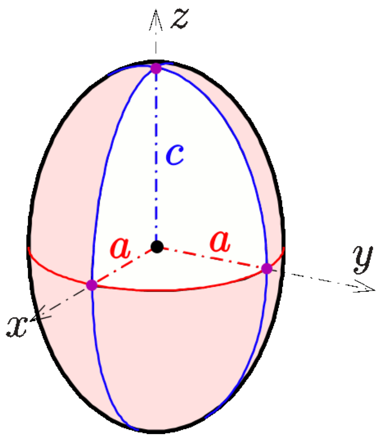
Bag valve mask mathematical model

volume calculation

It is assumed that the shape of the ambu bag will be that of a prolate or oblong spheroid, that is, an ellipse rotated around its major axis:



Its polar axis is greater than the equatorial, in our case the equator is a perfect circle before being deformed by the pressure of the robotic claw. The largest radius **c** lies on the **z** axis, the smallest radius of the circle on the x-y plane is **a**.

$V=\frac{4}{3}\pi c a^2$ spheroid volume

The movement of a pressure plate coming from the negative axis **y** is assumed, the volume differential at all times will be:

$$dv=\pi\, xz\, dy \quad (1)$$

$$v=\pi\int_{-a}^{a} xz\, dy \quad (2)$$

the equation of the circumference in the x-y plane:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \quad (3)$$

$$x^2+y^2=a^2$$

$$x=\sqrt{a^2-y^2} \quad (4)$$

the ellipse on the y-z plane:

$$\frac{y^2}{a^2}+\frac{z^2}{c^2}=1 \rightarrow z^2=c^2\left(1-\frac{y^2}{a^2}\right) \rightarrow z=c\sqrt{1-\frac{y^2}{a^2}} \quad (5)$$

$$z=\frac{c}{a}\sqrt{a^2-y^2} \quad (6)$$

replacing (4) and (6) in (2):

$$v(y)=\pi\int_{-a}^{a}\frac{c}{a}(a^2-y^2)\,dy=\pi\frac{c}{a}\int_{-a}^{a}(a^2-y^2)\,dy$$

$$v(y)=\pi\frac{c}{a}\left[a^2 y-\frac{y^3}{3}\right]_{-a}^{a} \quad (7)$$

This equation is what we are looking for, instead of integrating up to *a* we will only do it up to **h** which represents the movement of the pressure plate coming from the negative **y** axis:

$$v(y)=\pi\frac{c}{a}\left[a^2 y-\frac{y^3}{3}\right]_{-a}^{h} \quad (7b)$$

evaluating limits:

$$v(h)=\pi\frac{c}{a}\left[a^2 h-\frac{h^3}{3}-a^2(-a)+\frac{(-a)^3}{3}\right]=\pi\frac{c}{a}\left[a^2 h-\frac{h^3}{3}+a^3-\frac{a^3}{3}\right]$$

$$v(h)=\pi\frac{c}{a}\left[a^2 h-\frac{h^3}{3}+\frac{2}{3}a^3\right] \quad (7c)$$

(7c) calculates the volume as a function of the movement of the pressure plate on the bag, from it we start to obtain the inverse function, that is, how much the plate must move to obtain an **x** volume defined by the user through the potentiometer or other console input. Developing:

$$\frac{va}{\pi c}=a^2 h-\frac{h^3}{3}+\frac{2}{3}a^3$$

$$\frac{h^3}{3}-a^2 h-\frac{2}{3}a^3+\frac{av}{\pi c}=0$$

$$h^3-3a^2 h+3\frac{av}{\pi c}-2a^3=0 \quad (8)$$

(8) is a reduced third-degree equation in **h** of the form:

$$h^3+ph+q=0 \quad (9)$$

with
$$p=-3a^2$$
$$q(v)=3\frac{av}{\pi c}-2a^3$$

Graphing equation (9) in the Scilab with **h** on the X axis and **v** on the Y axis, assuming radius **a** = 5 cm and a greater radius **c** = 10 cm, the following solution curves are obtained for compression volumes **v** between 0 and 500 [cc]:

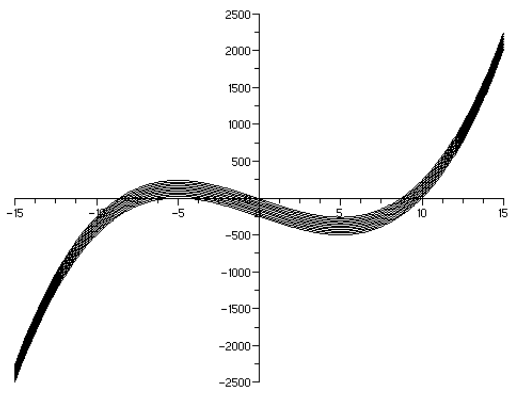


Fig 1

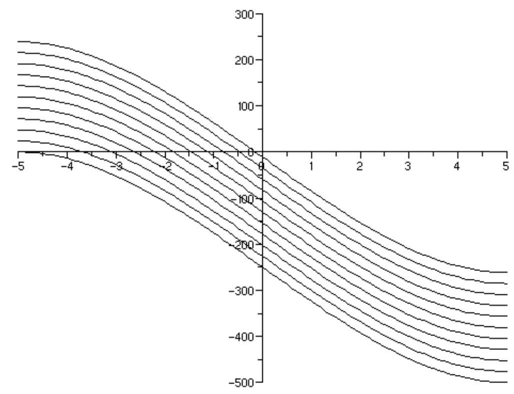


Fig 2

In *Fig 1*, 3 solutions are observed for each value **v** between [0,500] ml, all of them real, since the equation (7) starts at **a** = -5 and has a maximum of 0 -half of the bag- we recognize which of the 3 solutions in each curve is useful to control the pressure of the robotic claw. *Fig 2* is a zoom to these solutions.

There are several methods to find the 3 solutions of equation (9), but the most convenient as will be seen later is:

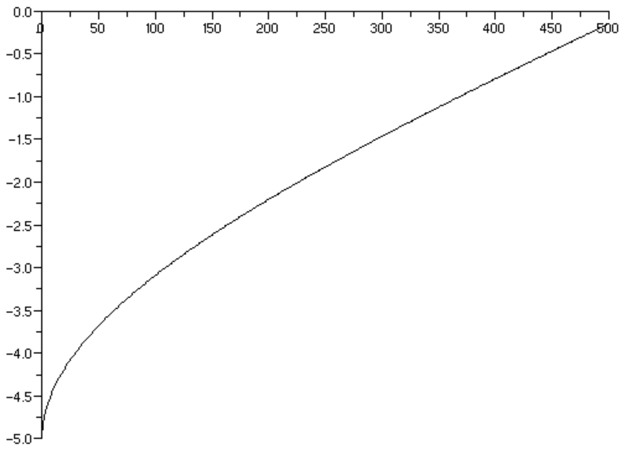
$$h=2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi k}{3}\right] \quad (10)$$
$$\textit{for } k=0,1,2$$

Testing in Scilab equation (10) it was found that the desired intermediate solution always occurs for k = 1, therefore the formula that relates the movement of the pressure plate as a function of volume is:

$$h(v)=2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi}{3}\right] \quad (11)$$

with
$$p=-3a^2$$
$$q(v)=3\frac{av}{\pi c}-2a^3$$

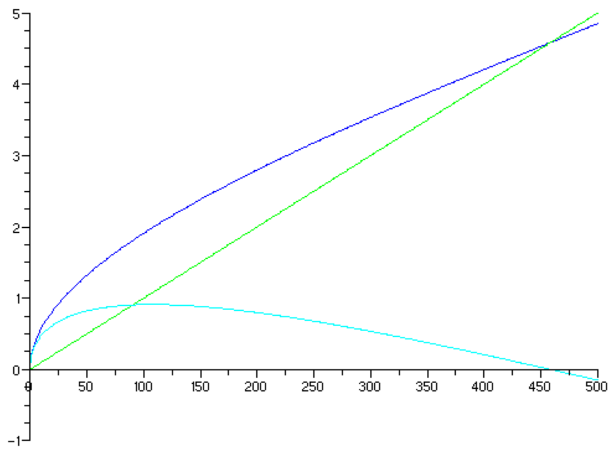
Graphing (11) for a volume **v** between 0 and 500 [cc] in the Scilab:



The values of **h** are negative, as one would expect having swept **y** from -5 towards the origin of coordinates. To have positive values, add the value of constant '*a*':

$$h_2(v)=h(v)+a=2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi}{3}\right]+a \quad (12)$$

Graphing (12) in blue, for comparison, a linear behavior is drawn in green and the resulting difference in light blue:



Then the equation (12) receives the input requested by the user and as a result the distance that must be pressed by one side of the robotic claw is returned, it is recalled that the contact surface must be flat to fit the equation, and that the real volume provided will be twice as the other side of the claw will be pressing from the **y** axis positive towards the origin of coordinates with the same absolute displacement. These considerations are used in the following subtitle.

Final form of the equation h(v):

From (11): $h(v)=2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)-\frac{2\pi}{3}\right]$

replacing $\begin{matrix} p=-3a^2 \\ q=\frac{3av}{\pi c}-2a^3 \end{matrix}$ in (11):

we have $h(v)=2\sqrt{\frac{3a^2}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{1}{2}(\frac{9av}{\pi c}-6a^3)\sqrt{\frac{-3}{(-3a^2)^3}}-\frac{2\pi}{3}\right)\right]$ (12)

$$h(v)=2\sqrt{\frac{3a^2}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{1}{2}(\frac{9av}{\pi c}-6a^3)\frac{1}{\sqrt{(-3)^2a^6}}-\frac{2\pi}{3}\right)\right] \quad (13)$$

We choose to preserve the negative sign of -3 in sqr ((- 3) ^ 2) because the positive solution does not provide the solutions in the range we need, this was verified in the Scilab.

$$h(v)=2a\cos\left[\frac{1}{3}\left(\arccos(\frac{9av-6a^3\pi c}{2\pi c}\frac{1}{-3a^3})-2\pi\right)\right] \quad (14)$$

$$h(v)=2a\cos\left[\frac{1}{3}\left(\arccos(\frac{2\pi a^2c-3v}{2\pi a^2c})-2\pi\right)\right]$$

$$h(v)=2a\cos\left[\frac{1}{3}\left(\arccos(1-\frac{3v}{2\pi a^2c})-2\pi\right)\right] \quad (15)$$

where **a** and **c** are the dimensions of the ambu:
a = smallest radius in cm
c = largest radius in cm

So the general formula is equation (15) but this must be modified before being inserted into the arduino code due to the following considerations:

1) Equation (15) receives the input of the required volume **v** and returns the distance that one side of the bag must be pressed, when using a robotic claw that exerts pressure on both sides, the formula must be modified assuming that each side will obtain only half the volume **Vi** required from user input. This is accomplished by doing v = Vi / 2
2) The integral was performed from -*a* to *h* so the formula responds to negative values [-a, 0], it is necessary to convert them to [0, h], this is achieved by adding the constant '*a'*
3) The number of steps (ticks) provided to the stepper motor must be included to reach its maximum

movement **Pmax** (= 180 in my case) divided by the maximum distance that one side of the robotic claw will press.

Applying the previous considerations to equation (15) we have:

$$P_c = \frac{P_{max}}{a}\left\{2a\cos\left[\frac{1}{3}\left(\arccos\left(1-\frac{3(v_i/2)}{2\pi a^2 c}\right)-2\pi\right)\right]+a\right\}$$

$$P_c = P_{max}\left\{2\cos\left[\frac{1}{3}\left(\arccos\left(1-\frac{3v_i}{4\pi a^2 c}\right)-2\pi\right)\right]+1\right\} \quad (16)$$

Which is the formula that receives the **Vi** input required by the user and returns the number of steps **Pc** that the motor must move to obtain said volume. This allows the claw movement to be accommodated to any flow waveform needed.

Final thoughts

This modeling assumes a perfect prolate ellipsoid that only approximates the different variants of each manufacturer, so it must be compared to the actual behavior to incorporate the required additions.
This equation can facilitate the correction process for closed cycle execution since the deviations of the ideal volume or flow will be smaller.

Greetings.
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