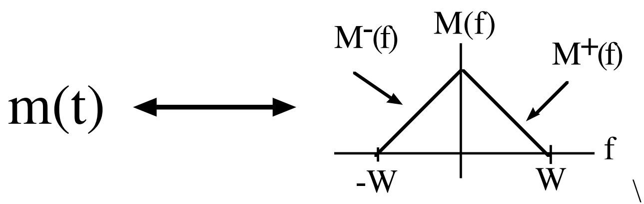
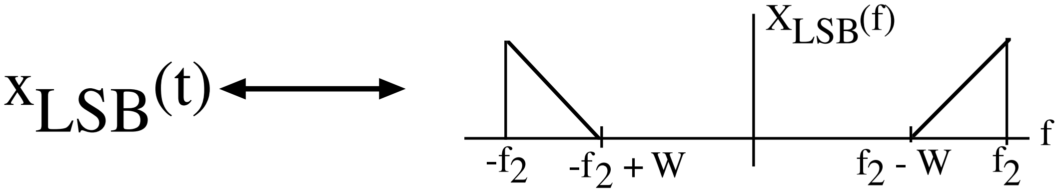
Weaver's SSB Demodulator

The GNU radio SSB demodulator in *ssb_rcv_file.py* uses Weaver's method to demodulate a SSB signal. The operation of this demodulator is described here along with an analysis of the signals at each point in the system. Additionally, the spectrum of an actual LSB signal is shown as it passes though the system.

In this example we assume our theoretical baseband signal, m(t) has a Fourier transform as shown below:



The actual baseband signal used in this example has a maximum frequency component W = 3KHz and is modulated using LSB up to a carrier frequency of f_2 = 50.353 MHz. The spectrum of the resulting LSB signal is:



In the time domain, we express the LSB signal as:

$$x_{LSB}(t) = m(t)\cos\omega_2 t + \hat{m}(t)\sin\omega_2 t$$

where $\hat{m}(t)$ is the Hilbert transform of m(t). As shown above this signal has a Fourier transform:

$$X_{LSB}(f) = M^-(f - f_2) + M^+(f + f_2)$$

<u>USRP Data</u>

The USRP samples the incoming LSB signal and uses an image reject mixer (multiplies the incoming signal by $e^{j\omega_1 t}$) to shift the signal by f_1 = 50.3 MHz. The output of the complex mixer is:

$$x_{LSB}(t)e^{j\omega_1 t} = x_{LSB}(t)\left[\cos\omega_1 t + j\sin\omega_1 t\right]$$

Thus, the Fourier transform of the mixer output is:

$$X_{LSB}(f) * \left[FT(\cos\omega_1 t) + jFT(\sin\omega_1 t)\right]$$

$$= \left[M^-(f-f_2) + M^+(f+f_2)\right] * \left[\left(\frac{1}{2}\right)\left(\delta(f-f_1)+\delta(f+f_1)\right) + j\left(\frac{j}{2}\right)\left(\delta(f+f_1)-\delta(f-f_1)\right)\right]$$

$$= \left(\frac{1}{2}\right)\left[M^-(f-f_1-f_2) + M^-(f+f_1-f_2) + M^+(f-f_1+f_2) + M^+(f+f_1+f_2)\right]$$

$$-\left(\frac{1}{2}\right)\left[M^-(f-f_2+f_1) - M^-(f-f_1-f_2) + M^+(f+f_1+f_2) - M^+(f-f_1+f_2)\right]$$

Note that the entire spectrum coming into the USRP is shifted by f_1 = 50.3 MHz. Thus, our signal of interest has a carrier frequency f_c = 53 KHz = $f_2 - f_1$ coming out of the USRP. Using this relationship to eliminate f_2 from the expression above:

$$X_{LSB}(f) * \left[FT(\cos\omega_1 t) + jFT(\sin\omega_1 t)\right]$$

$$= \left(\frac{1}{2}\right)\left[M^-(f-2f_1-f_c) + M^-(f-f_c) + M^+(f+f_c) + M^+(f+2f_1+f_c)\right]$$

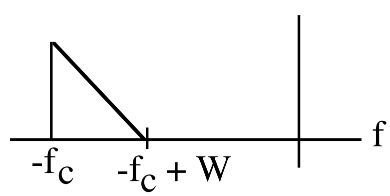
$$-\left(\frac{1}{2}\right)\left[M^-(f-f_c) - M^-(f-2f_1-f_c) + M^+(f+2f_1+f_c) - M^+(f-f_c)\right]$$

$$= \left[M^-(f-2f_1-f_c) + M^+(f+f_c)\right]$$

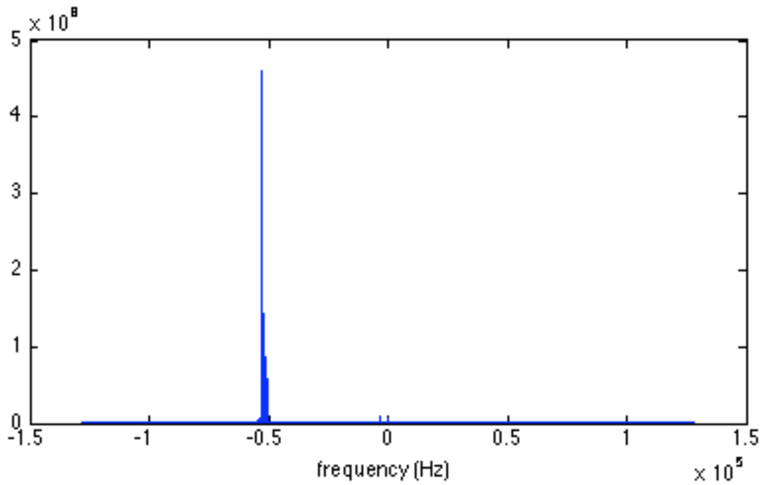
and after the low pass filter that is part of the DDC, we have a USRP output of:

$$M^+(f+f_c)$$

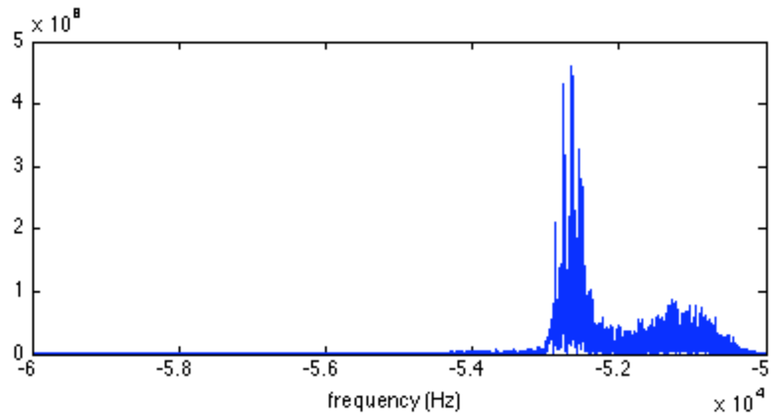
or



Note that due to the use of complex sampling we do not have the symmetry normally observed with real sampling. As the receiver was running a few seconds of this USRP output data was captured to a file. Using MATLAB to take the FFT of these samples we have the corresponding spectrum:



or expanding the frequency axis:



This is the spectrum predicted above. Similarly, we can determine the output of the USRP in the time domain. This information will be needed later. As stated above, in the time domain, we express the LSB signal as:

$$x_{LSB}(t) = m(t)\cos\omega_2 t + \hat{m}(t)\sin\omega_2 t$$

Thus, after complex mixing in the USRP:

$$x_{LSB}(t)\left[\cos\omega_1 t + j\sin\omega_1 t\right] = \left[m(t)\cos\omega_2 t + \hat{m}(t)\sin\omega_2 t\right]\left[\cos\omega_1 t + j\sin\omega_1 t\right]$$
$$= m(t)\cos\omega_1 t\cos\omega_2 t + \hat{m}(t)\cos\omega_1 t\sin\omega_2 t + jm(t)\sin\omega_1 t\cos\omega_2 t + j\hat{m}(t)\sin\omega_1 t\sin\omega_2 t$$
$$= \frac{m(t)}{2}\left[\cos\left(\omega_1 - \omega_2\right)t + \cos\left(\omega_1 + \omega_2\right)t\right] + \frac{\hat{m}(t)}{2}\left[\sin\left(\omega_1 + \omega_2\right)t - \sin\left(\omega_1 - \omega_2\right)t\right]$$
$$+ \frac{jm(t)}{2}\left[\sin\left(\omega_1 + \omega_2\right)t - \sin\left(\omega_2 - \omega_1\right)t\right] + \frac{j\hat{m}(t)}{2}\left[\cos\left(\omega_1 - \omega_2\right)t - \cos\left(\omega_1 + \omega_2\right)t\right]$$

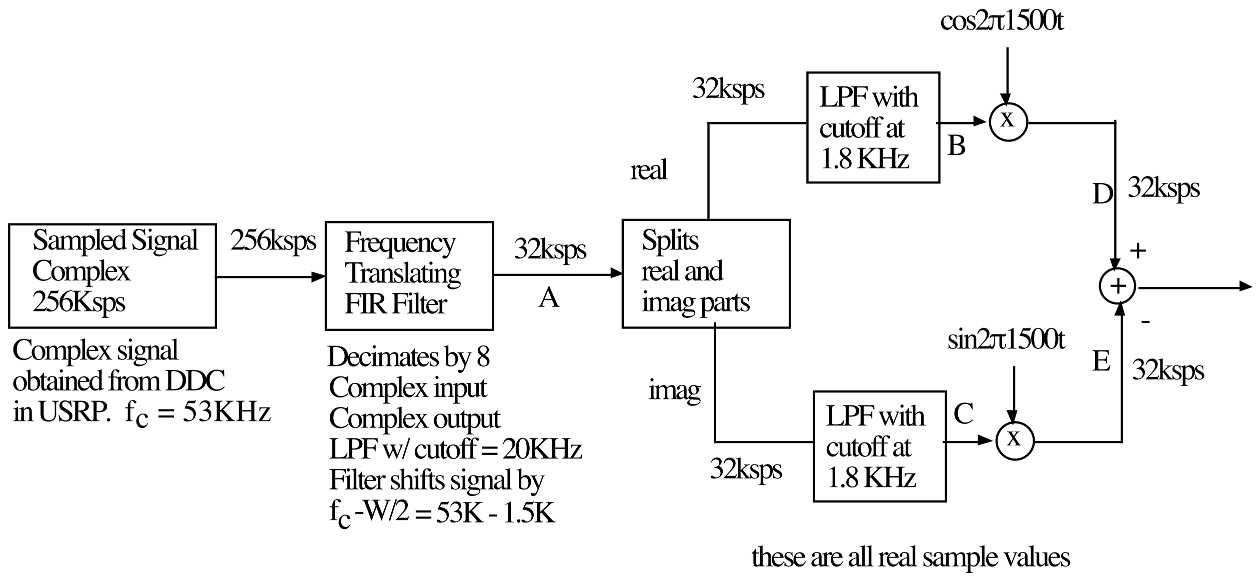
Substituting $f_c = f_2 - f_1$ and applying the low pass filter we have a USRP output of:

$$\frac{1}{2}\left[m(t)\cos\omega_c t + \hat{m}(t)\sin\omega_c t\right] + \frac{j}{2}\left[-m(t)\sin\omega_c t + \hat{m}(t)\cos\omega_c t\right]$$

Show that this time domain signal has the same spectrum as found above.

Block Diagram of GNU Radio Receiver

The complex samples are processed according to the block diagram shown below.



Frequency Translating Filter

The frequency translating filter uses complex mixing to shift the input signal by f = W/2. After the mixing (frequency translation) we have:

$$\left\{\frac{1}{2}\left[m(t)\cos\omega_c t+\hat{m}(t)\sin\omega_c t\right]+\frac{j}{2}\left[-m(t)\sin\omega_c t+\hat{m}(t)\cos\omega_c t\right]\right\}e^{j2\pi\left(f_c-\frac{W}{2}\right)t}$$

$$=\left\{\frac{1}{2}\left[m(t)\cos\omega_c t+\hat{m}(t)\sin\omega_c t\right]+\frac{j}{2}\left[-m(t)\sin\omega_c t+\hat{m}(t)\cos\omega_c t\right]\right\}*$$

$$\left[\cos 2\pi\left(f_c-\frac{W}{2}\right)t+j\sin 2\pi\left(f_c-\frac{W}{2}\right)t\right]$$

$$=\frac{1}{4}\left[m(t)\cos\omega_c t\cos 2\pi\left(f_c-\frac{W}{2}\right)t+jm(t)\cos\omega_c t\sin 2\pi\left(f_c-\frac{W}{2}\right)t\right.$$

$$+\hat{m}(t)\sin\omega_c t\cos 2\pi\left(f_c-\frac{W}{2}\right)t+j\hat{m}(t)\sin\omega_c t\sin 2\pi\left(f_c-\frac{W}{2}\right)t$$

$$-jm(t)\sin\omega_c t\cos 2\pi\left(f_c-\frac{W}{2}\right)t+m(t)\sin\omega_c t\sin 2\pi\left(f_c-\frac{W}{2}\right)t$$

$$\left.+j\hat{m}(t)\cos\omega_c t\cos 2\pi\left(f_c-\frac{W}{2}\right)t-\hat{m}(t)\cos\omega_c t\sin 2\pi\left(f_c-\frac{W}{2}\right)t\right]$$

Applying the appropriate trigonometric identities and filtering out the high frequency terms we are left with the following signal at point A in the diagram:

$$\frac{1}{2}\left[m(t)\cos 2\pi\frac{W}{2}t+\hat{m}(t)\sin 2\pi\frac{W}{2}t\right]+\frac{j}{2}\left[\hat{m}(t)\cos 2\pi\frac{W}{2}t-m(t)\sin 2\pi\frac{W}{2}t\right]$$

Taking the Fourier transform of this signal we have:

$$\frac{1}{4}\left\{\left[M^-(f)+M^+(f)\right]*\left[\delta\left(f-\frac{W}{2}\right)+\delta\left(f+\frac{W}{2}\right)\right]+\left[jM^-(f)-jM^+(f)\right]*\left[j\delta\left(f+\frac{W}{2}\right)-\delta\left(f-\frac{W}{2}\right)\right]\right\}$$

$$+\frac{j}{4}\left\{\left[jM^-(f)-jM^+(f)\right]*\left[\delta\left(f-\frac{W}{2}\right)+\delta\left(f+\frac{W}{2}\right)\right]-\left[M^-(f)+M^+(f)\right]*\left[j\delta\left(f+\frac{W}{2}\right)-\delta\left(f-\frac{W}{2}\right)\right]\right\}$$

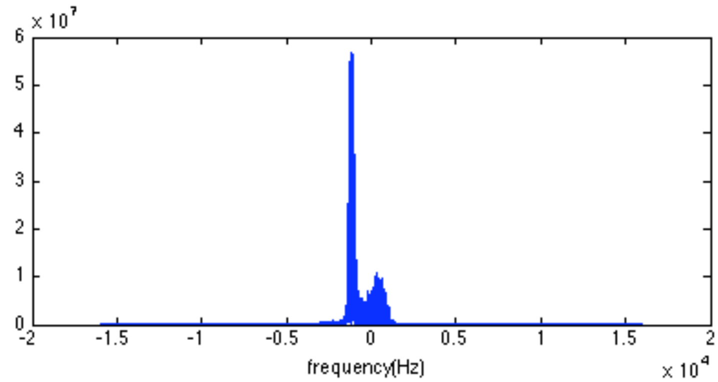
$$=\frac{1}{4}\left[M^-\left(f-\frac{W}{2}\right)+M^+\left(f-\frac{W}{2}\right)+M^-\left(f+\frac{W}{2}\right)+M^+\left(f+\frac{W}{2}\right)+M^+\left(f+\frac{W}{2}\right)-M^+\left(f-\frac{W}{2}\right)\right.$$

$$\left.-M^-\left(f+\frac{W}{2}\right)+M^-\left(f-\frac{W}{2}\right)\right]+\frac{j}{4}\left[-jM^+\left(f-\frac{W}{2}\right)-jM^+\left(f+\frac{W}{2}\right)+jM^-\left(f-\frac{W}{2}\right)+jM^-\left(f+\frac{W}{2}\right)\right.$$

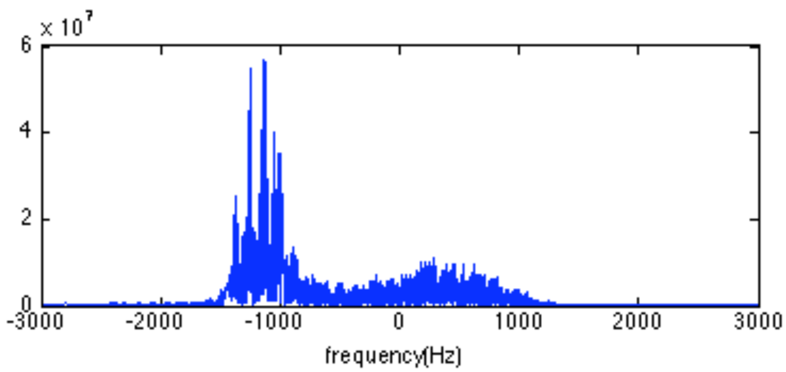
$$\left.-jM^+\left(f+\frac{W}{2}\right)+jM^+\left(f-\frac{W}{2}\right)-jM^-\left(f+\frac{W}{2}\right)+jM^-\left(f-\frac{W}{2}\right)\right]$$

$$=\frac{1}{2}\left[M^+\left(f+\frac{W}{2}\right)+M^-\left(f-\frac{W}{2}\right)-M^-\left(f-\frac{W}{2}\right)+M^+\left(f+\frac{W}{2}\right)\right]=M^+\left(f+\frac{W}{2}\right)$$

Using data captured from the output of the frequency translating filter with the actual signal we have an FFT of:



or on an expanded scale:



which does in fact correspond to the expected signal.

<u>Top Low Pass Filter</u>

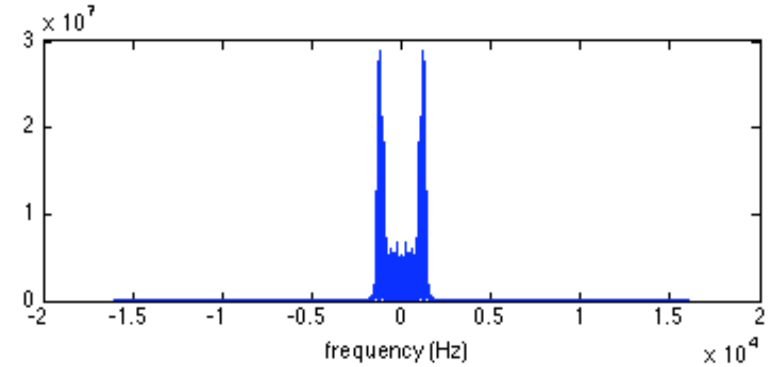
After the frequency translating filter the real and imaginary components of the signal are split and passed through separate low pass filters to eliminate any components above W/2 . Since these are no longer complex signals the Fourier transform now includes both the positive and negative components. The real signal leaving the top low pass filter is:

$$\frac{1}{2}\left[m(t)\cos 2\pi\frac{W}{2}t + \hat{m}(t)\sin 2\pi\frac{W}{2}t\right]$$

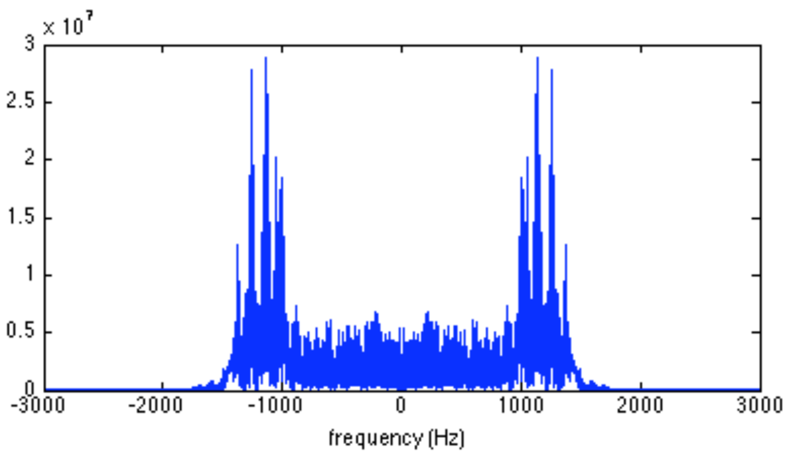
which is found to have a Fourier transform:

$$\frac{1}{2}\left[M^{+}\left(f+\frac{W}{2}\right)+M^{-}\left(f-\frac{W}{2}\right)\right]$$

This corresponds to the FFT of the actual signal at point B in the system:



or on an expanded scale:



Bottom Low Pass Filter

The imaginary signal leaving the bottom low pass filter is:

$$\frac{1}{2}\left[\hat{m}(t)\cos 2\pi\frac{W}{2}t - m(t)\sin 2\pi\frac{W}{2}t\right]$$

which is found to have a Fourier transform:

$$\frac{1}{2}\left[M^{+}\left(f+\frac{W}{2}\right)-M^{-}\left(f-\frac{W}{2}\right)\right]$$

This is the term at point C in the diagram.

Top Mixer Output

The mixer in the top branch multiplies the filtered I component:

$$\frac{1}{2}\left[m(t)\cos 2\pi\frac{W}{2}t+\hat{m}(t)\sin 2\pi\frac{W}{2}t\right]$$

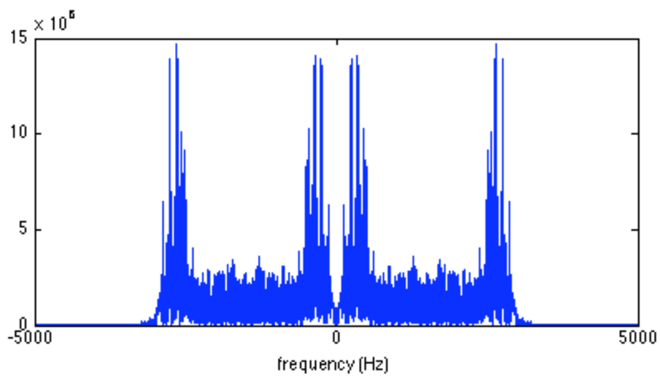
with a cosine at W/2. Previously the transform of the I component was found to:

$$\frac{1}{2}\left[M^{+}\left(f+\frac{W}{2}\right)+M^{-}\left(f-\frac{W}{2}\right)\right]$$

Thus, the output of the mixer is:

$$\frac{1}{4}\left[M^{+}\left(f+\frac{W}{2}\right)+M^{-}\left(f-\frac{W}{2}\right)\right]*\left[\delta\left(f+\frac{W}{2}\right)+\delta\left(f-\frac{W}{2}\right)\right]$$
$$=\frac{1}{4}\left[M^{+}(f)+M^{-}(f)+M^{+}(f+W)+M^{-}(f-W)\right]$$

The FFT of the data collected at point D is shown below and found to be in this form.



Bottom Mixer Output

The mixer in the bottom branch multiplies the filtered Q component:

$$\frac{1}{2}\left[\hat{m}(t)\cos 2\pi\frac{W}{2}t-m(t)\sin 2\pi\frac{W}{2}t\right]$$

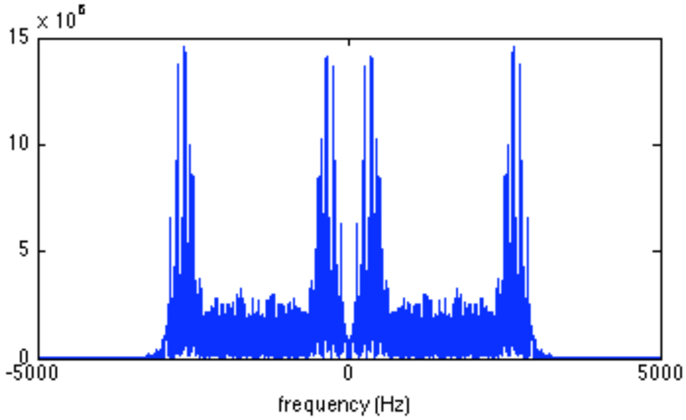
with a sine at W/2. Previously the transform of the Q component was found to:

$$\frac{1}{2}\left[M^{+}\left(f+\frac{W}{2}\right)-M^{-}\left(f-\frac{W}{2}\right)\right]$$

Thus, the output of the mixer is:

$$\frac{1}{4}\left[M^{+}\left(f+\frac{W}{2}\right)-M^{-}\left(f-\frac{W}{2}\right)\right]*\left[j\delta\left(f+\frac{W}{2}\right)-j\delta\left(f-\frac{W}{2}\right)\right]$$
$$=\frac{1}{4}\left[-jM^{+}(f)-jM^{-}(f)+jM^{+}(f+W)+jM^{-}(f-W)\right]$$

The FFT of the data collected at point E is shown below and found to be in this form.



Final Output

Keep in mind that the spectra shown in the previous two figures include amplitude only and not phase. Some of the components are negative. In the final block the term at E is subtracted from the term at D resulting in the correct baseband signal shown below.