

Chapter 12: Momentum and the Flow of Mass

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Chapter 12

Momentum and the Flow of Mass

*Even though the release was pulled, the rocket did not rise at first, but the flame came out, and there was a steady roar. After a number of seconds it rose, slowly until it cleared the flame, and then at express-train speed, curving over to the left, and striking the ice and snow, still going at a rapid rate. It looked almost magical as it rose, without any appreciably greater noise or flame, as if it said, “I’ve been here long enough; I think I’ll be going somewhere else, if you don’t mind.”*¹

Robert Goddard

12.1 The Challenger Flight

When the Rogers Commission in 1986 investigated the Challenger disaster, a commission member, physicist Richard Feynman, made an extraordinary demonstration during the hearings. measured with a thermometer. The mass, or number of moles or number of molecules, is a measure of the quantity of matter.

He (Feynman) also learned that rubber used to seal the solid rocket booster joints using O-rings, failed to expand when the temperature was at or below 32 degrees F (0 degrees C). The temperature at the time of the Challenger liftoff was 32 degrees F. Feynman now believed that he had the solution, but to test it, he dropped a piece of the O-ring material, squeezed with a C-clamp to simulate the actual conditions of the shuttle, into a glass of ice water. Ice, of course, is 32 degrees F. At this point one needs to understand exactly what role the O-rings play in the solid rocket booster (SRB) joints. When the material in the SRB start to heat up, it expands and pushes against the sides of the SRB. If there is an opening in a joint in the

¹describing the first rocket flight using liquid propellants at Aunt Effie’s farm, 17 March 1926.

SRB, the gas tries to escape through that opening (think of it like water in a tea kettle escaping through the spout.) This leak in the Challenger's SRB was easily visible as a small flicker in a launch photo. This flicker turned into a flame and began heating the fuel tank, which then ruptured. When this happened, the fuel tank released liquid hydrogen into the atmosphere where it exploded. As Feynman explained, because the O-rings cannot expand in 32 degree weather, the gas finds gaps in the joints, which led to the explosion of the booster and then the shuttle itself.”²

In the Report of the Presidential Commission on the Space Shuttle *Challenger* Accident (1986), Appendix F - *Personal observations on the reliability of the Shuttle*, Feynman wrote

The Challenger flight is an excellent example. . . . The O-rings of the Solid Rocket Boosters were not designed to erode. Erosion was a clue that something was wrong. Erosion was not something from which safety can be inferred. There was no way, without full understanding, that one could have confidence that conditions the next time might not produce erosion three times more severe than the time before. Nevertheless, officials fooled themselves into thinking they had such understanding and confidence, in spite of the peculiar variations from case to case. A mathematical model was made to calculate erosion. This was a model based not on physical understanding but on empirical curve fitting. To be more detailed, it was supposed a stream of hot gas impinged on the O-ring material, and the heat was determined at the point of stagnation (so far, with reasonable physical, thermodynamic laws). But to determine how much rubber eroded it was assumed this depended only on this heat by a formula suggested by data on a similar material. A logarithmic plot suggested a straight line, so it was supposed that the erosion varied as the .58 power of the heat, the .58 being determined by a nearest fit. At any rate, adjusting some other numbers, it was determined that the model agreed with the erosion (to depth of one-third the radius of the ring). There is nothing much so wrong with this as believing the answer! Uncertainties appear everywhere. How strong the gas stream might be was unpredictable, it depended on holes formed in the putty. Blow-by showed that the ring might fail even though not, or only partially eroded through. The empirical formula was known to be uncertain, for it did not go directly through the very data points by which it was determined. There were a cloud of points some twice above, and some twice below the fitted curve, so erosions twice predicted were reasonable from that cause alone. Similar uncertainties surrounded the other constants in the formula, etc., etc. When using a mathematical model careful attention must be given to uncertainties in the model. . . .

In any event this has had very unfortunate consequences, the most serious of which is to encourage ordinary citizens to fly in such a dangerous

²<http://www.fotuva.org/online/frameload.htm?/online/challenger.htm..>

machine, as if it had attained the safety of an ordinary airliner. The astronauts, like test pilots, should know their risks, and we honor them for their courage. Who can doubt that McAuliffe was equally a person of great courage, who was closer to an awareness of the true risk than NASA management would have us believe? Let us make recommendations to ensure that NASA officials deal in a world of reality in understanding technological weaknesses and imperfections well enough to be actively trying to eliminate them. For a successful technology, reality must take precedence over public relations, for nature cannot be fooled.

12.2 Introduction: Continuous Momentum Transfer

So far we have restricted ourselves to considering systems consisting of discrete objects or point-like objects that have fixed amounts of mass. We shall now consider systems in which material flows between the objects in the system, for example we shall consider coal falling from a hopper into a moving railroad car, sand leaking from railroad car fuel, grain moving forward into a railroad car, and fuel ejected from the back of a rocket. In each of these examples material is continuously flows into or out of an object. We have already shown that the total external force causes the momentum of a system to change,

$$\vec{F}_{\text{ext}}^{\text{total}} = \frac{d\vec{p}_{\text{system}}}{dt}. \quad (12.1)$$

We shall analyze how the momentum of the constituent elements our system change over a time interval $[t, t + \Delta t]$, and then consider the limit as $\Delta t \rightarrow 0$. We can then explicit calculate the derivative on the right hand side of Equation 12.1 which becomes

$$\vec{F}_{\text{ext}}^{\text{total}} = \frac{d\vec{p}_{\text{system}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{p}_{\text{system}}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{p}_{\text{system}}(t + \Delta t) - \vec{p}_{\text{system}}(t)}{\Delta t}. \quad (12.2)$$

We need to be very careful how we apply this generalized version of Newton's Second Law to systems in which mass flows between constituent objects. In particular, when we isolate elements as part of our system we must be careful to identify the mass Δm of the material that continuously flows in or out of an object that is part of our system during the time interval Δt under consideration.

We shall consider four categories of mass flow problems that are characterized by a continuous mass and momentum transfer.

12.2.1 Transfer of Material into an Object, but no Transfer of Momentum

Consider rain falling vertically downward with speed u into car of mass m moving forward with speed v . A small amount of falling rain Δm has no component of

³R. P. Feynman, *Appendix F - Personal observations on the reliability of the Shuttle*, Report of the PRESIDENTIAL COMMISSION on the Space Shuttle Challenger Accident (1986), <http://history.nasa.gov/rogersrep/genindex.htm>.

momentum in the direction of motion of the car. There is a transfer of rain into the car but no transfer of momentum in the direction of motion of the car (Figure 12.1).

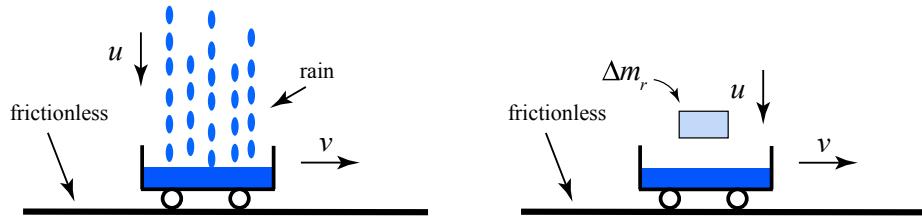


Figure 12.1: Transfer of rain mass into the car but no transfer of momentum in direction of motion.

12.2.2 Transfer of Material Out of an Object, but no Transfer of Momentum

Consider an ice skater gliding on ice at speed v holding a bag of sand that is leaking straight down with respect to the moving skater. The sand continually leaves the bag but it does not transport any momentum away from the bag in the direction of motion of the object. In Figure 12.2, sand of mass Δm_s leaves the bag.

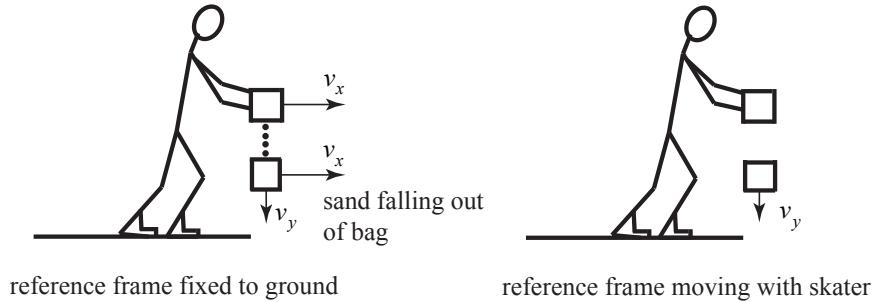


Figure 12.2: Transfer of mass out of object but no transfer of momentum in direction of motion.

12.2.3 Transfer of Material Impulses Object Via Transfer of Momentum

Suppose a fire hose is used to put out a fire on a boat of mass m_b . Assume the column of water moves horizontally with speed u . The incoming water continually hits the boat

propelling it forward. During the time interval Δt , a column of water of mass Δm_w will hit the boat that is moving forward with speed v increasing its speed (Figure 12.3).

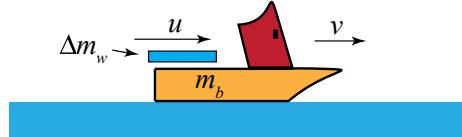
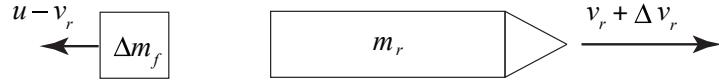


Figure 12.3: Transfer of mass of water increases speed of boat.

12.2.4 Material Continually Ejected From Object results in Recoil of Object

When fuel of mass Δm_f is ejected from the back of a rocket with speed u relative to the rocket, the rocket of mass m_r recoils forward. In a reference frame in which the rocket is moving forward with speed v_r , then the speed after recoil is $v_r + \Delta v_r$. The speed of the backwardly ejected fuel is $u - v_r$, (Figure 12.4a). Figure 12.4b shows the recoil of the rocket in the reference frame of the rocket. The rocket recoils forward with speed Δv_r . We must carefully identify the momentum of the object and the material transferred at



(a) reference frame in which rocket is moving with speed v_r



(b) reference frame of rocket

Figure 12.4: Transfer of mass out of rocket provides impulse on rocket in (a) reference frame in which rocket moves with speed v_r , (b) reference frame of rocket.

time t in order to determine $\vec{p}_{\text{system}}(t)$. We must also identify the momentum of the object and the material transferred at time $t + \Delta t$ in order to determine $\vec{p}_{\text{system}}(t + \Delta t)$ as well. Recall that when we defined the momentum of a system, we assumed that the mass of the system remain constant. Therefore we cannot ignore the momentum of the transferred material at time $t + \Delta t$ even though it may have left the object; it is still part of our system (or at time t even though it has not flowed into the object yet).

12.3 Worked Examples

12.3.1 Filling a Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F . At the same time coal begins to run into the car at a constant rate b from a coal hopper at rest along the track (Figure 12.5). Find the speed when a mass m_c of coal has been transferred.

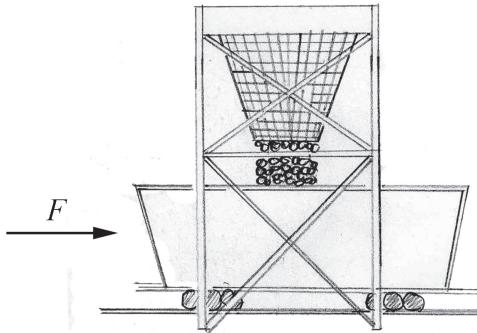


Figure 12.5: Filling a coal car.

Solution: We shall analyze the momentum changes in the horizontal direction, which we call the x -direction. Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x -direction to the coal car. So we shall take as our system the empty coal car and a mass m_c of coal that has been transferred. Our initial state at $t = 0$ is when the coal car is empty and at rest before any coal has been transferred. The x -component of the momentum of this initial state is zero,

$$p_x(0) = 0. \quad (12.3)$$

Our final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f . The x -component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f. \quad (12.4)$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x -direction is

$$\int_0^{t_f} F dt = \Delta p_x = p_x(t_f) - p_x(0). \quad (12.5)$$

Because the force is constant, the integral is simple and the momentum principle becomes

$$Ft_f = (m_0 + bt_f)v_f. \quad (12.6)$$

The final speed is

$$v_f = \frac{Ft_f}{(m_0 + bt_f)}. \quad (12.7)$$

12.3.2 Emptying a Freight Car

A freight car of mass m_c contains sand of mass m_s . At $t = 0$, the freight car is at rest and a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s/dt$. Find the speed of the freight car when all the sand is gone (Figure 12.6).

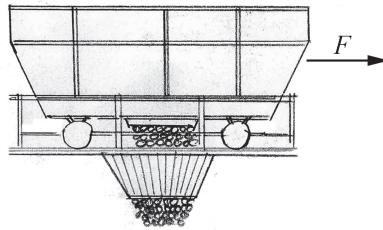


Figure 12.6: Emptying a freight car.

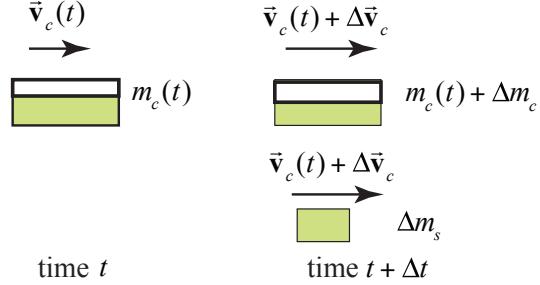
Solution: Choose the positive x -direction to point in the direction that the car is moving. Choose for the system the amount of sand in the freight car at time t , $m_c(t)$. At time t , the car is moving with velocity $\vec{v}_c(t) = v_c(t)\hat{i}$. The momentum diagram for the system at time t is shown in the diagram on the left in Figure 12.7. The momentum of the system at time t is given by

$$\vec{p}_{sys}(t) = m_c(t)\vec{v}_c(t). \quad (12.8)$$

During the time interval $t+\Delta t$, an amount of sand of mass Δm_s leaves the freight car and the mass of the freight car changes by $m_c(t+\Delta t) = m_c(t) + \Delta m_c$, where $\Delta m_c = -\Delta m_s$. At the end of the interval the car is moving with velocity $\vec{v}_c(t+\Delta t) = \vec{v}_c(t) + \Delta\vec{v}_c = (v_c(t) + \Delta v_c)\hat{i}$. The momentum diagram for the system at time $t+\Delta t$ is shown in the diagram on the right in Figure 12.7. The momentum of the system at time $t+\Delta t$ is given by

$$\vec{p}_{sys}(t+\Delta t) = (\Delta m_s + m_c(t) + \Delta m_c)(\vec{v}_c(t) + \Delta\vec{v}_c) = m_c(t)(\vec{v}_c(t) + \Delta\vec{v}_c). \quad (12.9)$$

Note that the sand that leaves the car is shown with velocity $\vec{v}_c(t) + \Delta\vec{v}_c$. This implies that all the sand leaves the car with the velocity of the car at the end of the interval.

Figure 12.7: Momentum diagrams at time t and at time $t + \Delta t$.

This is an approximation. Because the sand leaves continuous, the velocity will vary from $\vec{v}_c(t)$ to $\vec{v}_c(t) + \Delta\vec{v}_c$ but so does the change in mass of the car and these two contributions to the system's moment exactly cancel. The change in momentum of the system is then

$$\Delta\vec{p}_{sys} = \vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t) = m_c(t)(\vec{v}_c(t) + \Delta\vec{v}_c) - m_c(t)\vec{v}_c(t) = m_c(t)\Delta\vec{v}_c. \quad (12.10)$$

Throughout the interval a constant force $\vec{F} = F\hat{i}$ is applied to the system so the momentum principle becomes

$$\vec{F} = \lim_{\Delta t \rightarrow 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} m_c(t) \frac{\Delta\vec{v}_c}{\Delta t} = m_c(t) \frac{d\vec{v}_c}{dt}. \quad (12.11)$$

Because the motion is one-dimensional, Equation 12.11 written in terms of x -components becomes

$$F = m_c(t) \frac{dv_c}{dt}. \quad (12.12)$$

Denote the initial mass of the car by $m_{c,0} = m_c + m_s$ where m_c is the mass of the car and m_s is the mass of the sand in the car at time $t = 0$. The mass of the sand that has left the car at time t is given by

$$m_s(t) = \int_0^t \frac{dm_s}{dt} dt = \int_0^t b dt = bt. \quad (12.13)$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt. \quad (12.14)$$

Therefore Equation 12.12 becomes

$$F = (m_c + m_s - bt) \frac{dv_c}{dt}. \quad (12.15)$$

This equation can be solved for the x -component of the velocity at time t , $v_c(t)$, (which in this case is the speed) by the method of separation of variables. Rewrite Equation

12.15 as

$$dv_c = \frac{Fdt}{(m_c + m_s - bt)}. \quad (12.16)$$

Then integrate both sides

$$\int_{v'_c=0}^{v'_c=v_c(t)} dv'_c = \int_{t'=0}^{t'=t} \frac{Fdt'}{m_c + m_s - bt'}. \quad (12.17)$$

Note the agreement of limits on the two definite integrals. Integration yields the speed of the car as a function of time

$$\begin{aligned} v_c(t) &= -\frac{F}{b} \ln(m_c + m_s - bt')|_{t'=0}^{t'=t} \\ &= -\frac{F}{b} \ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right) = \frac{F}{b} \ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right), \end{aligned} \quad (12.18)$$

where we used the property that $\ln(a) - \ln(b) = \ln(a/b)$ and therefore $\ln(a/b) = -\ln(b/a)$. Note that $m_c + m_s \geq m_c + m_s - bt$, so the term $\ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right) \geq 0$, and the speed of the car increases as we expect.

12.3.3 Filling a Freight Car

Grain is blown into car A from car B at a rate of b kilograms per second. The grain leaves the chute vertically downward, so that it has the same horizontal velocity, u as car b, (Figure 12.8). Car A is initially at rest before any grain is transferred in and has mass $m_{A,0}$. At the moment of interest, car A has mass m_A and speed v . Determine an expression for the speed car A as a function of time t .

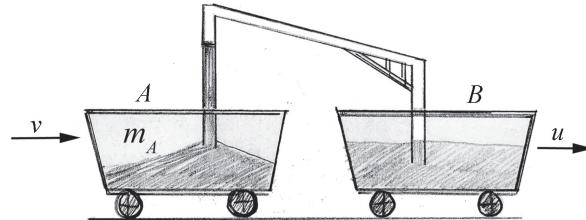


Figure 12.8: Filling a freight car.

Solution: Choose positive x -direction to the right in the direction the cars are moving. Define the system at time t to be the car and grain that is already in it, which together has mass $m_A(t)$, and the small amount of material of mass Δm_g that is blown into car A during the time interval $[t, t + \Delta t]$. At time t , car A is moving with velocity

$\vec{v}_A(t) = v_A(t)\hat{\mathbf{i}}$, and the material blown into car is moving with velocity $\vec{\mathbf{u}} = u\hat{\mathbf{i}}$. At time $t + \Delta t$, car A is moving with velocity $\vec{v}_A(t) + \Delta\vec{v}_A = (v_A(t) + \Delta v_A)\hat{\mathbf{i}}$, and the mass of car A is $m_A(t + \Delta t) = m_A(t) + \Delta m_A$, where $\Delta m_A = \Delta m_g$. The momentum diagrams for times t and $t + \Delta t$ are shown in Figure 12.9. The momentum at time t is

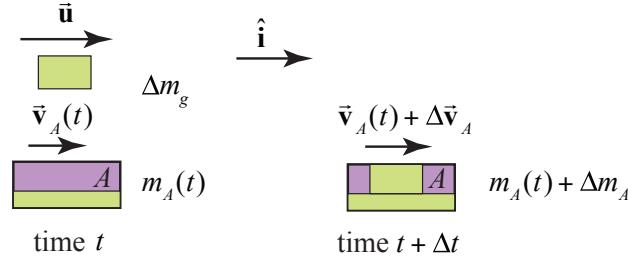


Figure 12.9: Momentum diagrams at times t and $t + \Delta t$.

$$\vec{p}_{sys}(t) = m_A(t)\vec{v}_A(t) + \Delta m_g\vec{\mathbf{u}}. \quad (12.19)$$

The momentum at time $t + \Delta t$ is

$$\vec{p}_{sys}(t + \Delta t) = (m_A(t) + \Delta m_A)(\vec{v}_A(t) + \Delta\vec{v}_A). \quad (12.20)$$

There are no external forces acting on the system in the x -direction and the external forces acting on the system perpendicular to the motion sum to zero, so the momentum principle becomes

$$\mathbf{0} = \lim_{\Delta t \rightarrow 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}. \quad (12.21)$$

Using the momentum results above, the momentum principle becomes

$$\mathbf{0} = \lim_{\Delta t \rightarrow 0} \frac{(m_A(t) + \Delta m_A)(\vec{v}_A(t) + \Delta\vec{v}_A) - (m_A(t)\vec{v}_A(t) + \Delta m_g\vec{\mathbf{u}})}{\Delta t}. \quad (12.22)$$

Apply the condition that $\Delta m_A = \Delta m_g$ and some rearrangement of terms:

$$\mathbf{0} = \lim_{\Delta t \rightarrow 0} \frac{m_A(t)\Delta\vec{v}_A}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\Delta m_A(\vec{v}_A(t) - \vec{\mathbf{u}})}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\Delta m_A\Delta\vec{v}_A}{\Delta t}. \quad (12.23)$$

In the limit as $\Delta t \rightarrow 0$, the product $\Delta m_A\Delta\vec{v}_A$ is a second order differential (the product of two first order differentials) and the term $\Delta m_A\Delta\vec{v}_A/\Delta t$ approaches zero, therefore the momentum principle yields the differential equation

$$\mathbf{0} = m_A(t) \frac{d\vec{v}_A}{dt} + \frac{dm_A}{dt}(\vec{v}_A(t) - \vec{\mathbf{u}}). \quad (12.24)$$

The x -component of Equation 12.24 is then

$$0 = m_A(t) \frac{dv_A}{dt} + \frac{dm_A}{dt}(v_A(t) - u). \quad (12.25)$$

Rearranging terms and using the fact that the material is blown into car A at a constant rate $b = dm_A/dt$, we have that the rate of change of the x -component of the velocity of car A is given by

$$\frac{dv_A(t)}{dt} = \frac{b(u - v_A(t))}{m_A(t)}. \quad (12.26)$$

We cannot directly integrate Equation 12.26 with respect to dt because the mass of the car A is a function of time. In order to find the x -component of the velocity of car A we need to know the relationship between the mass of car A and the x -component of the velocity of the car A. There are two approaches. In the first approach we separate variables in 12.24 where we have suppressed the dependence on t in the expressions for m_A and v_A yielding

$$\frac{dv_A}{u - v_A} = \frac{dm_A}{m_A}, \quad (12.27)$$

which becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{m'_A=m_{A,0}}^{m'_A=m_A(t)} \frac{dm'_A}{m'_A}. \quad (12.28)$$

where $m_{A,0}$ is the mass of the car before any material has been blown in. After integration we have that

$$\ln \frac{u}{u - v_A(t)} = \ln \frac{m_A(t)}{m_{A,0}}. \quad (12.29)$$

Exponentiate both side yields

$$\frac{u}{u - v_A(t)} = \frac{m_A(t)}{m_{A,0}}. \quad (12.30)$$

We can solve this equation for the x -component of the velocity of the car

$$v_A(t) = \frac{m_A(t) - m_{A,0}}{m_A(t)} u. \quad (12.31)$$

Because the material is blown into the car at a constant rate $b = dm_A/dt$, the mass of the car as a function of time is given by

$$m_A(t) = m_{A,0} + bt. \quad (12.32)$$

Therefore substituting Equation 12.32 into Equation 12.31 yields the x -component of the velocity of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u. \quad (12.33)$$

In a second approach, we substitute Equation 12.32 into Equation 12.26 yielding

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_{A,0} + bt}. \quad (12.34)$$

Separate variables in Equation 12.34:

$$\frac{dv_A}{u - v_A} = \frac{bdt}{m_{A,0} + bt}. \quad (12.35)$$

which then becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{t'=0}^{t'=t} \frac{dt'}{m_{A,0} + bt'}. \quad (12.36)$$

Integration yields

$$\ln \frac{u}{u - v_A(t)} = \ln \frac{m_{A,0} + bt}{m_{A,0}}. \quad (12.37)$$

Exponentiate both sides resulting in

$$\frac{u}{u - v_A(t)} = \frac{m_{A,0} + bt}{m_{A,0}}. \quad (12.38)$$

After some algebraic manipulation we can find the speed of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u. \quad (12.39)$$

in agreement with Equation 12.33.

Check result

We can rewrite Equation 12.39 as

$$(m_{A,0} + bt)v_A(t) = bu, \quad (12.40)$$

which illustrates the point that the momentum of the system at time t is equal to the momentum of the grain that has been transferred to the system during the interval $[0, t]$.

12.4 Rocket Propulsion

A rocket at time t is moving with velocity $\vec{v}_{r,i}$ with respect to a fixed reference frame. During the time interval $[t_i, t_f]$ the rocket continuously burns fuel that is continuously ejected backwards with velocity \vec{u} relative to the rocket. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction accelerating the rocket forward. The rocket velocity is a function of time, $\vec{v}_r(t)$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r/dt . Let \vec{F}_{ext} denote the total external force acting on the rocket. We shall use the momentum principle, to determine a differential equation that relates $d\vec{v}_r/dt$, dm_r/dt , \vec{u} , $\vec{v}_r(t)$ and

\vec{F}_{ext} , an equation known as the *rocket equation*.

We shall apply the momentum principle during the time interval $[t, t + \Delta t]$ with Δt taken to be a small interval with $t_i < t < t_f$. We shall eventually consider the limit that $\Delta t \rightarrow 0$. During this interval, choose as our system the mass of the rocket at time t ,

$$m_{\text{sys}} = m_r(t) = m_{r,d} + m_f(t). \quad (12.41)$$

where $m_{r,d}$ is the dry mass of the rocket and $m_r(t)$ is the mass of the fuel in the rocket at time t . During the time interval $[t, t + \Delta t]$, a small amount of fuel of mass Δm_f is ejected backwards with velocity \vec{u} to the rocket. Before the fuel is ejected, it is traveling at the velocity of the rocket and so during the time interval $[t, t + \Delta t]$, the ejected fuel undergoes a change in momentum and the rocket recoils forward. At time $t + \Delta t$ the rocket has velocity $\vec{v}_r(t + \Delta t)$. Although the ejected fuel continually changes its velocity, we shall assume that the fuel is all ejected at the instant $t + \Delta t$ and then consider the limit as $\Delta t \rightarrow 0$. Therefore the velocity of the ejected fuel with respect to the fixed reference frame is the vector sum of the relative velocity of the fuel with respect to the rocket and the velocity of the rocket, $\vec{u} + \vec{v}_r(t + \Delta t)$. Figure 12.10 represents momentum diagrams for our system at time t and $t + \Delta t$ relative to a fixed inertial reference frame in which velocity of the rocket at time t is $\vec{v}_r(t)$. The momentum of the system at time t is

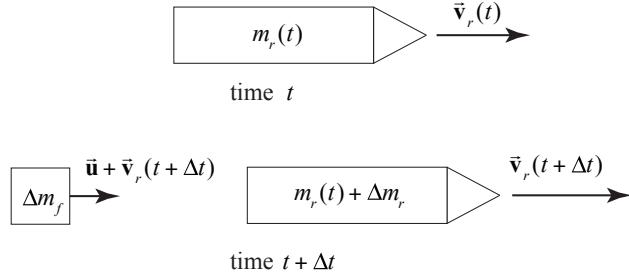


Figure 12.10: Momentum diagrams at times t and $t + \Delta t$.

$$\vec{p}_{\text{sys}}(t) = m_r(t) \vec{v}_r(t). \quad (12.42)$$

Note that the mass of the system at time t is

$$m_{\text{sys}} = m_r(t). \quad (12.43)$$

The momentum of the system at time $t + \Delta t$ is

$$\vec{p}_{\text{sys}}(t + \Delta t) = m_r(t + \Delta t) \vec{v}_r(t + \Delta t) + \Delta m_f (\vec{u} + \vec{v}_r(t + \Delta t)). \quad (12.44)$$

where $m_r(t + \Delta t) = m_r(t) + \Delta m_r$. With this notation the mass of the system at time $t + \Delta t$ is given by

$$m_{\text{sys}} = m_r(t + \Delta t) + \Delta m_f = m_r(t) + \Delta m_r + \Delta m_f. \quad (12.45)$$

Because the mass of the system is constant, setting Equation 12.43 equal to Equation 12.45 requires that

$$\Delta m_r = -\Delta m_f. \quad (12.46)$$

The momentum of the system at time $t + \Delta t$ (Equation 12.44) can be rewritten as

$$\begin{aligned} \vec{p}_{\text{sys}}(t + \Delta t) &= (m_r(t) + \Delta m_r)\vec{v}_r(t + \Delta t) - \Delta m_r(\vec{u} + \vec{v}_r(t + \Delta t))s \\ &= m_r(t)\vec{v}_r(t + \Delta t) - \Delta m_r\vec{u} \end{aligned}. \quad (12.47)$$

We can now apply Newton's Second Law in the form of the momentum principle,

$$\begin{aligned} \vec{F}_{\text{ext}} &= \lim_{\Delta t \rightarrow 0} \frac{(m_r(t)\vec{v}_r(t + \Delta t) - \Delta m_r\vec{u}) - m_r(t)\vec{v}_r(t)}{\Delta t} \\ &= m_r(t) \lim_{\Delta t \rightarrow 0} \frac{\vec{v}_r(t + \Delta t) - \vec{v}_r(t)}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta m_r}{\Delta t} \vec{u} \end{aligned}. \quad (12.48)$$

We now take the limit as

$$\vec{F}_{\text{ext}} = m_r(t) \frac{d\vec{v}_r}{dt} - \frac{dm_r}{dt} \vec{u}. \quad (12.49)$$

Equation 12.49 is known as the *rocket equation*.

Suppose the rocket is moving in the positive x -direction with an external force given by $\vec{F}_{\text{ext}} = F_x \hat{i}$. Then $\vec{u} = -u \hat{i}$, where $u > 0$ is the relative speed of the fuel and it is moving in the negative x -direction. The rocket has velocity $\vec{v}_r = v_{r,x} \hat{i}$. Then Equation 12.49 becomes

$$F_x = m_r(t) \frac{dv_{r,x}}{dt} + \frac{dm_r}{dt} u. \quad (12.50)$$

Note that the rate of decrease of the mass of the rocket, dm_r/dt , is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt}. \quad (12.51)$$

We can rewrite Equation 12.50 as

$$F_x - \frac{dm_r}{dt} u = m_r(t) \frac{dv_{r,x}}{dt}. \quad (12.52)$$

The second term on the left-hand-side of Equation 12.52 is called the *thrust*

$$F_{\text{thrust},x} = -\frac{dm_r}{dt} u = \frac{dm_f}{dt} u. \quad (12.53)$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f/dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x -direction.

12.4.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Equation 12.52 becomes

$$-\frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \quad (12.54)$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$, then multiply both sides by dt yielding

$$dv_{r,x} = -u\frac{dm_r}{m_r(t)}dt. \quad (12.55)$$

We now integrate both sides of Equation 12.55 with limits corresponding to the values of the x -component of the velocity and mass of the rocket at times t_i when the ejection of the burned fuel began and the time t_f when the burn ended,

$$\int_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = - \int_{m'_{r,i}=m_{r,i}}^{m'_{r,f}=m_{r,f}} \frac{u}{m'_{r,f}} dm'_{r,f}. \quad (12.56)$$

Performing the integration and substituting in the values at the endpoints yields

$$v_{r,x,f} - v_{r,x,i} = -u \ln \left(\frac{m_{r,f}}{m_{r,i}} \right). \quad (12.57)$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Equation 12.57 as

$$v_{r,x,f} - v_{r,x,i} = u \ln \left(\frac{m_{r,i}}{m_{r,f}} \right). \quad (12.58)$$

Note that $\ln(m_{r,i}/m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect. After a slight rearrangement of Equation 12.58, we have an expression for the x -component of the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u \ln \left(\frac{m_{r,i}}{m_{r,f}} \right). \quad (12.59)$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} = m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \quad (12.60)$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u \ln R. \quad (12.61)$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed than if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt}. \quad (12.62)$$

Then we can integrate Equation 12.62

$$\int_{m'_r=m_{r,i}}^{m'_r=m_r(t)} dm'_r = -b \int_{t'=t_i}^{t'=t} dt'. \quad (12.63)$$

yielding an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i). \quad (12.64)$$

For this special case, if we set $t_f = t$ in Equation 12.59, then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u \ln \left(\frac{m_{r,i}}{m_{r,i} - bt} \right). \quad (12.65)$$

12.4.2 Example: Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81 \times 10^7 \text{ kg}$, of which the mass of the fuel is $m_{f,i} = 2.46 \times 10^7 \text{ kg}$. The fuel is burned at a constant rate with total burn time is 510 s and ejected at a speed $u = 3000 \text{ m} \cdot \text{s}^{-1}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution:

The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35 \times 10^7 \text{ kg}$, hence $R = m_{r,i}/m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_r = u \ln R = 6250 \text{ m/s}. \quad (12.66)$$

12.4.3 Example: Two-Stage Rocket

Now suppose that the same rocket in Example 12.4.2 burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03 \times 10^7 \text{ kg}$ with burn time 150 s. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4 \times 10^6 \text{ kg}$. All the remaining fuel with mass is

burned during the second stage with burn time of 360 s. What is the final speed of the rocket after all the fuel has been burned?

Solution:

The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78 \times 10^7 \text{ kg}$ and $R_1 = m_{r,1,i}/m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u \ln R_1 = 3840 \text{ m/s.} \quad (12.67)$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1 \times 10^6 \text{ kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3 \times 10^6 \text{ kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4 \times 10^6 \text{ kg}$. Then $R_2 = m_{r,2,i}/m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u \ln R_2 = 3340 \text{ m/s.} \quad (12.68)$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u \ln R_1 + u \ln R_2 = u \ln(R_1 R_2) = 7190 \text{ m/s,} \quad (12.69)$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure 12.11.

12.4.4 Rocket in a Constant Gravitational Field

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field. Neglect air resistance then the external force is

$$\vec{F}_{\text{ext}} = m_r \mathbf{g}. \quad (12.70)$$

Choose the positive z -axis in the upward direction then $F_{\text{ext},z}(t) = -m_r(t)g$. Then the rocket equation (Equation 12.49) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t) \frac{dv_{r,z}}{dt}. \quad (12.71)$$

Multiply both sides of Equation ?? by dt , and divide both sides by $m_r(t)$, then

$$dv_{r,z} = -g dt - \frac{dm_r}{m_r(t)}u. \quad (12.72)$$

We now integrate both sides

$$\int_{v'_{r,x}=0}^{v'_{r,x}=v_{r,x}(t)} dv'_{r,x} = -u \int_{m'_{r,i}=m'_{r,i}}^{m'_{r}=m_r(t)} \frac{dm'_r}{m'_r} - g \int_{t'=0}^{t'=t} dt'. \quad (12.73)$$

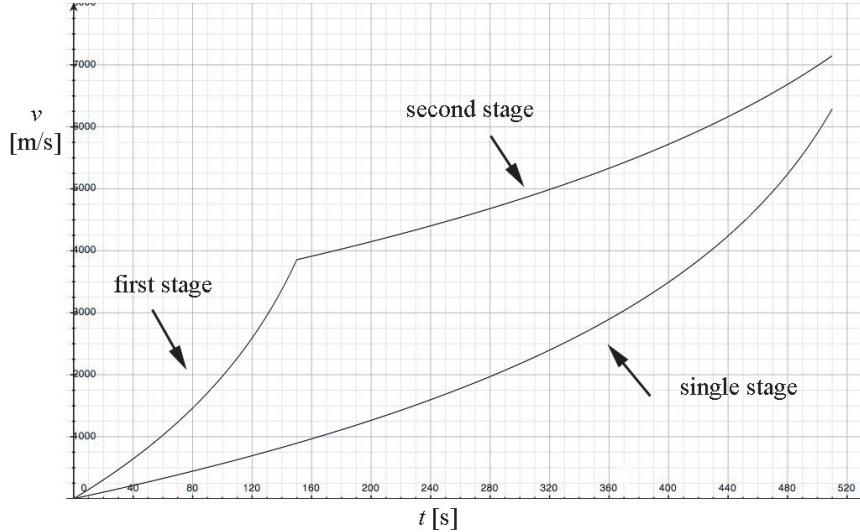


Figure 12.11: Plots of speed of rocket for both one-stage burn and two-stage burn.

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$\int_{v'_{r,x}=0}^{v'_{r,x}=v_{r,x}(t)} dv'_{r,x} = -u \int_{m'_{r}=m_{r,i}}^{m'_{r}=m_r(t)} \frac{dm'_{r}}{m'_{r}} - g \int_{t'=0}^{t'=t} dt'. \quad (12.74)$$

After all the fuel is burned at $t = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u \ln R - g t_f. \quad (12.75)$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third term, and hence the rocket ends up with a larger final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.