Table II Explanation

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The values in Table II are calculated similarly to Dilution of Precision (DOP), a common uncertainty metric in GPS [1]. Specifically, we are computing Position DOP (PDOP), the norm of the x, y, z covariances (i.e., σ_{xx}, σ_{yy}, σ_{zz}), after being propagated through the nonlinear geometry linearized with respect to position and then evaluated at some relative pose $\mathbf{T}_B^A$.

This document breaks down this computation and shows how these values are calculated in the accompanying Jupyter Notebook.

1 Problem Formulation:

Consider two agents, A and B, in world frame W with true state vectors:

$$\mathbf{x}_A^W = \begin{bmatrix} x_A^W & y_A^W & z_A^W & \alpha_A^W & \beta_A^W & \gamma_A^W \end{bmatrix}^\top = \mathbf{T}_A^W$$

$$\mathbf{x}_B^W = \begin{bmatrix} x_B^W & y_B^W & z_B^W & \alpha_B^W & \beta_B^W & \gamma_B^W \end{bmatrix}^\top = \mathbf{T}_B^W$$

where $\langle x, y, z \rangle$ refers to position and $\langle \alpha, \beta, \gamma \rangle$ refers to roll/pitch/yaw, superscript represents the frame, and subscript represents the target. While in general $\mathbf{x} \leftrightarrow \mathbf{T}$, $\mathbf{x}$ refers to a column state vector while $\mathbf{T}$ refers to $\mathbb{R}^{4\times4}$ transformation matrix. Additionally, let $\mathbf{T}(\cdot)$ refer to the function that constructs a transformation matrix. Specifically, the following notations and shorthands are equivalent:

$$\begin{aligned} \mathbf{T}(\mathbf{x}) &= \mathbf{T}(x, y, z, \alpha, \beta, \gamma) \\ \mathbf{T}(x, y, z) &= \mathbf{T}(x, y, z, 0, 0, 0) \\ \mathbf{T}(x, y, z, \gamma) &= \mathbf{T}(x, y, z, 0, 0, \gamma) \end{aligned}$$

Let the relative state from agent A to agent B be defined as:

$$\mathbf{x} = \mathbf{x}_B^A = \begin{bmatrix} x_B^A & y_B^A & z_B^A & \alpha_B^A & \beta_B^A & \gamma_B^A \end{bmatrix}^\top = \left(\mathbf{T}_A^W\right)^{-1} \mathbf{T}_B^W = \mathbf{T}_B^A$$

Relative coordinate frames are assumed to be oriented with respect to downward gravity.
Here, we have:

- $\hat{\mathbf{x}}$ are the estimated state parameters
- $\mathbf{z}$ are the current measurements

where:

- $\hat{\mathbf{x}} \in \mathbb{R}^3 \implies \hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_B^A & \hat{y}_B^A & \hat{z}_B^A \end{bmatrix}$
- $\mathbf{z} \in \mathbb{R}^{37}$ has measurement model:

$$\mathbf{h}(\mathbf{x}_B^A) = \begin{bmatrix} \mathbf{d} & z_B^A \end{bmatrix}^\top$$

where $z_B^A \in \mathbb{R}$ is the relative altitude and $\mathbf{d}$ is the vector of true UWB range. In this case, there are $\mathbf{d} \in \mathbb{R}^{36}$ since agent A and B have $N_A = 6$ and $N_B = 6$ antennas respectively. In general, $\mathbf{d}$ has the form:

$$\mathbf{d} = \begin{bmatrix} d_{11} & d_{12} & \quad \dots & d_{N_A N_B} \end{bmatrix}^\top$$

with each element defined as:

$$d_{ij} = d_{ij}(\mathbf{x}_B^A) = ||\mathbf{T}_B^A \mathbf{p}_j^B - \mathbf{p}_i^A||_2$$

where $\mathbf{p}_i^A$ and $\mathbf{p}_j^B$ is the homogeneous coordinate of agent A's ith and B's jth antenna in A's and B's frame respectively.

With this measurement model, we can convince ourselves the system is fully observable. Specifically, the direct inclusion of z_B^A into the measurement model resolves the usual $\pm z$ ambiguity stemming from antennas only spanning their relative xy-plane.

We note that while z_B^A is not directly measurable, since both local z-axes are aligned with gravity down, z_B^A can be calculated as the difference of two realizable measurements:

$$z_B^A = z_B^W - z_A^W$$

In practice, while we can always locally measure z_A^W, we do not want to transmit z_B^W. This will need to be considered in our error analysis.

2 Computing PDOP:

Consider following formulas from [1]:

$$\Sigma_x = (A^\top \Sigma_z^{-1} A)^{-1}$$

in general, which can be simplified to:

$$\Sigma_x = (A^\top A)^{-1} \sigma^2$$

when covariances are consistent across all measurements and independent. Here, we have:

- A is the linearized measurement model about some point $\mathbf{x}_0$
- Σ_z is the covariance matrix of the measurements $\mathbf{z}$
- Σ_x is the covariance matrix of the state variables $\mathbf{x}$ after being propagated through the linearized measurement model A

When calculating DOP, the diagonal values of $(A^\top A)^{-1}$ are `XDOP`, `YDOP`, `ZDOP` values (i.e., the variance amplification resulting from the linearized geometry in various state dimensions).

In general since d_{ij} is a UWB measurement and z_B^A is some calculated combination of a low variance LiDAR measurement and/or high variance assumption, we have:

$$\texttt{var}\,[d_{ij}] \neq \texttt{var}\left[z_B^A\right]$$

forcing us to use the former equation to calculate Σ_x.

In the paper (Figure 2a), we experimentally measured our UWB measurement variance to be:

$$\texttt{var}\,[d_{ij}] = 0.24\text{m}$$

For the purpose of this error analysis, we will assume the measurement error is modeled by a zero mean Gaussian.

Depending on our experimental setup, z_A^W and z_B^W are either measured by downward facing LiDAR (with a reported operating tolerance of $\pm$4cm in our operating range) or assumed by some pre-communicated (but locally monitored) global altitude tolerance (i.e., $\bar{z}_A^W \pm \tilde{z}_A^W$ and $\bar{z}_B^W \pm \tilde{z}_B^W$ respectively where $\bar{z}$ is the commanded altitude and $\tilde{z}$ is the altitude tolerance).

3 Jupyter Notebook:

Using the $\hat{\mathbf{x}} \in \mathbb{R}^3$ and $\mathbf{z} \in \mathbb{R}^{37}$ as defined above, we perform the above error analysis with different values for $\texttt{var}\left[z_B^A\right]$ (selected based on current communication/sensor assumptions and encoded in Σ_z) and different measurement model linearization points (select based on current position and encoded in A) and calculate the corresponding PDOPs. This implies A is of the form $\mathbb{R}^{37\times 3}$, making this a cumbersome computation to do by hand. Thus, we make the Jupyter Notebook used to compute Table II (and other relevant facts) available here:

- `https://github.com/mit-acl/murp-datasets`

References

[1] Richard B Langley et al. Dilution of precision. *GPS world*, 10(5):52–59, 1999.