Chapter 2 Units, Dimensional Analysis, and Estimation

Chapter 2 Units, Dimensional Analysis, Problem Solving, Estimation, and Error Analysis

But we must not forget that all things in the world are connected with one another and depend on one another, and that we ourselves and all our thoughts are also a part of nature. It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the change of things; made because we are not restricted to any one definite measure, all being interconnected. A motion is termed uniform in which equal increments of space described correspond to equal increments of space described by some motion with which we form a comparison, as the rotation of the earth. A motion may, with respect to another motion, be uniform. But the question whether a motion is in itself uniform, is senseless. With just as little justice, also, may we speak of an "absolute time" --- of a time independent of change. This absolute time can be measured by comparison with no motion; it has therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception.[1]

Ernst Mach

2.1 The Speed of light

When we observe and measure phenomena in the world, we try to assign numbers to the physical quantities with as much accuracy as we can possibly obtain from our measuring equipment. For example, we may want to determine the speed of light, which we can calculate by dividing the distance a known ray of light propagates over its travel time,

$$\text{speed of light} = \frac{\text{distance}}{\text{time}}. \tag{2.1.1}$$

In 1983 the General Conference on Weights and Measures defined the *speed of light* to be

$$c = 299{,}792{,}458 \text{ meters/second}. \tag{2.1.2}$$

This number was chosen to correspond to the most accurately measured value of the speed of light and is well within the experimental uncertainty.

2.2 International System of Units

[1] E. Mach, *The Science of Mechanics*, translated by Thomas J. McCormack, Open Court Publishing Company, La Salle, Illinois, 1960, p. 273.
[2] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew

The system of units most commonly used throughout science and technology today is the *Système International* (SI). It consists of seven *base quantities* and their corresponding *base units,* shown in Table 2.1.

Table 2.1 International System of Units

Base Quantity	Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electric Current	ampere (A)
Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

We shall refer to the *dimension* of the base quantity by the quantity itself, for example

$$\text{dim length} \equiv \text{length} \equiv \text{L}, \text{dim mass} \equiv \text{mass} \equiv \text{M}, \text{dim time} \equiv \text{time} \equiv \text{T}. \tag{2.2.1}$$

Mechanics is based on just the first three of these quantities, the MKS or meter-kilogram-second system. An alternative metric system, still widely used, is the CGS system (centimeter-gram-second).

2.2.1 Standard Mass

The unit of mass, the kilogram (kg), remains the only base unit in the International System of Units (SI) that is still defined in terms of a physical artifact, known as the “International Prototype of the Standard Kilogram.” George Matthey (of Johnson Matthey) made the prototype in 1879 in the form of a cylinder, 39 mm high and 39 mm in diameter, consisting of an alloy of 90 % platinum and 10 % iridium. The international prototype is kept in the Bureau International des Poids et Mèsures (BIPM) at Sevres, France, under conditions specified by the 1st Conférence Générale des Poids et Mèsures (CGPM) in 1889 when it sanctioned the prototype and declared “This prototype shall henceforth be considered to be the unit of mass.” It is stored at atmospheric pressure in a specially designed triple bell-jar. The prototype is kept in a vault with six official copies.

The 3rd Conférence Générale des Poids et Mèsures CGPM (1901), in a declaration intended to end the ambiguity in popular usage concerning the word “weight” confirmed that:

> *The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.*

There is a stainless steel one-kilogram standard that is used for comparisons with standard masses in other laboratories. In practice it is more common to quote a conventional mass value (or weight-in-air, as measured with the effect of buoyancy), than the standard mass. Standard mass is normally only used in specialized measurements wherever suitable copies of the prototype are stored.

Example 2.1 The International Prototype Kilogram

In order to minimize the effects of corrosion, the platinum-iridium prototype kilogram is a right cylinder with dimensions chosen to minimize the surface area for a given fixed volume. The standard kilogram is an alloy of 90 % platinum and 10 % iridium. The density of the alloy is $\rho = 21.56 \text{ g} \cdot \text{cm}^{-3}$. Based on this information, (i) determine the radius of the prototype kilogram, and (ii) the ratio of the radius to the height.

Solution: The volume for a cylinder of radius r and height h is given by

$$V = \pi r^2 h . \tag{2.2}$$

The surface area can be expressed as a function of the radius r and the constant volume V according to

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2V}{r} . \tag{2.3}$$

To find the smallest surface area for a fixed volume, minimize the surface area with respect to the radius by setting

$$0 = \frac{dA}{dr} = 4\pi r - \frac{2V}{r^2} , \tag{2.4}$$

which we can solve for the radius

$$r = \left(\frac{V}{2\pi} \right)^{1/3} . \tag{2.5}$$

Because we also know that $V = \pi r^2 h$, we can rewrite Eq. (2.5) as

$$r^3 = \frac{\pi r^2 h}{2\pi} , \tag{2.6}$$

which implies that ratio of the radius to the height is

$$\frac{r}{h} = \frac{1}{2} . \tag{2.7}$$

The standard kilogram is an alloy of 90% platinum and 10% iridium. The density of platinum is $21.45 \text{ g} \cdot \text{cm}^{-3}$ and the density of iridium is $22.55 \text{ g} \cdot \text{cm}^{-3}$. Thus the density of the standard kilogram is

$$\rho = (0.90)(21.45\ \text{g}\cdot\text{cm}^{-3}) + (0.10)(22.55\ \text{g}\cdot\text{cm}^{-3}) = 21.56\ \text{g}\cdot\text{cm}^{-3}, \tag{2.8}$$

and its volume is

$$V = m/\rho = (1000\ \text{g})/(21.56\ \text{g}\cdot\text{cm}^{-3}) = 46.38\ \text{cm}^3. \tag{2.9}$$

For the standard mass, the radius is

$$r = \left(\frac{V}{2\pi}\right)^{1/3} = \left(\frac{46.38\ \text{cm}^3}{2\pi}\right)^{1/3} \cong 1.95\ \text{cm}. \tag{2.10}$$

Because the prototype kilogram is an artifact, there are some intrinsic problems associated with its use as a standard. It may be damaged, or destroyed. The prototype gains atoms due to environment wear and cleaning, at a rate of change of mass corresponding to approximately $1\ \mu\text{g}/\text{year}$, ($1\ \mu\text{g} \equiv 1\,\text{microgram} \equiv 1\times 10^{-6}\ \text{g}$).

Several new approaches to defining the SI unit of mass [kg] are currently being explored. One possibility is to define the kilogram as a fixed number of atoms of a particular substance, thus relating the kilogram to an atomic mass. Silicon is a good candidate for this approach because it can be grown as a large single crystal, in a very pure form.

Example 2.2 Mass of a Silicon Crystal

A given standard unit cell of silicon has a volume V_0 and contains N_0 atoms. The number of molecules in a given mole of substance is given by Avogadro's constant $N_A =$ $6.02214129(27)\times 10^{23}\ \text{mol}^{-1}$. The molar mass of silicon is given by M_{mol}. Find the mass m of a volume V in terms of V_0, N_0, V, M_{mol}, and N_A.

Solution: The mass m_0 of the unit cell is the density ρ of the silicon cell multiplied by the volume of the cell V_0,

$$m_0 = \rho V_0. \tag{2.11}$$

The number of moles in the unit cell is the total mass, m_0, of the cell, divided by the molar mass M_{mol},

$$n_0 = m_0 / M_{\text{mol}} = \rho V_0 / M_{\text{mol}}. \tag{2.12}$$

The number of atoms in the unit cell is the number of moles times the Avogadro constant, N_A,

$$N_0 = n_0 N_A = \frac{\rho V_0 N_A}{M_{\text{mol}}} . \tag{2.13}$$

The density of the crystal is related to the mass m of the crystal divided by the volume V of the crystal,

$$\rho = m / V . \tag{2.14}$$

The number of atoms in the unit cell can be expressed as

$$N_0 = \frac{m V_0 N_A}{V M_{\text{mol}}} . \tag{2.15}$$

The mass of the crystal is

$$m = \frac{M_{\text{mol}}}{N_A} \frac{V}{V_0} N_0 \tag{2.16}$$

The molar mass, unit cell volume and volume of the crystal can all be measured directly. Notice that M_{mol} / N_A is the mass of a single atom, and $(V / V_0) N_0$ is the number of atoms in the volume. This accuracy of the approach depends on how accurate the Avogadro constant can be measured. Currently, the measurement of he Avogadro constant has a relative uncertainty of 1 part in 10^8, which is equivalent to the uncertainty in the present definition of the kilogram.

2.2.2 Atomic Clock and the Definition of the Second

Isaac Newton, in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), distinguished between time as duration and an absolute concept of time,

> "*Absolute true and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.* "[2].

The development of clocks based on atomic oscillations allowed measures of timing with accuracy on the order of 1 part in 10^{14}, corresponding to errors of less than one microsecond (one millionth of a second) per year. Given the incredible accuracy of this measurement, and clear evidence that the best available timekeepers were atomic in

[2] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 6.

nature, the *second* [s] was redefined in 1967 by the International Committee on Weights and Measures as a certain number of cycles of electromagnetic radiation emitted by cesium atoms as they make transitions between two designated quantum states:

> *The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.*

2.2.3 Meter

The meter [m] was originally defined as 1/10,000,000 of the arc from the Equator to the North Pole along the meridian passing through Paris. To aid in calibration and ease of comparison, the meter was redefined in terms of a length scale etched into a platinum bar preserved near Paris. Once laser light was engineered, the meter was redefined by the 17th Conférence Générale des Poids et Mèsures (CGPM) in 1983 to be a certain number of wavelengths of a particular monochromatic laser beam.

> *The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.*

Example 2.3 Light-Year

Astronomical distances are sometimes described in terms of *light-years* [ly]. A light-year is the distance that light will travel in one year [yr]. How far in meters does light travel in one year?

Solution: Using the relationship $\text{distance} = (\text{speed of light})\cdot(\text{time})$, one light year corresponds to a distance. Because the speed of light is given in terms of meters per second, we need to know how many seconds are in a year. We can accomplish this by converting units. We know that

$1 \text{ year} = 365.25 \text{ days},\ 1 \text{ day} = 24 \text{ hours},\ 1 \text{ hour} = 60 \text{ minutes},\ 1 \text{ minute} = 60 \text{ seconds}$

Putting this together we find that the number of seconds in a year is

$$1 \text{ year} = \left(365.25 \text{ day}\right)\left(\frac{24 \text{ hours}}{1 \text{ day}}\right)\left(\frac{60 \text{ min}}{1 \text{ hour}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 31{,}557{,}600 \text{ s}. \qquad (2.2.17)$$

The distance that light travels in a one year is

$$1 \text{ ly} = \left(\frac{299{,}792{,}458 \text{ m}}{1 \text{ s}}\right)\left(\frac{31{,}557{,}600 \text{ s}}{1 \text{ yr}}\right)\left(1 \text{ yr}\right) = 9.461\times 10^{15} \text{ m}. \qquad (2.2.18)$$

The distance to the nearest star, a faint red dwarf star, Proxima Centauri, is 4.24 ly.

2.2.4 Radians

Consider the triangle drawn in Figure 2.1. The basic trigonometric functions of an angle θ in a right-angled triangle ONB are $\sin(\theta) = y / r$, $\cos(\theta) = x / r$, and $\tan(\theta) = y / x$.

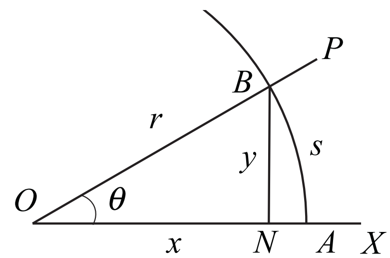

Figure 2.1 Trigonometric relations

It is very important to become familiar with using the measure of the angle θ itself as expressed in *radians* [rad]. Let θ be the angle between two straight lines OX and OP. Draw a circle of radius r centered at O. The lines OP and OX cut the circle at the points A and B where $OA = OB = r$. Denote the length of the arc AB by s, then the radian measure of θ is given by

$$\theta = s / r, \tag{2.2.19}$$

and the ratio is the same for circles of any radii centered at O -- just as the ratios y / r and y / x are the same for all right triangles with the angle θ at O. As θ approaches 360°, s approaches the complete circumference $2\pi r$ of the circle, so that $360^\circ = 2\pi \text{ rad}$.

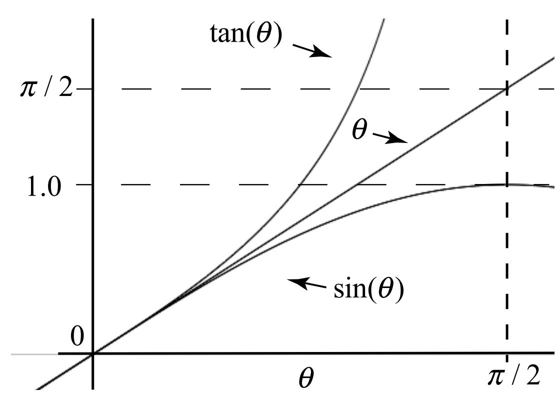

Figure 2.2 Radians compared to trigonometric functions.

Let's compare the behavior of $\sin(\theta)$, $\tan(\theta)$ and θ itself for small angles. One can see from Figure 2.1 that $s / r > y / r$. It is less obvious that $y / x > \theta$. It is very instructive to plot $\sin(\theta)$, $\tan(\theta)$, and θ as functions of θ [rad] between 0 and $\pi/2$ on the same graph (see Figure 2.2). For small θ, the values of all three functions are almost equal. But how small is "small"? An acceptable condition is for $\theta << 1$ in radians.

We can show this with a few examples. Recall that $360^\circ = 2\pi \text{ rad}$, $57.3^\circ = 1\text{ rad}$, so an angle $6^\circ \cong (6^\circ)(2\pi \text{ rad}/360^o) \cong 0.1 \text{ rad}$ when expressed in radians. In Table 2.2 we compare the value of θ (measured in radians) with $\sin(\theta)$, $\tan(\theta)$, $(\theta - \sin\theta)/\theta$, and $(\theta - \tan\theta)/\theta$, for $\theta = 0.1\text{ rad}$, 0.2 rad, 0.5 rad, and 1.0 rad.

Table 2.2 Small Angle Approximation

θ [rad]	θ [deg]	$\sin(\theta)$	$\tan(\theta)$	$(\theta-\sin\theta)/\theta$	$(\theta-\tan\theta)/\theta$
0.1	5.72958	0.09983	0.10033	0.00167	-0.00335
0.2	11.45916	0.19867	0.20271	0.00665	-0.01355
0.5	28.64789	0.47943	0.54630	0.04115	-0.09260
1.0	57.29578	0.84147	1.55741	0.15853	-0.55741

The values for $(\theta - \sin\theta)/\theta$, and $(\theta - \tan\theta)/\theta$, for $\theta = 0.2\text{ rad}$ are less than $\pm 1.4\%$. Provided that θ is not too large, the approximation that

$$\sin(\theta) \simeq \tan(\theta) \simeq \theta, \tag{2.2.20}$$

called the *small angle approximation*, can be used almost interchangeably, within some small percentage error. This is the basis of many useful approximations in physics calculations.

Example 2.4 Parsec

A standard astronomical unit is the parsec. Consider two objects that are separated by a distance of one astronomical unit, $1\text{AU} = 1.50 \times 10^{11}\text{ m}$, which is the mean distance between the earth and sun. (One astronomical unit is roughly equivalent to eight light minutes, $1\text{AU} = 8.3\text{light-minutes}$.) One parsec is the distance at which one astronomical unit subtends an angle $\theta = 1\text{ arcsecond} = (1/3600)\text{ degree}$. Suppose is a spacecraft is located in a space a distance 1 parsec from the Sun as shown in Figure 2.3. How far is the spacecraft in terms of light years and meters?

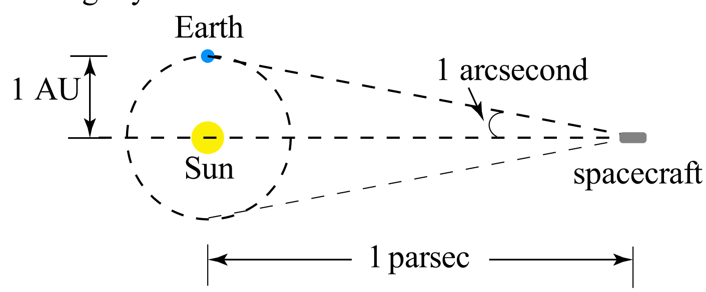

Figure 2.3 Example 2.4

Because one arc second corresponds to a very small angle, one parsec is therefore equal to distance divided by angle, hence

$$\begin{aligned} 1\,\text{pc} &= \frac{(1\,\text{AU})}{(1/3600)} = (2.06\times10^{5}\,\text{AU})\left(\frac{1.50\times10^{11}\,\text{m}}{1\,\text{AU}}\right) = 3.09\times10^{16}\ \text{m} \\ &= (3.09\times10^{16}\ \text{m})\left(\frac{1\,\text{ly}}{9.46\times10^{15}\,\text{m}}\right) = 3.26\ \text{ly} \end{aligned} \quad . \quad (2.2.21)$$

2.2.5 Steradians

The *steradian* [sr] is the unit of solid angle that, having its vertex in the center of a sphere, cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere. The conventional symbol for steradian measure is Ω, the uppercase Greek letter "Omega." The total solid angle Ω_{sph} of a sphere is then found by dividing the surface area of the sphere by the square of the radius,

$$\Omega_{\text{sph}} = 4\pi r^2 / r^2 = 4\pi \qquad (2.2.22)$$

This result is independent of the radius of the sphere.

2.2.6 Radiant Intensity

> *"The SI unit, candela, is the luminous intensity of a source that emits monochromatic radiation of frequency* $540\times10^{12}\ \text{s}^{-1}$*, in a given direction, and that has a radiant intensity in that direction of 1/683 watts per steradian."*

Note that "in a given direction" cannot be taken too literally. The intensity is measured per steradian of spread, so if the radiation has no spread of directions, the luminous intensity would be infinite.

2.3 Dimensions of Commonly Encountered Quantities

Many physical quantities are derived from the base quantities by a set of algebraic relations defining the physical relation between these quantities. The dimension of the derived quantity is written as a power of the dimensions of the base quantities. For example velocity is a derived quantity and the dimension is given by the relationship

$$\text{dim velocity} = (\text{length})/(\text{time}) = \text{L}\cdot\text{T}^{-1}. \qquad (2.3.1)$$

where $\text{L} \equiv \text{length}$, $\text{T} \equiv \text{time}$. Force is also a derived quantity and has dimension

$$\text{dim force} = \frac{\text{(mass)(dim velocity)}}{\text{(time)}} \quad . \tag{2.3.2}$$

where $M \equiv \text{mass}$. We can also express force in terms of mass, length, and time by the relationship

$$\text{dim force} = \frac{\text{(mass)(length)}}{\text{(time)}^2} = \text{M} \cdot \text{L} \cdot \text{T}^{-2} . \tag{2.3.3}$$

The derived dimension of kinetic energy is

$$\text{dim kinetic energy} = \text{(mass)(dim velocity)}^2 , \tag{2.3.4}$$

which in terms of mass, length, and time is

$$\text{dim kinetic energy} = \frac{\text{(mass)(length)}^2}{\text{(time)}^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} . \tag{2.3.5}$$

The derived dimension of work is

$$\text{dim work} = \text{(dim force)(length)} , \tag{2.3.6}$$

which in terms of our fundamental dimensions is

$$\text{dim work} = \frac{\text{(mass)(length)}^2}{\text{(time)}^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} . \tag{2.3.7}$$

So work and kinetic energy have the same dimensions. Power is defined to be the rate of change in time of work so the dimensions are

$$\text{dim power} = \frac{\text{dim work}}{\text{time}} = \frac{\text{(dim force)(length)}}{\text{time}} = \frac{\text{(mass)(length)}^2}{\text{(time)}^3} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-3} . \tag{2.3.8}$$

In Table 2.3 we include the derived dimensions of some common mechanical quantities in terms of mass, length, and time.

2.3.1 Dimensional Analysis

There are many phenomena in nature that can be explained by simple relationships between the observed phenomena.

Table 2.3 Dimensions of Some Common Mechanical Quantities

$$\mathrm{M} \equiv \text{mass}, \ \mathrm{L} \equiv \text{length}, \ \mathrm{T} \equiv \text{time}$$

Quantity	Dimension	MKS unit
Angle	dimensionless	Dimensionless = radian
Solid Angle	dimensionless	Dimensionless = sterradian
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	s^{-1} = hertz = Hz
Velocity	$\mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Acceleration	$\mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{m} \cdot \mathrm{s}^{-2}$
Angular Velocity	T^{-1}	$\mathrm{rad} \cdot \mathrm{s}^{-1}$
Angular Acceleration	T^{-2}	$\mathrm{rad} \cdot \mathrm{s}^{-2}$
Density	$\mathrm{M} \cdot \mathrm{L}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^{-3}$
Momentum	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$
Angular Momentum	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$
Force	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2} = \text{newton} = \mathrm{N}$
Work, Energy	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} = \text{joule} = \mathrm{J}$
Torque	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$ = watt = W
Pressure	$\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2} = \text{pascal} = \mathrm{Pa}$

Example 2.5 Period of a Pendulum

Consider a simple pendulum consisting of a massive bob suspended from a fixed point by a string. Let T denote the time interval (period of the pendulum) that it takes the bob to complete one cycle of oscillation. How does the period of the simple pendulum depend on the quantities that define the pendulum and the quantities that determine the motion?

Solution: What possible quantities are involved? The length of the pendulum l, the mass of the pendulum bob m, the gravitational acceleration g, and the angular amplitude of the bob θ_0 are all possible quantities that may enter into a relationship for the period of the swing. Have we included every possible quantity? We can never be sure but let's first work with this set and if we need more than we will have to think harder! Our problem is then to find a function f such that

$$T = f(l, m, g, \theta_0) \tag{2.3.9}$$

We first make a list of the dimensions of our quantities as shown in Table 2.4.

Table 2.4 Dimensions of Quantities Relevant to the Period of Pendulum

Name of Quantity	Symbol	Dimensional Formula
Time of swing	t	T
Length of pendulum	l	L
Mass of pendulum	m	M
Gravitational acceleration	g	$\mathrm{L} \cdot \mathrm{T}^{-2}$
Angular amplitude of swing	θ_0	No dimension

Our first observation is that the mass of the bob cannot enter into our relationship, as our final quantity has no dimensions of mass and no other quantity has dimensions of mass. Let's focus on the length of the string and the gravitational acceleration. In order to eliminate length, these quantities must divide each other when appearing in some functional relation for the period T. If we choose the combination l/g, the dimensions are

$$\dim[l\,/\,g] = \frac{\text{length}}{\text{length/(time)}^2} = (\text{time})^2 \qquad (2.3.10)$$

It appears that the time of swing may proportional to the square root of this ratio. Thus we have a candidate formula

$$T \sim \left(\frac{l}{g}\right)^{1/2}. \qquad (2.3.11)$$

(in the above expression, the symbol "$\sim$" represents a proportionality, not an approximation). Because the angular amplitude θ_0 is dimensionless, it may or may not appear. We can account for this by introducing some function $y(\theta_0)$ into our relationship, which is beyond the limits of this type of analysis. The period is then

$$T = y(\theta_0)\left(\frac{l}{g}\right)^{1/2}. \qquad (2.3.12)$$

We shall discover later on that $y(\theta_0)$ is nearly independent of the angular amplitude θ_0 for very small amplitudes and is equal to $y(\theta_0) = 2\pi$,

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2} \qquad (2.3.13)$$

2.4 Order of Magnitude Estimates - Fermi Problems

Counting is the first mathematical skill we learn. We came to use this skill by distinguishing elements into groups of similar objects, but counting becomes problematic when our desired objects are not easily identified, or there are too many to count. Rather than spending a huge amount of effort to attempt an exact count, we can try to estimate the number of objects. For example, we can try to estimate the total number of grains of sand contained in a bucket of sand. Because we can see individual grains of sand, we expect the number to be very large but finite. Sometimes we can try to estimate a number, which we are fairly sure but not certain is finite, such as the number of particles in the universe.

We can also assign numbers to quantities that carry dimensions, such as mass, length, time, or charge, which may be difficult to measure exactly. We may be interested in estimating the mass of the air inside a room, or the length of telephone wire in the United States, or the amount of time that we have slept in our lives. We choose some set of units, such as kilograms, miles, hours, and coulombs, and then we can attempt to estimate the number with respect to our standard quantity.

Often we are interested in estimating quantities such as speed, force, energy, or power. We may want to estimate our natural walking speed, or the force of wind acting against a bicycle rider, or the total energy consumption of a country, or the electrical power necessary to operate a university. All of these quantities have no exact, well-defined value; they instead lie within some range of values.

When we make these types of estimates, we should be satisfied if our estimate is reasonably close to the middle of the range of possible values. But what does "reasonably close" mean? Once again, this depends on what quantities we are estimating. If we are describing a quantity that has a very large number associated with it, then an estimate within an order of magnitude should be satisfactory. The number of molecules in a breath of air is close to 10^{22}; an estimate anywhere between 10^{21} and 10^{23} molecules is close enough. If we are trying to win a contest by estimating the number of marbles in a glass container, we cannot be so imprecise; we must hope that our estimate is within 1% of the real quantity. These types of estimations are called *Fermi problems*. The technique is named after the physicist Enrico Fermi, who was famous for making these sorts of "back of the envelope" calculations.

2.4.1 Methodology for Estimation Problems

Estimating is a skill that improves with practice. Here are two guiding principles that may help you get started.

(1) You must identify a set of quantities that can be estimated or calculated.

(2) You must establish an approximate or exact relationship between these quantities and the quantity to be estimated in the problem.

Estimations may be characterized by a precise relationship between an estimated quantity and the quantity of interest in the problem. When we estimate, we are drawing upon what we know. But different people are more familiar with certain things than others. If you are basing your estimate on a fact that you already know, the accuracy of your estimate will depend on the accuracy of your previous knowledge. When there is no precise relationship between estimated quantities and the quantity to be estimated in the problem, then the accuracy of the result will depend on the type of relationships you decide upon. There are often many approaches to an estimation problem leading to a reasonably accurate estimate. So use your creativity and imagination!

Example 2.6 Lining Up Pennies

Suppose you want to line pennies up, diameter to diameter, until the total length is 1 kilometer . How many pennies will you need? How accurate is this estimation?

Solution: The first step is to consider what type of quantity is being estimated. In this example we are estimating a dimensionless scalar quantity, the number of pennies. We can now give a precise relationship for the number of pennies needed to mark off 1 kilometer

$$\#\text{ of pennies} = \frac{\text{total distance}}{\text{diameter of penny}} . \tag{2.4.1}$$

We can estimate a penny to be approximately 2 centimeters wide. Therefore the number of pennies is

$$\begin{aligned}\#\text{ of pennies} &= \frac{\text{total distance}}{\text{length of a penny}} = \frac{(1\text{ km})}{(2\text{ cm})(1\text{ km}/10^5\text{ cm})} \\ &= 50{,}000\text{ pennies} = 5\times 10^4\text{ pennies.}\end{aligned} \tag{2.4.2}$$

When applying numbers to relationships we must be careful to convert units whenever necessary. How accurate is this estimation? If you measure the size of a penny, you will find out that the width is 1.9 cm, so our estimate was accurate to within 5%. This accuracy was fortuitous. Suppose we estimated the length of a penny to be 1 cm. Then our estimate for the total number of pennies would be within a factor of 2, a margin of error we can live with for this type of problem.

Example 2.7 Estimation of Mass of Water on Earth

Estimate the mass of the water on the Earth.

Solution: In this example we are estimating mass, a quantity that is a fundamental in SI units, and is measured in kg. We start by approximating that the amount of water on Earth is approximately equal to the amount of water in all the oceans. Initially we will try to estimate two quantities: the density of water and the volume of water contained in the oceans. Then the relationship we want is

$$\text{mass} = (\text{density})(\text{volume})\,. \tag{2.4.3}$$

One of the hardest aspects of estimation problems is to decide which relationship applies. One way to check your work is to check dimensions. Density has dimensions of mass/volume, so our relationship is correct dimensionally.

The density of fresh water is $\rho = 1.0\ \text{g}\cdot\text{cm}^{-3}$; the density of seawater is slightly higher, but the difference won't matter for this estimate. You could estimate this density by estimating how much mass is contained in a one-liter bottle of water. (The density of water is a point of reference for all density problems. Suppose we need to estimate the density of iron. If we compare iron to water, we might estimate that iron is 5 to 10 times denser than water. The actual density of iron is $\rho_{\text{iron}} = 7.8\ \text{g}\cdot\text{cm}^{-3}$).

Because there is no precise relationship, estimating the volume of water in the oceans is much harder. Let's model the volume occupied by the oceans as if the water completely covers the earth, forming a spherical shell of radius R_E and thickness d (Figure 2.4, which is decidedly not to scale), where R_E is the radius of the earth and d is the average depth of the ocean. The volume of that spherical shell is

$$\text{volume} \cong 4\pi R_{\text{earth}}^2 d\,. \tag{2.4.4}$$

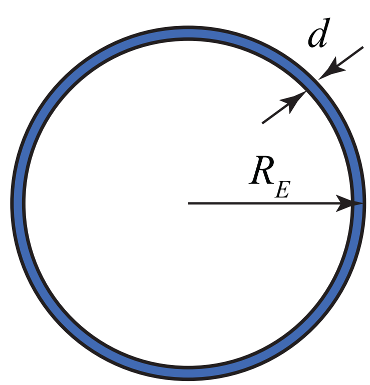

Figure 2.4 A model for estimating the mass of the water on Earth.

We also estimate that the oceans cover about 75% of the surface of the earth. So we can refine our estimate that the volume of the oceans is

$$\text{volume} \cong (0.75)(4\pi R_E{}^2 d)\,. \tag{2.4.5}$$

We therefore have two more quantities to estimate, the average depth of the ocean, which we can estimate as $d \cong 1\text{km}$, and the radius of the earth, which is approximately $R_E \cong 6\times 10^3\text{km}$. (The quantity that you may remember is the circumference of the earth, about 25,000 miles. Historically the circumference of the earth was defined to be 4×10^{7} m). The radius R_E and the circumference s are exactly related by

$$s = 2\pi R_E\,. \qquad (2.4.6)$$

Thus

$$R_E = \frac{s}{2\pi} = \frac{\left(2.5\times10^{4}\ \text{mi}\right)\left(1.6\ \text{km}\cdot\text{mi}^{-1}\right)}{2\pi} = 6.4\times10^{3}\ \text{km} \qquad (2.4.7)$$

We will use $R_E \cong 6\times10^{3}\,\text{km}$; additional accuracy is not necessary for this problem, since the ocean depth estimate is clearly less accurate. In fact, the factor of 75% is not needed, but included more or less from habit. Altogether, our estimate for the mass of the oceans is

$$\begin{aligned}
\text{mass} &= (\text{density})(\text{volume}) \cong \rho(0.75)(4\pi R_E^{\ 2} d)\\
\text{mass} &\cong \left(\frac{1\text{g}}{\text{cm}^3}\right)\left(\frac{1\,\text{kg}}{10^3\ \text{g}}\right)\left(\frac{(10^5\ \text{cm})^3}{(1\,\text{km})^3}\right)(0.75)(4\pi)(6\times10^3\ \text{km})^2(1\text{km}) \qquad (2.4.8)\\
\text{mass} &\cong 3\times10^{20}\ \text{kg} \cong 10^{20}\ \text{kg}.
\end{aligned}$$