# Optimizing Stream Programs Using Linear State Space Analysis

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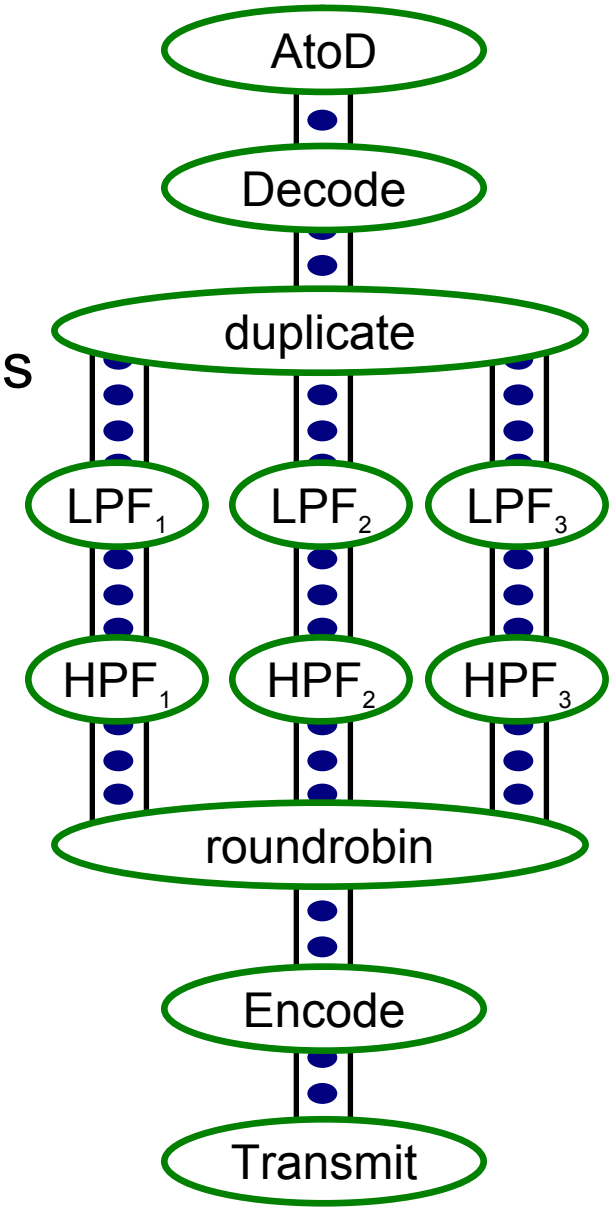
CASES 2005

StreamIt

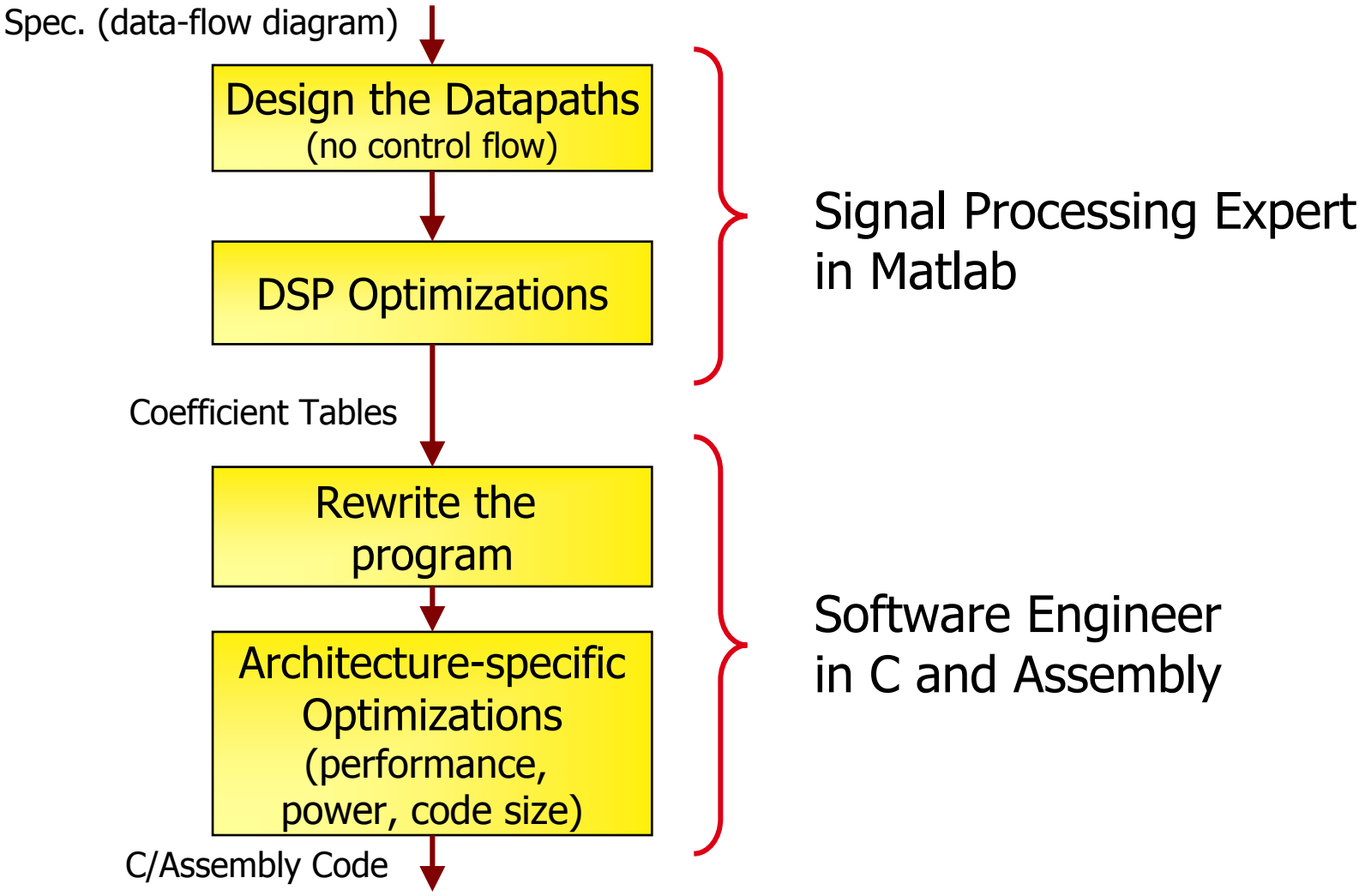
http://cag.lcs.mit.edu/streamit

# Streaming Application Domain

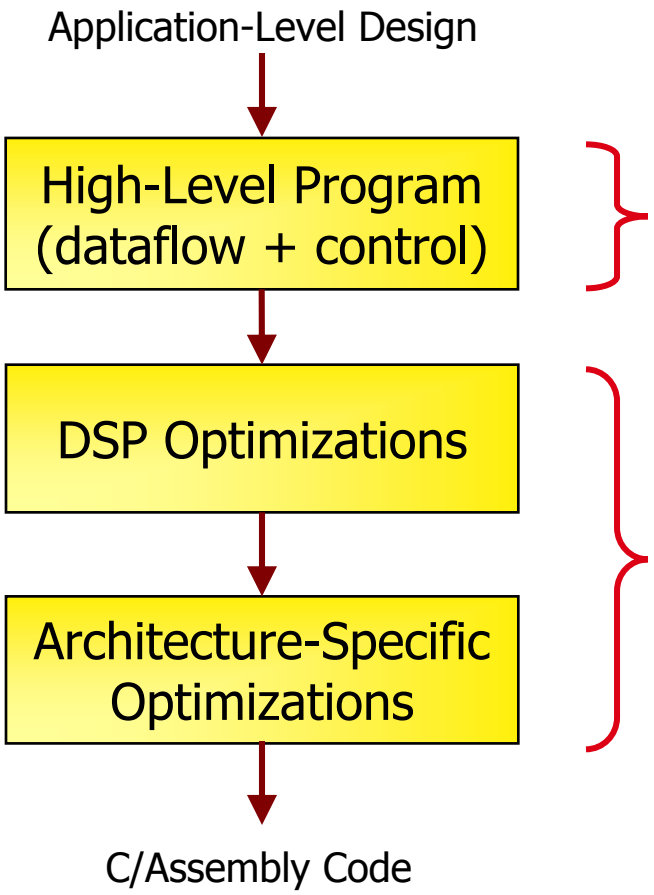
- Based on a stream of data
  - Graphics, multimedia, software radio
  - Radar tracking, microphone arrays, HDTV editing, cell phone base stations
- Properties of stream programs
  - Regular and repeating computation
  - Parallel, independent actors with explicit communication
  - Data items have short lifetimes



# Conventional DSP Design Flow



# Ideal DSP Design Flow




Application Programmer

Compiler

*Challenge: maintaining performance*

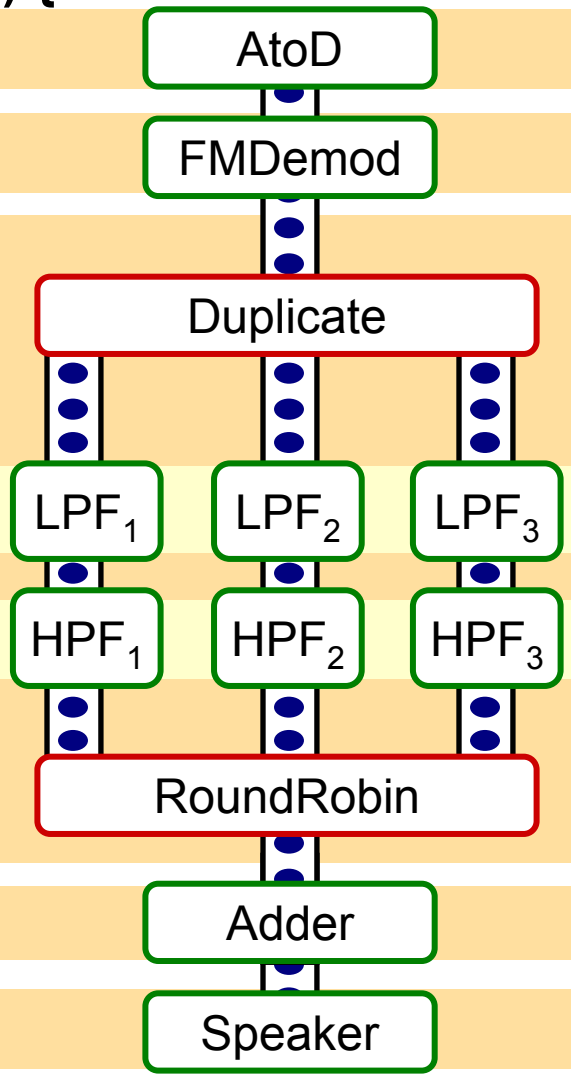
# The StreamIt Language

- Goals:
  - Provide a high-level stream programming model
  - Invent new compiler technology for streams

- Contributions:
  - Language design [CC '02, PPoPP '05]
  - Compiling to tiled architectures [ASPLOS '02, ISCA '04, Graphics Hardware '05]
  - Cache-aware scheduling [LCTES '03, LCTES '05]
  - Domain-specific optimizations [PLDI '03, CASES '05]

# Programming in StreamIt

```
void->void pipeline FMRadio(int N, float lo, float hi) {
  add AtoD();
  add FMDemod();
  add splitjoin {
    split duplicate;
    for (int i=0; i<N; i++) {
      add pipeline {
        add LowPassFilter(lo + i*(hi - lo)/N);
        add HighPassFilter(lo + i*(hi - lo)/N);
      }
    }
    join roundrobin();
  }
  add Adder();
  add Speaker();
}
```

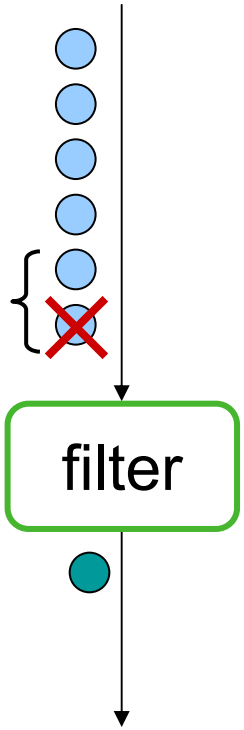


# Example StreamIt Filter

```
float->float filter LowPassButterWorth (float sampleRate, float cutoff) {
    float coeff;
    float x;

    init {
        coeff = calcCoeff(sampleRate, cutoff);
    }

    work peek 2 push 1 pop 1 {
        x = peek(0) + peek(1) + coeff * x;
        push(x);
        pop();
    }
}
```



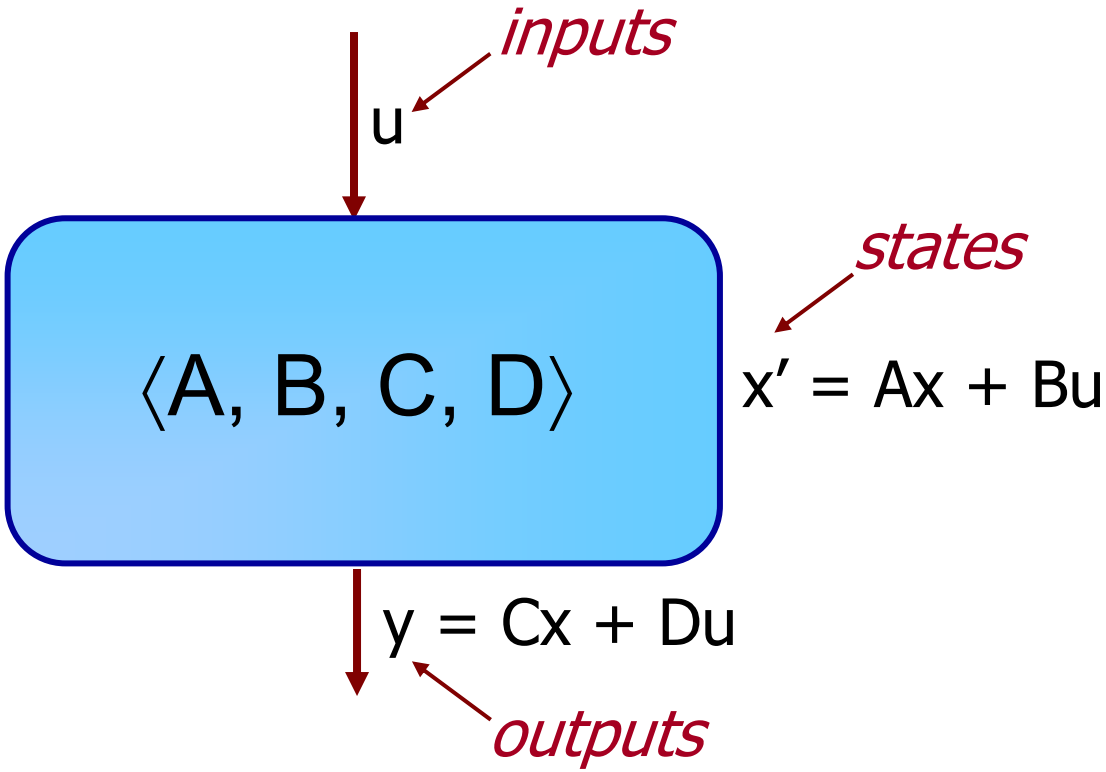
# Focus: Linear State Space Filters

- Properties:
  1. Outputs are linear function of inputs and states
  2. New states are linear function of inputs and states

- Most common target of DSP optimizations
  - FIR / IIR filters
  - Linear difference equations
  - Upsamplers / downsamplers
  - DCTs

# Representing State Space Filters

- A state space filter is a tuple ⟨A, B, C, D⟩



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float->float filter IIR {
  float x1, x2;
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    float u = pop();
    push(2*(x1+x2+u));
    x1 = 0.9*x1 + 0.3*u;
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} }
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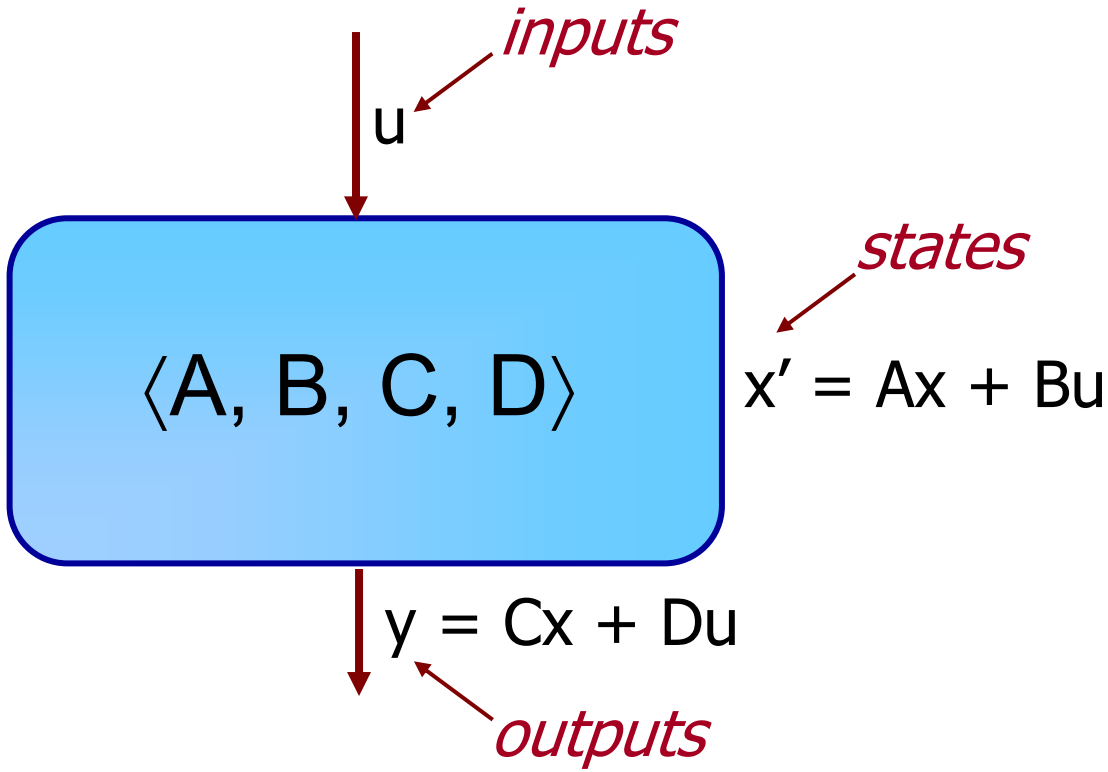

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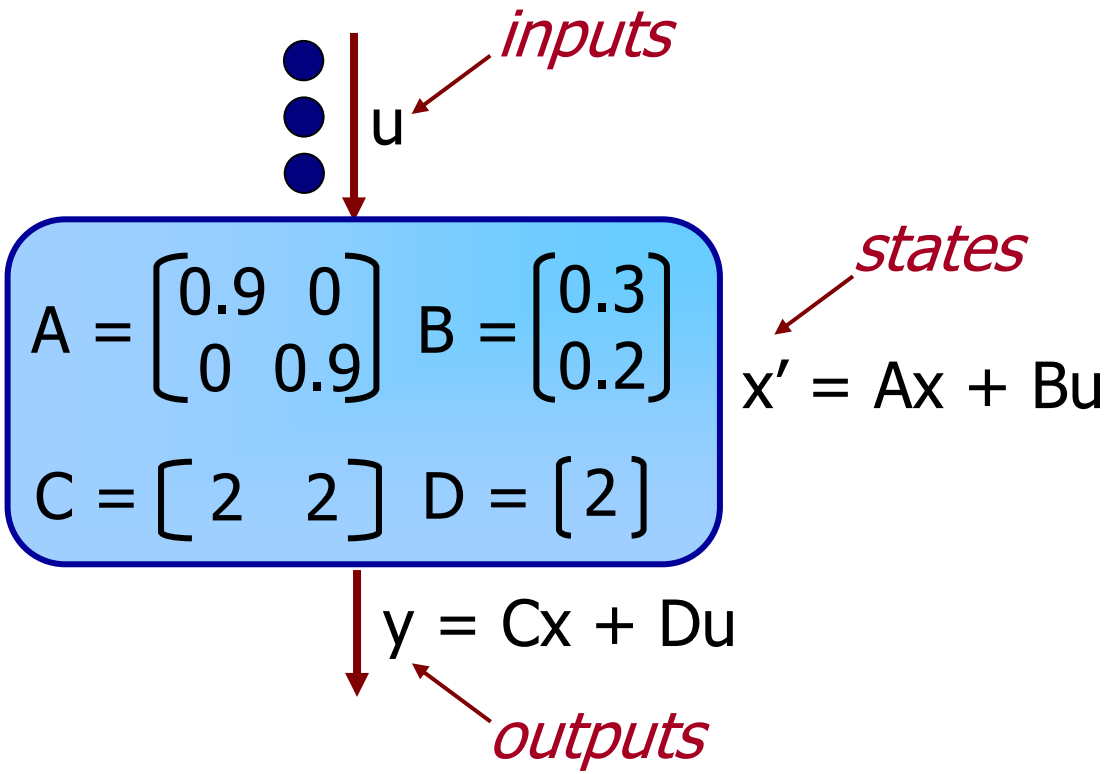

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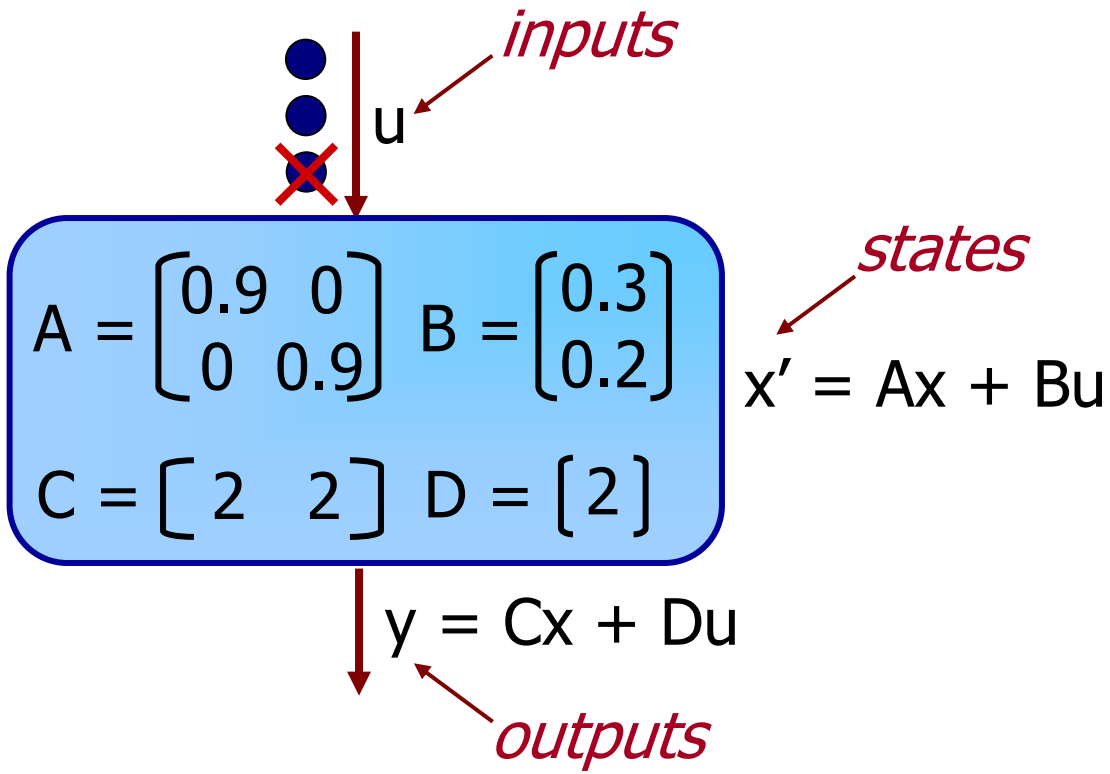

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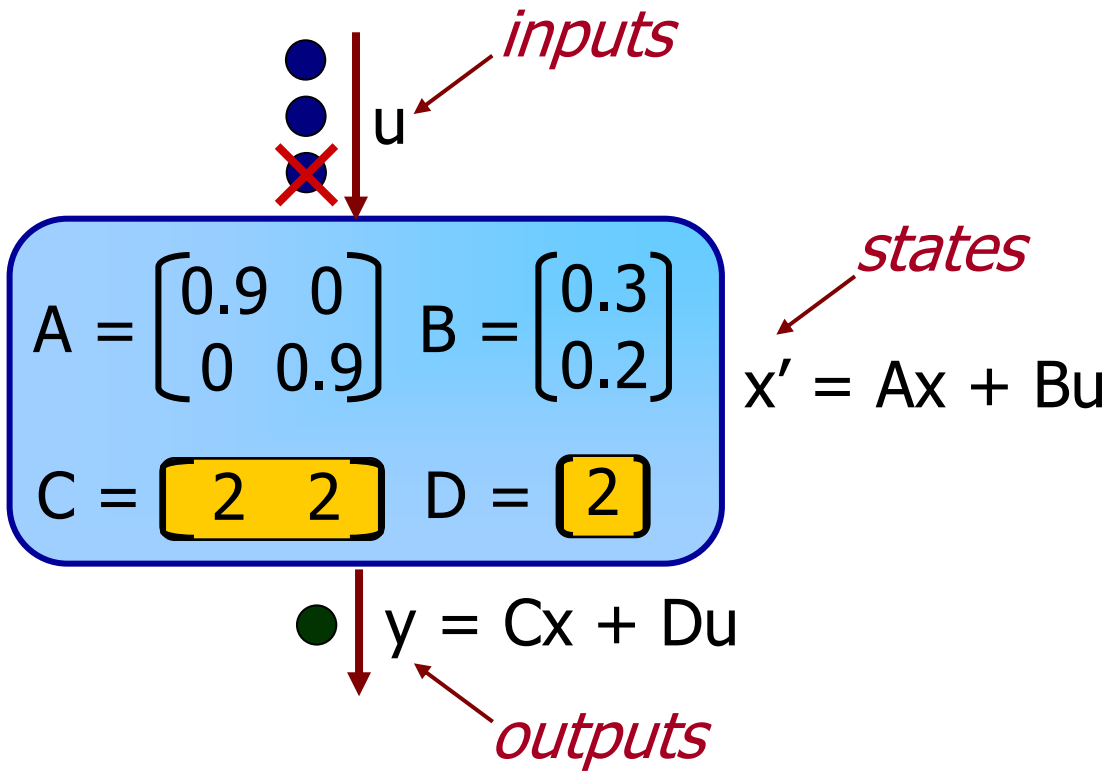

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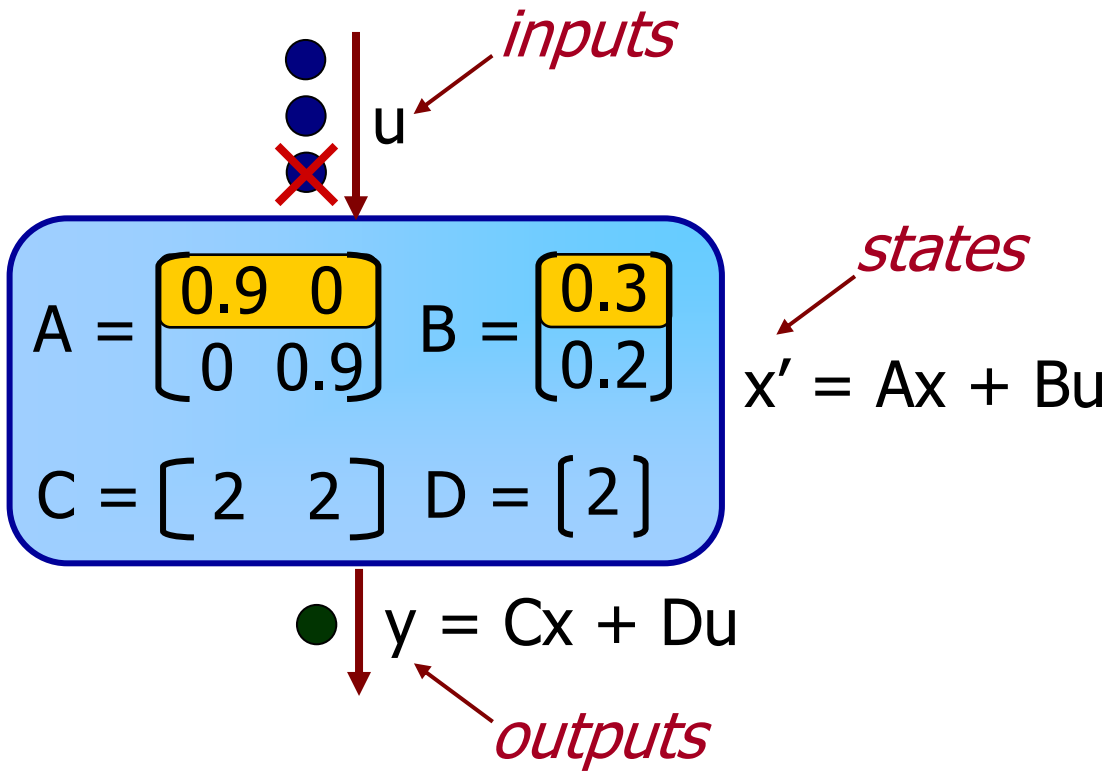

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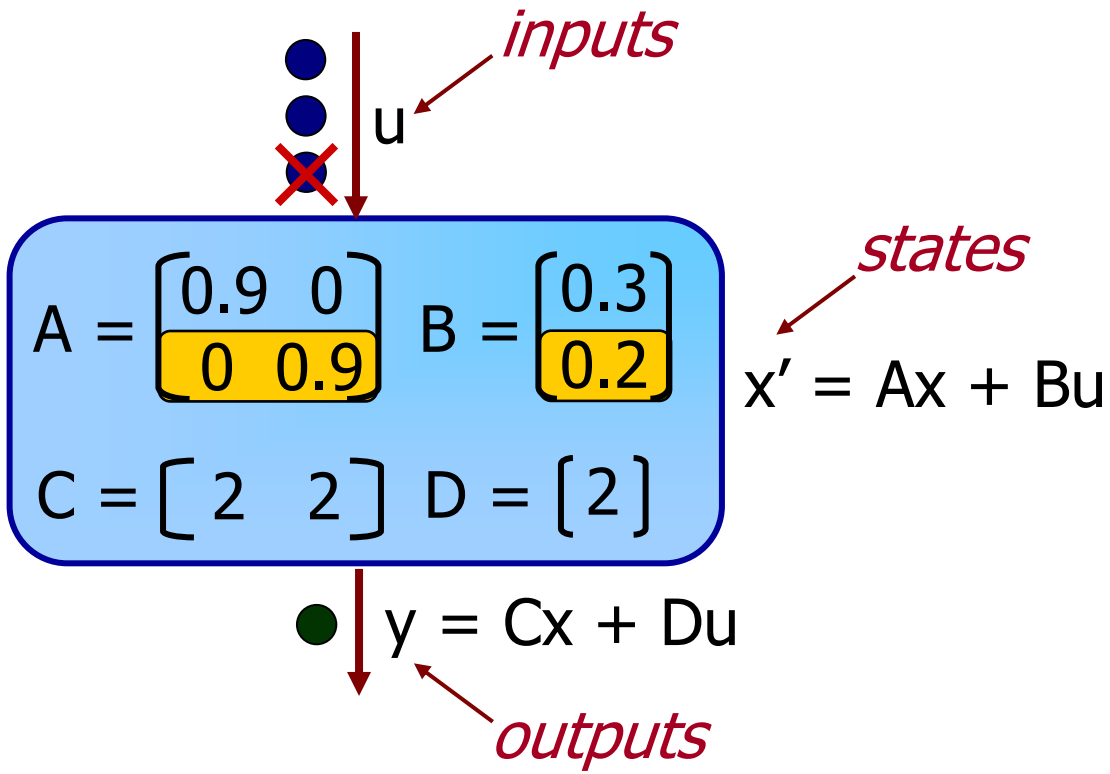

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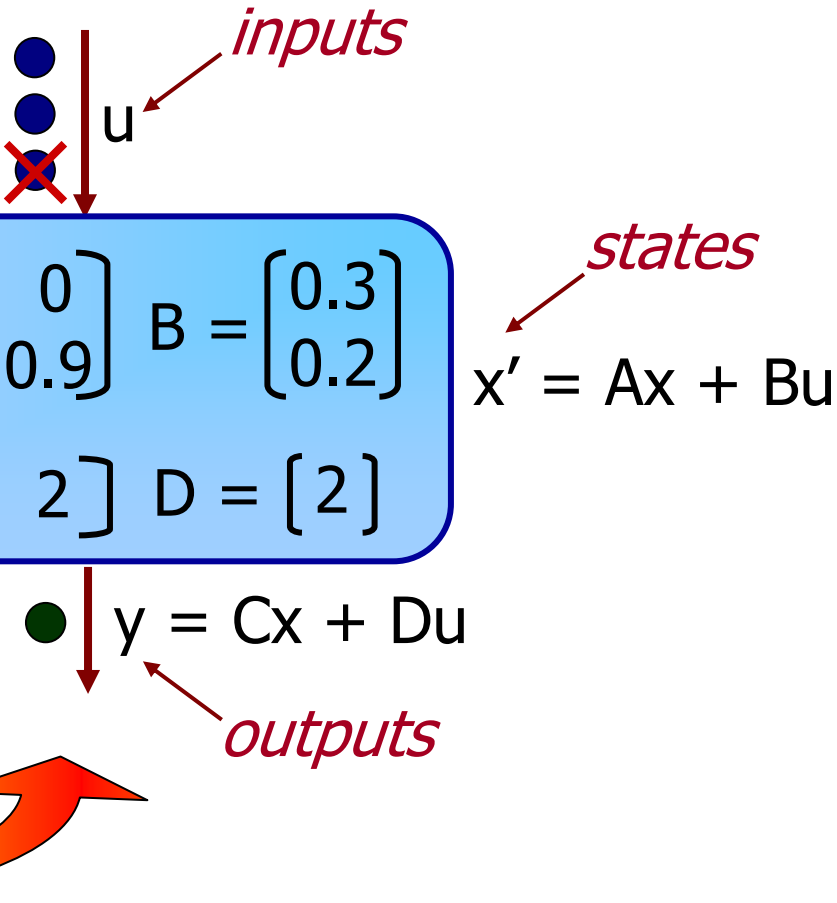


Linear dataflow analysis

# State Space Optimizations

1. State removal
2. Reducing the number of parameters
3. Combining adjacent filters

# Change-of-Basis Transformation

$$x' = Ax + Bu$$
$$y = Cx + Du$$

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T = invertible matrix

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$$x' = Ax + Bu$$
$$y = Cx + Du$$

T = invertible matrix

$$Tx' = TA(T^{-1}T)x + TBu$$
$$y = C(T^{-1}T)x + Du$$

# Change-of-Basis Transformation

$$x' = Ax + Bu$$
$$y = Cx + Du$$

T = invertible matrix

$$Tx' = TAT^{-1}(Tx) + TBu$$
$$y = CT^{-1}(Tx) + Du$$

# Change-of-Basis Transformation

$$x' = Ax + Bu$$
$$y = Cx + Du$$

T = invertible matrix, z = Tx

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$$x' = Ax + Bu$$
$$y = Cx + Du$$

T = invertible matrix, z = Tx

$$z' = A'z + B'u$$
$$y = C'z + D'u$$

$$A' = TAT^{-1} \quad B' = TB$$
$$C' = CT^{-1} \quad D' = D$$

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$$A' = TAT^{-1} \quad B' = TB$$
$$C' = CT^{-1} \quad D' = D$$

Can map original states x to transformed states z = Tx without changing I/O behavior

# 1) State Removal

- Can remove states which are:
  a. Unreachable – do not depend on input
  b. Unobservable – do not affect output

- To expose unreachable states, reduce [A | B] to a kind of row-echelon form
  – For unobservable states, reduce $[A^T \mid C^T]$
- Automatically finds minimal number of states

# State Removal Example

$$x' = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix} x + \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 \end{bmatrix} x + 2u$$

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$x' = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix} x + \begin{bmatrix} 0.3 \\ \mathbf{0.5} \end{bmatrix} u$$

$$y = \begin{bmatrix} \mathbf{0} & 2 \end{bmatrix} x + 2u$$

```
float->float filter IIR {
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$$y = \begin{bmatrix} \mathbf{0} & 2 \end{bmatrix} x + 2u$$

*x1 is unobservable*

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$$x' = 0.9x + 0.5u$$

$$y = 2x + 2u$$

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```

# State Removal Example

9 FLOPs
12 load/store

output

5 FLOPs
8 load/store

output

```
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} }
```

# 2) Parameter Reduction

- Goal:
  Convert matrix entries (parameters) to 0 or 1

- Allows static evaluation:

  1*x → x  Eliminate 1 multiply

  0*x + y → y  Eliminate 1 multiply, 1 add

- Algorithm (Ackerman & Bucy, 1971)
  - Also reduces matrices [A | B] and [$A^T$ | $C^T$]
  - Attains a canonical form with few parameters

# Parameter Reduction Example

$x' = 0.9x + 0.5u$
$y = 2x + 2u$

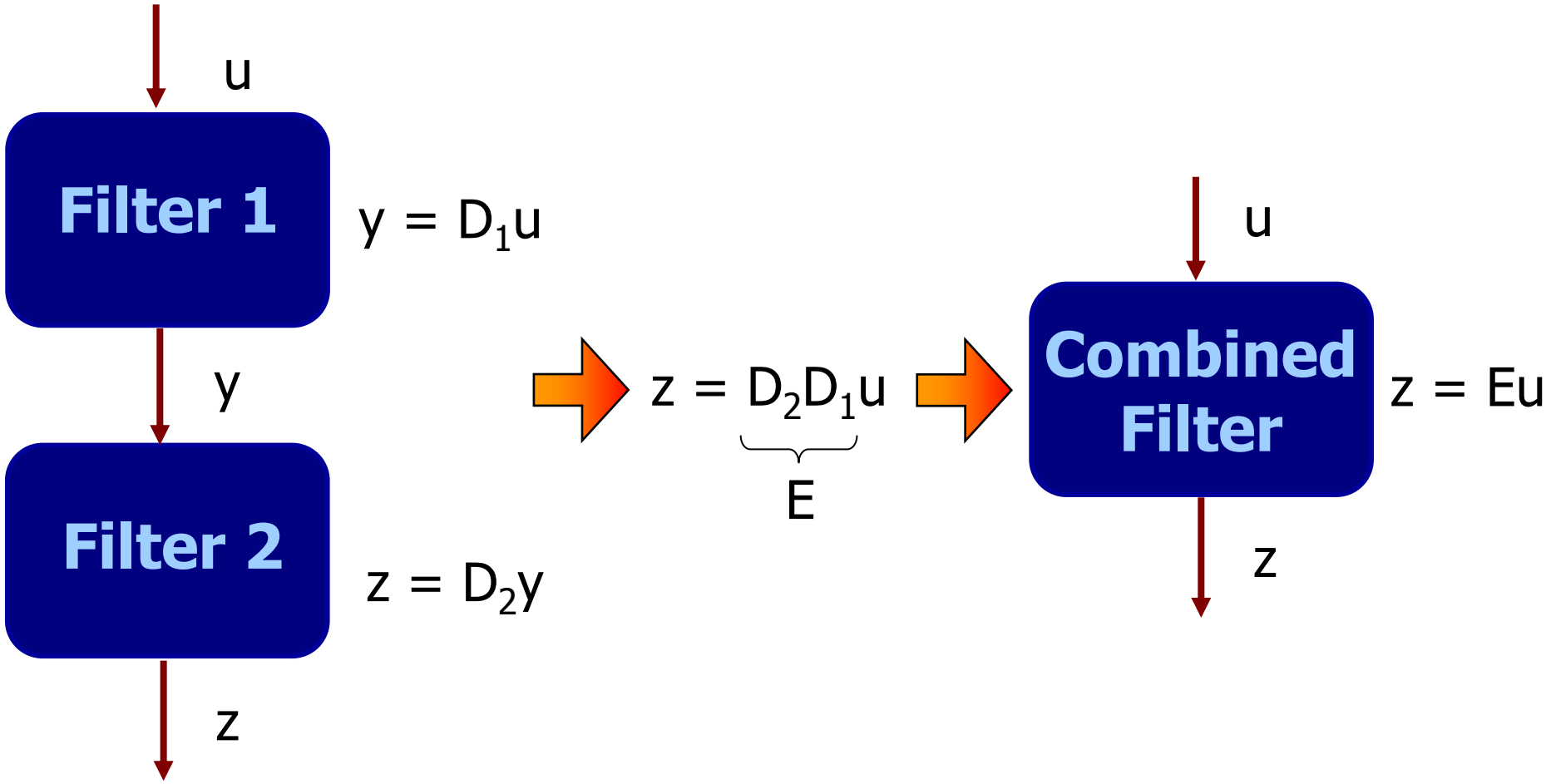
$T = [2]$

$x' = 0.9x + \mathbf{1}u$
$y = \mathbf{1}x + 2u$

$\frac{6 \text{ FLOPs}}{\text{output}}$ 

$\frac{4 \text{ FLOPs}}{\text{output}}$ 

# 3) Combining Adjacent Filters



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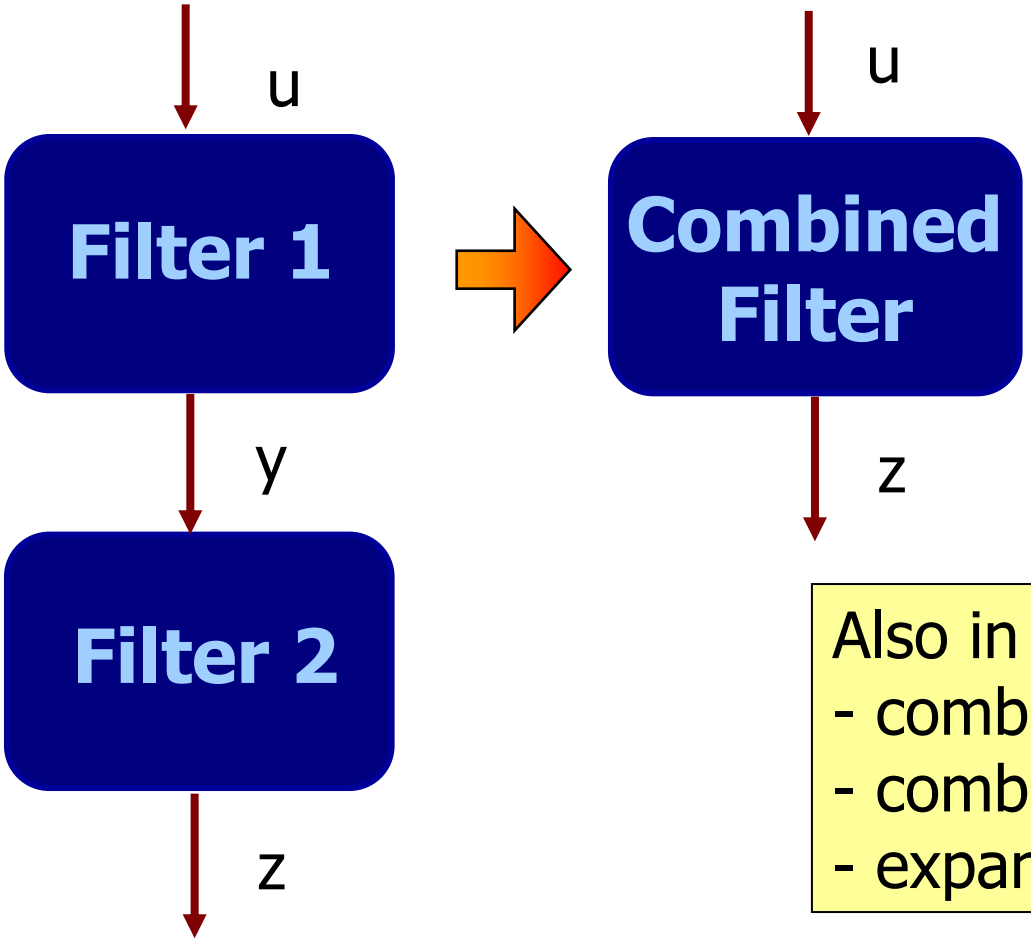
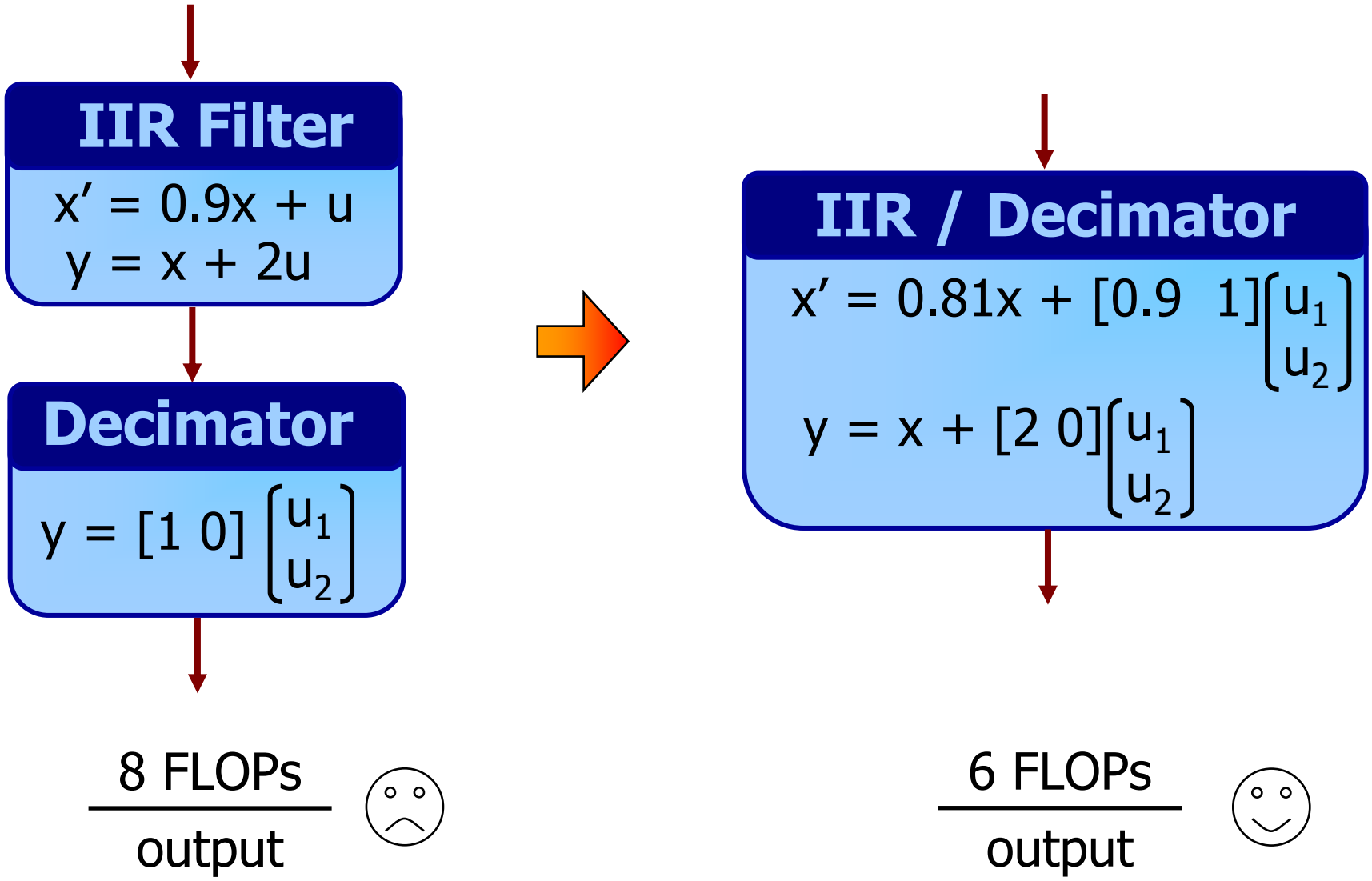



$$x' = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u$$

$$z = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} x + \begin{bmatrix} D_2D_1 \end{bmatrix} u$$

Also in paper:
- combination of parallel streams
- combination of feedback loops
- expansion of mis-matching filters

# Combination Example



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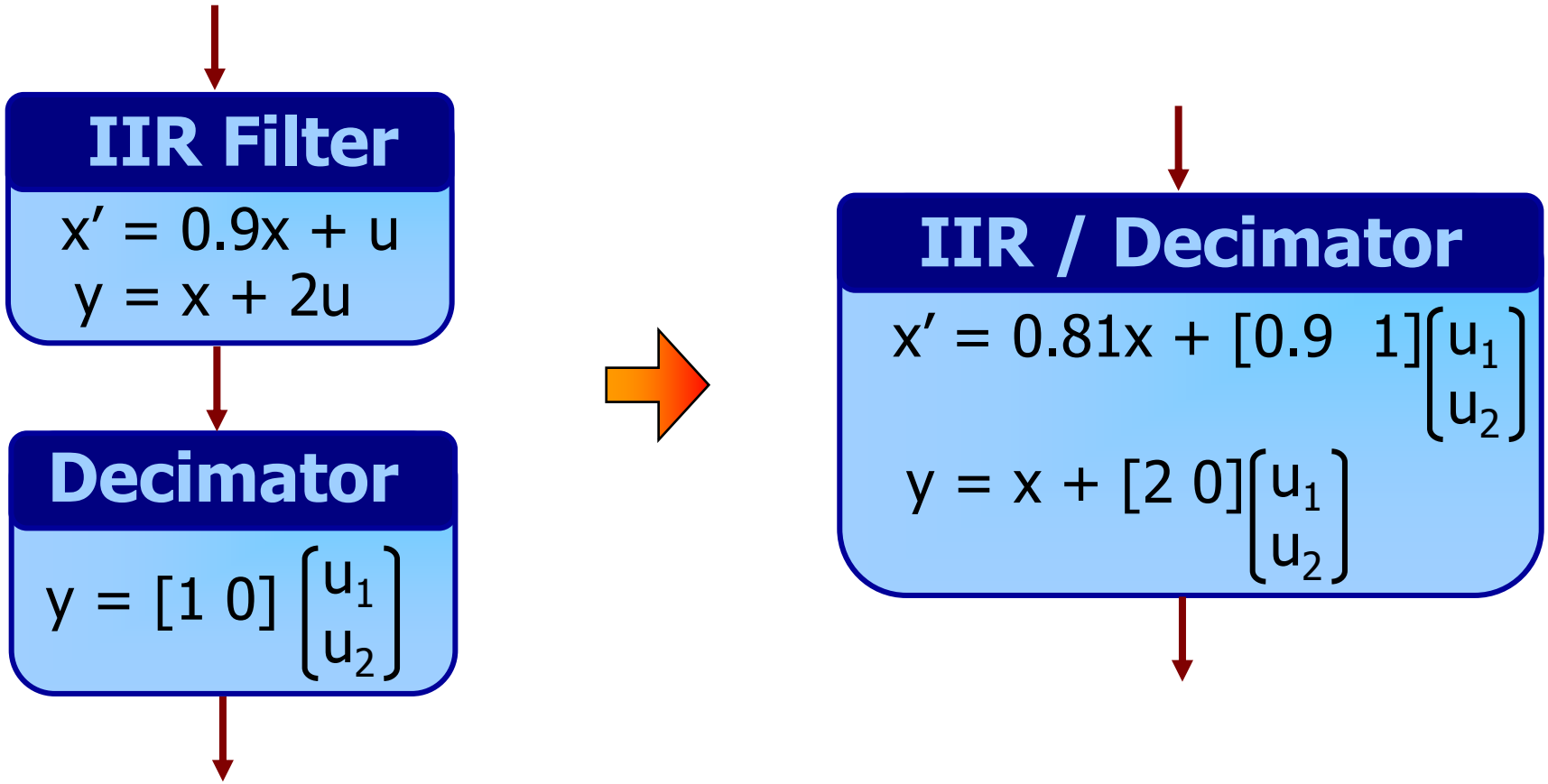


As decimation factor goes to ∞,
eliminate up to 75% of FLOPs.

# Combination Hazards

- Combination sometimes increases FLOPs
- Example: FFT
  - Combination results in DFT
  - Converts O(n log n) algorithm to $O(n^2)$
- Solution: only apply where beneficial
  - Operations known at compile time
  - Using selection algorithm, FLOPs never increase
    - See PLDI '03 paper for details

# Results

- Subsumes combination of linear components
  - Evaluated previously [PLDI '03]
    - **Applications**: FIR, RateConvert, TargetDetect, Radar, FMRadio, FilterBank, Vocoder, Oversampler, DtoA
  - Removed 44% of FLOPs
  - Speedup of 120% on Pentium 4
- Results using state space analysis

| | Speedup (Pentium 3) |
|---|---|
| IIR + 1:2 Decimator | 49% |
| IIR + 1:16 Decimator | 87% |

# Ongoing Work

- Experimental evaluation
  - Evaluate real applications on embedded machines
  - In progress:  MPEG2, JPEG, radar tracker
- Numerical precision constraints
  - Precision often influences choice of coefficients
  - Transformations should respect constraints

# Related Work

- Linear stream optimizations [Lamb et al. ’03]
  - Deals with stateless filters
- Automatic optimization of linear libraries
  - SPIRAL, FFTW, ATLAS, Sparsity
- Stream languages
  - Lustre, Esterel, Signal, Lucid, Lucid Synchrone, Brook, Spidle, Cg, Occam , Sisal, Parallel Haskell
- Common sub-expression elimination

# Conclusions

- Linear state space analysis:
  An elegant compiler IR for DSP programs
- Optimizations using state space representation:
  1. State removal
  2. Parameter reduction
  3. Combining adjacent filters
- Step towards adding efficient abstraction layers that remove the DSP expert from the design flow

StreamIt

http://cag.lcs.mit.edu/streamit