Chong, Edwin Kah Pin., and Stanislaw H. Żak. *An Introduction to Optimization*. Fourth edition., John Wiley & Sons, Inc., 2013.

1 BASICS OF SET CONTSTRAINED AND UNCONSTRAINED OPTIMIZATION

6.1 Consider the problem

$$\begin{aligned}&\text{minimize } f(\boldsymbol{x})\\&\text{subject to } \boldsymbol{x} \in \Omega,\end{aligned}$$

where $f \in C^2$. For each of the following specifications for Ω, $\boldsymbol{x}^*$ and f, determine if the given point $\boldsymbol{x}^*$ is: (i) definitely a local minimizer; (ii) definitely not a local minimizer; or (iii) possibly a local minimizer.

a. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1\}$, $\boldsymbol{x}^* = [1,2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1,1]^\top$.

b. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1,2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1,0]^\top$.

c. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 0, x_2 \geq 0\}$, $\boldsymbol{x}^* = [1,2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [0,0]^\top$, and Hessian $\boldsymbol{F}(\boldsymbol{x}^*) = \boldsymbol{I}$.

d. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1,2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [1,0]^\top$, and Hessian $\boldsymbol{F}(\boldsymbol{x}^*) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

a. $\boldsymbol{x}^*$ is definitely not a local minimizer. Note that $\boldsymbol{d} = [1,2]^\top$ is a feasible direction at x^*. However, $\boldsymbol{d}^\top \nabla f(\boldsymbol{x}^*) = [1,2][1,1]^\top = -1$, which violate the FONC.

b. $\boldsymbol{x}^*$ satisfies the FONC, and thus is possibly a local minimizer, but it is impossible to be definite based on the given information.

c. $\boldsymbol{x}^*$ satisfies the SOSC, and thus is definitely a (strict) local minimizer.

d. $\boldsymbol{x}^*$ is definitely not a local minimizer. To see this, note that $\boldsymbol{d} = [0,1]^\top$ is a feasible direction at $\boldsymbol{x}^*$, and $\boldsymbol{d}^\top \nabla f(\boldsymbol{x}^*) = 0$. However, $\boldsymbol{d}^\top \boldsymbol{F}(\boldsymbol{x}^*)\boldsymbol{d} = -1$, which violates the SONC.

6.9 Consider the following function:

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1.$$

a. In what direction does the function f decrease most rapidly at the point $\boldsymbol{x}^{(0)} = [2,1]^\top$?

b. What is the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction of maximum decrease of f?

c. Find the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction $\boldsymbol{d} = [3,4]^\top$.

a. A differentiable function f decreases most rapidly in the direction of the negative gradient. In our problem,

$$\nabla f(\boldsymbol{x}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]^\top = [2x_1x_2 + x_2^3, x_1^2 + 3x_1x_2^2]^\top.$$

Hence, the direction of most rapid decrease is

$$-\nabla f(\boldsymbol{x}^{(0)}) = -[5, 10]^\top.$$

b. The rate of increase of f at $\boldsymbol{x}^{(0)}$ in the direction $-\nabla f(\boldsymbol{x}^{(0)})$ is

$$\nabla f(\boldsymbol{x}^{(0)})^\top - \frac{\nabla f(\boldsymbol{x}^{(0)})}{\|\nabla f(\boldsymbol{x}^{(0)})\|} = \|\nabla f(\boldsymbol{x}^{(0)})\| = -\sqrt{125} = -5\sqrt{5}.$$

c. The rate of increase of f at $\boldsymbol{x}^{(0)}$ in the direction $\boldsymbol{d}$ is

$$\nabla f(\boldsymbol{x}^{(0)})^\top \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} = [5, 10] \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{1}{5} = 11.$$

6.11 Consider the problem

$$\begin{aligned} &\text{minimize } -x_2^2 \\ &\text{subject to } |x_2| \le x_1^2 \\ &\qquad\qquad\quad x_1 \ge 0, \end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$.

a. Does the point $[x_1, x_2]^\top = \mathbf{0}$ satisfy the first-order necessary condition for a minimizer? That is, if f is the objective function, is it true that $\boldsymbol{d}^\top \nabla f(\mathbf{0}) \ge 0$ for all feasible directions $\boldsymbol{d}$ at $\mathbf{0}$?

b. Is the point $[x_1, x_2]^\top = \mathbf{0}$ a local minimizer, a strict local minimizer, a local maximizer, a strict local maximizer, or none of the above?

a. Write the objective function as $f(\boldsymbol{x}) = -x_2^2$. In this problem the only feasible directions at $\mathbf{0}$ are of the form $\boldsymbol{d} = [d_1, 0]^\top$. Hence, $\boldsymbol{d}^\top \nabla f(\mathbf{0}) = 0$ for all feasible directions $\boldsymbol{d}$ at $\mathbf{0}$.

b. The point $\mathbf{0}$ is a local maximizer, because $f(\mathbf{0}) = 0$, while any feasible point $\boldsymbol{x}$ satisfies $f(\boldsymbol{x}) \le 0$. The point $\mathbf{0}$ is not a strict local maximizer because for any $\boldsymbol{x}$ of the form $\boldsymbol{x} = [x_1, 0]^\top$, we have $f(\boldsymbol{x}) = 0 = f(\mathbf{0})$, and there are such points in any neighborhood of $\mathbf{0}$.

The point $\mathbf{0}$ is not a local minimizer because for any point $\boldsymbol{x}$ of the form $\boldsymbol{x} = [x_1, x_1^2]$ with $x_1 > 0$, we have $f(\boldsymbol{x}) = -x_1^4 < 0$, and there are such points in any neighborhood of $\mathbf{0}$. Since $\mathbf{0}$ is not a local minimizer, it is also not a strict local minimizer.

6.23 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = (x_1 - x_2)^4 + x_1^2 - x_2^2 - 2x_1 + 2x_2 + 1,$$

where $\boldsymbol{x} = [x_1, x_2]^\top$. Suppose that we wish to minimize f over $\mathbb{R}^2$. Find all points satisfying the FONC. Do these points satisfy the SONC?

We have

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 4(x_1 - x_2)^3 + 2x_1 - 2 \\ -4(x_1 - x_2)^3 - 2x_2 + 2 \end{bmatrix}.$$

Setting $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ we get

$$\begin{aligned} 4(x_1 - x_2)^3 + 2x_1 - 2 &= 0 \\ -4(x_1 - x_2)^3 - 2x_2 + 2 &= 0 \end{aligned}$$

Adding the two equations, we obtain $x_1 = x_2$, and substituting back yields $x_1 = x_2 = 1$. Hence, the only point satisfying the FONC is $[1,1]^\top$.

We have

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} 12(x_1 - x_2)^2 + 2 & -12(x_1 - x_2)^2 \\ -12(x_1 - x_2)^2 & 12(x_1 - x_2)^2 - 2 \end{bmatrix}.$$

Hence,

$$\boldsymbol{F}([1,1]^\top) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Since $\boldsymbol{F}([1,1]^\top)$ is not positive semidefinite, the point $[1,1]^\top$ does not satisfy the SONC.