FM Transmitter and Receiver theory

A: FM with General Message

To derive the equation for an FM wave, we have 3 starting points:

1. The idea of frequency modulation is that the instantaneous frequency $f_i(t)$ is varied linearly with the baseband signal $m(t)$. Thus for a general message $m(t)$ the idea of FM is:

$$f_i(t) = f_c + k_f m(t) \tag{1}$$

   where the constant k_f represents the frequency sensitivity of the modulator, expressed in hertz per volt.

2. Recall that a general signal is written:

$$s(t) = a(t)\cos\theta(t) = a(t)cos[2\pi f_c t + \phi(t)] \tag{2}$$

   where $\theta(t) = 2\pi f_c t + \phi(t)$ has a linear variation at rate f_c and a time varying part $\phi(t)$

3. Recall the instantaneous frequency of a signal is:

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt} \tag{3}$$

Combining these 3 starting points, we can write:

$$\theta(t) = 2\pi\int_0^t f_i(\alpha)d\alpha = 2\pi\int_0^t \left[f_c + k_f m(\alpha)\right]d\alpha = 2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha \tag{4}$$

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right] \tag{5}$$

The FM signal $s(t)$ can be written in standard IQ format:

$$s(t) = i(t)\cos 2\pi f_c t - q(t) sin 2\pi f_c t \tag{6}$$

$$\begin{aligned}
s(t) &= \text{Re}\{a(t)e^{j\phi(t)}e^{j2\pi f_c t}\} = \\
&\text{Re}\{[I(t) + jQ(t)][\cos 2\pi f_c t + j sin 2\pi f_c t]\} \\
&= I(t)\cos 2\pi f_c t - Q(t) sin 2\pi f_c t \\
&= a(t)\cos[2\pi f_c t + \phi(t)]
\end{aligned}$$

Thus for FM:

$$a(t) = A_c$$
$$\phi(t) = 2\pi k_f \int_0^t m(\alpha)d\alpha$$
$$I(t) = A_c \cos 2\pi k_f \int_0^t m(\alpha)d\alpha$$
$$Q(t) = A_c \sin 2\pi k_f \int_0^t m(\alpha)d\alpha \qquad (7)$$

B: FM with Sinusoidal Message

Now consider a sinusoidal modulating wave defined by

$$m(t) = A_m \cos(2\pi f_m t) \qquad (8)$$

We re-derive the equation for the FM wave with the same 3 starting points.

1. The idea of FM is that the instantaneous frequency of the resulting FM wave equals:

   $$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned} \qquad (9)$$

   where $\Delta f = k_f A_m$. Thus the message causes the frequency to vary above and below the carrier frequency f_c. The quantity Δf is called the <u>frequency deviation</u>.

2. A general signal:

   $$s(t) = a(t)\cos\theta(t) \qquad (10)$$

3. The instantaneous frequency of a signal is:

   $$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} \qquad (11)$$

Combining these 3 starting points, we can write:

$$\begin{aligned} \theta(t) &= 2\pi f_c t + \frac{\Delta f}{f_m}\sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta\sin(2\pi f_m t) \end{aligned} \qquad (12)$$

Where $\beta = \frac{k_f A_m}{f_m} = \frac{\Delta f}{f_m}$ is called the <u>modulation index</u> of the FM wave.

Thus the FM wave itself is given In terms of βby:

$$s(t) = A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right] \qquad (13)$$

If β is small we have narrowband FM (NBFM) and if β is large (compared to one radian) we have wideband FM.

Thus for a sinusoidal modulating wave $m(t) = A_m \cos(2\pi f_m t)$ the FM wave can be written:

$$s(t) = I(t)\cos 2\pi f_c t - Q(t) sin 2\pi f_c t = a(t)\cos[2\pi f_c t + \phi(t)] \qquad (14)$$

where:

$$\begin{aligned} a(t) &= A_c \\ \phi(t) &= \beta sin 2\pi f_m t \\ i(t) &= A_c \cos \beta sin 2\pi f_m t \\ q(t) &= A_c \sin \beta sin 2\pi f_m t \end{aligned} \qquad (15)$$

C: Power Spectrum and Bandwidth of an FM Signal

Expanding $s(t)$ in the form of a Fourier series, we get:

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t] \qquad (16)$$

where $J_n(\beta)$ is the Bessel function of the first kind of order *n*.

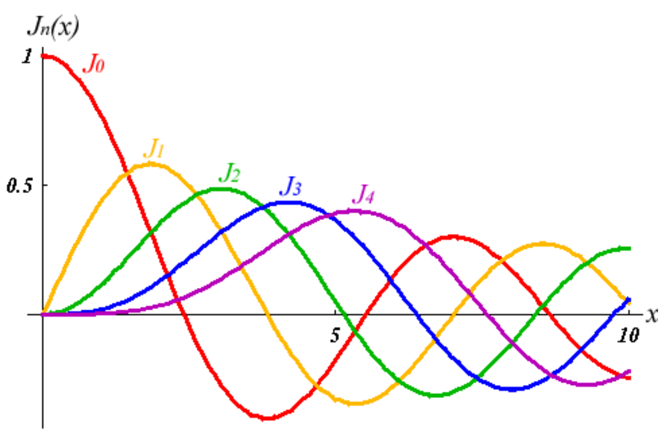
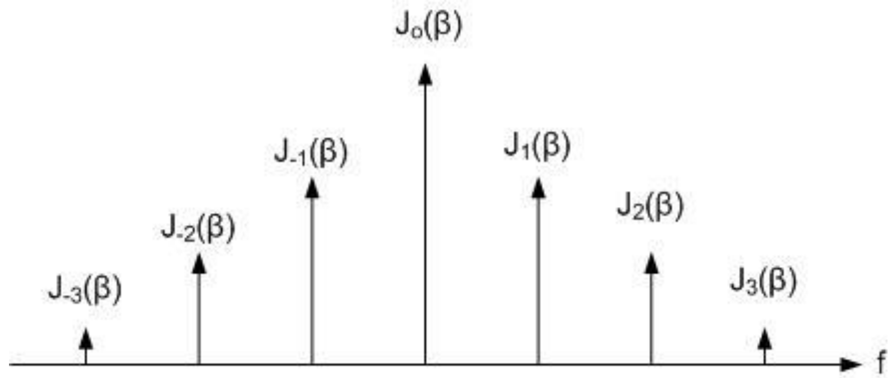


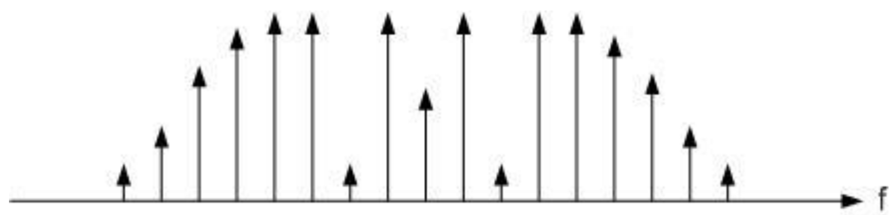
Figure 1 Bessel Functions

The discrete spectrum of the FM wave is obtained as

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \qquad (17)$$



Bessel Function value [$J_n(\beta)$] defining the FM spectrum depends upon β



Example of FM spectrum with relatively large β

To visualize the spectrum $S(f)$, we note that:

$$J_n\beta = (-1)^n J_{-n}(\beta) \tag{18}$$

and:

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1 \tag{19}$$

Also, for small values of β, we have:

$$J_0(\beta) \simeq 1, \; J_1(\beta) \simeq \frac{\beta}{2} \tag{20}$$

and:

$$J_n(\beta) \simeq 0, \quad n > 1 \tag{21}$$

The average power of an FM wave developed across a 1 ohm resistor is given by:

$$P = 1/2 \;\; A_c^2 \cdot \sum_{n=-\infty}^{\infty} J_n^2(\beta) \qquad (22)$$

For practical purposes, the bandwidth of the FM wave corresponds to the bandwidth containing 98% of the signal power.

The effective bandwidth of the FM signal is approximately given by Carson's formula:

$$B = 2(1+\beta) f_m \qquad (23)$$

D: Frequency Demodulation

Frequency demodulation extracts the original message wave from the frequency-modulated wave. We describe a digital FM demodulator.
A digital FM demodulator starts with the I and Q outputs of a general IQ receiver. Recall for an FM signal:

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\alpha)d\alpha\right] \qquad (24)$$

$$I(t) = A_c \cos 2\pi k_f \int_0^t m(\alpha)d\alpha$$

$$Q(t) = A_c \sin 2\pi k_f \int_0^t m(\alpha)d\alpha \qquad (25)$$

To extract $m(t)$ from $I(t), Q(t)$ we show consider *I(t)* and *Q(t)* as a complex signal.

$$s(t) = \mathrm{Re}\{a(t)e^{j\phi(t)}e^{j2\pi f_c t}\} = \mathrm{Re}\{[I(t) + jQ(t)]e^{j2\pi f_c t}\} = \mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\} \qquad (26)$$

where

$$\tilde{s}(t) = I(t) + jQ(t) = a(t)e^{j\phi(t)}$$

It can be shown that *m(t)* is obtained from the following formula:

$$m(t) = \arg[\tilde{s}(t-1)\tilde{s}^*(t)] \qquad (27)$$

Where:

$$(t-1) \rightarrow z^{-1} \qquad (28)$$

represents one sample delay

Mathematical Proof:

$$\arg[\tilde{s}(t-1)\tilde{s}^*(t)] = \arg[a(t-1)e^{j\phi(t-1)}a(t)e^{-j\phi(t)}]$$
$$= \phi(t-1) - \phi(t) \approx \frac{d\phi}{dt} = 2\pi k_f m(t)$$

Intuitive idea

The math shows that an FM demodulator does a differentiation $\arg[s(t-1)s^*(t)] \approx \frac{d\phi}{dt}$

We can see this intuitively as follows. We know that from the idea of FM $f_i(t) = f_c + k_f m(t) = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$ so that $m(t) = \frac{1}{2\pi k_f}\frac{d\phi(t)}{dt}$