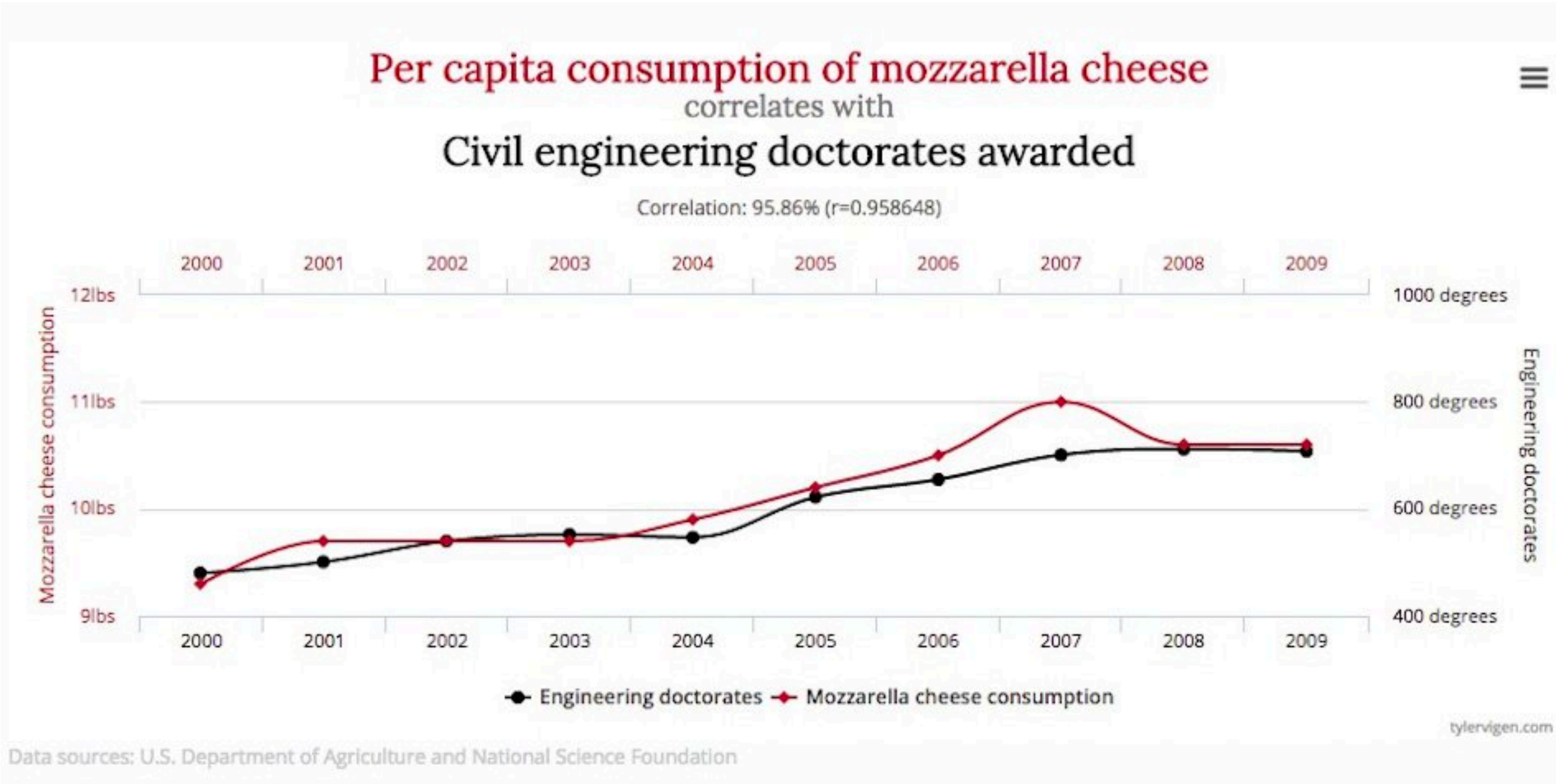
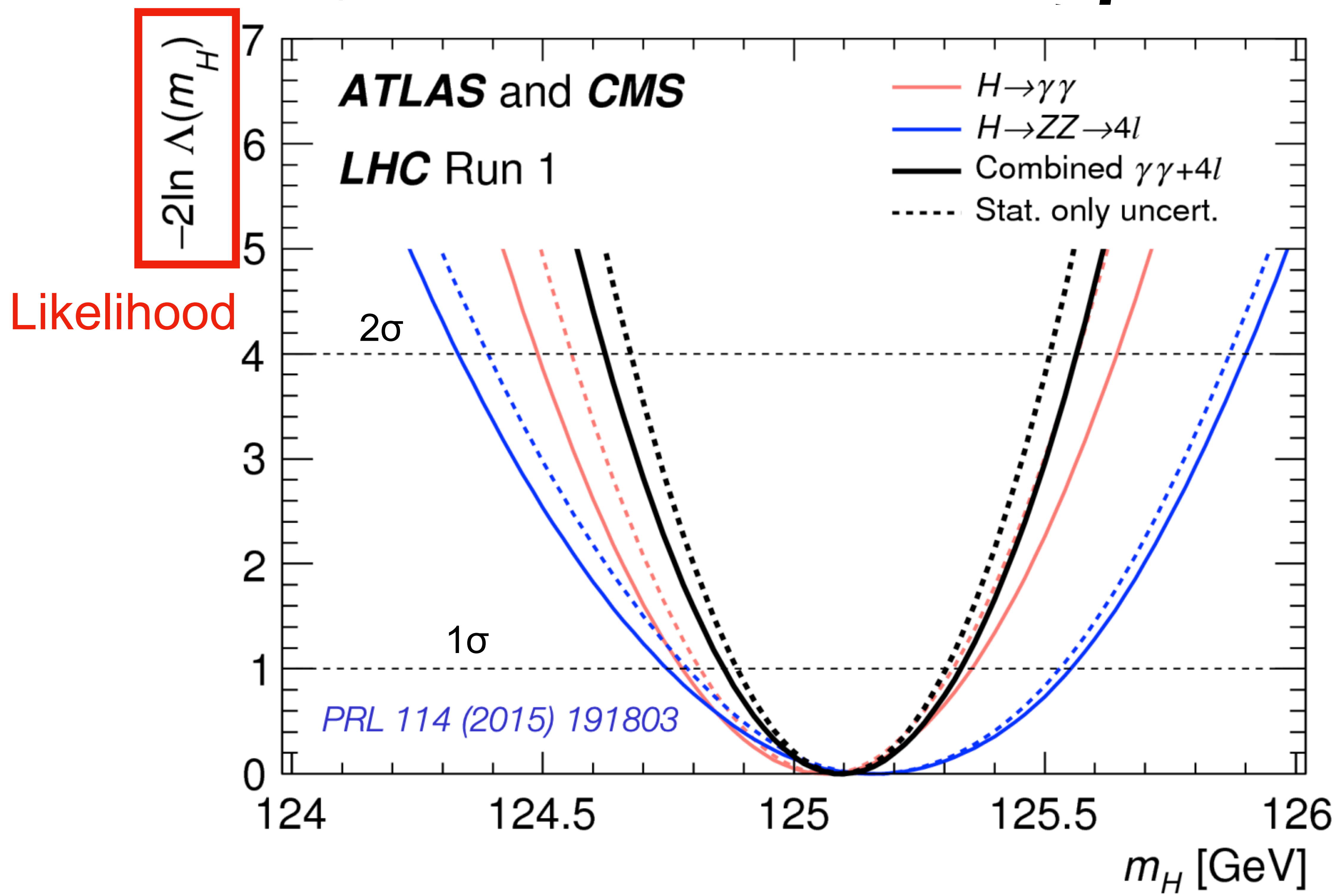
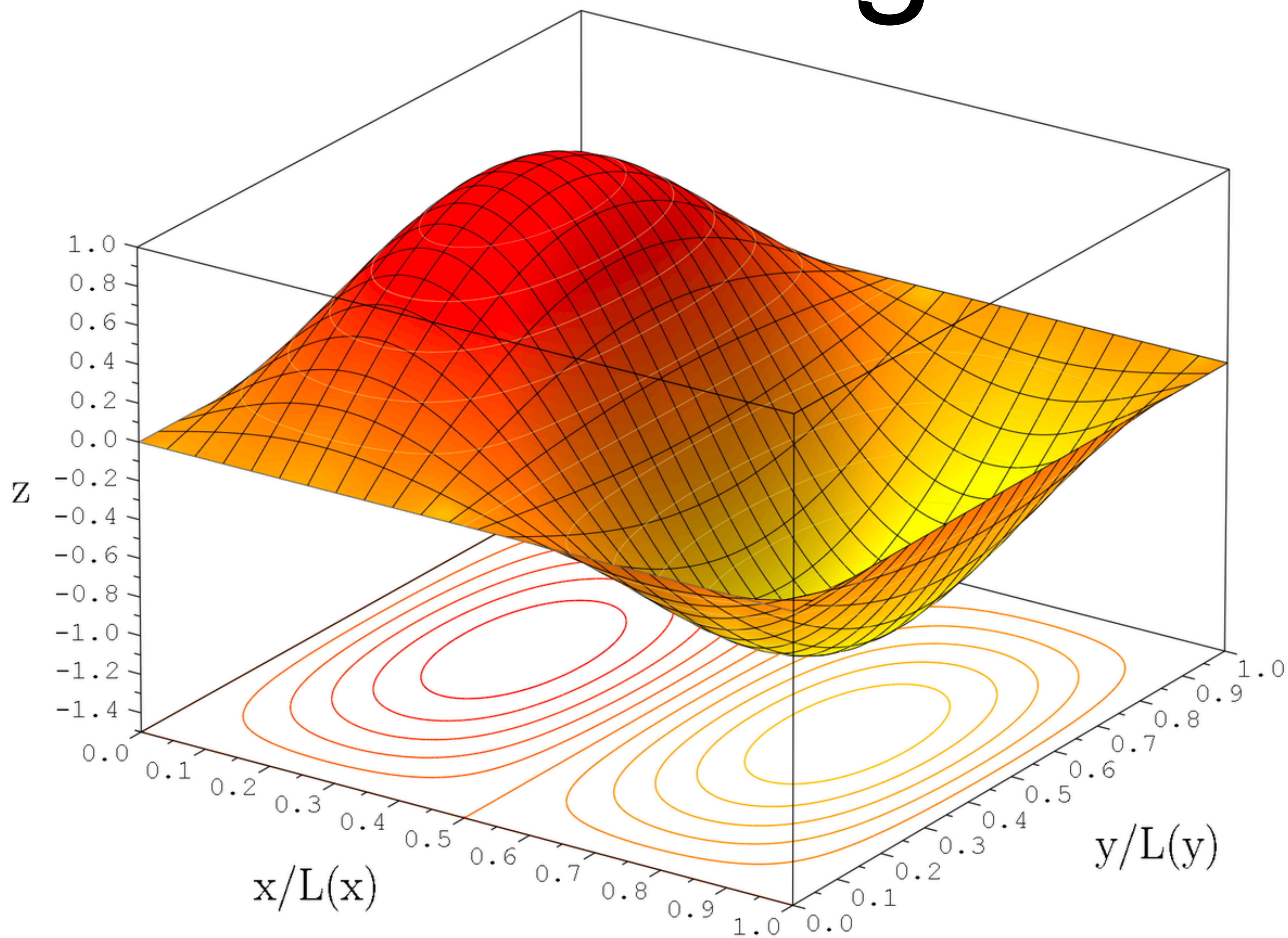


Lecture 7: Correlations

Understanding Best Fit

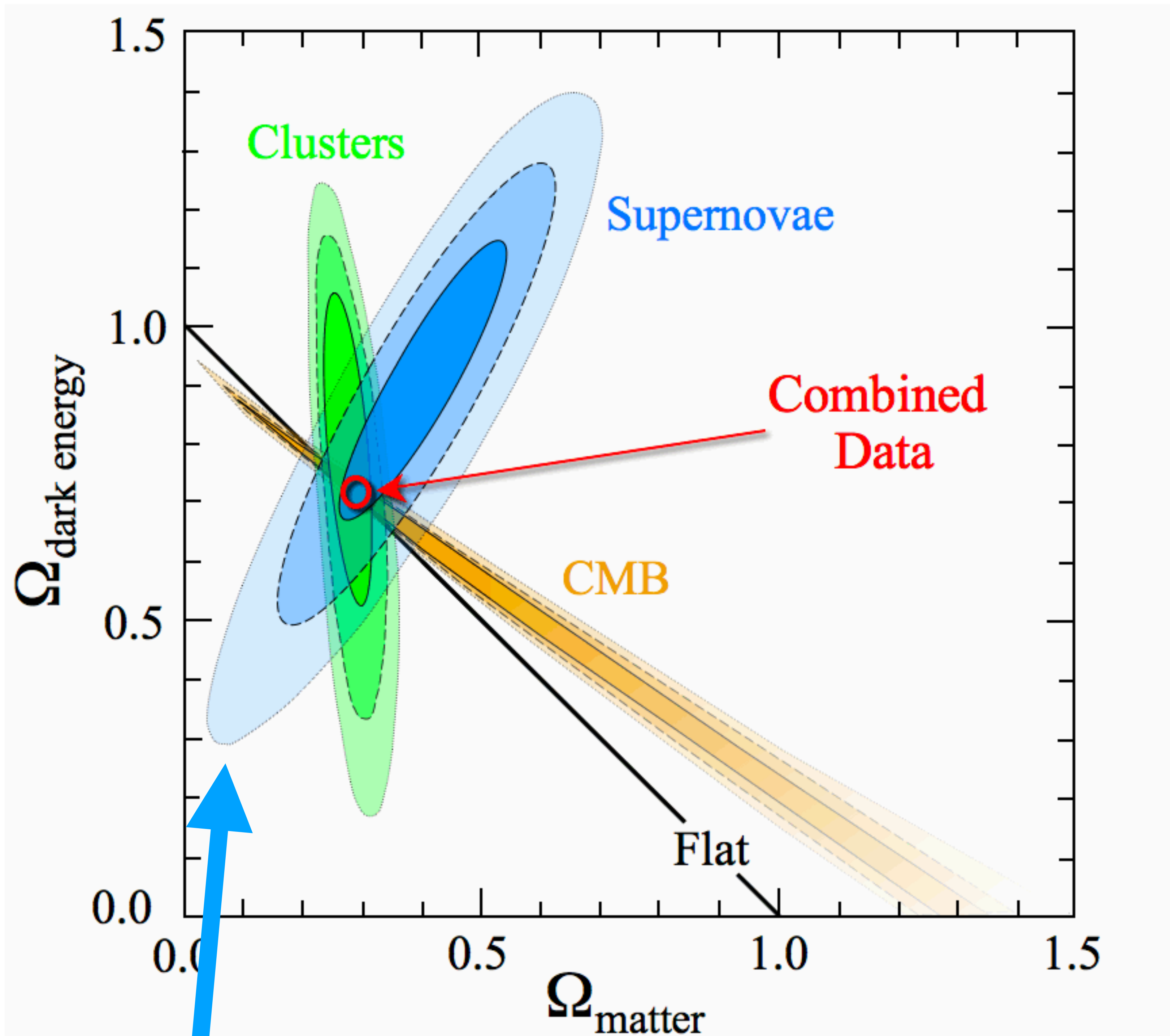


Minimizing A Surface



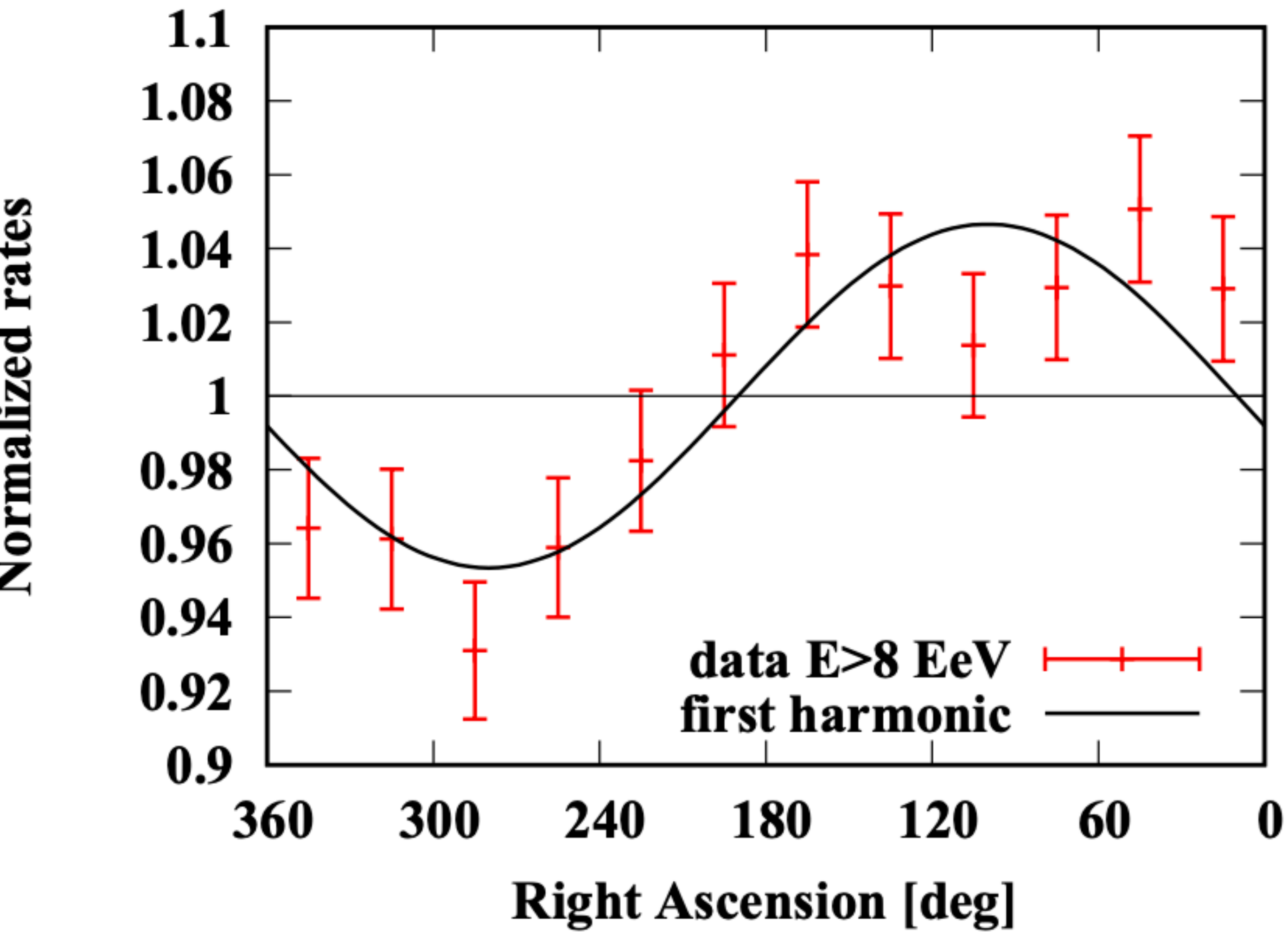
- How do we get to the minimum of that

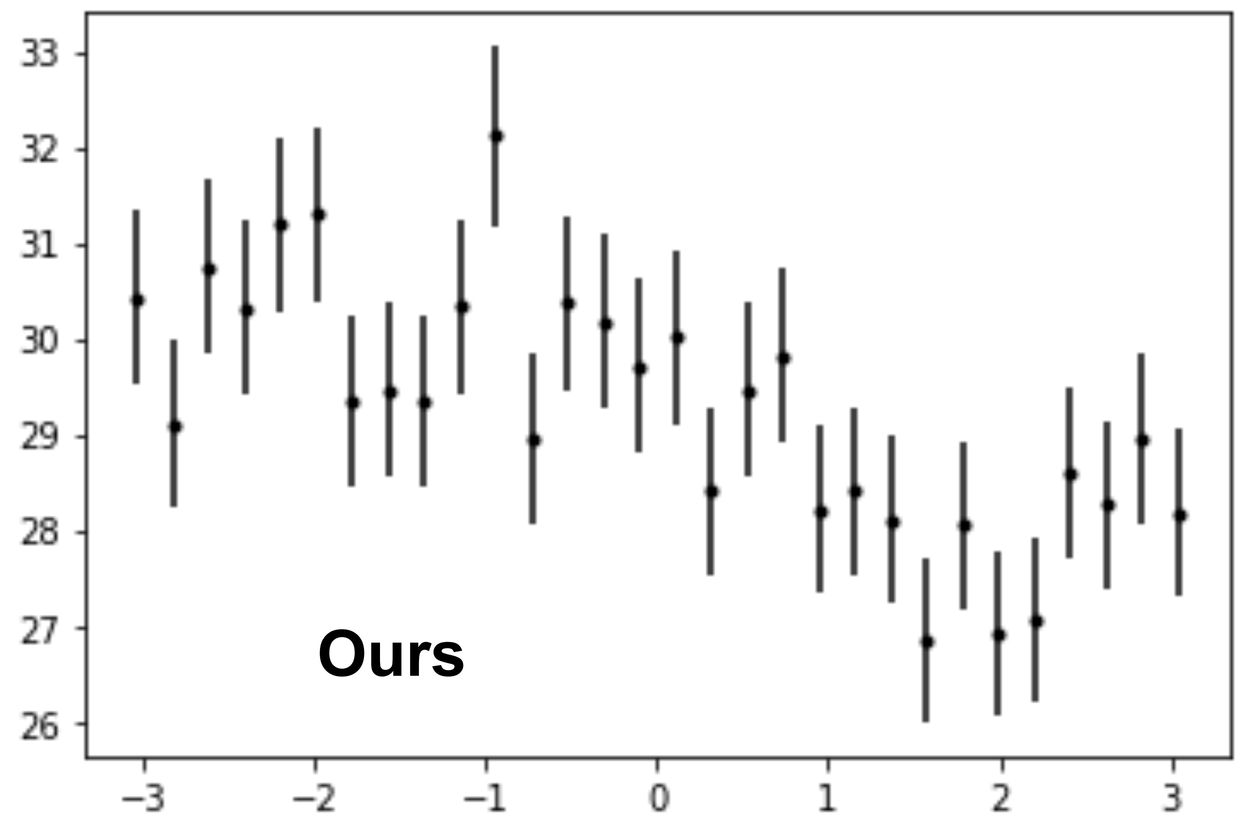
Making a 2D Minima



How Can we make the contours on the supernova data?

Cosmic Ray Data





Multiple Dimensions

- For N variables the expansion is the same

$$\chi^2(x_i,\vec{\theta}) = \chi^2_{min}(x_i,\vec{\theta}) + \boxed{\frac{1}{2}(\theta_i-\theta_0)^T \boxed{\frac{\partial^2}{\partial\theta_i\theta_j}\chi^2_{min}(x_i,\vec{\theta}_0)}(\theta_j-\theta_0)}$$

χ^2 distribution of 1 degree of freedom

V[$\chi^2(x)$]=1

Hessian of
the χ^2 distribution

$$\Delta\chi^2 = 2\Delta\log \mathrm{L} = 1$$

For one degree of freedom

This is known as Wilk's Theorem

$$\boxed{\sigma^2_{ij} = \left(\frac{\partial^2 \log L}{\partial\theta_i\theta_j}\right)^{-1}}$$

2D examples

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \theta_a - \theta_{a-min} & \theta_b - \theta_{b-min} \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \\ \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \theta_a - \theta_{a-min} \\ \theta_b - \theta_{b-min} \end{pmatrix}$$

$$\frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \approx 0$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \Delta\theta_a & \Delta\theta_b \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \Delta\theta_a \\ \Delta\theta_b \end{pmatrix}$$

$$\begin{aligned} \chi^2(x,\vec{\theta}) &= \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\left(\Delta\theta_a^2 \frac{\partial^2\chi^2}{\partial\theta_a^2} + \quad \Delta\theta_b^2 \frac{\partial^2\chi^2}{\partial\theta_b^2} \right) \\ &= \chi^2_{min}(x,\vec{\theta}) + \left(\frac{\Delta\theta_a^2}{\sigma^2_{\theta_a}} + \quad \frac{\Delta\theta_b^2}{\sigma^2_{\theta_b}} \right) \end{aligned}$$

2D examples

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \theta_a - \theta_{a-min} & \theta_b - \theta_{b-min} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial \theta_a^2} & \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} \\ \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} & \frac{\partial^2 \chi^2}{\partial \theta_b^2} \end{pmatrix} \begin{pmatrix} \theta_a - \theta_{a-min} \\ \theta_b - \theta_{b-min} \end{pmatrix}$$

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$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\left(\Delta\theta_a^2 \frac{\partial^2 \chi^2}{\partial \theta_a^2} + \Delta\theta_b^2 \frac{\partial^2 \chi^2}{\partial \theta_b^2} \right)$$

$$= \boxed{\chi^2_{min}(x,\vec{\theta}) + \left(\frac{\Delta\theta_a^2}{\sigma_{\theta_a}^2} + \frac{\Delta\theta_b^2}{\sigma_{\theta_b}^2} \right)}$$

Ellipse

Relating all the 2Ds

$$\frac{2}{\sigma^2} = \frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}$$

$$\sigma^2 = 2\left(\frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}\right)^{-1}$$

Wilk's Theorem

$$\begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{2}{\sigma_a^2} & 0 \\ 0 & \frac{2}{\sigma_b^2} \end{pmatrix}$$

Wilk's For Uncorrelated Parameters

For correlated Paramters Can always Diagonalize

$$A^{-1} 2 \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} \\ \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix}^{-1} \quad A = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

2D Terminology

Covariance Matrix

$$\begin{pmatrix} \sigma_a^2 & \mathrm{COV}(a,b) \\ \mathrm{COV}(a,b) & \sigma_b^2 \end{pmatrix} = \sum_{i=1}^{N} \begin{pmatrix} (a_i - \bar{a})^2 & (a_i - \bar{a})\left(b_i - \bar{b}\right) \\ (a_i - \bar{a})\left(b_i - \bar{b}\right) & \left(b_i - \bar{b}\right)^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & \frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b} \\ \boxed{\frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b}} & 1 \end{pmatrix}$$

Correlation Matrix

Correlation Coefficient

Relating all the 2Ds

$$\frac{2}{\sigma^2} = \frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}$$

$$\sigma^2 = 2\left(\frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}\right)^{-1}$$

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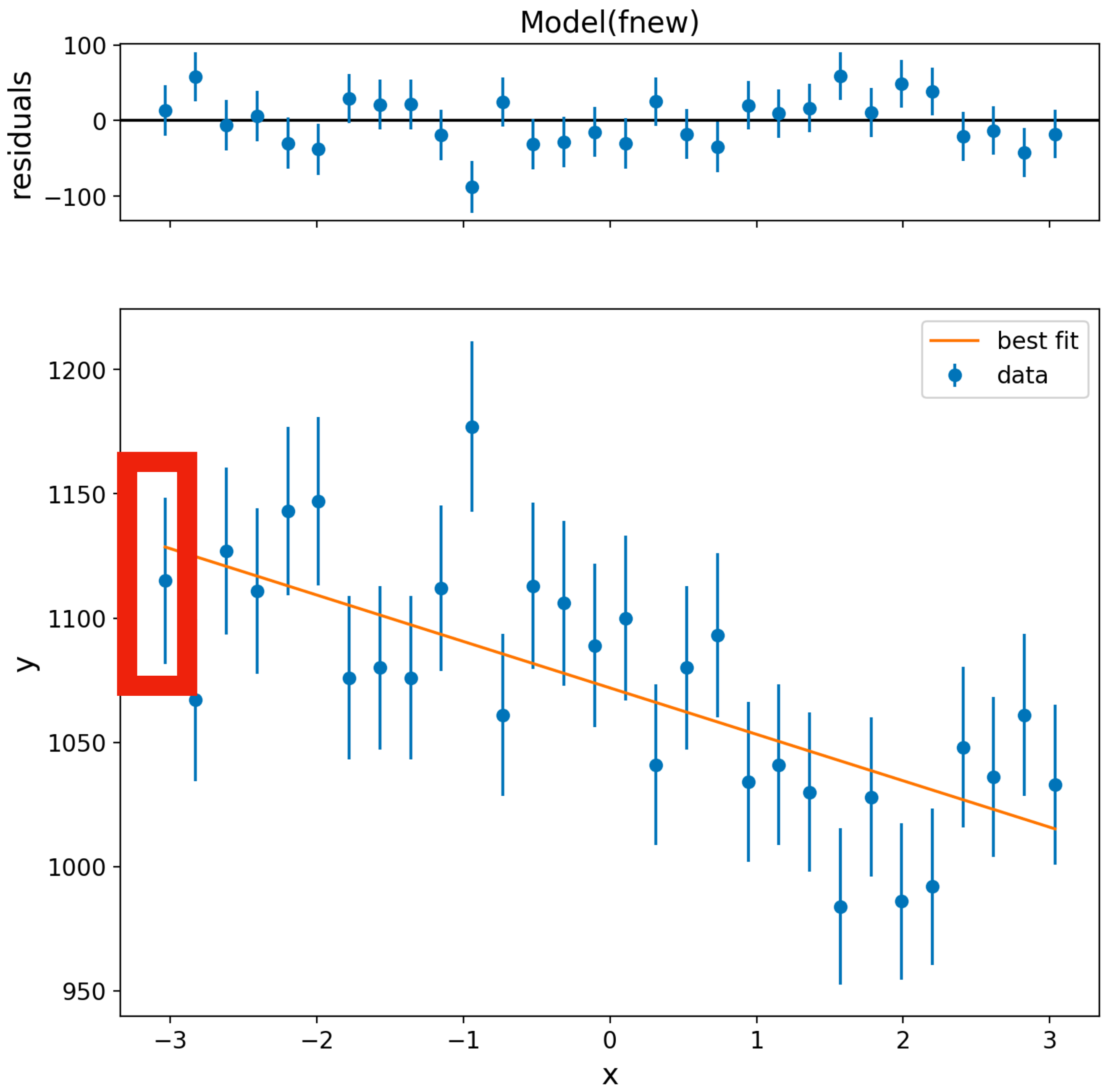
Correlation Matrix

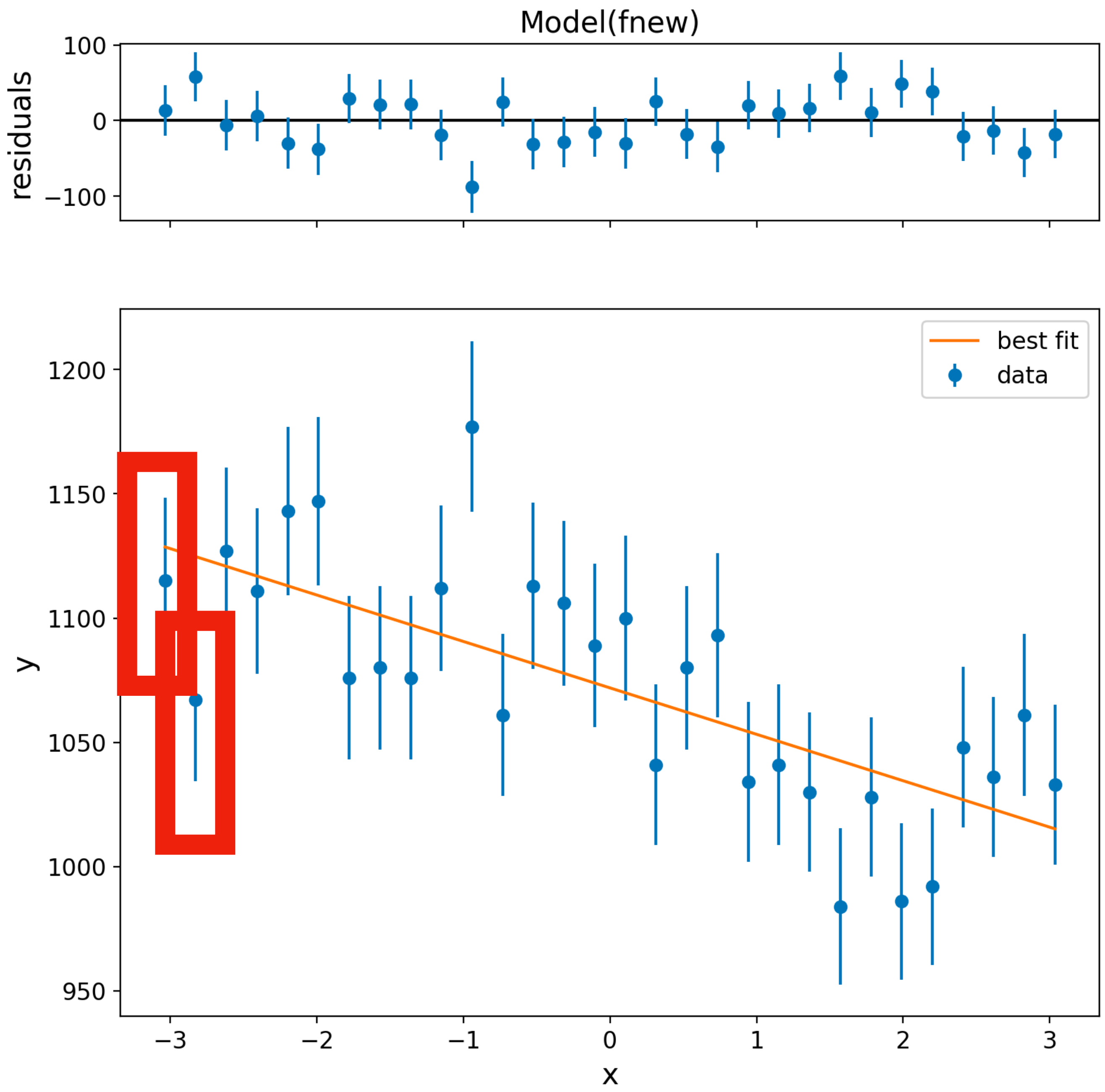
Correlation Coefficient

Toys



Poisson Fluctuate this bin

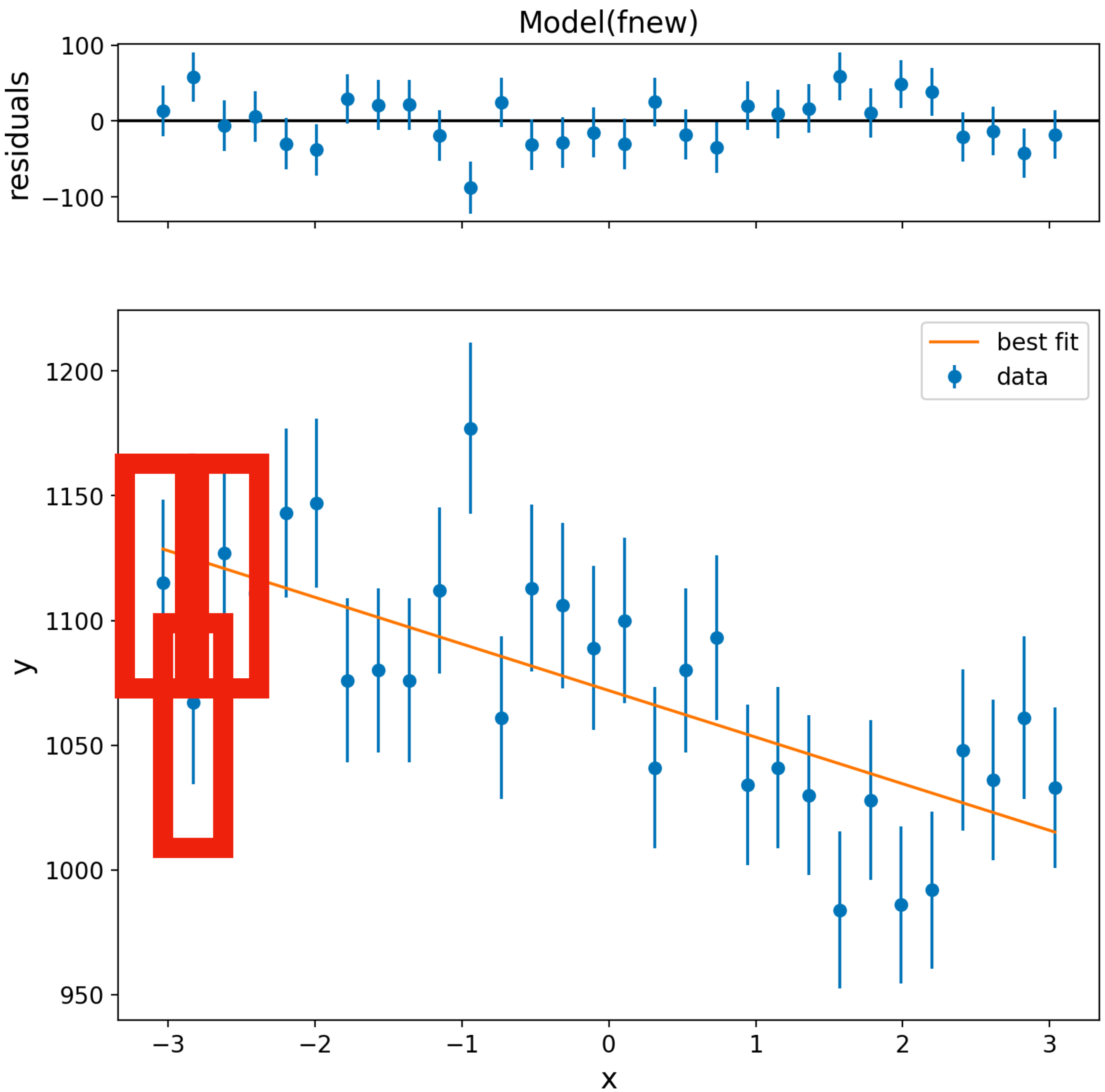
Toys



Poisson Fluctuate this bin

Poisson Fluctuate this bin

Toys

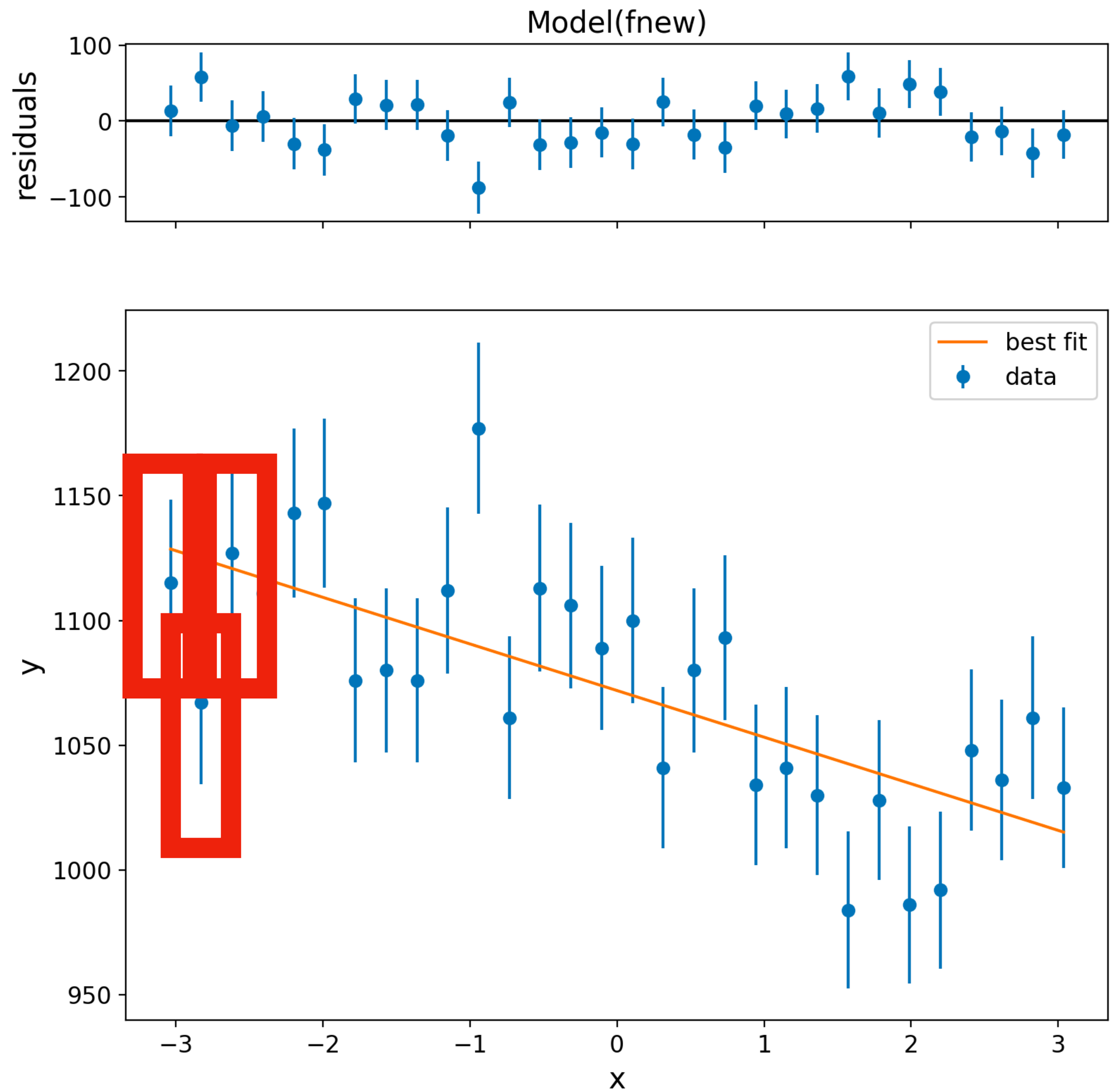


Poisson Fluctuate this bin

Poisson Fluctuate this bin

Poisson Fluctuate this bin

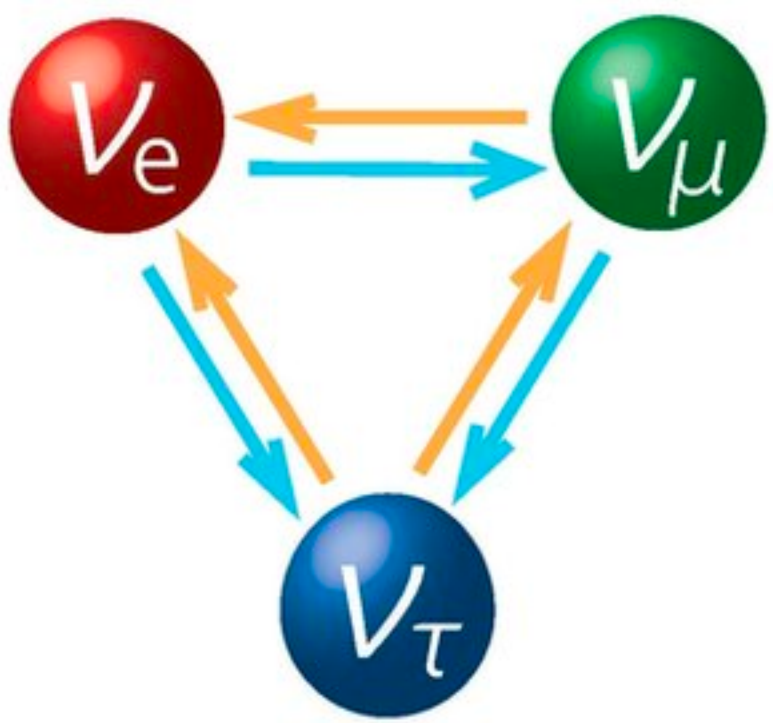
Toys



Poisson Fluctuate this bin

Poisson Fluctuate this bin ….

Poisson Fluctuate this bin



Neutrinos oscillate into other neutrinos

Nobel Prize

The Nobel Prize in Physics 2015

Takaaki Kajita
Arthur B. McDonald

Share this

The Nobel Prize in Physics 2015

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Takaaki Kajita

Prize share: 1/2

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Arthur B. McDonald

Prize share: 1/2

- Nobel prize was recently awarded for neutrino oscillations
  - This is a second award on neutrino properties

Neutrino Mixing

Particle Eigenstates

$$\boxed{|\nu_\alpha >} = \boxed{\sum_{i=1}^{3} U^*_{\alpha i} |\nu_i >}$$

Mass Eigenstates

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \quad s_{ij} \equiv \sin\theta_{ij},\ c_{ij} \equiv \cos\theta_{ij}$$

$$= \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Neutrino Mixing

$$P_{\alpha\to\beta} = |<\nu_\beta(t)|\nu_\alpha(t)>|^2$$

$$= \left(\sum_{j=1}^{3} U_{\beta j} U^*_{\alpha j} e^{-i\frac{m_j L}{2E}}\right)\left(\sum_{i=1}^{3} U^*_{\beta i} U_{\alpha i} e^{i\frac{m_j L}{2E}}\right)$$

As we evolve over time the particle eigenstates oscillate through the mass states

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\Re[U^*_{\alpha i}U_{\alpha j}U_{\beta i}U^*_{\beta j}]\sin^2\left(\frac{\Delta m^2_{ij}}{4E}L\right)$$

$$+ 2\sum_{i>j}\Im[U^*_{\alpha i}U_{\alpha j}U_{\beta i}U^*_{\beta j}]\sin^2\left(\frac{\Delta m^2_{ij}}{2E}L\right)$$

Neutrino Mixing

Master Formula

$$P_{\mu\to\mu} \simeq 1 - 4s_{23}^2 c_{23}^2 (s_{12}^2 + c_{12}^2) \sin^2\left(\frac{1.27\Delta m_{atm.}^2}{E} L\right)$$

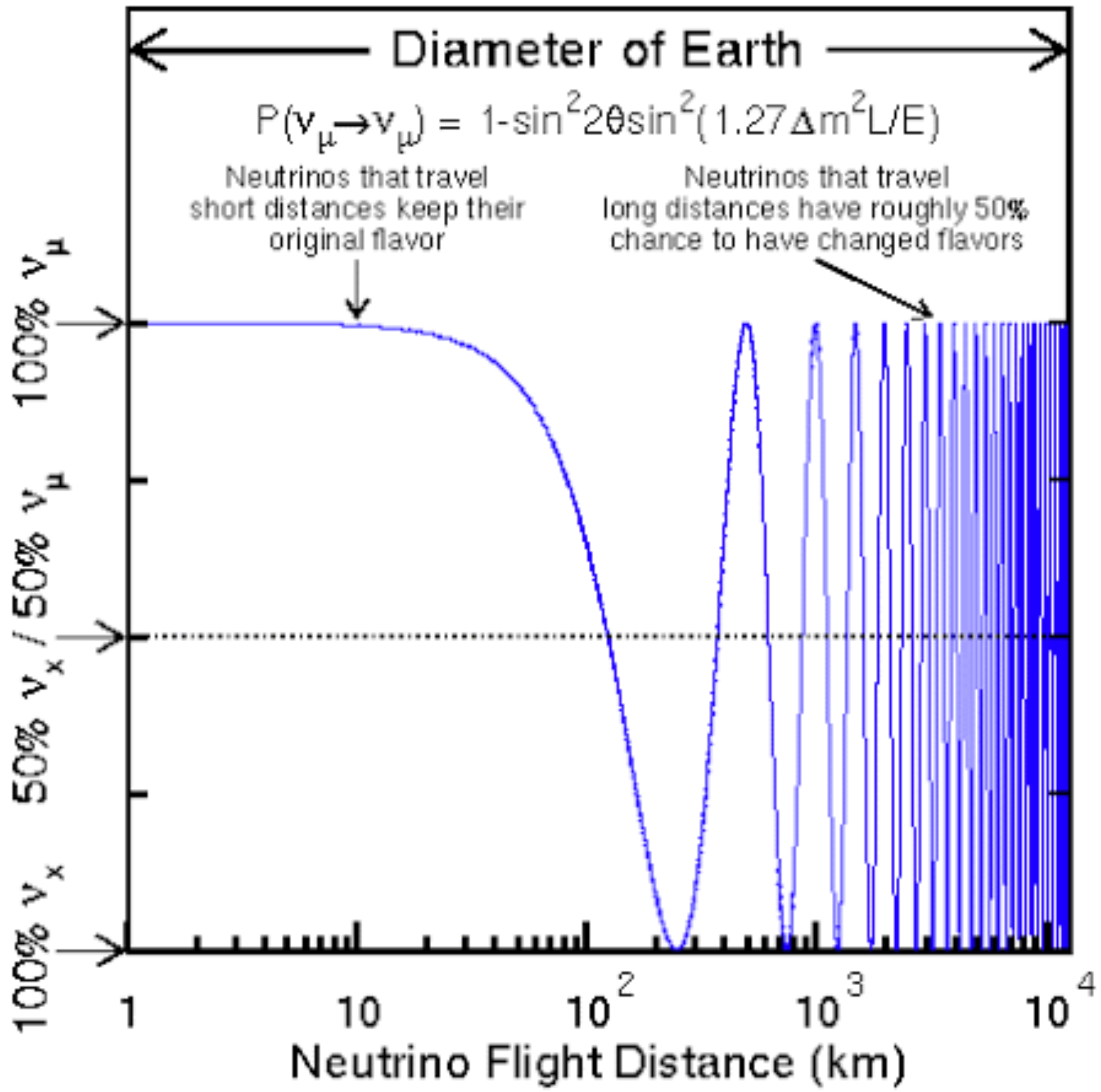
$$\simeq 1 - \sin^2 2\theta_{23} \sin^2\left(\frac{1.27\Delta m_{atm.}^2}{E} L\right),$$

As we evolve over time the particle eigenstates oscillate through the mass states

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathfrak{R}[U_{\alpha i}^* U_{\alpha j} U_{\beta i} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{ij}^2}{4E} L\right)$$

$$+ 2\sum_{i>j}\mathfrak{I}[U_{\alpha i}^* U_{\alpha j} U_{\beta i} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{ij}^2}{2E} L\right)$$

Observing Oscillations



Neutrinos oscillate distance

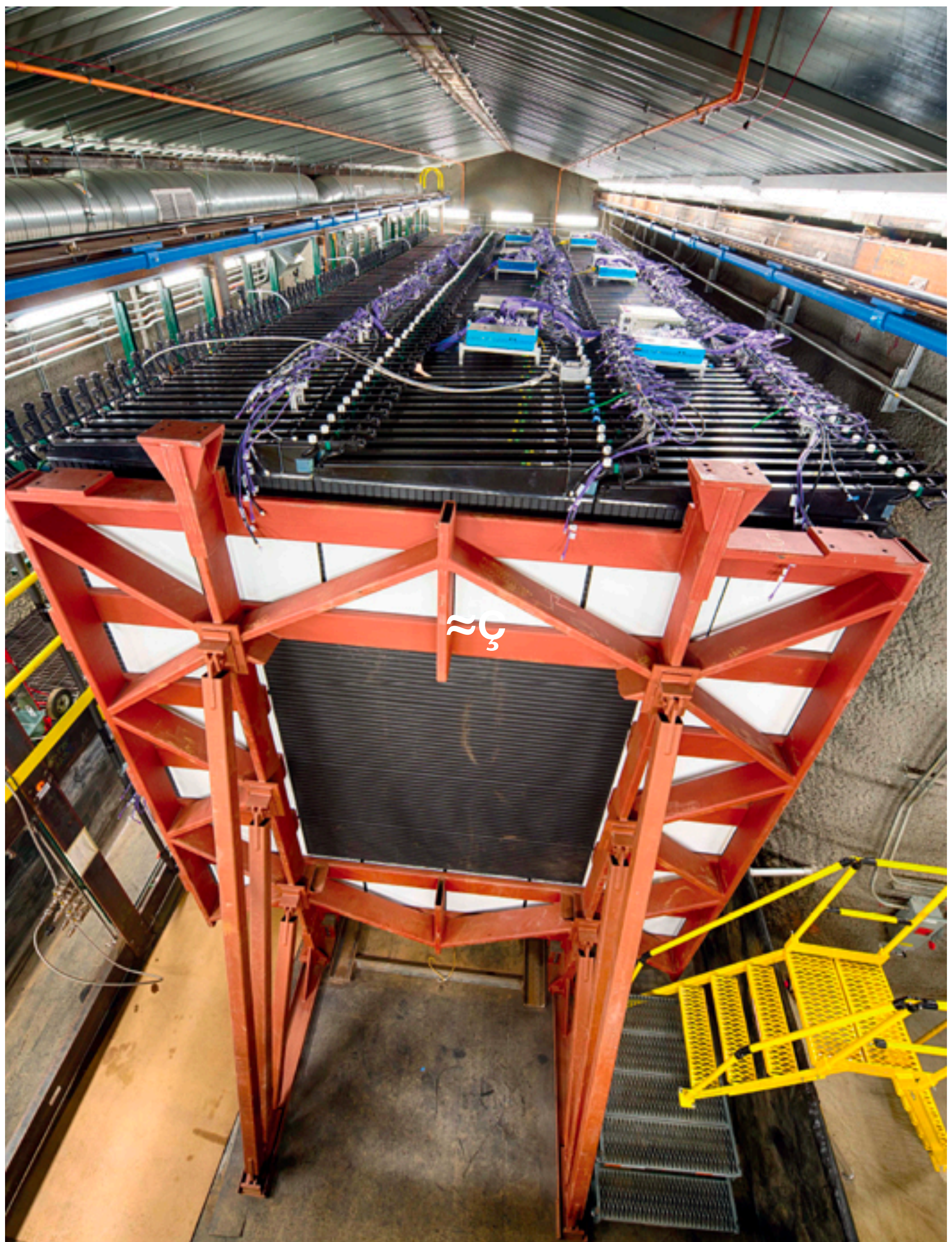
Muon neutrino beam

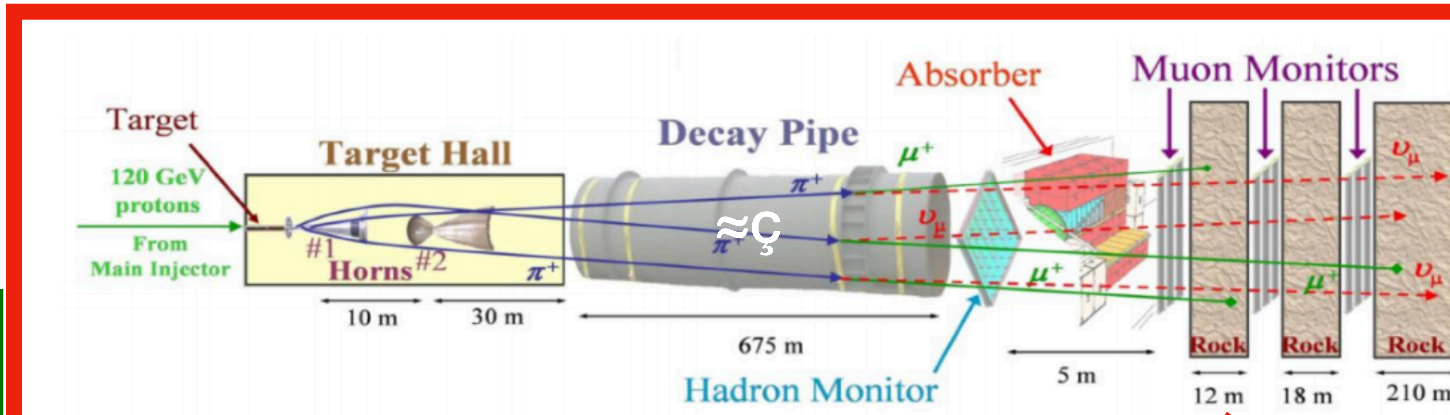
A long Distance

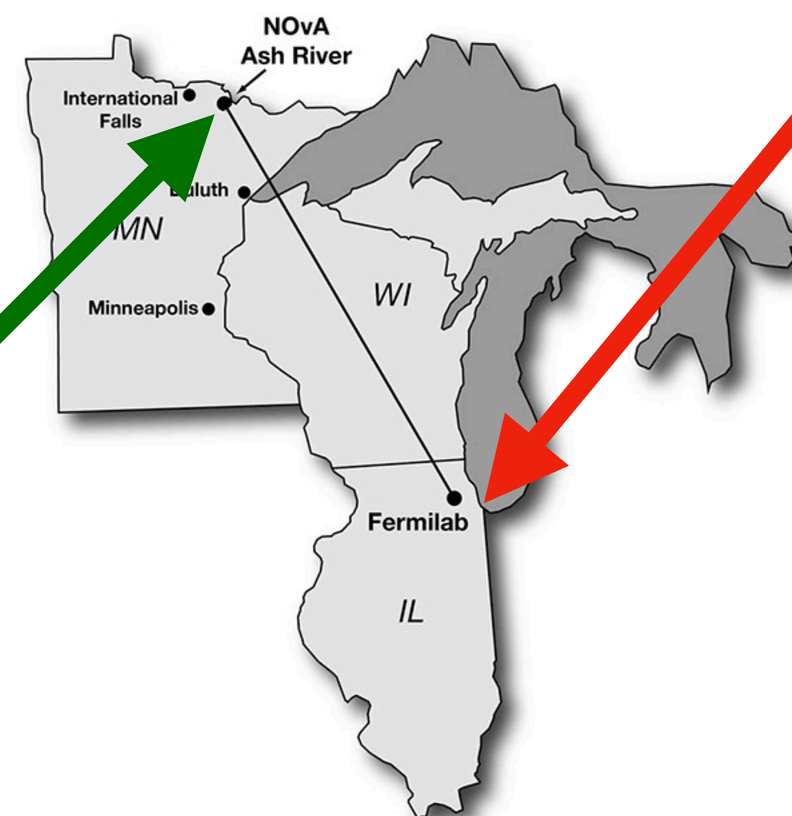
Muon+Electron+Tau Neutrinos

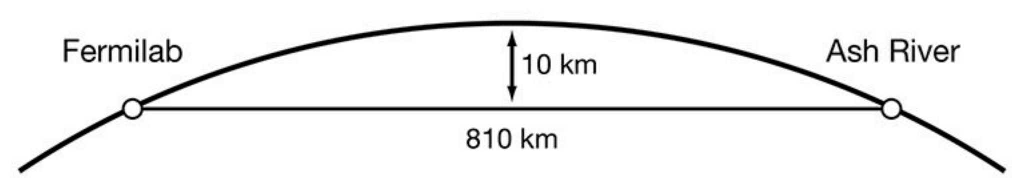
NOvA Detector

Neutrino Beam
From Fermilab (Near Chicago)
To Minnesota (Near Canada)

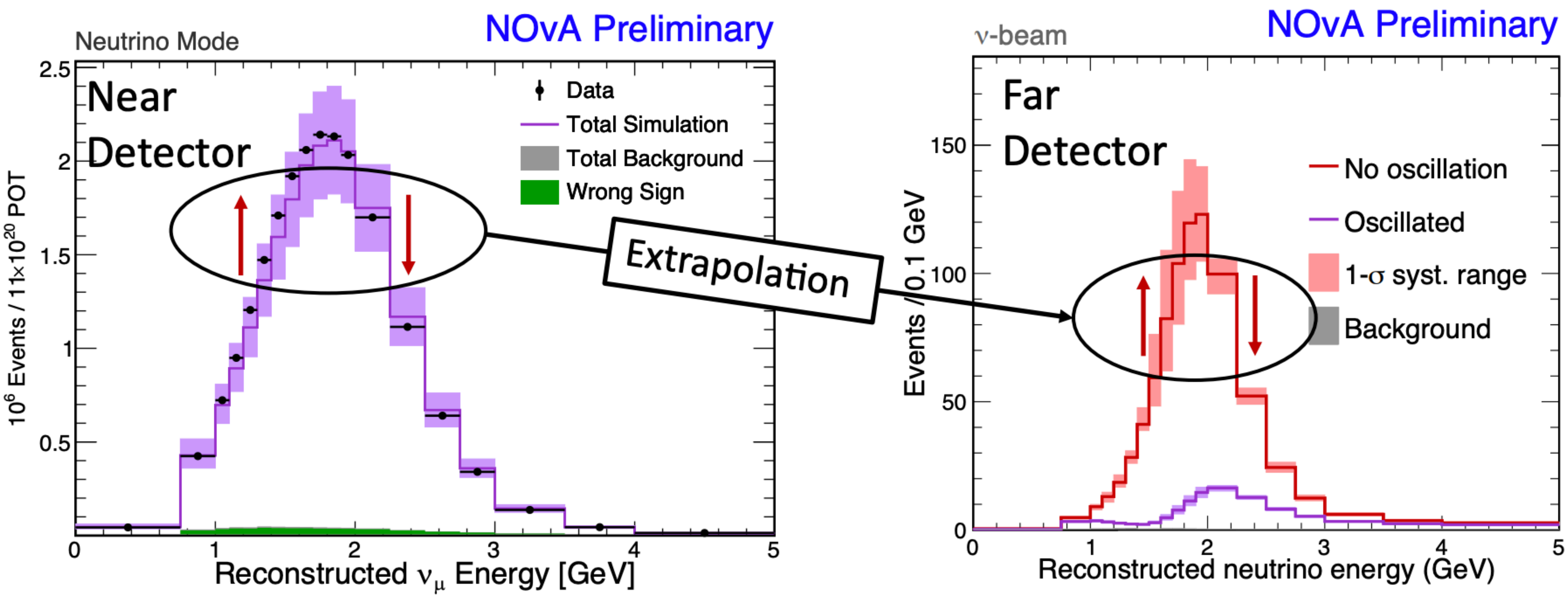








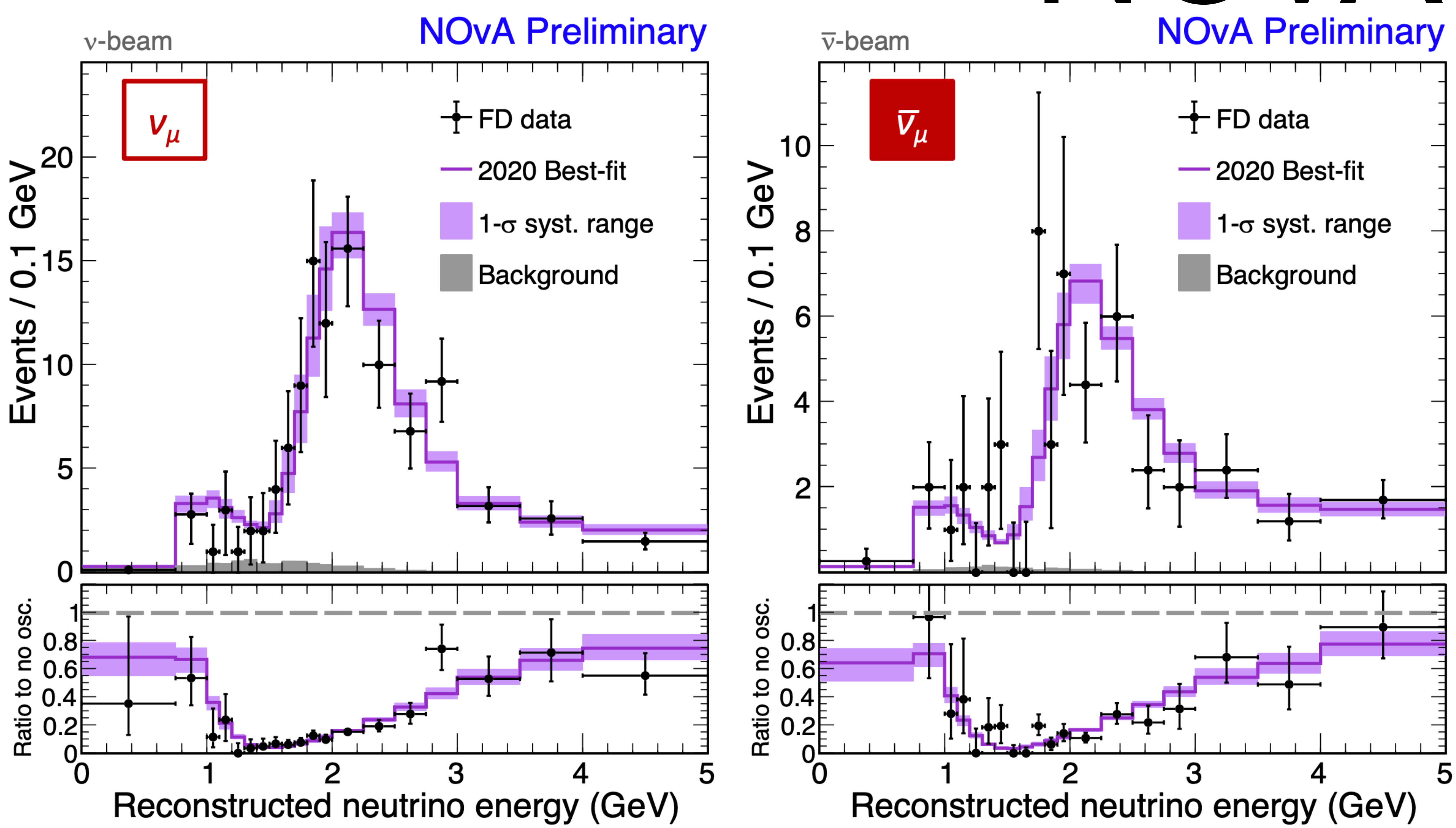
NOvA Data



- With the input detector we can predict the output dist
  - In this way we predict it without any oscillations
  - The oscillation depends on our formula

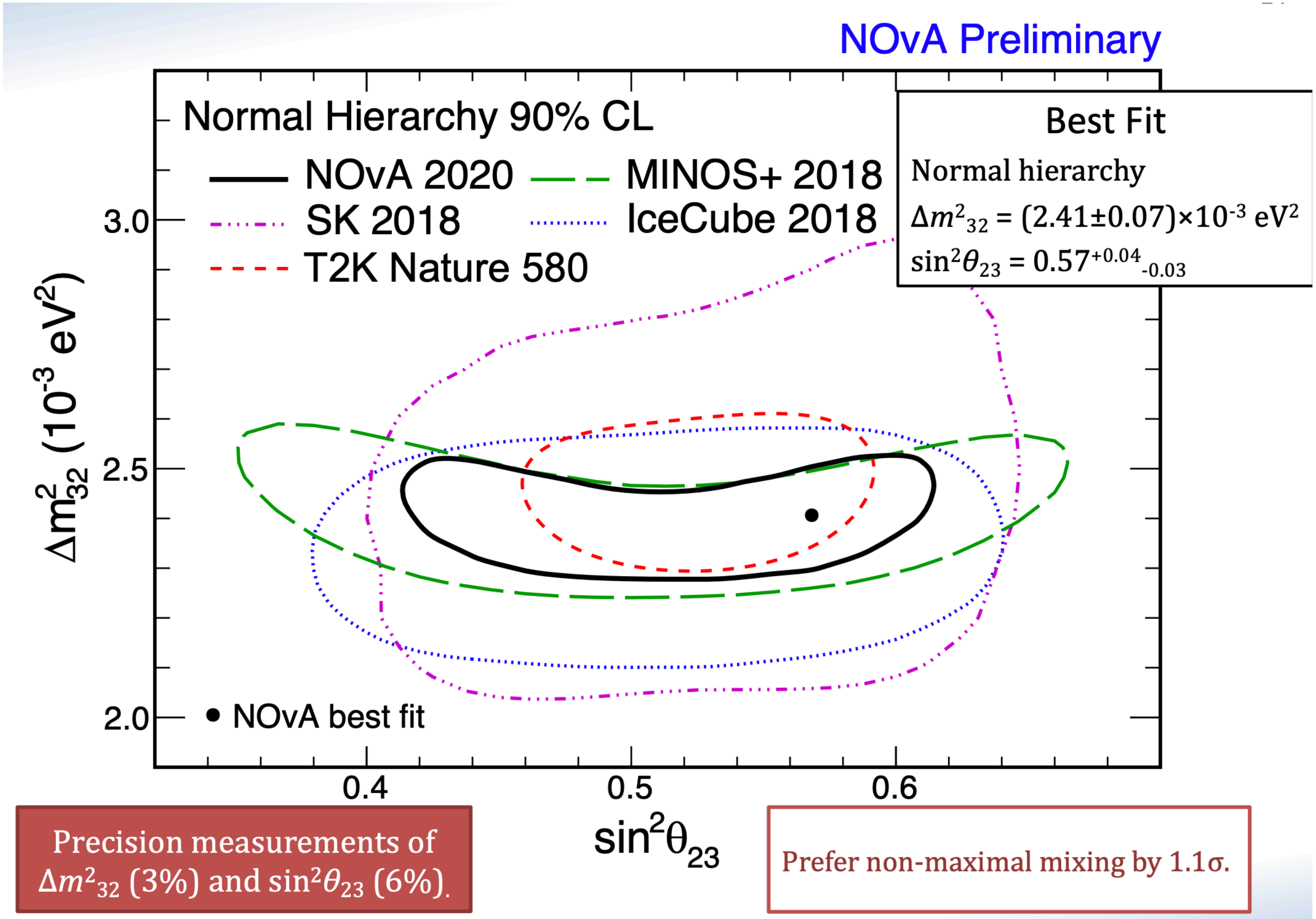
https://indico.fnal.gov/event/43209/contributions/187840/attachments/130740/159597/NOvA-Oscillations-NEUTRINO2020.pdf

NOvA Data



- With the input detector we can predict the output dist
  -

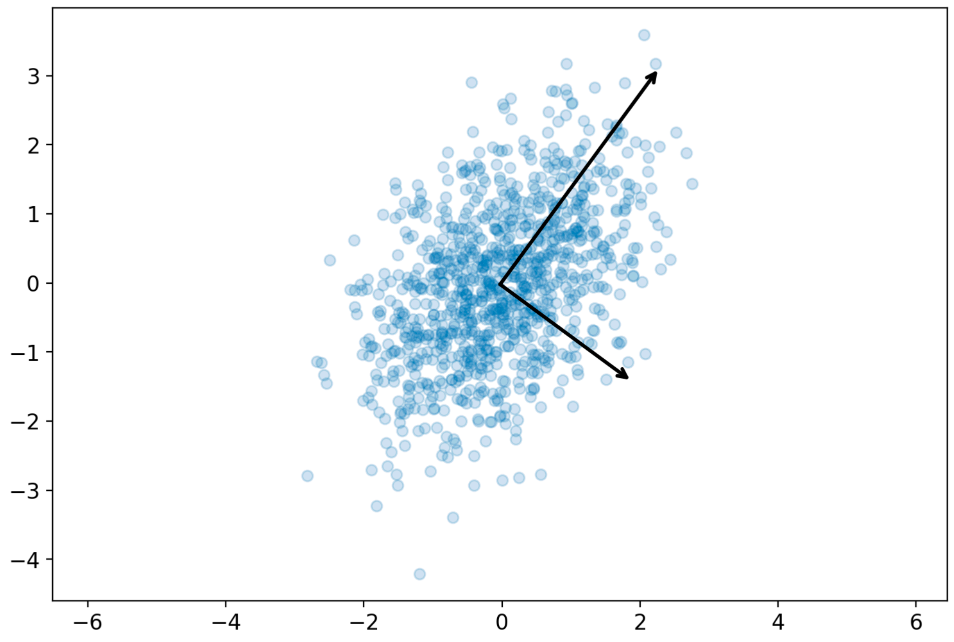
NOvA Result



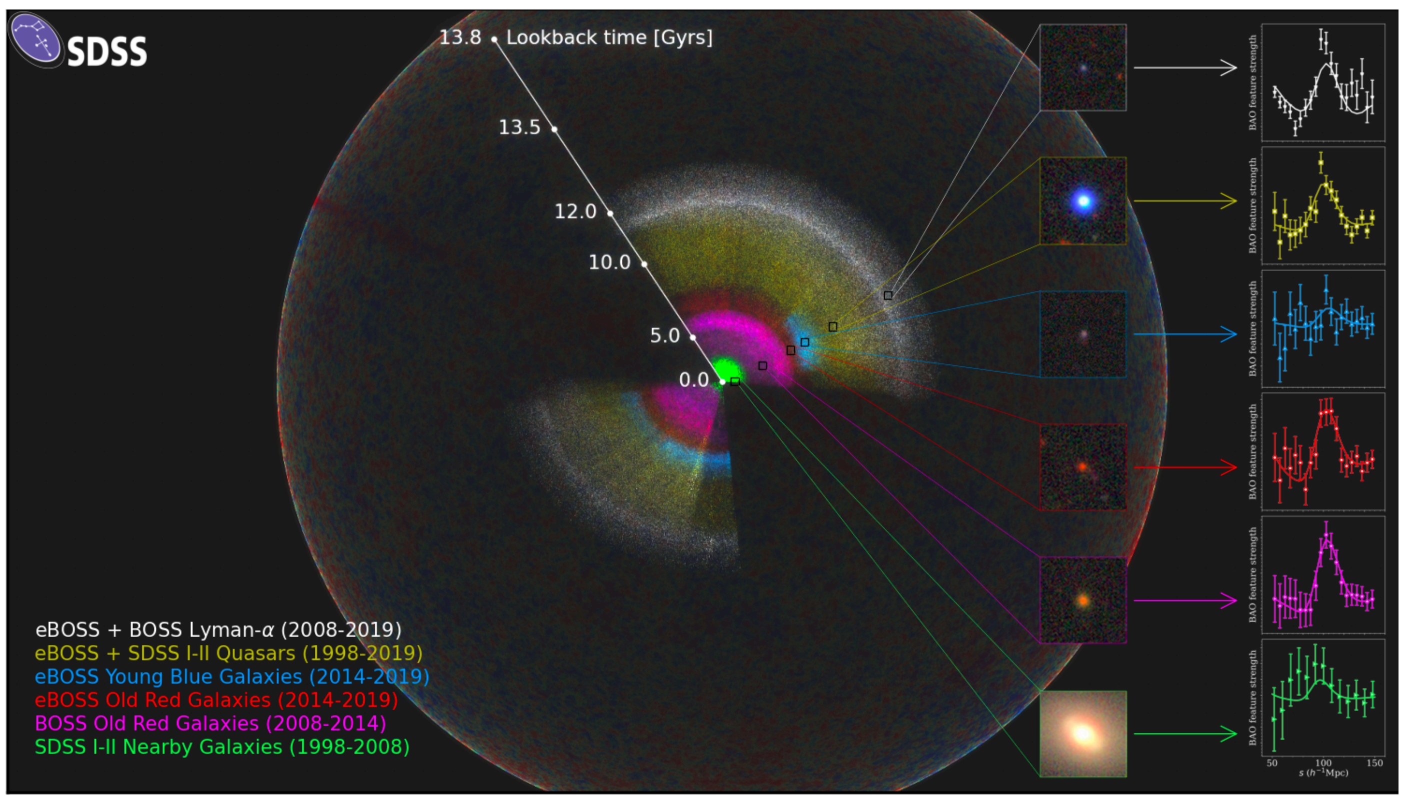
- From those plots: Lets look at those results

Principal Component Analysis

- Just taking the Eigen-vectors/values of an n-d space
  - Sorted by Eigenvalues, which give rank
  - Number of Eigenvalues above threshold is dimension



SDSS


SDSS
13.8 Lookback time [Gyrs]
13.5
12.0
10.0
5.0
0.0
eBOSS + BOSS Lyman-α (2008-2019)
eBOSS + SDSS I-II Quasars (1998-2019)
eBOSS Young Blue Galaxies (2014-2019)
eBOSS Old Red Galaxies (2014-2019)
BOSS Old Red Galaxies (2008-2014)
SDSS I-II Nearby Galaxies (1998-2008)