Tavernetti - UC Davis
Friday October 28

Lecture 15

"Humans see what they want to see."
— Rick Riordan, The Lightning Thief

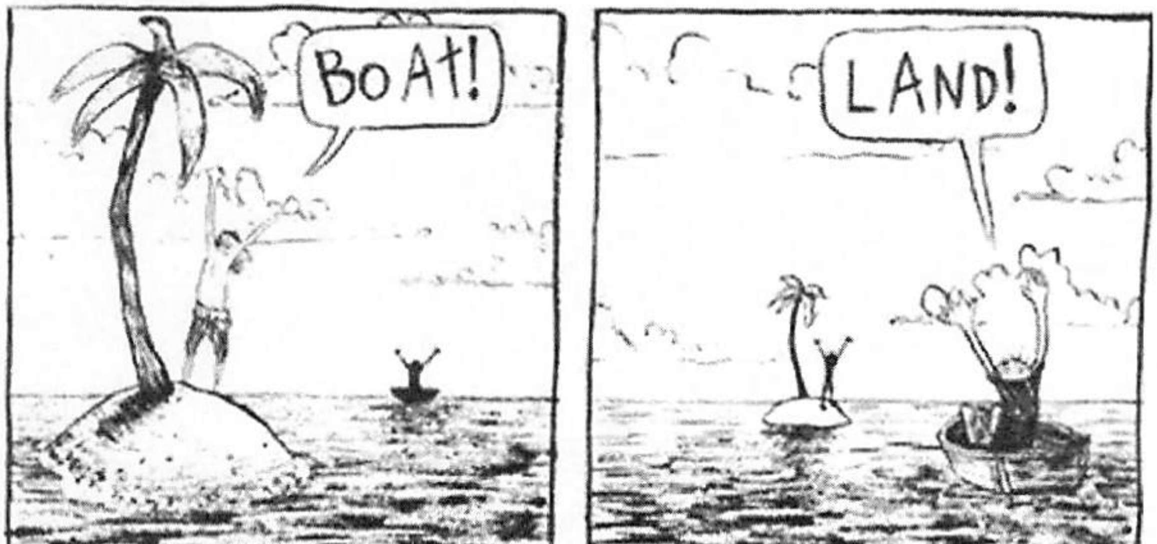
"We can complain because rose bushes have thorns, or rejoice because thorn bushes have roses." — Abraham Lincoln

per·spec·tive - noun
The art of drawing solid objects on a two-dimensional surface so as to give the right impression of their height, width, depth, and position in relation to each other when viewed from a particular point.

office hours start at $\boxed{\begin{array}{l}1:35 \\ \text{or } 1:40\end{array}}$ again

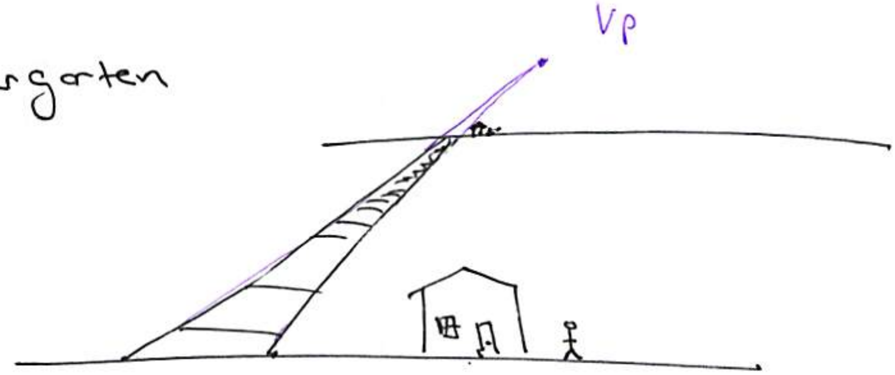
Today's Lecture

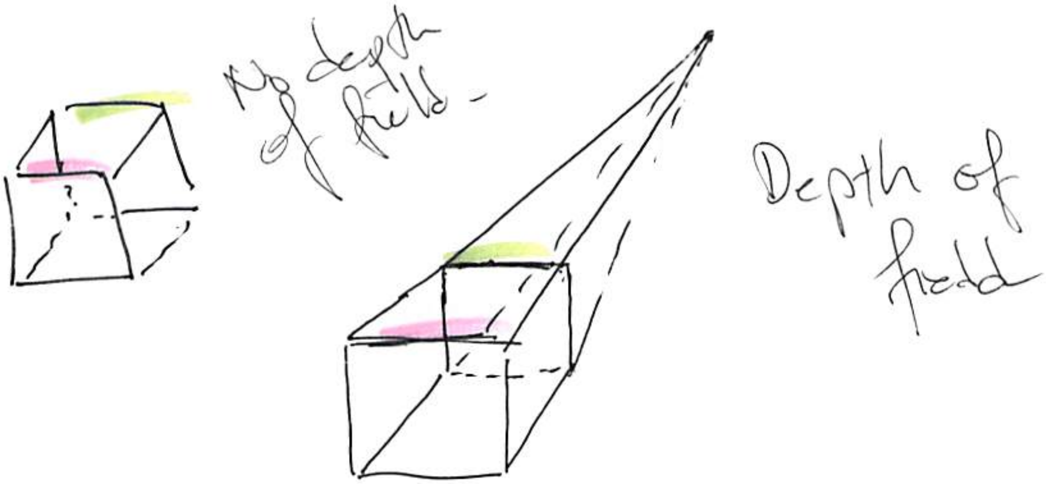
- Non-Homogeneous Equations
  - Examples
  - Discussion
  - Guessing the form of the particular solution!



Perspective...

Kindergarten





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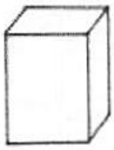

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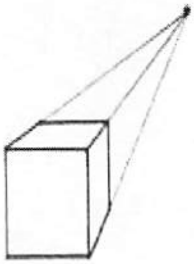

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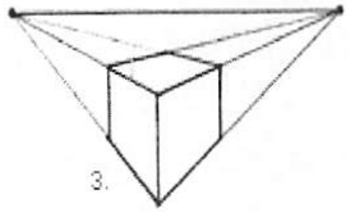

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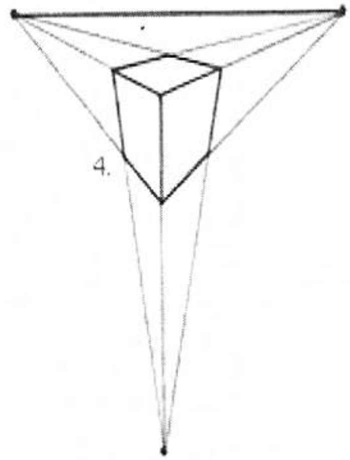
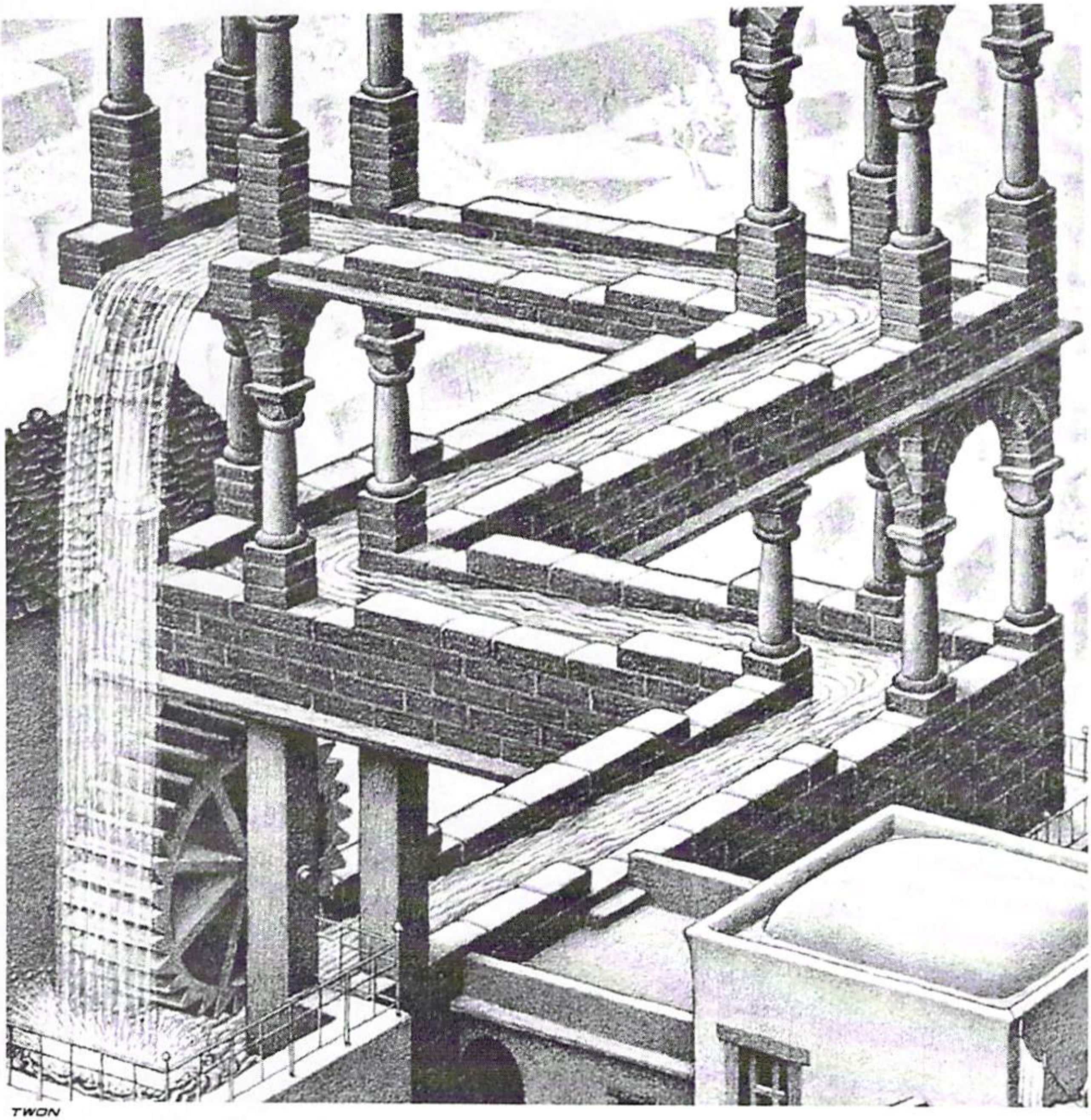
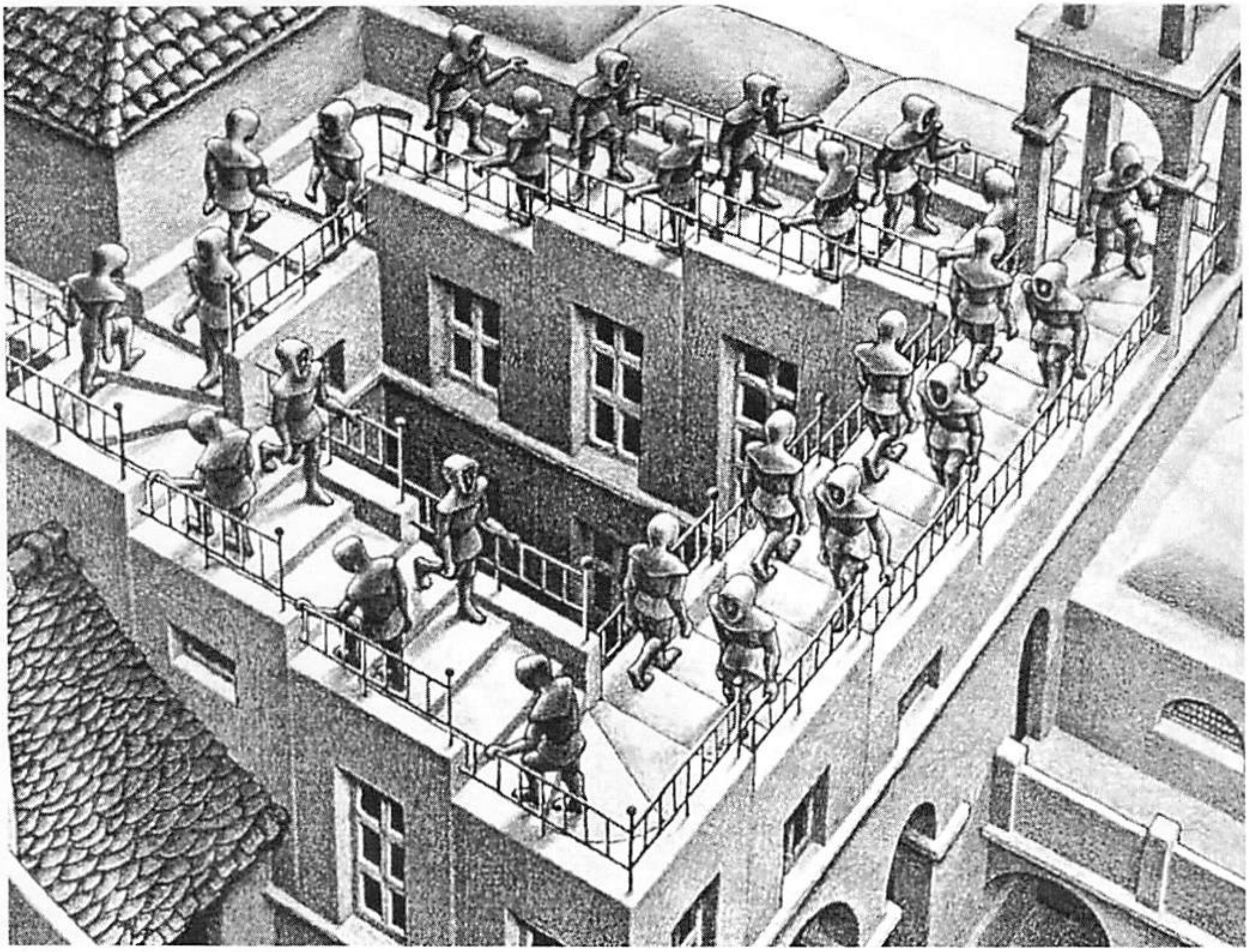
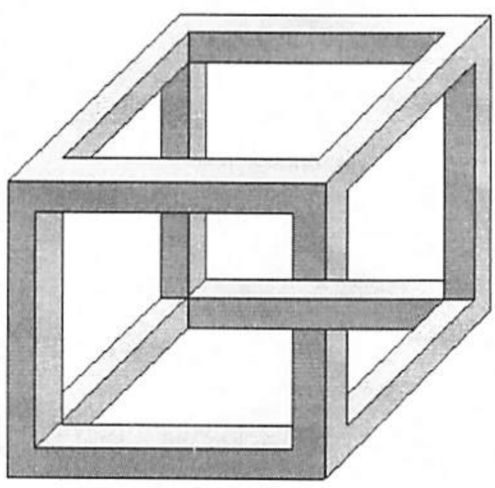




TWON

M.C. Escher.





§ 3.5 Method of Undetermined Coef.

- To solve: $y'' + p(t)y' + q(t)y = g(t)$

(Non-Homogenous) $g(t) \neq 0$

- we mostly restrict to

$$ay'' + by' + cy = g(t) \leftarrow g(t) \neq 0$$

- First solve y_h from $ay'' + by' + cy = 0$
- Then solve for $Y = y_p$ which solves

$$ay'' + by' + cy = g(t)$$

∴ Then the general solution is

$$y_{g.s.} = y_p + y_h = \underbrace{c_1 y_1 + c_2 y_2}_{\text{§ 3.1–3.4 Fundamental sets}} + Y$$

Remark: this whole technique is super weak.

only useful for $g(t) = e^x, \cos(x), P_n(x)$

- $y'' + 2y' = 3 + 4\sin(2t)$

① Solve for y_h from $y'' + 2y' = 0$

$$r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2$$

$$\Rightarrow y_h = c_1 e^{0t} + c_2 e^{-2t} = \boxed{c_1} + c_2 e^{-2t}$$

Y_1

② Guess form of Y the particular solution.

$$Y_{guess} = \boxed{A} + B\sin(2t) + C\cos(2t)$$

Y_1

$$= At + B\sin(2t) + C\cos(2t)$$

- $Y = A$, $Y' = 0$, $Y'' = 0$ $\Big\}$ $\underbrace{y'' + 2y'}_{} = 0 + 0 = \underbrace{3 + 4\sin(2t) = g(t)}_{}$

$0 = g(t)$ you have the wrong particular solution.

- Multiply A by t. , now

$$Y = At + B\sin(2t) + C\cos(2t)$$

$$Y' = A + B\cos(2t)\cdot 2 - C\sin(2t)\cdot 2$$

$$Y'' = 0 - B\sin(2t)\cdot 4 - C\cos(2t)\cdot 4$$

$$\Rightarrow y'' + 2y' \Big|_{y=Y} = g(t) = 3 + 4\sin(2t)$$

$$\Rightarrow -B\sin(2t)\cdot 4 - C\cos(2t)\cdot 4 + 2(A + B\cos(2t)\cdot 2 - C\cdot\sin(2t)\cdot 2) = 3 + 4\sin(2t)$$

$$2A + \sin(2t)[(-4B) - 4C] + \cos(2t)[-4C + 4B] =$$

$$2A = 3 \longrightarrow A = 3/2$$

$$-4B - 4C = 4 \longrightarrow B + C = -1 \Rightarrow 2B = -1 \Rightarrow B = -\tfrac{1}{2} = C$$

$$-4C + 4B = 0 \qquad 4B = 4C \rightarrow B = C$$

$$\Longrightarrow \quad Y = At + B\sin(2t) + C\cos(2t)$$

$$= 3/2\, t + -\tfrac{1}{2}\sin(2t) - \tfrac{1}{2}\cos(2t)$$

$$\Rightarrow \quad y_{g.s.} = y_h + Y \quad y_p$$

$$= c_1 + c_2 e^{-2t} + Y$$

$$\boxed{y = c_1 + c_2 e^{-2t} + \tfrac{3}{2}t - \tfrac{1}{2}\sin(2t) - \tfrac{1}{2}\cos(2t)}$$

Review and More Rules for Method of Undetermined Coefficients

Form is $ay'' + by' + cy = G(x)$

1. If $G(x)$ is a polynomial, use $y_p = Ax^n + Bx^{n-1} + \ldots + C.$ = Y
2. If $G(x) = Ce^{kx}$, use $y_p = Ae^{kx}$.
3. If $G(x) = C \sin kx$ or $C \cos kx$, use $y_p = A \cos kx + B \sin kx$
4. If $G(x)$ is a product of functions, multiply them for y_p.

   Example: $G(x) = x \cos 3x \rightarrow y_p = (Ax + B) \cos 3x + (Cx + D) \sin 3x$

Back to 3.5 $y'' - y' - 2y = 2e^{-t}$

By method of ~~other~~ undetermined coef.

① $y_h = c_1 e^{2t} + \boxed{c_2 e^{-t}}$

$r = -1$

② guess $Y = \boxed{Ae^{-t}}$ $\left(Ae^{rt}, g = 2e^{-t}\right)$

$\uparrow y_2$.

B/c this y_2, guess $s=1$

$$Y = Ate^{-t}$$
$$Y' = Ae^{-t} + Ate^{-t}(-1)$$
$$Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$
$$= -2Ae^{-t} + Ate^{-t}$$

$$y'' - y' - 2y \Big|_{y=Y} = g(t) = 2e^{-t}$$

$\Rightarrow -2Ae^{-t} + Ate^{-t} - (Ae^{-t} - Ate^{-t}) - 2Ate^{-t} = g$

$\Rightarrow e^{-t}\left(-2A + \cancel{At} - A + \cancel{At} - \cancel{2At}\right) = 2e^{-t}$

$\Rightarrow -3A = 2 \Rightarrow A = -\frac{2}{3}$

$\therefore \Rightarrow Y_p = -\frac{2}{3}te^{-t}$

③ $$\boxed{y = y_h + Y = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3}te^{-t}}$$

Summary. We now summarize the steps involved in finding the solution of an initial value problem consisting of a nonhomogeneous equation of the form

$$ay'' + by' + cy = g(t), \tag{23}$$

where the coefficients a, b, and c are constants, together with a given set of initial conditions:

1. Find the general solution of the corresponding homogeneous equation.
2. Make sure that the function $g(t)$ in Eq. (23) belongs to the class of functions discussed in this section, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the next section).
3. If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation
$$ay'' + by' + cy = g_i(t),$$
where i runs from 1 to n.
4. For the ith subproblem assume a particular solution $Y_i(t)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (found in step 1), then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. See Table 3.6.1.
5. Find a particular solution $Y_i(t)$ for each of the subproblems. Then the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (23).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

TABLE 3.6.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t$ $+(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

Theorem

If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation (18) corresponding to the nonhomogeneous equation (16),

$$y'' + p(t)y' + q(t)y = g(t), \quad \leftarrow y_1, y_2$$

then a particular solution of Eq. (16) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt, \tag{28}$$

and the general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t), \tag{29}$$

as prescribed by Theorem 3.6.2.

$$ay'' + \cancel{p(t)}\, by' + cy = g$$

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = g/a$$

$$\uparrow \qquad\quad \uparrow$$

$$p \qquad\quad q$$

§ 3.6 — variation of parameters.

(Ex) $y'' - y' - 2y = 2e^{-t} = g(t)$

① y_h: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2, -1$

$\boxed{y_h = c_1 e^{2t} + c_2 e^{-t}} = c_1 y_1 + c_2 y_2$

② $Y = -y_1 \cdot \int_{t_0}^{t} \frac{y_2 \cdot g}{W(y_1, y_2)}\, dt + y_2 \cdot \int_{t_0}^{t} \frac{y_1 \cdot g}{W(y_1, y_2)}\, dt$

$Y = -e^{2t} \cdot \int \frac{e^{-t} \cdot 2e^{-t}}{-3e^{t}}\, dt + e^{-t} \cdot \int \frac{e^{2t} \cdot 2e^{-t}}{-3e^{t}}\, dt$

$= e^{-t} e^{-t} e^{-t}$

$= e^{-3t}$

$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{2t} e^{-t} - e^{-t} \cdot 2e^{2t}$

$= -e^{t} - 2e^{t} = -3e^{t}$

$= -e^{2t} \cdot \left(\int -\frac{2}{3} e^{-3t}\, dt \right) + e^{-t} \left(\int -\frac{2}{3}\, dt \right)$

=

$= -e^{+2t} \left(+\frac{2}{9} e^{-3t} + c_1 \right) + e^{-t} \left(-\frac{2}{3} t + c_2 \right)$

" Y_p " $= \cancel{-c_1 e^{2t}} - \cancel{\frac{2}{9} e^{-t}} - \frac{2}{3} t e^{-t} + \cancel{c_2 e^{-t}}$ ✓

$= c_1 y_1 + c_2 y_2 + Y = \cancel{c_1 e^{2t}} + c_2 e^{-t} + Y$

$= -\frac{2}{3} t e^{-t}$

③ $y = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3} t e^{-t}$. ← Final Answer.