MINVO Basis: Finding Simplexes with Minimum Volume Enclosing Polynomial Curves

Jesus Tordesillas, and Jonathan P. How

***Abstract*—Applications such as computer graphics rendering, path planning for robotics and finite element simulations greatly benefit from having an n-simplex that encloses a given polynomial curve. Bézier curves (which use the Bernstein basis) are probably the most common choice for this, since it is guaranteed that each interval of the curve is contained inside the convex hull of its control points. However, it is known that this convex hull is not the one with smallest volume, causing therefore great conservatism in all the applications mentioned before. This paper derives and presents the MINVO basis, a polynomial basis that generates the smallest n-simplex that encloses any given polynomial curve. We show that the MINVO basis is able to obtain a volume that is 2.36 times smaller when $n = 3$, and 903 times smaller when $n = 7$. Global optimality is discussed and proven for some n using Sum-Of-Squares (SOS) Programming and moment relaxations.**

SUPPLEMENTARY MATERIAL

The MATLAB code used for this paper can be downloaded from the following link:

https://www.dropbox.com/s/lro345qygg7bd7j/code_project_Jesus_Tordesillas_Torres.zip?dl=0

TODOS

Say that all the solutions satisfy Klee's theorem (see paper FACET-CENTROIDS AND VOLUME MINIMIZATION)

Cite these ones: https://scholar.google.com/scholar?cites=57810[illegible]&as_sdt=[illegible]005&sciodt=0,22&hl=en

I. INTRODUCTION

Bézier curves are extensively used in many different applications, which include computer aided design ([1], [2], [3], [4], [5]), simulations and animations ([6], [7]), and robotics ([8], [9], [10], [11], [12], [13], [14]), just to mention some. The ability to approximate the space occupied by a given polynomial curve with a polyhedron is crucial for many of these applications, specially those that need to check the intersection between two objects in real time. For instance, many path planning algorithms that use Bézier curves can easily ensure safety by ensuring that the control points of the trajectory are inside the sequence of overlapping polyhedra that define the free space (see [12], [13], [14] for instance). This allows to impose a finite number of constraints only on the vertexes of this outer polyhedral representation of the interval, instead of infinite number of constraints over all the points of the interval of the polynomial.

The authors are with the Aerospace Controls Laboratory, MIT, 77 Massachusetts Ave., Cambridge, MA, USA {jtorde, jhow}@mit.edu

The basis for the Bézier curves are the Bernstein polynomials, but, while having some other useful properties, they are not designed to minimize the volume of the simplex that encloses a given interval of the curve. This directly translates into an unnecessary conservatism in many of the applications mentioned above. The goal of this paper is to obtain a polynomial basis that can be used to generate the simplex with minimum volume that encloses a polynomial curve.

The contributions are summarized as follows:

- Formulation of the optimization problem that solves, without loss of generality, for the smallest n-simplex that encloses a given polynomial curve. We show that this problem is also equivalent to find the polynomial curve whose convex hull has maximum volume, while being enclosed in a given simplex. Global optima for $n = 1$ is obtained using moment relaxations and semidefinite programming. Local optima for $n = 2, 3, 4$ and feasible solutions for $n = 5, 6, 7$ are also obtained.
- Say sth about the improvement wrt Bernstein/B-Splines
- A more tractable formulation that imposes a specific structure in the polynomials. With this formulation, [illegible] moment relaxations and semidefinite programming), and local optima are obtained for $n = 4, 5, 6, 7$.

II. RELATED WORK

A work that attempted to solve exactly the same problem proposed in this paper was the work by Gary Herron [15]. In this work, the author imposed a specific structure in the form for the polynomials of the basis, and then solved the associated nonconvex optimization problem over the roots of those polynomials. For this specific form of the polynomials, a global minimizer was found for $n = 2$, and local minimizer was found for $n = 3$, but no global optimality was proven for this. These paper goes further, and using SOS programming, it first proposes the most general formulation that does not impose any specific structure in the form of the polynomials (apart from being SOS), proving global and local optimality for some n. Then, by imposing a structure in the polynomials, we are able to able to find global optima for $n = 1, 2, 3$ and local optima $n = 4, 5, 6, 7$.

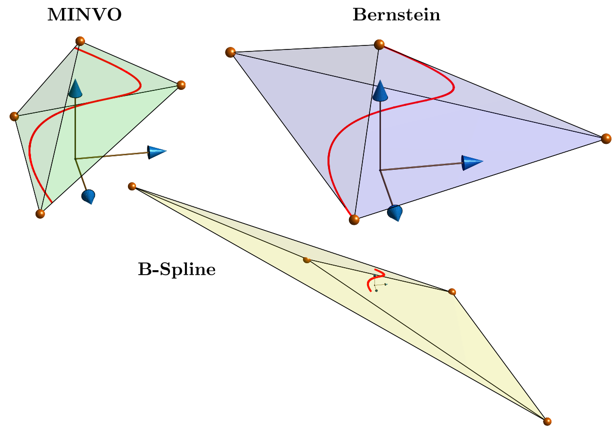

Figure 1: Comparison between the MINVO basis and the Bernstein basis for $n = 3$. The simplex found using the MINVO basis (left) encloses the 3^{rd}-order polynomial curve in red. For any 3^{rd}-order polynomial curve, the MINVO basis generates a simplex that is 2.36 times smaller that the one found using the Bernstein basis (right).

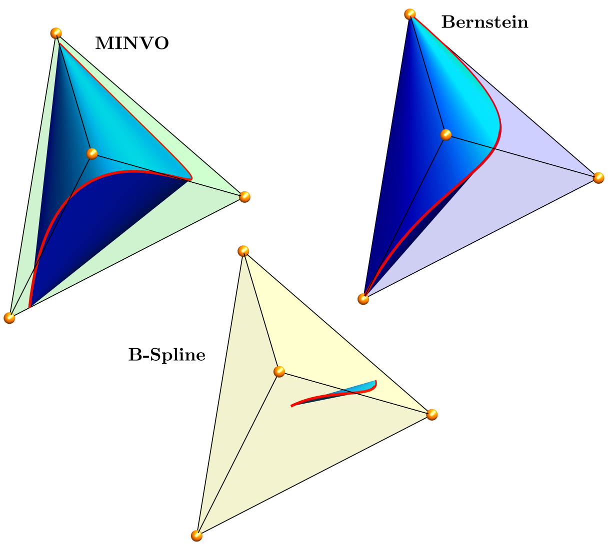

Figure 2: Comparison between the MINVO basis and the Bernstein basis for $n = 3$. The simplex found using the MINVO basis (left) encloses the 3^{rd}-order polynomial curve in red. For any 3^{rd}-order polynomial curve, the MINVO basis generates a simplex that is 2.36 times smaller that the one found using the Bernstein basis (right).

A similar problem has also been studied in the hyperspectral unmixing problem inside the remote sensing community, which consists of trying to find, in a given image pixel, the proportions (or abundances) of each macroscopic materials (endmembers) contained in that pixel [16]. One way to address this problem is to find the simplex with minimum volume that encloses a set of points (see [17] for instance). However, the fact of dealing with many data points, and not necessarily distributed along a polynomial curve, leads to the need of different iterative algorithms to solve it ([18], [19], [20]).

III. Notation and Definitions

Unless otherwise noted, all the indexes in this paper will start at 0. For instance, $M_{0,3}$ is the fourth element of the first row of the matrix $\boldsymbol{M}$. Let us also introduce the two following common definitions and their respective notations. This paper will use the following notation:

Symbol	Meaning
a	Scalar
$\boldsymbol{a}$	Column vector
$\boldsymbol{A}$	Matrix
$\mathcal{A}$	Set of points
conv($\boldsymbol{p}(t)$)	Convex hull of a curve $\boldsymbol{p}(t)$
$\lvert\boldsymbol{A}\rvert$	Determinant of $\boldsymbol{A}$
abs(a)	Absolute value of a
$\propto$	Proportional to
$\cdot_{m\times n}$	Size of a matrix (m rows $\times$ n columns)
$\boldsymbol{a} \geq \boldsymbol{b}$	Element-wise inequality
$\boldsymbol{1}$	Column vector of ones
$\boldsymbol{0}$	Column vector of zeros
$\boldsymbol{e}_i$	$i-$th vector of the standard basis (i.e. vector with all zeros except a 1 in its $i-$th entry). Its size is determined by the context.
$\boldsymbol{M}_{:,c:d}$	Matrix formed by the rows $c, c+1, ..., d$ of $\boldsymbol{M}$
$\boldsymbol{t}$	$\begin{bmatrix} t^r & t^{r-1} \cdots & 1 \end{bmatrix}^T$ (r given by the context)
$\hat{\boldsymbol{t}}$	$\begin{bmatrix} 1 & \cdots & t^{r-1} & t^r \end{bmatrix}^T$ (r given by the context)
$\mathbb{S}_+^m$	Positive semidefinite cone (set of all symmetric positive semidefinite $m \times m$ matrices)
vol($\cdot$)	Volume
dist($\boldsymbol{a}, \boldsymbol{\pi}$)	Distance between the point $\boldsymbol{a}$ and the plane $\boldsymbol{\pi}$
Pf($\boldsymbol{A}$)	Pfaffian of $\boldsymbol{A}$, as defined by [21]

n-**simplex**: Convex hull of $n+1$ points $\boldsymbol{v}_0, ..., \boldsymbol{v}_n \in \mathbb{R}^n$. These points are **vertexes** of the simplex, and will be stacked in the **matrix of vertexes** $\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_0 & \cdots & \boldsymbol{v}_n \end{bmatrix}$. A simplex with $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{e}_0 & \cdots & \boldsymbol{e}_n \end{bmatrix}$ is the **standard simplex**. The letter S will denote the simplex, while S^n will denote the set of all possible n-simplexes.

Polynomial curve of degree n and dimension k: Parametric curve $\boldsymbol{p}(t) := \begin{bmatrix} p_0(t) \\ \vdots \\ p_{k-1}(t) \end{bmatrix} := \boldsymbol{P}\boldsymbol{t}$, where $p_i(t)$ is a polynomial of degree n. The matrix $\boldsymbol{P}$ is the coefficient matrix, and will be denoted as $\boldsymbol{P} := \begin{bmatrix} \boldsymbol{c}_0^T \\ \vdots \\ \boldsymbol{c}_{k-1}^T \end{bmatrix}_{k\times(n+1)} \equiv \begin{bmatrix} \boldsymbol{p}_n & \cdots & \boldsymbol{p}_0 \end{bmatrix}_{k\times(n+1)}$. In this paper, we will use the intervals $t \in [-1,1]$ of polynomial curves of the form $k = n$ and we will refer to these intervals simply as n-th order polynomial curves.

IV. Problem definition

As discussed in the introduction, the goal of this paper is to investigate the following optimization problem:

Problem 1: Given an n-th order polynomial curve $\boldsymbol{p}(t)$, find the n-simplex S with minimum volume that contains $\boldsymbol{p}(t)$ $\forall t \in [-1,1]$. In other words:

$$\min_{S\in S^n} \quad f_1 := \text{vol}(S) \propto \text{abs}\left(\left|\begin{bmatrix} \boldsymbol{V}^T & \boldsymbol{1} \end{bmatrix}\right|\right)$$

$$s.t. \quad \boldsymbol{p}(t) \in S \quad \forall t \in [-1,1]$$

For $n = 2$, Problem 1 tries to find the triangle with the smallest area that contains the curve $\begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$, where $x(t)$ and $y(t)$ are 2^{nd}-degree polynomials. For $n = 3$, it tries to find the tetrahedron with the smallest volume that contains the curve $\begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$(all 3^{rd}-degree polynomials). Without loss of generality, we will assume throughout the paper that the polynomial curve $\boldsymbol{p}(t)$ is not contained in a subspace $\mathbb{R}^m$, with $m < n$. In other words, $|\boldsymbol{P}_{:,0:n}| \neq 0$. If $\boldsymbol{p}(t)$ is contained in a subspace $\mathbb{R}^m$ $(m < n)$, then we can solve the problem by simply using the basis of that subspace $\mathbb{R}^m$. Is this true if, for instance, m=1 and n=3??

As we will prove later in the report, Problem 1 is closely connected this other optimization problem:

Theorem 1: The volume of the convex hull of an n-th order polynomial curve $\boldsymbol{p}(t)$, with $t \in [-1,1]$, is given by:

$$\text{vol}\left(\text{conv}\left(\boldsymbol{p}(t)\right)\right) = \frac{\text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right)}{n!} 2^{\frac{n(n+1)}{2}} \prod_{0\leq i<j\leq n} \left(\frac{j-i}{j+i}\right)$$

Proof: See Appendix (Sec. IX). □

Problem 2: Given the vertexes $\boldsymbol{v}_0, ..., \boldsymbol{v}_n$ of an n-simplex S, find the n-th order polynomial curve $\boldsymbol{p}(t)$ contained in S, whose convex hull has maximum volume ($t \in [-1,1]$):

$$\min_{\boldsymbol{p}(t)} \quad f_2 := -\text{vol}(\text{conv}(\boldsymbol{p}(t)) \propto -\text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right)$$

$$s.t. \quad \boldsymbol{p}(t) \in S \quad \forall t \in [-1,1]$$

Note that another geometric intepretation of Problem 2 is that we are tying to find the n-th order polynomial curve $\boldsymbol{p}(t)$ contained in S, whose coefficients vectors $\boldsymbol{p}_n, ..., \boldsymbol{p}_1$ span a parallelogram with largest volume.

Similar to the previous case, we will assume that the matrix of vertexes of the simplex S satisfy $\left|\begin{bmatrix} \boldsymbol{V}^T & \boldsymbol{1} \end{bmatrix}\right| \neq 0$. In other words, the simplex S is not contained in $\mathbb{R}^m$, with $m < n$.

V. Equivalent Formulation

Let us now study the constraints and the objective functions of Problems 1 and 2.

A. Constraints of Problem 1 and 2:

Both problems share the same constraint $\boldsymbol{p}(t) \in S \quad \forall t \in [-1,1]$, which is equivalent to $\boldsymbol{p}(t)$ being a convex combination of the vertexes $\boldsymbol{v}_i$ of the simplex for $t \in [-1,1]$:

$$\left.\begin{array}{c} \boldsymbol{p}(t) = \sum_{i=0}^{n} \lambda_i(t)\boldsymbol{v}_i \\ \sum_{i=0}^{n} \lambda_i(t) = 1 \quad \forall t \\ \lambda_i(t) \geq 0 \quad \forall i = 0,..,n \quad \forall t \in [-1,1] \end{array}\right\}$$

Taking $\lambda_i(t)$ as an n-th degree polynomial $\lambda_i(t) := \boldsymbol{\lambda}_i^T\boldsymbol{t}$, and matching the coefficients we have that:

$$\boldsymbol{P} = \boldsymbol{V}\underbrace{\begin{bmatrix} \boldsymbol{\lambda}_0^T \\ \boldsymbol{\lambda}_1^T \\ \vdots \\ \boldsymbol{\lambda}_n^T \end{bmatrix}}_{:=\boldsymbol{A}}$$

where we have also defined the $(n+1)\times(n+1)$ matrix $\boldsymbol{A}$. The i-th row of this matrix $\boldsymbol{A}$ contains the coefficients of the polynomial $\lambda_i(t)$. We can also write $\boldsymbol{V}^T = \left(\boldsymbol{P}\boldsymbol{A}^{-1}\right)^T = \boldsymbol{A}^{-T}\boldsymbol{P}^T$.

The geometric interpretation of each $\lambda_i(t)$ is as follows. Let us define $\boldsymbol{\pi}_i$ as the hyperplane that passes through the points $\{\boldsymbol{v}_0, \boldsymbol{v}_1, ..., \boldsymbol{v}_n\}\backslash\boldsymbol{v}_i$, and $\boldsymbol{n}_i$ as the normal vector of the hyperplane $\boldsymbol{\pi}_i$ that satisfies $\boldsymbol{n}_i^T\boldsymbol{p}(t) \geq 0$ (i.e. pointing towards the interior of the simplex). Choosing now a vertex $k \in \{0, ..., n\}\backslash i$ we have that:

$$\boldsymbol{p}(t) = \sum_{j=0}^{n} \lambda_j(t)\boldsymbol{v}_j = \boldsymbol{v}_k + \sum_{\substack{j=0 \\ j\neq k}}^{n} \lambda_j(t)\left(\boldsymbol{v}_j - \boldsymbol{v}_k\right)$$

And therefore:

$$dist\left(\boldsymbol{p}(t), \boldsymbol{\pi}_i\right) := \left(\boldsymbol{p}(t) - \boldsymbol{v}_k\right)^T\boldsymbol{n} =$$

$$= \sum_{j=0}^{n-1} \lambda_j(t)\left(\boldsymbol{v}_i - \boldsymbol{v}_k\right)^T\boldsymbol{n} = \lambda_i(t)\, dist\left(\boldsymbol{v}_i, \boldsymbol{\pi}_i\right)$$

And therefore:

$$\lambda_i(t) = \frac{dist\left(\boldsymbol{p}(t), \boldsymbol{\pi}_i\right)}{dist\left(\boldsymbol{v}_i, \boldsymbol{\pi}_i\right)}$$

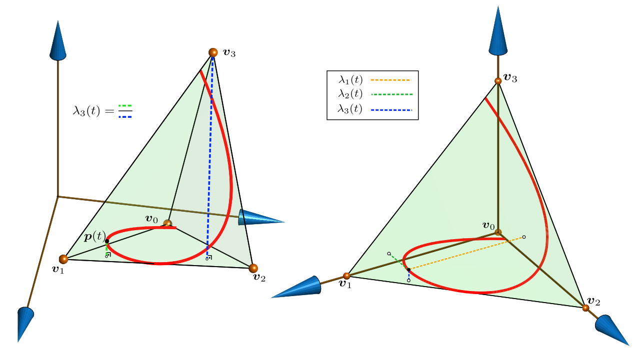

Figure 3: Geometric interpretation of $\lambda_i(t)$. Each $\lambda_i(t)$ represents the normalized distance between the curve $\boldsymbol{p}(t)$ and the plane formed by the vertexes $\boldsymbol{v}_i$, $i \neq j$, normalized with the distance from the vertex $\boldsymbol{v}_i$ to that plane (left). For a standard simplex in 3D (i.e. $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{bmatrix}$) (right), the curve in red is $\boldsymbol{p}(t) = \begin{bmatrix} \lambda_1(t) & \lambda_2(t) & \lambda_3(t) \end{bmatrix}^T$.

Hence, each λ_i represents the ratio between the distance of the curve to the hyperplane $\boldsymbol{\pi}$ (formed by the points $\boldsymbol{v}_j$, $j \neq i$) and the distance from $\boldsymbol{v}_i$ to that hyperplane $\boldsymbol{\pi}$ (see Fig. 3 for the case $n = 3$).

With respect to the second constraint $\sum_{i=0}^{n} \lambda_i(t) = 1$, it can be rewritten as $\boldsymbol{A}^T\boldsymbol{1} = \boldsymbol{e}_n$ (or equivalently $\boldsymbol{e}_n^T\boldsymbol{A}^{-1} = \boldsymbol{1}^T$).

B. Objective Function of Problem 1

Given the previous relationships, we have that

$$\left|\begin{bmatrix} \boldsymbol{V}^T & \boldsymbol{1} \end{bmatrix}\right| = \left|\begin{bmatrix} \boldsymbol{A}^{-T}\boldsymbol{P}^T & \boldsymbol{1} \end{bmatrix}\right| =$$

$$= \left|\boldsymbol{A}^{-T}\right| \left|\begin{bmatrix} \boldsymbol{P}^T & \underbrace{\boldsymbol{A}^T\boldsymbol{1}}_{=\boldsymbol{e}_n} \end{bmatrix}\right| \propto \left|\boldsymbol{A}^{-1}\right| = \frac{1}{|\boldsymbol{A}|}$$

where we have used the fact that everything inside $\begin{bmatrix} \boldsymbol{P}^T\boldsymbol{e}_n \end{bmatrix}$ is given (i.e. not a decision variable of the optimization problem), and the fact that $|\boldsymbol{A}| = \left|\boldsymbol{A}^T\right|$. Therefore, we conclude that $f_1 \propto \frac{1}{abs(|\boldsymbol{A}|)}$. We can therefore maximize $abs(|\boldsymbol{A}|)$, or, equivalently, minimize $-abs\left(|\boldsymbol{A}|\right)$.

C. Objective Function of Problem 2

Similarly to the previous subsection, we have that

$$|\boldsymbol{P}_{:,0:n}| = \left|\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{e}_n^T \end{bmatrix}\right| = \left|\begin{bmatrix} \boldsymbol{V} \\ \underbrace{\boldsymbol{e}_n^T\boldsymbol{A}^{-1}}_{=\boldsymbol{1}^T} \end{bmatrix} \boldsymbol{A}\right| \propto |\boldsymbol{A}|$$

And therefore, $f_2 \propto -abs\left(|\boldsymbol{A}|\right)$.

D. Equivalent Formulation for Problem 1 and 2

Therefore, we can solve Problem 1 (or Problem 2 respectively) for **any** given polynomial curve (**any** given simplex respectively) by solving the following problem:

Problem 3:

$$\min_{\boldsymbol{A}\in\mathbb{R}^{n+1}} \quad f_3 := -abs\left(|\boldsymbol{A}|\right)$$

$$s.t. \quad \boldsymbol{A}\boldsymbol{t} \geq \boldsymbol{0} \quad \forall t \in [-1, 1]$$
$$\boldsymbol{A}^T\boldsymbol{1} = \boldsymbol{e}_n$$

As the objective function of Problem 3 is the determinant of a nonsymmetric matrix $\boldsymbol{A} := \begin{bmatrix} \boldsymbol{\lambda}_0 & \cdots & \boldsymbol{\lambda}_n \end{bmatrix}^T$, it is clearly a nonconvex problem. For the first constraint of Problem 3, we can use the following result of Sum-Of-Squares programming, to rewrite that constraint and leave positive semidefinite matrices as decision variables [22]:

- If n is odd, $\lambda_i(t) \geq 0 \ \ \forall t \in [-1, 1]$ if and only if
$$\begin{cases} \lambda_i(t) = \hat{\boldsymbol{t}}^T\left((t+1)\boldsymbol{W}_i + (1-t)\boldsymbol{V}_i\right)\hat{\boldsymbol{t}} \\ \boldsymbol{W}_i \in \mathbb{S}_+^{\frac{n+1}{2}}, \boldsymbol{V}_i \in \mathbb{S}_+^{\frac{n+1}{2}} \end{cases}$$
- If n is even, $\lambda_i(t) \geq 0 \ \ \forall t \in [-1, 1]$ if and only if
$$\begin{cases} \lambda_i(t) = \hat{\boldsymbol{t}}^T\boldsymbol{W}_i\hat{\boldsymbol{t}} + \hat{\boldsymbol{t}}^T(t+1)(1-t)\boldsymbol{V}_i\hat{\boldsymbol{t}} \\ \boldsymbol{W}_i \in \mathbb{S}_+^{\frac{n}{2}+1}, \boldsymbol{V}_i \in \mathbb{S}_+^{\frac{n}{2}} \end{cases}$$

Note that the *if and only if* condition applies because it is a univariate polynomial [22]. We end up therefore with a nonconvex finite-dimensional optimization problem, where the decision variables are the positive semidefinite matrices $\boldsymbol{W}_i$ and $\boldsymbol{V}_i$, $i = 0, ..., n$.

No generality has been lost in Problem 3. However, this problem easily becomes intractable for large n. To make it more tractable, we can try to reduce the number of decision variables of Problem 3 by imposing a structure in $\lambda_i(t)$. As Problem 1 is trying to minimize the volume of the simplex, we can impose that the planes of the n-simplex be tangent to the curve [23], [24]. Using the geometric interpretation of the $\lambda_i(t)$ given in subsection V-A, this means that each $\lambda_i(t)$ should have either real double roots (where the curve is tangent to the plane), and/or roots at $t = \pm 1$. This translates into the formulation shown in Problem 4.

REVISAR LAS CONSTRAINTS $\lambda_i(t) = \lambda_{n-i+2}(-t)$ DebajoCREO QUE ALGUN INDICE PUEDE ESTAR MAL

The relationship between Problems 1, 2, 3 and 4 is given in Fig. 4. First note that the structure imposed for $\lambda_i(t)$ in Problem 4 guarantees that they are nonnegative for $t \in [-1, 1]$. Hence the feasible set of Problem 4 is contained in the feasible set of Problem 3, and therefore, $f_3^* \leq f_4^*$ holds. The matrix $\boldsymbol{A}^*$ found in Problem 3 or 4 can be used to find the vertexes of the simplex in Problem 1 (by simply using $\boldsymbol{V}^* = \boldsymbol{P}\left(\boldsymbol{A}^*\right)^{-1}$, where $\boldsymbol{P}$ is the coefficient matrix of the polynomial given), or to find the coefficient matrix

Problem 4:

$$\min_{\boldsymbol{A}} \quad f_4 := -abs\left(|\boldsymbol{A}|\right)$$

$$s.t:$$

If n is odd:

$$\begin{gathered}
\lambda_i(t) = -b_i(t-1)\prod_{j=1}^{\frac{n-1}{2}}(t-t_j)^2 \quad i = 0,2,...,n-1 \\
\lambda_i(t) = \lambda_{n-i+2}(-t) \qquad i = 1,3,...,n \\
b_i \geq 0 \quad i = 0,2,...,n-1 \\
\sum_{i=0}^{n}\lambda_i(t) = 1
\end{gathered}$$

If n is even: define $\mathcal{I}_a$ the set of odd integers $\in \left[0, \frac{n}{2}-1\right]$, and $\mathcal{I}_b$ the set of even integers $\in \left[0, \frac{n}{2}-1\right]$:

$$\begin{gathered}
\lambda_i(t) = -b_i(t+1)(t-1)\prod_{j=1}^{\frac{n-2}{2}}(t-t_j)^2 \quad i \in \mathcal{I}_a \\
\lambda_i(t) = b_i\prod_{j=1}^{\frac{n}{2}}(t-t_j)^2 \qquad i \in \mathcal{I}_b \\
\lambda_j(t) = \lambda_{n-i+1}(-t) \qquad i = \frac{n}{2}+1,...,n \\
b_i \geq 0 \quad i \leq \frac{n}{2} \\
\sum_{i=0}^{n}\lambda_i(t) = 1
\end{gathered}$$

- If $\frac{n}{2}$ is odd ($n = 2, 6, 10...$):

$$\lambda_i(t) = -b_i(t+1)(t-1)\prod_{j=1}^{\frac{n-2}{4}}(t-t_j)^2(t+t_j)^2 \qquad i = \frac{n}{2}$$

- If $\frac{n}{2}$ is even ($n = 4, 8, 12...$):

$$\lambda_i(t) = b_i\prod_{j=1}^{\frac{n}{4}}(t-t_j)^2(t+t_j)^2 \qquad i = \frac{n}{2}$$

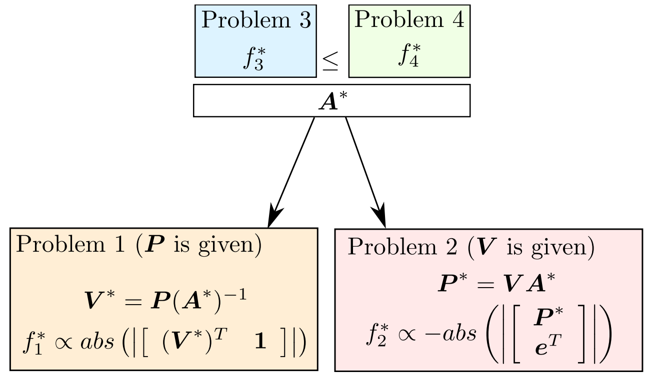

Figure 4: Relationship between Problems 1, 2, 3 and 4: Problem 4 and Problem 3 have the same objective function, but the feasible set of Problem 4 is contained in the feasible set of Problem 3, and therefore $f_3^* \leq f_4^*$. Both Problem 3 and 4 generate a (potentially locally) optimal solution $\boldsymbol{A}^*$, which can be applied to any polynomial curve to find the simplex in Problem 1, or to any simplex to find the polynomial curve in Problem 2.

of the polynomial in Problem 2 (by using $\boldsymbol{P}^* = \boldsymbol{V}\boldsymbol{A}^*$, where $\boldsymbol{V}$ contains the vertexes of the simplex given).

VI. Results

The optimization Problems 3 and 4 have been solved using the nonconvex solvers *IPOPT* [25] and *Knitro* [26], [27] (for Problem 3), the *fmincon* nonconvex solver [28] (for Problem 4), and the *Yalmip* interface [29], [30].

We were able to find local minima for Problem 4 for $n = 1, 2, 3, 4, 5, 6, 7$, and the same local minima were found in Problem 3 for $n = 1, 2, 3, 4$ (Problem 3 becomes intractable for higher n). The optimal matrices $\boldsymbol{A}$ found are shown in Table I, and denoted as $\boldsymbol{A}_{\text{MV}}$. Note that any permutation in the rows of $\boldsymbol{A}_{\text{MV}}$will not change the objective value, since only the sign of the determinant is affected. Its determinant $|\boldsymbol{A}_{\text{MV}}|$ is also compared with the one of the Bernstein and B-Spline matrices (denoted as $\boldsymbol{A}_{\text{Be}}$ and $\boldsymbol{A}_{\text{BS}}$ respectively) The corresponding plots of the MINVO basis functions are shown in Fig. 6, together with the Bernstein, B-Spline, and Lagrange bases for comparison.

One natural question to ask is whether the basis found constitutes a global minimizer either for Problem 3 or Problem 4. To answer this, first note that both Problem 3 and Problem 4 are polynomial optimization problems. Therefore, we can make use of Lasserre's moment method [31], and increase the order of the moment relaxation to find tighter lower bounds of the original nonconvex polynomial optimization problem. Using this technique, we were able to obtain the same objective value (proving therefore global optimality) for $n = 1, 2, 3$ (in Problem 4), and for $n = 1$ (in Problem 3).

All these results lead us to the following conclusions, which are also summarized in Table I:

- The matrix $\boldsymbol{A}$ found for $n = 1$ is a global minimizer of both Problem 3 and Problem 4.
- The matrices $\boldsymbol{A}$ found for $n = 2, 3$ are global minimizers for Problem 4, and at least local minimizers for Problem 3.
- The matrix $\boldsymbol{A}$ found for $n = 4$ is at least a local minimizer of both Problem 3 and Problem 4.
- The matrices $\boldsymbol{A}$ found for $n = 5, 6, 7$ are at least local minimizers for Problem 4, and are at least feasible solutions for Problem 3.

In Table I, it is also shown the ratio

$$\frac{abs\left(|\boldsymbol{A}_{\text{Be}}|\right)}{abs\left(|\boldsymbol{A}_{\text{MV}}|\right)}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}_B$ denote the coefficient matrices for the MINVO and Bernstein bases respectively. Let us now analyze the meaning of these results for Problems 1 and 2.

A. From the point of view of Problem 1

Problem 1 attempts to find the simplex with smallest volume the encloses a given polynomial curve. In Fig. 5

n	$\boldsymbol{A}_{\text{MV}}$	$abs\left(\lvert\boldsymbol{A}_{\text{MV}}\rvert\right)$	$\frac{abs(\lvert\boldsymbol{A}_{\text{MV}}\rvert)}{abs(\lvert\boldsymbol{A}_{\text{Be}}\rvert)}$	$\frac{abs(\lvert\boldsymbol{A}_{\text{MV}}\rvert)}{abs(\lvert\boldsymbol{A}_{\text{BS}}\rvert)}$	**Problem 3**	**Problem 4**
1	$\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$\frac{1}{2}=0.5$	$=1$	$=1$	Global Opt.	Global Opt.
2	$\frac{1}{8}\begin{bmatrix} 3 & -2\sqrt{3} & 1 \\ -6 & 0 & 6 \\ 3 & 2\sqrt{3} & 1 \end{bmatrix}$	$\frac{3\sqrt{3}}{16}\approx 0.3248$	≈ 1.299	≈ 5.196	Local Opt. (at least)	Global Opt.
3	$\begin{bmatrix} -0.4302 & 0.4568 & -0.02698 & 0.0004103 \\ 0.8349 & -0.4568 & -0.7921 & 0.4996 \\ -0.8349 & -0.4568 & 0.7921 & 0.4996 \\ 0.4302 & 0.4568 & 0.02698 & 0.0004103 \end{bmatrix}$	≈ 0.3319	≈ 2.360	≈ 254.9	Local Opt. (at least)	Global Opt.
4	$\begin{bmatrix} 0.5255 & -0.5758 & -0.09435 & 0.1381 & 0.03023 \\ -1.108 & 0.8108 & 0.9602 & -0.8108 & 0.1483 \\ 1.166 & 0 & -1.732 & 0 & 0.643 \\ -1.108 & -0.8108 & 0.9602 & 0.8108 & 0.1483 \\ 0.5255 & 0.5758 & -0.09435 & -0.1381 & 0.03023 \end{bmatrix}$	≈ 0.5678	≈ 6.057	$\approx 1.675\cdot 10^{5}$	Local Opt. (at least)	Local Opt. (at least)
5	$\begin{bmatrix} -0.7392 & 0.7769 & 0.3302 & -0.3773 & -0.0365 & 0.04589 \\ 1.503 & -1.319 & -1.366 & 1.333 & -0.121 & 0.002895 \\ -1.75 & 0.5424 & 2.777 & -0.9557 & -1.064 & 0.4512 \\ 1.75 & 0.5424 & -2.777 & -0.9557 & 1.064 & 0.4512 \\ -1.503 & -1.319 & 1.366 & 1.333 & 0.121 & 0.002895 \\ 0.7392 & 0.7769 & -0.3302 & -0.3773 & 0.0365 & 0.04589 \end{bmatrix}$	≈ 1.6987	≈ 22.27	$\approx 1.924\cdot 10^{9}$	Feasible (at least)	Local Opt. (at least)
6	$\begin{bmatrix} 1.06 & -1.134 & -0.7357 & 0.8348 & 0.1053 & -0.1368 & 0.01836 \\ -2.227 & 2.055 & 2.281 & -2.299 & -0.08426 & 0.2433 & 0.0312 \\ 2.59 & -1.408 & -4.27 & 2.468 & 1.58 & -1.081 & 0.152 \\ -2.844 & 0 & 5.45 & 0 & -3.203 & 0 & 0.5969 \\ 2.59 & 1.408 & -4.27 & -2.468 & 1.58 & 1.081 & 0.152 \\ -2.227 & -2.055 & 2.281 & 2.299 & -0.08426 & -0.2433 & 0.0312 \\ 1.06 & 1.134 & -0.7357 & -0.8348 & 0.1053 & 0.1368 & 0.01836 \end{bmatrix}$	≈ 9.1027	≈ 117.8	$\approx 4.750\cdot 10^{14}$	Feasible (at least)	Local Opt. (at least)
7	$\begin{bmatrix} -1.637 & 1.707 & 1.563 & -1.682 & -0.3586 & 0.4143 & -0.006851 & 2.854\cdot 10^{-5} \\ 3.343 & -3.285 & -3.947 & 4.173 & 0.6343 & -0.9385 & -0.02111 & 0.05961 \\ -4.053 & 2.722 & 6.935 & -4.96 & -2.706 & 2.269 & -0.2129 & 0.00535 \\ 4.478 & -1.144 & -9.462 & 2.469 & 6.311 & -1.745 & -1.312 & 0.435 \\ -4.478 & -1.144 & 9.462 & 2.469 & -6.311 & -1.745 & 1.312 & 0.435 \\ 4.053 & 2.722 & -6.935 & -4.96 & 2.706 & 2.269 & 0.2129 & 0.00535 \\ -3.343 & -3.285 & 3.947 & 4.173 & -0.6343 & -0.9385 & 0.02111 & 0.05961 \\ 1.637 & 1.707 & -1.563 & -1.682 & 0.3586 & 0.4143 & 0.006851 & 2.854\cdot 10^{-5} \end{bmatrix}$	≈ 89.0191	≈ 902.7	$\approx 2.997\cdot 10^{21}$	Feasible (at least)	Local Opt. (at least)

Table I: Creo que Puedo demostrar facilmente que el caso n=2 es global optimum para el problem 3 by simply imposing the centroid necessary condition optimi EL CASO N=2 DEBERIA SER GLOBAL OPTIMIMUM?? (I.E. LO PUEDO RESOLVER ANALITICAMENTE O NO?)Lo puedo resolver analiticamente para el problema 4 for sure, pero not sure para el problema 3...Results for the MINVO basis. $\boldsymbol{A}_{\text{MV}}$ denotes the coefficient matrix of the MINVO basis while $\boldsymbol{A}_{\text{Be}}$ represents the matrix of the Bernstein basis ($t\in[-1,1]$ for both). The bigger the absolute value of the determinant, the smaller the associated simplex (for Problem 1) and the bigger the convex hull of the curve (for Problem 2). These matrices $\boldsymbol{A}$ are independent of the polynomial curve given (in Problem 1), or of the simplex given (in Problem 2).

it is shown the simplex found for $n=2$, and in Fig. 1 for $n=3$. The simplex found using the Bernstein basis is also shown for comparison. Let S_{MV} and S_{Be} denote the simplexes that the MINVO and Bernstein bases generate respectively, we have that:

$$\frac{vol\left(S_{\text{MV}}\right)}{vol\left(S_{\text{Be}}\right)}=\frac{abs\left(\lvert\boldsymbol{A}_{\text{Be}}\rvert\right)}{abs\left(\lvert\boldsymbol{A}_{\text{MV}}\rvert\right)}$$

This means, for instance, that for $n=3$, the MINVO basis finds a simplex that is ≈ 2.36 smaller than the one the Bernstein basis finds. For $n=7$, the MINVO basis is able to obtain a simplex that is ≈ 903 times smaller than the one achieved using the Bernstein basis.

B. From the point of view of Problem 2

In problem 2, the n-simplex S is given, and we obtain the polynomial curve that is contained in S, whose coefficients vectors $\boldsymbol{p}_n,\ldots,\boldsymbol{p}_1$ span a parallelogram with largest volume. The results for a regular triangle ($n=2$) and a tetrahedron ($n=3$) are shown in Figs. 7 and 8 respectively. Denoting P_{MV} and P_B as the parallelograms that the MINVO and Bernstein bases generate respectively, the ratio of their volumes is:

$$\frac{vol\left(P_{\text{Be}}\right)}{vol\left(P_{\text{MV}}\right)}=\frac{abs\left(\lvert\boldsymbol{A}_{\text{Be}}\rvert\right)}{abs\left(\lvert\boldsymbol{A}_{\text{MV}}\rvert\right)}$$

This means that, for $n=3$, the MINVO basis finds a parallelogram ≈ 2.36 times bigger than the Bernstein basis (≈ 903 times bigger for $n=7$).

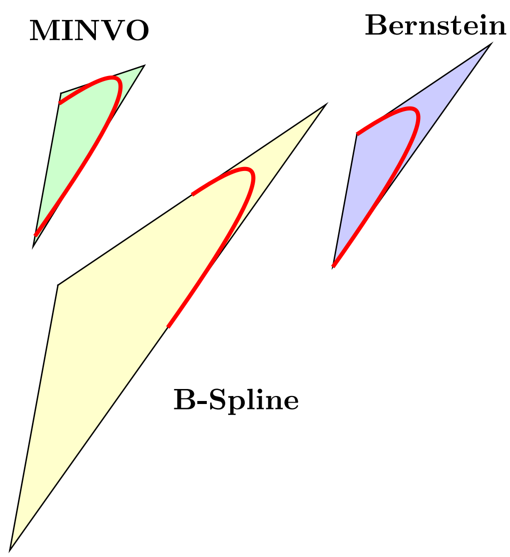

Figure 5: Comparison between the MINVO basis and the Bernstein basis for $n = 2$. The simplex found using the MINVO basis (left) encloses the 2^{nd}-order polynomial curve in red, and, for any 2^{nd}-order polynomial curve, the MINVO basis generates a simplex that is ≈ 1.3 times smaller that the one found using the Bernstein basis (right).

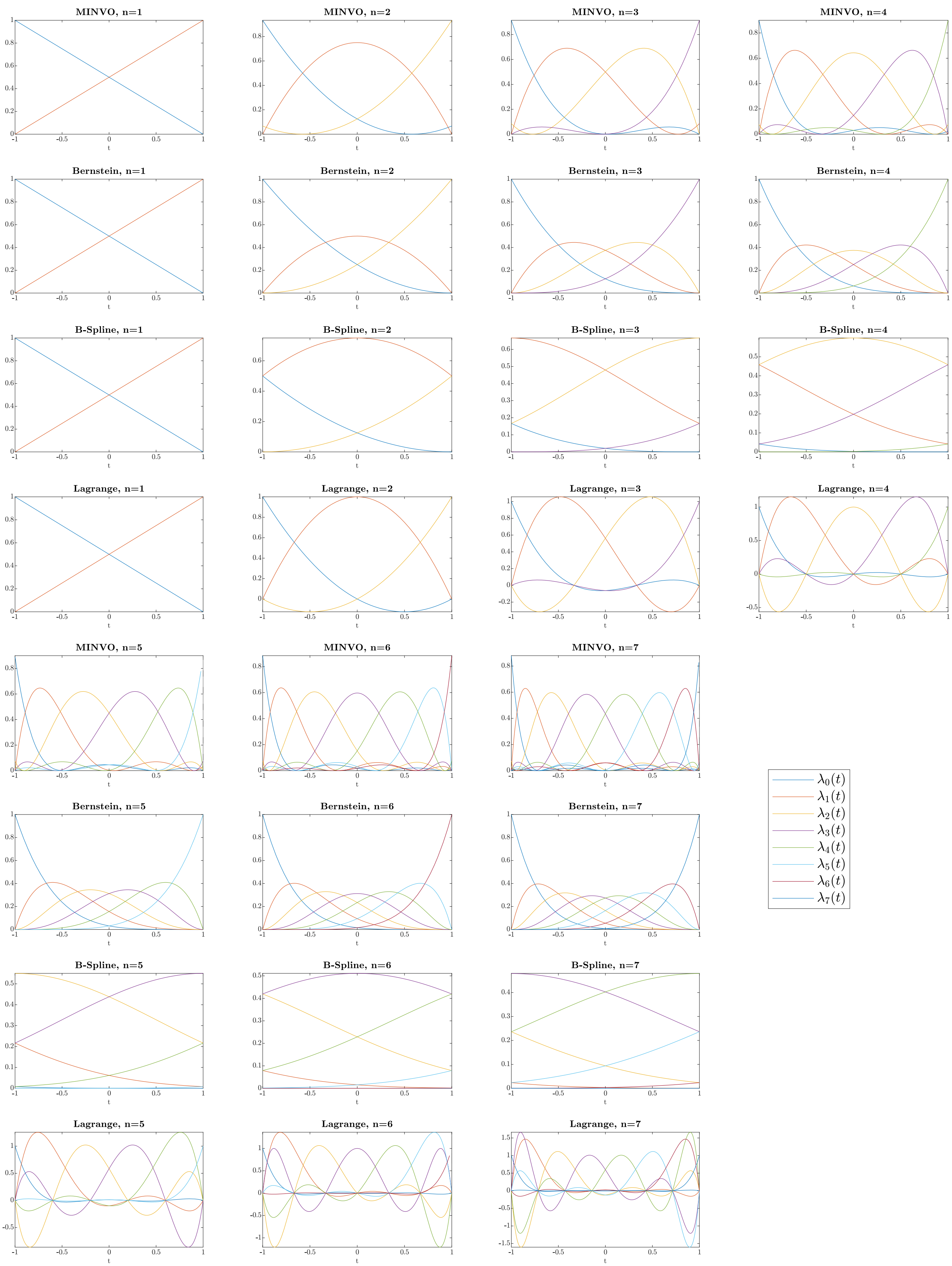

Figure 6: Comparison between the MINVO, Bernstein, B-Spline, and Lagrange bases for $n = 1 : 7$. All these bases satisfy $\sum_{i=0}^{n} \lambda_i(t) = 1$, and the MINVO, Bernstein and B-Spline bases also satisfy $\lambda_i(t) \geq 0 \ \ \forall t \in [-1, 1]$

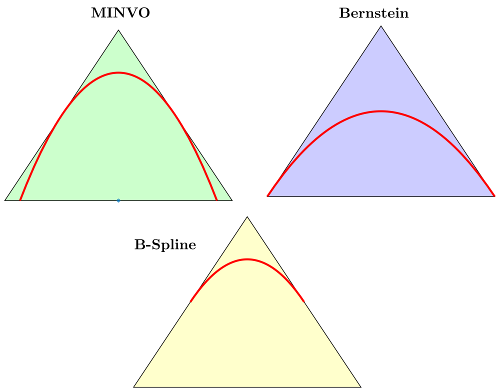

Figure 7: Optimal solution for Problem 2 with $n = 2$. The triangle is given, and the red polynomial curve using the MINVO basis is the one whose coefficients vectors $\boldsymbol{p}_2, \boldsymbol{p}_1$ span a rectangle with largest area, while being contained in the triangle. On the right, the curve obtained using the Bernstein basis.

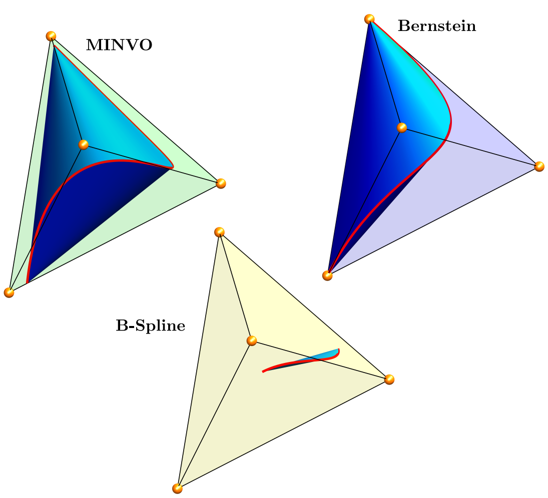

Figure 8: Optimal solution for Problem 2 with $n = 3$. The tetrahedron is given, and the red polynomial curve using the MINVO basis is the curve inside the tetrahedron whose coefficients vectors $\boldsymbol{p}_3, \boldsymbol{p}_2, \boldsymbol{p}_1$ span a paralellogram with largest volume. On the right, the curve obtained using the Bernstein basis.

VII. Conversion between MINVO, Bernstein/Bézier and B-Spline

Given an n-th order polynomial curve $\boldsymbol{p}(t) =: \boldsymbol{P}\boldsymbol{t}$, the MINVO, Bézier or B-Spline control points (i.e. the vertexes of the simplex that enclose that curve) can be obtained using the formula

$$\boldsymbol{V}_x = \boldsymbol{P}\left(\boldsymbol{A}_x\right)^{-1} = \boldsymbol{V}_y \boldsymbol{A}_y \left(\boldsymbol{A}_x\right)^{-1}$$

where $x, y \in \{\text{MV}, \text{Be}, \text{BS}\}$ (MINVO, Bernstein, B-Spline respectively). For instance to convert from MINVO control points to Bernstein control points we have that $\boldsymbol{V}_{\text{Be}} = \boldsymbol{P}\left(\boldsymbol{A}_{\text{Be}}\right)^{-1} = \boldsymbol{V}_{\text{MV}}\boldsymbol{A}_{\text{MV}}\left(\boldsymbol{A}_{\text{Be}}\right)^{-1}$. The matrices $\boldsymbol{A}_{\text{MV}}$ are the ones shown in table I, while the matrices $\boldsymbol{A}_{\text{BS}}$ and $\boldsymbol{A}_{\text{Be}}$ are available in [32]. Note that all the matrices need to be expressed in the same interval ($t \in [-1, 1]$ in this paper). The code provided with this paper contains functions that generate all these matrices.

==

Note that all the solutions found satisfy the neccesary condition found by [24] (the centroid of each face belongs to the convex body. In our case, to the convex hull of the curve)

Citar [33] (obtains same result for $n = 2$)

VIII. Conclusions and Future Work

This work derived and presented the MINVO basis. The key features of this basis is that it finds the smallest n-simplex that encloses a given polynomial curve (Problem 1), and also finds the polynomial curve that is inside a given simplex, and whose coefficient vectors $\boldsymbol{p}_n, ..., \boldsymbol{p}_1$ span a parallelogram with largest volume (Problem 2). Global optimality was also proven for some n.

The exciting results of this work naturally lead to the following questions and conjectures, that we leave as future work:

- Is the global optimum of Problem 3 the same as the global optimum of Problem 4? In other words, are we losing optimality by imposing the specific structure in $\lambda_i(t)$? The results seem to indicate that it is likely that no optimality is lost.
- Does there exist an analytical solution (i.e. not numerical) for Problem 3 for any degree n?
- Does there exist a recursive formula to obtain the solution of problem 3 for n given the previous solutions $1, ..., n-1$? Would this recursive formula allow to easily prove global optimality $\forall n$ of Problem 3?

IX. Acknowledgements

The authors would like to thank Prof. Gilbert Strang, Prof. Johan Löfberg, Ashkan M. Jasour, Kasra Khosoussi, Juan José Madrigal and Marc Adillón for helpul insights and discussions.

References

[1] E Dimas and D Briassoulis. 3d geometric modelling based on nurbs: a review. *Advances in Engineering Software*, 30(9-11):741–751, 1999.

[2] Eric A De Kemp. Visualization of complex geological structures using 3-d bezier construction tools. *Computers & Geosciences*, 25(5):581–597, 1999.

[3] Autodesk® Inventor®, 2020.

[4] Autodesk® Autocad®, 2020.

[5] Autodesk® Maya®, 2020.

[6] Adam W Bargteil and Elaine Cohen. Animation of deformable bodies with quadratic bézier finite elements. *ACM Transactions on Graphics (TOG)*, 33(3):1–10, 2014.

[7] SH Roth, Markus H Gross, Silvio Turello, and Friedrich R Carls. A bernstein-bézier based approach to soft tissue simulation. In *Computer Graphics Forum*, volume 17, pages 285–294. Wiley Online Library, 1998.

[8] FC Park and Bahram Ravani. Be´ zier curves on riemannian manifolds and lie groups with kinematics applications. 1995.

[9] Julian J Faraway, Matthew P Reed, and Jing Wang. Modelling three-dimensional trajectories by using bézier curves with application to hand motion. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 56(5):571–585, 2007.

[10] Igor Škrjanc and Gregor Klančar. Optimal cooperative collision avoidance between multiple robots based on bernstein–bézier curves. *Robotics and Autonomous systems*, 58(1):1–9, 2010.

[11] KG Jolly, R Sreerama Kumar, and R Vijayakumar. A bezier curve based path planning in a multi-agent robot soccer system without violating the acceleration limits. *Robotics and Autonomous Systems*, 57(1):23–33, 2009.

[12] Jesus Tordesillas, Brett T Lopez, and Jonathan P How. FASTER: Fast and safe trajectory planner for flights in unknown environments. In *2019 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2019.

[13] James A Preiss, Karol Hausman, Gaurav S Sukhatme, and Stephan Weiss. Trajectory optimization for self-calibration and navigation. In *Robotics: Science and Systems*, 2017.

[14] Ozgur Koray Sahingoz. Generation of Bézier curve-based flyable trajectories for multi-UAV systems with parallel genetic algorithm. *Journal of Intelligent & Robotic Systems*, 74(1-2):499–511, 2014.

[15] Gary Herron. Polynomial bases for quadratic and cubic polynomials which yield control points with small convex hulls. *Computer aided geometric design*, 6(1):1–9, 1989.

[16] Tatsumi Uezato, Mathieu Fauvel, and Nicolas Dobigeon. Hyperspectral unmixing with spectral variability using adaptive bundles and double sparsity. *IEEE Transactions on Geoscience and Remote Sensing*, 57(6):3980–3992, 2019.

[17] Eligius MT Hendrix, Inmaculada García, Javier Plaza, and Antonio Plaza. On the minimum volume simplex enclosure problem for estimating a linear mixing model. *Journal of Global Optimization*, 56(3):957–970, 2013.

[18] Derek M Rogge, Benoit Rivard, Jinkai Zhang, and Jilu Feng. Iterative spectral unmixing for optimizing per-pixel endmember sets. *IEEE Transactions on Geoscience and Remote Sensing*, 44(12):3725–3736, 2006.

[19] Marian-Daniel Iordache, José Bioucas-Dias, and António Plaza. Unmixing sparse hyperspectral mixtures. In *2009 IEEE International Geoscience and Remote Sensing Symposium*, volume 4, pages IV–85. IEEE, 2009.

[20] Miguel Velez-Reyes, Angela Puetz, Michael P Hoke, Ronald B Lockwood, and Samuel Rosario. Iterative algorithms for unmixing of hyperspectral imagery. In *Algorithms and Technologies for Multispectral, Hyperspectral, and Ultraspectral Imagery IX*, volume 5093, pages 418–429. International Society for Optics and Photonics, 2003.

[21] NG De Bruijn. On some multiple integrals involving determinants. *J. Indian Math. Soc*, 19:133–151, 1955.

[22] Pablo Parrilo. Lecture notes in algebraic techniques and semidefinite optimization, 206.

[23] Yunhong Zhou and Subhash Suri. Algorithms for a minimum volume enclosing simplex in three dimensions. *SIAM Journal on Computing*, 31(5):1339–1357, 2002.

[24] Victor Klee. Facet-centroids and volume minimization. *Studia Scientiarum Mathematicarum Hungarica*, 21:143–147, 1986.

[25] Andreas Wächter and Lorenz T Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming*, 106(1):25–57, 2006.

[26] Richard H Byrd, Jorge Nocedal, and Richard A Waltz. Knitro: An integrated package for nonlinear optimization. In *Large-scale nonlinear optimization*, pages 35–59. Springer, 2006.

[27] Richard H Byrd, Mary E Hribar, and Jorge Nocedal. An interior point algorithm for large-scale nonlinear programming. *SIAM Journal on Optimization*, 9(4):877–900, 1999.

[28] Matlab optimization toolbox, 2020. The MathWorks, Natick, MA, USA.

[29] J. Löfberg. Yalmip : A toolbox for modeling and optimization in matlab. In *In Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[30] Johan Löfberg. Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5):1007–1011, 2009.

[31] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[32] Kaihuai Qin. General matrix representations for b-splines. *The Visual Computer*, 16(3-4):177–186, 2000.

[33] József Kuti, Póter Galambos, and Póter Baranyi. Minimal volume simplex (mvs) approach for convex hull generation in tp model transformation. In *IEEE 18th International Conference on Intelligent Engineering Systems INES 2014*, pages 187–192. IEEE, 2014.

[34] Kostyantyn Mazur. Convex hull of $\left(t, t^2, \cdots, t^N\right)$. *arXiv preprint arXiv:1706.02060*, 2017.

[35] Jürgen Eckhoff. Helly, radon, and carathéodory type theorems. In *Handbook of convex geometry*, pages 389–448. Elsevier, 1993.

[36] Shui-Hung Hou and Edwin Hou. On a recursion formula related to confluent vandermonde. *The American Mathematical Monthly*, 122(8):766–772, 2015.

[37] kimchi lover (https://math.stackexchange.com/users/457779/kimchi lover). Closed-form solution for the determinant of a vandermonde-like matrix. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3743664 (version: 2020-07-03).

[38] metamorphy (https://math.stackexchange.com/users/543769/metamorphy). Closed-form solution for nested integrals of this polynomial? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3754782 (version: 2020-07-12).

Appendix: Proof of Theorem 1

For this proof, we will denote $\phi_j(t) := t^j \quad \psi_j(t) := jt^{j-1}$ and, for $i \neq j$, define also $K_{ij} := \int_0^1 \phi_i(t)\psi_j(t) = j\int_0^1 t^{i+j-1} = \frac{j}{i+j}$. Let us also define the $(n+1)\times(n+1)$ matrix $\hat{\boldsymbol{K}}$ with zeros in the diagonal and

$$\hat{K}_{ij} := \frac{1}{2}\left(K_{ij}-K_{ji}\right) = \frac{1}{2}\left(\frac{j-i}{j+i}\right) \quad i \neq j$$

We will also define $\boldsymbol{f}(t) := \begin{bmatrix} t^n & t^{n-1} & \cdots & t \end{bmatrix}^T$.

- **If n is odd** ($n = 3,5,7,...$). Let us define $a := \frac{n+1}{2}$, and the parameters $\alpha_0, \alpha_1, ..., \alpha_{a-2}$ and $t_0, t_1, ..., t_{a-1}$. Moreover, we will use $i \in \{0,1,...,a-1\}$ and $j \in \{0,1,...,a-1\}$ Using the corollary of [34](Sec. 1), every point in $conv(\boldsymbol{p}(t))$ can be expressed as a convex combination of a points of the polynomial curve. (This paper provides, for a polynomial curve, a tighter bound than [35]. Denote these points as $\boldsymbol{p}(t_0), ..., \boldsymbol{p}(t_{a-1})$. We have then that:

$$\boldsymbol{p}(t) = \alpha_0\boldsymbol{p}(t_0) + ... + \alpha_{a-2}\boldsymbol{p}(t_{a-2}) + \left(1-\sum_{i=0}^{a-2}\alpha_i\right)\boldsymbol{p}(t_{a-1}) \quad \alpha_i > 0, 1-\sum_{i=0}^{a-2}\alpha_i > 0, t_i \in [0,1], t_i > t_{i-1}$$

And therefore, the volume is given by:

$$Vol\left(conv\left(\boldsymbol{p}(t)\right)\right) = \overbrace{\int_0^1\int_0^{1-\alpha_0}\int_0^{1-\alpha_0-\alpha_1}...\int_0^{1-\sum_{i=0}^{a-3}\alpha_i}}^{a-1 \text{ integrals}}\left[\overbrace{\int_{-1}^1\int_{t_0}^1\int_{t_1}^1...\int_{t_{a-2}}^1}^{a \text{ integrals}} J dt_{a-1}dt_{a-2}...dt_0\right] d\alpha_{a-2}d\alpha_{a-3}...d\alpha_0$$

where J is the absolute value of the Jacobian:

$$J := \text{abs}\left(\begin{vmatrix} \frac{\partial \boldsymbol{p}(t)}{\alpha_0} & \cdots & \frac{\partial \boldsymbol{p}(t)}{\alpha_{a-2}} & \frac{\partial \boldsymbol{p}(t)}{t_0} & \cdots & \frac{\partial \boldsymbol{p}(t)}{t_{a-1}} \end{vmatrix}\right) \overset{\text{(a)}}{=} \text{abs}\left(\left|\boldsymbol{P}_{:,0:n}\right|\left|\boldsymbol{R}\right|\left(\prod_{i=0}^{a-2}\alpha_i\right)\left(1-\sum_{i=0}^{a-2}\alpha_i\right)\right)$$

where

$$\boldsymbol{R} := \left[\begin{array}{c:c:c:c:c:c:c:c} \boldsymbol{f}(t_0)-\boldsymbol{f}(t_{a-1}) & \boldsymbol{f}(t_1)-\boldsymbol{f}(t_{a-1}) & \cdots & \boldsymbol{f}(t_{a-2})-\boldsymbol{f}(t_{a-1}) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \cdots & \frac{\partial \boldsymbol{f}(t_{a-1})}{\partial t_{a-1}} \end{array}\right]$$

Now, note that:

$$\text{abs}\left(|\boldsymbol{R}|\right) = \text{abs}\left(\begin{vmatrix} \boldsymbol{R} & \boldsymbol{f}(t_{a-1}) \\ \boldsymbol{0}^T & 1 \end{vmatrix}\right) = \text{abs}\left(\begin{vmatrix} \boldsymbol{f}(t_0) & \boldsymbol{f}(t_1) & \cdots & \boldsymbol{f}(t_{a-2}) & \boldsymbol{f}(t_{a-1}) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \cdots & \frac{\partial \boldsymbol{f}(t_{a-1})}{\partial t_{a-1}} \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{vmatrix}\right) =$$

$$= abs\left(\begin{vmatrix} \boldsymbol{f}(t_0) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \boldsymbol{f}(t_1) & \frac{\partial \boldsymbol{f}(t_1)}{\partial t_1} & \cdots & \boldsymbol{f}(t_{a-1}) & \frac{\partial \boldsymbol{f}(t_{a-1})}{\partial t_{a-1}} \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}\right) \overset{\text{(b)}}{=} \prod_{i<j}(t_i-t_j)^4 = \prod_{i\neq j}(t_i-t_j)^2$$

a: Expressing the matrix as the product of two matrices and applying the properties of the determinants.
b: See [36] and [37].
c: See ??
d: Using simmetry.
e: Using the change of variables $t_i = 2q_i - 1$ [38].
Now, note that:

- $I_1 := \overbrace{\int_0^1\int_0^{1-\alpha_0}...\int_0^{1-\sum_{i=0}^{a-3}\alpha_i}}^{a-1 \text{ integrals}}\left(\prod_{i=0}^{a-2}\alpha_i\right)\left(1-\sum_{i=0}^{a-2}\alpha_i\right)d\alpha_{a-2}d\alpha_{a-3}...d\alpha_0 \overset{\text{(c)}}{=} \frac{1}{n!}$
- $I_2 := \overbrace{\int_{-1}^1\int_{t_0}^1...\int_{t_{a-2}}^1}^{a \text{ integrals}} \text{abs}\left(|\boldsymbol{R}|\right)dt_{a-1}dt_{a-2}...dt_0 \overset{\text{(d)}}{=} \frac{1}{a!}\int_{[-1,1]^a}\text{abs}\left(|\boldsymbol{R}|\right)dV \overset{\text{(e)}}{=} \frac{2^{a(2a-1)}}{a!}\int_{[0,1]^a}\prod_{i\neq j}(q_i-q_j)^2 dV$

$$\left|\hat{K}\right| = \pm\frac{1}{2^{n+1}}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)^2 \implies \text{Pf}(\hat{K}) = \pm\frac{1}{2^{\frac{n+1}{2}}}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)$$

And therefore, we have that $I_2 = \frac{2^{a(2a-1)}}{a!}2^a(a!)\frac{1}{2^a}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right) = 2^{a(2a-1)}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)$. Finally:

$$\boxed{\omega = \text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right) I_1 I_2 = \frac{\text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right)}{n!}2^{\frac{n(n+1)}{2}}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)}$$

- **If** n **is even** (n=2,4,6,...): Let us define $a := \frac{n}{2}-1$, and the parameters $\alpha_0, ..., \alpha_a$ and $t_0, ..., t_a$. Moreover, and unless otherwise noted, we will use $i \in \{0, 1, ..., a\}$ and $j \in \{0, 1, ..., a\}$. Using the corollary of [34](Sec. 1), every point in $conv(\boldsymbol{p}(t))$ can be expressed as a convex combination of $a+2$ points of the polynomial curve, with one point fixed (that we will take as $\boldsymbol{p}(1)$). Denote these points as $\boldsymbol{p}(1), \boldsymbol{p}(t_0), ..., \boldsymbol{p}(t_a)$. We have then that:

$$\boldsymbol{p}(t) = \alpha_0\boldsymbol{p}(1) + \alpha_1\boldsymbol{p}(t_0) + ... + \alpha_a\boldsymbol{p}(t_{a-1}) + \left(1-\sum_{i=0}^{a}\alpha_i\right)\boldsymbol{p}(t_a) \quad \alpha_i > 0, 1-\sum_{i=0}^{a}\alpha_i > 0, t_i \in [0,1], t_i > t_{i-1}$$

And therefore, the volume is given by:

$$\omega := \overbrace{\int_0^1\int_0^{1-\alpha_0}...\int_0^{1-\sum_{i=0}^{a-1}\alpha_i}}^{a+1 \text{ integrals}}\left[\overbrace{\int_{-1}^1\int_{t_0}^1...\int_{t_{a-1}}^1}^{a+1 \text{ integrals}} J dt_a dt_{a-1}...dt_0\right] d\alpha_a d\alpha_{a-1}...d\alpha_0$$

where J is the absolute value of the Jacobian:

$$J := abs\left(\begin{vmatrix} \frac{\partial \boldsymbol{p}(t)}{\alpha_0} & \cdots & \frac{\partial \boldsymbol{p}(t)}{\alpha_a} & \frac{\partial \boldsymbol{p}(t)}{t_0} & \cdots & \frac{\partial \boldsymbol{p}(t)}{t_a}\end{vmatrix}\right) = \text{abs}\left(|\boldsymbol{P}_{:,0:n}|\,|\boldsymbol{R}|\left(\prod_{i=1}^{a}\alpha_i\right)\left(1-\sum_{i=0}^{a-1}\alpha_i\right)\right)$$

where

$$\boldsymbol{R} := \left[\begin{array}{c:c:c:c:c:c:c:c} \boldsymbol{1}-\boldsymbol{f}(t_a) & \boldsymbol{f}(t_0)-\boldsymbol{f}(t_a) & \cdots & \boldsymbol{f}(t_{a-1})-\boldsymbol{f}(t_a) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \cdots & \frac{\partial \boldsymbol{f}(t_a)}{\partial t_a}\end{array}\right]$$

Now, note that:

$$\text{abs}\left(|\boldsymbol{R}|\right) = \text{abs}\left(\begin{vmatrix}\boldsymbol{R} & \boldsymbol{f}(t_a)\\ \boldsymbol{0}^T & 1\end{vmatrix}\right) = \text{abs}\left(\begin{vmatrix}\boldsymbol{1} & \boldsymbol{f}(t_0) & \cdots & \boldsymbol{f}(t_{a-1}) & \boldsymbol{f}(t_a) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \cdots & \frac{\partial \boldsymbol{f}(t_a)}{\partial t_a}\\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0\end{vmatrix}\right) =$$

$$= \text{abs}\left(\begin{vmatrix}\boldsymbol{f}(t_0) & \frac{\partial \boldsymbol{f}(t_0)}{\partial t_0} & \cdots & \boldsymbol{f}(t_a) & \frac{\partial \boldsymbol{f}(t_a)}{\partial t_a} & \boldsymbol{1}\\ 1 & 0 & \cdots & 1 & 0 & 1\end{vmatrix}\right) = \left(\prod_{i\neq j}(t_i-t_j)^2\right)\left(\prod_i (t_i-1)^2\right)$$

Now note that:

- $I_1 := \overbrace{\int_0^1\int_0^{1-\alpha_0}...\int_0^{1-\sum_{i=0}^{a-1}\alpha_i}}^{a+1 \text{ integrals}}\left(\prod_{i=1}^{a}\alpha_i\right)\left(1-\sum_{i=0}^{a}\alpha_i\right)d\alpha_a d\alpha_{a-1}...d\alpha_0 = \frac{1}{n!}$
- $I_2 := \overbrace{\int_{-1}^1\int_{t_0}^1...\int_{t_{a-1}}^1}^{a+1 \text{ integrals}}\text{abs}\left(|\boldsymbol{R}|\right)dt_a dt_{a-1}...dt_0 = \frac{1}{(a+1)!}\int_{[-1,1]^{a+1}}\text{abs}\left(|\boldsymbol{R}|\right)dV = \frac{2^{a+1}2^{2a(a+1)}2^{2(a+1)}}{(a+1)!}\int_{[0,1]^{a+1}} .. = \frac{2^{2a^2+5a+3}}{(a+1)!}\int_{[0,1]^{a+1}}\left(\prod_{i\neq j}(q_i-q_j)^2\right)\left(\prod_i(q_i-1)^2\right)dV$

where we have used the sustitution $t_i = 2q_i - 1$ and used the simmetry. Now, note that:

$$\text{Pf}(\hat{\boldsymbol{K}}) = \text{Pf}\left(\begin{bmatrix}\hat{\boldsymbol{K}} & \boldsymbol{1}\\ -\boldsymbol{1}^{\boldsymbol{T}} & 0\end{bmatrix}\right) = \frac{1}{2^{a+1}}\text{Pf}\left(\begin{bmatrix}2\hat{\boldsymbol{K}} & \boldsymbol{1}\\ -\boldsymbol{1}^{\boldsymbol{T}} & 0\end{bmatrix}\right)$$

and therefore we have that $I_2 = \frac{2^{2a^2+5a+3}}{(a+1)!}2^{a+1}(a+1)!\text{abs}\left(\text{Pf}(\hat{\boldsymbol{K}})\right) = 2^{2a^2+5a+3}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)$. Finally:

$$\boxed{\omega = \text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right) I_1 I_2 = \frac{\text{abs}\left(|\boldsymbol{P}_{:,0:n}|\right)}{n!}2^{\frac{n(n+1)}{2}}\prod_{0\leq i<j\leq n}\left(\frac{j-i}{j+i}\right)}$$

$\square$

(for the second step see Proposition 2.6 of https://d31kydh6n6r5j5.cloudfront.net/uploads/sites/66/2019/04/pfaffian_comps_06_07.pdf)

For the third step, see properties in wikipedia

For the first step, see paper.

Citar stackoverflow??

Cuidado, acordarse de poner valores absolutos everywhere!!

Citar theorem 2.2 y 2.3 de de http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.973.6094&rep=rep1&type=pdf (aunque ellos parece que lo cojen de Karlin/Shapley)

Comprobar que las formulas estan bien cuando P es una moment curve (ver formula en “Geometry of Moment Spaces, page 51”)