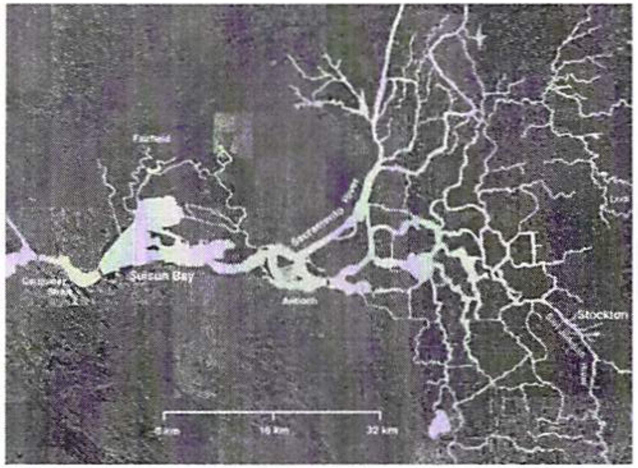
Lecture 4 – Wednesday Sept 28, 2016

"He who asks a question is a fool for 5 minutes;
he who does not ask a question remains a fool forever"
Confucius

**<u>Schedule for Lecture</u>**

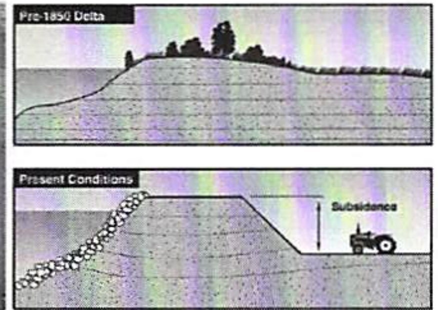
- Warm-up
- Review of I.F.M.
- Separable Equations
  - Examples
  - Implicit vs. Explicit Form
  - Intervals of Validity

Consider the San Joaquin delta area of California






Levee walls break and the cost is tremendous




- The problem is both interesting and practical
- Need appropriate equations and model problem
- Need to solve them and verify the solution



Water flow in two dimensions can be modeled by these equations

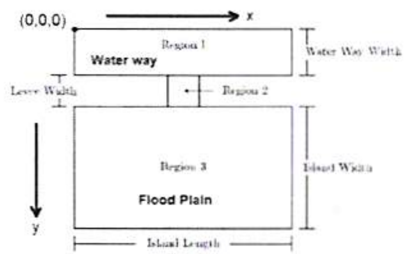
$$h_t + (hu)_x + (hv)_y = 0$$
$$(hu)_t + (hu^2 + 0.5gh^2)_x + (huv)_y = 0$$
$$(hv)_t + (huv)_x + (hv^2 + 0.5gh^2)_y = 0$$

Where $h$ = height, $v$ = velocity in $y$, $u$ = velocity in $x$.

- All it says is that mass and momentum are conserved



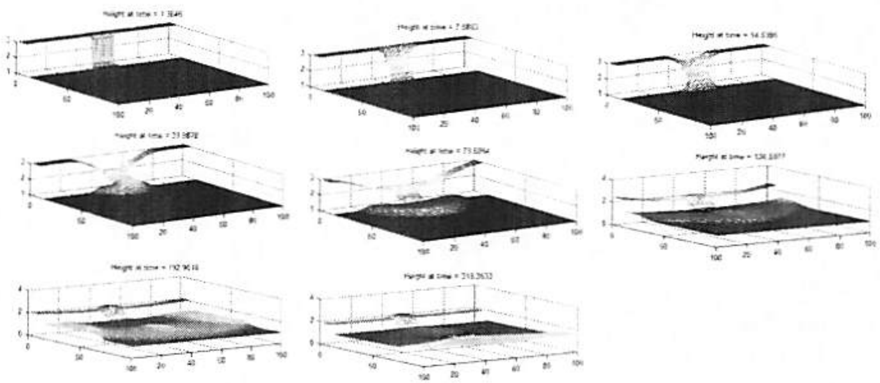
We need to think about how we are going to consider the problem




**Often times we need a greatly simplified geometry**

**A real problem may call for very irregular boundaries**

Now we use the model setup and the equations to compute the water flooding through a break in a levee wall



The water flows in from the main water way and eventually fills up the flood plain



Can also address some other questions like mixing with the dirt as the water flows using a concentration model

$$(ch)_t + (uch)_x + (vch)_y = \alpha\left(|u|^2 + |v|^2\right) - \kappa ch + \epsilon\Delta u$$

- $\alpha$ is tunable to soil diffusivity... (dusty or rocky)
- $\alpha\left(|u|^2 + |v|^2\right)$ says forcing is proportional to the kinetic energy of water locally
- If $\kappa = 0$ then $c \to \infty$ for $\alpha > 0$
- $\kappa > 0$ limit the total allowable concentration to account for sedimentation (saturation)

These are things that are open to discussion / debate



When you put it all together, you would then have to solve 4 equations together

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + (hu^2 + 0.5gh^2)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + (hv^2 + 0.5gh^2)_y &= 0 \\ (ch)_t + (uch)_x + (vch)_y &= \alpha\left(|u|^2 + |v|^2\right) - \kappa ch + \epsilon\Delta u \end{aligned}$$

So while this might answer more interesting questions, it becomes more complicated to study.



In applied mathematical modeling

- Have a well defined problem
- Think about the questions that you want to answer
- Think critically about what approximations can be made
- Choose mathematical equations suited to the problem and your questions
- Setup up a geometric model suited to the equations and the problem
- Solve the equations and study the results

Solve the IVP

$$ty' + 2y = \sin t \quad , \; t > 0 \text{ and } y(\tfrac{\pi}{2}) = 1.$$

① $\xrightarrow{\text{S.F.}}$ $y' + \frac{2}{t}y = \frac{\sin t}{t}$

1 $\boxed{\frac{dy}{dt} + p(t)y = g(t)}$ ← $\mu(t) = e^{\int p(t)\,dt}$.

② $\xrightarrow{\mu}$ $\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t| + \overset{0}{\cancel{C}}} = e^{\ln|t^2|}\underbrace{C_1}_{e^C}$

$= C \cdot t^2$

③ $\xrightarrow{y(t)}$ $\mu \cdot y' + \mu\frac{2}{t}y = \mu\frac{\sin t}{t}$

$$\frac{d}{dt}(\mu y) = \mu\frac{\sin t}{t}$$

$\Rightarrow \quad \mu y = C_1\int t^2\frac{\sin t}{t}\,dt + C = \int t\sin t\,dt + C$

$\cancel{C_1}\, t^2 y = \cancel{C_1}\left[\int t\sin t\,dt + \tilde{C}\right]$

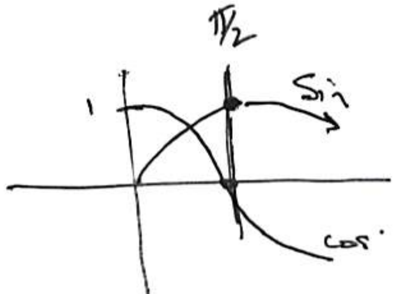
$u = t \quad dv = \sin t\,dt$

$du = dt \quad v = -\cos t$

$y = \frac{1}{t^2}\cdot\left[\int t\sin t\,dt + C\right]$

$= \frac{1}{t^2}\left[-t\cos t + \int \cos t\,dt + C\right]$

$= \frac{1}{t^2}\cdot\left[-t\cos t + \sin t + C\right]$




$$\boxed{y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}}$$ ← General Soln.

④ $\xrightarrow{\text{I.C.}}$

$y(\pi/2) = 1$

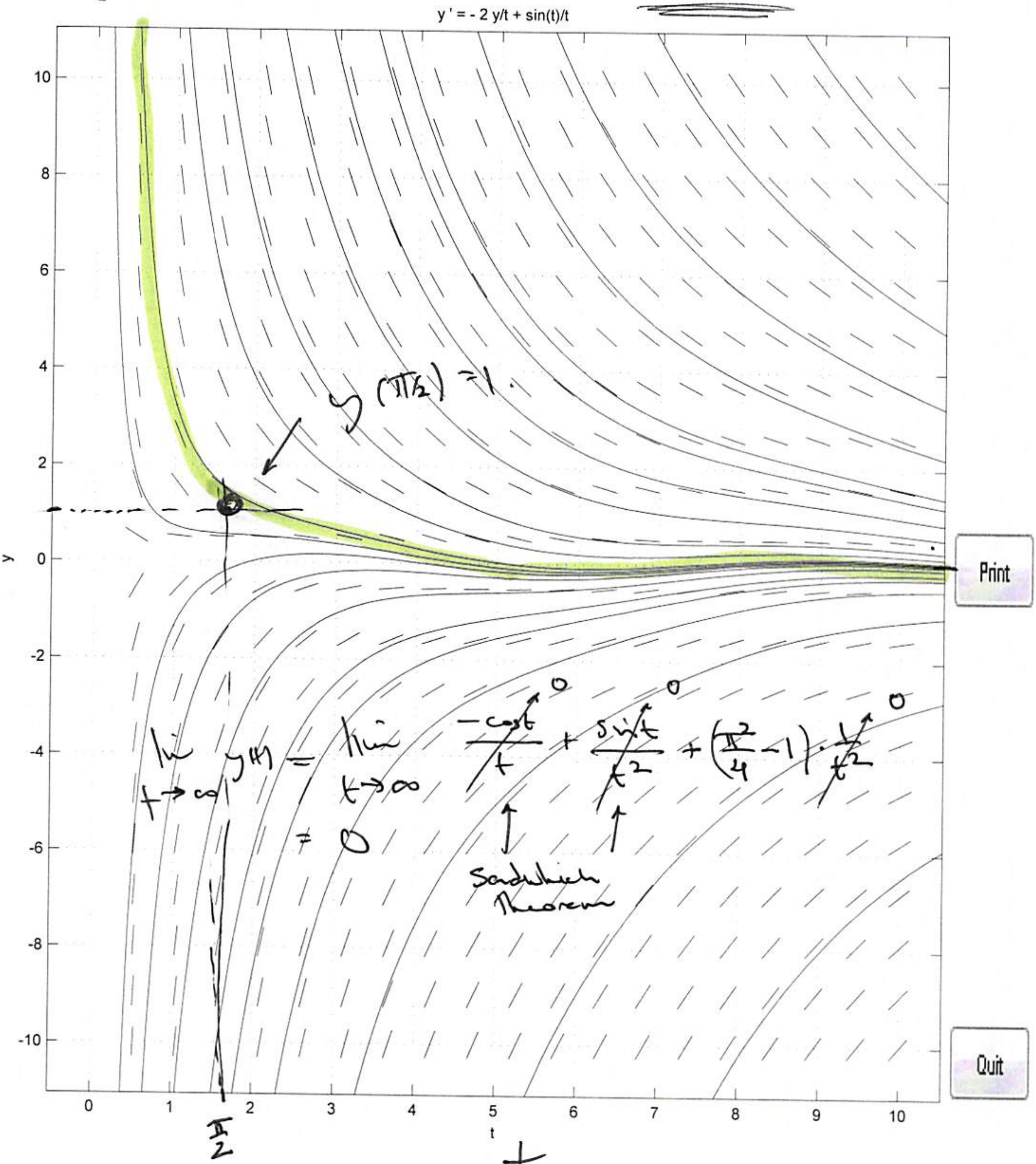
$1 = -\frac{\overset{0}{\cancel{\cos(\pi/2)}}}{\pi/2} + \frac{\overset{1}{\cancel{\sin(\pi/2)}}}{(\frac{\pi}{2})^2} + \frac{C}{(\pi/2)^2}$

$\Rightarrow 1 = \frac{4}{\pi^2} + \frac{4}{\pi^2}\cdot C \Rightarrow \frac{\pi^2}{4}\left(1 - \frac{4}{\pi^2}\right) = C$

$$\Rightarrow \quad y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{\pi^2}{4}\left(1 - \underline{\frac{4}{\pi^2}}\right)\frac{1}{t^2}$$

particular soln.

$\lim_{t\to\infty} y(t) = 0$ $\qquad y(\pi/2) = 1;\ t>0$



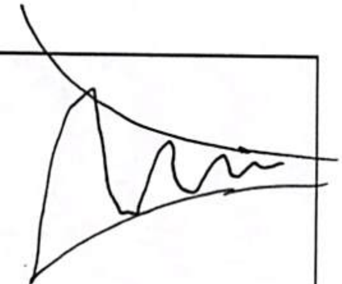

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{-\cos t}{t} + \frac{\sin t}{t^2} + \left(\frac{\pi^2}{4}-1\right)\cdot\frac{1}{t^2}$$
$$= 0$$

Sandwich Theorem

Ready.
Computing the field elements.
Ready. Cursor position: (1.6, 3.86)
Computing the field elements.
Ready.

$$\lim_{t\to\infty} \frac{\sin t}{t} = 0.$$

$$\lim_{t\to\infty} \frac{\cos(t)}{t^2} = 0$$



Not L'Hopital Rule, These need "Sandwich Theorem." Student Version of MATLAB

## S.O.V Separation of Variables §2.2

$$\frac{dy}{dt} = \frac{f(t)}{g(y)} \rightarrow dy = \frac{f(t)}{g(y)}\,dt$$

$$\rightarrow \int g(y)\,dy = \int f(t)\,dt$$

**ex1** $y' - ty = 0.$

$$\boxed{y' + (p(t))y = g(t)}$$

$p(t) = -t \qquad g(t) = 0$

**I.F.M** $\mu = e^{\int -t\,dt} = e^{-t^2/2}$

$$\Rightarrow y = \frac{1}{\mu}\left[\int \mu\cdot \overset{0}{g}\,dt + C\right] = e^{t^2/2}\left[0 + C\right]$$

$$\boxed{y(t) = Ce^{t^2/2}} \checkmark$$

**S.O.V** $\dfrac{dy}{dt} = ty \Rightarrow \displaystyle\int \frac{dy}{y} = \int t\,dt$

$$\Rightarrow \ln|y| = t^2/2 + C$$

$$\Rightarrow y(t) = e^{t^2/2 + C}$$

$$\Rightarrow \boxed{y(t) = Ce^{t^2/2}}.$$

Ex. Explicit vs. Implicit forms.

$$\frac{dy}{dx} = \frac{3x^2-1}{3+2y}$$

$$\Rightarrow \int (3+2y)\,dy = \int (3x^2-1)\,dx$$

$$\Rightarrow 3y + y^2 = x^3 - x + C \quad \leftarrow \text{Implicit}$$

$$\Rightarrow y^2 + 3y - x^3 + x = C \quad \leftarrow f(x,y) = C$$

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How to get "$y(x) = f(x)$" $\leftarrow$ Explicit

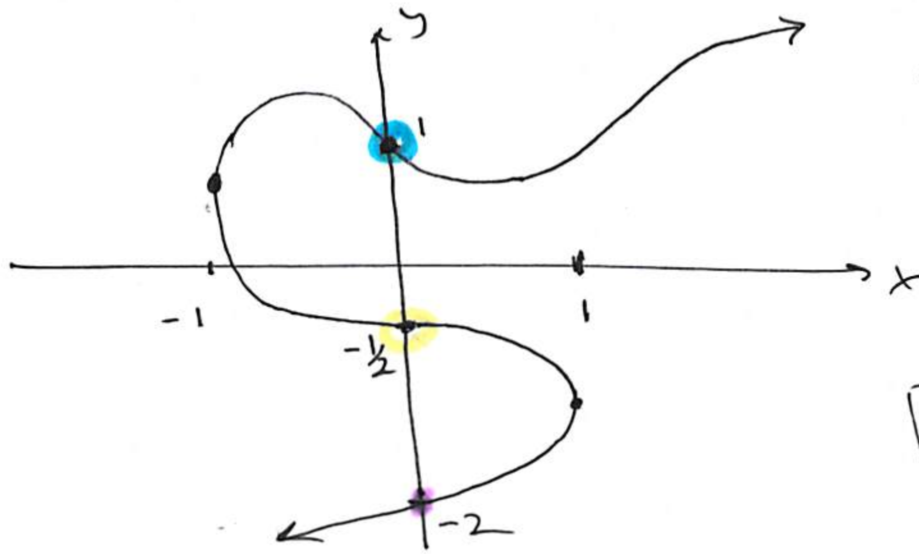
Complete $\square$ $\Rightarrow y^2 + 3y + \frac{9}{4} = x^3 - x + (C + 9/4)$

$$\Rightarrow \left(y + \frac{3}{2}\right)^2 = x^3 - x + C$$

$$\Rightarrow y + \frac{3}{2} = \pm\sqrt{x^3 - x + C}$$

$$\Rightarrow y = -\frac{3}{2} \pm \sqrt{x^3 - x + C} \quad \leftarrow \text{Explicit Form.}$$

# Interval of Validity (concept)




$$\frac{dy}{dx} = f(x,y)$$

solve

$$\boxed{H_1(x) + H_2(y) = c}$$

3 I.C's

| $y(0) = 1$ | $y(0) = -\frac{1}{2}$ | $y(0) = -2$ |
|---|---|---|
| Interval of Validity | Interval of Validity | Interval of V. |
| $\boxed{x > -1}$ | $\boxed{-1 < x < 1}$ | $\boxed{x < 1}$ |

$\boxed{\text{Ex}}$ Solve $y' = 2\overset{=0}{(1+x)}\overset{>0}{(1+y^2)}$, $y(0)=0$

Find the interval of validity <u>and</u> determine where $y(x)$ is a minimum.

- Notice $y'=0$ at $x=-1$. (candidate local min)

$$y' \quad \overset{-}{\phantom{xx}} \underset{\text{local min}}{\overset{-1}{|}} \overset{+}{\phantom{xx}}$$

✓

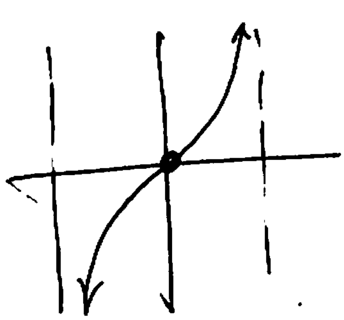
- Sov. to solve the I.V.P.

$$\frac{dy}{dx} = 2(1+x)(1+y^2)$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int 2(1+x)\,dx$$

$\text{atan}(y) = \tan^{-1}(y)$ $\Rightarrow$ $\text{atan}(y) = 2x + x^2 + c$

$$\Rightarrow \quad y(t) = \tan(2x + x^2 + c)$$



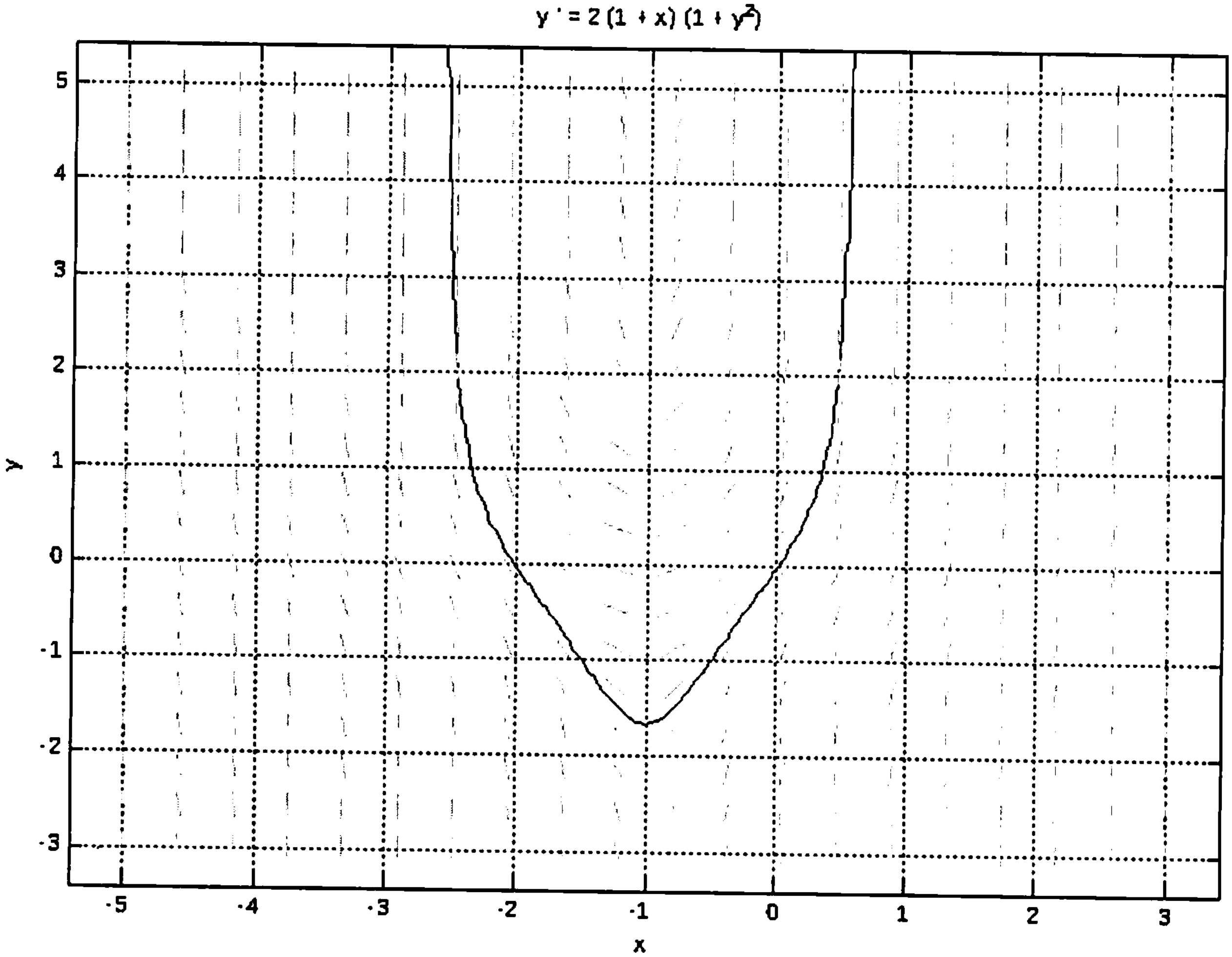
- I.C. $y(0)=0$

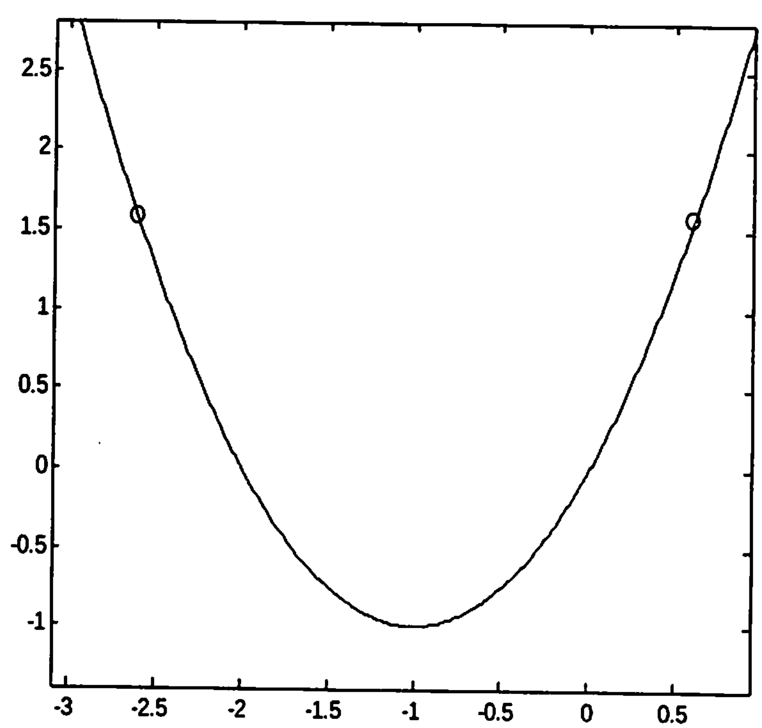
$$\Rightarrow \quad 0 = \tan(c) \Rightarrow c = 0$$

$$\Rightarrow \quad y(t) = \tan(2x + x^2).$$

<u>TBD</u> Interval of validity

$y(x)$ is a minimum. $(-1, \tan(-1))$


y ' = 2 (1 + x) (1 + y²)
5
4
3
2
1
0
-1
-2
-3
y
-5
-4
-3
-2
-1
0
1
2
3
x


2.5
2
1.5
1
0.5
0
-0.5
-1
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5