typing

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Below, W_1 is used for $\langle w_1, l_1\rangle$, W_2 for $\langle w_2, l_2\rangle$, and U for $\langle u, l\rangle$. Naming conventions are

- W_i for starting worlds
- U_i for future worlds (usually introduced by a $\forall$)
- V for "inner" worlds (usually under a $\triangleright$)

preworld

0.1 $preworld : K_0$

Showing $\Gamma \vdash preworld : K_0$ for any Γ
Apply 40 to get

1. $\Gamma, w : \underline{K_0} \vdash \mathbb{N} \to (\triangleright w \to_K \triangleright \mathbb{N} \to U_0) : K_0$
   Apply 8 to get
   (a) $\Gamma \cdots \vdash \mathbb{N} : U_0$
       basic types
   (b) $\Gamma, w : \underline{K_0} \vdash (\triangleright w \to_K \triangleright \mathbb{N} \to U_0)$
       Apply 13 to get
       i. $\Gamma, w : \underline{K_0} \vdash \triangleright w : K_0$
          32
       ii. $\Gamma, w : \underline{K_0} \vdash \triangleright \mathbb{N} \to U_0 : K_0$
           Apply 8
           A. $\Gamma, w : \underline{K_0} \vdash \triangleright \mathbb{N} : U_0$
              32
           B. $\Gamma, w : \underline{K_0} \vdash U_0 : K_0$
              97
2. $0 : level$
   done

Showing $\Gamma \vdash \triangleright preworld : K_0$.
32 on above. Need lemma allowing weakening to promote(Γ).

0.2 $\Gamma \vdash w_1\, i\, u\, l : U_0$

Given $\Gamma \vdash W_1 : \mathit{world}$, $\Gamma \vdash i : \mathbb{N}$, $\Gamma \vdash u : \triangleright \mathit{preworld}$, $\Gamma \vdash l : \triangleright\mathbb{N}$.
Apply 6 to get

$$\Gamma \vdash w_1 i u : \Pi(y : \triangleright\mathbb{N}).U_0$$

Apply 108, 11 to get

$$\Gamma \vdash w_1 i u : \triangleright\mathbb{N} \to U_0$$

Applying 6, 108, (11 and 16 with transitivity) gives

$$\Gamma \vdash w_1 i : \triangleright pw \to_k \triangleright\mathbb{N} \to U_0$$

Applying 6, 108, 11 again gives

$$\Gamma \vdash w_1 : \mathbb{N} \to \triangleright pw \to_k \triangleright\mathbb{N} \to U_0$$

Applying 108, 43 gives

- $\Gamma \vdash w_1 : \mu w.\mathbb{N} \to \triangleright w \to_k \triangleright\mathbb{N} \to U_0$
  Beta reduction on $\pi_1 W_1$, using $\Gamma \vdash \langle w_1, l_1\rangle : \mathit{world}$.
- $\Gamma, w : \underline{\mathit{type}} \vdash \mathbb{N} \to \triangleright w \to_k \triangleright\mathbb{N} \to U_0 : \mathit{type}$
  preworld, 101, 105

0.3 cons_b

(cons on the back)
$\mathit{cons}_b\ w_1\ l_1\ x$ is defined as

$$\lambda i.\mathit{ite}(i <_b l_1)(w_1 i)(x)$$

Showing that if $\Gamma \vdash w_1 : \mathit{preworld}$ and and $\Gamma \vdash l_1 : \mathbb{N}$ and $\Gamma \vdash x : \triangleright pw \to_k \triangleright\mathbb{N} \to U_0$, then $\Gamma \vdash \mathit{cons}_b\ w_1\ l_1\ x : pw$.

subseq

$\Gamma \vdash W_1 \leq W_2 : U_0$

For Γ where $\Gamma \vdash W_i : \mathit{world}$. Apply 27 to get

- $\Gamma \vdash l_1 \leq l_2 : U_0$
  Beta reduction on $\pi_1 W_1$, using $\Gamma \vdash \langle w_1, l_1\rangle : \mathit{world}$. Beta reduction on $\pi_2 W_j$, using $\Gamma \vdash \langle w_j, l_j\rangle : \mathit{world}$, gives $\Gamma \vdash l_j : \mathit{nat}$ for $j \in \{1, 2\}$. Then use basic types.
- $\Gamma \vdash \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})\Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type}) : U_0$
  Apply 66 to get

  $$\Gamma, u : \triangleright pw \vdash \Pi(l : \triangleright\mathbb{N})\Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type}) : U_0$$

  Apply 4 to get

  $$\triangleright\mathbb{N} : U_0$$

  which is easy by 34 and basic types as well as

  $$\Gamma, u : \triangleright pw, l : \triangleright\mathbb{N} \vdash \Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type}) : U_0$$

  Apply 4 again

  $$\Gamma, u : \triangleright pw, l : \triangleright\mathbb{N}, i : \mathbb{N}, m : i \leq l_1 \vdash (w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type}) : U_0$$

(where $\Gamma, u : \rhd pw, l : \rhd \mathbb{N}, i : \mathbb{N} \vdash m : i \leq l_1 : U_0$ follows from basic types and the fact that $\Gamma \vdash l_i : nat$).
Apply 107

$$\Gamma, u : \rhd pw, l : \rhd \mathbb{N}, i : \mathbb{N}, m : i \leq l_1 \vdash w_j\ i\ u\ l : U_i$$

This follows from preworld

0.4 transitivity

$\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3 \vdash \langle *, \lambda l.\lambda i.\lambda m.* \rangle : W_1 \leq W_3$
Apply (108 with 28) then 20 to get

- $$\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3 \vdash, * : l_1 \leq l_3$$

  Apply 119, taking advantage of rule 28 and the fact that if $\Gamma \vdash A = B$ then $\Gamma \vdash A \leq B$ (basic types)

  $$\Gamma, m_1 : \Sigma(l_1 \leq l_2).B, m_2 : \Sigma(l_2 \leq l_3).B \vdash, * : l_1 \leq l_3$$

  From here, I have by 21 that

  $$\Gamma, m_1 : \Sigma(l_1 \leq l_2).B, m_2 : \Sigma(l_2 \leq l_3).B \vdash\vdash \pi_1 m_1 : l_1 \leq l_2$$

  and

  $$\Gamma, m_1 : \Sigma(l_1 \leq l_2).B, m_2 : \Sigma(l_2 \leq l_3).B \vdash\vdash \pi_1 m_2 : l_2 \leq l_3$$

  I can then apply transitivity of $\leq$ (basic types) to get that

  $$\Gamma \cdots \vdash, * : l_1 \leq l_3$$

- $$\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3 \vdash \lambda l.\lambda i.\lambda m.* : \forall(u : \rhd pw)\Pi(l : \rhd \mathbb{N}).\Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1 iul = w_3 iul : type)$$

  Apply 67

  $$\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3, u : \rhd pw \vdash \lambda l.\lambda i.\lambda m.* : \Pi(l : \rhd \mathbb{N}).\Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1 iul = w_3 iul : type)$$

  Apply 5 multiple times

  $$\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3, u : \rhd pw, l : \rhd \mathbb{N}, i : \mathbb{N}, m : i \leq l_1 \vdash * : (w_1\ i\ u\ l = w_3\ i\ u\ l : type)$$

  I denote $\Gamma, m_1 : W_1 \leq W_2, m_2 : W_2 \leq W_3, u : \rhd pw, l : \rhd \mathbb{N}, i : \mathbb{N}, m : i \leq l_1$ as Γ'. It suffices to show there is an x, y where

  $$\Gamma' \vdash x : (w_1\ i\ u\ l = w_2\ i\ u\ l : type)$$

  and

  $$\Gamma' \vdash y : (w_2\ i\ u\ l = w_3\ i\ u\ l : type)$$

  - I assume the existence of such an x, y. Then,
    Result 5 of basic types gives

    $$\Gamma' \vdash w_1\ i\ u\ l = w_2\ i\ u\ l : type$$

    and

    $$\Gamma' \vdash (w_2\ i\ u\ l = w_3\ i\ u\ l : type)$$

    Then 110 gives

    $$\Gamma' \vdash w_1\ i\ u\ l = w_3\ i\ u\ l : type$$

    which is equivalent to the desired

    $$\Gamma' \vdash * : (w_1\ i\ u\ l = w_3\ i\ u\ l : type)$$

– I show

$$\Gamma' \vdash (\pi_2\, m_1) l\, i\, m : (w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type}) \ (1)$$

and

$$\Gamma' \vdash (\pi_2\, m_2) l\, i\, ((\pi_1 m_1) \circ m) : (w_2\, i\, u\, l = w_3\, i\, u\, l : \mathit{type}) \ (2)$$

The proofs go similarly, so I show only 1 for now. Apply 119 with 28 and result 4 of basic types.

$\Gamma, \ldots, m_1 : \Sigma(l_1 \leq l_2).B, u : \triangleright pw, l : \triangleright\mathbb{N}, i : \mathbb{N}, m : i \leq l_1 \vdash (\pi_2\, m_1) l\, i\, m : (w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type})$

Applying 6 multiple times gives

$$\Gamma, \ldots, m_1 : \Sigma(l_1 \leq l_2).B, u : \triangleright pw, l : \triangleright\mathbb{N}, i : \mathbb{N}, m : i \leq l_1 \vdash (\pi_2\, m_1) :$$
$$\Pi(l : \triangleright\mathbb{N})\Pi(i : \mathbb{N})\Pi(m : i \leq l_1)(w_1\, i\, u\, l = w_2\, i\, u\, l : \mathit{type})$$

But this follows from 22.
The proof of 2 requires only the additional explanation that $(\pi_1 m_1) \circ m : i \leq l_2$. But, this comes from transitivity of $\leq$ and the typing of m_1 and rule 21.

0.5 reflexivity

if $\Gamma \vdash W : \mathit{world}$ then $\Gamma \vdash * : W \leq W$.

0.6 $cons_b$

if $\Gamma \vdash W_1 : \mathit{world}$ and $\Gamma \vdash x : \triangleright pw \to_k \to \triangleright\mathbb{N} \to U_0$, then
$\Gamma \vdash \langle *, \lambda l_v.\lambda i.\lambda m_i.* \rangle : W_1 \leq \langle (cons_b\, w_1\, l_1\, x), succ(l_1) \rangle$.
I abbreviate $cons_b\, w_1\, l_1\, x$ as w_2.
Applying 108 with 28, then 20 yields $\Gamma \vdash * : l_1 \leq succ(l_1)$ (which is done in basic types) as well as

$$\Gamma \vdash \lambda l_v.\lambda i.\lambda m_i.* :$$
$$\forall(v : \triangleright pw)\Pi(l_v : \triangleright\mathbb{N})\Pi(i : \mathbb{N})\Pi(m : i < l_1).(w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type}$$

Apply 67

$$\Gamma, v : \triangleright pw \vdash \lambda l_v.\lambda i.\lambda m_i.* :$$
$$\Pi(l_v : \triangleright\mathbb{N})\Pi(i : \mathbb{N})\Pi(m : i < l_1).(w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type}$$

Applying 5 x3 gives

$$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : i < l_1 \vdash * : (w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type}$$

I apply 119, taking advantage of the reflection lemma between $i <_b l_1 = \mathit{true} : \mathit{bool}$ and $i < l_1$ to get, amongst goals easily solved,

$$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}) \vdash * : (w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type} \ (*)$$

It suffices to show that

$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}) \vdash \lambda\_.* : \Pi(m' : i <_b l_1 = \mathit{true} : \mathit{bool})((w_1 i v l_v) = (w_2 i v l_v) : \mathit{type})(**)$

- I assume $(**)$ and show $(*)$
  By $(**)$ and 6, I have that

  $$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}) \vdash (\lambda\_.*) m : ((w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type})$$

  β reduction gives

  $$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}) \vdash * : ((w_1\, i\, v\, l_v) = (w_2\, i\, v\, l_v) : \mathit{type})$$

  But, this is exactly (*)

- Showing (**)
I apply 121 with $M := i <_b l_1$ to generate
$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}) \vdash i <_b l_1 : \mathit{bool}$ (basic types) and

$$\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}), b : \mathit{bool} \vdash$$

$$\lambda\_.* : \Pi(m' : b = \mathit{true} : \mathit{bool})((w_1\ i\ v\ l_v) = ((\lambda i.\mathit{ite}(b)(w_1 i)(x))\ i\ v\ l_v) : \mathit{type})$$

I denote $\Gamma, v : \triangleright pw, l_v : \triangleright\mathbb{N}, i : \mathbb{N}, m : (i <_b l_1 = \mathit{true} : \mathit{bool}), b : \mathit{bool}$ as $\Gamma_1, b : \mathit{bool}$. Applying 5 yields $\Gamma_1, b : \mathit{bool} \vdash (b = \mathit{true}) : \mathit{bool} : \mathit{type}$ (which is straightforward) as well as

$$\Gamma_1, b : \mathit{bool}, m' : b = \mathit{true} : \mathit{bool} \vdash * : \big((w_1\ i\ v\ l_v) = ((\lambda i.\mathit{ite}(b)(w_1 i)(x))\ i\ v\ l_v) : \mathit{type}\big)$$

Applying 120 gives

  - $\Gamma_1, b : \mathit{bool}, m' : b = \mathit{true} : \mathit{bool} \vdash \big((w_1\ i\ v\ l_v) = ((\lambda i.\mathit{ite}(b)(w_1 i)(x))\ i\ v\ l_v) : \mathit{type}\big) : \mathit{type}$
  Similar reasoning to $\mathit{cons}_b : w_1\ l_1\ x : pw$ to show that the λ has type pw, then similar reasoning to section 0.2 to show that both applications have type U_0.
  - $\Gamma_1, b : \mathit{bool}, m' : b = \mathit{true} : \mathit{bool} \vdash b = \mathit{true} : \mathit{bool}$
  basic types
  - $FV(\mathit{true}) \cap \Gamma_2, b : \mathit{bool} = \emptyset$
  nice
  -
  $$\Gamma_1, m' : \mathit{true} = \mathit{true} : \mathit{bool} \vdash * : \big((w_1\ i\ v\ l_v) = ((\lambda i.\mathit{ite}(\mathit{true})(w_1 i)(x))\ i\ v\ l_v) : \mathit{type}\big)$$
  After β reduction, this is
  $$\Gamma_1 \cdots \vdash * : \big(w_1\ i\ v\ l_v = w_1 i\ v\ l_v : \mathit{type}\big)$$
  which is to say
  $$\Gamma_1 \cdots \vdash w_1\ i\ v\ l_v : \mathit{type}$$
  This follows from the fact that $\Gamma_1 \vdash w_1 : pw$ and 0.2 of preworld.

store

Showing $\Gamma, w_1 : pw, l_1 : \mathbb{N} \vdash \mathit{store}(W_1) : U_0$
Apply 66 to get

$$\Gamma, w_1 : pw, l_1 : \mathbb{N}, u : pw \vdash \Pi(l : \mathbb{N})\langle w_1, l_1\rangle \leq \langle u, l\rangle \to *w_1/\langle u, l\rangle$$

Apply 4, 10, subseq to get

$$\Gamma, w_1 : pw, l_1 : \mathbb{N}, u : pw, l : \mathbb{N} \vdash *w_1/\langle u, l\rangle : U_0$$

which is to say

$$\Gamma, w_1 : pw, l_1 : \mathbb{N}, u : pw, l : \mathbb{N} \vdash \pi(i : \mathbb{N}).w_1\ i(\mathit{next}\ u)\ (\mathit{next}\ l) : U_0$$

Apply 4 to get

$$\Gamma, w_1 : pw, l_1 : \mathbb{N}, u : pw, l : \mathbb{N}, i : \mathbb{N} \vdash w_1\ i(\mathit{next}\ u)\ (\mathit{next}\ l) : U_0$$

This follows from preworld and 35.

types

For any τ in source language and Γ where $\Gamma \vdash W_1 : \mathit{world}$, $\Gamma \vdash \tau @ W_1 : U_0$. Induction on τ.

0.7 nat

basic types

0.8 $\tau_1 \to \tau_2$

Apply 66 to get

1. $\Gamma \vdash pw : K_0$
   preworld
2. $\Gamma, u : pw \vdash \Pi(l : \mathbb{N}) \cdots : U_0$
   Apply 4
   (a) $\mathbb{N} : U_0$
       basic types
   (b) $\Gamma, u : pw, l : \mathbb{N} \vdash W_1 \leq U \to \tau_1 @ U \to \tau_2 @ U$
       Apply 10
       i. $\Gamma, u : pw, l : \mathbb{N} \vdash W_1 \leq U : U_0$
          subseq
       ii. $\Gamma, u : pw, l : \mathbb{N} \vdash \tau_1 @ U \to \tau_2 @ U : U_0$
          apply 10, by IH suffices to show that $\Gamma, u : pw, l : \mathbb{N} \vdash U : \mathit{world}$. Apply (28 and 108), then 20. Suffices to show that $\Gamma \cdots \vdash \mathbb{N} : \mathit{type}$.
3. $\Gamma \vdash 0 : \mathit{level}$
   done
4. $0 \leq 0$
   basic types

0.9 $\bigcirc\tau$

Apply 66 to get, amongst other goals solved before,

$$\Gamma, u : pw \vdash \Pi(l : \mathbb{N}) W_1 \leq U \to \mathit{store}(U) \to \rhd \rhd \exists(v : pw)\Sigma(l' : \mathbb{N})(U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau @ \langle v, l' \rangle) : U_0$$

Apply 4 to get $\Gamma, u : pw \vdash \mathbb{N} : U_0$ and

$$\Gamma, u : pw, l : \mathbb{N} \vdash W_1 \leq U \to \mathit{store}(U) \to \rhd \rhd \exists(v : pw)\Sigma(l' : \mathbb{N})(U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau @ \langle v, l' \rangle) : U_0$$

Apply 10 repeatedly to get

1. $\Gamma, u : pw, l : \mathbb{N} \vdash W_1 \leq U : U_0$
   subseq
2. $\Gamma, u : pw, l : \mathbb{N} \vdash \mathit{store}(U) : U_0$
   store
3. $\Gamma, u : pw, l : \mathbb{N} \vdash \rhd \rhd \exists(v : pw)\Sigma(l' : \mathbb{N})(U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau @ \langle v, l' \rangle) : U_0$
   Apply the bar rule from bar types to get
   $$\Gamma, u : pw, l : \mathbb{N} \vdash \exists(v : pw)\Sigma(l' : \mathbb{N})(U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau @ \langle v, l' \rangle) : U_0$$
   Apply 70 to get
   $$\Gamma, u : pw, l : \mathbb{N}, v : pw \vdash \Sigma(l' : \mathbb{N})(U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau @ \langle v, l' \rangle) : U_0$$

Apply 19 to get

$$\Gamma, u : pw, l : \mathbb{N}, v : pw, l' : \mathbb{N} \vdash U \leq \langle v, l' \rangle \times \mathit{store}\langle v, l' \rangle \times \tau@\langle v, l' \rangle : U_0$$

Apply 27 twice to get

- $\Gamma, u : pw, l : \mathbb{N}, v : pw, l' : \mathbb{N} \vdash U \leq \langle v, l' \rangle : U_0$
  subseq
- $\Gamma, u : pw, l : \mathbb{N}, v : pw, l' : \mathbb{N} \vdash \mathit{store}\langle v, l' \rangle : U_0$
  store
- $\Gamma, u : pw, l : \mathbb{N}, v : pw, l' : \mathbb{N} \vdash \tau@\langle v, l' \rangle : U_0$
  induction

0.10 $ref \tau$

Apply 19 to get

$$\Gamma, i : \mathbb{N} \vdash (i < l_1) \times \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w_1 iul = \triangleright(\tau@U) : \mathit{type})$$

Apply 27 to get

- $\Gamma, i : \mathbb{N} \vdash (i < l_1) : U_0$
  basic types (to get $l_1 : \mathbb{N}$, β reduce $\pi_2\langle w_1, l_1 \rangle$ using $\Gamma \vdash \langle w_1, l_1 \rangle : \mathit{world}$)
- $\Gamma, i : \mathbb{N} \vdash \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w_1 iul = \triangleright(\tau@U) : \mathit{type}) : U_0$
  Apply 66 to get

  $$\Gamma, i : \mathbb{N}, u : (\triangleright pw) \vdash \Pi(l : \triangleright\mathbb{N})(w_1 iul = \triangleright(\tau@U) : \mathit{type}) : U_0$$

  Apply 4 to get

  $$\Gamma, i : \mathbb{N}, u : \triangleright pw, l : \triangleright\mathbb{N} \vdash w_1 iul = \triangleright(\tau@U) : \mathit{type} : U_0$$

  Apply 107
  - $\Gamma, i : \mathbb{N}, u : \triangleright pw, l : \triangleright\mathbb{N} \vdash w_1 iul : U_0$
    preworld
  - $\Gamma, i : \mathbb{N}, u : \triangleright pw, l : \triangleright\mathbb{N} \vdash \triangleright(\tau@U) : \mathit{type} : U_0$
    34, induction (with weakening)

move

Given $\Gamma \vdash \langle w_1, l_1 \rangle, \langle w_2, l_2 \rangle : \mathit{world}$, $\Gamma \vdash m_0 : \langle w_1, l_1 \rangle \leq \langle w_2, l_2 \rangle$. For source type τ, showing

$$\Gamma \vdash \mathit{move}_\tau m_0 : \tau@W_1 \to \tau@W_2$$

.

0.11 move $\tau_1 \to \tau_2$

- $\Gamma \vdash move_{\tau_1 \to \tau_2} m_0 : (\tau_1 \to \tau_2)@W_1 \to (\tau_1 \to \tau_2)@W_2$ (5)
  - $\Gamma, f : \tau_1 \to \tau_2@W_1 \vdash \lambda(l : \mathbb{N}) \cdots : \tau_1 \to \tau_2@W_2$(67)
    - $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw \vdash \lambda(l : \mathbb{N}) \cdots : \Pi(l : \mathbb{N}) \ldots$ (5x3)
      - one
      - $\Gamma \ldots u : pw, l : \mathbb{N} \vdash W_2 \leq U : U_0$
        - subseq
      - $\Gamma \ldots u : pw, l : \mathbb{N} \vdash \tau_1@U : U_0$
        - types
    - $\Gamma \cdots \vdash pw : K_0$
      - preworld
  - $\Gamma \vdash \tau_1 \to \tau_2@W_1 : U_0$
    - types

(one)

Showing $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, x : \tau_1@U \vdash fl(m \circ m_0)x : \tau_2@U$
Repeated applications of (6,11, 108) yield the following goals

- $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, x : \tau_1@U \vdash x : \tau_1@U$
  Done
- $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U \vdash (m \circ m_0) : W_1 \leq U$
  Done by subseq
- $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, x : \tau_1@U \vdash l : \mathbb{N}$
  done
- $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, x : \tau_1@U \vdash f : \Pi(l : \mathbb{N}).W_1 \leq U \to \tau_1@U \to \tau_2@U$
  I denote $\Gamma, f : \tau_1 \to \tau_2@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, x : \tau_1@U$ as Γ'. Apply 68 to get
  - $\Gamma' \vdash f : \forall(x : pw)\Pi(l : \mathbb{N}).W_1 \leq \langle x, l\rangle \to \tau_1@\langle x, l\rangle \to \tau_2@\langle x, l\rangle$
    done by def $\tau_1 \to \tau_2@W_1$
  - $\Gamma' \vdash u : pw$
    done
  - $\Gamma', x : pw \vdash \Pi(l : \mathbb{N}).W_1 \leq \langle x, l\rangle \to \tau_1@\langle x, l\rangle \to \tau_2@\langle x, l\rangle : type$
    By types I have $\Gamma' \vdash \tau_1 \to \tau_2@W_1 : U_0$. Similar reasoning to the body of that proof gives that $\Gamma', x : pw \vdash \Pi(l : \mathbb{N}).W_1 \leq \langle x, l\rangle \to \tau_1@\langle x, l\rangle \to \tau_2@\langle x, l\rangle : U_0$

move $\bigcirc\tau$

Apply 5, then 67, then 5 multiple times to get, amongst goals already solved by types, preworld, basic types, subseq, and store

$$\Gamma, c : \bigcirc(\tau)@W_1, u : pw, l : \mathbb{N}, m : W_2 \leq U, s : storeU \vdash \quad (1)$$

$$c\,l\,(m \circ m_0)\,s : \triangleright \triangleright \exists(u_3 : pw)\Sigma(l_3 : \mathbb{N})(\langle u_2, l_2\rangle \leq \langle u_3, l_3\rangle \times store\langle u_3, l_3\rangle \times \tau@\langle u_3, l_3\rangle)$$

I denote context (1) as Γ'. Repeated applications of 6 and (108 with 11) give, amongst goals solved immediately by Γ' or transitivity of subseq,

$\Gamma' \vdash c : \Pi(l : \mathbb{N})(W_1 \leq U \to store(U) \to \triangleright\triangleright\exists(u_3 : pw)\Sigma(l_3 : \mathbb{N})(\langle u_2, l_2\rangle \leq \langle u_3, l_3\rangle \times store\langle u_3, l_3\rangle \times \tau@\langle u_3, l_3\rangle)$

Apply 68 to this to get, amongst goals solved by preworld and types,

$\Gamma' \vdash c : \forall(u : pw)\Pi(l : \mathbb{N})(W_1 \leq U \to store(U) \to \triangleright\triangleright\exists(u_3 : pw)\Sigma(l_3 : \mathbb{N})(\langle u_2, l_2\rangle \leq \langle u_3, l_3\rangle \times store\langle u_3, l_3\rangle \times \tau@\langle u_3, l_3\rangle)$

But, this is $c : \bigcirc(\tau) \in \Gamma'$.

move $ref\tau$

Apply 5 to get

$$\Gamma, R : ref(\tau)@W_1 \vdash \langle \pi_1 R, \langle (\pi_1\, m_0) \circ (\pi_1\pi_2 R), \lambda\_.*\rangle\rangle : ref(\tau)@W_2$$

Apply 20

- $\Gamma, R : ref(\tau)@W_1, i : \mathbb{N} \vdash (i < l_2 \times \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\; i\; u\; l = \triangleright(\tau@U) : type) : type$
  Similar reasoning to the body of $\Gamma, R : ref(\tau)@W_1 \vdash ref(\tau)@W_2 : U_0$
- $\Gamma, R : ref(\tau)@W_1 \vdash \pi_1 R : \mathbb{N}$
  Apply 21
- $\Gamma, R : ref(\tau)@W_1 \vdash$
  $\langle (\pi_1\, m_0) \circ (\pi_1\pi_2 R), \lambda\_.*\rangle : (\pi_1 R < l_2 \times \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type)$
  Apply 108 with 28 to get
  $\Gamma, R : ref(\tau)@W_1 \vdash$
  $\langle (\pi_1\, m_0) \circ (\pi_1\pi_2 R), \lambda\_.*\rangle : \Sigma(\pi_1 R < l_2).\forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type).$
  Apply 20 to get
  - $\Gamma, R : ref(\tau)@W_1 \vdash \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type) : type$
    Use 121 with $\Gamma, R : ref(\tau)@W_1 \vdash \pi_1 R : \mathbb{N}$ to get
    $$\Gamma, R : ref(\tau)@W_1, i : \mathbb{N} \vdash \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\; i\; u\; l = \triangleright(\tau@U) : type) : type$$
    .
    Then, invert on $\Gamma, R : ref(\tau)@W_1 \vdash ref(\tau) : U_0$ to get
    $$\Gamma, R : ref(\tau)@W_1 \vdash \_ \times \forall(u : \triangleright pw)\Pi(l : \triangleright\mathbb{N})(w\; i\; u\; l = \triangleright(\tau@U) : type) : U_0$$
  - $\Gamma, R : ref(\tau)@W_1 \vdash (\pi_1\, m_0) \circ (\pi_1\pi_2 R) : (\pi_1 R < l_2)$
    by transitivity of $<$, $\leq$, it suffices to show
    - $\Gamma, R : ref(\tau)@W_1 \vdash (\pi_1\pi_2 R) : \pi_1 R < l_1$
      Rule 21, rule (108 with 28),
    - $\Gamma, R : ref(\tau)@W_1 \vdash \pi_1\, m_0 : l_1 \leq l_2$
      rule (108 with 28), rule 21
  - $\Gamma, R : ref(\tau)@W_1 \vdash \lambda\_.* : \Pi(l : \triangleright\mathbb{N})(w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type)$
    Apply 5
    $\Gamma, R : ref(\tau)@W_1, l : \triangleright\mathbb{N} \vdash * : (w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type)$.
    By basic types 5) it suffices to show that
    $$\Gamma, R : ref(\tau)@W_1, l : \triangleright\mathbb{N} \vdash \pi_2\pi_2 R : (w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type)$$
    . Apply 22, rule (108 with 28) to get
    $$\Gamma, R : ref(\tau)@W_1, l : \triangleright\mathbb{N} \vdash \pi_2 R : ((\pi_1 R) < l_1) \times (w\;(\pi_1 R)\; u\; l = \triangleright(\tau@U) : type)$$
    Apply 22.

$move_\Gamma m$

For Γ a context in the source language and W_1, W_2 where $\Delta \vdash W_1, W_2 : world$, if $\Delta, \Gamma@W_1 \vdash m : W_1 \leq W_2$ and $\Delta, \Gamma@W_2 \vdash e : B$ and $\Gamma \cap FV(B) = \emptyset$, then $\Delta, \Gamma@W_1 \vdash move_\Gamma\ m\ e : B$.
Recall first that

$$move_\Gamma\ m\ e \equiv e[i := move_{\tau_i}\ m\ i]_{i:\tau_i \in \Gamma}$$

I can rewrite this as

$$move_\Gamma\ m\ e \equiv e[i := move_{\tau_i}\ m\ i]_{i:\tau_i \in \Gamma}[j := j]_{j \in \Delta}$$

To show

$$\Delta, \Gamma@W_1 \vdash e[i := move_{\tau_i}\ m\ i]_{i:\tau_i \in \Gamma}[j := j]_{j \in \Delta} : B$$

I rewrite as

$$\Delta, \Gamma@W_1 \vdash e[i := move_{\tau_i}\ m\ i]_{i:\tau_i \in \Gamma}[j := j]_{j \in \Delta} : B[i := move_{\tau_i}\ m\ i]_{i:\tau_i \in \Gamma}[j := j]_{j \in \Delta}$$

(allowed as $\Gamma \cap FV(B) = \emptyset$). I apply rule 121 $|\Delta, \Gamma|$ times to get

- $\Delta, \Gamma@W_1 \cdots \vdash move_{\tau_i} m\ i : \tau_i@W_2$ for $i : \tau_i \in \Gamma$ (where $\ldots$ denotes some extension to $\Delta, \Gamma@W_1$)
  Apply 6 to get $\Delta, \Gamma@W_1 \vdash i : \tau_i@W_1$ (def $\Gamma@W_1$) and $\Delta, \Gamma@W_1 \vdash move_{\tau_i} m : \Pi(\_ : \tau_i@W_1).\tau_i@W_2$.
  This follows from move, (108 with 11), and the fact that $\Delta, \Gamma@W_1 \vdash m : W_1 \leq W_2$.
- $\Delta, \Gamma@W_1 \cdots \vdash j : \tau_j$ for $j : \tau_j \in \Delta$
  Def Δ
- $\Delta, \Gamma@W_2 \vdash e : B$
  given

translation

When I write $\lambda(W_1 : world)$, it is an abuse. I really mean $\lambda(l_1 : \mathbb{N})$ (the function really only takes the nat part of the world). I include the preworld part just to keep track. The same goes for $f\ W_1$. I really mean $f\ l_1$ but pass in the W_1 to keep track.
Showing that for any $\Gamma \vdash e : \tau$ in the source language and $\Delta \vdash \langle w_1, l_1 \rangle : world$ in the target language, $\Delta, \Gamma@W_1 \vdash \bar{e}\ W_1 : \tau@W_1$. I proceed by induction on $\Gamma \vdash e : \tau$.

0.12 ap

Have: $\Gamma \vdash e_1\ e_2 : \tau_2$.
Showing: $\Delta, \Gamma@W_1 \vdash \big(\lambda W_1.\overline{e_1} W_1\ l\ refl_{W_1}\ (\bar{(}e_2)W_1)\big) W_1 : \tau_2@W_1$.
β reduce to get $\Delta, \Gamma@W_1 \vdash \overline{e_1} W_1\ l_1\ refl_{W_1}\ (\bar{(}e_2)W_1) : \tau_2@W_1$. Apply 6 to get

- $\Delta, \Gamma@W_1 \vdash \overline{e_1} W_1\ l_1\ refl_{W_1} : \Pi(e_2 : \tau_1@W_1)(\tau_2@W_1)$
  108 with 11 gives

  $$\Delta, \Gamma \vdash \overline{e_1} W_1\ l_1\ refl_{W_1} : \tau_1@W_1 \to \tau_2@W_1$$

  6 gives

  - $\Delta, \Gamma@W_1 \vdash \overline{e_1} W_1\ l_1 : \Pi(W_1 \leq W_1).\tau_1@W_1 \to \tau_2@W_1$ 108 with 11 gives
    $\Delta, \Gamma@W_1 \vdash \overline{e_1} W_1\ l_1 : W_1 \leq W_1 \to \tau_1@W_1 \to \tau_2@W_1$
    6 gives $\Delta, \Gamma@W_1 \vdash l_1 : \mathbb{N}$ (which follows from the fact that $\Delta, \Gamma@W_1 \vdash W_1 : world$) and

    $$\Delta, \Gamma@W_1 \vdash \overline{e_1} W_1 : \Pi(l : \mathbb{N}) W_1 \leq \langle w_1, l \rangle \to \tau_1@W_1 \to \tau_2@W_1$$

    Apply 68 to get

* $\Delta, \Gamma@W_1 \vdash \overline{e_1}W_1 : \forall(u : preworld)\Pi(l : \mathbb{N})W_1 \leq U \rightarrow \tau_1@U \rightarrow \tau_2@U$
  induction on $\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2$.
* $\Delta, \Gamma@W_1 \vdash w_1 : preworld$
  follows from the fact that $\Delta, \Gamma \vdash W_1 : world$
* $\Delta, \Gamma@W_1, u : pw \vdash \Pi(l : \mathbb{N})W_1 \leq U \rightarrow \tau_1@U \rightarrow \tau_2@U : type$
  Shown in proof that $(\tau_1 \rightarrow \tau_2@W_1) : type$

– $\Delta, \Gamma \vdash refl_{W_1} : W_1 \leq W_1$
  subseq, as $\Delta, \Gamma \vdash W_1 : world$

- $\Delta, \Gamma \vdash (\overline{e_2}W_1) : \tau_1@W_1$
  IH on $\Gamma \vdash e_2 : \tau_1$

0.13 lam

Have: $\Gamma \vdash \lambda x.e : \tau_1 \rightarrow \tau_2$.
Showing: $\Delta, \Gamma@W_1 \vdash \big(\lambda W_1.\lambda l.\lambda m.\lambda x.(move_\Gamma\ m\ (\overline{e}\ U))\big)W_1 : (\tau_1 \rightarrow \tau_2)@W_1$.
β- reduce to get $\Delta, \Gamma@W_1 \vdash \lambda l.\lambda m.\lambda x.(move_\Gamma\ m\ (\overline{e}\ U)) : (\tau_1 \rightarrow \tau_2)@W_1$.
Apply 67 to get

$$\Delta, \Gamma@W_1, u : pw \vdash \lambda l.\lambda m.\lambda x.(move_\Gamma\ m\ (\overline{e}\ U)) : \Pi(l : \mathbb{N}).W_1 \leq U \rightarrow \tau_1@U \rightarrow \tau_2@U$$

Apply 5 and (108 with 11) repeatedly to get

- $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N} \vdash W_1 \leq U : type$
  subseq, as $\Delta, u : pw, l : \mathbb{N} \cdots \vdash W_1, U : world$

- $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U \vdash \tau_1@U : type$
  types

- $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, x : \tau_1@U \vdash (move_\Gamma\ m\ (\overline{e}\ U)) : \tau_2@U$
  After applying result $move_\Gamma m$ with $\Delta := \Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U$ and $\Gamma := \Gamma, x : \tau_1$ I need show that

  – $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U \vdash W_1, U : world$ Follows pretty fast from assumption that $\Delta \vdash W_1 : world$.

  – $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, \Gamma@W_1, x : \tau_1@W_1 \vdash m : W_1 \leq U$
    Easy

  – $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, \Gamma@U, x : \tau_1@U \vdash \overline{e}\ U : \tau_2@U$
    I have $\Gamma, x : \tau_1 \vdash e : \tau_2$. My IH on this with $\Delta := \Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U$ gives exactly

    $$\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, \Gamma@U, x : \tau_1@U \vdash \overline{e}\ U : \tau_2@U$$

  – $\Gamma \cap FV(\tau_2@U) = \emptyset$
    By the above bullet and rule 129, I have

    $$\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, \Gamma@U, x : \tau_1@U \vdash \tau_2@U : typ$$

    I have by structural rules that for any $i \in \Gamma@U$, $i \notin FV(\tau_2@U)$.

0.14 ret

Have: $\Gamma \vdash return(e) : \bigcirc\tau$.
Showing:

$$\Delta, \Gamma@W_1 \vdash \big(\lambda W_1.\lambda l.\lambda m.\lambda s.return_{A\ target}\langle l, \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}\ W_1)\rangle\rangle\big)W_1 : (\bigcirc\tau)@W_1$$

where

$$A := \exists(v : pw)\Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times \tau@\langle v, l_v\rangle$$

β-reduce, apply 67

$\Delta, \Gamma@W_1, u : pw \vdash \lambda l.\lambda m.\lambda s.return_{A\ target}\langle l, \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}W_1)\rangle\rangle : \Pi(l : \mathbb{N})W_1 \leq U \to store(U) \to \dots$

Apply 5 and (108 with 11) repeatedly

- $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N} \vdash W_1 \leq U : type$
  subseq, as $\Delta, u : pw, l : \mathbb{N} \cdots \vdash W_1, U : world$
- $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U \vdash store(U) : type$
  store
- $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : store(U) \vdash$$
  $return_{A\ target}\langle l, \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}W_1)\rangle\rangle : \triangleright\triangleright\exists(v : pw)\Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times \tau@\langle v, l_v\rangle$

  I denote $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : store(U)$ as Γ'.
  Apply 6 to get

  - $\Gamma' \vdash return_{A\ target} : \Pi(\_ : A).\triangleright\triangleright A$
    apply return rule from bar types. suffices to show that $\Gamma' \vdash A : U_0$. as A is a (structural) subterm of $\bigcirc\tau@W_1$, this follows similar reasoning to the proof that $\Delta, \Gamma \vdash \tau@W_1 : U_0$.
  - $\Gamma' \vdash \langle l, \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}\ W_1)\rangle\rangle : A$
    Apply 71 to get $\Gamma' \vdash u : pw$ (easy by def Γ') and

    * $$\Gamma' \vdash \langle l, \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}\ W_1)\rangle\rangle : \Sigma(l_v : \mathbb{N})U \leq \langle u, l_v\rangle \times store\langle u, l_v\rangle \times \tau@\langle u, l_v\rangle$$

      Apply 20 to get $\Gamma' \vdash l : \mathbb{N}$ (easy by def Γ') and
      $\Gamma', l_v : \mathbb{N} \vdash U \leq \langle u, l_v\rangle \times store\langle u, l_v\rangle \times \tau@\langle u, l_v\rangle : type$ (which follows the (subterm of $\bigcirc\tau@W_1$) argument) and

      $$\Gamma' \vdash \langle refl_{u,l}, s, move_\tau\ m\ (\bar{e}\ W_1)\rangle : U \leq U \times store(U) \times \tau@U$$

      Applying 20 multiple times gives

      · $\Gamma' \vdash refl_{u,l} : U \leq U$
        subseq
      · $\Gamma' \vdash s : store(U)$
        def Γ'
      · $\Gamma' \vdash move_\tau\ m\ (\bar{e}\ W_1) : \tau@U$
        Applying 6 gives
      · $\Gamma' \vdash move_\tau\ m : \Pi(\_ : \tau@W_1).\tau@U$
        result move gives that it suffices to show that $\Gamma' \vdash W_1, U : world$ and $\Gamma' \vdash m : W_1 \leq U$. The former follows by weakening and the latter by definition of Γ'.

· $\Gamma' \vdash \overline{e}\, W_1 : \tau @ W_1$
Induction on the typing $\Gamma \vdash e : \tau$ gives $\Delta, \Gamma @ W_1 \vdash \overline{e}\, W_1 : \tau @ W_1$. Weakening then gives $\Gamma' \vdash \overline{e}\, W_1 : \tau @ W_1$

*

$$\Gamma', v : pw \vdash \Sigma(l_v : \mathbb{N}) U \leq \langle v, l_v \rangle \times store\langle v, l_v \rangle \times \tau @ \langle v, l_v \rangle$$

As $\Sigma(l_v : \mathbb{N}) U \leq \langle v, l_v \rangle \times store\langle v, l_v \rangle \times \tau @ \langle v, l_v \rangle$ is a subterm of $\bigcirc\tau @ W_1$, this follows similar reasoning to the proof that $\Gamma' \vdash \tau @ W_1 : U_0$.

0.15 bind

Have: $\Gamma \vdash bind(e_1,\ x.e_2) : \bigcirc\tau_2$.
Let

$$A := \exists(v : pw)\Sigma(l_v : \mathbb{N})(U \leq \langle v, l_v \rangle \times store\langle v, l_v \rangle \times \tau_1 @ \langle v, l_v \rangle)$$
$$B := \exists(v' : pw)\Sigma(l'_v : \mathbb{N})(U \leq \langle v', l'_v \rangle \times store\langle v', l'_v \rangle \times \tau_2 @ \langle v', l'_v \rangle)$$
$$C := \exists(v' : pw)\Sigma(l'_v : \mathbb{N})(\langle v, \pi_1 z \rangle \leq \langle v', l'_v \rangle \times store\langle v', l'_v \rangle \times \tau_2 @ \langle v', l'_v \rangle)$$

$$\overline{e_2}' := \lambda x.\big(move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\langle v, \pi_1 z \rangle)\big)(\pi_4\ z)\ \pi_1 z\ refl_{v,\, \pi_1\, z}\ \pi_3 z$$

I am showing

$$\Delta, \Gamma @ W_1 \vdash \lambda l.\lambda m.\lambda s.bind_{target}\ (\overline{e_1}\ W_1\ l\ m\ s)\ \lambda(z_1 : A).\Big(bind_{target}\ \overline{e_2}'\ (\dots)\Big)[z := z_1] : (\bigcirc\tau_2) @ W_1$$

Apply 67, then (5 and (108 with 11)) repeatedly to get

- $\Delta, \Gamma @ W_1, u : pw, l : \mathbb{N} \vdash W_1 \leq U : type$
  subseq, similar to ret proof
- $\Delta, \Gamma @ W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U \vdash store(U) : type$
  store
- I denote $\Delta, \Gamma @ W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : store(U)$ as Γ'. I need show

  $$\Gamma' \vdash bind_{target}\ (\overline{e_1}\ W_1\ l\ m\ s)\ \lambda(z_1 : \exists v \dots).\Big(bind_{target}\ \overline{e_2}'\ (\dots)\Big)[z := z_1] : \rhd \rhd B$$

  Apply the bind rule to get

  - $\Gamma' \vdash \overline{e_1}\ W_1\ l\ m\ s : \rhd \rhd A$
    Apply 6 and (108 with 11) multiple times to get

    $$\Gamma' \vdash \overline{e_1}\ W_1 : \Pi(l : \mathbb{N}).W_1 \leq U \to store(U) \to \rhd \rhd A$$

    Apply 68 to get (amongst goals solved before)

    $$\Gamma' \vdash \overline{e_1}\ W_1 : \forall(u : pw)\Pi(l : \mathbb{N}).W_1 \leq U \to store(U) \to \rhd \rhd A$$

    I have by induction on $\Gamma \vdash e_1 : \bigcirc\tau_1$ that $\Delta, \Gamma \vdash \overline{e_1}\ W_1 : (\bigcirc\tau_1) @ W_1$. The desired result follows by weakening.

  -

    $$\Gamma' \vdash \lambda z_1.\Big(bind_{target}\ \overline{e_2}'\ \big(\lambda z_2.\big(return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z' \rangle\rangle\big)[z' := z_2]\big)\Big)[z := z_1] : A \to \rhd\rhd B$$

    Apply 5 to get (amongst goals solved before)

    $$\Gamma', z_1 : A \vdash \Big(bind_{target}\ \overline{e_2}'\ \big(\lambda z_2.\big(return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z' \rangle\rangle\big)[z' := z_2]\big)\Big)[z := z_1] : \rhd\rhd B$$

    Apply 72 to get

* $\Gamma', z_1 : A \vdash z_1 : A$
  done
* $\Gamma' \vdash pw : type$
  done
* $\Gamma', v : pw \vdash \Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times \tau_1@\langle v, l_v\rangle : type$
  Since the type above is a subterm of $\bigcirc\tau_1@U$ and since $\Gamma' \vdash \tau_1@U : U_0$ by types, I can use similar reasoning to that proof. Below, I denote $\Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times \tau_1@\langle v, l_v\rangle$ as A'.

as well as

$$\Gamma', v : pw, z : A' \vdash bind_{target}\ \overline{e_2}'\ \big(\lambda z_2.\big(return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z'\rangle\rangle\big)[z' := z_2]\big) : \triangleright\triangleright B$$

Applying the bind rule gives

* $\Gamma', v : pw, z : A' \vdash \overline{e_2}' : \overline{C}$
  Recalling the definition of $\overline{e_2}'$, I am showing that

  $$\Gamma', v : pw, z : A' \vdash \lambda x.\big(move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\langle v, \pi_1 z\rangle)\big)(\pi_4 z)\ \pi_1 z\ refl_{v,\, \pi_1\, z}\ \pi_3 z : \overline{C}$$

  Applying 6 and (108 with 11) to this multiple times gives the following goals
  - $\Gamma', v : pw, z : A' \vdash \pi_3 z : store(\langle v, \pi_1 z\rangle)$
    def A', some wiggling involving the "third projection"
  - $\Gamma', v : pw, z : A' \vdash refl_{v,\, \pi_1\, z} : \langle v, \pi_1\ z\rangle \leq \langle v, \pi_1\ z\rangle$ by the reflexivity result in subseq it suffices to show that $\Gamma', v : pw, z : A' \vdash \pi_1 z : \mathbb{N}$. But, this follows from rule 21.
  - $\Gamma', v : pw, z : A' \vdash \pi_1\ z : \mathbb{N}$
    see above bullet

  as well as, with V_{mid} abbreviating $\langle v, l_{mid}\rangle$ and V abbreviating $\langle v, \pi_1\ z\rangle$

  $$\Gamma', v : pw, z : A' \vdash \lambda x.\big(move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\langle v, \pi_1 z\rangle)\big)(\pi_4\ z) :$$

  $$\Pi(l_{mid} : \mathbb{N})V \leq V_{mid} \to store(V_{mid}) \to \triangleright\triangleright\exists(v' : pw)\Sigma(l'_v : \mathbb{N})(V_{mid} \leq \langle v', l'_v\rangle \times store\langle v', l'_v\rangle \times \tau_2@\langle v, l'_v\rangle)$$

  Apply rule 68 to get (amongst goals solved before)

  $$\Gamma', v : pw, z : A' \vdash \lambda x.\big(move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\langle v, \pi_1 z\rangle)\big)(\pi_4\ z) : \bigcirc\tau_2@V$$

  Apply rule 6 to get $\Gamma', v : pw, z : A' \vdash \pi_4\ z : \tau_1@V$ (easy by def A') and

  $$\Gamma', v : pw, z : A' \vdash \lambda x.move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\ V) : \Pi(\_ : \tau_1@V \to).\ \bigcirc\ \tau_2@V$$

  Apply rule 5 to get $\Gamma' \cdots \vdash \tau_1@V : type$ (done in types) and

  $$\Gamma', v : pw, z : A', x : \tau_1@V \vdash move_\Gamma\ (\pi_2 z \circ m)\ (\overline{e_2}\ V) : \bigcirc\tau_2@V$$

  By $move_\Gamma$ it suffices to show
  - $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, v : pw, z : A' \vdash W_1, V : world$
    easy
  - $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, v : pw, z : A' \vdash \pi_2\ z \circ m : W_1 \leq V$
    transitivity of subseq
  - $\Delta, u : pw, l : \mathbb{N}, m : W_1 \leq U, v : pw, z : A', \Gamma@V, x : \tau_1@V \vdash \overline{e_2}\ V \bigcirc \tau_2@V$
    I have that $\Gamma, x : \tau_1 \vdash e_2 : \bigcirc\tau_2$. IH on this gives exactly the result desired.

* $\Gamma', v : pw, z : A' \vdash \lambda z_2.\big(return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z'\rangle\rangle\big)[z' := z_2] : C \to \overline{B}$
  Below, I denote $\Sigma(l'_v : \mathbb{N})(\langle v, \pi_1 z\rangle \leq \langle v', l'_v\rangle \times store\langle v', l'_v\rangle \times \tau_2@\langle v', l'_v\rangle)$ as C'.
  Apply 5 to get

$$\Gamma', v : pw, z : A', z_2 : C \vdash \big(return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z'\rangle\rangle\big)[z' := z_2]\overline{B}$$

  Apply 72 to get (amongst goals solved before)

$$\Gamma', v : pw, z : A', v' : pw, z' : C' \vdash return_{target}\langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z'\rangle\rangle : \overline{B}$$

  Apply the return rule to get (as well as a goal solved before)

$$\Gamma', v : pw, z : A', v' : pw, z' : C' \vdash \langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z, \pi_3 z', \pi_4 z'\rangle\rangle : B$$

  Apply 71 with v' := v' to get

$$\Gamma', v : pw, z : A', v' : pw, z' : C' \vdash \langle \pi_1 z', \langle \pi_2 z' \circ \pi_2 z,\ \pi_3 z', \pi_4 z'\rangle\rangle :$$
$$\Sigma(l'_v : \mathbb{N})(U \leq \langle v', l'_v\rangle \times store\langle v', l'_v\rangle \times \tau_2@\langle v', l'_v\rangle)$$

  Multiple applications of rule 20 (with rule 108 and 28) gives
  - $\Gamma', v : pw, z : A', v' : pw, z' : C' \vdash \pi_1 z' : \mathbb{N}$
  - $\pi_2 z' \circ \pi_2 z : U \leq \langle v', l'_v\rangle$
    Follows from def C', A' (some wiggling with projections) that $\pi_2 z' : \langle v, \pi_1 z\rangle \leq \langle v', l'_v\rangle$ and $\pi_2 z : U \leq \langle v, \pi_1 z\rangle$
  - $\pi_3 z' : store\langle v', l'_v\rangle$
    def C'
  - $\pi_4 z' : \tau_2@\langle v', l'_v\rangle$ def C'

0.16 ref

Have: $\Gamma \vdash ref(e) : \bigcirc(ref\tau)$.
Let m_1 denote $\langle *, \lambda l_v.\lambda i.\lambda m_i.*\rangle$.
Let p_1 denote

$$\langle\langle *, \lambda l_v.\lambda i.\lambda m_i.*\rangle, \lambda l_2.\lambda m_2.\lambda i.ite\ (i <_b l)\ ((s\ l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\bar{e}W_1)))\rangle$$

Let u_1 be as written in the jamboard.
Let x denote

$$\text{let next } v' = v \text{ in let next } l'_v = l_v \text{ in } \rhd(\tau@\langle v', l'_v\rangle)$$

Showing:

$$\Delta, \Gamma@W_1 \vdash \lambda(l : \mathbb{N}).\lambda(m : W_1 \leq U).\lambda(s : store(U)).return_{target}\langle succ(l), \langle p_1, \langle l, *, \lambda\_.*\rangle\rangle\rangle : \bigcirc(ref\tau)@W_1$$

Apply 67, then (5 and (108 with 11)) repeatedly to get (amongst goals solved before)

$$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : store(U) \vdash return_{target}\langle succ(l), \langle p_1, \langle l, *, \lambda\_.*\rangle\rangle\rangle :$$
$$\rhd \rhd \exists(v : pw)\Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times (ref\tau)@\langle v, l_v\rangle$$

Apply the return rule

$$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : store(U) \vdash \langle succ(l), \langle p_1, \langle l, *, \lambda\_.*\rangle\rangle\rangle :$$
$$\exists(v : pw)\Sigma(l_v : \mathbb{N})U \leq \langle v, l_v\rangle \times store\langle v, l_v\rangle \times (ref\tau)@\langle v, l_v\rangle$$

Apply 71 to get

- $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash u_1 : pw$$

  By cons_b suffices to show $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash u : pw, l : \mathbb{N}$ (easy) and

  $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash \lambda v.\lambda l_v.x :$$

  $$\triangleright pw \to_k \triangleright \mathbb{N} \to U_0$$

  Apply (108 with 16), then (108 with 11), then 5, then (108 with 11), then 5 again to get

  $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U), v : \triangleright pw, l_v : \triangleright \mathbb{N} \vdash x : U_0$$

  By 122

  $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U), v : \triangleright pw, l_v : \triangleright \mathbb{N} \vdash x : \text{let next v' = v in } U_0$$

  Then by 36

  $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U), v : \triangleright pw, l_v : \triangleright \mathbb{N}, v' : \underline{pw} \vdash \text{let next } l'_v = l_v \text{ in } \triangleright(\tau@\langle v', l'_v\rangle) : U_0$$

  Similar reasoning gives

  $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U), v : \triangleright pw, l_v : \triangleright \mathbb{N}, v' : \underline{pw}, l'_v : \underline{\mathbb{N}} \vdash \triangleright(\tau@\langle v', l'_v\rangle) : U_0$$

  Apply 34

  $$\overline{\Delta}, \overline{\Gamma@W_1}, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U), v : \triangleright pw, l_v : \triangleright \mathbb{N}, v' : pw, l'_v : \mathbb{N} \vdash \tau@\langle v', l'_v\rangle : U_0$$

  Follows from types

- $$\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash \langle \mathit{succ}(l), \langle p_1, \langle l, *, \lambda\_.*\rangle\rangle\rangle :$$

  $$\Sigma(l_v : \mathbb{N})U \leq \langle u_1, l_v\rangle \times \mathit{store}\langle u_1, l_v\rangle \times (\mathit{ref}\,\tau)@\langle u_1, l_v\rangle$$

  Let $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U)$ be denoted Γ'.
  I abbreviate $\langle u_1, \mathit{succ}(l)\rangle$ as U_1 and $\langle u_2, l_2\rangle$ as U_2
  Multiple applications of 20 yield $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash \mathit{succ}(l) : \mathbb{N}$, as well as

  - $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash p_1 : U \leq \langle u_1, \mathit{succ}(l)\rangle \times \mathit{store}\langle u_1, \mathit{succ}(l)\rangle$

    Recall the definition of p_1, apply 20.

    1st projection

    Showing

    $$\Gamma' \vdash m_1 : U \leq U_1$$

    Recalling that $u_1 := \langle \mathit{cons}_b\ u\ l\ \lambda v.\lambda l_v.x, \mathit{succ}(l)\rangle$, it follows from result cons_b in subseq that I need only show $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash U : \mathit{world}$ (easy) and $\Delta, \Gamma@W_1, u : pw, l : \mathbb{N}, m : W_1 \leq U, s : \mathit{store}(U) \vdash \lambda v.\lambda l_v.x : \triangleright pw \to_k \triangleright N \to U_0$ (done in round bullet above).

2nd projection

Showing

$$\Gamma' \vdash \lambda l_2.\lambda m_2.\lambda i.ite\ (i <_b l)\ ((s\, l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\overline{e}W_1))) : store(U_1)$$

Apply 67 to get

$$\Gamma', u_2 : pw \vdash \lambda l_2.\lambda m_2.\lambda i.ite\ (i <_b l)\ ((s\, l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\overline{e}W_1))) :$$

$$\Pi(l_2 : \mathbb{N})U_1 \leq U_2 \to *u_1/U_2$$

Applying 5 x3 (along with 108 and 11) gives

$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash$$

$$ite\ (i <_b l)\ ((s\, l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1))) : u_1\ i\ (next\, u_2)(next\, l_2)$$

Apply 121 with $M := i <_b l$ to get

$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N}, b : bool \vdash$$

$$ite\ b\ ((s\, l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1))) :$$

$$\Big(\lambda i.ite\ b\ (u\ i)(\lambda v.\lambda l_v.x)i\Big)\ i\ (next\, u_2)(next\, l_2)$$

β reduce to get

$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N}, b : bool \vdash$$

$$ite\ b\ ((s\, l_2)(m_2 \circ m_1)\ i)\ (next(move_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1))) :$$

$$\Big(ite\ b\ (u\ i)(\lambda v.\lambda l_v.x)i\Big)\ (next\, u_2)(next\, l_2)$$

Apply 57 to get

- $$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash$$

  $$(s\, l_2)(m_2 \circ m_1)\ i : \Big(ite\ true\ (u\ i)(\lambda v.\lambda l_v.x)i\Big)\ (next\, u_2)(next\, l_2)$$

  which β reduces to

  $$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash (s\, l_2)(m_2 \circ m_1)\ i : u\ i\ (next\, u_2)(next\, l_2)$$

  Apply 6 twice to get (amgonst goals easily solved)

  - $$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash (m_2 \circ m_1) : U \leq U_2$$

    In '1st projection', I showed that $\Gamma' \vdash m_1 : U \leq U_1$. This and transitivity of subseq gives the above.

  - $$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash (s\, l_2) : \Pi(U \leq U_2).\Pi(i : \mathbb{N})\big(u\, i\, (next\, u_2)(next\, l_2)\big)$$

    Apply 108 with 11, then apply 6 to get $\Gamma' \cdots \vdash l_2 : \mathbb{N}$ (easy) as well as

    $$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \leq U_2, i : \mathbb{N} \vdash s : \Pi(l_2 : \mathbb{N}).U \leq U_2 \to \Pi(i : \mathbb{N})\big(ui(nextu_2)(nextl_2)\big)$$

Apply 68 to get

$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N} \vdash s : \forall(u_2 : pw)\Pi(l_2 : \mathbb{N}).U \le U_2 \to \Pi(i : \mathbb{N})\big(ui(nextu_2)(nextl_2)\big)$

In other terms

$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N} \vdash s : \mathit{store}(U)$$

This follows immediately from the def of Γ'.

*
$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash \mathit{next}(\mathit{move}_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1)) : \Big(ite\, false\ (u\, i)(\lambda v.\lambda l_v.x)i\Big)\ (\mathit{next}\, u_2)(\mathit{next}\, l_2)$$

which β reduces to

$$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash \mathit{next}(\mathit{move}_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1)) : \text{let next } v' = (\mathit{next}\, u_2) \text{ in let next } l'_v = (\mathit{next}\, l_2) \text{ in } \rhd(\tau@\langle v', l'_v\rangle)$$

This reduces further to

$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash \mathit{next}(\mathit{move}_\tau(m_2 \circ m_1 \circ m)(\overline{e}W_1)) : \rhd(\tau@\langle u_2, l_2\rangle)$

Apply 35 (and the weaking rule for promotion)

$\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash (\mathit{move}_\tau(m_2 \circ m_1 \circ m)(\overline{e}\, W_1)) : \tau@\langle u_2, l_2\rangle$

By *move*, 6, and (108 with 11) it suffices to show that

- $\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash m_2 \circ m_1 \circ m : W_1 \le U_2$
  By def Γ' I have that $m : W_1 \le U$. I showed in 1st projection that $\Gamma' \vdash m_1 : U \le U_1$. The result then follows from transitivity of $\le$.
- $\Gamma', u_2 : pw, l_2 : \mathbb{N}, m_2 : U_1 \le U_2, i : \mathbb{N}, b : \mathit{bool} \vdash \overline{e}\, W_1 : \tau@W_1$
  I have that $\Gamma \vdash e : \tau$. My IH then gives that $\Delta, \Gamma \vdash \overline{e}\, W_1 : \tau@W_1$. The desired result comes from weakening.

– $\Gamma' \vdash \langle l, *, \lambda\_.*\rangle\rangle : (ref\tau)@\langle u_1, succ(l)\rangle$
Apply 20 twice (along with 108 and 28) to generate

* $\Gamma' \vdash l : \mathbb{N}$
  by def Γ'
* $\Gamma' \vdash * : l < \mathit{succ}(l)$
  basic types
* $\Gamma' \vdash \lambda\_.* : \forall(v : pw)\Pi(l_v : \mathbb{N})((u_1\, l\, (\mathit{next}\, v)\, (\mathit{next}\, l_v)) = \rhd(\tau@\langle v, l_v\rangle) : \mathit{type})$
  Apply 67 and 5 to get

$$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : (u_1\, l\, (\mathit{next}\, v)\, (\mathit{next}\, l_v)) = \rhd(\tau@\langle v, l_v\rangle) : \mathit{type}$$

Recalling the definition of u_1, this is equivalent to

$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : \big((\lambda i.ite\,(i <_b l)\,(u\, i)\, \lambda v.\lambda l_v.x)\, l\, (\mathit{next}\, v)\, (\mathit{next}\, l_v)\big) = \rhd(\tau@\langle v, l_v\rangle) : \mathit{type}$

β reduction gives

$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : \big(ite\ (l <_b l)\ (u\, l)\ \lambda v.\lambda l_v.x\big)\ (\mathit{next}\, v)\ (\mathit{next}\, l_v) = \rhd(\tau@\langle v, l_v\rangle) : \mathit{type}\ \ (*)$

It suffices to show that

$$(**)\ \ \Gamma', v : \rhd pw, l_v : \rhd\mathbb{N} \vdash \lambda\_.* : \Pi(m' : l <_b l = false : \mathit{bool})\Big(\big(ite(l <_b l)(ul)\lambda v.\lambda l_v.x\big)(nextv)(nextl_v) = \rhd(\tau@\langle v, l_v\rangle) : \mathit{type}\Big)$$

·

· I assume $(**)$ and show $(*)$
By basic types I have that $\Gamma' \cdots \vdash * : (l <_b l = false : bool)$. So, by $(**)$ and 6, I have that

$$\Gamma' \cdots \vdash \lambda\_. * * : \big(ite\,(l <_b l)\,(u\,l)\,\lambda v.\lambda l_v.x\big)\,(next\,v)\,(next\,l_v) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

β reduction gives exactly (*).

· I show $(**)$
I apply 121 with $M := l <_b l$ to generate

$$\Gamma' \ldots b : bool \vdash \lambda\_.* : \Pi(m' : b = false : bool)\Big(\big(iteb(ul)\lambda v.\lambda l_v.x\big)(nextv)(nextl_v) = \triangleright(\tau@\langle v, l_v\rangle) : type\Big)$$

Apply 5 for

$$\Gamma' \ldots b : bool, m' : b = false : bool \vdash * : \big(iteb(ul)\lambda v.\lambda l_v.x\big)(nextv)(nextl_v) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

Apply 120 with $M := false$ to get, amongst goals solved before

$$\Gamma' \ldots m' : false = false : bool \vdash$$

$$* : \big(ite\,false\,(u\,l)\,\lambda v.\lambda l_v.x\big)\,(next\,v)\,(next\,l_v) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

After β reduction, this is

$$\Gamma' \vdash * : \big(\lambda v.\lambda l_v.x\big)\,(next\,v)\,(next\,l_v) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

Definition of x and more β reduction gives

$$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : \text{let next } v' = (nextv) \text{ in let next } l'_v = (nextl_v) \text{ in } \triangleright(\tau@\langle v', l'_v\rangle) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

β reduction again gives

$$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : \triangleright(\tau@\langle v, l_v\rangle) = \triangleright(\tau@\langle v, l_v\rangle) : type$$

Apply 33 and use the promotion-weakening lemma to get

$$\Gamma', v : pw, l_v : \mathbb{N} \vdash * : \tau@\langle v, l_v\rangle = \tau@\langle v, l_v\rangle : type$$

This follows from types and the fact that $\Gamma', v : pw, l_v : \mathbb{N} \vdash \langle v, l_v\rangle : world$.

bar types

bind rule

$\Gamma \vdash M_0 : \overline{A}$ and $\Gamma \vdash M_1 : A \to \overline{B}$ means that $\Gamma \vdash bind_{target\ A\ B} M_0\, M_1 : \overline{B}$.

0.17 return rule

If $\Gamma \vdash A : U_0$ then $\Gamma \vdash return_{A\ target} : \Pi(\_ : A). \triangleright \triangleright A$.

0.18 bar rule

If $\Gamma \vdash A : U_i$ then $\Gamma \vdash \triangleright \triangleright A : U_i$

basic types

1. $\mathbb{N} : U_0$
2. $l_i : \mathbb{N}$ gives $l_1 \leq l_2 : U_0$
3. Transitivity of $\leq$
   $\Gamma \vdash q_1 : l_1 \leq l_2$ and $\Gamma \vdash q_2 : l_2 \leq l_3$ gives $\Gamma \vdash q : l_1 \leq l_3$
4. $\Gamma \vdash A = B$ gives that $\Gamma \vdash A \leq B$
5. If $\Gamma \vdash x : (A = B) : type$, then $\Gamma \vdash A = B : type$
   I'm showing $\Gamma \vdash * = * : (A = B : type)$. I have

$$\Gamma \vdash (x = *) :(A = B) : type \qquad 111 \qquad (1)$$
$$\Gamma \vdash (* = x) :(A = B) : type \qquad 124 \qquad (2)$$
$$\Gamma \vdash (* = *) :(A = B) : type \qquad 125 \qquad (3)$$

6. refl of $\leq$
7. trans of $<$ to $\leq$ (in that order) $\Gamma \vdash i < j$, $\Gamma \vdash j \leq k$ gives $\Gamma \vdash i < k$
8. Boolean lt for nats
9. reflection of $=_{nat}$ with $==_{nat}$
   $\Gamma \vdash i, j : \mathbb{N}$ means $\Gamma \vdash (i = j : \mathbb{N}) = ((i == j) = true : bool) : type$
10. reflection of $<$ with $<_b$
    $\Gamma \vdash i, j : \mathbb{N}$ means $\Gamma \vdash (i < j) = ((i == j) = true : bool) : type$
11. $\Gamma \vdash l : \mathbb{N}$ gives $\Gamma \vdash * : l \leq succ(l)$
12. $A = B : type$ means $A \subseteq B$ and $B \subseteq A$
13. $l < succ(l)$
14. $l < l = false : bool$
15. $\Gamma \vdash m : (M = true : bool)$ and $\Gamma \vdash P : type$ means that $\Gamma \vdash ite(M)\ P\ Q = P : type$.

structural rules

1. if $\Gamma \vdash \tau_1 @ W_1 : typ$ and $x : \tau_x @ W_x \in \Gamma$, then $x \notin FV(\tau_1 @ W_1)$
   induction on τ_1. need some concept of what it means to be a free variable.