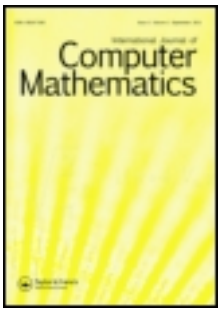

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Aggregation based on graph matching and inexact coarse grid solve for algebraic two grid

Pawan Kumar [a]

[a] Department of Computer Science , KU Leuven, Leuven , Belgium

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Aggregation based on graph matching and inexact coarse grid solve for algebraic two grid

Pawan Kumar*

Department of Computer Science, KU Leuven, Leuven, Belgium

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A graph matching is used to construct aggregation-based coarsening for an algebraic two-grid method. Effects of inexact coarse grid solve is analysed numerically for a highly discontinuous convection–diffusion coefficient matrix, and for problems from the Florida matrix market collection. The proposed strategy is found to be more robust compared to a classical algebraic multi-grid approach based on strength of connections. Basic properties of two-grid method are outlined.

Keywords: algebraic multi-grid; iterative method; preconditioners; eigenvalues; eigenvectors; two grid

2010 AMS Subject Classifications: 15A06; 65F08; 65F10

1. Introduction

We concern ourselves with the problem of solving large sparse linear systems of the form

$$Ax = b, \tag{1}$$

arising from the cell-centred finite volume discretization of the convection–diffusion equation as follows:

$$\begin{aligned} \operatorname{div}(\mathbf{a}(x)u) - \operatorname{div}(\kappa(x)\nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega_{\mathrm{D}}, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega_{\mathrm{N}}, \end{aligned} \tag{2}$$

where $\Omega = [0,1]^n$ ($n = 2$ or 3), $\partial\Omega_{\mathrm{N}} = \partial\Omega/\partial\Omega_{\mathrm{D}}$. The vector field $\mathbf{a}$ and the tensor κ are the given coefficients of the partial differential operator. In 2D case, we have $\partial\Omega_{\mathrm{D}} = [0,1] \times \{0,1\}$, and in 3D case, we have $\partial\Omega_{\mathrm{D}} = [0,1] \times \{0,1\} \times [0,1]$. We consider the heterogeneous coefficients case where κ has jumping values. Other sources include problems from the Florida matrix market collection [8]; see Table 2 for a list of problems considered in this paper.

Currently, one of the most successful methods are the multi-grid (MG) methods [21,25,26]. The robustness of a MG method is often significantly improved when used as a preconditioner in

*Emails: pawan.kumar@cs.kuleuven.be; kumar.lri@gmail.com

a Krylov subspace method [22]. If B denotes a MG preconditioner then a preconditioned linear system is a transformation of the linear system (1) to $B^{-1}Ax = B^{-1}b$. Here, B is an approximation to the matrix A such that the spectrum of $B^{-1}A$ is 'favourable' for achieving faster convergence of preconditioned Krylov subspace methods. For example, if the eigenvalues are clustered and are sufficiently close to one, then a fast convergence is observed in practice. Furthermore, the preconditioner B should be cheap to build and apply. With the advent of modern day multi-processor and multi-core era, the proposed method should have sufficient parallelism as well.

In MG-like methods, the problem is solved using a hierarchy of discretizations; the finest grid is at the top of the hierarchy followed by coarser grids. The two complementary processes are: smoothening and coarse grid correction. The smoothers are usually chosen to be one of the classical relaxation methods such as Jacobi, Gauss–Seidel, or incomplete LU (ILU) methods [22]. Analysis for model problems (e.g. Laplacian) reveals that the smoothers efficiently eliminates the high-frequency part of the error, while the global correction which is obtained by solving a restricted problem on the coarser grid, damps the low-frequency part of the error [30]. The low-frequency errors on the fine grid becomes high-frequency error on the coarse grid leading to their efficient resolution on the coarser grid. It is therefore crucial to choose efficient smoothers and a coarse grid solver. The classical geometric MG methods require information on the grid geometry and construct a restriction operator and a coarse grid. Since a geometric MG method is closely related to the grid, a problem with nonlinearity can be resolved efficiently. But, for an unstructured grid, applicability of the method becomes increasingly complicated. On the contrary, algebraic MG (AMG) methods define the necessary ingredients based solely on the coefficient matrix. Much research has been devoted to algebraic MG methods and several variants exist [7,9,17,25,26].

In this paper, an aggregation-based algebraic two grid is proposed. The aggregation is based on graph matching. This is achieved by partitioning the adjacency graph of the matrix such that the partitioned subgraphs are assumed to be the aggregates. Once a set of aggregates is found, a coarse grid is constructed from the Galerkin formula. In [12], authors use graph matching to form aggregates, and forward Gauss–Seidel with downwind numbering is used as pre- and post-smoother with the usual recursive MG scheme, where the coarsening is continued until the number of unknowns in the coarse grid are less than 10. This approach may lead to a deep hierarchy of grids, thus making the method very recursive and less adapted to modern day multi-processor or multi-core environment. In [20], similar graph-based matching is used to form a coarse grid, and the classical recursive smoothed AMG approach is followed, however, here, incomplete LU with threshold (ILUT) [22] is used for pre- and post-smoothing.

Our aim in this work is to propose a strategy that tries to avoid deep recursion but combines several different approaches as above. The strategy we adopt has the following ingredients:

- Coarsening based on graph matching.
- An incomplete LU with no-fill (ILU(0)) is the smoother with natural or nested dissection reordering.
- Coarse grid equation is solved inexactly using ILUT.

We show that the strategy proposed above is simple, easy to implement, and works well in practice for symmetric positive definite (SPD) systems with large jumps in the coefficients. Solving a coarse grid inexactly leads to a faster and cheaper method. We provide an estimate of heuristic coarse grid size and an estimate of a parameter involved in the inexact coarse grid solve. We show that the proposed strategy is more robust than the classical AMG based on strength of connection [18]. We also outline the similarities and differences of the proposed two-grid approach from the filtering preconditioners.

The rest of this paper is organized as follows. In Section 2, we discuss the classical coarsening strategy based on strength of connection, and the one based on graph matching. The numerical

experiments are presented in Section 3; the proposed method is compared to a classical AMG method on discontinuous convection–diffusion problems and some problems from the Florida matrix market collection [8]. Finally, Section 4 concludes this paper.

2. Graph matching-based aggregation for AMG

In classical AMG, a set of coarse grid unknowns is selected and the matrix entries are used to build interpolation rules that define the prolongation matrix P, and the coarse grid matrix A_c is computed from the following Galerkin formula

$$A_c = P^{\mathrm{T}} A P. \tag{3}$$

In contrast to the classical AMG approach, in aggregation-based MG, first a set of aggregates G_i is defined. Let N_c be the total number of such aggregates, then the interpolation matrix P is defined as follows:

$$P_{ij} = \begin{cases} 1 & \text{if } i \in G_j, \\ 0 & \text{otherwise,} \end{cases}$$

Here, $1 \le i \le N$, $1 \le j \le N_c$, N being the size of the original coefficient matrix A. Further, we assume that the aggregates G_i are such that

$$G_i \cap G_j = \phi, \quad \text{for } i \ne j \text{ and } \cup_i G_i = [1, N]. \tag{4}$$

Here, $[1, N]$ denotes the set of integers from 1 to N. Notice that the matrix P defined above is a $N \times N_c$ matrix, but since it has only one non-zero entry (which are 'one') per row, the matrix can be defined by a single array of length N storing the location of the non-zero entry on each row. The coarse grid matrix A_c may be computed as follows:

$$(A_c)_{ij} = \sum_{k \in G_i} \sum_{l \in G_j} a_{kl},$$

where $1 \le i, j \le N_c$, and a_{kl} is the (k, l)th entry of A.

Numerous aggregation schemes have been proposed in the literature, but in this paper, we consider two aggregation schemes as follows:

- *Aggregation based on strength of connection:* This approach is closely related to the classical AMG [26], where one first defines a set of nodes S_i to which i is strongly negatively coupled, using the strong/weak coupling threshold β:

  $$S_i = \{ j \ne i \mid a_{ij} < -\beta \max |a_{ik}| \}.$$

  Then, an unmarked node i is chosen such that priority is given to a node with minimal M_i, here M_i being the number of unmarked nodes that are strongly negatively coupled to i. For a complete algorithm of aggregation, the reader is referred to Notay [18].
- *Aggregation based on graph matching:* Several graph partitioning methods exists, notably, in software form [11,14,24]. Aggregation for AMG is achieved by calling a graph partitioner with required number of aggregates as an input parameter. The subgraph being partitioned are then considered as aggregates. For instance, in this paper, we use this approach by calling METIS_PartGraphKway of METIS with the adjacency graph of the matrix and required number of partitions as input parameters. The partitioning information is obtained in the output

argument, say, 'part'. The part array maps a given node to its partition, i.e. part$(i) = j$ means that the node i is mapped to the jth partition. This part array essentially determines the interpolation operator P. For instance, we observe that the 'part' array is a discrete many-to-one map. Thus, the ith aggregate $G_i = \text{part}^{-1}(i)$, where

$$\text{part}^{-1}(i) = \{j \in [1, N] \mid \text{part}(j) = i\}.$$

Similar graph matching techniques were explored in [5,12,20]. For notational convenience, the method introduced in this paper will be called GMG (graph matching-based aggregation MG).

Let S denote the operator that acts as a smoother in the GMG method. The typical choice of S is a Gauss–Siedel method [22]. However, in this paper, we choose ILU(0) as a smoother, we find that the choice of ILU(0) as a smoother is more robust compared to Gauss–Siedel method, however, at an additional storage cost. Another aspect that we explore is to use only two-grid approach but with an incomplete coarse grid solve. That is, we use an incomplete ILU(t), where t is the tolerance for dropping the entries [22]. The approximation $\tilde{A}_c$ of the coarse grid operator A_c is given as follows:

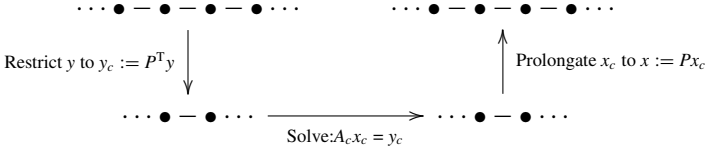
$$\tilde{A}_c = \tilde{L}_c \tilde{U}_c, \quad \text{where } [\tilde{L}_c, \tilde{U}_c] = \text{ILUT}(A_c),$$

where ILUT stands for ILU(t). The reason for using only two grid, and using an incomplete (and possibly parallel) coarse grid solve is to avoid the recursion in the typical AMG method. It may be profitable to solve the coarse grid problem in parallel and inexactly, when the problem size becomes large. This may be achieved by a call to one of the several hybrid incomplete solvers based on ILU [2] like approximation or by using a sparse approximate inverse [4]. The investigation with the parallel inexact approximation of the coarse solver may be a subject of future research. In this paper, we shall understand the qualitative behaviour such as the convergence and robustness of the proposed strategy compared to a classical AMG approach found in [17].

Let the two-grid preconditioner without post-smoothing be defined as follows:

$$B = (S^{-1} + M_+^{-1} - M_+^{-1} A S^{-1})^{-1}, \tag{5}$$

where $M_+^{-1} = P A_c^{-1} P^{\mathrm{T}}$, thus, an equation of the form $x = M_+^{-1} y$ is solved by first restricting y to $y_c = P^{\mathrm{T}} y$, then solving with the coarse matrix A_c the following linear system: $A_c x_c = y_c$. Finally, prolongating the coarse grid solution x_c to $x = P x_c$. Following diagram illustrates a simple two-grid hierarchy.



Here, we have considered the exact coarse grid solve, the inexact version is similar to the exact two-grid preconditioner (5) defined above except that M_+^{-1} is replaced by $\tilde{M}_+^{-1} = P \tilde{A}_c^{-1} P^{\mathrm{T}}$, and we denote the inexact two-grid preconditioner by $\tilde{B}$. In Algorithm (1), we present the pseudocode for generalized minimal residual (GMRES) method preconditioned by $\tilde{B}$. This algorithm is a slightly modified form of algorithm presented in [3, Figure 2.6].

2.1 *Analysis of graph-based two-grid method*

Let $\mathbb{R}$ denote the universal set of real numbers. For any matrix K, let $K \succ 0$ denote that the matrix K is SPD. We use the notation $K(:, j)$ to denote the jth column of K. If $A \succ 0$, then the inner

Algorithm 1 PSEUDOCODE TO SOLVE $Ax = b, A \in \mathbb{R}^{N\times N}, x, b \in \mathbb{R}^N$

OBJECTIVE: To solve Ax = b

SETUP PHASE

Call graph partitioner to get partitions in an array, say, part.
Use part array to form aggregates G_i and the prolongation matrix P (subgraphs are aggregates)
Create coarse grid matrix $A_c \in \mathbb{R}^{N_c\times N_c}$ as follows:

$$(A_c)_{ij} = \sum_{k\in G_i}\sum_{l\in G_j} a_{kl}.$$

Factor the coarse grid matrix inexactly: $\tilde{A}_c = \text{ILUT}(A_c)$. Here, ILUT is incomplete LU with tolerance.
Setup smoother: $S = L_0U_0 = \text{ILU0}(A)$. Here, ILU0 is incomplete LU with zero fill-in

Define (not to be formed explicitly) two-grid preconditioner $\tilde{B}$ as follows:

$$\tilde{B} = (S^{-1} + \tilde{M}_+^{-1} - \tilde{M}_+^{-1}AS^{-1})^{-1}, \quad \tilde{M}_+^{-1} = P\tilde{A}_c^{-1}P^{\mathrm{T}}$$

PRECONDITIONED GMRES ITERATION

x_0 is an initial guess
for $j = 1, 2, \ldots$ **do**
  Solve r from $\tilde{B}r = b - Ax_0$ (See SOLVE $\tilde{B}q = z$ function below)
  $v^{(1)} = r/\|r\|_2, \quad s := \|r\|_2 e_1$
  for $i = 1, 2, \ldots, m$ **do**
    Solve w from $\tilde{B}w = Av^{(i)}$ (See SOLVE $\tilde{B}q = z$ function below)
    for $k = 1, \ldots, i$ **do**
      $h_{k,i} = (w, v^{(k)}), \quad w = w - h_{k,i}v^{(k)}$
    end for
    $h_{i+1,i} = \|w\|_2, \quad v^{(i+1)} = w/h_{i+1,i}$
    apply $J_1, \ldots, J_{i-1}$ on $(h_{1,i}, \ldots, h_{i+1,i})$
    construct J_i, acting on the ith and $(i + 1)$st component of $h_{\cdot,i}$,
    such that $(i + 1)$st component of $J_ih_{\cdot,i}$ is 0
    set $s := J_is$
    if $s(i + 1)$ is small enough **then**
      UPDATE$(\tilde{x}, i)$ and quit
    end if
  end for
  UPDATE$(\tilde{x}, m)$
end for

UPDATE$(\tilde{x}, m)$

Solve for y in $Hy = \tilde{s}$. Here, upper $i \times i$ part of H has $h_{i,j}$ as its element. $\tilde{s}$ represents the first i components of s

$$\tilde{x} = x^{(0)} + y_1v^{(1)} + y_2v^{(2)} + \cdots + y_iv^{(i)}, \quad s^{(i+1)} = \|b - A\tilde{x}\|_2$$

If $\tilde{x}$ is accurate enough then quit else $x^{(0)} = \tilde{x}$

SOLVE $\tilde{B}q = z$

Solve $St = z$ (use $S = L_0U_0$), solve $f = \tilde{M}_+^{-1}z$ (See SOLVE $g = \tilde{M}_+^{-1}h$ function below), solve $q = \tilde{M}_+^{-1}(At)$, set $q = t + f - q$

SOLVE $g = \tilde{M}_+^{-1}h$

set $h_c = P^{\mathrm{T}}h$, Solve $\tilde{A}_cg_c = h_c$ (use $\tilde{A}_c = \tilde{L}\tilde{U}$), $g = Pg_c$

product $(\ ,\ )_A$ defined by $(u, v)_A = u^{\mathrm{T}}Av$ is a well-defined inner product, and it induces the energy norm $\|\ \|_A$ defined by $\|v\|_A = (v, v)_A^{1/2}$ for any vector v. A matrix K is called A-self-adjoint if

$$(Ku, v)_A = (u, Kv)_A,$$

or equivalently if

$$A^{-1}K^{\mathrm{T}}A = K.$$

For any matrix K, let span(K) denote a set of all possible linear combinations of the columns of the matrix K. Let $\|x\|$ denote the Euclidean norm $(\sum_{i=1}^{n} x_i^2)^{1/2}$. In what follows, we assume that the matrix P is orthonormalized, such that $P^{\mathrm{T}}P = I$. Let $\kappa(K)$ denote the condition number of matrix K. The basic linear fixed point method for solving the linear systems (1) is given as follows:

$$x^{n+1} = x^n + B^{-1}(b - Ax^n) = (I - B^{-1}A)x^n + B^{-1}b.$$

Subtracting the above equation with the identity $x = x - B^{-1}Ax + B^{-1}b$ yields the following equation for the error $e^{n+1} = x - x^{n+1}$

$$e^{n+1} = (I - B^{-1}A)e^n = (I - B^{-1}A)^2e^{n-1} = \cdots = (I - B^{-1}A)^{n+1}e^0.$$

Choosing B as in Equation (5), we have the following relation:

$$e^{n+1} = (I - M_+^{-1}A)^{n+1}(I - S^{-1}A)^{n+1}e^0.$$

Thus, the quality of the preconditioner B depends on how well the smoother S and the coarse grid preconditioner M_+ act on the error.

In [1,13,27–29], a combination preconditioner similar to the one in Equation (5) is proposed, where the matrix M_+^{-1} is replaced by a preconditioner, say M_f^{-1} that deflates the eigenvector corresponding to smaller eigenvalue. The preconditioner M_f is constructed such that it satisfies a 'filtering property' as follows:

$$M_f t = At,$$

where t is a filter vector. In [1], the filter vector is chosen to be a Ritz vector corresponding to smallest Ritz value in magnitude obtained after a few iterations of ILU(0) preconditioned matrix. In [29], the iteration is first started with a fixed set of filter vector, and later the filter vector is changed adaptively using error vector. In [13], authors fixed the filter vector to be the vector of all ones, and show that the combination preconditioner is efficient for a range of convection–diffusion type problems.

The two-grid preconditioner B (5) is similar to the combination preconditioner defined in [1,13], where instead of defining a coarse grid operator, a deflation preconditioner is used. Thus, rather than satisfying a 'filtering property', the coarse grid operator satisfies the following relation

$$(I - M_+^{-1}A)P = 0,$$

where columns of interpolation matrix P spans a subspace of dimension N_c. Thus, any vector in the column space of P is filtered out. This is proved later.

THEOREM 1 *If $A \succ 0$, then $A_c \succ 0$.*

Proof We have $A_c = P^{\mathrm{T}}AP$ and

$$\begin{aligned}(A_c x, x) &= (P^{\mathrm{T}}APx, x) \quad \text{for } x \neq 0\\ &= (APx, Px) \quad \text{for } x \neq 0\\ &> 0 \quad \text{for } x \neq 0.\end{aligned}$$

Notice that we use the fact that P is a Boolean matrix, i.e. it has one and only one non-zero entry equal to 'one' per row, and it has full column rank. Thus, $Px \neq 0$ for $x \neq 0$, $x \in \mathbb{R}^{N_c}$. Hence the theorem. ■

However, the global preconditioner corresponding to the coarse grid solve represented by $M_+^{-1} = PA_c^{-1}P^{\mathrm{T}}$ or $M_+^{-1}A$ is not necessarily SPD. We have the following counter examples.

THEOREM 2 *$A \succ 0$ does not imply that $M_+^{-1} \succ 0$ or $M_+^{-1}A \succ 0$.*

Proof Let $N = 4$ be the size of A. Let there be two aggregates, $G_1 = \{1, 3\}$ and $G_2 = \{2, 4\}$, then the restriction operator P^{T} is defined as follows $P^{\mathrm{T}} = \left[\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right]$, choosing $x^{\mathrm{T}} = [1, 0, -1, 0]$, we have $P^{\mathrm{T}}x = 0$. Thus, we have

$$(M_+^{-1}x, x) \not\succ 0, \quad (M_+^{-1}Ax, x) \not\succ 0 \text{ for all } x \neq 0.$$

■

It is to be noted that unlike the coarse grid preconditioner, some of the filtering preconditioners such as the low-frequency filtering decomposition (LFTFD) [1] are SPD whenever the coefficient matrix is SPD, moreover LFTFD is convergent. On contrary, two-grid scheme is significantly rank deficient. However, as a stand-alone preconditioner, the convergence is slow, and it is applied in combination with ILU(0).

THEOREM 3 *Let $A \succ 0$, then following holds:*

(1) $(I - M_+^{-1}A)P = 0$,
(2) *If $t = [1, 1, \ldots, 1]$, then $t \in \mathrm{span}(P)$ and $\dim(\mathrm{span}(P)) = N_c$.*

Proof We have

$$\begin{aligned}(I - M_+^{-1}A)P &= P - PA_c{}^{-1}P^{\mathrm{T}}AP,\\ &= P - PA_c{}^{-1}A_c,\\ &= 0.\end{aligned}$$

From Equations (2) and (4), we find that P is an $N \times N_c$ matrix, and the jth column of the matrix P has a non-zero entry $P(i, j) = 1$ if and only if $i \in G_j$. Since the aggregates $G_j's$ cover all the nodes in the set $[1, N]$, for all $i \in [1, N]$, there exists an aggregate G_j such that $i = G_j(k)$ for some k, and consequently $P(i, j) = 1$. Moreover, since the aggregates $G_j's$ do not intersect, such j is unique. In other words, for each $i \in [1, N]$, there exists *one and only one* column $P(:, j)$ of P such that the ith entry of column $P(:, j)$ is 1. Hence, we have

$$Pt = \sum_{1 \leq i \leq N_c} P(:, j) = t$$

and since each columns of P are linearly independent we have $\dim(\mathrm{span(P)}) = N_c$. ■

For an effective method, the columns of the interpolation matrix P should approximate well the eigenvectors corresponding to low eigenvalues.

In the literature, much results have been proved when the coefficient matrix is a diagonally dominant M-matrix. We collect relevant results, and use them to understand the proposed method.

DEFINITION 1 *Let $G(A) = (V, E)$ be the adjacency graph of a matrix $A \in \mathbb{R}^{N\times N}$. The matrix A is called irreducible if any vertex $i \in V$ is connected to any vertex $j \in V$. Otherwise, A is called reducible.*

DEFINITION 2 *A matrix $A \in \mathbb{R}^{N\times N}$ is called an M-matrix if it satisfies the following three properties*:

(1) $a_{ii} > 0$ *for* $i = 1, \ldots, N$,
(2) $a_{ij} \leq 0$ *for* $i \neq j$, $i,j = 1, \ldots, N$,
(3) A *is non-singular and* $A^{-1} \geq 0$.

DEFINITION 3 *A square matrix A is strictly diagonally dominant if the following holds*

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \quad i = 1, \ldots, N$$

and it is called irreducibly diagonally dominant if A is irreducible and the following holds

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|, \quad i = 1, \ldots, N$$

where strict inequality holds for at least one i.

A simpler criteria for an M-matrix property is given by the following theorem.

THEOREM 4 [12] *If the coefficient matrix A is strictly or irreducibly diagonally dominant and satisfies the following conditions*

(1) $a_{ii} > 0$ *for* $i = 1, \ldots, N$,
(2) $a_{ij} \leq 0$ *for* $i \neq j$, $i,j = 1, \ldots, N$,

then A is an M-matrix.

THEOREM 5 [12] *If $A \in \mathbb{R}^{N\times N}$ is a strictly or irreducibly diagonally dominant M-matrix, then so is the coarse grid matrix $A_c = P^{\mathrm{T}}AP$.*

THEOREM 6 [15] *If the coefficient matrix A is symmetric M-matrix, and let $S = \tilde{L}\tilde{L}^{\mathrm{T}}$ be the incomplete Cholesky factorization, then the fixed point iteration with the error propagation matrix $I - S^{-1}A$ is convergent.*

Theorem 6 tells us that for an M-matrix, ILU(0) preconditioned method will be convergent by itself. However, the convergence is usually slow due to large iteration count with increasing problem size. Combining ILU(0) with a coarse grid correction leads to convergence rate which depends mildly on the problem size. Following result shows that the inexact factorization is as stable as the exact factorization of the coarse grid operator.

THEOREM 7 *If the given coefficient matrix A is a symmetric irreducibly diagonally dominant M-matrix, and if the inexact coarse grid operator $\tilde{A}_c$ is based on incomplete LU factorization as follows:*

$$\tilde{A}_c = \text{CHOLINC}(A_c),$$

where CHOLINC *is the incomplete Cholesky factorization, then the construction of $\tilde{A}_c$ is at least as stable as the construction of an exact decomposition of A_c without pivoting.*

Proof If the original matrix A is symmetric and irreducibly diagonally dominant M-matrix, then Theorem 5 tells us that the coarse grid operator A_c obeys the same property. Now, A_c being an M-matrix, [15, Theorem 3.2] tells us that $\tilde{A}_c$ defined above is as stable as the exact Cholesky factorization of A_c. ■

For a diagonally dominant M-matrix, pivoting is rarely needed. However, pivoting generally improves the stability of incomplete LU type factorizations. This is the reason why we use incomplete LU with pivoting, namely, ILUT function of MATLAB. Moreover, using ILUT suitable for unsymmetric matrices that are not necessarily diagonally dominant. We refer the curious reader to [10] for a small 2×2 example where pivoting would be essential to obtain stable triangular factorization.

For problems with jumping coefficients, the ratio of maximum and minimum entry of the coefficient matrix can provide useful bounds as shown in the following theorem.

LEMMA 1 [12, p. 7] *Let A be a symmetric $N \times N$ matrix with eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_N(A)$ arranged in nondecreasing order, then the following holds*

$$\lambda_1(A) \leq \min_i\{a_{ii}\} \leq \max_i\{a_{ii}\} \leq \lambda_N(A).$$

In particular, if $A \succ 0$, then cond(A) *is bounded below by* $\max_i\{a_{ii}\}/\min_i\{a_{ii}\}$.

Proof The proof follows by writing the following expression

$$\lambda_1(A) = \min_{\|x\|=1}\{x^{\mathrm{T}}Ax\}, \quad \lambda_N(A) = \max_{\|x\|=1}\{x^{\mathrm{T}}Ax\},$$

and by setting x as the ith column of the identity matrix I. ■

The theorem above gives an estimate of the lower bound condition number of the coarse and fine grids. This lower bound is useful for problems with large jumps in the coefficients such as those considered in this paper. The condition number of a coarse grid is always bounded above by the condition number of the fine grid. This is easily proved by using the following Poincare's separation theorem.

THEOREM 8 (Poincaré [19]) *Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and let P be a semi-orthogonal $n \times k$ matrix with the property that $P^{\mathrm{T}}P = I_k$. The eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$ of $P^{\mathrm{T}}AP$ are separated by the eigenvalues of A as follows:*

$$\lambda_i \leq \mu_i \leq \lambda_{n-k+i}. \tag{6}$$

Following result follows trivially form Theorem 8.

THEOREM 9 *Let $A_{N\times N} \succ 0$, let $P_{N\times N_c}$ be any semi-orthogonal interpolation operator, and let $A_c = P^{\mathrm{T}}AP$, then we have*

$$\kappa(A_c) \leq \kappa(A). \tag{7}$$

Proof From Theorem 8, we have $\lambda_{\min}(A_c) \geq \lambda_{\min}(A)$, and $\lambda_{\max}(A_c) \leq \lambda_{\max}(A)$. Thus, we have

$$\kappa(A_c) = \frac{\lambda_{\max}(A_c)}{\lambda_{\min}(A_c)} \leq \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} = \kappa(A).$$

■

In [16], convergence analysis of perturbed two-grid and MG methods was done. In the context of domain decomposition methods, in [6], numerical and theoretical analysis suggests the advantages of using inexact solves.

3. Numerical experiments

All the numerical experiments were performed in a 64 bit MATLAB version 7.10, R2010a in double-precision arithmetic on Intel core i7 (720QM) with 6 GB RAM. For comparison, we use the aggregation-based AMG (AGMG) software [18]. The AGMG software is written in Fortran 90 with amex interface compiled using mex command with -O flag of mex, on the other hand, the AMG method introduced in this paper, namely, GMG, is written completely in MATLAB. Since test cases also include nonsymmetric matrices, for GMG, the iterative accelerator used is GMRES available at [23], the code was changed such that the stopping criteria is based on the decrease of the 2-norm of the relative residual. For AGMG, GCR method is used [17]. For both GMRES and GCR, the maximum number of iterations allowed is 600, and no restart is done. The stopping criteria is the decrease of the relative residual below 10^{-7}, i.e. when

$$\frac{\|b - Ax_k\|}{\|b\|} < 10^{-7}.$$

Here, b is the right-hand side and x_k is an approximation to the solution at the kth step. A list of shorthand notations are shown in Table 1.

Table 1. Notations used in tables of numerical experiments.

Notations	Meaning
h	Discretization step
N	Size of the original matrix
N_c	Size of the coarse grid matrix
its	Iteration count
time	Total CPU time (setup plus solve) in seconds
ME	Memory allocation problem
F1	AGMG returned flag 1, see [18]
SPD	Symmetric positive definite
NA	Not applicable
NC	Did not converged
–	Data not available
GMG-NO	Graph-based matching for AMG, smoother has natural ordering
GMG-ND	Graph-based matching for AMG, smoother has nested dissection ordering
EGMG-NO	Graph-based matching for AMG, exact coarse grid, smoother has natural ordering
EGMG-ND	Graph-based matching for AMG, exact coarse grid, smoother has nested dissection ordering
AGMG	Classical AMG, see [18]
cf	Coarsening factor, cf $= (N/N_c)^{1/2}$ for the 2D case and cf $= (N/N_c)^{1/3}$ for the 3D case

Table 2. Florida matrix market matrices.

Matrices	Kind	SPD	Size	Non-zeros
gyro_m	Model reduction problem	Yes	17,361	340K
bodyy4	Structural problem	Yes	17,546	121K
nd6k	2D/3D problem	Yes	18,000	6.8M
bodyy5	Structural problem	Yes	18,589	128K
wathen100	Random 2D/3D problem	Yes	30,401	471K
wathen120	Random 2D/3D problem	Yes	36,441	565K
torsion1	Duplicate optimization problem	Yes	40,000	197K
obstclae	Optimization problem	Yes	40,000	197K
jnlbrng1	Optimization problem	Yes	40,000	199K
minsurfo	Optimization problem	Yes	40,806	203K
gridgena	Optimization problem	Yes	48,962	512K
crankseg_1	Structural problem	Yes	52,804	10M
qa8fk	Acoustic problem	Yes	66,127	1M
cfd1	Computational fluid dynamics	Yes	70,656	1.8M
finan512	Economic problem	Yes	74,752	596K
shallow_water1	Computational fluid dynamics	Yes	81,920	327K
2cubes_sphere	Electromagnetic problem	Yes	101,492	1.6M
Thermal_TC	Thermal problem	Yes	102,158	711K
Thermal_TK	Thermal problem	Yes	102,158	711K
G2_circuit	Circuit simulation	Yes	150,102	726K
bcsstk18	Structural problem	Yes	11,948	149K
cbuckle	Structural problem	Yes	13,681	676K
Pres_Poisson	Computational fluid dynamics	Yes	14,822	715K

Note: SPD, symmetric positive definite.

3.1 *Test cases*

Convection–Diffusion: Our primary test case is the convection–diffusion equation (2) defined earlier. We use the notation DC to indicate that the problems are discontinuous. We consider the following test cases.

DC1, 2D case: The tensor κ is isotropic and discontinuous. The domain contains many zones of high permeability that are isolated from each other. Let $[x]$ denote the integer value of x. For 2D case, we define $\kappa(x)$ as follows:

$$\kappa(x) = \begin{cases} 10^3 * ([10 * x_2] + 1) & \text{if } [10 * x_i] \equiv 0 \pmod 2,\ i = 1, 2, \\ 1 & \text{otherwise.} \end{cases}$$

The velocity field **a** is kept zero. We consider a $n \times n$ uniform grid where n is the number of discrete points along each spatial directions.

DC1, 3D case: For 3D case, $\kappa(x)$ is defined as follows:

$$\kappa(x) = \begin{cases} 10^3 * ([10 * x_2] + 1) & \text{if } [10 * x_i] \equiv 0 \pmod 2,\ i = 1, 2, 3, \\ 1 & \text{otherwise.} \end{cases}$$

Here again, the velocity field **a** is kept zero. We consider a $n \times n \times n$ uniform grid. The jump in the diagonal entries of the coefficient matrix is shown in Figure 1.

DCC1, 2D case: Same as DC1, 2D case above, except that the velocity is non-zero and it is given as $a(x) = (1000, 1000)$.

DCC1, 3D case: Same as DC1, 3D case above, except that the velocity $a(x) = (1000, 1000, 1000)$.

Florida matrix market collection: The list of matrices from the Florida matrix collection are shown in Table 2. As we observe, all the problems are SPD steaming from a wide range of applications. For more on the properties of these matrices, the reader is referred to [8].

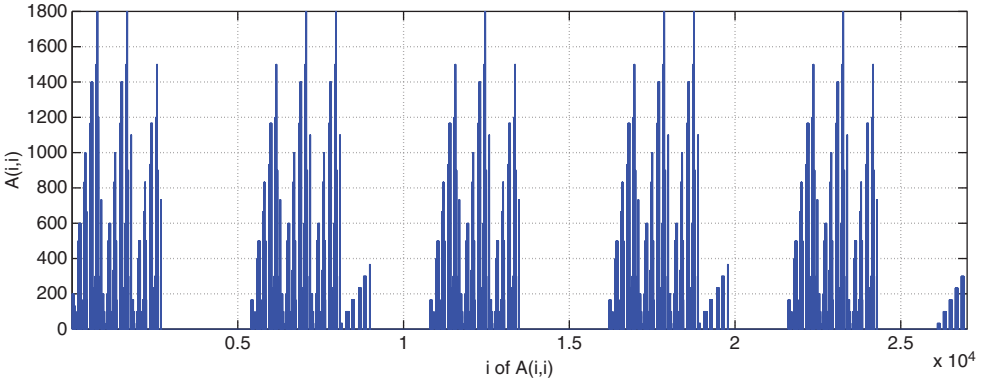


Figure 1. Jump in the diagonal entries of DC3D $30 \times 30 \times 30$ matrix when $\kappa(x)$ values are kept as 10^3.

3.2 *Comments on numerical results*

Two versions of GMG are shown, namely, GMG-NO which stands for GMG where smoother has natural ordering, and GMG-ND stands for GMG with smoother having nested dissection ordering. In particular, for GMG-ND, we first apply the nested dissection reordering and then the smoother is defined. We observe that after applying nested dissection reordering, the smoother which is ILU(0) in our case can be computed and applied in parallel. Since, in ILU(0), no pivoting is done, parallelizing ILU(0) after ND ordering leads to a parallel smoother. Certainly, not much parallelism is expected when the smoother is applied with natural ordering of unknowns. As mentioned before, for the coarse grid solve, we use ILU(10^{-4}) to solve it inexactly. We do this inexact solve to see the effect of inexact solve in the iteration count and time. For AGMG, Gauss–Seidel smoothing is used, and the choice of the coarse grid is based on the strength of connection between nodes. Moreover, in AGMG, usual multi-level recursive approach is followed, i.e. going down the grid hierarchy until the coarse grid is small enough (or it stops when it does not satisfy certain criteria) to be solved exactly.

We recall here that our aim is to compare the classical MG approach implemented in AGMG with the two-grid approach of GMG with the following ingredients:

- Coarse grid based on graph matching (call to `METIS`).
- ILU(0) is the smoother (built in `MATLAB`).
- Coarse grid equation is solved inexactly (using built in ILU(t) routine in `MATLAB`).

First, we present the test results for DC1 for 2D and 3D case. In Tables 3–12, we have shown the iteration count and the total CPU time: setup time plus solve time, for values of cf ranging from 2.5 to 8. For 2D problems, we find that the AGMG method is several times faster than the GMG-based methods. Even though, the iteration count for AGMG is significantly higher compared to GMG-NO, the small CPU time is due to an efficient implementation in Fortran 90 and use of MG. However, for 3D problems, the AGMG method does not converge at all. In contrast, GMG methods converge and show mesh independent convergence rates for all values of cf. Considering 2D case first: the least CPU time for GMG-NO method is observed for cf $= 3$. For GMG-ND method, the least CPU time is observed for cf $= 2.5$ for 1000×1000, and for 800×800 and 1200×1200 problem the least CPU time is observed for cf $= 3$. On the other hand, for EGMG-NO method, the least CPU time was observed when we had cf $= 4$ for 800×800, when cf $= 3$ for 1000×1000, and when cf $= 5$ for 1200×1200 problem.

Table 3. Numerical results for DC1 problem with cf = 2.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	23	25.5	17	17.4	16	38.7	27	6.3
	1/1000	23	45.5	18	30.3	18	82.0	37	14.1
	1/1200	24	70.8	20	48.2	18	159.5	37	18.2
3D	1/70	145	65.8	21	21.0	14	163.0	NC	NA
	1/100	191	306.5	26	141.0	ME	NA	NC	NA
	1/120	147	598.0	45	379.2	ME	NA	NC	NA

Table 4. Numerical results for DC1 problem with cf = 3 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	28	24.7	20	16.8	19	22.8	27	6.3
	1/1000	29	50.4	19	25.6	19	25.5	37	14.1
	1/1200	26	65.8	19	37.6	20	70.8	37	18.2
3D	1/70	165	66.0	20	14.3	15	47.5	NC	NA
	1/100	191	314.1	25	94.2	16	839.2	NC	NA
	1/120	165	417.0	35	236.8	ME	ME	NC	NA

Table 5. Numerical results for DC1 problem with cf = 3.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	33	30.6	28	22.7	22	19.9	27	6.3
	1/1000	36	50.9	23	28.6	22	33.8	37	14.1
	1/1200	34	72.7	23	42.2	22	54.8	37	18.2
3D	1/70	153	55.7	23	11.6	17	21.5	NC	NA
	1/100	200	221.8	32	58.6	23	254.8	NC	NA
	1/120	153	339.5	33	119.5	19	740.2	NC	NA

Table 6. Numerical results for DC1 problem with cf = 4 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	44	34.8	26	20.6	24	19.7	27	6.3
	1/1000	37	49.8	25	30.1	25	33.2	37	14.1
	1/1200	37	72.3	28	51.1	27	53.8	37	18.2
3D	1/70	164	57.6	84	29.5	55	26.3	NC	NA
	1/100	212	235.8	27	44.5	19	116.3	NC	NA
	1/120	168	346.2	31	94.8	21	326.3	NC	NA

For 3D problems, the least CPU time is again obtained with GMG-NO. For a smaller 3D $70 \times 70 \times 70$ problem, the iteration count of GMG-NO rather increases rapidly with the increasing value of cf. The least CPU time for this problem is for cf = 3.5. For smaller problems, not keeping the coarse grid fine enough could be costly. For larger problems, for $100 \times 100 \times 100$, the least CPU time is obtained for cf = 4.5, and for $120 \times 120 \times 120$ the least CPU time is obtained for

Table 7. Numerical results for DC1 problem with cf = 4.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	43	32.7	27	20.5	27	21.1	27	6.3
	1/1000	44	53.9	28	33.9	27	33.4	37	14.1
	1/1200	41	74.0	27	47.5	29	55.6	37	18.2
3D	1/70	170	58.7	89	31.0	70	26.6	NC	NA
	1/100	205	215.4	28	40.9	22	63.3	NC	NA
	1/120	157	302.3	34	83.5	24	165.0	NC	NA

Table 8. Numerical results for DC1 problem with cf = 5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	46	34.4	32	24.4	30	24.2	27	6.3
	1/1000	47	56.8	32	37.9	34	39.2	37	14.1
	1/1200	51	86.8	30	53.8	29	53.6	37	18.2
3D	1/70	284	94.2	199	63.3	193	62.9	NC	NA
	1/100	206	280.9	30	55.2	23	60.5	NC	NA
	1/120	168	335.3	40	92.2	26	122.7	NC	NA

Table 9. Numerical results for DC1 problem with cf = 5.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	51	37.0	33	23.5	33	23.9	27	6.3
	1/1000	52	61.3	34	37.6	33	37.8	37	14.1
	1/1200	50	84.2	33	54.3	32	55.3	37	18.2
3D	1/70	469	148.1	162	50.9	160	51.5	NC	NA
	1/100	219	222.9	148	146.2	109	117.0	NC	NA
	1/120	221	393.7	42	83.4	28	95.4	NC	NA

Table 10. Numerical results for DC1 problem with cf = 6 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	55	41.4	36	25.2	36	25.2	27	6.3
	1/1000	56	65.8	36	40.0	36	40.0	37	14.1
	1/1200	55	92.9	36	56.1	36	56.8	37	18.2
3D	1/70	568	183.2	316	98.9	354	115.9	NC	NA
	1/100	208	213.5	103	100.4	94	101.1	NC	NA
	1/120	167	308.1	43	83.7	41	95.5	NC	NA

cf = 5.5. In general, the pattern suggests that as the problem size increases, it is worth taking smaller coarse grid (i.e. larger cf value) to obtain faster convergence. For GMG-ND, the number of iterations are very large, thus it converges slowly. The classical coarsening-based AMG fails to converge within 600 iterations.

Table 11. Numerical results DC1 problems with cf = 7 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	68	41.9	51	25.8	41	27.2	27	6.3
	1/1000	68	73.5	41	36.9	41	40.9	37	14.1
	1/1200	73	112.0	41	59.4	41	60.0	37	18.2
3D	1/70	NC	NA	462	140.9	277	86.6	NC	NA
	1/100	566	542.2	266	249.9	255	239.0	NC	NA
	1/120	173	314.5	72	127.5	57	114.5	NC	NA

Table 12. Numerical results for DC1 problems with cf = 8 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	77	48.3	48	30.9	48	30.9	27	6.3
	1/1000	76	74.6	46	46.8	46	47.2	37	14.1
	1/1200	77	108.6	46	67.0	45	65.9	37	18.2
3D	1/70	NC	NA	360	120.2	381	125.1	NC	NA
	1/100	NC	NA	454	441.4	440	426.5	NC	NA
	1/120	486	831.2	186	319.5	187	326.2	NC	NA

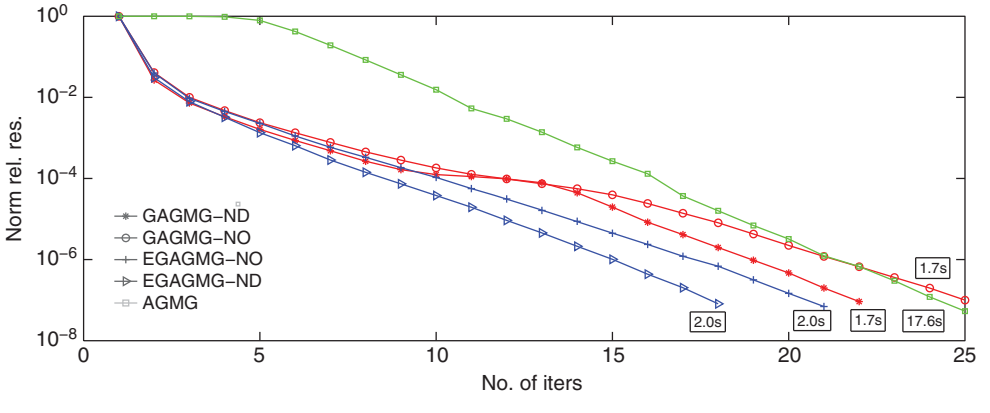


Figure 2. Convergence curve for Pres_Poisson for exact and inexact coarse grid solves. CPU time indicated inside small box.

In Figures 2 and 3, we find that the convergence curve for the respective exact and inexact methods are very similar, this also suggests a similar spectrum and probably a similar condition number. To find how close the approximated coarse matrix is to the exact coarse operator, in Table 23, we compare the relative error $\|LU - \tilde{L}\tilde{U}\|/\|LU\|$ for both natural and ND ordering. We find that the relative error is quite small. Since the inexact methods are relatively fast to build and apply, we save significant amount of CPU time and storage requirement, see Figure 4 where an inexact solve needs about 10 times less storage compared to the exact solver.

In Tables 13–22, we have similar plots for the test case DCC1. For this problem, we find that the AGMG method converges much faster compared to the GMG methods for both 2D and 3D problems, exception being the $120 \times 120 \times 120$ problem where the method fails to converge. However, we remind ourselves that GMG methods are implemented in Matlab, and thus they

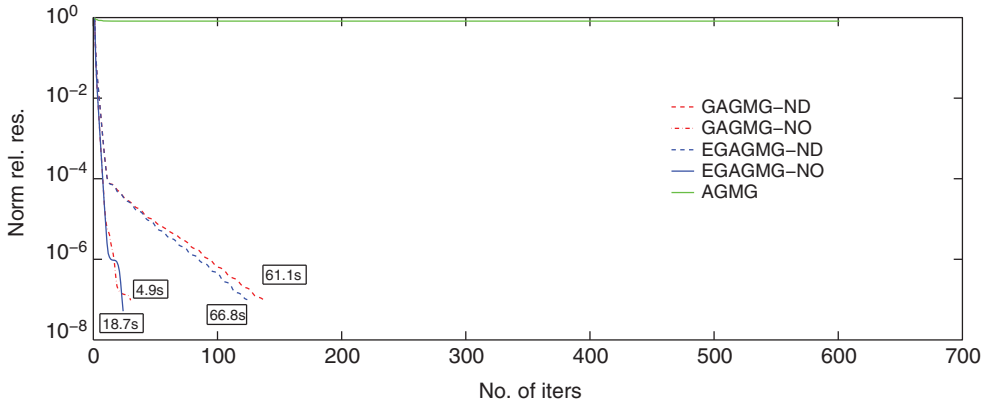


Figure 3. Convergence curve for DC3D $50 \times 50 \times 50$ for exact and inexact coarse grid solves. CPU time indicated inside small box.

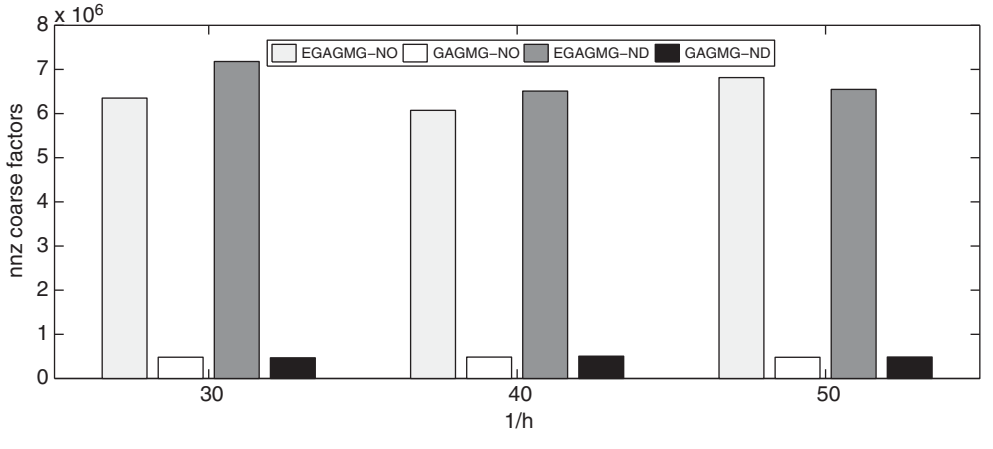


Figure 4. Comparison of no. of non-zeros for exact and inexact coarse grid solves for DC1, 3D.

Table 13. Numerical results for DCC1 problem with cf = 2.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	54	45.3	49	36.6	18	14.2	27	6.3
	1/1000	75	90.4	60	73.7	19	83.0	37	14.1
	1/1200	51	101.4	40	61.1	21	165.6	37	18.2
3D	1/70	100	50.3	14	17.4	14	161.6	16	3.4
	1/100	139	223.8	15	100.8	ME	NA	15	8.8
	1/120	123	471.3	15	264.2	ME	NA	NC	NA

are expected to be faster when they are implemented in lower level languages such as Fortran or C. Thus, our prediction is that even for these problems where GMG shows larger CPU time, an implementation in Fortran may have the convergence time comparable with that of AGMG. Notably, for this test case, the iteration count decreases even more rapidly (compared to test case DC1) with the increase in the size of the problem.

The rule of thumb in the choice of coarse grid size is to increase the cf value proportionally with the increasing size of the problem. For a smaller size problem with discontinuous

Table 14. Numerical results for DCC1 problem with cf = 3 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	28	24.7	20	16.8	19	22.8	27	6.3
	1/1000	29	50.4	19	25.6	19	25.5	37	14.1
	1/1200	26	65.8	19	37.6	20	70.8	37	18.2
3D	1/70	113	48.7	15	10.6	15	46.9	16	3.4
	1/100	153	207.3	17	53.2	17	681.8	15	8.8
	1/120	135	337.8	17	119.7	ME	ME	NC	NA

Table 15. Numerical results for DCC1 problem with cf = 3.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	47	36.5	32	25.1	27	24.7	27	6.3
	1/1000	52	64.0	41	44.4	28	42.9	37	14.1
	1/1200	44	80.8	30	57.8	27	65.5	37	18.2
3D	1/70	129	48.6	16	8.7	16	21.0	16	3.4
	1/100	171	194.2	18	36.5	18	248.0	15	8.8
	1/120	150	325.0	18	78.3	18	744.7	NC	NA

Table 16. Numerical results for DCC1 problems with cf = 4 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
J2D	1/800	57	43.8	32	24.8	31	26.2	27	6.3
	1/1000	53	56.4	38	41.7	35	43.7	37	14.1
	1/1200	52	81.3	39	61.9	38	66.1	37	18.2
3D	1/70	138	52.0	26	12.4	26	15.8	16	3.4
	1/100	182	191.2	19	31.1	19	109.9	15	8.8
	1/120	170	308.8	20	61.2	19	312.9	NC	NA

Table 17. Numerical results for DCC1 problems with cf = 4.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
J2D	1/800	76	53.9	39	27.6	37	27.5	27	6.3
	1/1000	74	82.2	39	43.9	38	44.4	37	14.1
	1/1200	64	112.6	39	64.1	37	65.6	37	18.2
3D	1/70	151	53.5	38	16.1	38	18.1	16	3.4
	1/100	210	234.6	21	31.1	21	66.0	15	8.8
	1/120	180	363.6	22	61.5	22	169.2	NC	NA

coefficients such as DC1 and DCC1, it is good to keep the cf value small. The choice of drop tolerance in ILUT to be 10^{-4} worked well in practice and we did not encounter any breakdown. However, it may be beneficial to tune these parameters depending on the problem to obtain optimal performance.

Table 18. Numerical results for DCC1 problems with cf = 5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	88	60.95	45	29.0	43	28.1	27	6.1
	1/1000	80	77.2	48	48.6	49	50.8	37	13.2
	1/1200	81	114.0	43	64.4	42	64.7	37	18.3
3D	1/70	169	59.4	60	22.8	60	24.0	16	3.4
	1/100	224	219.8	22	29.7	22	45.3	15	8.8
	1/120	195	340.35	23	56.8	23	111.0	NC	NA

Table 19. Numerical results for DCC1 problems with cf = 5.5 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	107	73.0	50	33.7	48	32.7	27	6.1
	1/1000	100	107.8	50	55.3	49	52.7	37	13.2
	1/1200	90	148.7	48	74.2	47	72.8	37	18.3
3D	1/70	282	100.6	65	24.5	70	26.7	16	3.4
	1/100	240	260.6	44	52.0	45	63.2	15	8.8
	1/120	223	405.1	24	56.9	24	84.5	NC	NA

Table 20. Numerical results for DCC1 problems with cf = 6 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	132	79.7	56	36.7	56	37.0	27	6.3
	1/1000	117	112.4	55	56.1	54	55.2	37	14.1
	1/1200	111	152.5	55	80.9	53	77.9	37	18.2
3D	1/70	261	88.9	90	32.2	88	31.5	16	3.4
	1/100	256	250.4	41	47.1	41	51.3	15	8.8
	1/120	225	384.6	26	61.7	27	78.9	NC	NA

Table 21. Numerical results for DCC1 problem with cf = 7 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	184	112.3	67	42.7	65	42.0	27	6.3
	1/1000	173	162.6	69	68.1	67	67.5	37	14.1
	1/1200	397	530.4	67	97.6	64	95.2	37	18.2
3D	1/70	351	117.5	109	32.6	109	36.3	16	3.4
	1/100	301	296.2	80	82.8	78	81.8	15	8.8
	1/120	257	440.0	29	69.7	29	74.5	NC	NA

Finally, in Table 24, we show some experiments with the Florida matrix market problems. We fixed the coarse grid size to be 4096. In general, for most of the problems, we find that the two-grid method is faster compared to AGMG, exception being, torsion1, obstclae, jnlbrng1, minsurfo, qa8fk, and shallow_water, where AGMG is about five times faster. For rest of the problems, GMG

Table 22. Numerical results for DCC1 problem with cf = 8 using GMRES(30).

		GMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	h	Its	Time	Its	Time	Its	Time	Its	Time
2D	1/800	266	171.3	82	51.5	83	54.2	27	6.3
	1/1000	229	225.6	84	81.0	83	79.8	37	14.1
	1/1200	218	297.7	81	112.4	78	107.8	37	18.2
3D	1/70	502	161.0	123	41.0	115	37.9	16	3.4
	1/100	449	445.9	105	102.4	104	101.8	15	8.8
	1/120	294	517.2	57	109.8	57	112.2	NC	NA

Table 23. Comparison of exact and inexact coarse operators.

Problem	$1/h$	$\frac{\|LU-\tilde{L}\tilde{U}\|}{\|LU\|}$ for NO	$\frac{\|LU-\tilde{L}\tilde{U}\|}{\|LU\|}$ for ND
DC1, 3D	1/30	$7.5e^{-5}$	$7.8e^{-5}$
	1/40	$8.3e^{-5}$	$7.2e^{-5}$
	1/50	$7.0e^{-5}$	$1.1e^{-4}$

Here, NO stands for natural ordering and ND stands for nested dissection ordering.

Table 24. Numerical results for Florida matrix market collection.

		GMG-ND		GMG-NO		AGMG	
Matrix	N_c	Its	Time	Its	Time	Its	Time
gyto_m	4096	111	5.2	137	7.0	> 600	NA
bodyy4	4096	54	1.8	19	0.4	> 600	NA
nd6k	4096	NC	NA	89	21.3	MEM	NA
bodyy5	4096	115	4.6	30	0.7	–	–
wathen100	4096	12	1.4	9	0.7	11	4.1
wathen120	4096	12	1.3	9	0.9	11	35.0
torsion1	4096	13	0.99	9	0.6	6	0.1
obstclae	4096	13	0.9	9	0.6	6	0.1
jnlbrng1	4096	22	1.5	11	0.7	11	0.1
minsurfo	4096	16	1.0	11	0.6	8	0.1
gridgena	4096	310	81.4	189	34.0	> 600	NA
crankseg_1	4096	60	22.3	76	26.2	334	69.1
qa8fk	4096	11	4.9	12	3.6	15	1.3
cfd1	4096	145	45.1	127	32.0	MEM	NA
finan512	4096	7	1.8	7	1.2	4	106.0
shallow_water1	4096	6	1.3	6	0.8	4	0.1
2cubes_sphere	4096	6	3.9	6	2.5	7	8.6
Thermal_TC	4096	6	2.03	7	1.4	MEM	NA
Thermal_TK	4096	22	3.6	23	3.5	80	5.0
G2_circuit	4096	32	7.7	24	4.2	91	5.9
bcsstk18	4096	232	11.6	111	3.7	> 600	NA
cbuckle	4096	41	2.3	62	3.0	493	18.7
Pres_Poisson	4096	21	1.7	24	1.7	24	16.8

methods show more robustness compared to AGMG. Comparing GMG-NO to GMG-ND, we find that GMG-NO converges faster with few exceptions. In Table 25, we present the numerical results with cf = 2.5. A detailed investigation of the best coarse grid size for these problems deserves more effort and detailed study.

Table 25. Numerical results for Florida matrix market collection with cf = 2.5.

	GMG-ND		EGMG-ND		GMG-NO		EGMG-NO		AGMG	
Matrix	Its	Time	Its	Time	Its	Time	Its	Time	Its	Time
gyto_m	271	4.1	319	4.6	465	6.9	562	7.3	NC	NA
bodyy4	56	0.9	56	0.9	19	0.3	19	0.4	NC	NA
nd6k	NC	NA	NC	NA	NC	NA	NC	NA	ME	NA
bodyy5	135	1.8	137	1.8	31	0.6	31	0.6	1	245.4
wathen100	12	1.0	12	1.1	9	0.6	9	0.7	11	4.3
wathen120	12	1.2	12	1.3	9	0.8	9	0.8	11	7.0
torsion1	14	0.9	14	1.0	10	0.5	10	0.6	6	0.1
obstclae	14	0.9	14	1.0	10	0.5	10	0.6	6	0.1
jnlbrng1	25	1.3	24	1.3	12	0.6	12	0.6	11	0.2
minsurfo	18	1.0	18	1.2	11	0.5	12	0.6	7	0.2
gridgena	NC	NA	NC	NA	307	10.3	308	9.2	NC	NA
crankseg_1	74	21.4	109	31.6	99	25.5	87	29.0	468	96.0
qa8fk	12	4.9	11	5.8	11	3.6	10	4.4	15	1.3
cfd1	439	37.0	442	38.3	259	23.0	255	30.5	ME	NA
finan512	6	1.8	6	1.9	7	1.2	7	1.3	4	106.0
shallow_water1	6	1.4	6	2.3	6	0.9	6	1.7	4	0.1
2cubes_sphere	6	4.4	7	8.6	5	3.0	5	15.2	7	8.6
Thermal_TC	6	2.2	6	2.5	6	1.5	6	1.9	ME	NA
Thermal_TK	NC	NA	NC	NA	NC	NA	NC	NA	NC	NA
G2_circuit	28	7.3	21	7.7	21	4.7	15	5.0	74	4.8
bcsstk18	NC	NA	NC	NA	166	1.9	169	2.0	F1	NA
cbuckle	54	1.5	54	1.5	247	4.0	198	3.7	F1	NA
Pres_Poisson	24	1.0	22	1.0	26	1.0	26	1.0	24	21.2

4. Conclusion

We have proposed a two-grid approach GMG with following ingredients

- Coarse grid based on graph matching.
- ILU(0) is the smoother with natural or nested dissection reordering.
- Coarse grid equation is solved inexactly.

We compared our approach with the classical AGMG scheme based on strength of connection. On comparison, we found that the new strategy (GMG) is more robust with a very modest coarse grid size which is further solved cheaply by performing an inexact solve. The proposed approach is easy to implement, yet robust for problems with highly heterogeneous coefficients. The two-grid approach is also compared with filtering preconditioners and basic properties are outlined.

We have tried only the sequential version of our method, in future, we would like to implement the method in parallel with a parallel inexact coarse grid solver.

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