Lecture 6– Monday October 3, 2016

Sonder (son-der)

(n) the realization that each person is living a life as vivid and complex as your own.

Schedule for Lecture

- Warm-up
- Existence and uniqueness of solutions for first order linear equations (2.4)
  - Theorem 2.4.1
- Existence and uniqueness of solutions for first order nonlinear equations (2.4)
  - Theorem 2.4.2
- Examples
- Modeling with first order equations (2.3)
- Autonomous Equations and Population Dynamics (2.5)

Theorem 2.4.1 – Linear 1^{st} order D.E.

If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\boxed{y' + p(t)y = g(t)} \tag{1}$$

for each t in I, and that also satisfies the initial condition

$$\boxed{y(t_0) = y_0.} \tag{2}$$

where y_0 is an arbitrary prescribed initial value.

Basically says… for a problem where you can apply the integrating factor method… the solution exists and it's unique in some neighborhood of the initial time value for which the p(t) and g(t) are continuous.

Theorem 2.4.2 – nonlinear D.E.'s. (1^{st} order)

region

Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$\boxed{y' = f(t, y), \qquad y(t_0) = y_0.} \tag{8}$$

Important Remarks:

The existence of solution(s) can be established from the continuity of f(t,y) alone, but not the uniqueness.

We will discuss this theorem and it's proof in detail in section 2.8 (next Friday).

$\boxed{\text{IVP}}$ $\quad y' = f(t,y) \quad + \quad y(t_0) = y_0$

$$y' = y^2 + t$$

$$\sin(y)\, y^3 \cdot y' = \sin t$$

$$y' = \frac{\sin t}{\sin(y)\, y^3}.$$

In each of Problems 1 through 6 determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

2.4.1

1. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$
2. ~~$t(t-4)y'' + (t-2)y' + y = 0, \quad y(2) = 1$~~ ?
3. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2, \quad y(-3) = 1$
5. $(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \quad y(2) = 3$

$y' + p(t)y = g(t)$

In each of Problems 7 through 12 state the region in the ty-plane where the hypotheses of Theorem 2.4.2 are satisfied. Thus there is a unique solution through each given initial point in this region.

2.4.2

$y' = f(t,y)$

7. $y' = \dfrac{t-y}{2t+5y}$

8. $y' = (1-t^2-y^2)^{1/2}$

9. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$

10. $y' = (t^2+y^2)^{3/2}$

11. $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$

12. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

13. $y' = -4t/y, \quad y(0) = y_0$
14. $y' = 2ty^2, \quad y(0) = y_0$
15. $y' + y^3 = 0, \quad y(0) = y_0$
16. $y' = t^2/y(1+t^3), \quad y(0) = y_0$

In each of Problems 17 through 20 draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases, and how their behavior depends on the initial value y_0 when $t = 0$.

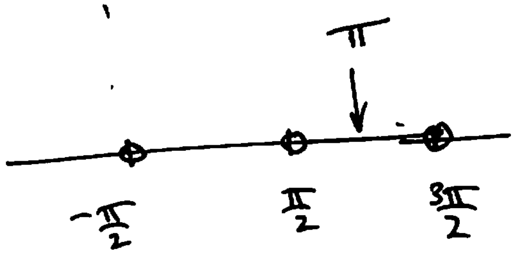
17. $y' = ty(3-y)$
18. ▶ $y' = y(3-ty)$
19. $y' = -y(3-ty)$
20. ▶ $y' = t-1-y^2$

21. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.
    (a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

3. $y' + (\tan t)y = \sin t$ $\qquad y(\pi) = 0.$

$y' + p(t)y = g(t)$ $\qquad t = \pi$

always cts.



disc. for $t = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

~~Solution~~

- A unique sdn must exist for $\frac{\pi}{2} < t < \frac{3\pi}{2}$. b/c by Thm 2.4.1 $p(t)$, $g(t)$ are cts on this interval.

- Remark: how does the answer change if $y(0) = 3$? $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$.

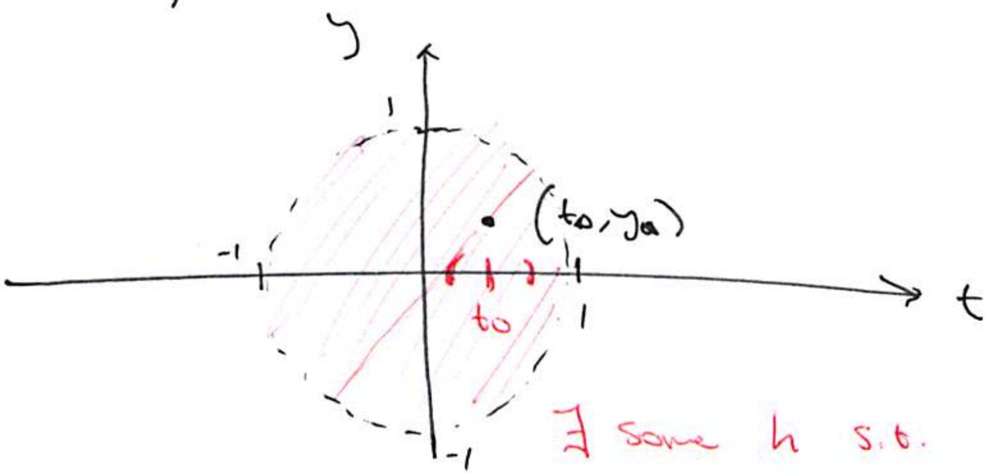
(8) $$y' = (1-t^2-y^2)^{1/2}.$$ apply 2.4.2

$$y' = f(t,y)$$

$\Longrightarrow$ $f(t,y) = \sqrt{1-t^2-y^2}$ cts for $1-t^2-y^2 \geq 0$

and $\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{1-t^2-y^2}}(-2y)$ cts for $1-t^2-y^2 > 0$

$\Longrightarrow$ $f, \dfrac{\partial f}{\partial y}$ are cts $1 > t^2+y^2$ (circle)



$\exists$ some h s.t.
$t_0 - h < t < t_0 + h$.
where a unique soln
is known to exist

(9) $y' = \dfrac{\ln|ty|}{1-t^2+y^2} = f(t,y)$

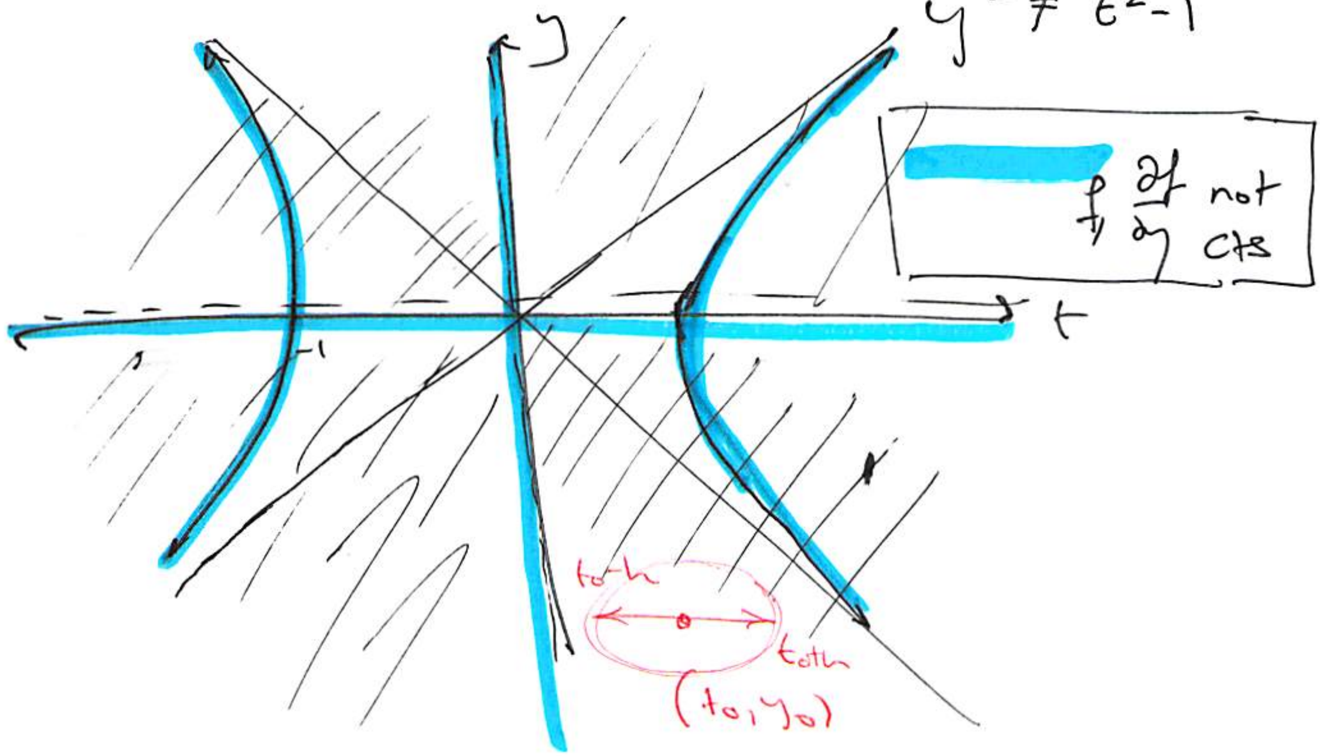
- $f(t,y) = \dfrac{\ln|ty|}{1-t^2+y^2} \Rightarrow$ cts for $|ty|>0$ and $1-t^2+y^2 \neq 0$

- $\dfrac{\partial f}{\partial y} = \dfrac{(1-t^2+y^2)\cdot\frac{1}{ty}\cdot t - \ln|ty|\,2y}{(1-t^2+y^2)^2}$

cts for $|ty|>0$, $1-t^2+y^2 \neq 0$.

~~$t,y>0$ or $t,y<0$~~, but $t\neq 0, y\neq 0$

$y^2 \neq t^2-1$



(10) $y' = (t^2+y^2)^{3/2} = f(t,y)$

f is cts for all $t, y \in \mathbb{R}^2$

$\frac{\partial f}{\partial y} = \frac{3}{2}(t^2+y^2)^{1/2} \cdot 2y$ is also cts for all $\mathbb{R}^2$.

By Thm 2.4.2 there exists a unique solution for every (t_0, y_0) in $\mathbb{R}^2$.

13 $y' = -\frac{4t}{y}$ $y(0) = y_0$

$\int y\,dy = -4\int t\,dt$

$\Rightarrow \frac{y^2}{2} = -4\frac{t^2}{2} + c$

$\Rightarrow y^2 = -4t^2 + c \Rightarrow \boxed{y^2 = -4t^2 + y_0^2}$

$\Rightarrow y_0^2 = c$

Notice $y = \pm\sqrt{y_0^2 - 4t^2}$

we need $y_0^2 - 4t^2 \geq 0$

$\Rightarrow y_0^2 \geq 4t^2$

$\sqrt{X^2} = |X|$

$\Rightarrow \sqrt{\left(\frac{y_0}{2}\right)^2} \geq \sqrt{t^2}$

$\Rightarrow \left|\frac{y_0}{2}\right| \geq |t|$ ✓

Tells us how $y = \pm\sqrt{y_0^2 - 4t^2}$ existence depends on y_0.

Applications for first order equations (2.3)

- Fluid mixing problems (1-6, 19)
    - TBD
- Compounded interest (7-12)
    - TBD
- Carbon dating and radioactive decay (13)
    - Already discussed in brief
- Population Growth (14-15) § 2.6
- Newton's law of convective cooling (16, 18)
    - Already discussed in brief
- Boltzmann's law of radiative cooling (17)
    - Same idea as Newton's law, just more dramatic because it's u^4 instead of u
- Fall in gravitational field and orbital mechanics (20-23,25-31)
    - Already discussed in brief
- Rocket sled in water (24)
    - Already discussed in brief, it's similar to the parachutist problem
- Efficient motion of particles - Brachistochrone Problem (32)
    - TBD

§ 2.6 Qualitative Analysis.

i.e $\frac{dy}{dt} > 0 \Rightarrow y$ increasing

$\frac{dy}{dt} = 0 \Rightarrow y$ is steady.

For an <u>autonomous equation</u>

$$\frac{dy}{dt} = f(y)$$

no explicit t on right hand side.

example...

$$\frac{dy}{dt} = ky \longrightarrow y(t) = Ce^{kt}$$

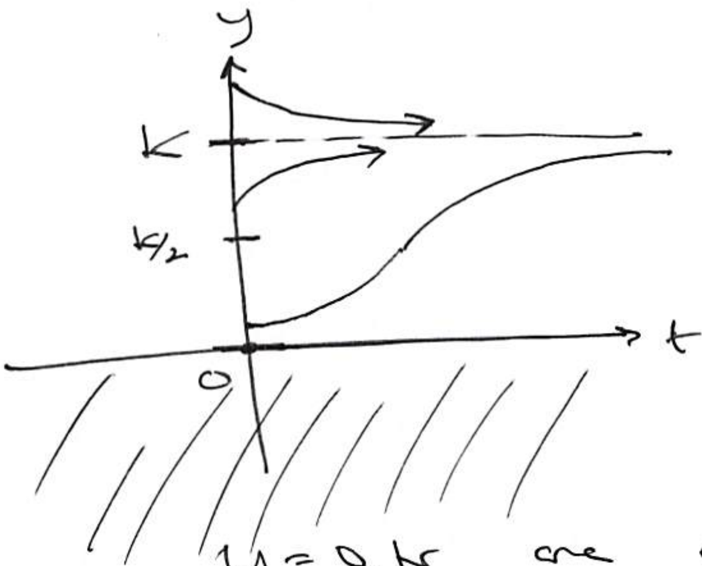
consider for pop growth $(k > 0)$, ~~—~~
not possible to grow forever.....

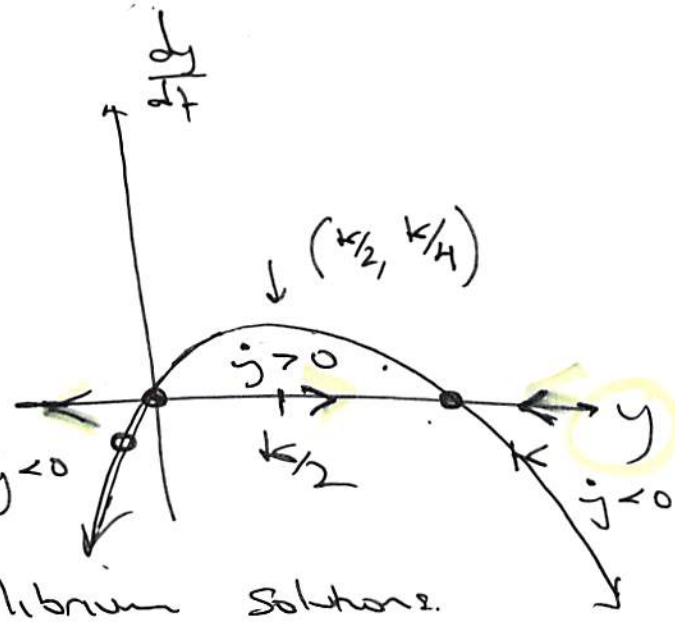
A better model takes into account growth limits

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) \cdot y$$

$r, K > 0$, this is <u>logistic growth</u>.

Qualitative analysis





$y = 0, K$ are equilibrium solutions.