Chapter 9: Static fluids

Peter Dourmashkin

September 6, 2023

Contents

Chapter 9

Applications of Newton's Second Law

Those who are in love with practice without knowledge are like the sailor who gets into a ship without rudder or compass and who never can be certain whether he is going. Practice must always be founded on sound theory... [1]

Leonardo da Vinci

9.1 Introduction

Water is everywhere around us, covering 71% of the Earth's surface. The water content of a human being can vary between 45%and 70% of body weight. Water can exist in three states of matter: solid (ice), liquid, or gas. Water flows through many objects: through rivers, streams, aquifers, irrigation channels, and pipes to mention a few. Humans have tried to control and harness this flow through many different technologies such as aqueducts, Archimedes' screw, pumps, and water turbines. Water in the gaseous state also flows. Water vapor, lighter than air, can cause convection currents that form clouds. In the liquid state, the density of water molecules is greater than the gaseous state but in both states water can flow. Liquid water forms a surface while water vapor does not. Water in both the liquid and gaseous state is classified as a fluid to distinguish it from the solid state.

At the macroscopic scale, matter can be roughly grouped into two classes, solids and fluids. There is some ambiguity in the use of the term fluid. In ordinary language, the term fluid is used to describe the liquid state of matter. More technically, a fluid is a state of matter that, when at rest, cannot sustain a shear stress and hence will flow. A solid, when at rest, can sustain a shear stress and although it may deform it will remain at rest. However there is some ambiguity in this description. Glacier ice will flow but

[1] *Notebooks of Leonardo da Vinci Complete*, tr. Jean Paul Richter, 1888, Vol.1.

very slowly. So for a time interval that is small compared to the time interval involved in the flow, glacial ice can be thought of as a solid. This description of a fluid applies to both liquids and gases. A gas will expand to fill whatever volume it is confined in, while a liquid placed in a container will have a well-defined volume with a surface layer separated the liquid and vapor phases of the substance. The viscosity of a fluid is a measure of its resistance to gradual deformation by shear stress or tensile stress.

9.2 Density

The *density* of a small amount of matter is defined to be the amount of mass ΔM divided by the volume ΔV of that element of matter,

$$\rho = \Delta M/\Delta V. \tag{9.1}$$

The SI unit for density is the kilogram per cubic meter, $\mathrm{kg \cdot m^{-3}}$. If the density of a material is the same at all points, then the density is given by

$$\rho = M/V, \tag{9.2}$$

where M is the mass of the material and V is the volume of material. A material with constant density is called *homogeneous*. For a homogeneous material, density is an *intrinsic* property. If we divide the material in two parts, the density is the same in both parts,

$$\rho = \rho_1 = \rho_2. \tag{9.3}$$

However mass and volume are *extrinsic* properties of the material. If we divide the material into two parts, the mass is the sum of the individual masses

$$M = M_1 + M_2, \tag{9.4}$$

as is the volume

$$V = V_1 + V_2, \tag{9.5}$$

The density is tabulated for various materials in Table 19.1.

If we examine a small volume element of a fluid, it consists of molecules interacting via intermolecular forces. If we are studying the motion of bodies placed in fluids or the flow of the fluid at scales that are large compared to the intermolecular forces then we can consider the fluid to be *continuous* and quantities like density will vary smoothly from point to point in the fluid.

9.3 Pressure in a Fluid

When a tangential (shear) force is applied to the surface of fluid, the fluid will undergo flow. When a fluid is static, the force on any surface within the fluid must be perpendicular (normal) to each side of that surface. This force is due to the collisions between the molecules of the fluid on one side of the surface with molecules on the other side. For a static fluid, these forces must sum to zero. Consider a small portion of a static fluid

Table 9.1: Density for Various Materials (Unless otherwise noted, all densities given are at standard conditions for temperature and pressure, that is, 273.15 K (0.00 °C) and 100 kPa (0.987 atm).

Material	**Density** ρ, $[\mathrm{kg}\cdot\mathrm{m}^{-3}]$
Helium	0.179
Air (at sea level)	1.20
Styrofoam	75
Wood Seasoned, typical	0.7×10^{3}
Ethanol	0.81×10^{3}
Ice	0.92×10^{3}
Water	1.00×10^{3}
Seawater	1.03×10^{3}
Blood	1.06×10^{3}
Aluminum	2.70×10^{3}
Iron	7.87×10^{3}
Copper	8.94×10^{3}
Lead	11.34×10^{3}
Mercury	13.55×10^{3}
Gold	19.32×10^{3}
Plutonium	19.84×10^{3}
Osmium	22.57×10^{3}

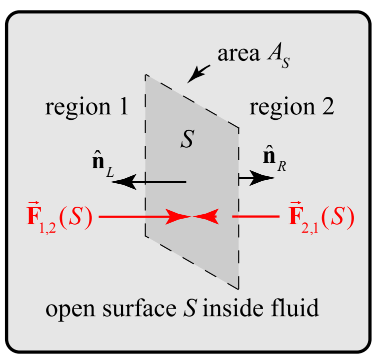

Figure 9.1: Forces on a surface within a fluid.

shown in Figure 19.1 That portion of the fluid is divided into two parts, which we shall designate 1 and 2, by a small mathematical shared surface element S of area A_S. The force $\vec{\mathbf{F}}_{1,2}(S)$ on the surface of region 2 due to the collisions between the molecules of 1 and 2 is perpendicular to the surface. The force $\vec{\mathbf{F}}_{2,1}(S)$ on the surface of region 1 due to the collisions between the molecules of 1 and 2 by Newton's Third Law satisfies

$$\vec{\mathbf{F}}_{1,2}(S) = -\vec{\mathbf{F}}_{2,1}(S) \tag{9.6}$$

Denote the magnitude of these forces that form this interaction pair by

$$F_{\perp}(S) = \left|\vec{\mathbf{F}}_{1,2}(S)\right| = \left|\vec{\mathbf{F}}_{2,1}(S)\right|. \tag{9.7}$$

Define the *hydrostatic pressure* at those points within the fluid that lie on the surface S by

$$P \equiv \frac{F_{\perp}(S)}{A_S}. \tag{9.8}$$

The pressure at a point on the surface S is the limit

$$P = \lim_{A_S \to 0} \frac{F_{\perp}(S)}{A_S}. \tag{9.9}$$

The SI units for pressure are $[\mathrm{N} \cdot \mathrm{m}^2]$ and is called the *pascal* (Pa), where

$$1 \text{ Pa} = 1 \text{ N} \cdot \mathrm{m}^{-2} = 10^{-5} \text{ bar}. \tag{9.10}$$

Atmospheric pressure at a point is the force per unit area exerted on a small surface containing that point by the weight of air above that surface. In most circumstances atmospheric pressure is closely approximated by the hydrostatic pressure caused by the weight of air above the measurement point. On a given surface area, low-pressure areas have less atmospheric mass above their location, whereas high-pressure areas have more atmospheric mass above their location. Likewise, as elevation increases, there is less overlying atmospheric mass, so that atmospheric pressure decreases with increasing elevation. On average, a column of air one square centimeter in cross-section, measured from sea level to the top of the atmosphere, has a mass of about 1.03 kg and weight of about 10.1 N. (A column one square inch in cross-section would have a weight of about 14.7 lbs, or about 65.4 N). The *standard atmosphere* $[\mathrm{atm}]$ is a unit of pressure such that

$$1 \text{ atm} = 1.01325 \times 10^5 \text{ Pa} = 1.01325 \text{ bar}. \tag{9.11}$$

9.4 Compressibility of a Fluid

When the pressure is uniform on all sides of an object in a fluid, the pressure will squeeze the object resulting in a smaller volume. When we increase the pressure by ΔP on a material of volume V_0, then the volume of the material will change by $\Delta V < 0$ and consequently the density of the material will also change. Define the *bulk stress* by the change in pressure

$$\sigma_B \equiv \Delta P. \tag{9.12}$$

Define the *bulk strain* by the ratio

$$\varepsilon_B \equiv \frac{\Delta V}{V_0}. \tag{9.13}$$

For many materials, for small pressure changes, the bulk stress is linearly proportional to the bulk strain,

$$\Delta P = -B\frac{\Delta V}{V_0}. \tag{9.14}$$

where the constant of proportionality B is called the *bulk modulus*. The SI unit for bulk modulus is the pascal, $[Pa]$. If the bulk modulus of a material is very large, a large pressure change will result in only a small volume change. In that case the material is called *incompressible*. In Table 27.2, the bulk modulus is tabulated for various materials.

Table 9.2: Bulk Modulus for Various Materials

Material	**Bulk Modulus**, B, [Pa]
Diamond	4.4×10^{11}
Nickel	1.7×10^{11}
Iron	1.6×10^{11}
Steel	1.6×10^{11}
Copper	1.4×10^{11}
Aluminum	7.5×10^{10}
Brass	6.0×10^{10}
Crown Glass	5.0×10^{10}
Lead	4.1×10^{10}
Water (value increases at higher pressure)	2.2×10^{9}
Air (adiabatic bulk modulus)	1.42×10^{5}
Air (isothermal bulk modulus)	1.01×10^{5}

9.4.1 Example: Compressibility of water

Determine the percentage decrease in a fixed volume of water at a depth of 4 km where the pressure difference is $40\mathrm{MPa}$, with respect to sea level. **Answer**

The bulk modulus of water is $2.2\times10^9\mathrm{Pa}$. Apply Equation 19.14,

$$\frac{\Delta V}{V_0} = -\frac{\Delta P}{B} = -\frac{40\times10^6\mathrm{Pa}}{2.2\times10^9\ \mathrm{Pa}} = -0.018.$$

There is only a 1.8% decrease in volume. Water is essentially incompressible even at great depths in ocean.

9.5 Pascal's Law: Pressure as a Function of Depth in a Fluid of Uniform Density in a Uniform Gravitational Field

Consider a column of seawater of cross-sectional area A, with the top of the column at sea level and the bottom of the column at a depth d. You may assume that column of seawater is at rest (there are no bulk motions). Assume the density ρ of seawater is constant. The atmospheric pressure at sea level is P_0, and the acceleration of gravity is g. Choose a coordinate system such that the z-axis points vertically downward and $z = 0$ is at sea level, (Figure 19.2(a)).

We shall do a force analysis on an smalll cylindrical volume element of the fluid at a depth z , cross-sectional area A and thickness Δz as shown in Figure 19.2(b). The volume of the element is $\Delta V = A\Delta z$ and the mass of the fluid contained within the element is $\Delta m = \rho A \Delta z$.

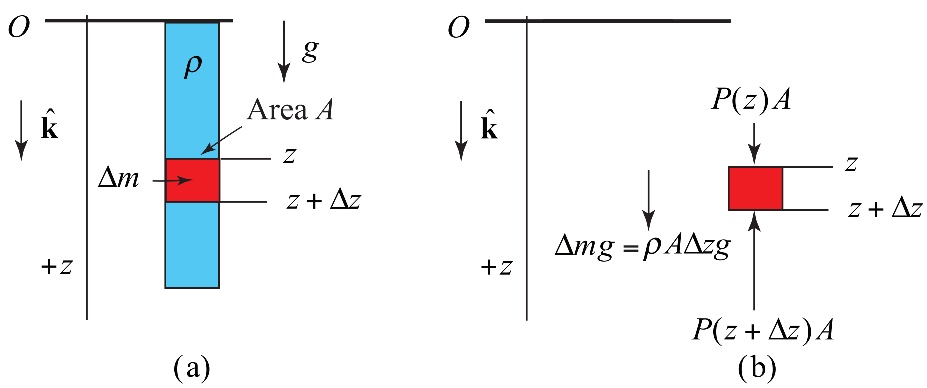

Figure 9.2: (a) Column of seawater; (b) force diagram on infinitesimal mass element.

The surface of the small fluid cylindrical element has three faces, two caps and the cylindrical body. Because the fluid is static, the force due to the fluid pressure points inward on each of these three faces. The distributed pressure forces on the cylindrical surface sum to zero. On the end-cap, at a distance $z + \Delta z$ below the surface, the force due to the pressure of the fluid above the end-cap is downward, $\vec{\mathbf{F}}(z) = F(z)\hat{\mathbf{k}}$. On the end-cap, at a distance $z+\Delta z$ below the surface, the force due to the pressure of the fluid below the end-cap is upward, $\vec{\mathbf{F}}(z+\Delta z) = -F(z+\Delta z)\hat{\mathbf{k}}$. The gravitational force acting on the element is given by $\vec{\mathbf{F}}^g = -\Delta m g \hat{\mathbf{k}} = -\rho \mathrm{A} \Delta \mathrm{z g} \hat{\mathbf{k}}$. The free body force diagram on the element is shown in Figure 19.2(b).

The vector sum of the forces is zero because the fluid is static (Newton's Second Law). Therefore in the $+\hat{\mathbf{k}}$-direction

$$F(z) - F(z+\Delta z) + \rho A \Delta z g = 0. \tag{9.15}$$

We use the relationship between pressure and force (Equation 19.8) to rewrite

Equation 19.15 in terms of pressures

$$(P(z) - P(z+\Delta z))A + \rho A \Delta z g = 0. \tag{9.16}$$

Divide through by the area a and Δz and rewrite Equation 19.16 as

$$\frac{P(z+\Delta z - P(z)}{\Delta z} = \rho g. \tag{9.17}$$

Now take the limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{P(z+\Delta z) - P(z)}{\Delta z} = \rho g. \tag{9.18}$$

The left-hand side of Equation 19.18 is the definition of the derivative of pressure with respect to z, therefore the differential equation describing how pressure varies with depth is given by

$$\frac{dP}{dz} = \rho g. \tag{9.19}$$

In order to find the pressure as a function of depth, integrate Equation 19.19 using the technique of separation of variables. first rewrite the differential equation in terms of differentials and two integration variables P' and z'

$$dP' = \rho g dz'. \tag{9.20}$$

Now integrate both sides, with the following limits for the definite integrals: integrate the pressure from $P' = P_0$ to $P' = P(z)$ and the depth vawriable $z' = 0$ to $z' = z$,

$$\int_{P'=P_0}^{P'=P(z)} dP' = \int_{z'=0}^{z'=z} \rho g dz'. \tag{9.21}$$

Perform the integrals on both sides of Equation 19.21:

$$P(z) - P_0 = \rho g z \qquad \text{(Pascal's Law)}, \tag{9.22}$$

describes the change in pressure between a depth z and the surface of a fluid, a result known as *Pascal's Law*.

9.5.1 Example: Pressure in Earth's ocean

What is the change in pressure between a depth of $z = 4\mathrm{km}$ and the surface $z = 0$ in Earth's ocean? **Answer**

Assume the density of water is uniform in the ocean, and so we can use Pascal's Law, Equation 19.22 to determine the pressure, where we use $\rho = 1.03 \times 10^3\ \mathrm{kg \cdot m^{-3}}$ for the density of seawater (see Table 19.1). Then

$$\begin{aligned}
&P(z) - P(z=0) = \rho g z \\
&= (1.03 \times 10^3 \mathrm{kg \cdot m^{-3}})(9.8\ \mathrm{m \cdot s^{-2}})(4 \times 10^3\ \mathrm{m}) \\
&= 40 \times 10^6\ \mathrm{Pa}.
\end{aligned}$$

9.5.2 Example: Pressure in an ideal gas atmosphere surrounding a planet

In an ideal gas atmosphere surrounding a planet, find a relationship for the pressure P in the atmosphere as a function of height above the surface of the planet. The magnitude of the acceleration due to gravity is g

Answer

Model the atmosphere as an ideal gas of density ρ in static equilibrium at constant temperature T. The atmospheric pressure at the surface of the planet is P_0.

Choose a coordinate system such that the z-axis points vertical upward and the plane $z = 0$ is at the surface of the planet. Choose an small cylindrical volume element of the atmosphere at a height z, of cross-sectional area A, thickness Δz and mass $\Delta m = \rho A \Delta z$ as shown in the Figure 19.3.

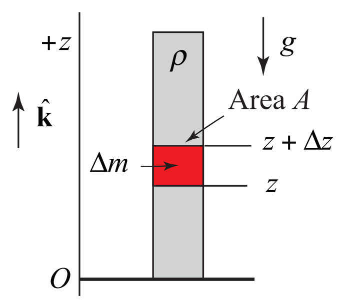

Figure 9.3: Coordinate system and small mass element in an ideal gas atmosphere.

The cylindrical column of the atmosphere has now been divided into three pieces, an upper piece, the small mass element, and a lower piece.

The free-body force diagram on the small mass element of the atmosphere of thickness is shown in Figure 19.4. The pressure force of the atmosphere on the bottom surface of the mass element is directed away from the surface and is given by $\vec{\mathbf{F}}(z) = F(z)\hat{\mathbf{k}}$. The pressure force of the atmosphere on the upper surface of the mass element is directed towards the surface and is given by $\vec{\mathbf{F}}(z+\Delta z) = F(z+\Delta z)\hat{\mathbf{k}}$. The magnitudes of the two forces are related to the pressure by $F(z) = P(z)A$ and $F(z + \Delta z) = P(z + \Delta)A$. The gravitational force is directed downward and given by $\vec{\mathbf{F}}^g = -\Delta m g\hat{\mathbf{k}} = -\rho \mathrm{A} \Delta \mathrm{z} \mathrm{g} \hat{\mathbf{k}}$.

We now apply Newton's Second Law to the small element noting that the acceleration is zero

$$-(P(z) + \Delta P)A + P(z)A - \rho A \Delta z\, g = 0. \tag{9.23}$$

The difference in the pressure $\Delta P \equiv P(z + \Delta z) - P(z)$ divided by the thickness Δz is then

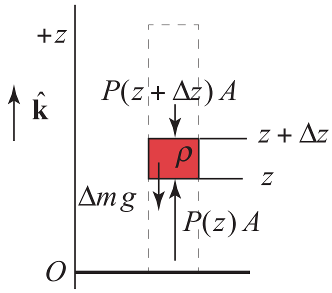

Figure 9.4: Force diagram for small mass element in an ideal gas atmosphere.

$$\frac{\Delta P}{\Delta z} = -\rho g. \tag{9.24}$$

Now take the limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{|DeltaP}{\Delta z} = \rho g, \tag{9.25}$$

therefore the differential equation describing how pressure varies with depth is given by

$$\frac{dP}{dz} = -\rho g. \tag{9.26}$$

The *Ideal Gas Law* is

$$P = \frac{n_m RT}{V}, \tag{9.27}$$

where n_m is the number of moles and R is the universal gas constant. The density of the gas is

$$\rho = \frac{M_{tot}}{V} \tag{9.28}$$

where the total mass $M_{tot} = n_m M$, where M is the molar mass, and V is the volume of the gas. Therefore rewrite the ideal gas law as

$$P = \frac{\rho RT}{M}, \tag{9.29}$$

The density of the gas is then

$$\rho = \frac{MP}{RT}. \tag{9.30}$$

Substitute the equation of state for density, Equation 19.30, into the differential equation 19.26: The density of the gas is then

$$\frac{dP}{dz} = -\frac{Mg}{RT} P. \tag{9.31}$$

In order to find the pressure as a function of height from tthe surface, integrate Equation 19.31 using the technique of separation of variables. first rewrite the differential equation in terms of differentials and two integration variables P' and z'

$$\frac{dP'}{P'} = -\frac{Mg}{RT}z'. \tag{9.32}$$

Now integrate both sides, with the following limits for the definite integrals: integrate the pressure from $P' = P_0$ to $P' = P(z)$ and the depth vawriable $z' = 0$ to $z' = z$,

$$\int_{P'=P_0}^{P'=P(z)} \frac{dP'}{P'} = -\int_{z'=0}^{z'=z} \frac{Mg}{RT}\,dz'. \tag{9.33}$$

Perform the integrals on both sides of Equation 19.21:

$$\ln\left(\frac{P(z)}{P_0}\right) = -\frac{Mg}{RT}z. \tag{9.34}$$

Exponentiate both sides of this equation using the fact that $e^{\ln z} = z$ yielding

$$\frac{P(z)}{P_0} = e^{-(Mg/RT)z}. \tag{9.35}$$

The pressure at a height z s then

$$P(z) = P_0 e^{-(Mg/RT)z}. \tag{9.36}$$

9.6 Archimedes' Principle: Buoyant Force

When we place a piece of solid wood in water, the wood floats on the surface. The density of most woods is less than the density of water, and so the fact that wood floats does not seem so surprising. However, objects like ships constructed from materials like steel that are much denser than water also float. In both cases, when the floating object is at rest, there must be some other force that exactly balances the gravitational force. This balancing of forces also holds true for the fluid itself.

Consider a static fluid with uniform density ρ_f. Consider an arbitrary volume element of the fluid with volume V and mass M. The gravitational force acts on the volume element, pointing downwards, and is given by $\vec{\mathbf{F}}^g = -\rho_f V g\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector pointing in the upward direction. The pressure on the surface is perpendicular to the surface (Figure 19.5). Therefore on each area element of the surface there is a perpendicular force on the surface.

Let $\vec{\mathbf{F}}^B$ denote the resultant force, called the *buoyant force*, on the surface of the volume element due to the pressure of the fluid. The buoyant force must exactly balance the gravitational force because the fluid is in static equilibrium (Figure 19.6),

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B + \vec{\mathbf{F}}^g = \vec{\mathbf{F}}^B - \rho_f V g\hat{\mathbf{k}}. \tag{9.37}$$

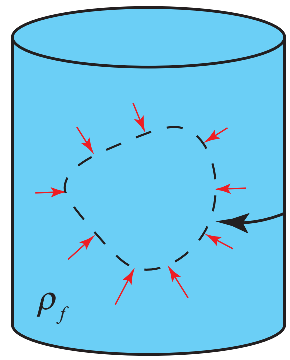

Figure 9.5: Pressure forces are perpendicular to an arbitrary region of the fluid..

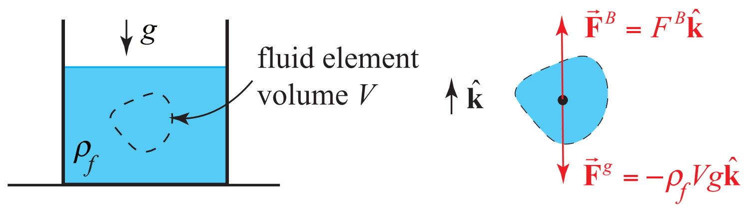

Figure 9.6: Free-body force diagram on volume element showing gravitational force and buoyant force.

Therefore the buoyant force is

$$\vec{\mathbf{F}}^B = \rho_f V g\hat{\mathbf{k}}. \tag{9.38}$$

The buoyant force depends on the density of the fluid, the gravitational constant, and the volume of the fluid element. This macroscopic description of the buoyant force that results from a very large number of collisions of the fluid molecules is called *Archimedes' Principle*.

We can now understand why when we place a stone in water it sinks. The density of the stone is greater than the density of the water, and so the buoyant force on the stone is less than the gravitational force on the stone and so it accelerates downward.

We also can understand why a steel ship of mass M and volume V floats: the average density of the boat ρ_{ave} is much less than the density of water ρ_w. Suppose that a ship is designed to displaces a volume of water V_w in calm water.. The buoyant force on the ship is then

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B + \vec{\mathbf{F}}^g = \rho_w V_w g\hat{\mathbf{k}} - Mg\hat{\mathbf{k}}. \tag{9.39}$$

Thus when the boat floats in calm waters, the mass of the boat is given by

$$M = \rho_w V_w. \tag{9.40}$$

Often ballast is added to a boat to insure that the total mass satisfies Equation 19.43.

9.6.1 Floating wooden block

Place a uniform wooden rectangular solid block in a cylindrical container of water. The block will float because the density of the block is less than the density of water, $\rho_b < \rho_w$. The block has thickness h, cross-sectional area A_b, and mass m_b hence the density is $\rho_b = m/hA_b$. The thickness of the block that is a beneath the surface of the water is h_1, and displaces a volume $V_1 = h_1 A_b$ of the water The thickness of the block that is above the surface displaces a volume $V_2 = (h - h_1)A_b$ of air. , (Figure 19.7).

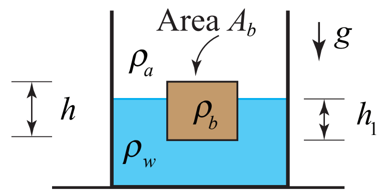

Figure 9.7: Object floating on the surface of water.

(a) How far below the surface h_1 is the bottom of the block?

(b) what is the ratio of the thickness of the exposed and submerged parts of the blocks?

(c)How much did the height of the water in the beaker rise when the block was placed in the beaker? The cross sectional area of he container is A_c.

Answer

Because the density of the air is much less that the density of water, we can neglect the buoyant force of the air on the object. The buoyant force of the water on the block, $\vec{\mathbf{F}}^B = \rho_w h_1 A_b \; g\hat{\mathbf{k}}$, must exactly balance the gravitational force on the block, $\vec{\mathbf{F}}^g = -\rho_b h A_b \; g\hat{\mathbf{k}}$.(Figure 19.8). Newton's Second Law is then

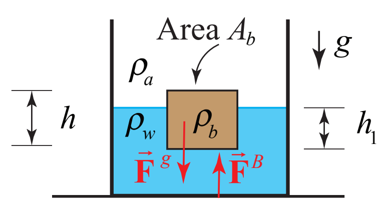

Figure 9.8: Free-body force diagram on floating object.

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B + \vec{\mathbf{F}}^g = \rho_w h_1 A_b\, g\hat{\mathbf{k}} - \rho_b h g\hat{\mathbf{k}}. \tag{9.41}$$

Therefore the distance between the bottom of the block and the surface is

$$h_1 = \frac{\rho_b}{\rho_w} h. \tag{9.42}$$

(b) The ratio of the thickness of the exposed and submerged parts of the blocks is then Therefore the distance between the bottom of the block and the surface is

$$\frac{h - h_1}{h_1} = \frac{h(1 - \dfrac{\rho_b}{\rho_w})}{\dfrac{\rho_b}{\rho_w} h} = \frac{\rho_w - \rho_b}{\rho_b} \tag{9.43}$$

Let s_i denote the initial height of water in the container before the block was added. When the wood is floating in the container, the volume of water in the container is equal to $A_c\, s_f - A_b\, h_1$, where s_f is the final height of the water in the container and $A_b\, h_1$ is the volume of the submerged portion of block. Because the volume of water has not changed

$$A_c\, s_i = A_c\, s_f - A_b\, h_1 = A_c\, s_f - A_b\, \frac{\rho_b}{\rho_w} h. \tag{9.44}$$

The final height is then

$$s_f = s_i + \frac{A_b}{A_c}\frac{\rho_b}{\rho_w} h. \tag{9.45}$$

Therefore the change in height of the water in the container is

$$\Delta s = s_f - s_i = \frac{A_b}{A_c}\frac{\rho_b}{\rho_w} h = \frac{m_b}{A_c\, \rho_w}. \tag{9.46}$$

9.6.2 Example: Rock inside a floating bowl

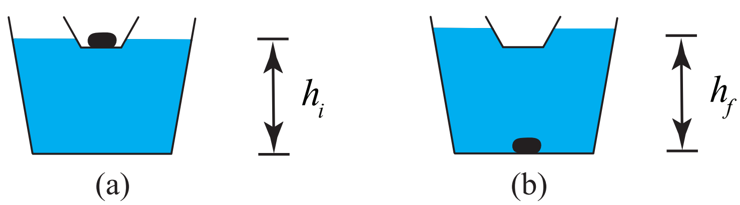

Figure 9.9: (a) rock inside floating boat; (b) rock on bottom of container.

A rock of mass m_r and density ρ_r is placed in a bowl of mass m_b. The bowl and rock float in a container of water of density ρ_w. The containerr has cross sectional area A_c. The rock is then removed from the bowl and allowed to sink to the bottom of the container. Does the water level rise or fall when the rock is resting on the bottom of the container?

Answer

When the rock is placed in the floating bowl, a volume V of water is displaced. The buoyant force $\vec{\mathbf{F}}^B = \rho_w V g \hat{\mathbf{k}}$ balances the gravitational force on the rock and bowl,

$$(m_r + m_b)g = \rho_w V g = \rho_w (V_1 + V_2)g, \tag{9.47}$$

where V_1 is the portion of the volume of displaced water that is necessary to balance just the gravitational force on the rock, $m_r g = \rho_w V_1 g$, Therefore $V_1 = m_r/\rho_w$. The volume of displaced water V_2 that is necessary to balance just the gravitational force on the bowl is $V_2 = m_b/\rho_w V$. The mass of the rock is given by $m_r = \rho_r V_r$. In particular

$$V_1 = \frac{m_r}{\rho_w} = \frac{\rho_r}{\rho_w} V_r. \tag{9.48}$$

Because the density of the rock is greater than the density of the water, $\rho_r > \rho_w$, the rock displaces more water when it is floating than when it is immersed in the water, $V_1 > V_r$. Therefore the water level drops when the rock is dropped into the water from the bowl.

9.6.3 Example:Block floating between oil and water

A cubical block of wood, each side of length $l = 10\ \mathrm{cm}$, floats at the interface between air and water. A thickness h of the block is submerged in the water.The air is then replaced with $d = 15\ \mathrm{cm}$ of oil that floats on top of the water.

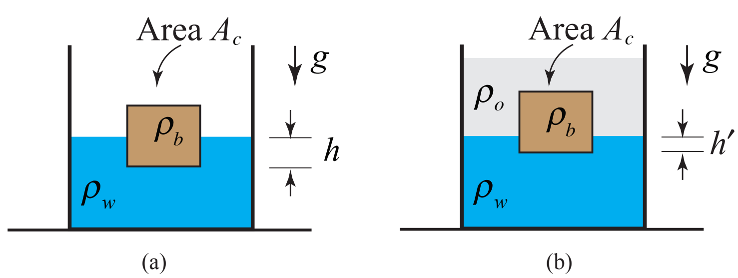

Figure 9.10: (a) Block floating on water, (b) Block floating on oil-water interface.

(a) Will the block rise or fall? Explain your reasoning.

(b) After the oil has been added and equilibrium established, the cubical block of wood floats at the interface between oil and water with its lower surface $h = 10\ \mathrm{cm}$ below the oil and water interface The density of the oil is $\rho_o = 6.5 \times 10^2\ \mathrm{kg} \cdot \mathrm{m}^{-3}$, the density of water is $\rho_w = 1.0\times 10^3 \text{ kg}\cdot\text{m}^{-3}$ and the density of air is $\rho_a = 1.225 \text{ kg}\cdot\text{m}^{-3}$

(b) What is the density of the block of wood?

Answer

(a) After the oil iss added, the thickness of the block that is submerged in the water decreases. A simple way to understand this without an extensive calculation is that the density of oli is greater than the density of air, so in order to keep the buoyant force unchanged less volume of water has to be displaced.

(b) The buoyant force is equal to the gravitational force on the block. Therefore before the oil is added

$$\rho_b g V = \rho_w g V_1 + \rho_a g (V - V_1)$$

where V is the volume of the block, V_1 is the volume of water displaced by the block, $V_2 = V - V_1$ is the volume of air displaced by the block, ρ_b is the density of the block of wood, ρ_w is the density of water and ρ_a is the density of air (Figure 19.10(a)).

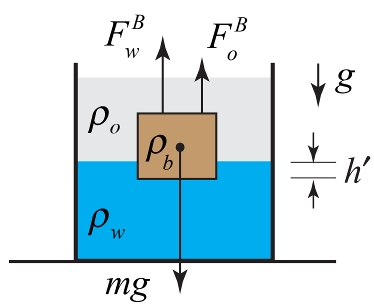

Figure 9.11: (a) Block floating on water, (b) Block floating on oil-water interface.

The volume of water displaced by the block is then

$$V_1 = \frac{(\rho_b - \rho_a)}{(\rho_w - \rho_a)} V. \tag{9.49}$$

When the oil is added, we can repeat the argument leading up to Equation 19.49 replacing ρ_a by ρ_o, (Figure 19.10(b)), yielding for the volume V_1' of water now displaced

$$V_1' = \frac{(\rho_b - \rho_o)}{(\rho_w - \rho_o)} V. \tag{9.50}$$

Substitute $V_1' = l^2 h'$ and $V = l^3$, into Equation 19.50 yielding

$$l^2 h' = \frac{(\rho_b - \rho_o) l^3}{(\rho_w - \rho_o)}. \tag{9.51}$$

which we can solve for the density of the block

$$\rho_b = \rho_o + (\rho_w - \rho_o)\frac{h'}{l}. \tag{9.52}$$

We now substitute the given values from the problem statement and find that the density of the block is

$$\begin{aligned}\rho_b &= ((1.0 \times 10^3\,\mathrm{kg\cdot m^{-3}}) - (6.5 \times 10^2\,\mathrm{kg\cdot m^{-3}}))\frac{(2.0\times 10^{-2}\mathrm{m})}{(1.0\times 10^{-1}\mathrm{m})} \\ &+ (6.5\times 10^2\,\mathrm{kg\cdot m^{-3}}) = 7.2\times 10^2\,\mathrm{kg\cdot m^{-3}}.\end{aligned} \tag{9.53}$$

The displaced volume before the oil was added is equal to

$$\begin{aligned}V_1 &= \frac{(\rho_b - \rho_a)}{(\rho_w - \rho_a)}V = \frac{((7.2\times 10^2\,\mathrm{kg\cdot m^{-3}}) - (1.225\,\mathrm{kg\cdot m^{-3}}))}{((1.0\times 10^3\,\mathrm{kg\cdot m^{-3}}) - (1.225\,\mathrm{kg\cdot m^{-3}}))}(1.0\times 10^{-1}\mathrm{m})^3 \\ &= 7.19\times 10^{-4}\mathrm{m}^3\end{aligned}. \tag{9.54}$$

The displaced volume after the oil was added is equal to

$$\begin{aligned}V_1' &= \frac{(\rho_b - \rho_o)}{(\rho_w - \rho_o)}V = \frac{((7.2\times 10^2\,\mathrm{kg\cdot m^{-3}}) - (6.5\times 10^2\,\mathrm{kg\cdot m^{-3}}))}{((1.0\times 10^3\,\mathrm{kg\cdot m^{-3}}) - (6.5\times 10^2\,\mathrm{kg\cdot m^{-3}}))}(1.0\times 10^{-1}\mathrm{m})^3 \\ &= 2.00\times 10^{-4}\mathrm{m}^3.\end{aligned} \tag{9.55}$$

Thus $V_1' < V_1$, in agreement with our previous reasoning that less water needs to be displaced when the oil is added.

9.7 Drag Forces in Fluids

When a solid object moves through a fluid it will experience a resistive force, called the *drag force*, opposing its motion. The fluid may be a liquid or a gas. This force is a very complicated force that depends on both the properties of the object and the properties of the fluid. The force depends on the speed, size, and shape of the object. It also depends on the density, viscosity and compressibility of the fluid. For objects moving in air, the air drag is still quite complicated but for rapidly moving objects the resistive force is roughly proportional to the square of the speed v, the cross-sectional area A of the object in a plane perpendicular to the motion, the density ρ of the air, and independent of the viscosity of the air. Traditionally the magnitude of the air drag for rapidly moving objects is written as

$$F_{\mathrm{drag}} = \frac{1}{2}C_D A\rho v^2. \tag{9.56}$$

The coefficient C_D is called the *drag coefficient*, a dimensionless number that is a property of the object. Figure 19.12 lists the drag coefficient for some simple shapes, (each of these objects has a Reynolds number of order 10^4).

The above model for air drag does not extend to all fluids. An object dropped in oil, molasses, honey, or water will fall at different rates due to the different viscosities of

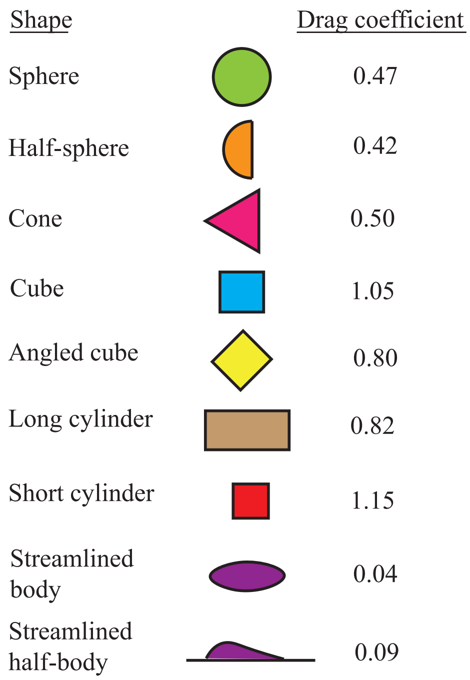

Figure 9.12: Drag Coefficients for various geometric shapes.

the fluid. For very low speeds, the drag force depends linearly on the speed and is also proportional to the viscosity η of the fluid. For the special case of a sphere of radius R, the drag force law can be exactly deduced from the principles of fluid mechanics and is given by

$$\vec{\mathbf{F}}_{\text{drag}} = -6\pi\eta R\vec{\mathbf{v}} \qquad (\text{sphere}). \tag{9.57}$$

This force law is known as *Stokes' Law*. The coefficient of viscosity η has SI units of $[\mathrm{N \cdot m^{-2} \cdot s}] = [\mathrm{Pa \cdot s}] = [\mathrm{kg \cdot m^{-1} \cdot s^{-1}}]$; a cgs unit called the *poise* is often encountered . Some typical coefficients of viscosity are listed in Table 19.4.

This law can be applied to the motion of slow moving objects in a fluid, for example: very small water droplets falling in a gravitational field, grains of sand settling in water, or the sedimentation rate of molecules in a fluid. In the later case, If we model a molecule as a sphere of radius R, the mass of the molecule is proportional to R^3 and the drag force is proportion to R, therefore different sized molecules will have different rates of acceleration. This is the basis for the design of measuring devices that separate molecules of different molecular weights.

In many physical situations the force on an object will be modeled as depending on the object's velocity. We have already seen static and kinetic friction between surfaces modeled as being independent of the surfaces' relative velocity. Common experience (swimming, throwing a Frisbee) tells us that the frictional force between an object and

Table 9.3: Coefficients of Viscosity for Various Materials

Material	**Temperature** °C	**Coefficient of viscosity** η; $[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}]$
Acetone	25	3.06×10^{-4}
Air	15	1.81×10^{-5}
Benzene	25	6.04×10^{-4}
Blood	37	$(3-4) \times 10^{-3}$
Castor oil	25	0.985
Corn syrup	25	1.3806
Ethanol	25	1.074×10^{-3}
Glycerol	20	1.412
Methanol	25	5.45×10^{-4}
Motor oil (SAE 10W)	20	6.5×10^{-2}
Olive oil	25	8.1×10^{-2}
Water	10	1.308×10^{-3}
Water	20	1.002×10^{-3}
Water	60	0.467×10^{-3}
Water	100	0.28×10^{-3}

a fluid can be a complicated function of velocity. Indeed, these complicated relations are an important part of such topics as aircraft design.

9.7.1 Example: Drag force at low speeds

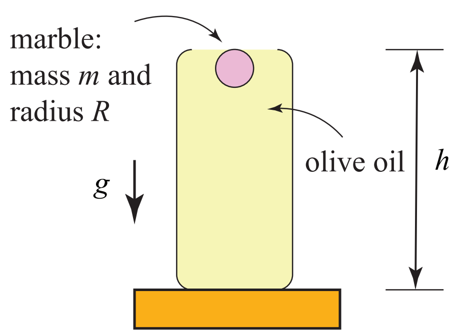

Figure 9.13: (a) Coordinate system for marble; (b) free body force diagram on marble.

A spherical marble of radius R and mass m is released from rest and falls under the influence of gravity through a jar of olive oil of viscosity η. The marble is released from rest just below the surface of the olive oil, a height h from the bottom of the jar. The gravitational acceleration is g (Figure 8.31). Neglect any force due to the buoyancy of the olive oil.

(a) Determine the velocity of the marble as a function of time.

(b) What is the maximum possible velocity $\vec{\mathbf{v}}_\infty = \vec{\mathbf{v}}(t = \infty)$ (terminal velocity), that the marble can obtain?

(c) Determine an expression for the viscosity η of olive oil in terms of g, m, R and $v_\infty = |\vec{\mathbf{v}}_\infty|$.

(d) Determine an expression for the position of the marble from the surface of the olive oil as a function of time.

Answer

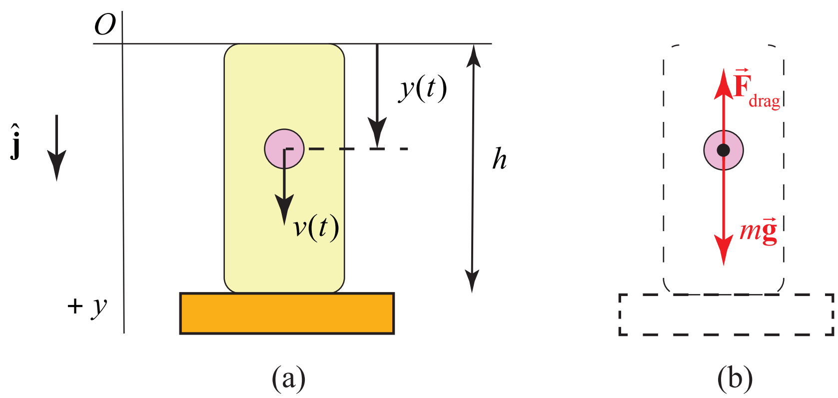

Figure 9.14: (a) Coordinate system for marble; (b) free body force diagram on marble.

Choose the positive y-direction downwards with the origin at the initial position of the marble as shown in Figure 19.14(a). There are two forces acting on the marble: the gravitational force, and the drag force which is given by Equation 19.57. The free body diagram is shown in the Figure 19.14(b). Newton's Second Law is then

$$mg - 6\pi\eta R v = m\frac{dv}{dt}, \tag{9.58}$$

where v is the y-component of the velocity of the marble. Let $\gamma = 6\pi\eta R/m$; the SI units of γ are $[\mathrm{s}^{-1}]$. Then Equation 19.58 becomes

$$g - \gamma v = \frac{dv}{dt}. \tag{9.59}$$

Suppose the object is released from rest, $v(t = 0) = 0$. We shall solve Equation

19.59 using the method of separation of variables. The differential equation may be rewritten as

$$\frac{dv}{(v - g/\gamma)} = -\gamma dt. \tag{9.60}$$

The integral version is then

$$\int_{v'=0}^{v'=v(t)} \frac{dv'}{v' - g/\gamma} = -\gamma \int_{t'=0}^{t'=t} dt' \quad . \tag{9.61}$$

Integrating both sides yields

$$\ln\left(\frac{v(t) - g/\gamma}{-g/\gamma}\right) = -\gamma\, t. \tag{9.62}$$

Recall that $e^{\ln x} = x$, therefore =exponentiation of Equation 19.62 yields

$$\frac{v(t) - g/\gamma}{-g/\gamma} = e^{-\gamma\, t}. \tag{9.63}$$

Thus the y-component of the velocity as a function of time is given by

$$v(t) = \frac{g}{\gamma}(1 - e^{-\gamma\, t}) = \frac{mg}{6\pi\eta R}(1 - e^{-(6\pi\eta R/m)\, t}). \tag{9.64}$$

A plot of $v(t)$ vs. t is shown in Figure 8.31 with parameters $R = 5.00 \times 10^{-3}\mathrm{m}$, $\eta = 8.10 \times 10^{-2}\ \mathrm{kg \cdot m^{-1} \cdot s^{-1}}$, $m = 4.08 \times 10^{-3}\ \mathrm{kg}$ and $g/\gamma = 1.87\ \mathrm{m \cdot s^{-1}}$. For large

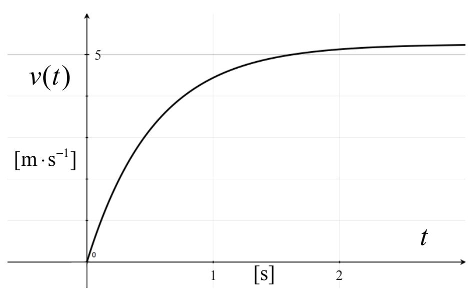

Figure 9.15: Plot of y-component of the velocity $v(t)$ vs.t for marble falling through oil with $g/\gamma = 1.87\ \mathrm{m \cdot s^{-1}}$.

values of t, the term $e^{-(6\pi\eta R/m)\, t}$ approaches zero, and the marble reaches a terminal velocity

$$v_\infty = v(t = \infty) = \frac{mg}{6\pi\eta R}. \tag{9.65}$$

The coefficient of viscosity can then be determined from the terminal velocity by the condition that

$$\eta = \frac{mg}{6\pi R v_{ter}}. \tag{9.66}$$

Let ρ_m denote the density of the marble. The mass of the spherical marble is $m = (4/3)\rho_m R^3$. The terminal velocity is then

$$v_\infty = \frac{2\rho_m R^2 g}{9\eta}. \tag{9.67}$$

The terminal velocity depends on the square of the radius of the marble, indicating that larger marbles will reach faster terminal speeds.

The position of the marble as a function of time is given by the integral expression

$$y(t) - y(t=0) = \int_{t'=0}^{t'=t} v(t')dt', \tag{9.68}$$

which after substitution of Equation 19.64 and integration using the initial condition that $y(t=0) = 0$, becomes

$$y(t) = \frac{g}{\gamma}t + \frac{g}{\gamma^2}\left(e^{-\gamma t} - 1\right). \tag{9.69}$$

9.7.2 Example:Drag forces at high speeds

An object of mass m at time $t = 0$ is moving rapidly with velocity $\vec{\mathbf{v}}_0$ through a fluid of density ρ. Let A denote the cross-sectional area of the object in a plane perpendicular to the motion. The object experiences a retarding drag force whose magnitude is given by Equation 19.56. Determine an expression for the velocity of the object as a function of time.

Answer

Choose a coordinate system such that the object is moving in the positive x-direction with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{i}}$. Set $\beta = (1/2)C_D A\rho$,. Newton's Second Law is then

$$-\beta v^2 = \frac{dv}{dt}. \tag{9.70}$$

Use separation of variables to write an integral equation

$$\int_{v'=v_0}^{v'=v(t)} \frac{dv'}{v'^2} = -\beta \int_{t'=0}^{t'=t} dt'. \tag{9.71}$$

Integration yields

$$-\left(\frac{1}{v(t)} - \frac{1}{v_0}\right) = -\beta t. \tag{9.72}$$

After some algebraic rearrangement the x-component of the velocity as a function of time is given by

$$v(t) = \frac{v_0}{1 + v_0 \beta t} = \frac{1}{1 + t/\tau} v_0, \tag{9.73}$$

where $\tau = 1/v_0\beta$. A plot of $v(t)$ vs. t is shown in Figure 8.34 with initial conditions $v_0 = 20\ \mathrm{m \cdot s^{-1}}$ and $\beta = 0.5\ \mathrm{s^{-1}}$.

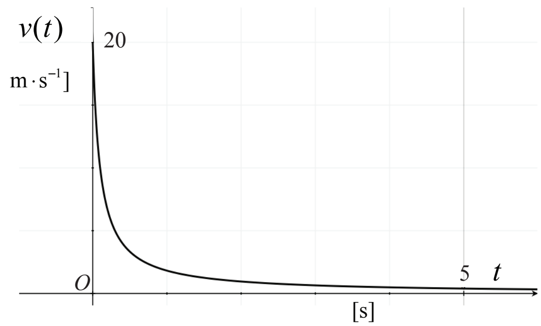

Figure 9.16: Plot of $v(t)$ vs. t for damping force $F_{\mathrm{drag}} = \frac{1}{2} C_D A \rho v^2$.

9.7.3 Example: Free Fall with Air Drag

Consider an object of mass m that is in free fall but experiencing air resistance. The magnitude of the drag force is given by Equation 19.56, where ρ is the density of air, A is the cross-sectional area of the object in a plane perpendicular to the motion, and C_D is the drag coefficient. Assume that at $t = 0$ the object is moving downward with speed v_0 at which Equation 19.56 applies.

(a) Determine the terminal velocity.

(b) Determine the velocity of the object as a function of time.

Answer

Choose positive y-direction downwards with the origin at the initial position of the object as shown in Figure 19.17(a).

There are two forces acting on the object: the gravitational force, and the drag force which is given by Equation 19.56. The free body diagram is shown in the Figure 19.17(b). Newton's Second Law is then

$$mg - (1/2) C_D A \rho v^2 = m \frac{dv}{dt}. \tag{9.74}$$

Set $\beta = (1/2) C_D A \rho$. Then

$$mg - \beta v^2 = m \frac{dv}{dt}. \tag{9.75}$$

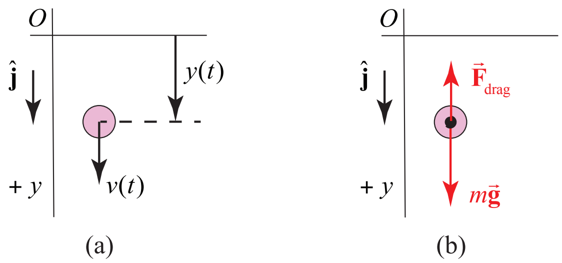

Figure 9.17: (a) Coordinate system for falling object; (b) free body force diagram on falling object.

Initially when the object is just released with speed v_0 , the air drag is non-zero and the acceleration $a = dv/dt$ is maximum. As the object increases its speed, the air drag becomes larger and dv/dt decreases until the object reaches terminal velocity when $dv/dt = 0$. Set $dv/dt = 0$ in Equation 19.75 and solve for the terminal velocity yielding.

$$v_\infty = \sqrt{\frac{mg}{\beta}} = \sqrt{\frac{2mg}{C_D A \rho}}. \tag{9.76}$$

Values for the magnitude of the terminal velocity is shown in Table 19.4 for a variety of objects with the same drag coefficient $C_D = 0.5$, and air density $\rho = 1.225\ \mathrm{kg \cdot m^{-3}}$.

Table 9.4: Terminal Velocities for Different Sized Objects with $C_D = 0.5$, and air density $\rho = 1.225\ \mathrm{kg \cdot m^{-3}}$,

Object	**Mass** m kg	**Area** A m^2	**Terminal Velocity** v_∞ $\mathrm{m \cdot s^{-1}}$
Rain drop	4×10^{-6}	3×10^{-6}	6.5
Hailstone	4×10^{-3}	3×10^{-4}	20
Osprey	1.5	2.5×10^{-1}	14
Human being	7.5×10^{1}	6×10^{-1}	60

In order to integrate Equation 19.75, we shall apply the technique of separation of variables and integration by partial fractions. First rewrite Equation 19.75 as

$$\frac{-\beta}{m} dt = \frac{dv}{\left(v^2 - \frac{mg}{\beta}\right)} = \frac{dv}{\left(v^2 - v_\infty^2\right)} = \left(-\frac{1}{2v_\infty(v + v_\infty)} + \frac{1}{2v_\infty(v - v_\infty)}\right) dv. \tag{9.77}$$

An integral equation is then

$$-\int_{v'=v_0}^{v'=v(t)} \frac{dv'}{2v_\infty(v' + v_\infty)} + \int_{v'=v_0}^{v'=v(t)} \frac{dv'}{2v_\infty(v' - v_\infty)} = -\frac{\beta}{m} \int_{t'=0}^{t'=t} dt'. \tag{9.78}$$

Integration yields

$$\frac{1}{2v_\infty}\left(-\ln\left(\frac{v(t)+v_\infty}{v_0+v_\infty}\right)+\ln\left(\frac{v(t)-v_\infty}{v_0-v_\infty}\right)\right)=-\frac{\beta}{m}t. \tag{9.79}$$

After some algebraic manipulations, Equation 19.79 can be rewritten as

$$(-\ln(v(t)+v_\infty)+\ln(v_0+v_\infty)+\ln(v_\infty-v(t))-\ln(v_\infty-v_0))=-\frac{2v_\infty\beta}{m}t \tag{9.80}$$

Thus

$$\ln\left(\frac{v_\infty-v(t)}{v_\infty+v(t)}\right)+\ln\left(\frac{v_0+v_\infty}{v_\infty-v_0}\right)=-\frac{2v_\infty\beta}{m}t\ \ . \tag{9.81}$$

Set $b=\ln\left(\frac{v_0+v_\infty}{v_\infty-v_0}\right)$.Then

$$\ln\left(\frac{v_\infty-v(t)}{v_\infty+v(t)}\right)=-(\frac{2v_\infty\beta}{m}t+b). \tag{9.82}$$

Exponentiate yields

$$\left(\frac{v_\infty-v(t)}{v(t)+v_\infty}\right)=e^{-\left(\frac{2v_\infty\beta}{m}t+b\right)}. \tag{9.83}$$

After some more algebraic rearrangement the y-component of the velocity as a function of time is given by

$$v(t)=v_\infty\left(\frac{1-e^{-(\frac{2v_\infty\beta}{m}t+b)}}{1+e^{-(\frac{2v_\infty\beta}{m}t-+b)}}\right)=v_\infty tanh\left(\frac{v_\infty\beta}{m}t+b\right), \tag{9.84}$$

where

$$\frac{v_\infty\beta}{m}=\sqrt{\frac{\beta g}{m}}=\sqrt{\frac{(1/2)C_DA\rho g}{m}}. \tag{9.85}$$