Chong, Edwin Kah Pin., and Stanislaw H. Żak. *An Introduction to Optimization*. Fourth edition., John Wiley & Sons, Inc., 2013.

1 BASICS OF SET CONTSTRAINED AND UNCONSTRAINED OPTIMIZATION

6.1 Consider the problem

$$\begin{aligned}&\text{minimize } f(\boldsymbol{x})\\&\text{subject to } \boldsymbol{x} \in \Omega,\end{aligned}$$

where $f \in C^2$. For each of the following specifications for Ω, $\boldsymbol{x}^*$ and f, determine if the given point $\boldsymbol{x}^*$ is: (i) definitely a local minimizer; (ii) definitely not a local minimizer; or (iii) possibly a local minimizer.

a. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1\}$, $\boldsymbol{x}^* = [1, 2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1, 1]^\top$.

b. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1, 2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1, 0]^\top$.

c. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 0, x_2 \geq 0\}$, $\boldsymbol{x}^* = [1, 2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [0, 0]^\top$, and Hessian $\boldsymbol{F}(\boldsymbol{x}^*) = \boldsymbol{I}$.

d. $f: \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1, 2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [1, 0]^\top$, and Hessian $\boldsymbol{F}(\boldsymbol{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

6.9 Consider the following function:

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1.$$

a. In what direction does the function f decrease most rapidly at the point $\boldsymbol{x}^{(0)} = [2, 1]^\top$?

b. What is the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction of maximum decrease of f?

c. Find the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction $\boldsymbol{d} = [3, 4]^\top$.

6.11 Consider the problem

$$\begin{aligned}&\text{minimize } -x_2^2\\&\text{subject to } |x_2| \leq x_1^2\\&\qquad\qquad\quad x_1 \geq 0,\end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$.

a. Does the point $[x_1, x_2]^\top = \boldsymbol{0}$ satisfy the first-order necessary condition for a minimizer? That is, if f is the objective function, is it true that $\boldsymbol{d}^\top \nabla f(\boldsymbol{0}) \geq 0$ for all feasible directions $\boldsymbol{d}$ at $\boldsymbol{0}$?

b. Is the point $[x_1, x_2]^\top = \boldsymbol{0}$ a local minimizer, a strict local minimizer, a local maximizer, a strict local maximizer, or none of the above?

6.23 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

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$$f(x) = (x_1 - x_2)^4 + x_1^2 - x_2^2 - 2x_1 + 2x_2 + 1,$$

where $\boldsymbol{x} = [x_1, x_2]^\top$. Suppose that we wish to minimize f over $\mathbb{R}^2$. Find all points satisfying the FONC. Do these points satisfy the SONC?