Lecture 5– Friday Sept 30, 2016

"A mind is a fire to be kindled, not a vessel to be filled." -- Plutarch

"Nothing was ever achieved without enthusiasm." - Emerson

Schedule for Lecture

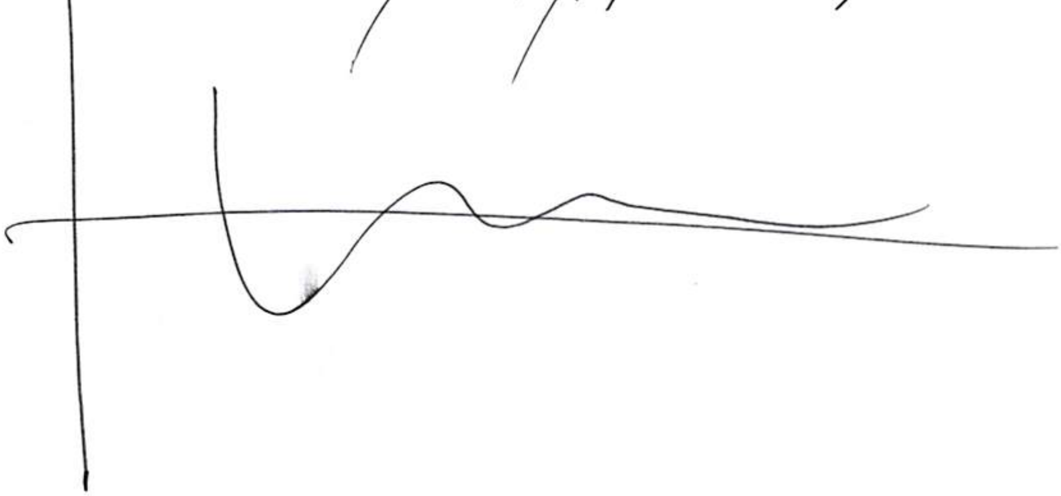
- Warm-up
- Finish example from last class (2.2)
- Second interval of validity example (2.2)
- Existence and uniqueness of solutions for first order linear equations (2.4)
  - Theorem 2.4.1
- Existence and uniqueness of solutions for first order nonlinear equations (2.4)
  - Theorem 2.4.2
- Examples
- Modeling with first order equations (2.3)

Bolzano-Weierstrass Theorem

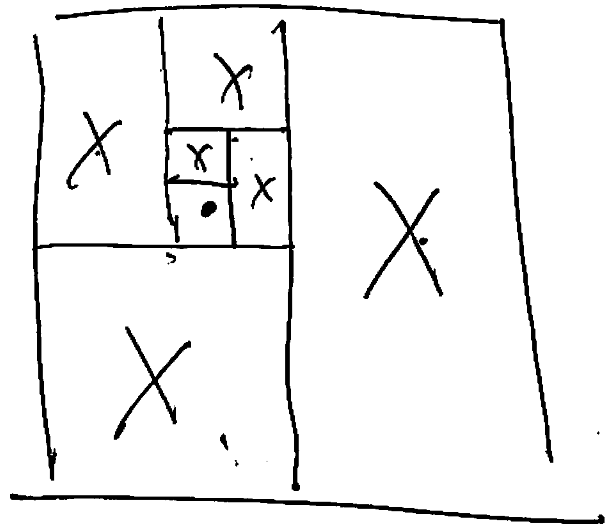
- Every sequence that is bounded has a convergent subsequence.

1D

$x_1, \cancel{x_2}, x_3, \cancel{x_4}, \ldots x_n, \ldots = \{x_n\}_{n=1}^{\infty}$



2D



$=0$ >0 §2.2

Ex 26. Solve $y' = 2(1+x)(1+y^2)$, $y(0)=0$ ✓

Find the interval of validity and determine where $y(x)$ is a minimum.

- Notice $y'=0$ at $x=-1$. (candidate local min)

y': $-$ | $+$ at -1 — local min ✓

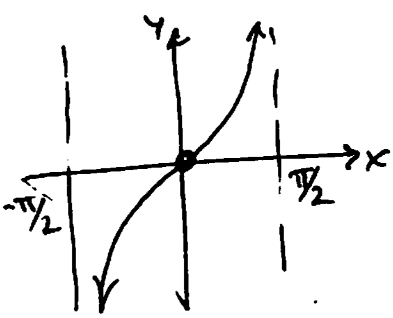
- Sov. to solve the I.V.P.

$$\frac{dy}{dx} = 2(1+x)(1+y^2)$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int 2(1+x)\,dx$$

$\operatorname{atan}(y) = \tan^{-1}(y)$ $\Rightarrow$ $\operatorname{atan}(y) = 2x + x^2 + c$

$$\Rightarrow \boxed{y(x) = \tan(2x+x^2+c)}$$



- I.C. $y(0)=0$

$$\Rightarrow 0 = \tan(c) \Rightarrow c=0$$

$$\Rightarrow \boxed{y(x) = \tan(2x+x^2)}.$$

P.V. tangent

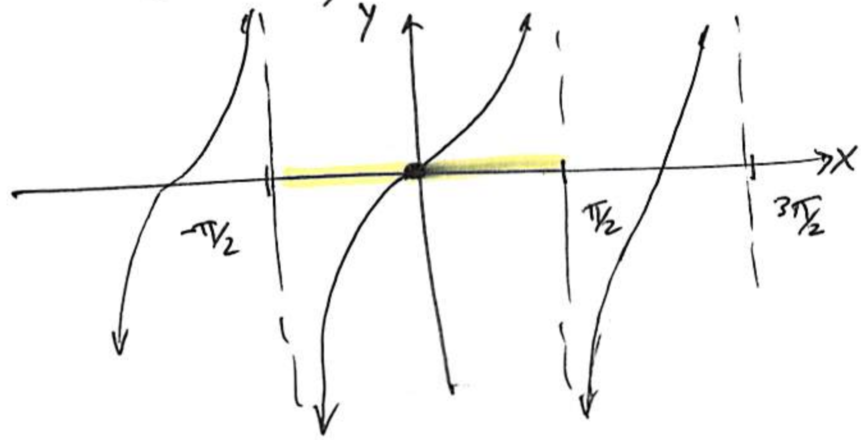
TBD Interval of validity

$y(x)$ is a minimum. $(-1, \tan(-1))$

$y(x) = \tan(2x+x^2)$ $\quad\quad$ $y(0)=0$:

$\uparrow$

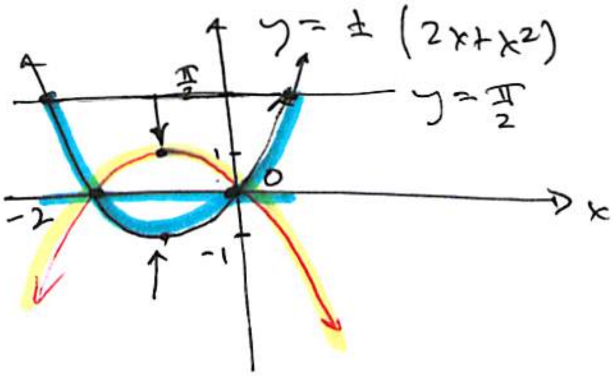
$2x+x^2 = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}.$



$|2x+x^2| < \frac{\pi}{2}$ $\leftarrow$

$2x+x^2 < \frac{\pi}{2}$

or $\quad -2x-x^2 < \pi/2$



Solve for $\; 2x+x^2 < \frac{\pi}{2}$

$\Rightarrow 2x+x^2-\frac{\pi}{2} < 0$

at $\quad 2x+x^2-\frac{\pi}{2} = 0$

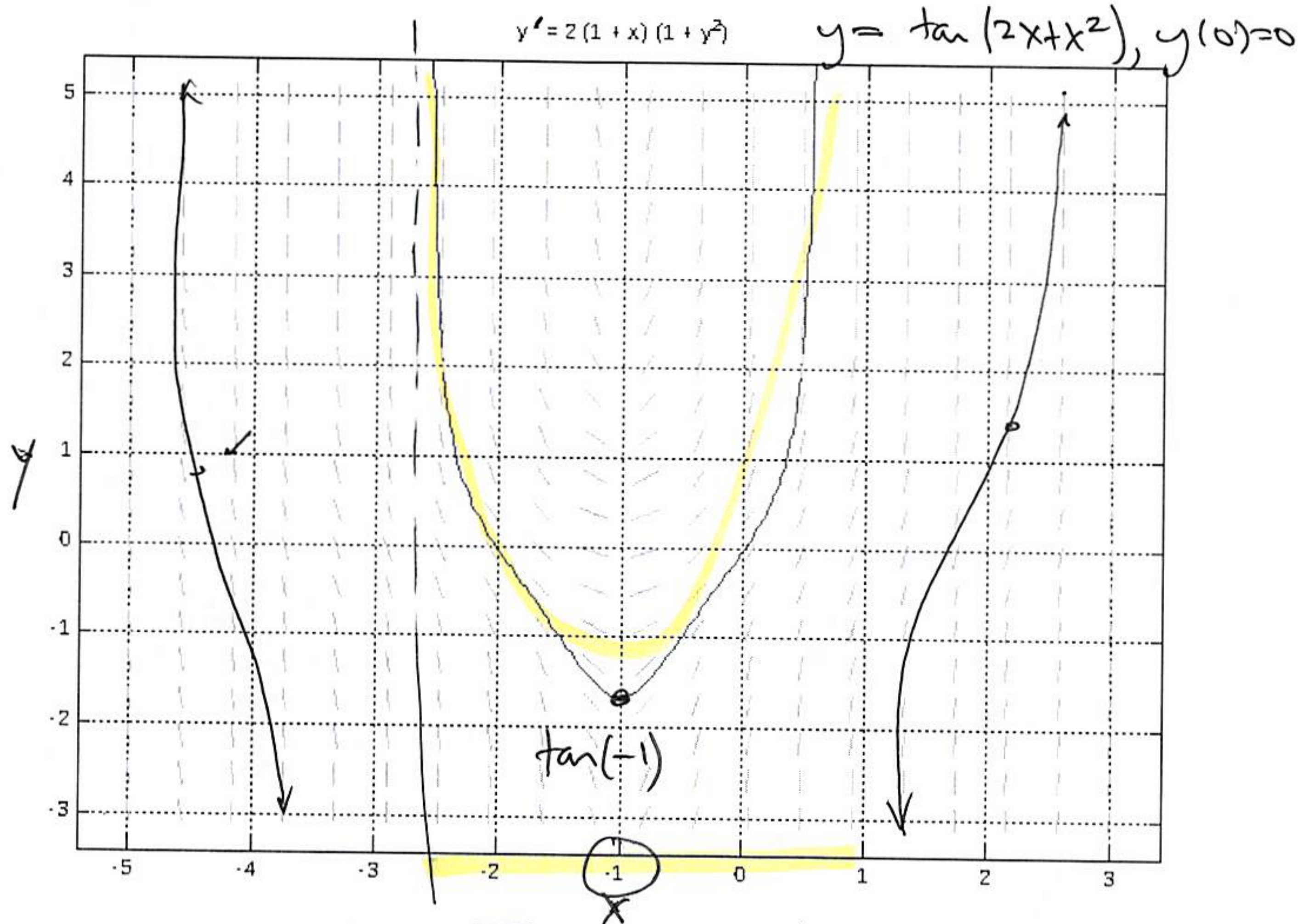
$$x_{1,2} = \frac{-\cancel{2} \pm \sqrt{\cancel{4}+\cancel{4}\cdot\frac{\pi}{2}}}{\cancel{2}}$$

$$= -1 \pm \sqrt{1+\frac{\pi}{2}}$$

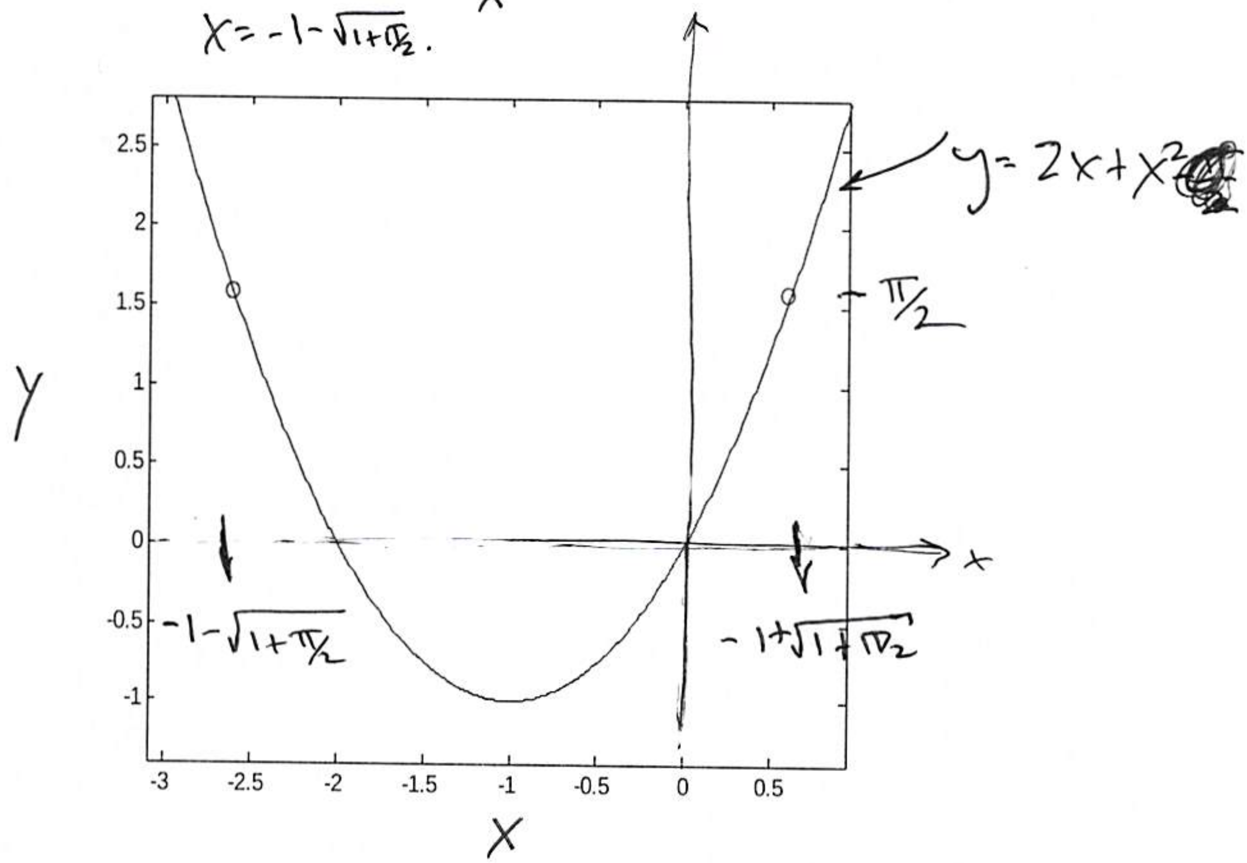
$$\approx -2.603, \; 0.603$$

Interval of validity is

$$-1-\sqrt{1+\frac{\pi}{2}} < x < -1+\sqrt{1+\pi/2}$$



$y' = 2(1+x)(1+y^2)$
$y = \tan(2x+x^2),\ y(0)=0$
$\tan(-1)$
$x = -1-\sqrt{1+\pi/2}$.
y
x



$y = 2x+x^2$
$\pi/2$
$-1-\sqrt{1+\pi/2}$
$-1+\sqrt{1+\pi/2}$
y
x

22. $y' = \dfrac{3x^2}{3y^2-4}$ $y(1)=0$

- Determine the interval of validity.

Notice that y' DNE (infinite discontinuity)

at $3y^2-4=0$

$\Rightarrow y^2 = \frac{4}{3} \Rightarrow y = \pm \frac{2}{\sqrt{3}}$.

- Solve the IVP.

$\dfrac{dy}{dx} = \dfrac{3x^2}{3y^2-4} \longrightarrow \int 3y^2-4\, dy = \int 3x^2\, dx$

$\Rightarrow y^3 - 4y = x^3 + C \quad \leftarrow$ I.C. $y(1)=0$

$\Rightarrow 0^3 - 4\cdot 0 = 1^3 + C \Rightarrow C=-1.$

$\Rightarrow \boxed{y^3 - 4y = x^3 - 1}$

- I.O.V. need $a < x < b$, find a, b.

We can't have $y = \pm \frac{2}{\sqrt{3}}$!

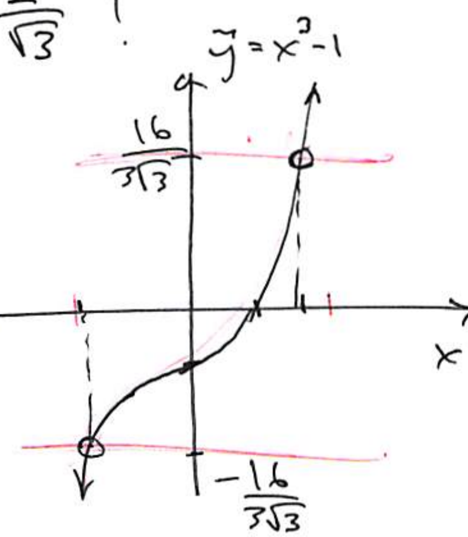
- Notice: $y(y^2-4) = x^3-1$

at $y = \pm \frac{2}{\sqrt{3}}$

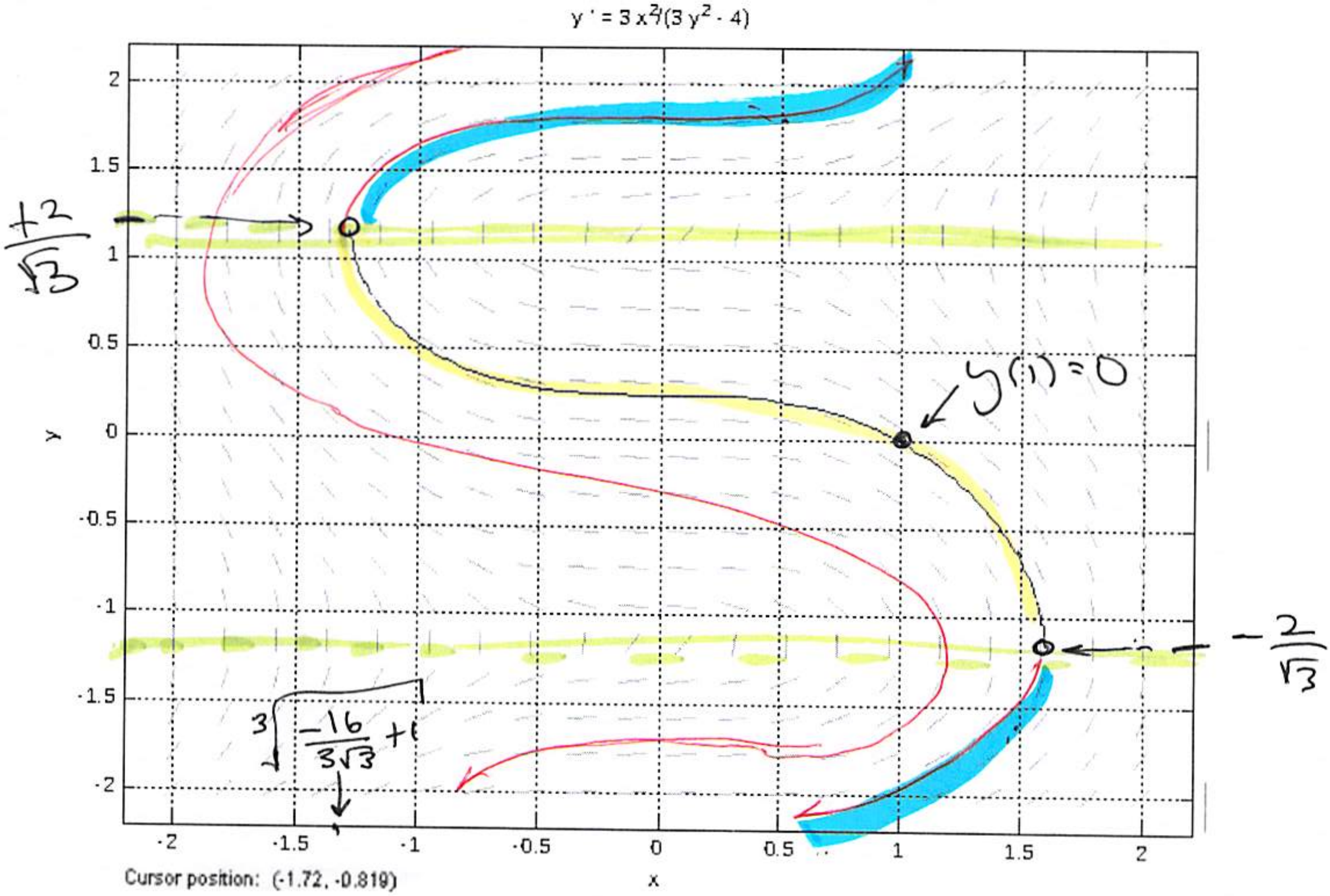
$y(y^2-4) = \pm \frac{2}{\sqrt{3}}\left(\frac{4}{3}-4\right)$

$= \pm \frac{2}{\sqrt{3}} \cdot \frac{8}{3}$

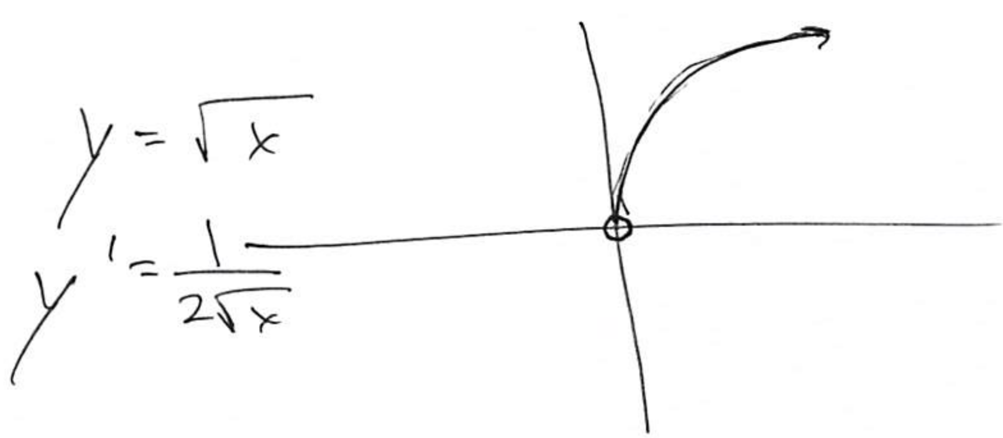
$= \pm \frac{16}{3\sqrt{3}}$



$\Rightarrow \boxed{-\frac{16}{3\sqrt{3}} < x^3 - 1 < \frac{16}{3\sqrt{3}}}$ ✓



y ' = 3 x²/(3 y² - 4)
$\frac{+2}{\sqrt{3}}$
$-\frac{2}{\sqrt{3}}$
y(1) = 0
$\sqrt[3]{\frac{-16}{3\sqrt{3}}+(}$
Cursor position: (-1.72, -0.819)
x
y



$y=\sqrt{x}$
$y'=\frac{1}{2\sqrt{x}}$

In each of Problems 1 through 6 determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2$
2. $t(t-4)y'' + (t-2)y' + y = 0, \qquad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1$
5. $(4-t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \qquad y(2) = 3$

In each of Problems 7 through 12 state the region in the ty-plane where the hypotheses of Theorem 2.4.2 are satisfied. Thus there is a unique solution through each given initial point in this region.

7. $y' = \dfrac{t-y}{2t+5y}$
8. $y' = (1-t^2-y^2)^{1/2}$
9. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$
10. $y' = (t^2+y^2)^{3/2}$
11. $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$
12. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

13. $y' = -4t/y, \qquad y(0) = y_0$
14. $y' = 2ty^2, \qquad y(0) = y_0$
15. $y' + y^3 = 0, \qquad y(0) = y_0$
16. $y' = t^2/y(1+t^3), \qquad y(0) = y_0$

In each of Problems 17 through 20 draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases, and how their behavior depends on the initial value y_0 when $t = 0$.

17. $y' = ty(3-y)$
18. $y' = y(3-ty)$
19. $y' = -y(3-ty)$
20. $y' = t - 1 - y^2$

21. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.
    (a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

Theorem 2.4.1

If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + \boxed{p(t)}y = \boxed{g(t)} \tag{1}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0, \tag{2}$$

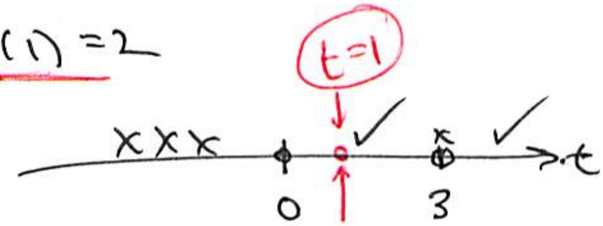
where y_0 is an arbitrary prescribed initial value.

Basically says… for a problem where you can apply the integrating factor method… the solution exists and it's unique in some neighborhood of the initial time value for which the p(t) and g(t) are continuous.

$(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$

$$y' + \underbrace{\frac{\ln(t)}{t-3}}_{p(t)} y = \underbrace{\frac{2t}{t-3}}_{g(t)}$$

not cts at $t \le 0$, $t = 3$ | not cts at $t = 3$.



$(0,3)$ ✓ $\cup$ $(3,\infty)$ ✓

where $p(t), g(t)$ are both cts.

- Thm 2.4.1 says there exists a unique solution $y(t) = f(t,y)$ on $0 < t < 3$.

- If $y(5) = 2$ was the initial condition then Thm 2.4.1 $\Rightarrow \exists!$ solution on $t > 3$

21B

$\frac{dy}{dx} = y$ ⊕ $y(0) = 1 \rightarrow y = e^x$.

$\int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + c$

$\Rightarrow \boxed{y = e^x}$