Finite Element Methods with Applications in Mathematical Finance

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Introduction

There are many types of PDE's that we are interested in solving, such as:

- 1D ODE: Poisson Equation $-u''(x) = f$ on $\Omega \subset \mathbb{R}$
- 2D PDE: Poisson Equation $-\nabla u(x,y) = f$ on $\Omega \subset \mathbb{R}^2$
- Parabolic PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ on $\Omega \times I$, $\Omega = [0, L]$, $I = (0, T]$

Since many PDE's do not have a closed form solution, a popular method with practical use in areas such as Engineering and Physics is the Finite Element Method, which is used to approximate complex continuous functions as discrete models. In this project, we shall explore the theory behind the method, examine in detail each of the steps in the numerical method and successfully apply the approach to option pricing problems.

The Weak Formulation and Finite Dimensional Subspace

To help introduce the Finite Element Method, we focus on the 1D Poisson equation with Dirichlet boundary conditions, which is:

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \qquad (D)$$

The equation above has a second order derivative, which can be difficult to work with. We can "reduce the restrictions" by transforming our equation to one involving first derivatives. This is where we convert our equation to the **weak formulation**:

$$\int_0^1 u' \cdot v' dx = \int_0^1 f \cdot v \, dx \quad \text{for any test function} \quad v \in V = H_0^1(0,1) \qquad (V)$$

We can now **construct** a *subspace* V_h , which is **finite dimensional** ($V_h \subset V$), in which a good approximate solution exists [1].

$$V_h = \left\{ v_h(x), \; v_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x) \right\}$$

Finite Element Discretisation - 1D Case

Our next step involves discretising our interval into **elements** $[x_i, x_{i+1}]$, where each element can be described by **piece-wise linear basis functions** $\phi_i(x)$ such that $\phi_i(x_j) \equiv \delta_{ij}$, and our finite element approximation will be a **linear combination** of the basis functions: $u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x) \in V_h$.

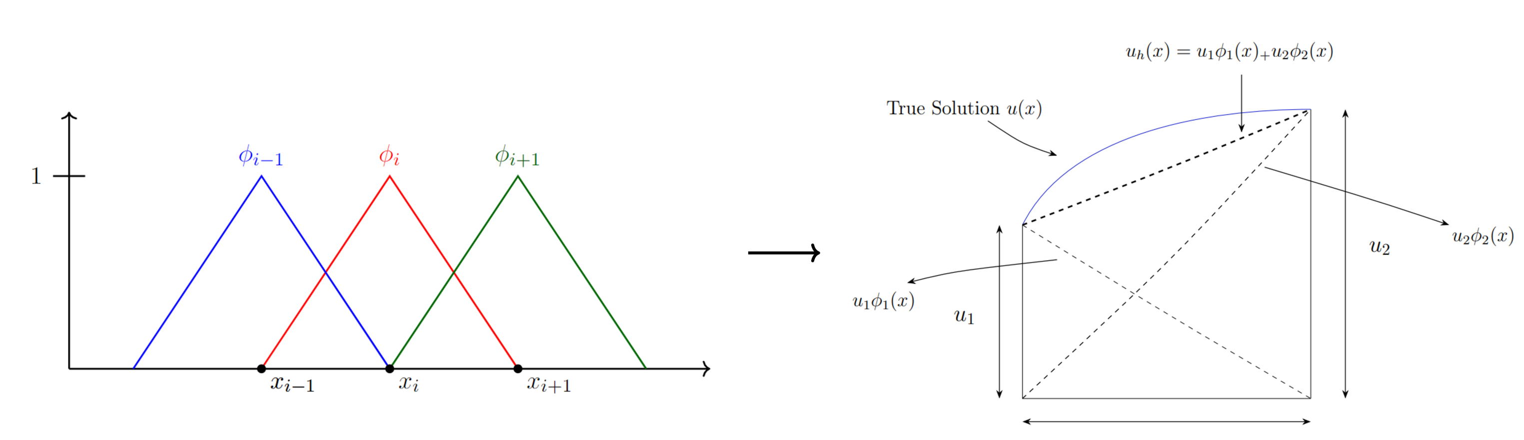

Figure 1. A visualisation of the 1D discretisation

Finite Element Discretisation - 2D Case

The 2D case is quite similar to the 1D case, except for a few key differences:

- We will be using a **2D grid** (be it uniform/non-uniform) instead of a 1D discretised domain.
- We shall use **2D triangle elements** (triangulation) instead of 1D sub-interval elements. The domain will be $\Omega = \bigcup_j K_j$ where K_j are non-overlapping triangles and N_i are the nodal points.
- Our piece-wise linear basis functions will be ϕ_i such that $\phi_i(N_j) \equiv \delta_{ij}$.

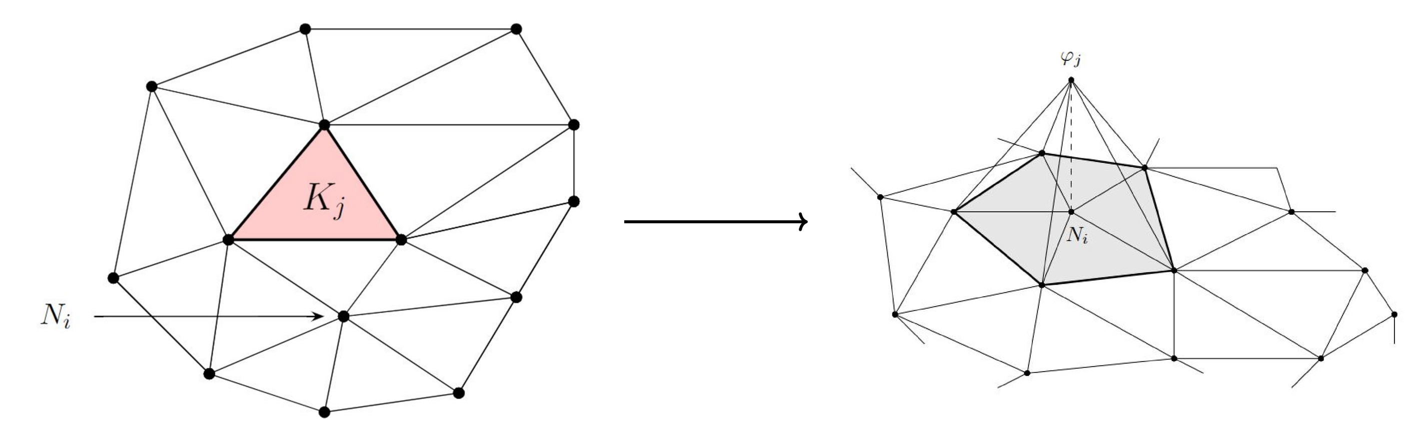

Figure 2. A visualisation of the 2D discretisation

Example in 1D and 2D

1D : $-u''(x) = f = 12x^2 - 36x + 18$, $u(0) = u(3) = 0$. Below we can see the FEM approximate the solution for increasing values of $numElem$.

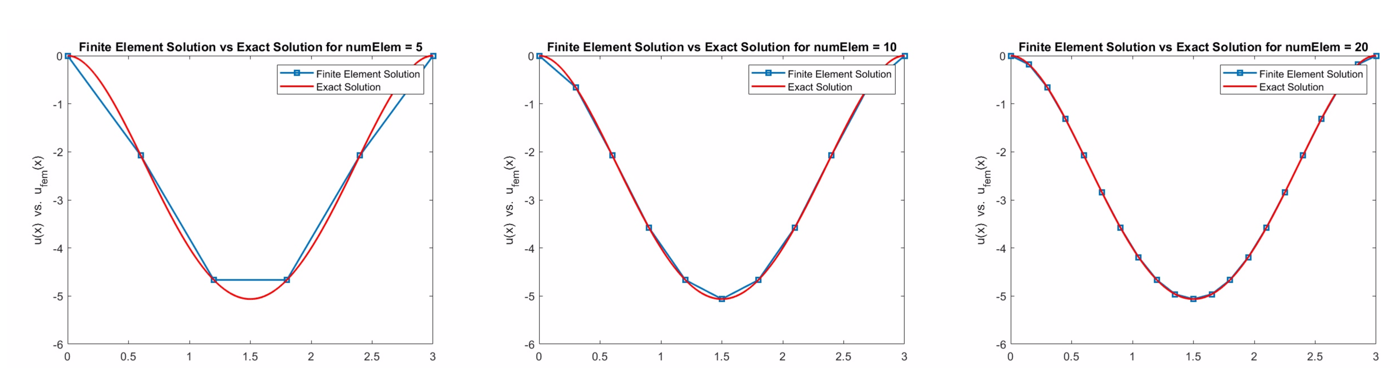

Figure 3. FEM Approximation for 1D ODE

2D : $-\nabla u(x,y) = f = 2\pi^2 sin(\pi x) sin(\pi y)$, $\partial\Omega = 0$ for $\Omega = [0,1] \times [0,1]$. Below we can see the FEM approximate the solution.

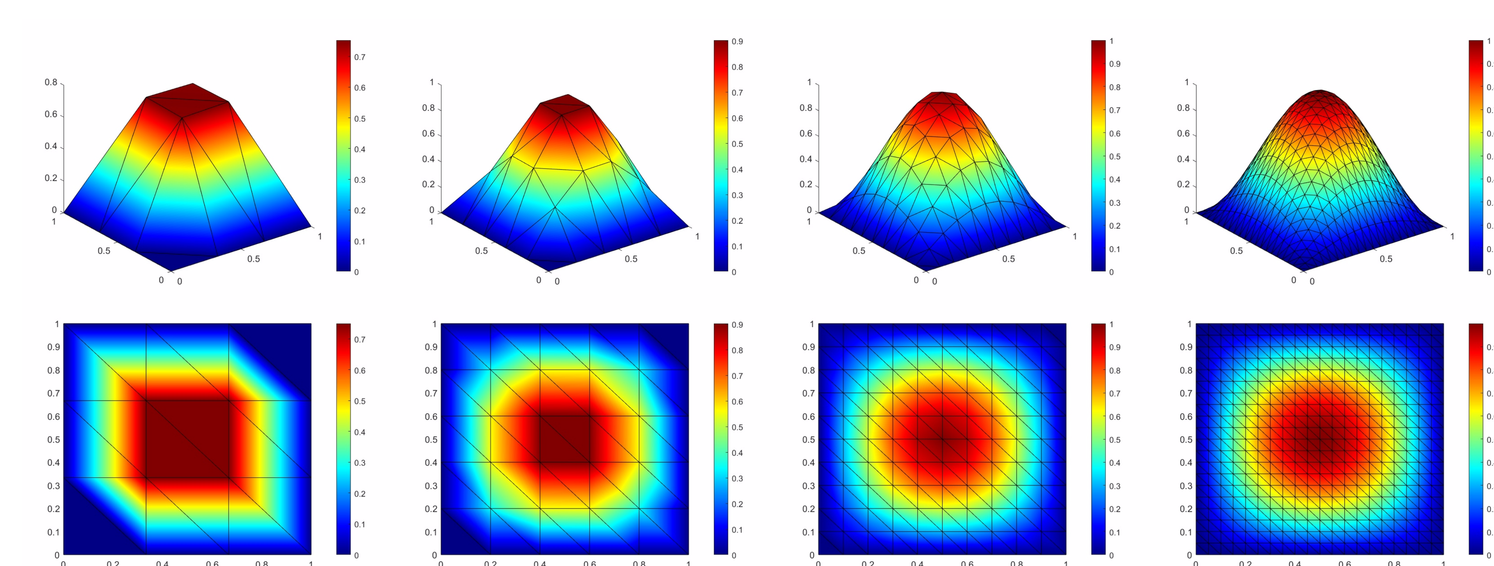

Figure 4. FEM Approximation for 2D PDE

Error Analysis

We let u be the analytical solution and u_h be the FEM approximation. Then

$$|u - u_h|_{H^1(\Omega_h)} \leq ch|u''|_{H^2(\Omega_h)} = O(h),$$
$$||u - u_h||_{L^2(\Omega_h)} \leq ch^2|u''|_{H^2(\Omega_h)} = O(h^2) .$$

We also consider the affect of **numerical quadrature** on the error [3]. Let $\tilde{u}_h$ be the approximation of u_h . Then,

$$||u - \tilde{u}_h||_{H^1(\Omega)} \leq ||u - u_h||_{H^1(\Omega)} + ||u_h - \tilde{u}_h||_{H^1(\Omega)} = O(h^m + h^r)$$

Application to Black-Scholes Equation

The Finite Element Method can be applied to approximate the Black-Scholes equation, a famous PDE that is used to price a variety of financial derivatives:

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 .$$

This PDE can be transformed into a 1-dimensional heat equation which is a time-dependent PDE. One can then apply the FEM to discretise the spatial domain and finally use a time-stepping scheme to discretise the time domain [2].

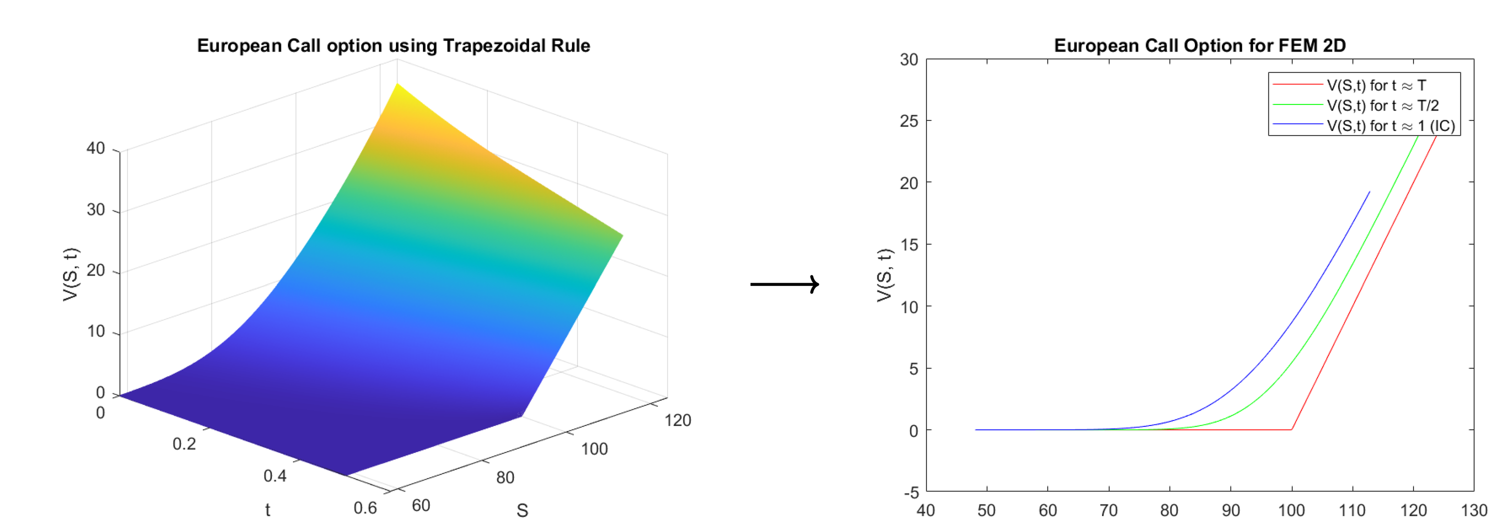

Figure 5. FEM approximation for the Black-Scholes equation on a European Call option

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