Lecture 14 – Wednesday 10/26

When it is so obvious that goals cannot be reached, don't adjust the goals, but adjust the action steps.

Confucius

It is never too late to be what you might have been.

George Eliot

Schedule for lecture

- **Method of ~~Undetermined Coefficients~~** order reduction **(3.4)**
- **Non-Homogeneous Equations (3.5)**

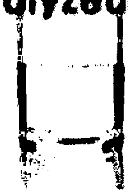
A psychologist walked around a room while teaching stress management to an audience. As she raised a glass of water, everyone expected they'd be asked the "half empty or half full" question. Instead, with a smile on her face, she inquired: "How heavy is this glass of water?"

Answers called out ranged from 8 oz. to 20 oz.

She replied, "The absolute weight doesn't matter. It depends on how long I hold it. If I hold it for a minute, it's not a problem. If I hold it for an hour, I'll have an ache in my arm. If I hold it for a day, my arm will feel numb and paralyzed. In each case, the weight of the glass doesn't change, but the longer I hold it, the heavier it becomes."

She continued, "The stresses and worries in life are like that glass of water. Think about them for a while and nothing happens. Think about them a bit longer and they begin to hurt. And if you think about them all day long, you will feel paralyzed – incapable of doing anything."

Remember to put the glass down.



§ 3.4 – Method of ~~Undetermined Coefficients~~ order reduction.

35. $(x-1)y'' - xy' + y = 0$, $x>1$, $y_1 = e^x$ is a soln

Use M.O.U.D.C. to find $\boxed{y_2 = x}$ ← Proven ↓

Guess. $\boxed{1}$ $y_2 = y = v(x)\cdot y_1 = ve^x$

Differentiate $\boxed{2}$

$$y' = v'e^x + ve^x$$

$$y'' = v''e^x + \underline{v'e^x + v'e^x} + ve^x$$

$$= v''e^x + 2v'e^x + ve^x$$

↓

plug into the eq. $\boxed{3}$ $(x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x = 0$

collect terms in v. $\boxed{4}$ $v''((x-1)e^x) + v'((x-1)\cdot 2e^x - xe^x) + v($[illegible] $(x-1)e^x - e^x\cdot x + e^x)$

$$+v(\underbrace{xe^x - e^x - xe^x + e^x}_{0}) = 0.$$

eq. in $v' = w$ $\boxed{5}$ $v''((x-1)e^x) + v'((x-1)\cdot 2e^x - xe^x) = 0$

$$w'((x-1)e^x) + w((x-1)\cdot 2e^x - xe^x) = 0$$

SOV and Solve for $w = v'$ $\boxed{6}$

$$\frac{dw}{dx} = -\frac{w((x-1)\cdot 2e^x - xe^x)}{(x-1)e^x}$$

$$\int \frac{dw}{w} = \int -\frac{(x-1)2\cancel{e^x} + x\cancel{e^x}}{(x-1)\cancel{e^x}}\,dx$$

$2(-x+1)+x$
$= -2x+2+x$
$= 2-x$

$$= \int -\frac{(x-1)\cdot 2 + x}{x-1}\,dx = \int \frac{2-x}{x-1}\,dx$$

$$\ln|w| = \ln|x-1| - x + c$$

$$e \qquad e$$

get w. $\boxed{7}$ $w = e^{\ln|x-1|}e^{-x}e^{c} = (x-1)e^{-x}\cdot K$

D.E in v. $\boxed{8}$ $\frac{dv}{dx} = (x-1)e^{-x}\cdot K$

↓

sov. $\int dv = \int \underset{\int u}{\underset{\uparrow}{(x-1)}}\, e^{-x} \cdot \underset{\cdot\, dv}{\underset{\uparrow}{k}}\, dx$

$\int u \cdot dv = uv - \int v\,du$

$= k(x-1)$

$= -k(x-1)e^{-x} + \int e^{-x} \cdot k\, dx$

$v = -k(x-1)e^{-x} - ke^{-x} + \boxed{C}$

$y_2 = y_1 v$ $\boxed{9}$ ↓ $y_2 = ve^{x} \leftarrow y_1 = e^{x}$

$e^0 = e^{x} \cdot e^{-x} = 1$

$= e^{x-x}$

$= \left(-k(x-1)e^{-x} - ke^{-x} + C\right)e^{x}$

$= -k(x-1) - k + Ce^{x}$

$= -kx + k - k + Ce^{x}$

$$\boxed{y_{g.s.} = -\underset{y_2}{\underset{\uparrow}{c_1 x}} + \underset{y_1}{\underset{\uparrow}{c_2 e^{x}}}}$$

2 part verification to see $y_{g.s.} = c_1 x + c_2 e^{x}$

① Show $L(y_{g.s.}) = 0$.

② Show linear independence. by $W(\cdot,\cdot) \neq 0$.

for any $x > 1$

$$W(\underset{y_2}{\underset{\uparrow}{x}}, \underset{y_1}{\underset{\uparrow}{e^{x}}}) = \begin{vmatrix} x & e^{x} \\ 1 & e^{x} \end{vmatrix} = xe^{x} - e^{x}$$

$= (x-1)e^{x} \neq 0$

Yes linearly independent. if $x > 1$.

① & ② $\Rightarrow$ $y_{g.s.}$ is a fundamental solution sets

Case study $\int \frac{2-x}{x-1}\,dx$

<u>u-sub</u>

$$u = x-1 \rightarrow -u-1 = -x$$
$$du = dx \qquad \underline{\quad +2 \quad\; +2 \quad}$$
$$-u+1 = 2-x$$
$$1-u = 2-x$$

$$\int \frac{2-x}{x-1}\,dx = \int \frac{1-u}{u}\,du = \int \frac{1}{u} - 1\,du$$
$$= \ln|u| - u + c$$
$$= \ln|x-1| - (x-1) + c \quad ✓$$

<u>long Division</u>

$$x-1\overline{\big)\,2-x} = x-1\overset{-1}{\overline{\big)\,-x+2}}$$
$$\underline{-(-x+1)}$$
$$0+1$$

$$\Rightarrow \frac{2-x}{x-1} = -1 + \frac{1}{x-1}$$

$$\int \frac{2-x}{x-1}\,dx = \int -1 + \frac{1}{x-1}\,dx$$
$$= -x + \ln|x-1| + c \quad ✓$$

$$= \ln|x-1| - x + C$$

Summary of §3.1-3.4.

① We studied $ay'' + by' + cy = 0$

Guess:
$$\left.\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2e^{rt} \end{aligned}\right\} \quad (\underset{\uparrow}{ar^2+br+c})e^{rt} = 0$$

this must be zero

$$\Rightarrow \underset{\nearrow}{ar^2+br+c=0.}$$

Find roots of the Characteristic Equation

- $r_1 \neq r_2 \in \mathbb{R}$ (§3.1)

  $y_{\text{general}} = c_1e^{r_1t} + c_2e^{r_2t}$

- $r_1 = r_2 \in \mathbb{R}$ (§3.4) $\quad \swarrow y_2 = t \cdot y_1$

  $y_{\text{general}} = c_1e^{r_1t} + \boxed{c_2te^{r_2t}}$

- $r_{1,2} = \lambda \pm i\mu.$ (§3.3)

  $y_{\text{general}} = \left[c_1e^{\lambda t}\cos(\mu t)\right] + \left[c_2e^{\lambda t}\sin(\mu t)\right]$

  $= e^{\lambda t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right).$

② We also studied $p(t)y'' + q(t)y' + r(t)y = 0$ (§3.4)

- given y_1 find $y_2 = v(t) \cdot y_1$

  → Called Method of undetermined coefficients.

③ Finally we studied linear independence of functions by the: $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ Wronskian Determinant. §3.2

- $W(f,g) \underset{\uparrow}{\neq} 0 \Leftrightarrow f, g$ are linearly independent

  for any t

We can now summarize the facts about fundamental sets of solutions, Wronskians, and linear independence in the following way. Let y_1 and y_2 be solutions of Eq. (7),

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then the following four statements are equivalent, in the sense that each one implies the other three:

1. The functions y_1 and y_2 are a fundamental set of solutions on I.
2. The functions y_1 and y_2 are linearly independent on I.
3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.
4. $W(y_1, y_2)(t) \neq 0$ for all t in I.

It is interesting to note the similarity between second order linear homogeneous differential equations and two-dimensional vector algebra. Two vectors **a** and **b** are said to be linearly dependent if there are two scalars k_1 and k_2, not both zero, such that $k_1\mathbf{a} + k_2\mathbf{b} = \mathbf{0}$; otherwise, they are said to be linearly independent. Let **i** and **j** be unit vectors directed along the positive x and y axes, respectively. Since $k_1\mathbf{i} + k_2\mathbf{j} = \mathbf{0}$ only if $k_1 = k_2 = 0$, the vectors **i** and **j** are linearly independent. Further, we know that any vector **a** with components a_1 and a_2 can be written as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, that is, as a linear combination of the two linearly independent vectors **i** and **j**. It is not difficult to show that any vector in two dimensions can be expressed as a linear combination of any two linearly independent two-dimensional vectors (see Problem 14). Such a pair of linearly independent vectors is said to form a basis for the vector space of two-dimensional vectors.

The term **vector space** is also applied to other collections of mathematical objects that obey the same laws of addition and multiplication by scalars that geometric vectors do. For example, it can be shown that the set of functions that are twice differentiable on the open interval I forms a vector space. Similarly, the set V of functions satisfying Eq. (7) also forms a vector space.

Since every member of V can be expressed as a linear combination of two linearly independent members y_1 and y_2, we say that such a pair forms a basis for V. This leads to the conclusion that V is two-dimensional; therefore, it is analogous in many respects to the space of geometric vectors in a plane.

We can summarize the discussion in this section as follows. To find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$ where $W \neq 0$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

§ 3.5 : $ay'' + by' + cy = \underbrace{g(x) \text{ or } g(t)}_{\text{non-homogeneous.}} \neq 0$

How to solve.

$\boxed{1}$ Find the ~~general so~~ homogeneous solution to

$$ay'' + by' + cy = 0$$ by methods of § 3.1-3.4.

$\boxed{2}$ Find the particular solution, a function y_p such that :

requires a guess. $L[y_p] = g(x)$

by the method of undetermined coef.

$\boxed{3}$ $y_{general} = y_{homogeneous} + y_{particular}.$

$= y_h + y_p.$

Review and More Rules for Method of Undetermined Coefficients

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Form is $ay'' + by' + cy = G(x)$

1. If $G(x)$ is a polynomial, use $y_p = Ax^n + Bx^{n-1} + \ldots + C$.
2. If $G(x) = Ce^{kx}$, use $y_p = Ae^{kx}$.
3. If $G(x) = C \sin kx$ or $C \cos kx$, use $y_p = A \cos kx + B \sin kx$
4. If $G(x)$ is a product of functions, multiply them for y_p.

   Example: $G(x) = x \cos 3x \rightarrow y_p = (Ax + B) \cos 3x + (Cx + D) \sin 3x$

Ex. Find the general solution to

$$y'' - y' - 2y = \underbrace{-2t + 4t^2}_{g(t) \neq 0}$$

$\boxed{1}$ Solve for y_h : $y'' - y' - 2y = 0$

$$\Rightarrow \quad (r^2 - r - 2)e^{rt} = 0$$

$$\Rightarrow \quad (r-2)(r+1) = 0$$

$$\Rightarrow \quad r = 2, -1$$

$$\therefore \quad y_h = c_1 e^{2t} + c_2 e^{-t}.$$

$\boxed{2}$ Find y_p. We have $g(t) = -2t + 4t^2$

Guess $y_p = Y = C + Bt + At^2$

$$Y' = B + 2At$$

$$Y'' = 2A$$

$$Y'' - Y' - 2Y = -2t + 4t^2$$

$$2A - (B + 2At) - 2(C + Bt + At^2) = -2t + 4t^2$$

$$\underset{\overset{\|}{0}}{\underline{[2A - (2A - B - 2C)]}} + t(-2A - 2B) + t^2(-2A)$$

$$-2A - 2B = -2 \qquad -2A = 4$$

$$A = -2 \qquad B = 3 \qquad C = -\frac{7}{2}.$$

$$y_p = -\frac{7}{2} + 3t - 2t^2$$

$\boxed{3}$ $y_{general} = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - \frac{7}{2} + 3t - 2t^2$

Summary. We now summarize the steps involved in finding the solution of an initial value problem consisting of a nonhomogeneous equation of the form

$$ay'' + by' + cy = g(t), \tag{23}$$

where the coefficients a, b, and c are constants, together with a given set of initial conditions:

1. Find the general solution of the corresponding homogeneous equation.
2. Make sure that the function $g(t)$ in Eq. (23) belongs to the class of functions discussed in this section, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the next section).
3. If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation

   $$ay'' + by' + cy = g_i(t),$$

   where i runs from 1 to n.
4. For the ith subproblem assume a particular solution $Y_i(t)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (found in step 1), then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. See Table 3.6.1.
5. Find a particular solution $Y_i(t)$ for each of the subproblems. Then the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (23).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

TABLE 3.6.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t$ $+(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.