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# An efficient method for computing the inverse of arrowhead matrices

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## ARTICLE INFO

## ABSTRACT

In this paper we propose a simple and effective method to find the inverse of arrowhead matrices which often appear in wide areas of applied science and engineering such as wireless communications systems, molecular physics, oscillators vibrationally coupled with Fermi liquid, and eigenvalue problems. A modified Sherman–Morrison inverse matrix method is proposed for computing the inverse of an arrowhead matrix. The effectiveness of the proposed method is illustrated and numerical results are presented along with comparative results.

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## 1. Introduction

Computing the inverse of important matrices for example tridiagonal matrices, $M$-matrices, etc., have been extensively studied and described in the literature (see [1–4] and references therein). Meanwhile, arrowhead matrices are an important example of matrices occurring in wide area of applications. Arrowhead matrices often arise in many topics coming from mathematics, physical problems and engineering applications, such as boundary value problems [5], solving the eigenvalue problem for large sparse matrices [6], modeling of radiationless transitions in isolated molecules [7], applications to oscillators vibrationally coupled with a Fermi liquid [8], modeling of multiple input and multiple output (MIMO) wireless communication systems [9], multiple-output neural-network (NN) models [10] and so on. Therefore, research about such matrices attracts the attention of many authors. An arrowhead matrix is a square matrix which is zero except for its main diagonal and one row and column.

This matrix is introduced in various books; see for example [11]. In [12] Walter, Cederbaum and Schirmer have presented an algorithm for computing the eigenvalues and eigenvectors of this matrix under certain conditions. O'Leary and Stewart [13], proposed an efficient method for computing eigenvalues and eigenvectors of symmetric arrowhead matrices by reduction of an arrowhead matrix to tridiagonal form using orthogonal similarity transformations combined with QR algorithms. Similar methods for reducing to tridiagonal or bidiagonal forms have been introduced in [14–18]. Arbenz and Golub [19] proved that no QR-like algorithm exists for symmetric arrow matrices. Morandi Cecchi and Di Nardo in [20], have presented a new method for computing the eigenvalues and eigenvectors of Hermitian matrices by finding the spectral decomposition of an arrowhead matrix. Additionally, recently a method has been proposed for computing the eigenvalues and corresponding eigenvectors for real symmetric arrowhead matrices [21]. Gravvanis in [22], has proposed

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a new class of approximate inverse matrix techniques based on the concept of sparse LU-type factorization procedures for computing explicitly inverses of arrowhead matrices without inverting the decomposition factors for solving arrowhead linear system [23,24]. Furthermore, in [25], he obtained the solution of symmetric arrowhead linear systems based on the concept of sparse approximate Cholesky-type factorization procedures and presented the explicit preconditioned iterative methods in conjunction with approximate inverse matrix techniques for the efficient solution of this symmetric linear system. Galantai [26] scrutinized the implementation of the block implicit LU ABS methods on linear and nonlinear systems with block arrowhead coefficient matrix and obtained a useful method for solving algebraic systems. In [27], Schäfer, presented interval arrowhead matrices for which the feasibility of the interval Gaussian algorithm was shown. Recently, discussions on this type of matrices have been given by many researchers in [28–31].

By all of these various applications on the arrowhead matrices, an interesting question is: how to invert arrowhead matrices? Also we know that it is an important topic in parallel computations since by computing efficiently the inverse of an arrowhead matrix, then can be used in conjunction with iterative methods to solve large linear systems on parallel and vector processors [32,33].

In this article, we respond to this question by proposing a simple and efficient method for computing the inverse of arrowhead matrices. Numerical experiments illustrate the effectiveness of the proposed method.

## 2. Modified Sherman–Morrison inverse of arrowhead matrices

The arrowhead matrix has the following form:

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & 0 & \cdots & 0 & b_2 \\ 0 & 0 & a_2 & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & a_n \end{bmatrix}, \tag{1}$$

and without loss of generality, we assume that $a_i = 1, i \in [1, n]$. Then we have

$$A = I + S_1 + S_2 \tag{2}$$

where $I$ is the identity matrix, $S_1$ and $S_2$ are strictly lower and strictly upper triangular parts of $A$, respectively.

In the following theorem we obtain the inverse of an arrowhead matrix, based on Sherman–Morrison formula [33,34] and $(I + S)$-type preconditioners ([34–37] and references therein).

**Theorem 2.1.** *Let $A$ be an arrowhead matrix in the form Eq. (1). Then the Modified Sherman–Morrison Inverse (MSMI) of $A$ may be obtained by the following matrix formulae:*

$$A^{-1} = (I - S_1)\left(I - \frac{1}{1+\alpha}(S_2(I - S_1))\right), \tag{3}$$

*where $1 + \alpha \neq 0$, $\alpha = \sum_{i=1}^{n-1}(-b_i)c_i$.*

**Proof.** By Sherman–Morrison formula we know that: If $B_{n \times n}$ is an invertible matrix and $\det(B + uv^T) \neq 0$, where $u$ and $v$ are $(n \times 1)$ vectors and the outer product of $u$ and $v$ is an $(n \times n)$ matrix of rank one.

Then we have

$$(B + uv^T)^{-1} = B^{-1} - \frac{B^{-1}uv^T B^{-1}}{1 + v^T B^{-1} u}. \tag{4}$$

Furthermore, if $B = I + S_1$, then

$$B^{-1} = I - S_1. \tag{5}$$

Further let us consider that

$$S_2 = uv^T, \tag{6}$$

where

$$u_{n \times 1} = (b_1, b_2, b_3, \ldots, b_{n-1}, 0)^T \quad \text{and} \quad v_{n \times 1} = (0, 0, 0, \ldots, 0, 1)^T.$$

Then,

$$A^{-1} = (I + S_1 + uv^T)^{-1} = (I - S_1) - \frac{(I - S_1)uv^T(I - S_1)}{1 + v^T(I - S_1)u}. \tag{7}$$

Moreover, for the bilinear form $v^T(I - S_1)u$, we have that

$$v^T(I - S_1)u = \operatorname{trac}(uv^T(I - S_1)) = \sum_{i=1}^{n-1}(-a_{ni})a_{in}.$$

Hence, it is evident that

$$A^{-1} = (I - S_1)\left(I - \frac{1}{1+\alpha}(S_2(I - S_1))\right). \quad \blacksquare$$

## 3. Numerical results

In this section the effectiveness and applicability of the new proposed schemes is examined for both symmetric and nonsymmetric matrices. The experimental results were obtained using an Intel Core I7-3370 at 3.4 GHz–3.9 GHz, 64-bit processor with 8 GB RAM memory running Windows 7. The computational platform used was the MATLAB environment. The execution time is given in "seconds. hundreds" using the "tic, toc" command from the MATLAB interface. The execution time presented in Tables 1–3 is the mean value of the execution time carried out 20 times for each case.

*Model problem 1: the symmetric case*

Let us consider a coefficient matrix $A$ of (1) where $a_i = 1.0$, $i \in [1, n]$ and $b_i = c_i = 0.9$, $i \in [1, n-1]$.

By considering a symmetric coefficient matrix, i.e. $b_i = c_i = t$, based on Theorem 2.1, it is evident that after some manipulation we have

$$A^{-1} = \begin{bmatrix} 1 + \gamma t^2 & \gamma t^2 & \gamma t^2 & \cdots & \gamma t^2 & -\gamma t \\ \gamma t^2 & 1 + \gamma t^2 & \gamma t^2 & \cdots & \gamma t^2 & -\gamma t \\ \vdots & \gamma t^2 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma t^2 & \gamma t^2 & \gamma t^2 & \cdots & 1 + \gamma t^2 & -\gamma t \\ -t - \gamma(n-1)t^3 & -t - \gamma(n-1)t^3 & -t - \gamma(n-1)t^3 & \cdots & -t - \gamma(n-1)t^3 & 1 - \gamma\alpha \end{bmatrix},$$

where $\gamma = \frac{1}{1+\alpha}$ and since

$$1 - \gamma\alpha = 1 - \frac{\alpha}{1+\alpha} = \gamma \quad \text{and} \quad -t - \gamma(n-1)t^3 = -t\gamma\left(\frac{1}{\gamma} + (n-1)t^2\right) = -t\gamma,$$

we obtain

$$A^{-1} = \begin{bmatrix} 1 + \gamma t^2 & \gamma t^2 & \gamma t^2 & \cdots & \gamma t^2 & -\gamma t \\ \gamma t^2 & 1 + \gamma t^2 & \gamma t^2 & \cdots & \gamma t^2 & -\gamma t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma t^2 & \gamma t^2 & \gamma t^2 & \cdots & 1 + \gamma t^2 & -\gamma t \\ -\gamma t & -\gamma t & -\gamma t & \cdots & -\gamma t & \gamma \end{bmatrix}.$$

In Table 1, the execution time for the Arrowhead Exact Inverse Matrix (AEIM) based on Arrowhead Approximate Root-Free Cholesky-type Factorization (AHARFCFA) algorithm (see [25]) and the MSMI algorithm for various values of $n$ are presented.

*Model problem 2: the nonsymmetric case*

Let us consider a coefficient matrix $A$ of (1) where $a_i = 1.0$, $i \in [1, n]$ and $b_i = 0.9$; $c_i = 0.1$, $i \in [1, n-1]$.

By considering a nonsymmetric coefficient matrix, based on Theorem 2.1, it is evident that after some manipulation we obtain

$$A^{-1} = \begin{bmatrix} 1 + \gamma b_1 c_1 & \gamma b_1 c_2 & \gamma b_1 c_3 & \cdots & \gamma b_1 c_{n-1} & -\gamma b_1 \\ \gamma b_2 c_1 & 1 + \gamma b_2 c_2 & \gamma b_1 c_3 & \cdots & \gamma b_2 c_{n-1} & -\gamma b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma b_{n-1} c_1 & \gamma b_{n-1} c_2 & \gamma b_{n-1} c_3 & \cdots & 1 + \gamma b_{n-1} c_{n-1} & -\gamma b_{n-1} \\ -\gamma c_1 & -\gamma c_2 & -\gamma c_3 & \cdots & -\gamma c_{n-1} & \gamma \end{bmatrix}.$$

**Table 1**
The execution time for symmetric matrices.

| $n$ | AHARFCFA–AEIM | MSMI |
|---|---|---|
| 1 000 | 0.005 | 0.003 |
| 2 000 | 0.013 | 0.007 |
| 4 000 | 0.043 | 0.021 |
| 6 000 | 0.087 | 0.044 |
| 8 000 | 0.149 | 0.075 |
| 10 000 | 0.227 | 0.111 |
| 15 000 | 0.499 | 0.238 |
| 20 000 | 0.881 | 0.405 |
| 25 000 | 1.377 | 0.601 |

**Table 2**
The execution time for nonsymmetric matrices.

| $n$ | ALUFA–AGEIM | MSMI |
|---|---|---|
| 1 000 | 0.012 | 0.007 |
| 2 000 | 0.044 | 0.027 |
| 4 000 | 0.170 | 0.117 |
| 6 000 | 0.367 | 0.261 |
| 8 000 | 0.664 | 0.479 |
| 10 000 | 1.049 | 0.773 |
| 15 000 | 2.453 | 1.849 |
| 20 000 | 4.579 | 3.465 |
| 25 000 | 6.873 | 5.183 |

**Table 3**
The execution time for nonsymmetric matrices using the "\" operator.

| $n$ | Computation of the inverse using "\" operator |
|---|---|
| 1 000 | 0.089 |
| 2 000 | 0.421 |
| 4 000 | 1.266 |
| 6 000 | 4.250 |
| 8 000 | 5.428 |
| 10 000 | 10.592 |
| 15 000 | 20.940 |
| 20 000 | Out of memory |

In Table 2, the execution time for the Arrowhead Generalized Exact Inverse Matrix (AGEIM) based on Arrow Approximate LU-type Factorization (ALUFA) algorithm (see [22]) and the MSMI algorithm for various values of $n$ are given. In Table 3, the execution time for computing the inverse using the "\" operator (MATLAB) for various values of $n$ are presented.

It should be noted that the Modified Sherman–Morrison Inverse (MSMI) algorithm adopts the "fish-bone" computational approach and remains the same level of inherent parallelism, cf. [32,33].

We should state that, the proposed MSMI method performs much better than existing methods for computing the inverse of an arrowhead matrix.

## 4. Conclusions

In this paper, we have proposed a modified Sherman–Morrison inverse matrix method for computing the inverse of an arrowhead matrix. The performance of the modified Sherman–Morrison inverse method is much better in comparison with the existing methods, as illustrated by the numerical results presented.

## References

[1] G. Meurant, A review on the inverse of symmetric tridiagonal and block tridiagonal matrices, SIAM J. Matrix Anal. Appl. 13 (1992) 707–728.
[2] P.D. Robinson, A.J. Wathen, Variational bounds on the entries of the inverse of a matrix, IMA J. Numer. Anal. 12 (1992) 463–486.
[3] C.M. da Fonseca, J. Petronilho, Explicit inverses of some tridiagonal matrices, Linear Algebra Appl. 325 (2001) 7–21.
[4] H. Saberi Najafi, M. Shams Solary, Computational algorithms for computing the inverse of a square matrix quasi inverse of a non-square matrix and block matrices, Appl. Math. Comput. 183 (2006) 539–550.
[5] G.A. Gravvanis, On the solution of boundary value problems by using fast generalized approximate inverse banded matrix techniques, J. Supercomput. 25 (2003) 119–129.
[6] B. Parlett, B. Nour-Omid, The use of refined error bound when updating eigenvalues of tridiagonals, Linear Algebra Appl. 68 (1985) 179–219.
[7] M. Bixon, J. Jortner, Intramolecular radiationless transitions, J. Chem. Phys. 48 (1968) 715–726.
[8] J.W. Gadzuk, Localized vibrational modes in Fermi liquids, general theory, Phys. Rev. B 24 (4) (1981) 1651–1663.
[9] H.T. Kung, B.W. Suter, A Hub matrix theory and applications to wireless communications, EURASIP J. Adv. Signal Process. 2007 (2007) 8. http://dx.doi.org/10.1155/2007/13659. Article ID 13659.

[10] E. Mizutani, J.W. Demmel, On structure-exploiting trust-region regularized nonlinear least squares algorithms for neural-network learning, Neural Netw. 16 (2003) 745–753.
[11] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.
[12] O. Walter, L.S. Cederbaum, J. Schirmer, The eigenvalue problem for arrow matrices, J. Math. Phys. 25 (4) (1984) 729–737.
[13] D.P. O'Leary, G.W. Stewart, Computing the eigenvalues and eigenvectors of symmetric arrowhead matrices, J. Comput. Phys. 90 (2) (1990) 497–505.
[14] H.Y. Zha, A two-way chasing scheme for reducing a symmetric arrowhead matrix to tridiagonal form, J. Numer. Linear Algebra Appl. 1 (1) (1992) 49–57.
[15] S. Van Huffel, H. Park, Efficient reduction algorithms for bordered band matrices, Numer. Linear Algebra Appl. 2 (2) (1995) 95–113.
[16] D.E. Stewart, A graph-theoretic model of symmetric givens operations and its implications, Linear Algebra Appl. 257 (1997) 311–320.
[17] S. Oliveira, A new parallel chasing algorithm for transforming arrowhead matrices to tridiagonal form, Math. Comp. 67 (221) (1998) 221–235.
[18] N. Mastronardi, S. Chandrasekaran, S. van Huffel, Fast and stable reduction of diagonal plus semi-separable matrices to tridiagonal and bidiagonal form, BIT 41 (2001) 149–157.
[19] P. Arbenz, G.H. Golub, QR-like algorithms for symmetric arrow matrices, SIAM J. Matrix Anal. Appl. 13 (2) (1992) 655–658.
[20] M. Morandi Cecchi, E. Di Nardo, The modified bordering method to evaluate eigenvalues and eigenvectors of normal matrices, Numer. Algorithms 11 (1–4) (1996) 285–309.
[21] N.J. Stor, I. Slapnicar, J.L. Barlow, Accurate eigenvalue decomposition of real symmetric arrowhead matrices and applications, Linear Algebra Appl. (2013) in press http://dx.doi.org/10.1016/j.laa.2013.10.007.
[22] G.A. Gravvanis, An approximate inverse matrix technique for arrowhead matrices, Int. J. Comput. Math. 70 (1) (1998) 35–45.
[23] I.S. Duff, A.M. Erisman, J.K. Reid, Direct Methods for Sparse Matrices, Oxford University Press, London, 1986.
[24] J.J. Dongarra, I.S. Duff, D.C. Sorensen, H.A. van der Vorst, Solving Linear Systems on Vector and Shared Memory Computers, SIAM, Philadelphia, 1991.
[25] G.A. Gravvanis, Solving symmetric arrowhead and special tridiagonal linear systems by fast approximate inverse preconditioning, J. Math. Model. Algorithms 1 (2002) 269–282.
[26] A. Galantai, Parallel ABS projection methods for linear and nonlinear systems with block arrowhead structure, Comput. Math. Appl. 38 (1999) 11–17.
[27] U. Schäfer, The feasibility of the interval Gaussian algorithm for arrowhead matrices, Reliab. Comput. 7 (1) (2001) 59–62.
[28] D.A. Bini, L. Gemignani, V.Y. Pan, Fast and stable QR eigenvalue algorithms for generalized companion matrices and secular equations, Numer. Math. 100 (3) (2005) 373–408.
[29] J. Pickmann, J.C. Egaña, R.L. Soto, Two inverse eigenproblems for symmetric doubly arrow matrices, Electron. J. Linear Algebra 18 (2009) 700–718.
[30] J.H. Brandts, R. Reis da Silva, Computable eigenvalue bounds for rank-$k$ perturbations, Linear Algebra Appl. 432 (12) (2010) 3100–3116.
[31] V. Kalofolias, E. Gallopoulos, Computing symmetric nonnegative rank factorizations, Linear Algebra Appl. 436 (2) (2012) 421–435.
[32] K.M. Giannoutakis, G.A. Gravvanis, High performance finite element approximate inverse preconditioning, Appl. Math. Comput. 201 (1–2) (2008) 293–304.
[33] G.A. Gravvanis, High performance inverse preconditioning, Arch. Comput. Methods Eng. 16 (1) (2009) 77–108.
[34] J.P. Milaszewicz, Improving Jacobi and Gauss–Seidel iterations, Linear Algebra Appl. 93 (1987) 161–170.
[35] H. Saberi Najafi, S.A. Edalatpanah, Some improvements in PMAOR method for solving linear systems, J. Inf. Comput. Sci. 6 (2011) 15–22.
[36] H. Saberi Najafi, S.A. Edalatpanah, Iterative methods with analytical preconditioning technique to linear complementarity problems: application to obstacle problems, RAIRO Oper. Res. 47 (2013) 59–71.
[37] H. Saberi Najafi, S.A. Edalatpanah, Comparison analysis for improving preconditioned SOR-type iterative method, Numer. Anal. Appl. 6 (2013) 62–70.