Chapter 10: Circular Motion Dynamics

Peter Dourmashkin

September 6, 2023

Contents

Chapter 10

Applications of Newton's Second Law and Circular Motion

I shall now recall to mind that the motion of the heavenly bodies is circular, since the motion appropriate to a sphere is rotation in a circle[1]

Nicholas Copernicus

10.1 Introduction: Newton's Second Law and Circular Motion

We have already shown that when an object moves in a circular orbit of radius r with angular velocity $\vec{\omega}$ (Figure 10.1(a)) it is most convenient to choose cylindrical coordinates to describe the position, velocity and acceleration vectors. In particular for the case in which $d\theta/dt > 0$ and $d^2\theta/dt^2 > 0$, the acceleration vector is given by (Figure 10.1(b))

$$\vec{\mathbf{a}}(t) = -r\left(\frac{d\theta}{dt}\right)^2 \hat{\mathbf{r}}(t) + r\,\frac{d^2\theta}{dt^2}\,\hat{\boldsymbol{\theta}}(\mathbf{t}). \tag{10.1}$$

Then Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, can be decomposed into radial and tangential components

$$F_r = -mr\left(\frac{d\theta}{dt}\right)^2 \ \text{(circular motion)}, \tag{10.2}$$

$$F_\theta = mr\,\frac{d^2\theta}{dt^2} \qquad \text{(circular motion)}. \tag{10.3}$$

For the special case of uniform circular motion, $d\theta/dt = 0$, the sum of the tangential components of the force acting on the object must therefore be zero, $F_\theta = 0$.

[1] *Dedicatory Letter to Pope Paul III.*

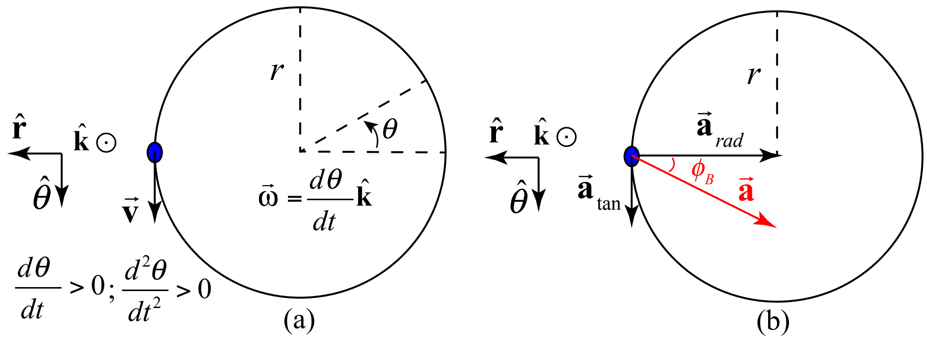

Figure 10.1: (a) Circular motion with angular velocity $\vec{\omega}$; (b) Components of acceleration.

10.2 Universal Law of Gravitation and the Circular Orbit of the Moon

An important example of (approximate) circular motion is the orbit of the Moon around Earth. We can approximately calculate the time T the oon takes to complete one circle around Earth (a calculation of great importance to early lunar calendar systems.) Denote the distance from the Moon to the center of Earth by $R_{e,m}$.

Because the Moon moves nearly in a circular orbit with angular speed $\omega = 2\pi/T$, it is accelerating towards Earth. The radial component of the acceleration (centripetal acceleration) is

$$a_r = -\frac{4\pi^2 R_{e,m}}{T^2}. \tag{10.4}$$

According to Newton's Second Law, there must be a centripetal force acting on the Moon directed towards the center of Earth that accounts for this inward acceleration.

10.2.1 Universal Law of Gravitation

Newton's Universal Law of Gravitation describes the gravitational force between two bodies 1 and 2 with masses m_1 and m_2 respectively. This force is a radial force (always pointing along the radial line connecting the masses) and the magnitude is proportional to the inverse square of the distance that separates the bodies. Then the force on object 2 due to the gravitational interaction between the bodies is given by

$$\vec{\mathbf{F}}_{1,2} = -G\,\frac{m_1\, m_2}{{r_{1,2}}^2}\,\hat{\mathbf{r}}_{1,2}. \tag{10.5}$$

where $r_{1,2}$ is the distance between the two bodies and $\hat{\mathbf{r}}_{1,2}$ is the unit vector located at the position of object 2 and pointing from object 1 towards object 2. The Universal Gravitation Constant is $G = 6.67\times10^{-11}\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$. Figure 10.2 shows the direction of the forces on bodies 1 and 2 along with the unit vectors. Newton realized that there

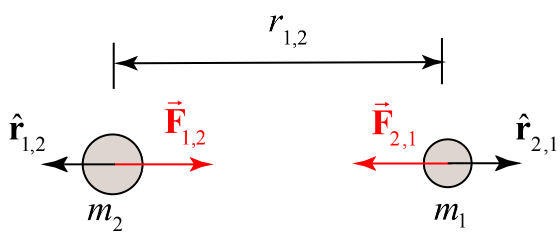

Figure 10.2: Gravitational force of interaction between two bodies.

were still some subtleties involved. First, why should the mass of Earth act as if it were all placed at the center of Earth? Newton showed that for a perfect sphere with uniform mass distribution, all the mass may be assumed to be located at the center. (This calculation is difficult and can be found in Appendix 9A to this chapter.) We assume for the present calculation that Earth and the Moon are perfect spheres with uniform mass distribution.

Second, does this gravitational force between Earth and the Moon form a Third Law interaction pair? When Newton first explained the Moon's motion in 1666, he had still not formulated the Third Law, which accounted for the long delay in the publication of the *Principia*. The link between the concept of force and the concept of a Third Law interaction pair of forces was the last piece needed to solve the puzzle of the effect of gravity on planetary orbits. Once Newton realized that the gravitational force between any two bodies forms a Third Law interaction pair, and satisfies both the Universal Law of Gravitation and his newly formulated Third Law, he was able to solve the oldest and most important physics problem of his time, the motion of the planets.

The test for the Universal Law of Gravitation was performed through experimental observation of the motion of planets, which turned out to be resoundingly successful. For almost 200 years, Newton's Universal Law was in excellent agreement with observation. As a signal of more complicated physics ahead, the first discrepancy only occurred when a slight deviation of the motion of Mercury was experimentally confirmed in 1882. The prediction of this deviation was the first success of Einstein's Theory of General Relativity (formulated in 1915).

We can apply this Universal Law of Gravitation to calculate the period of the Moon's orbit around Earth. The mass of the Moon is $m_1 = 7.36\times10^{22}$ kg and the mass of Earth is $m_2 = 5.98\times10^{24}$ kg. The distance from Earth to the Moon is [$R_{e,m} = 3.82\times10^8$ m. We show the force diagram in Figure 10.3.

Newton's Second Law of motion for the radial direction becomes

$$- G\,\frac{m_1\,m_2}{R_{E,M}^2} = -m_1\frac{4\pi^2 R_{E,M}}{T^2}, \tag{10.6}$$

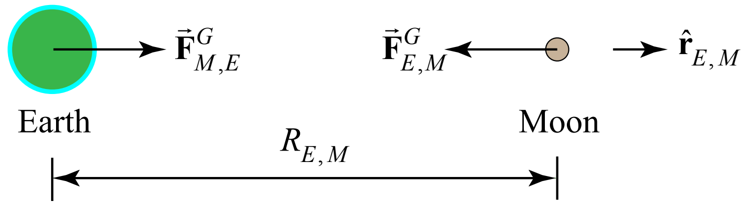

Figure 10.3: Gravitational interaction between the Moon and Earth.

which can be solved for the period of the orbit

$$T = \sqrt{\frac{4\pi^2 R_{E,M}^3}{G\, m_2}}. \tag{10.7}$$

Substitute the given values for the radius of the orbit, the mass of the earth, and the universal gravitational constant. The period of the orbit is

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2 (3.82 \times 10^8 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2})\,(5.98 \times 10^{24} \text{ kg})}} = 2.35 \times 10^6 \text{ s} \\ &= (2.35 \times 10^6 \text{ s}) \left(\frac{1 \text{ day}}{8.64 \times 10^4 \text{ s}} \right) = 27.2 \text{ days}. \end{aligned} \tag{10.8}$$

This period is called the *sidereal month* because it is the time that it takes for the Moon to return to a given position with respect to the stars.

The actual time T_1 between full moons, called the *synodic month* (the average period of the Moon's revolution with respect to Earth is 29.53 days, although it may range between 29.27 days and 29.83 days), is longer than the sidereal month because Earth is traveling around the Sun. For the next full moon, the Moon must travel a little farther than one full circle around Earth in order to be on the other side of Earth from the Sun (Figure 10.4). Therefore the time T_1 between consecutive full moons is approximately $T_1 + \Delta T$ where $\Delta T \approx T/12 = 2.3$ days. So the period of the synodic month is $T_1 = 29.5\text{days}$.

10.2.2 Kepler's Third Law for Circular Motion.

The first thing that we notice from the above solution is that the period does not depend on the mass of the Moon. We also notice that the square of the period is proportional to the cube of the distance between the Earth and the Moon,

$$T^2 = \frac{4\pi^2 R_{E,M}^3}{G\, m_2}. \tag{10.9}$$

This is an example of Kepler's Third Law, of which Newton was aware. This confirmation was convincing evidence to Newton that his Universal Law of Gravitation was

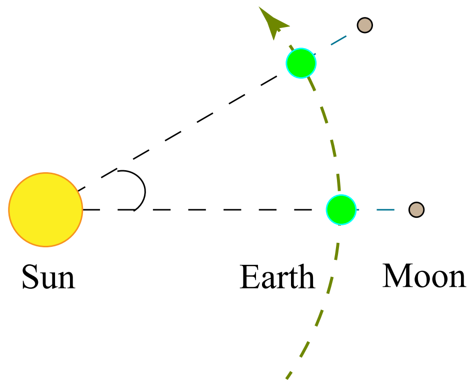

Figure 10.4: Orbital motion between full moons.

the correct mathematical description of the gravitational force law, even though he still could not explain what "caused" gravity.

10.3 Worked Examples Circular Motion

10.3.1 Example Geosynchronous Orbit

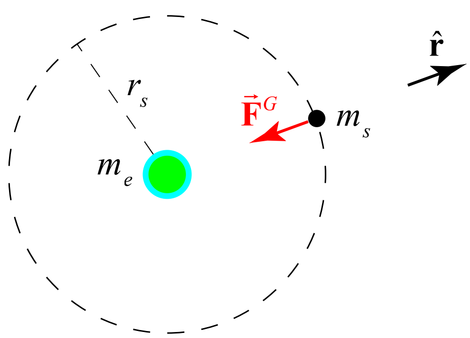

Figure 10.5: Geostationary satellite orbit (close to a scale drawing of orbit).

A geostationary satellite goes around Earth once every 23 hours 56 minutes and 4 seconds, (a sidereal day, shorter than the noon-to-noon solar day of 24 hours) so that its position appears stationary with respect to a ground station. The mass of Earth is $m_e = 5.98 \times 10^{24}$ kg. The mean radius of Earth is $R_e = 6.37 \times 10^6\ \mathrm{m}$. The universal constant of gravitation is $G = 6.67 \times 10^{-11}\ \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$. What is the radius of the orbit of a geostationary satellite? Approximately how many earth radii is this distance?

Answer

The satellite's motion can be modeled as uniform circular motion. The gravitational force between Earth and the satellite keeps the satellite moving in a circle (In Figure 10.5, the orbit is close to a scale drawing of the orbit). The acceleration of the satellite is directed towards the center of the circle, that is, along the radially inward direction.

Choose the origin at the center of the earth, and the unit vector $\hat{\mathbf{r}}$ along the radial direction. This choice of coordinates makes sense in this problem since the direction of acceleration is along the radial direction.

Let $\vec{\mathbf{r}}$ be the position vector of the satellite. The magnitude, $r_e = |\vec{\mathbf{r}}|$, is the distance from the center of the earth to the satellite, and hence the radius of its circular orbit. Let ω be the angular speed of the satellite, and T the period.. The acceleration is directed inward (centripetal), and is given by

$$\vec{\mathbf{a}} = -r_s \omega^2 \hat{\mathbf{r}}. \tag{10.10}$$

Apply Newton's Second Law to the satellite in the radial direction. The only force in this direction is the gravitational force due to the Earth, therefore

$$-G\frac{m_s m_e}{{r_s}^2}\hat{\mathbf{r}} = -m_s \omega^2 r_s\ \hat{\mathbf{r}}. \tag{10.11}$$

Equating components yields

$$G\frac{m_s m_e}{{r_s}^2} = m_s \omega^2 r_s, \tag{10.12}$$

which can be solved for the radius of the orbit

$$r_s = \left(\frac{G\,m_e}{\omega^2}\right)^{1/3}. \tag{10.13}$$

The period T of the satellite's orbit in seconds is 86164 s and so the angular speed is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86164\ \mathrm{s}} = 7.2921 \times 10^{-5}\ \mathrm{s}^{-1}. \tag{10.14}$$

Using the values of ω, G and m_e in Equation 10.13), the radius of the orbit is

$$r_s = 4.22 \times 10^7 \mathrm{m} = 6.62\ R_e. \tag{10.15}$$

10.3.2 Example Double Star System

Consider a double star system under the influence of gravitational force between the stars. Star 1 has mass m_1 and star 2 has mass m_2. Assume that each star undergoes uniform circular motion such that the stars are always a fixed distance s apart, rotating counterclockwise in Figure 10.6. What is the period of the orbit?

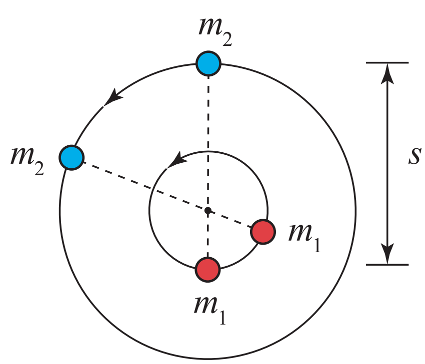

Figure 10.6: Two stars undergoing circular orbits about each other.

Answer

Because the distance between the two stars doesn't change as they orbit about each other, there is a central point where the lines connecting the two objects intersect as the objects move, as can be seen in the Figure 10.6, a point called the *center of mass of the system*. Choose radial coordinates for each star with origin at that central point. Let $\hat{\mathbf{r}}_1$ be a unit vector at star 1 pointing radially away from the center of mass. The position vector for star 1 is then $\vec{\mathbf{r}}_1 = r_1\hat{\mathbf{r}}_1$, where r_1 is the distance from the central point. Let $\hat{\mathbf{r}}_2$ be a unit vector at star 2 pointing radially away from the center of mass. The position vector for star 2 is then $\vec{\mathbf{r}}_2 = r_2\hat{\mathbf{r}}_2$, where r_2 is the distance from the central point. Because the distance between the two stars is fixed, the constraint condition is

$$s = r_1 + r_2. \tag{10.16}$$

The coordinate system is shown in Figure 10.7 The gravitational force on star 1 is

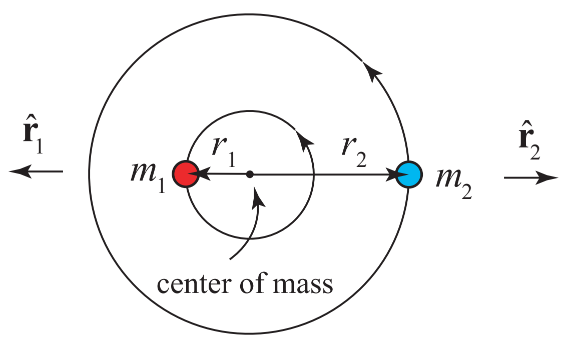

Figure 10.7: Coordinate system for double star orbits.

$$\vec{\mathbf{F}}^{G}_{2,1} = -\frac{Gm_1m_2}{s^2}\hat{\mathbf{r}}_1 \tag{10.17}$$

The gravitational force on star 2 is

$$\vec{\mathbf{F}}^{G}_{1,2} = -\frac{Gm_1m_2}{s^2}\hat{\mathbf{r}}_2. \tag{10.18}$$

The force diagrams on the two stars are shown in Figure 10.8.

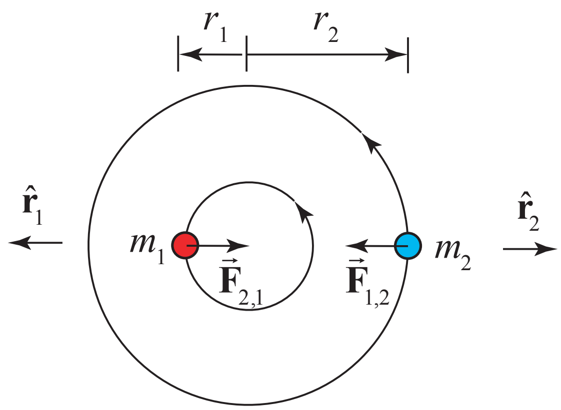

Figure 10.8: Force diagrams on stars 1 and 2.

Let ω denote the magnitude of the angular velocity of each star about the central point. Newton's Second Law for Star 1 in the radial direction $\hat{\mathbf{r}}_1$ is

$$-G\frac{m_1\,m_2}{s^2} = -m_1\,r_1\,\omega^2, \tag{10.19}$$

which can be solved for the distance r_1:

$$r_1 = G\frac{m_2}{\omega^2 s^2}. \tag{10.20}$$

Newton's Second Law for Star 2 in the radial direction $\hat{\mathbf{r}}_2$ is

$$-G\frac{m_1\,m_2}{s^2} = -m_2\,r_2\,\omega^2, \tag{10.21}$$

and so r_2 is

$$r_2 = G\frac{m_2}{\omega^2 s^2}. \tag{10.22}$$

Because the distance s between the two stars, is constant, set

$$s = r_1 + r_2 = G\frac{m_2}{\omega^2 s^2} + G\frac{m_1}{\omega^2 s^2} = G\frac{(m_2+m_1)}{\omega^2 s^2}. \tag{10.23}$$

Thus the magnitude of the angular velocity is

$$\omega = \left(G\frac{(m_2+m_1)}{s^3}\right)^{1/2}, \tag{10.24}$$

and the period is then

$$T = \frac{2\pi}{\omega} = \left(\frac{4\pi^2 s^3}{G(m_2+m_1)}\right)^{1/2}. \tag{10.25}$$

Note that both masses appear in the above expression for the period unlike the expression for Kepler's Law for circular orbits, (Equation 10.9). The reason is that in the argument leading up to Equation 10.9), we assumed that $m_M << m_E$, this was equivalent to assuming that the central point was located at the center of Earth. If we used Equation 10.25 instead we would find that the orbital period $T_{E,M}$ for the circular motion of Earth and the Moon about each other is

$$\begin{aligned} T_{E,M} &= \sqrt{\frac{4\pi^2(3.82\times10^8\ \mathrm{m})^3}{(6.67\times10^{-11}\ \mathrm{N\cdot m^2\cdot kg^{-2}})\,(5.98\times10^{24}\ \mathrm{kg} + 7.36\times10^{22}\ \mathrm{kg})}}\,. \\ &= 2.33\times10^6\ \mathrm{s} \end{aligned} \tag{10.26}$$

which is $1.43\times10^4\ \mathrm{s} = 0.17\ \mathrm{d}$ shorter than our previous calculation (Equation 10.8).

10.3.3 Tension in strings connected to two rotating blocks

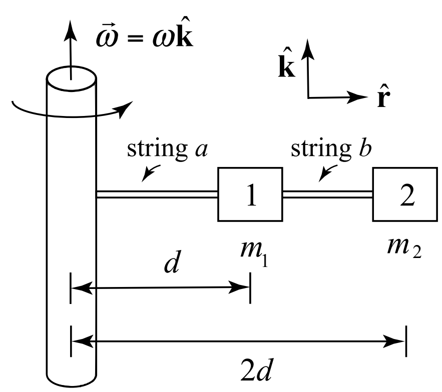

Figure 10.9: Objects attached to a rotating shaft.

Two objects 1 and 2 of mass m_1]and m_2 are undergoing circular motion around a shaft with a constant angular speed ω . The first object is a distance d from the central axis, and the second object is a distance $2d$ from the axis. You may ignore the mass of the strings and neglect the effect of gravity.

(a) What is the tension in the string between the inner object and the outer object?

(b) What is the tension in the string between the shaft and the inner object?

Answer

We begin by drawing separate force diagrams for each object. Let $T_1 \equiv T_{a,1}$ denote the force on object 1 due due to the inner string a. Let $T_{b,1}$ denote the force on object 1 due due to the outer string b. Let $T_{b,2}$ be the force on object 2 due to the outer string b. Because we are assuming that the strings are massless, the tension in each string is constant so $T_2 \equiv T_{b,1} = T_{b,2}$. Note that even though the magnitude are equal and their direction opposite, this is not a Third Law pair.

Newton's Second Law on object 1 in the radial direction is

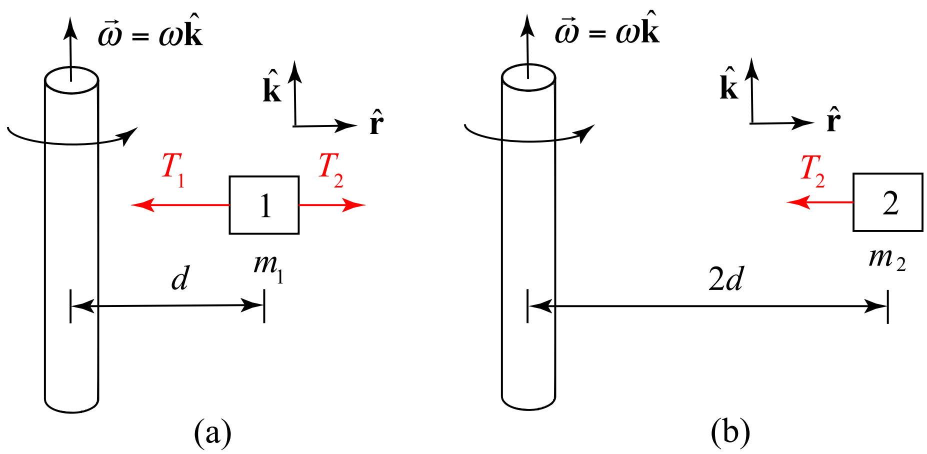

Figure 10.10: Free-body force diagrams: (a) object 1, (b) object 2.

$$\hat{\mathbf{r}} : \; T_2 - T_1 = -m_1 \, d \, \omega^2. \tag{10.27}$$

Newton's Second Law on object 2 in the radial direction is

$$\hat{\mathbf{r}} : \; -T_2 = -m_2 \, 2d \, \omega^2. \tag{10.28}$$

Therefore $T_2 = m_2 \, 2d \, \omega^2$, which upon substitution into Equation 10.27, we can solve for the tension in string a

$$T_1 = (m_1 + 2m_2) d \, \omega^2. \tag{10.29}$$

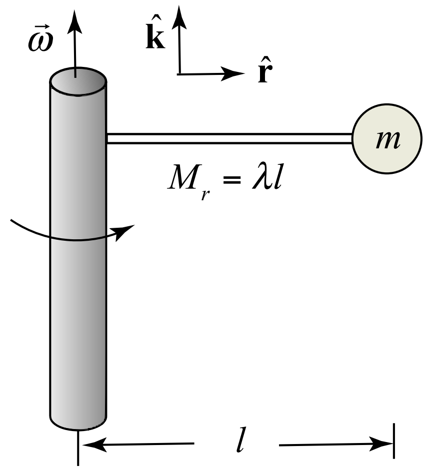

Figure 10.11: Spinning massive rope with point mass attached at end.

10.3.4 Tension in a massive rotating rope with a point mass at one end

A point particle of mass m is attached to one end of a massive rope of length l whose other end is attached to a rotating cylindrical shaft of radius a. The shaft is rotating at constant angular velocity $\vec{\omega} = \omega\hat{k}$ where $\omega > 0$. The rope has a uniform linear mass density λ. You may ignore the effect of gravitation. You may assume that the shaft is spinning fast enough that the rope and point mass undergo circular motion. For simplicity, assume that the shaft is very thin and the rope is attached at the center of the shaft.

(a) Does the magnitude of tension increase or decrease as the distance from the center of the shaft, r, increases? Why? [Briefly explain in words]

(b) Find an expression for the tension $T(r)$ in the rope as a function of r.

Answer

(a) Apply Newton's Second Law to a short segment of the string. Because the segment has non-zero mass and is travellling along a circular path, it is accelerating inward, there must be an inward net force on the segment. Consequently, the tension pulling inward on the segment must be greater than the tension pulling outward, so the tension must decrease with increasing distance from the center.

(b) Divide the rope into small pieces of length Δr with mass Δm. Because the mass is distributed uniformly along the rope, the mass in the small piece is $\Delta m = \lambda \Delta r$ In Figure 10.12 a small piece of mass Δm is shown. Its inner end is located at a distance r from the shaft, and its outer end at a distance $r + \Delta r$.The tensile forces acting on that

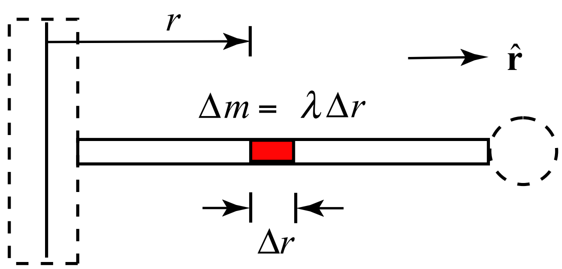

Figure 10.12: Small segment of rope.

piece are show in Figure 10.13. The sum of the radial forces is then is the difference

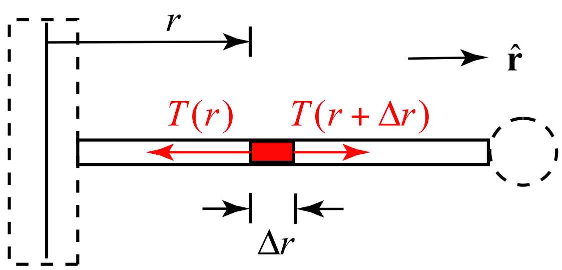

Figure 10.13: Free-body force diagram on small segment of rope.

between the tensions evaluated at the ends of the piece,

$$F_r = T(r + \Delta r) - T(r). \tag{10.30}$$

The piece is accelerating inward with a radial component $a_r = -r\omega^2$. Newton's Second Law is then

$$T(r + \Delta r) - T(r) = -\Delta m r\omega^2 = \lambda \Delta r r\omega^2. \tag{10.31}$$

Denote the difference in the tension by $\Delta T = T(r + \Delta r) - T(r)$. After dividing through by Δr, Equation 10.31 becomes

$$\frac{\Delta T}{\Delta r} = -\lambda r\omega^2. \tag{10.32}$$

Now take the limit as $\Delta r \longrightarrow 0$. Equation 10.32 is now the differential equation

$$\frac{dT}{dr} = -\lambda r\omega^2. \tag{10.33}$$

The differential equation immediately illustrates that the tension decreases with increasing radius. We can write an integral equation in which our integration variable r' ranges from the endpoint l to an arbitrary point r. The definite integral is then

$$T(r) - T(l) = \int_{r'=l}^{r'=r} \frac{dT}{dr'} dr' = -\lambda\omega^2 \int_{r'=l}^{r'=r} r' dr'. \tag{10.34}$$

The integral on the RHS of Equation 10.34 is straight forward yielding:

$$T(r) - T(l) = -\frac{\lambda\omega^2}{2}(r^2 - l^2). \tag{10.35}$$

In order to find the tension $T(l)$ where the object is attached to the end of the rope, we will apply Newton's Second Law on the object. The force diagram is shown in Figure . Newton's Second Law on the object is

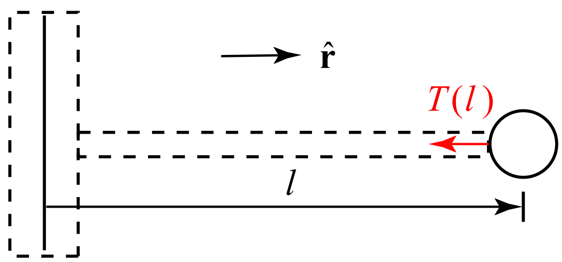

Figure 10.14: Free-body force diagram on object.

$$-T(l) = -ml\omega^2. \tag{10.36}$$

Using this boundary condition for tension, we can solve Equation 10.35 for the tension as a function of r,

$$T(r) = ml\omega^2 - \frac{\lambda\omega^2}{2}(r^2 - l^2) = \left(\frac{\lambda}{2}\left(l^2 - r^2\right) + ml\right)\omega^2. \tag{10.37}$$

We could have taken alternatively limits when we set up our integral equation, integrating r' from 0 to r. Then the definite integral becomes

$$T(r) - T(0) = \int_{r'=0}^{r'=r} \frac{dT}{dr'} dr' = -\lambda\omega^2 \int_{r'=0}^{r'=r} r' dr' = -\frac{\lambda\omega^2 r^2}{2}. \tag{10.38}$$

In order to find the tension $T(0)$, we start by considering the force diagram on the segment of the rope from 0 to r, (Figure 10.15(a)). We have already applied Newton's Second Law to a system consisting of multiple objects, where we only consider external forces acting on the system. How do we analyze a rotating system where the acceleration depends on the distance from the origin? The answer is that we treat the system of rope and object as a single point mass located at the *center of mass* of the system. (We shall

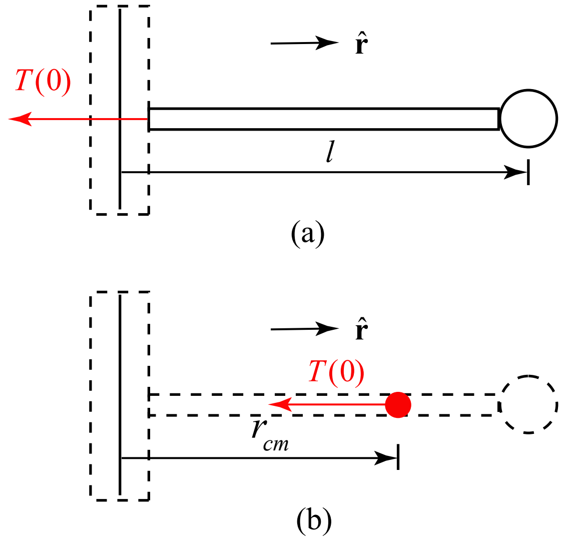

Figure 10.15: (a) Free-body force on rope and object; (b) free-body force diagram on center of mass of rope and object.

consider this in more detail when we study the translation and rotation of rigid bodies.) The center of mass of the rope is located at the midpoint of the rope $r_{cm,r} = l/2$. The center of mass of the system is located at

$$r_{cm} = \frac{M_r r_{cm,r} + ml}{M_r + m} = \frac{(\lambda l^2/2) + ml}{M_{rope} + m}. \tag{10.39}$$

The external force acting on the system is just $T(0)$. Figure 10.15(b) illustrates the force diagram on the system treated as a point mass located at the center of mass. Newton's Second Law on the combined mass is now

$$T(0) = (M_r + m) r_{cm} \omega^2 = \left(\frac{\lambda l^2}{2} + ml\right)\omega^2. \tag{10.40}$$

Substitute this result for $T(0)$ into Equation 10.38 and solve for $T(r)$:

$$T(r) = \left(\frac{\lambda}{2}(l^2 - r^2) + ml\right)\omega^2. \tag{10.41}$$

in agreement with Equation 10.37.

10.3.5 Example: Object sliding in a circular orbit on the inside of a cone

Consider an object of mass m that slides without friction on the inside of a cone moving in a circular orbit with velocity $\vec{\mathbf{v}}$. You may neglect friction and assume the speed v is

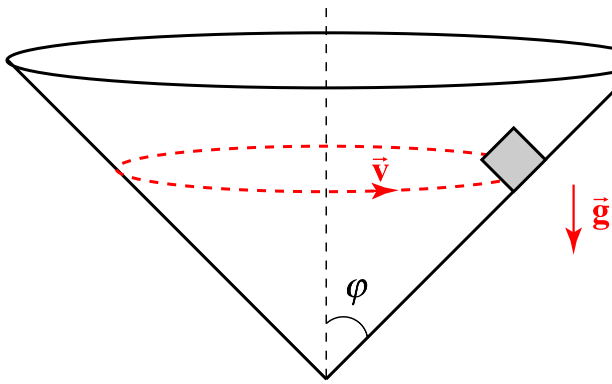

Figure 10.16: Block in circular motion on inside of cone.

constant. The apex half-angle of the cone is φ (the angle with respect to the vertical axis of the cone), (Figure 10.16). The acceleration of gravity is g. Find the radius of the circular path and the time it takes to complete one circular orbit in terms of the given quantities.

Answer

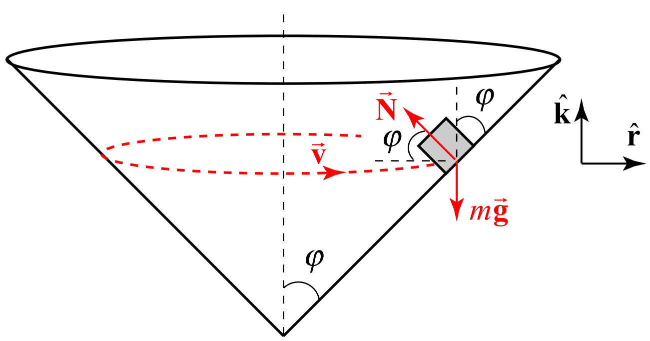

Figure 10.17: Cooridnate system and force diagram on block.

Choose cylindrical coordinates with unit vector $\hat{r}$ pointing in the radial outward direction and $\hat{k}$ pointing upwards. The two forces acting on the block are the normal force $\vec{\mathbf{N}} = -N\cos\varphi\hat{r} + N\sin\varphi\hat{k}$ and the gravitational force $m\vec{\mathbf{g}} = -mg\hat{k}$. The coordinate system and free-body force diagram are shown in Figure 10.17.

Newton's Second Law in the $\hat{r}$-direction is

$$-N\cos\varphi = \frac{-m\,v^2}{r}, \tag{10.42}$$

and in the $\hat{k}$-direction:

$$N\sin\varphi - m\,g = 0. \tag{10.43}$$

These equations can be re-expressed as

$$\begin{aligned} N\sin\varphi &= m\,g \\ N\cos\varphi &= \frac{mv^2}{r}, \end{aligned} \quad . \tag{10.44}$$

which upon dividing the two equations yields

$$\tan\varphi = \frac{r\,g}{v^2}. \tag{10.45}$$

The radius of the circular orbit is then

$$r = \frac{v^2\tan\varphi}{g}. \tag{10.46}$$

The time it takes the block to complete one revolution is then

$$T = \frac{2\pi r}{v} = \frac{2\pi v\tan\varphi}{g}. \tag{10.47}$$

The centripetal force in this problem is the vector component of the contact force that is pointing radially inwards:

$$F_r = -N\cos\varphi = -mg\cot\varphi \tag{10.48}$$

where $N = mg/\sin\varphi$ has been used to eliminate N in terms of m, g and φ. The radius is independent of the mass because the component of the normal force in the vertical direction must balance the gravitational force, and so the normal force is proportional to the mass.

10.3.6 Coin on a rotating turntable

A coin of mass m (which you may treat as a point object) lies on a turntable, exactly at the rim, a distance R from the center. The turntable turns at constant angular speed ω and the coin is not slipping with respect to the turntable. The coefficient of static friction between the turntable and the coin is given by μ_s. The acceleration of gravity is g. What is the maximum angular speed ω_m that the turntable can rotate such that the coin does not slip?

Answer

Choose a cylindrical coordinate system shown in Figure .The coin undergoes circular

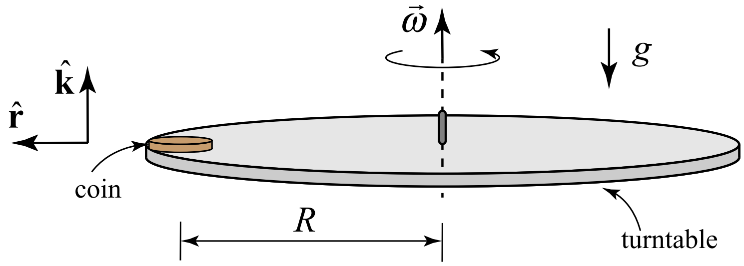

Figure 10.18: Coin rotating without slipping on a turntable.

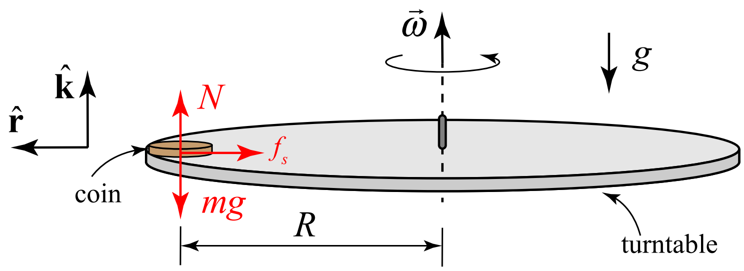

Figure 10.19: Free-body force diagram on coin.

motion at constant speed so it is accelerating inward. The force inward is static friction $\vec{\mathbf{f}}_s$ and at the just slipping point it has reached its maximum value, $f_{s,max} = \mu_s N$. We can use Newton's Second Law to find the maximum angular speed ω_m. The free-body force diagram is shown in Figure . The contact force between the coin and the turntable acting on the coin is given by $\vec{\mathbf{C}} = \vec{\mathbf{N}} + \vec{\mathbf{f}}_s = N\hat{k} - f_s\hat{r}$ and the gravitational force is $-mg\hat{k}$. Newton's Second Law in the $\hat{r}$-direction is then

$$- f_s = -m R\omega^2 \tag{10.49}$$

Newton's Second Law $\hat{k}$-direction is

$$N - mg = 0 \tag{10.50}$$

The static friction is a function of the angular speed ω. The maximum value of the static friction occurs when $f_{s,max} = \mu_s N = \mu_s mg$. At the just slipping angular speed ω_m, Equation 10.49 becomes

$$f_{s,max} = \mu_s mg = m R\omega_m^2, \tag{10.51}$$

which we can solve for maximum angular speed ω_m:

$$\omega_m = \sqrt{\frac{\mu g}{R}}. \tag{10.52}$$

Appendix A

The Gravitational Field of a Spherical Shell of Matter

A.1 Introduction

When analyzing gravitational interactions between uniform spherical bodies we assumed that we could treat each sphere as a point-like mass located at the center of the sphere and then use the Universal Law of Gravitation to determine the force between any spherical bodies. We shall now justify that assumption. For simplicity we only need to consider the interaction between a spherical object and a point-like mass. We would like to determine the gravitational force on the point-like object of mass m_1 due to the gravitational interaction with a solid uniform sphere of mass m_2 and radius R. In order to determine the force law we shall first consider the interaction between the point-like object and a uniform spherical shell of mass m_s and radius r. We will show that:

(a) The gravitational force acting on a point-like object of mass m_1 located a distance $r > R$ from the center of a uniform spherical shell of mass m_2 and radius r is the same force that would arise if all the mass of the shell were placed at the center of the shell.

(b) The gravitational force on an object of mass m_1 placed inside a spherical shell of matter is zero.

The force law summarizes these results:

$$\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_s m_1}{r^2}\hat{\mathbf{r}}, & r > R \\ \\ \mathbf{0}, & r < R \end{cases} \tag{A.1}$$

where *hatr*is the unit vector located at the position of the point-like object and pointing radially away from the center of the shell.

For a uniform spherical distribution of matter, we can divide the sphere into thin shells. Then the force between the point-like object and each shell is the same as if all the mass of the shell were placed at the center of the shell. Then we add up all the contributions of the shells (integration), with the result that the spherical distribution of matter can be treated as point-like object located at the center of the sphere.

Thus it suffices to analyze the case of the spherical shell. We shall first divide the shell into small area elements and calculate the gravitational force on the point-like object due to one element of the shell and then add the forces due to all these elements via integration.

A.2 Gravitational field of a spherical shell

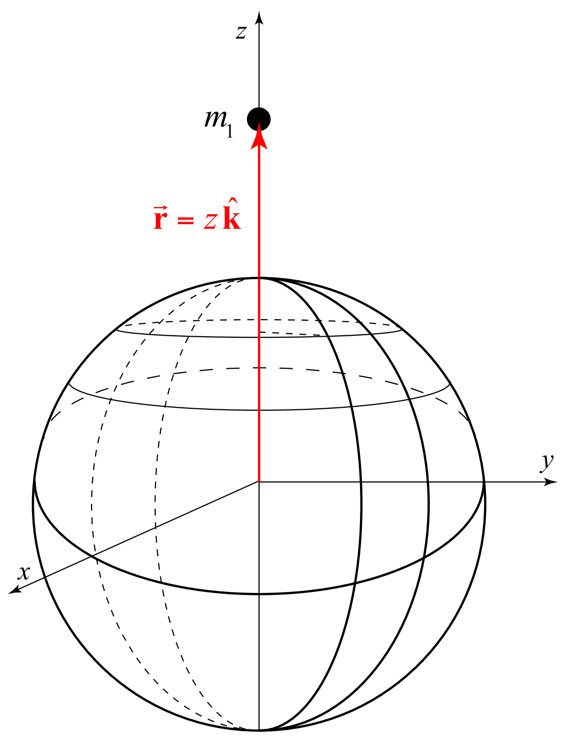

Figure A.1: Spherical shell of matter and a point mass.

We begin by choosing a coordinate system. Choose the z -axis to be directed from the center of the sphere to the position of the object, which has a position vector $\vec{\mathbf{r}} = z\hat{k}$, with $z > R$. (Figure A.1 shows the object lying outside the shell.). Choose spherical coordinates as shown in Figure A.2. For a point on the surface of a sphere of radius R ,

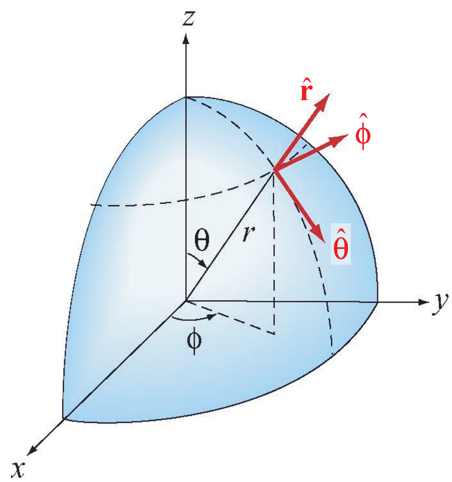

Figure A.2: Spherical coordinate system for the interaction of a spherical shell of matter and a point mass.

the Cartesian coordinates are related to the spherical coordinates by

$$\begin{aligned} x &= R\,\sin\theta\,\cos\phi, \\ y &= R\,\sin\theta\,\sin\phi, \\ z &= R\,\cos\theta, \end{aligned} \tag{A.2}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Note that the angle $]\theta$ in Figure A.2 and Equations A.2 is not the same as that in plane polar coordinates or cylindrical coordinates. The angle θ is known as the *co-latitude*, the complement of the latitude. We now choose an infinitesimal area element shown in Figure A.3, note the area element is enlarged to illustrate the dimensions. The infinitesimal area element on the surface of the shell is given by

$$da = R^2 \sin\theta d\theta d\phi. \tag{A.3}$$

The mass dm contained in that element is

$$dm = \sigma da = \sigma R^2 \sin\theta d\theta d\phi. \tag{A.4}$$

where σ is the surface mass density given by

$$\sigma = m_s/4\pi R^2. \tag{A.5}$$

The gravitational force $\vec{\mathbf{F}}_{dm,m_1}$ on the object of mass m_1 that lies outside the shell due to the infinitesimal piece of the shell (with mass dm) is shown in Figure A.4.

The contribution from the piece with mass dm to the gravitational force on the object of mass m_1 that lies outside the shell has a component pointing in the negative

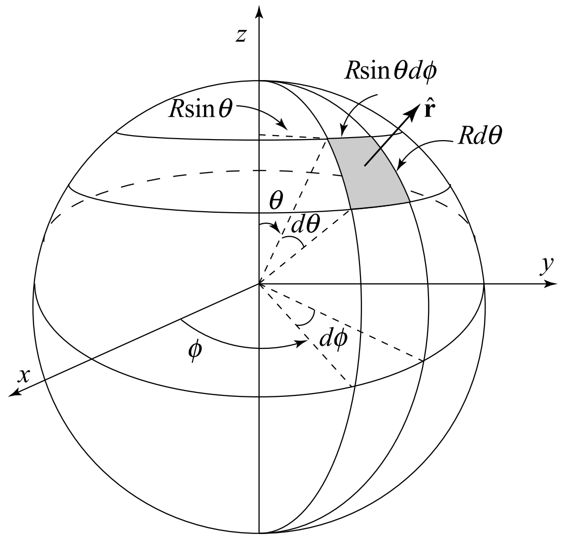

Figure A.3: Infinitesimal area element on spherical shell.

$\hat{k}$-direction and a component pointing radially away from the z-axis. By symmetry there is another mass element with the same differential mass $dm' = dm$ on the other side of the shell with same co-latitude θ but with ϕ replaced by $\phi \pm \pi$; this replacement changes the sign of x and y in Equations A.2 but leaves z unchanged. This other mass element produces a gravitational force that exactly cancels the radial component of the force pointing away from the z-axis. Therefore the sum of the forces of these differential mass elements on the object has only a component in the negative $\hat{k}$-direction (Figure A.5). Therefore we need only the z-component vector of the force due to the piece of the shell on the point-like object. From the geometry of the set-up (Figure A.6) the non-zero component fo the force is

$$(d\mathbf{F}_{s,1})_z \equiv dF_z\hat{\mathbf{k}} = -G\frac{m_1 dm}{{r_{s1}}^2}\cos\alpha\hat{\mathbf{k}} \tag{A.6}$$

Thus

$$dF_z = -G\frac{m_1 dm}{{r_{s1}}^2}\cos\alpha = -\frac{Gm_s m_1}{4\pi}\frac{\cos\alpha\sin\theta d\theta d\phi}{{r_{s1}}^2}, \tag{A.7}$$

where $r_{s,1}$ is the distance from dm to m_1. The integral of the force over the surface is then

$$F_z = -Gm_1\int\limits_{\theta=0}^{\theta=\pi}\int\limits_{\phi=0}^{\phi=2\pi}\frac{dm\cos\alpha}{{r_{s1}}^2} = -\frac{Gm_s m_1}{4\pi}\int\limits_{\theta=0}^{\theta=\pi}\int\limits_{\phi=0}^{\phi=2\pi}\frac{\cos\alpha\sin\theta d\theta d\phi}{{r_{s1}}^2}. \tag{A.8}$$

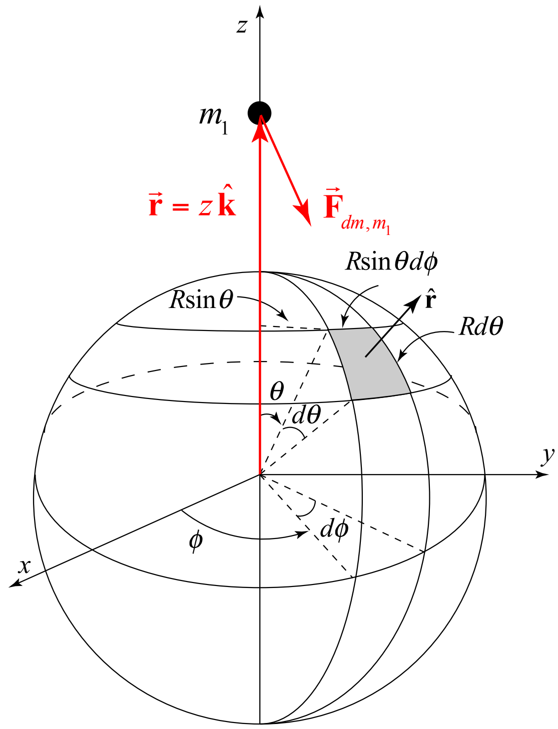

Figure A.4: Force on point-like object due to Infinitesimal mass element on spherical shell.

The ϕ-integral is straightforward yielding

$$F_z = -\frac{G m_s m_1}{2} \int_{\theta=0}^{\theta=\pi} \frac{\cos\alpha \sin\theta d\theta}{{r_{s1}}^2}. \tag{A.9}$$

From the geometry represented in Figure A.6 we can use the law of cosines in two different ways

$$\begin{array}{c} {r_{s,1}}^2 = R^2 + z^2 - 2\,R\,z\cos\theta \\ R^2 = z^2 + {r_{s1}}^2 - 2\,r_{s,1}\,z\cos\alpha. \end{array} \quad . \tag{A.10}$$

Differentiating the first expression in Equations A.10, with R and z constant yields,

$$2\,r_{s,1}\,dr_{s,1} = 2\,R\,z\sin\theta\,d\theta. \tag{A.11}$$

Hence

$$\sin\theta\,d\theta = \frac{r_{s,1}}{R\,z} dr_{s,1} \tag{A.12}$$

From the second equation in Equations A.10 we have that

$$\cos\alpha = \frac{1}{2\,z r_{s,1}} \left[(z^2 - R^2) + {r_{s1}}^2 \right]. \tag{A.13}$$

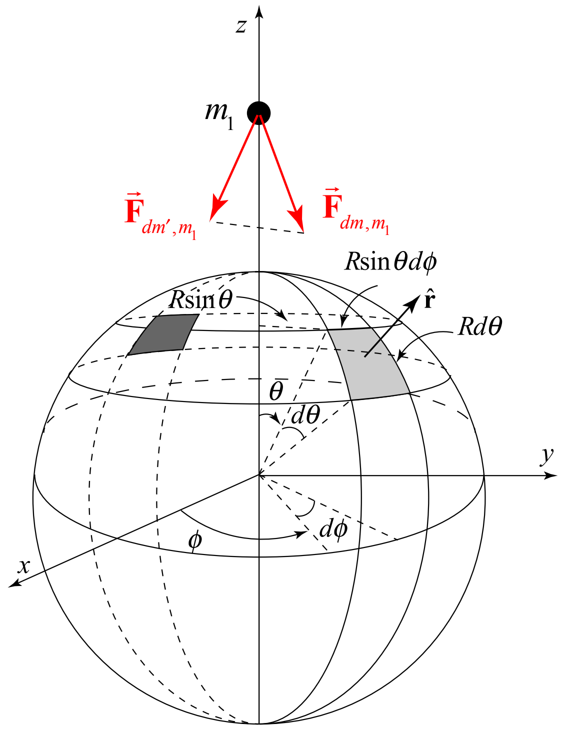

Figure A.5: Symmetric cancellation of components of force.

We now have everything we need in terms of $r_{s,1}$. For the case when $z > R$, $r_{s,1}$ varies from $z-R$ to $z+R$. Substituting Equations A.12 and A.12 into Equation A.9 and using the limits for the definite integral yields

$$\begin{aligned} F_z &= -\frac{Gm_s m_1}{2} \int_{\theta=0}^{\theta=\pi} \frac{\cos\alpha\, \sin\theta}{r^2{}_{s,1}}\, d\theta \\ &= -\frac{Gm_s m_1}{2} \frac{1}{2\,z} \int_{z-R}^{z+R} \frac{1}{r_{s,1}} \left[\left(z^2 - R^2\right) + r^2{}_{s,1}\right] \frac{1}{r^2{}_{s,1}} \frac{r_{s,1}\, dr_{s,1}}{R\, z} \quad . \\ &= -\frac{Gm_s m_1}{2} \frac{1}{2\,R\,z^2} \left[\left(z^2 - R^2\right) \int_{z-R}^{z+R} \frac{dr_{s,1}}{r^2{}_{s,1}} + \int_{z-R}^{z+R} dr_{s,1}\right]. \end{aligned} \tag{A.14}$$

The result is

$$\begin{aligned} F_z &= -\frac{Gm_s m_1}{2} \frac{1}{2\,R\,z^2} \left[-\frac{\left(z^2 - R^2\right)}{r_{s,1}} + r_{s,1}\right]_{z-R}^{z+R} \\ &= -\frac{Gm_s m_1}{2} \frac{1}{2\,R\,z^2} \left[-\left(z - R\right) + \left(z + R\right) + 2\,R\right] \quad . \\ &= -\frac{Gm_s m_1}{z^2}. \end{aligned} \tag{A.15}$$

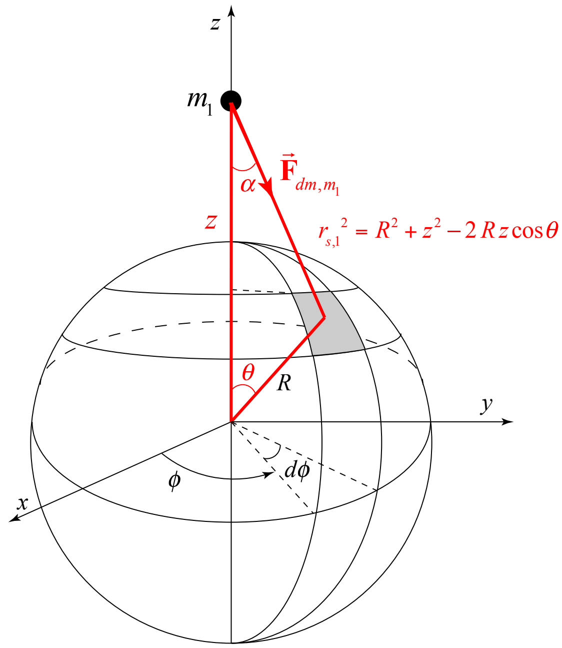

Figure A.6: Geometry for calculating the force due to piece of shell..

For the case when $z < R$, $r_{s,1}$ varies from $R - z$ to$R = Z$. Then the integral is

$$
\begin{aligned}
F_z &= -\frac{G m_s m_1}{2} \frac{1}{2\,R\,z^2} \left[-\frac{\left(z^2 - R^2\right)}{r_{s,1}} + r_{s,1} \right]_{R-z}^{R+z} \\
&= -\frac{G m_s m_1}{2} \frac{1}{2\,R\,z^2} \left[-(z-R) - (z+R) + 2\,z \right] \\
&= 0.
\end{aligned}
\quad . \tag{A.16}
$$

We have demonstrated the proposition that for a point-like object located on the z-axis, a distance z from the center of a spherical shell, the gravitational force on the point like object is given by

$$
\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_s m_1}{z^2}\hat{\mathbf{k}}, & z > R \\ \\ \vec{\mathbf{0}}, & z < R \end{cases} \tag{A.17}
$$

This proves the result that the gravitational force inside the shell is zero and the gravitational force outside the shell is equivalent to putting all the mass at the center of the shell.