Lecture 7– Wednesday October 5, 2016

"There is great intrinsic value in maths: Nothing is as straightforward and honest."

"Mathematics is the door and key to the sciences."
Roger Bacon

"Mathematics is the science of what is clear by itself."
Carl Gustav Jacob Jacobi

Schedule for Lecture

- Autonomous Equations and Population Dynamics (2.5)
  - Bifurcations
- Fluid Mixing (Back to an application from 2.3)
- Numerical Methods (2.7)

Ex §2.5

$$\frac{dy}{dt} = y(1-y^2) = y - y^3$$

① Determine all equilibria ← steady state, rest points, fixed points

② Classify them.

③ Draw phase plane $\left(\frac{dy}{dt}, y\right)$

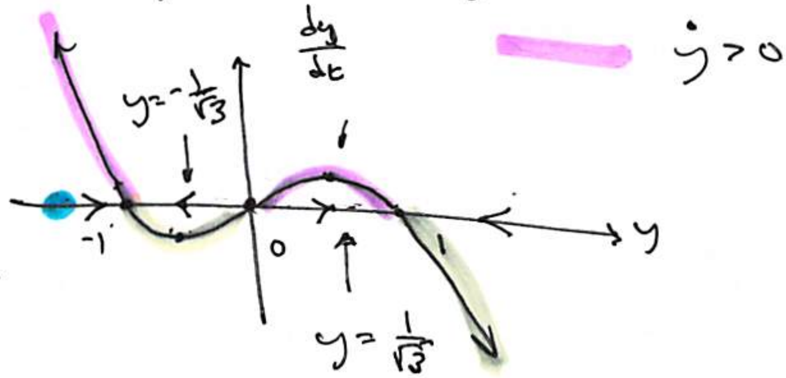
④ Sketch the solution. (y, t)

① $\dot{y} = \frac{dy}{dt} = 0 = y(1-y^2) \Rightarrow y = 0$ or ± 1.

③ Phase plane.

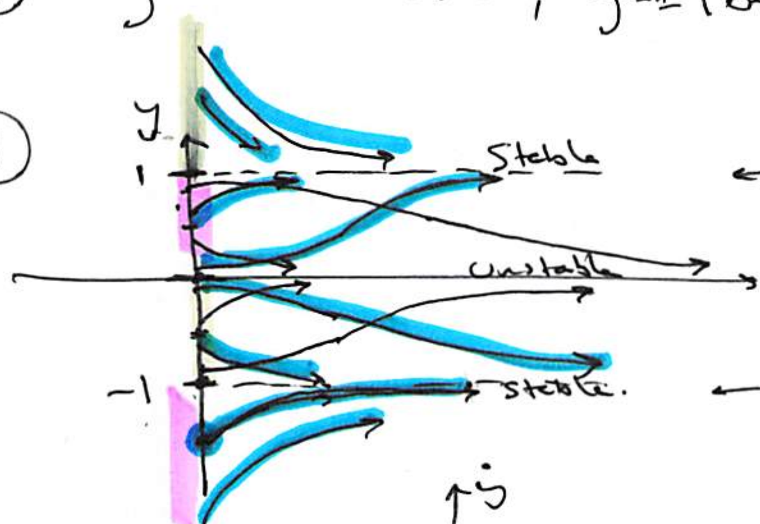
$y'' = 1 - 3y^2 = 0$

$\Rightarrow y = \pm\sqrt{1/3}$



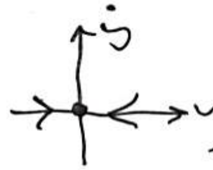
② $y=0$ is unstable, $y=\pm 1$ are both stable.

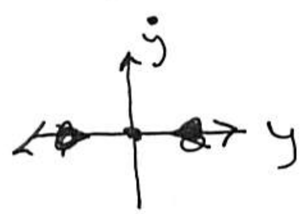
④

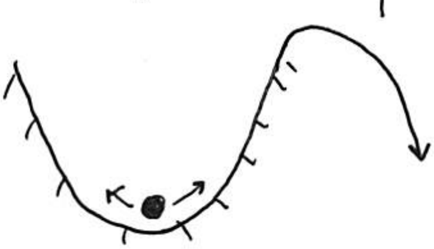


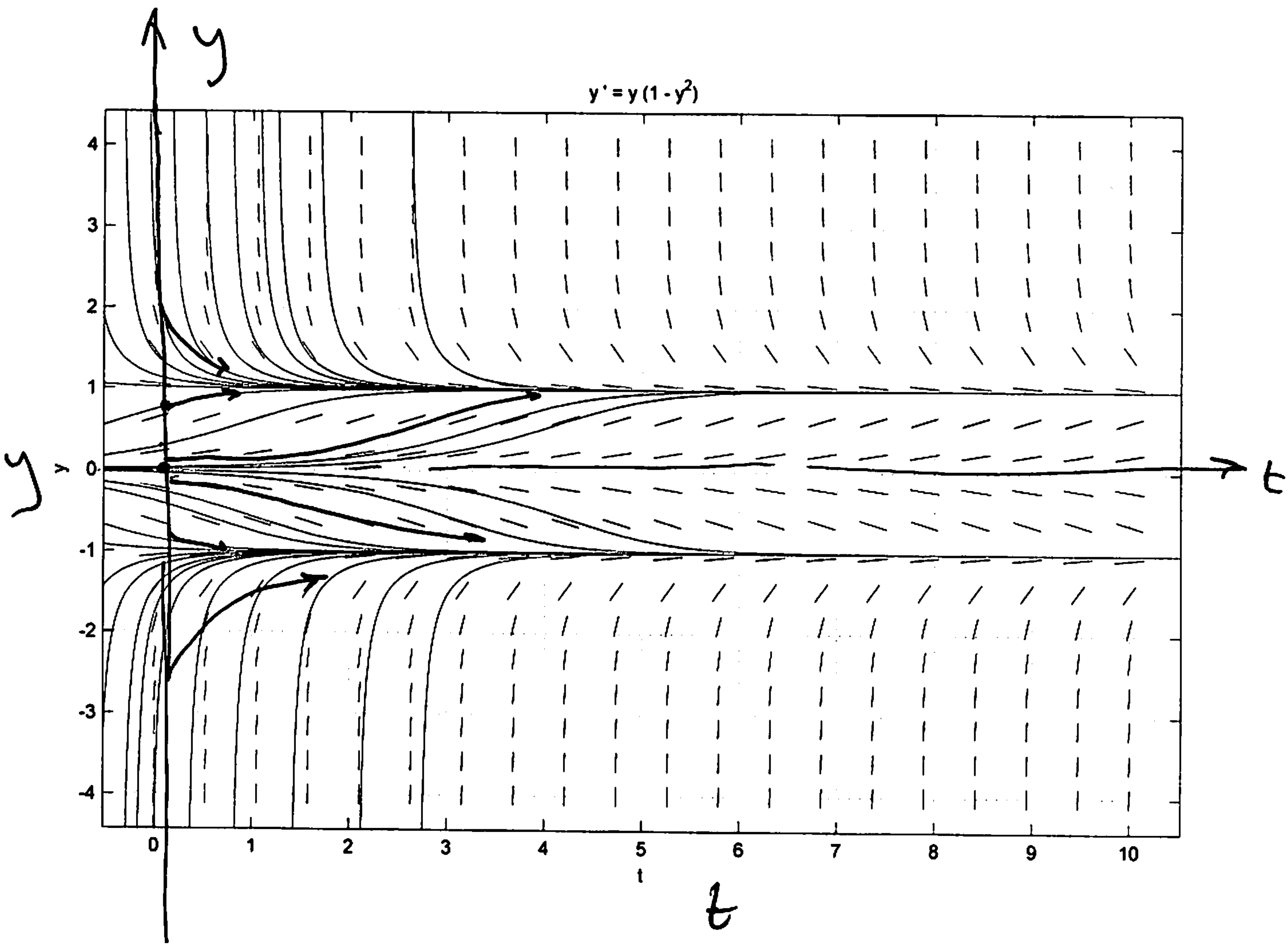
← $y(0) = 1 \Rightarrow y(t) = 1$

← $y(0) = -1 \Rightarrow y(t) = -1$

Stable Equilibria 

Unstable Equilibria 





Student Version of MATLAB

Ex

$$y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$\frac{dy}{dt} = y^2(4 - y^2) = 4y^2 - y^4$$

1. Determine all equilibria
2. Classify them
3. Draw phase plane ($\frac{dy}{dt}$ vs. y)
4. Sketch the solutions. (y vs. t)

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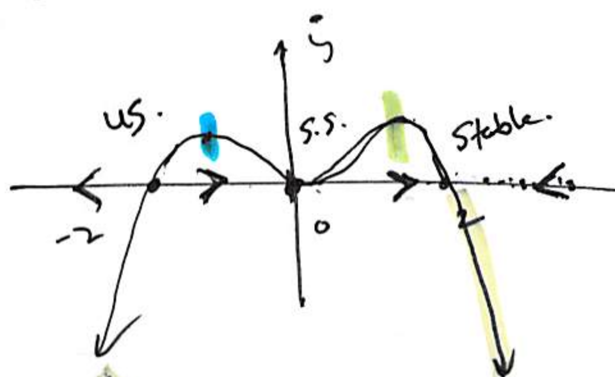
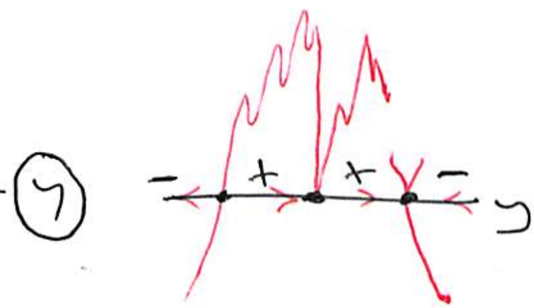
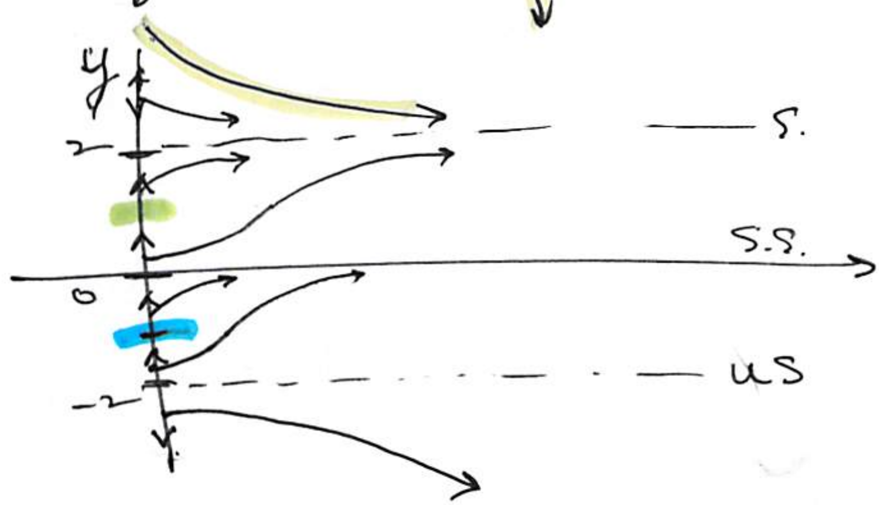
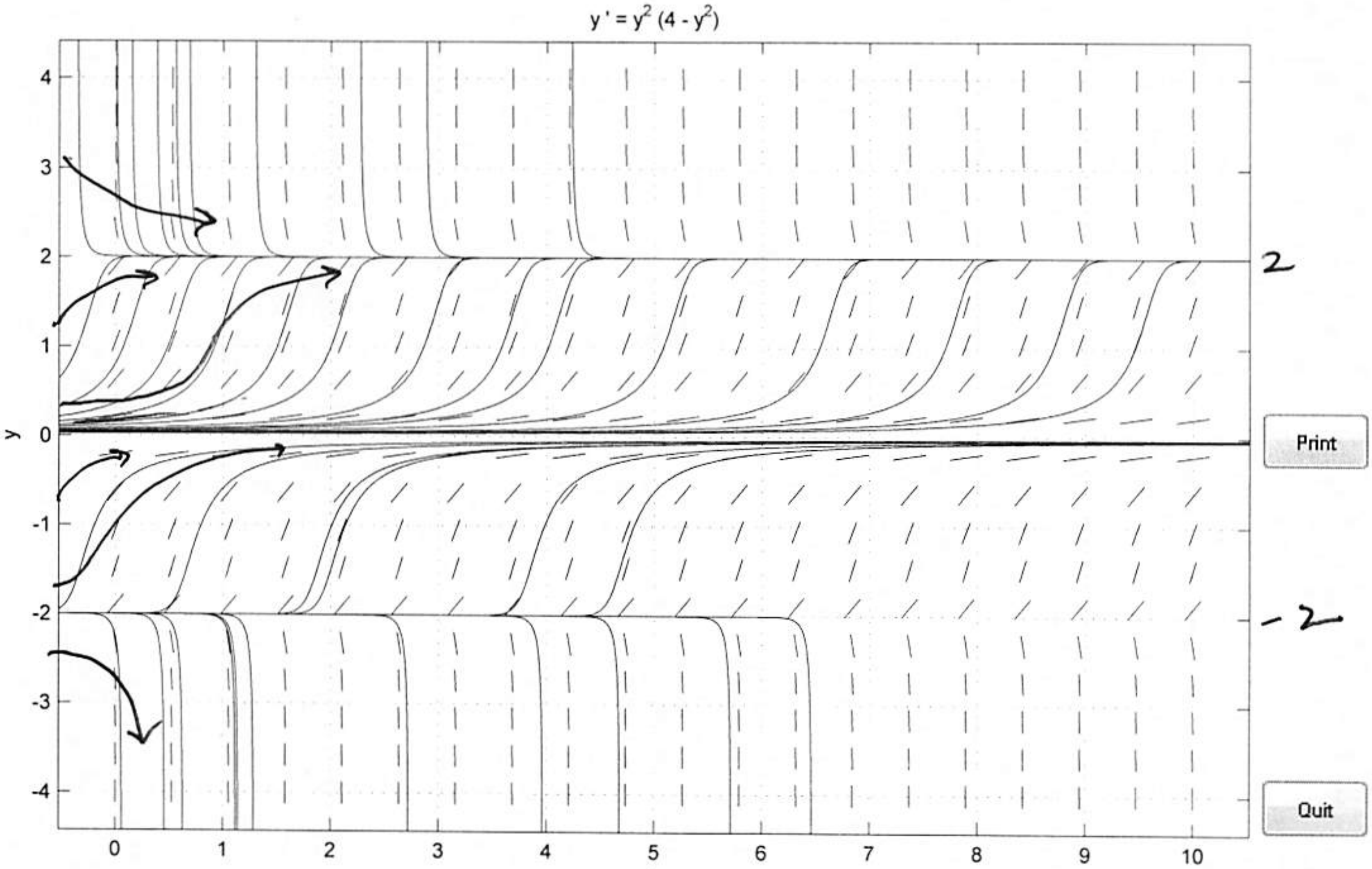
1. $\dot{y} = 0$ when $y = 0$, $y = \pm 2$.
2. $y = 0$ (semi-stable), $y = -2$ (unstable) $y = 2$ (stable)
3. 



4. 



Cursor position: (0.254, 1.85)

The backward orbit from (8.3, 0.39)
Ready.
The forward orbit from (9, 0.39)
The backward orbit from (9, 0.39)
Ready.

Ex Transcritical Bifurcation.

Consider $\dfrac{dy}{dt} = y\,(r - e^y)$, $r \in \mathbb{R}$.

- Sketch qualitatively different solutions and phase portraits.
- Sketch bifurcation diagram. ↙ ?

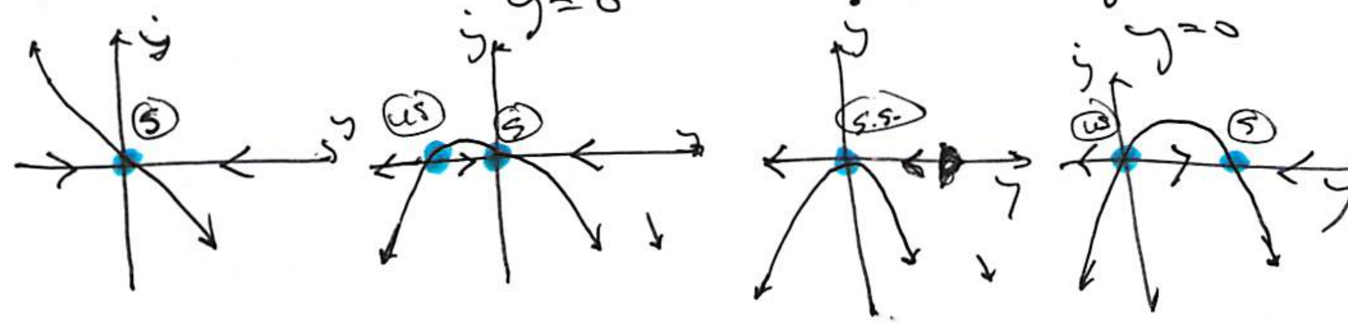
① Fixed points $\dot{y} = 0 \Rightarrow y = 0$ or $r = e^y$

$\rightarrow$ ~~$y = \ln(r)$~~ $y = \ln(r)$ $(r > 0)$

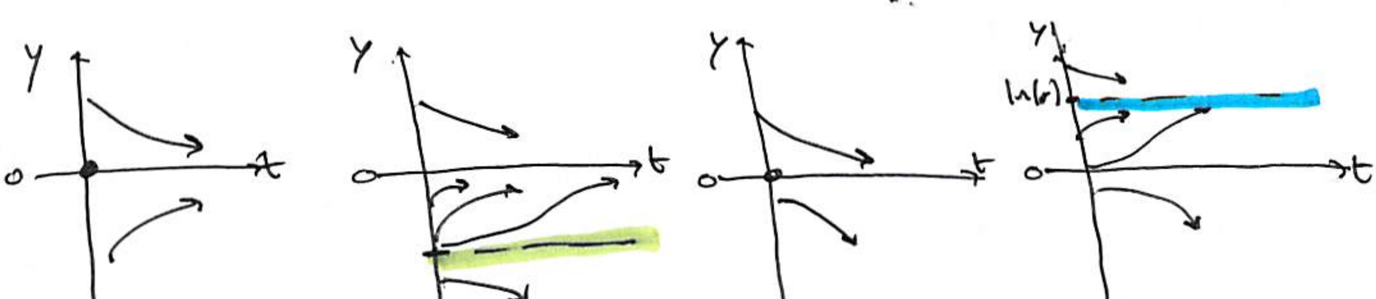
- $\lim_{y \to \infty} y(r - e^y) = -\infty$
- $\lim_{y \to -\infty} y(r - e^y) = -\infty$ $(r > 0)$

$(r \le 0)$	$(0 < r < 1)$	$(r = 1)$	$(r > 1)$
y^*: $y = 0$	$y = \ln(r) < 0$, $y = 0$	$y = 0$	$y = \ln(r) > 0$, $y = 0$

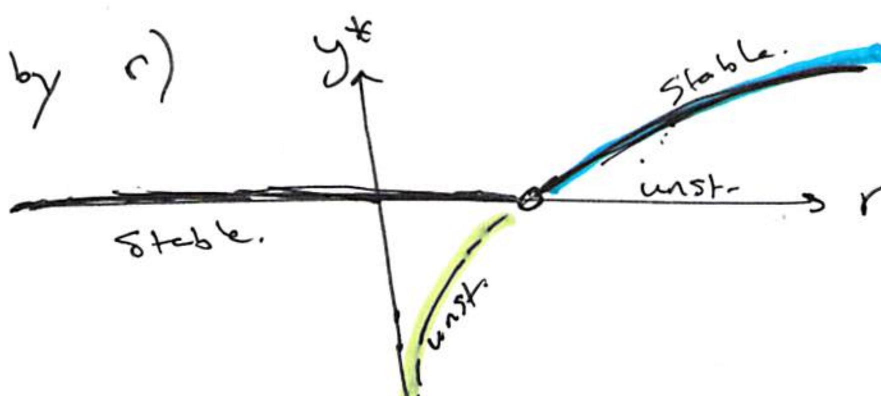
phase planes

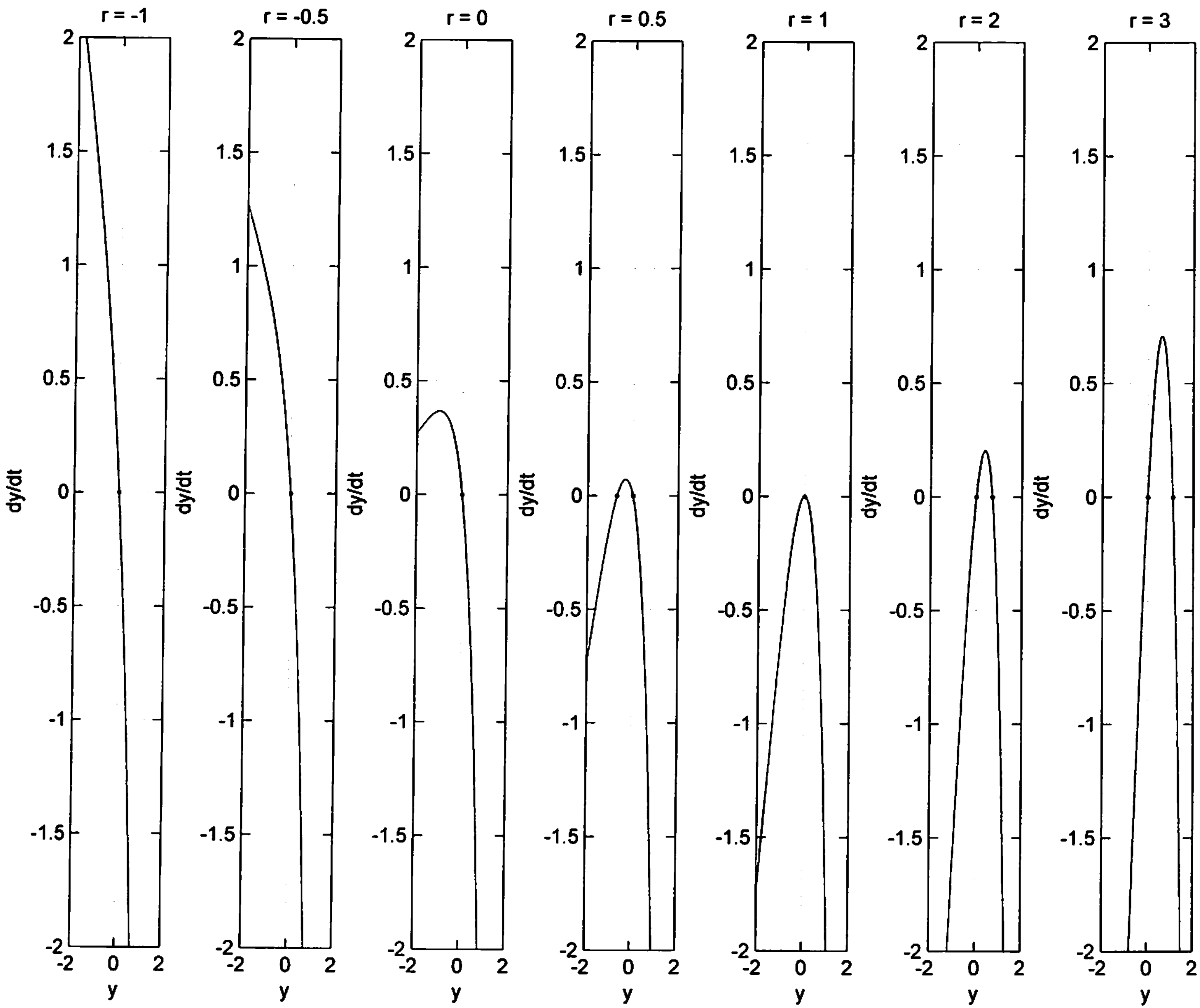


Solutions



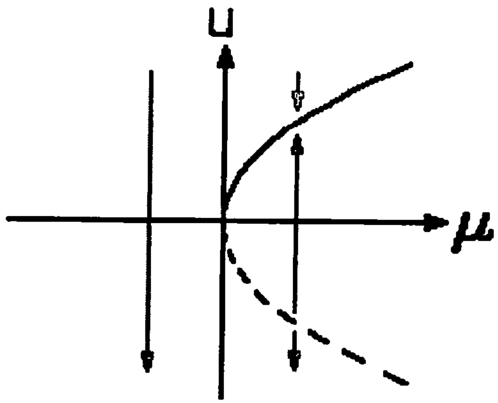
Bifurcation (y^* by r)





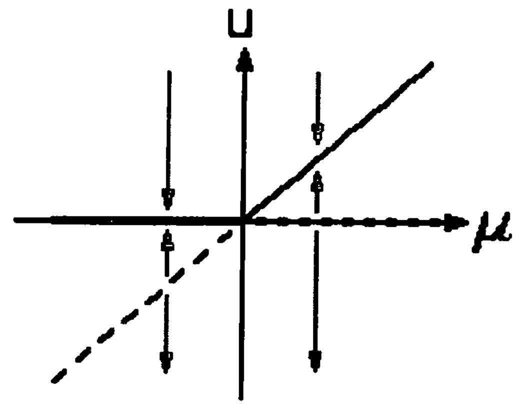
Student Version of MATLAB

saddle-node bifurcation



$$\dot{u} = \mu - u^2$$

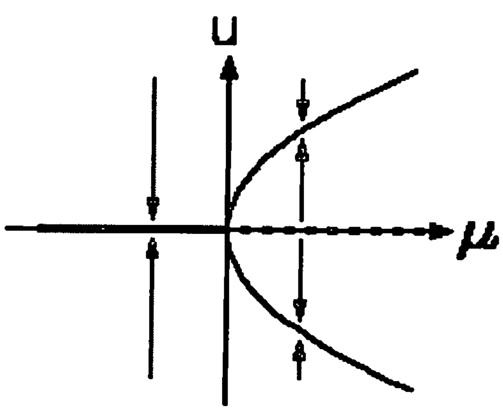
transcritical bifurcation



$$\dot{u} = (\mu - u)u$$

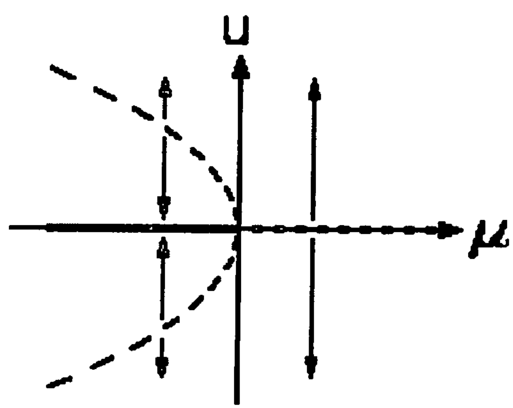
pitchfork bifurcation

supercritical



$$\dot{u} = (\mu - u^2)u$$

subcritical



$$\dot{u} = (\mu + u^2)u$$