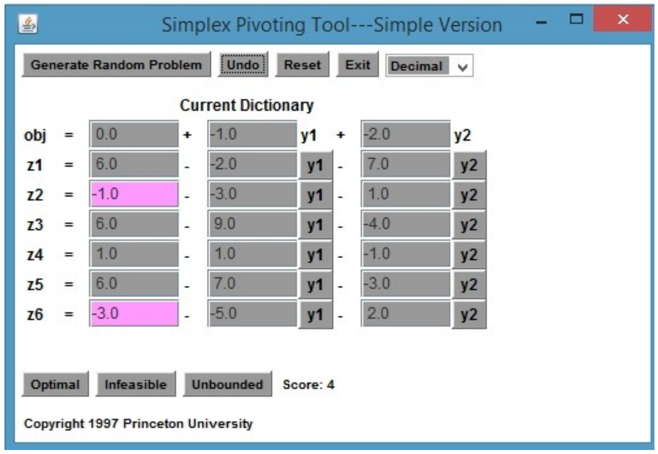
(f) Yes, Simplex method always satisfies complementary slackness.

(g) No, there are positive coefficient on the objective, you can pivot a few more steps. The optimal objective values is 8.73.

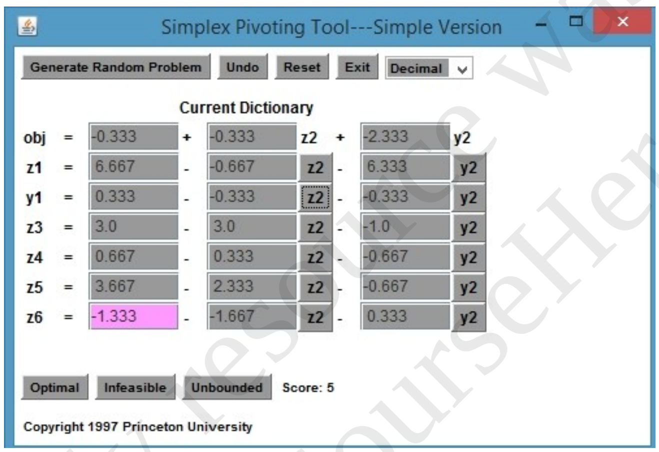
(h) x_2 will enter and w_2 will leave. The pivot will be degenerate.

Exercise 5.6

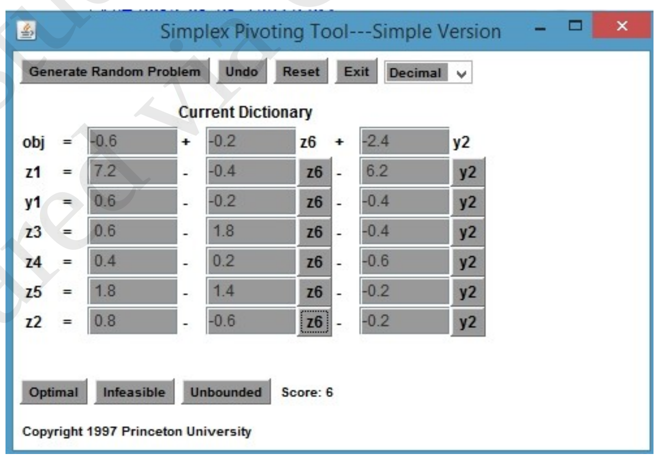
We set up the problem and use the dual simplex method directly:



By Bland's rule, we choose z_2 as the leaving variable and y_1 as the entering variable:



Next, we have to choose z_6 as the leaving variable and z_2 as the entering variable:



Now the dictionary is optimal and we read off the solution (Note the variable y in the pivot tool represents x in our problem): $x_1 = 0.6$ and $x_2 = 0$.

Exercise 5.10

The solution depends on the size of the problem.

Exercise 5.13

Each transformation contains two steps: 1. Find the dual of the primal problem 2. Change the "minimization" on the dual objective to "maximization". After four transformations, you get back to the original problem and it seems that at each step, the objective value is only increasing (or at least keeping the same).

It is important to note that when a linear programming problem has a minimum value, it does not necessary have a maximum value. That means while the dual problem has a optimal value on the minimization, the corresponding "maximization" problem can be unbounded. When the primal problem is unbounded, duality theory tells us that the dual problem is infeasible. Now when you are trying to maximize an infeasible problem, the objective will be negative infinity. This tells us that during one of the transformation, the objective value actually drops from ∞ to $-\infty$.

The theorem could be "More than half of the LP problems do not have optimal values."

Of course, there are examples where the four inequalities are indeed all equalities. For example, you can let the first objective be a constant and let $b_i = 0$ as well, then the objective value never changes.

Exercise 5.15

Since the primal and dual solutions are given, we only need to check the primal feasibility, the dual feasibility and complementary slackness to verify the optimality of the solution. We first write down the primal and dual of the problem with slack variables. Assuming all primal and dual variables and slack variables are nonnegative, the primal is given by

$$\max \sum_{j=1}^{n} p_j x_j$$

$$\text{subject to:} \quad \sum_{j=1}^{n} q_j x_j + w_0 = \beta$$

$$x_j + w_j = 1,\ j = 1, \cdots, n,$$

and the dual is

$$\min \beta y_0 + \sum_{j=1}^{n} y_j$$

$$\text{subject to:} \quad q_j y_0 + y_j - z_j = p_j,\ j = 1, \cdots, n.$$

We first check the primal feasibility. Note $x_k = \frac{\beta - q_{k+1} - \cdots - q_n}{q_k}$, so $\sum_{j=1}^{n} q_j x_j = q_k x_k + \sum_{j=k+1}^{n} q_j = \beta - q_{k+1} - \cdots - q_n + \sum_{j=k+1}^{n} q_j = \beta \le \beta$. Therefore, $w_0 = 0$. Also $w_j = 1, \forall j < k$ and $w_j = 0$, $\forall j > k$. Now we only need to check $0 \le x_k \le 1$. By the definition of $k = \min\{j:\ q_{j+1} + \cdots + q_n \le \beta\}$, we have $\sum_{k+1}^{n} q_k \le \beta$ and $\sum_{k}^{n} q_k \ge \beta$, substitute these into the definition of x_k, we have $0 \le x_k \le 1$. The slack variable $w_k = \frac{\beta - q_{k+1} - \cdots - q_n}{q_k} - 1$.
We then check the dual feasibility. $y_0 = \frac{p_k}{q_k}$ is a fixed number. According to the assumption, $\frac{p_j}{q_j} - \frac{p_k}{q_k} \ge 0$, $\forall j > k$, so we have $y_j \ge 0, \quad \forall j$. Now we only needs to verify $q_j y_0 + y_j \ge p_j$, $\forall j$. When $j \le k$, $q_j y_0 + y_j = q_j \frac{p_k}{q_k} \ge p_j$ by the assumption. $z_j = q_j \frac{p_k}{q_k} - p_j$, $\forall j \le k$. When $j > k$, $q_j y_0 + y_j = q_j \frac{p_k}{q_k} + q_j(\frac{p_j}{q_j} - \frac{p_k}{q_k}) = p_j \ge p_j$. $z_j = 0$, $\forall j > k$.
Finally we check the complementary slackness. We need $x_j z_j = 0$, $\forall j$. When $j < k$, $x_j = 0$. When $j > k$, $z_j = 0$. When $j = k$, $z_k = q_k \frac{p_k}{q_k} - p_k = 0$. We also need $y_j w_j = 0$, $\forall j$. When $j = 0$, $w_0 = 0$. When $j > k$, $w_j = 0$. When $1 \le j \le k$, $y_j = 0$. This shows the optimality of the given primal and dual solutions.

AMPL Problem

Part 1.

```
reset;

set STUDENTS;
set COURSES;
set GRADES within {STUDENTS, COURSES};

param grade {GRADES};

var aptitude {STUDENTS};
var easiness {COURSES};
var dev {GRADES} >= 0;

minimize sum_dev: sum {(s,c) in GRADES} dev[s,c];
```

```
subject to def_pos_dev {(s,c) in GRADES}:
aptitude[s] + easiness[c] - grade[s,c] <= dev[s,c];
subject to def_neg_dev {(s,c) in GRADES}:
-dev[s,c] <= aptitude[s] + easiness[c] - grade[s,c];
subject to normalized_easiness: sum {c in COURSES} easiness[c] = 0;

data;

set STUDENTS :=
include "names.txt";
set COURSES :=
include "courses.txt";
param: GRADES: grade :=
include "grade.txt";

solve;

display aptitude;
display easiness;
display dev;
```

The output is

```
The result is aptitude [*] :=
George  3.19091
  John  3.79091
  Paul  3.49091
 Ringo  2.79091
  Yoko  3.79091;

easiness [*] :=
AST305  -0.0909091
BIO201   0.209091
COS127  -0.490909
ECO101   0.209091
ECO221  -0.0909091
```

```
ECO307  -0.0909091
HIS203  -0.0909091
HIS411   0.209091
PHY211  -0.190909
PSY327   0.209091
REL242   0.209091;
```

John and Yoko did the best since they have the highest aptitude. Ringo performed the worst with a lowest aptitude score. BIO201, ECO101, HIS411, PSY327 and REL242 are equally inflated. COS127 is the course with the most deflated grades.
part 2.

```
reset;

set STUDENTS;
set COURSES;
set GRADES within {STUDENTS, COURSES};

param grade {GRADES};

var aptitude {STUDENTS};
var easiness {COURSES};
var dev {GRADES};

minimize sum_dev: sum {(s,c) in GRADES} dev[s,c]^2;

subject to def_dev {(s,c) in GRADES}:
aptitude[s] + easiness[c] - grade[s,c] = dev[s,c];
subject to normalized_easiness: sum {c in COURSES} easiness[c] = 0;

data;
set STUDENTS :=
include "names.txt";
set COURSES :=
include "courses.txt";
param: GRADES: grade :=
```

```
include "grade.txt";

solve;

display aptitude;
display easiness;
display dev;
```

The output is

```
aptitude [*] :=
George  3.28246
  John  3.77983
  Paul  3.5443
 Ringo  2.80517
  Yoko  3.80923;

easiness [*] :=
AST305  -0.00294001
BIO201   0.121338
COS127  -0.532287
ECO101   0.14706
ECO221  -0.0195652
ECO307   0.0311364
HIS203  -0.0691737
HIS411   0.190367
PHY211  -0.210642
PSY327   0.18888
REL242   0.155826;
```

Yoko did the best with an highest aptitude. Ringo performed the worst with a lowest aptitude score. HIS411 has the most inflated grades and COS127 has the most deflated grades.