1 FM with General Message

To derive the equation for an FM wave, we have 3 starting points:

1. The idea of frequency modulation is that the instantaneous frequency $f_i(t)$ is varied linearly with the baseband signal $m(t)$. Thus for a general message $m(t)$ the idea of FM is:

   $$f_i(t) = f_c + k_f m(t)$$

   where the constant k_f represents the frequency sensitivity of the modulator, expressed in hertz per volt.

2. Recall that a general signal is written:

   $$s(t) = a(t)\cos\theta(t) = a(t)cos[2\pi f_c t + \phi(t)]$$

   where $\theta(t) = 2\pi f_c t + \phi(t)$ has a linear variation at rate f_c and a time varying part $\phi(t)$

3. Recall the instantaneous frequency of a signal is:

   $$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

Combining these 3 starting points, we can write:

$$\theta(t) = 2\pi\int_0^t f_i(\alpha)d\alpha = 2\pi\int_0^t [f_c + k_f m(\alpha)]d\alpha$$

$$= 2\pi f_c t + k_f \int_0^t m(\alpha)d\alpha$$

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right]$$

The FM signal $s(t)$ can be written in standard IQ format:

$$s(t) = i(t)\cos 2\pi f_c t - q(t)sin2\pi f_c t$$

$$\begin{aligned}
s(t) &= \mathrm{Re}\{a(t)e^{j\phi(t)}e^{j2\pi f_c t}\} = \\
&\mathrm{Re}\{[I(t) + jQ(t)][\cos 2\pi f_c t + jsin2\pi f_c t]\} \\
&= I(t)\cos 2\pi f_c t - Q(t)sin2\pi f_c t \\
&= a(t)\cos[2\pi f_c t + \phi(t)]
\end{aligned}$$

Thus for FM:

$$a(t) = A_c$$
$$\phi(t) = 2\pi k_f \int_0^t m(\alpha)d\alpha$$
$$I(t) = A_c \cos 2\pi k_f \int_0^t m(\alpha)d\alpha$$
$$Q(t) = A_c \sin 2\pi k_f \int_0^t m(\alpha)d\alpha$$

2 FM with Sinusoidal Message

Now consider a sinusoidal modulating wave defined by

$$m(t) = A_m \cos(2\pi f_m t)$$

We re-derive the equation for the FM wave with the same 3 starting points.

1. The idea of FM is that the instantaneous frequency of the resulting FM wave equals:

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

   where $\Delta f = k_f A_m$. Thus the message causes the frequency to vary above and below the carrier frequency f_c. The quantity Δf is called the <u>frequency deviation</u>.

2. A general signal:

$$s(t) = a(t) \cos \theta(t)$$

3. The instantaneous frequency of a signal is:

$$f_i(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt}$$

Combining these 3 starting points, we can write:

$$\begin{aligned} \theta(t) &= 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta \sin(2\pi f_m t) \end{aligned}$$

Where $\beta = \frac{k_f A_m}{f_m} = \frac{\Delta f}{f_m}$ is called the <u>modulation index</u> of the FM wave.

Thus the FM wave itself is given In terms of β by:

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

If β is small we have <u>narrowband FM</u> (NBFM) and if β is large (compared to one radian) we have <u>wideband FM</u>.

Thus for a sinusoidal modulating wave $m(t) = A_m \cos(2\pi f_m t)$ the FM wave can be written:

$$s(t) = I(t)\cos 2\pi f_c t - Q(t) sin 2\pi f_c t = a(t)\cos[2\pi f_c t + \phi(t)]$$

where:

$$a(t) = A_c$$
$$\phi(t) = \beta sin 2\pi f_m t$$
$$i(t) = A_c \cos \beta sin 2\pi f_m t$$
$$q(t) = A_c \sin \beta sin 2\pi f_m t$$

3 FM with Sinusoidal Message and $\beta << 1$

When β is small compared to one radian, i.e. $\beta << 1$, the FM wave may be approximated using $\cos x \approx 1, \sin x \approx x$ for $x << 1$ to obtain

$$s(t) \simeq A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)$$

Thus for $\beta << 1$

$$i(t) = A_c$$
$$q(t) = \beta A_c sin 2\pi f_m t$$

Comparing the waveforms of NBFM and AM, we observe that a NBFM signal wave requires essentially the same transmission bandwidth (i.e., $2f_m$) as an AM wave.

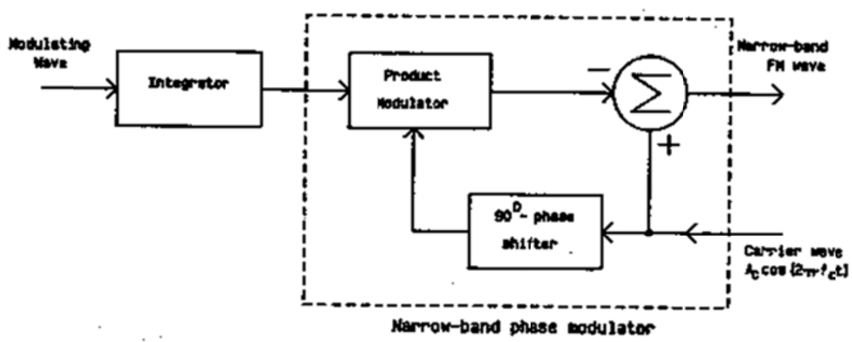


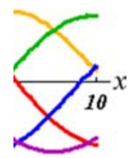
Figure 1 Narrow-band Phase Modulator

4 Power Spectrum and Bandwidth of an FM Signal

Expanding $s(t)$ in the form of a Fourier series, we get:

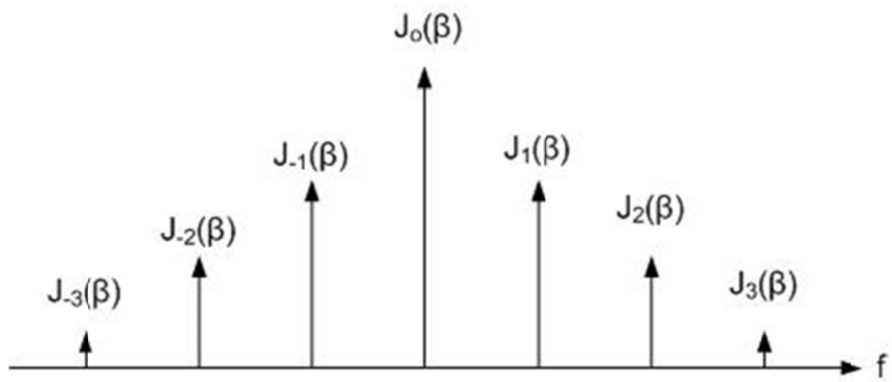
$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$$

where $J_n(\beta)$ is the Bessel function of the first kind of order *n.*

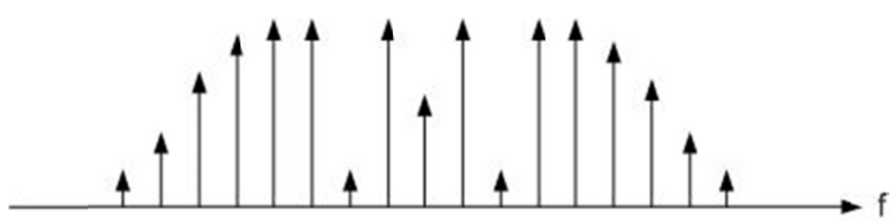


The discrete spectrum of the FM wave is obtained as

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$



Bessel Function value [$J_n(\beta)$] defining the FM spectrum depends upon β



Example of FM spectrum with relatively large β

To visualize the spectrum $S(f)$, we note that:

$$J_n \beta = (-1)^n J_{-n}(\beta)$$

and:

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

Also, for small values of β, we have:

$$J_0(\beta) \simeq 1, \ J_1(\beta) \simeq \frac{\beta}{2}$$

and:

$$J_n(\beta) \simeq 0, \quad n > 1$$

The average power of an FM wave developed across a 1 ohm resistor is given by:

$$P = 1/2 \ A_c^2 \cdot \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

For practical purposes, the bandwidth of the FM wave corresponds to the bandwidth containing 98% of the signal power.

The effective bandwidth of the FM signal is approximately given by Carson's formula:

$$B = 2(1+\beta) f_m$$

5 Frequency Demodulation

Frequency demodulation extracts the original message wave from the frequency-modulated wave. We describe two basic devices, the analog-like frequency discriminator and a digital FM demodulator.

6 Frequency Discriminator

How to extract the message from an FM signal? We consider four intuitive approaches.

Intuition 1: Differentiation

We know that from the idea of FM $f_i(t) = f_c + k_f m(t) = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$ so that $m(t) = \frac{1}{2\pi k_f}\frac{d\phi(t)}{dt}$

Since $s(t) = a(t) cos[2\pi f_c t + \phi(t)]$ with $a(t) = A_c$ (constant) we see that if we differentiate $s(t)$ we will get a term in $\frac{ds(t)}{dt}$ that looks like $\frac{d\phi(t)}{dt}$

Thus we differentiate $s(t)$ as a first step to extract the message $m(t)$. Assuming A_c is constant we have:

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha) d\alpha\right]$$

$$\frac{ds(t)}{dt} = -A_c\left[2\pi f_c + 2\pi k_f m(t)\right]\sin\left[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt\right]$$

This expression is in the form of an AM signal

$$2\pi f_c A_c[1+k_a m(t)]\cos[2\pi f_c t+\phi(t)] = a(t)\cos[2\pi f_c t+\phi(t)]$$

with envelope $2\pi A_c f_c\left[1+\frac{k_j}{f_c}m(t)\right]$ with $k_a = k_f / f_c$ and phase $2\pi k_f \int_0^t m(\alpha)d\alpha - \pi/2$

The resulting AM signal can be demodulated by an envelope detector to obtain DC plus the message. The envelope detector ignores the phase.

Note that if the FM signal has $a(t) \neq A_c$ is not constant (e.g. due to channel fading or noise), then differentiation will not work, since for this case

$$\frac{ds(t)}{dt} = -a(t)\left[2\pi f_c + 2\pi k_f m(t)\right]\sin\left[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt\right] + \frac{da(t)}{dt}\cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right]$$

To make $a(t) = A_c$ is constant, we can use a limiter (e.g. with back to back diodes) ahead of the discriminator. The limiter limits (clips) the input signal so that it is of constant amplitude. The output of the limiter looks like a square wave with changing frequency, and will contain harmonics at odd multiples of f_c. The limiter must be followed by a bandpass filter at f_c to restore the signal $s(t) = A_c\cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right]$

A block diagram of this FM demodulator is

$$s(t) \rightarrow \text{ LIMITER } \rightarrow \text{ BPF } \rightarrow \text{ DISCRIMINATOR } \rightarrow \text{ ENVELOPE DETECTOR } \rightarrow \text{ DC BLOCK} \rightarrow m(t)$$

Intuition 2: Linear Amplitude vs. Frequency Characteristic

We know that from the idea of FM $f_i(t) = f_c + k_f m(t)$. If we have a circuit that has output amplitude that increases linearly with frequency, then the circuit output amplitude will vary in step with the message.

One such circuit is the ideal differentiator, with a transfer function given by $H(f) = j2\pi f$.

Proof: If $s(t) \rightarrow S(f)$ then $ds(t)/dt \rightarrow j2\pi f S(f)$

The action of an ideal differentiator (Figure 3(a)) can be approximated by any device whose magnitude transfer function is reasonably linear, within the range of frequencies of interest. In Figure 3(b) an RL circuit approximation to a differentiator is used followed by an envelope detector. A bandpass version of this circuit is shown in Figure 3(c). These discriminators are known as slope detectors. A more linear response can be obtained by taking the difference between two bandpass magnitude responses, as is done by the balanced discriminator shown in Figure 3(d).

In all cases (a)-(d), a limiter and bandpass filter is required ahead of the discriminator

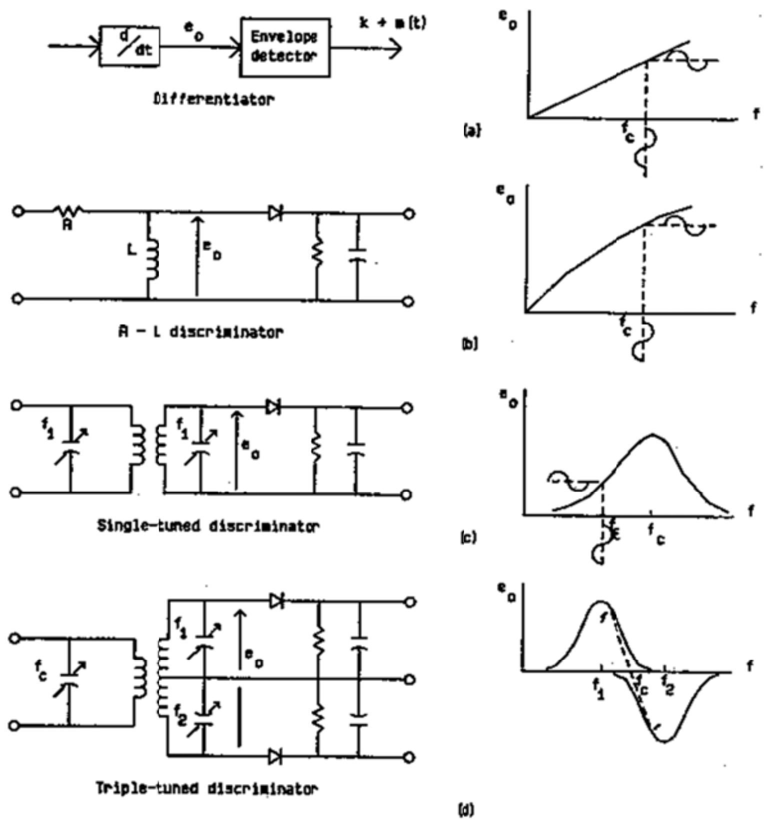


Figure 3 FM Detectors

Intuition 3: Zero Crossing Counter

The message information is contained in the time (location) of the zero crossings, and the amplitude can be ignored.

Intuition 4: Phase Locked Loop

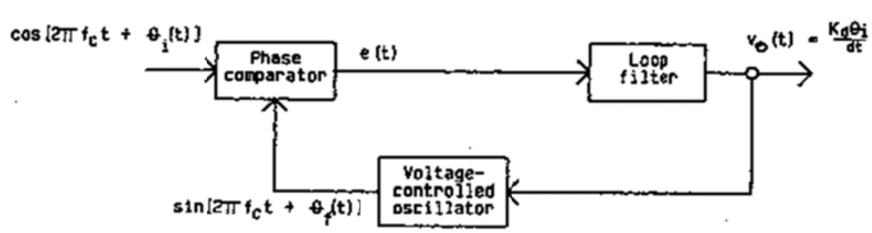


Figure 4 Phase-Locked Loop

This circuit uses ideas from the Costas loop receiver used for DSB. The circuit comprises a phase comparator (multiplier), low pass (loop) filter and voltage controlled oscillator (equivalent to an FM modulator).

With signal input $s(t)=\cos[2\pi f_c t+\theta_i(t)]$ and FM modulator (VCO) output $\sin[2\pi f_c t+\theta_f(t)]$, the phase comparator (multiplier) output signal $e(t)=k_c\left[\theta_i(t)-\theta_f(t)\right]$ plus double frequency terms. With high gain in the loop filter, $e(t)\approx 0$ and the VCO output frequency is the same as the input signal frequency. Thus the VCO input must be the same as the message. The loop filter output voltage is proportional to the instantaneous frequency of the input, and FM demodulation is achieved.

7 Digital FM Demodulator

A digital FM demodulator starts with the I and Q outputs of a general IQ receiver. Recall for an FM signal:

$$s(t)=A_c\cos\left[2\pi f_c t+2\pi k_f\int_0^t m(\alpha)d\alpha\right]$$

$$I(t)=A_c\cos 2\pi k_f\int_0^t m(\alpha)d\alpha$$

$$Q(t)=A_c\sin 2\pi k_f\int_0^t m(\alpha)d\alpha$$

To extract $m(t)$ from $I(t),Q(t)$ we show two methods that can be implemented in software.

8 I(t), Q(t) as Real Signals

Formula: $$m(t)=\frac{d}{dt}arctan\left\{\frac{Q(t)}{I(t)}\right\}$$

Block diagram: $I(t),Q(t)\rightarrow DIVIDE\rightarrow ARCTAN\rightarrow d/dt\rightarrow m(t)$

Proof:

$$\frac{d}{dt}\arctan\left\{\frac{Q(t)}{I(t)}\right\}=\frac{d}{dt}\arctan\frac{A_c\sin 2\pi k_f\int_0^t m(\alpha)d\alpha}{A_c\cos 2\pi k_f\int_0^t m(\alpha)d\alpha}$$

$$=\frac{d}{dt}\arctan\{\tan 2\pi k_f\int_0^t m(\alpha)d\alpha\}=$$

$$=\frac{d}{dt}2\pi k_f\int_0^t m(\alpha)d\alpha=2\pi k_f m(t)$$

9 I(t) + jQ(t) as a Complex Signal

$$s(t)=\mathrm{Re}\{a(t)e^{j\phi(t)}e^{j2\pi f_c t}\}=\mathrm{Re}\{[I(t)+jQ(t)]e^{j2\pi f_c t}\}=\mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\}$$

Formula:

$m(t) = \arg[\tilde{s}(t-1)\tilde{s}*(t)]$

Where:

$(t-1) \to z^{-1}$

represents one sample delay

Block diagram:

$$\tilde{s}(t) \to \tilde{s}(t-1), \tilde{s}^{*}(t) \to MULTIPLY \to ARG$$

Proof:

$$\arg[s(t-1)s*(t)] = \arg[a(t-1)e^{j\phi(t-1)}a(t)e^{-j\phi(t)}]$$

$$= \phi(t-1) - \phi(t) \approx \frac{d\phi}{dt} = 2\pi k_f m(t)$$