Optimization (168)

Lecture 3-4-5

Jesús De Loera

UC Davis, Mathematics

Wednesday, April 2, 2012

THE SIMPLEX METHOD

LINEAR PROGRAMS (LPs)

We wish to optimize subject to linear constraints ($=$ or $\geq$ is special case)

$$\text{maximize (minimize) } C_1 x_1 + C_2 x_2 + \cdots + C_d x_d$$

$$\text{among all } \; x_1, x_2, \ldots, x_d, \text{satisfying:}$$

$$\begin{gathered}
a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,d} x_d \leq b_1 \\
a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,d} x_d \leq b_2 \\
\vdots \\
a_{k,1} x_1 + a_{k,2} x_2 + \cdots + a_{k,d} x_d \leq b_k
\end{gathered}$$

- A vector satisfying all constraints is called a **feasible solution**
- A feasible solution is **Optimal** if it attains a maximum or minimum.
- Some problems have no solutions, so they are called **infeasible** for example

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 2 \\ & -2x_1 - 2x_2 \le -9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- An LP can be **unbounded** when there are arbitrarily large feasible solutions (say in Euclidean norm).

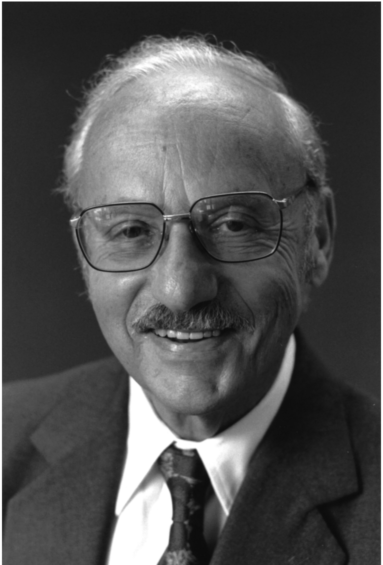
$$\begin{aligned} \max \quad & x_1 - 4x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \le -1 \\ & -x_1 - 2x_2 \le -2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- **GOAL:** Find optimal solution OR detect when LP infeasible or unbounded!

The simplex method

George Dantzig invented the SIMPLEX METHOD in the 1940's.

Lemma The set of all feasible solutions is a convex polyhedron. Let us use geometry!



LINEAR PROGRAMS (LPs)

The general LP problem can be reduced to the case

$$\text{maximize (minimize) } C_1 x_1 + C_2 x_2 + \cdots + C_d x_d$$

$$\text{subject to } \; x_1, x_2, \ldots, x_d, \text{satisfying:}$$

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d &\leq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d &\leq b_2 \\
\vdots \qquad\qquad & \vdots \\
a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_d &\leq b_k
\end{aligned}
$$

with the condition $\quad x_1, x_2, \ldots, x_d \geq 0.$

WHY??

Replace a non-restricted x_i by $x_i = x_i^+ - x_i^-$, with $x_i^+ \geq 0, x_i^- \geq 0.$

Next we can rewrite this as

$$\text{maximize (minimize) } C_1 x_1 + C_2 x_2 + \cdots + C_d x_d$$

$$\text{subject to } \; x_1, x_2, \ldots, x_d, \text{satisfying:}$$

$$a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,d} x_d + s_1 = b_1$$

$$a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,d} x_d + s_2 = b_2$$

$$\vdots \qquad \vdots$$

$$a_{k,1} x_1 + a_{k,2} x_2 + \cdots + a_{k,d} x_d + s_k = b_k$$

with the condition $\quad x_1, x_2, \ldots, x_d, s_1, s_2, \ldots s_k \geq 0.$

WHY??

The variables s_i are capture the slack, **slack variables**. Original variables **decision variables**. Looks more like linear algebra.

From this form we follow an iterative procedure!

How the simplex Method works

$$
\begin{aligned}
\max \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\
& 4x_1 + x_2 + 2x_3 \leq 11 \\
& 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Rewrite problem using the slack variables (we call them s_1, s_2, s_3

$$
\begin{aligned}
z &= 5x_1 + 4x_2 + 3x_3 \\
s_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
s_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
s_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

The problem is now the same as $\max z$ subject to $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

- **Key idea 1** There is a one-to-one correspondence of the feasible solutions of the original LP to the feasible solutions of the LP with slacks (**canonical form**)!
- **Key idea 2** Carry on successive improvements, starting with a feasible of LP in canonical form proceed to a **better** feasible solution, one with better value of z.
- Given

$$
\begin{aligned}
z &= 5x_1 + 4x_2 + 3x_3 \\
s_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
s_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
s_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

  Set $x_1 = x_2 = x_3$ so we have a solution but $z = 0$.
- NOTE: if we keep $x_2 = x_3 = 0$ but increase x_1 we increase z.
- Just how much do we increase x_1 (while keeping $x_2 = x_3 = 0$) and still maintain feasibility?

- Look at first equation! $s_1 = 5 - 2x_1 - 3x_2 - x_3$, s_1 must be non-negative! Thus $x_1 \leq 5/2$.
- From second equation $s_2 = 11 - 4x_1 - x_2 - 2x_3 \geq 0$ Thus $x_1 \leq 11/4$.
- From third equation $s_3 \geq 0$ implies $x_1 \leq 8/3$.
- Of these 3 bounds which is more restrictive? The first is the most restrictive!!
- Keep $x_2 = x_3 = 0$ but increase x_1 we increase to $\frac{5}{2}$, thus now $s_1 = 0, s_2 = 1, s_3 = \frac{1}{2}$. Now $z = \frac{25}{2}$
- Need to rewrite the system in such a way that variables that are positive are given in term of those that are zero, rewrite
- $x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1$, use it to rewrite the system!!! Replace x_1 everywhere in

$$
\begin{aligned}
z &= 5x_1 + 4x_2 + 3x_3 \\
s_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
s_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
s_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- From

$$
\begin{aligned}
z &= 5x_1 + 4x_2 + 3x_3 & \rightarrow \quad z &= 5(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1) + 4x_2 + 3x_3 \\
s_1 &= 5 - 2x_1 - 3x_2 - x_3 & \rightarrow \quad x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1 \\
s_2 &= 11 - 4x_1 - x_2 - 2x_3 & \rightarrow \quad s_2 &= 11 - 4(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1) - x_2 - 2x_3 \\
s_3 &= 8 - 3x_1 - 4x_2 - 2x_3 & \rightarrow \quad s_3 &= 8 - 3(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1) - 4x_2 - 2x_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

We get a new system is

$$
\begin{aligned}
z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}s_1 \\
x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1 \\
s_2 &= 1 + 5x_2 \quad + 2s_1 \\
s_3 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}s_1 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- $$
\begin{aligned}
z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}s_1 \\
x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1 \\
s_2 &= 1 + 5x_2 \quad + 2s_1 \\
s_3 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}s_1 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- We shall again increase value of z by increasing the value of a variable on the right side, while keeping at zero the others.
- NOTE: increasing x_2 or s_1 would give a **decrease!**. Thus we must increase x_3! How much?
- Read answer from new system: While $x_2 = s_1 = 0$ we have $x_3 \leq 1$ is the most we can increase!!! (WHY?? Which equation says that?).
- Rewrite the system so that the positive value variables are on the left (x_1, x_3, s_2) and the right-side has only zero-valued variables (x_2, s_1, s_3).
- From third equation we have $x_3 = 1 + x_2 + 3s_1 - 2s_3$. Substitute in the system again. We get new system!!!!

- After substitution

$$
\begin{aligned}
z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}s_1 \\
x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1 \\
s_2 &= 1 + 5x_2 \quad + 2s_1 \\
s_3 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}s_1 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

  turns into

- $$
\begin{aligned}
z &= 13 - 3x_2 - s_1 - s_3 \\
x_1 &= 2 - 2x_2 - 2s_1 + s_3 \\
s_2 &= 1 + 5x_2 \quad + 2s_1 \\
x_3 &= 1 + x_2 + 3s_1 - 2s_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- $$
\begin{aligned}
z &= 13 - 3x_2 - s_1 - s_3 \\
x_1 &= 2 - 2x_2 - 2s_1 + s_3 \\
s_2 &= 1 + 5x_2 \quad + 2s_1 \\
x_3 &= 1 + x_2 + 3s_1 - 2s_3 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- To repeat the process we need to find a variable from the right side to go to the left!!! One that when we increase it will increase z!!!
- BUT if we increase either x_2, s_1, s_3 we will **decrease** z!!!!
- We have reached a stand-still. Claim: the answer we have now is OPTIMAL!!
- Why Look at the first row. Setting $x_2 = s_1 = s_3 = 0$, yields $z = 13$, but since all feasible solutions MUST satisfy $x_2, s_1, s_3 \geq 0$ this must be the best possible value!!

General Description

- We start with a problem in the form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1 \dots m$$

$$x_j \geq 0$$

- Turn it into a **DICTIONARY** by adding slack variables (for each inequality).

$$z = \sum_{j=1}^{n} c_j x_j$$

$$s_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad \text{for} \quad i = 1 \dots m$$

$$x_j, s_i \geq 0$$

- The simplex method moves from one dictionary to the next.

- We call the variables in the left **BASIC** variables and the variables in the right **NON-BASIC** variables.
- If N denotes the NON-BASIC variables, B denotes the basic variables

$$z = \bar{z} + \sum_{j \in N} \bar{c}_j x_j$$
$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for} \quad i \in B$$
$$x_j, s_i \geq 0$$

- The current solution when we set the non-basic variables to zero is a **BASIC FEASIBLE SOLUTION**.
- Exactly one variable goes from NON-BASIC to become BASIC (moves from RIGHT to LEFT) and vice versa.
- Pick a non-basic variable whose coefficient in the objective function is > 0. Candidates are $\{j \in N, \bar{c}_j > 0\}$.
- If Candidate set is empty, the we have an OPTIMAL SOLUTION!!!! WHY? Current basic feasible solution is best possible.
- Otherwise if Candidates has more than one element, choose one. HOW? Many options!!!

- Say x_k is chosen to leave (currently non-basic, turned into basic). We require $\bar{b}_i - \bar{a}_{ik} x_k \geq 0$. This implies
$$x_k \leq \frac{\bar{b}_i}{\bar{a}_{ik}}$$
Pick index l where $\frac{\bar{b}_l}{\bar{a}_{lk}}$ is smallest possible (NOTE: book talks about maximum reciprocal).
- Next, pivot, using the equation
$$x_l = \bar{b}_l - \sum_{j \in N} \bar{a}_{lj} x_j$$
we rewrite, put x_k in the left, x_l in the right.
- REPEAT!!!

BUT there are several questions? What if the initial b_i are not all positive? How to recognize unboundedness? How do I know that the process will terminate?

Initialization: PHASE I

- Suppose the original b_i are not all non-negative. Then we cannot right away find a feasible solution!!!!
- Consider instead of the original problem:

$$\begin{aligned} &\max -x_0 \\ &\text{s.t.} \sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i \quad i = 1 \ldots m \\ &x_j \geq 0 \end{aligned}$$

- **Lemma** Easy to find a feasible solution: Set $x_j = 0$ for $j = 1, \ldots, n$ and set x_0 large enough positive.
- **Lemma** Original problem has a feasible solution if and only if an optimal solution of the auxiliar problem has objective value ZERO.

$$
\begin{aligned}
\max\quad & x_1 - x_2 + x_3 \\
\text{s.t.}\quad & 2x_1 - x_2 + 2x_3 \le 4 \\
& 2x_1 - 3x_2 + x_3 \le -5 \\
& -x_1 + x_2 - 2x_3 \le -1 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

We need to put it in the auxiliary form, where a feasible solution is easy to find:

$$
\begin{aligned}
\max\quad & -x_0 \\
\text{s.t.}\quad & 2x_1 - x_2 + 2x_3 - x_0 \le 4 \\
& 2x_1 - 3x_2 + x_3 - x_0 \le -5 \\
& -x_1 + x_2 - 2x_3 - x_0 \le -1 \\
& x_1, x_2, x_3, x_0 \ge 0
\end{aligned}
$$

We put it in dictionary form by adding slacks.

$$
\begin{aligned}
z &= -x_0 \\
x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

With a single pivot, x_0 enters basics, x_5 leaves basics, we get

$$
\begin{aligned}
z &= -5 - 2x_1 + 3x_2 - x_3 - x_5 \\
x_4 &= \quad 9 \qquad\qquad - 2x_2 - x_3 + x_5 \\
x_0 &= \quad 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_6 &= \quad 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

Starts normal pivoting!!!

$$\begin{aligned}
z &= -2 + 0.25x_1 + 1.25x_3 - 0.25x_5 - 0.75x_6 \\
x_2 &= 1 + 0.75x_1 + 0.75x_3 + 0.25x_5 - 0.25x_6 \\
x_0 &= 2 - 0.25x_1 - 1.25x_3 + 0.25x_5 + 0.75x_6 \\
x_4 &= 7 - 1.5x_1 - 2.5x_3 + 0.5x_5 + 0.5x_6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}$$

Pivot again!? x_3 enters basics, x_0 leaves.

$$\begin{aligned}
z &= \quad - x_0 \\
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\
x_4 &= 3 \quad - x_1 \quad - x_6 + 2x_0 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}$$

WE have an optimal dictionary!! and $x_0 = 0$.

We are (almost) ready to solve the original problem!!!

$$
\begin{aligned}
z &= x_1 - x_2 + x_3 \\
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\
x_4 &= 3 \quad - x_1 \qquad - x_6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

WHAT IS THE PROBLEM NOW? WHY CAN'T WE START?

Need to rewrite the objective function in terms of the non-basic variables x_1, x_5, x_6. The correct z is

$$
\begin{aligned}
z &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6 \\
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\
x_4 &= 3 - x_1 \quad - x_6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

PHASE I is finished!! Now we have a feasible solution for original problem!

Starting from this feasible solution we find the optimum (PHASE II)

UNBOUNDEDNESS

- What if all the ratios we test are negative or there is division by zero?
- Sometimes the non-basic variable can be increased indefinitely!!! Producing an arbitrarily large objective value.
- EXAMPLE

$$
\begin{aligned}
z &= 5 + x_3 - x_1 \\
x_2 &= 5 + 2x_3 - 3x_1 \\
x_4 &= 7 \qquad\; - 4x_1 \\
x_5 &= \qquad\qquad x_1
\end{aligned}
$$

SUMMARY

- Given a feasible dictionary we have to select an entering variable, find a leaving variable and to construct the new dictionary by pivoting!!!
- **Choosing an entering variable:** The entering variable is a non-basic variable x_j with a positive coefficient in the objective function row.
- **NOTE:** The rule is ambiguous, we may have more than one candidate!!! (SO FAR we choose x_j with largest cofficient!).
- **Finding the leaving variable:** The leaving variable is that basic variable whose non-negativity imposes the most stringent upper bound on the increase of the entering variable.
- **NOTE:** Again the rule is ambiguous, we may have more than one candidate!!! This has VERY important consequences!!!
- If there is no candidate for leaving the basis, then we can make the value of the entering variable as large as we wish!! **UNBOUNDED!**

DEGENERACY

$$
\begin{aligned}
z &= 2x_1 - x_2 + 8x_3 \\
x_4 &= 1 \qquad\qquad - 2x_3 \\
x_5 &= 3 - 2x_1 + 4x_2 - 6x_3 \\
x_6 &= 2 + x_1 - 3x_2 - 4x_3 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

Clearly x_3 enters the basis, but who leaves? All variables x_4, x_5, x_6 give the same increase! Choose any!! Say x_4 pivot.

$$
\begin{aligned}
z &= 4 + 2x_1 - x_2 - 4x_4 \\
x_3 &= 0.5 \qquad\qquad - 0.5x_4 \\
x_5 &= -2x_1 + 4x_2 + 3x_4 \\
x_6 &= x_1 - 3x_2 - 2x_4 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

NOTE: x_5, x_6 are basic, but they are also equal to ZERO! DEGENERATE PROBLEM!

This has some annoying consequences. For example if we pivot again, x_1 enters the basis and x_5 leaves (limit of increment is zero!).

$$
\begin{aligned}
z =& 4 + 3x_2 - x_4 - x_5 \\
x_1 =& 2x_2 + 1.5x_4 - 0.5x_5 \\
x_3 =& 0.5 \qquad\quad - 0.5x_4 \\
x_6 =& - x_2 + 3.5x_4 - 0.5x_5 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_0 \geq 0
\end{aligned}
$$

This does not change the solution at all!!!
Sometimes the simplex method goes through a few degenerate iterations one after the other, sometimes **CYCLING CAN HAPPEN!!**