Chapter 12: Inertial Reference Fames

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Contents

Chapter 12

Momentum and the Flow of Mass

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium" suggest that the phenomena of electromagnetism as well as mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, ..., the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, ..., namely that light is always propagated in empty space with a definite velocity c, which is independent of the state of motion of the emitting body [1]

Albert Einstein

12.1 Introduction

In order to describe physical events that occur in space and time such as the motion of objects, and only use physical forces we introduced an *inertial reference frame* in Chapter 8. In that reference frame, we introduced a coordinate system with unit vectors at every point that enabled us to mathematically define the position, velocity and acceleration vectors for any object. The position vector of an object depends on the choice of origin but the displacement, velocity, and acceleration vectors are independent of the spatial location of the origin.

We can always choose a second inertial reference frame that is moving with respect

[1]A. Einstein, Zur Elektrodynamik begetter Körper, (On the Electrodynamics of Moving Bodies), Ann. Physik, 17, 891 (1905); translated by W. Perrett and G.B. Jeffrey, 1923, in The Principle of Relativity, Dover, New York.

to the first inertial reference frame. Two reference frames are relatively inertial Then the position, velocity and acceleration of bodies as seen by the different observers do depend on the relative motion of the two reference frames. The relative motion can be described in terms of the relative position, velocity, and acceleration of the observer at the origin, , in reference frame with respect to a second observer located at the origin, , in reference frame .

11.2 Galilean Coordinate Transformations

Let the vector point from the origin of frame to the origin of reference frame . Suppose an object is located at a point 1. Denote the position vector of the object with respect to origin of reference frame by . Denote the position vector of the object with respect to origin of reference frame by (Figure 11.1).

Figure 11.1 Two reference frames and .

The position vectors are related by (11.2.1)

These coordinate transformations are called the Galilean Coordinate Transformations. They enable the observer in frame to predict the position vector in frame , based only on the position vector in frame and the relative position of the origins of the two frames.

The relative velocity between the two reference frames is given by the time derivative of the vector , defined as the limit as of the displacement of the two origins divided by an interval of time, as the interval of time becomes infinitesimally small,

(11.2.2)

11.2.1 Relatively Inertial Reference Frames and the Principle of Relativity

If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

(11.2.3)

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are called relatively inertial reference frames.

We can reinterpret Newton's First Law

Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

as the Principle of Relativity:

In relatively inertial reference frames, if there is no net force impressed on an object at rest in frame S, then there is also no net force impressed on the object in frame .

11.3 Law of Addition of Velocities: Newtonian Mechanics

Suppose the object in Figure 11.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of the object in frame by , and the velocity of the object in frame by . Since the derivative of the position is velocity, the velocities of the object in two different reference frames are related according to , (11.3.1) (11.3.2)

This is called the Law of Addition of Velocities.

11.4 Worked Examples

Example 11.1 Relative Velocities of Two Moving Planes

An airplane A is traveling northeast with a speed of . A second airplane B is traveling southeast with a speed of . (a) Choose a coordinate system and write down an expression for the velocity of each airplane as vectors, and . Carefully use unit vectors to express your answer. (b) Sketch the vectors and on your coordinate system. (c) Find a vector expression that expresses the velocity of aircraft A as seen from an observer flying in aircraft B. Calculate this vector. What is its magnitude and direction? Sketch it on your coordinate system.

Solution: From the information given in the problem we draw the velocity vectors of the airplanes as shown in Figure 11.2a.

(a) (b)

Figure 11.2 (a): Motion of two planes Figure 11.2 (b): Coordinate System

An observer at rest with respect to the ground defines a reference frame . Choose a coordinate system shown in Figure 11.2b. According to this observer, airplane is moving with velocity , and airplane is moving with velocity . According to the information given in the problem airplane A flies northeast so and airplane B flies southeast east so . Thus and

Consider a second observer moving along with airplane B, defining reference frame . What is the velocity of airplane A according to this observer moving in airplane ? The velocity of the observer moving along in airplane with respect to an observer at rest on the ground is just the velocity of airplane and is given by . Using the Law of Addition of Velocities, Equation (11.3.2), the velocity of airplane with respect to an observer moving along with Airplane is given by

. (11.4.1)

Figure 11.3 shows the velocity of airplane A with respect to airplane B in reference frame .

Figure 11.3 Airplane A as seen from observer in airplane B

The magnitude of velocity of airplane as seen by an observer moving with airplane B is given by

. (11.4.2)

The angle of velocity of airplane as seen by an observer moving with airplane B is given by,

. (11.4.3)

Example 11.2 Relative Motion and Polar Coordinates

Particles and are moving in opposite directions around a circle with angular velocity of magnitude , as shown in Figure 11.4. At they are both at the top of the circle with position vector point , where is the radius of the circle. Find the velocity of relative to .

Figure 11.4: Particles and moving relative to each other; Fig 11.5: position functions in different reference frames and for particle

Solution: Let denote the reference with the origin fixed at the center of the circle. Let be a reference frame moving with particle . Let the vector point from the origin of frame to the origin of reference frame . Denote by the position vector of particle with respect to origin of reference frame . Denote by the position vector of particle with respect to the origin of reference frame by (Figure 11.5).

The position vectors are related by

. (11.4.4)

The relative position vector between the origins of the two frames is given by

, (11.4.6)

where is a radially outward pointing unit vector located at particle . The position vector of particle relative to frame is given by

, (11.4.7)

where is a radially outward pointing unit vector located at particle . The position vector of particle relative to reference frame is then

. (11.4.8)

We can decompose each of the unit vectors and with respect to the Cartesian unit vectors and (see Figure 11.5),

(11.4.9) . (11.4.10)

Then Eq. (11.4.8) becomes

. (11.4.11)

The velocity vector of particle relative to reference frame (i.e. with respect to particle), is then found by differentiating Eq. (11.4.11)

. (11.4.12)

Example 11.3 Recoil in Different Frames

A person of mass is standing on a cart of mass . Assume that the cart is free to move on its wheels without friction. The person throws a ball of mass at an angle of with respect to the horizontal as measured by the person in the cart. The ball is thrown with a speed with respect to the cart (Figure 11.6). (a) What is the final velocity of the ball as seen by an observer fixed to the ground? (b) What is the final velocity of the cart as seen by an observer fixed to the ground? (c) With respect to the horizontal, what angle the fixed observer see the ball leave the cart?

Figure 11.6 Recoil of a person on cart due to thrown ball

Solution: a), b) Choose a reference frame fixed to the ground. We shall take as our initial state that before the ball is thrown (cart, ball, throwing person stationary) and our final state after the ball is thrown. We are assuming that there is no friction, and so there are no external forces acting in the horizontal direction. The initial -component of the total momentum is zero, . (11.4.13)

After the ball is thrown, the cart and person have a final momentum

(11.4.14)

as measured by the person on the ground, where is the speed of the person and cart. (The person's center of mass will move with respect to the cart while the ball is being thrown, but since we're interested in velocities, not positions, we need only assume that the person is at rest with respect to the cart after the ball is thrown.)

The ball is thrown with a speed and at an angle with respect to the horizontal as measured by the person in the cart. Therefore the person in the cart throws the ball with velocity . (11.4.15).

Because the cart is moving in the negative -direction with speed just as the ball leaves the person's hand, the -component of the velocity of the ball as measured by an observer on the ground is given by

. (11.4.16)

The ball appears to have a smaller -component of the velocity according to the observer on the ground. The velocity of the ball as measured by an observer on the ground is

. (11.4.17)

The final momentum of the ball according to an observer on the ground is

. (11.4.18)

The momentum flow diagram is shown in (Figure 11.7).

Figure 11.7 Momentum flow diagram for recoil

Because the -component of the momentum of the system is constant, we have that (11.4.19)

We can solve Equation (11.4.19) for the final speed and velocity of the cart as measured by an observer on the ground, , (11.4.20) . (11.4.21)

Note that the -component of the momentum is not constant because as the person is throwing the ball he or she is pushing off the cart and the normal force with the ground exceeds the gravitational force so the net external force in the -direction is non-zero.

Substituting Equation (11.4.20) into Equation (11.4.17) gives

(11.4.22)

As a check, note that in the limit , has speed and is directed at an angle above the horizontal; the fact that the much more massive person-cart combination is free to move doesn't affect the flight of the ball as seen by the fixed observer. Also note that in the unrealistic limit the ball is moving at a speed much smaller than as it leaves the cart.

c) The angle at which the ball is thrown as seen by the observer on the ground is given by (11.4.23)

For arbitrary values for the masses, the above expression will not reduce to a simplified form. However, we can see that for arbitrary masses, and that in the limit , and in the unrealistic , . Can you explain this last odd prediction?

12.2 Introduction: Continuous Momentum Transfer

So far we have restricted ourselves to considering systems consisting of discrete objects or point-like objects that have fixed amounts of mass. We shall now consider systems in which material flows between the objects in the system, for example we shall consider coal falling from a hopper into a moving railroad car, sand leaking from railroad car fuel, grain moving forward into a railroad car, and fuel ejected from the back of a rocket, In each of these examples material is continuously flows into or out of an object. We have already shown that the total external force causes the momentum of a system to change,

$$\vec{\mathbf{F}}^{\text{total}}_{\text{ext}} = \frac{d\,\vec{\mathbf{p}}_{\text{system}}}{dt}. \tag{12.1}$$

We shall analyze how the momentum of the constituent elements our system change over a time interval $[t, t + \Delta t]$, and then consider the limit as $\Delta t \to 0$. We can then explicit calculate the derivative on the right hand side of Equation **??** which becomes

$$\vec{\mathbf{F}}^{\text{total}}_{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{p}}_{\text{system}}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{\mathbf{p}}_{\text{system}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{system}}(t)}{\Delta t}. \tag{12.2}$$

We need to be very careful how we apply this generalized version of Newton's Second Law to systems in which mass flows between constituent objects. In particular, when we isolate elements as part of our system we must be careful to identify the mass Δm of the material that continuously flows in or out of an object that is part of our system during the time interval Δt under consideration.

We shall consider four categories of mass flow problems that are characterized by a continuous mass and momentum transfer.

12.2.1 Transfer of Material into an Object, but no Transfer of Momentum

Consider rain falling vertically downward with speed u into car of mass m moving forward with speed v. A small amount of falling rain Δm has no component of momentum in the direction of motion of the car. There is a transfer of rain into the car but no transfer of momentum in the direction of motion of the car (Figure **??**).

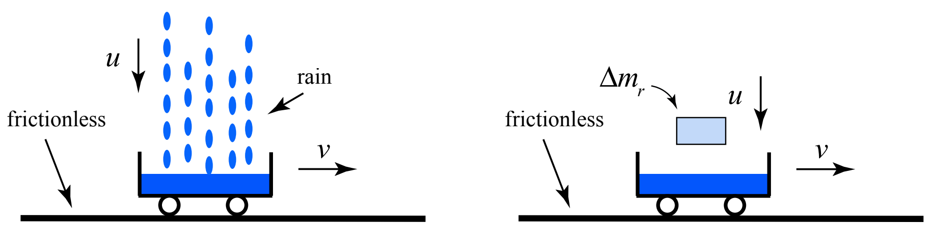

Figure 12.1: Transfer of rain mass into the car but no transfer of momentum in direction of motion.

12.2.2 Transfer of Material Out of an Object, but no Transfer of Momentum

Consider an ice skater gliding on ice at speed v holding a bag of sand that is leaking straight down with respect to the moving skater. The sand continually leaves the bag but it does not transport any momentum away from the bag in the direction of motion of the object. In Figure **??**, sand of mass Δm_s leaves the bag.

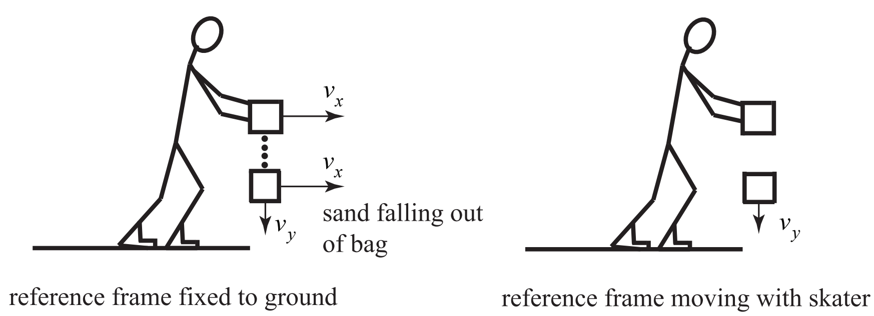

Figure 12.2: Transfer of mass out of object but no transfer of momentum in direction of motion.

12.2.3 Transfer of Material Impulses Object Via Transfer of Momentum

Suppose a fire hose is used to put out a fire on a boat of mass m_b. Assume the column of water moves horizontally with speed u. The incoming water continually hits the boat propelling it forward. During the time interval Δt, a column of water of mass Δm_w will hit the boat that is moving forward with speed v increasing it's speed (Figure **??**).

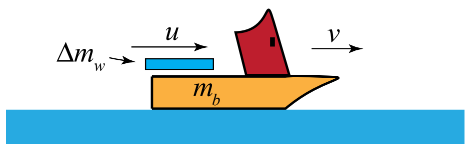

Figure 12.3: Transfer of mass of water increases speed of boat.

12.2.4 Material Continually Ejected From Object results in Recoil of Object

When fuel of mass Δm_f is ejected from the back of a rocket with speed u relative to the rocket, the rocket of mass m_r recoils forward. In a reference frame in which the rocket is moving forward with speed v_r, then the speed after recoil is $v_r + \Delta v_r$. The speed of the backwardly ejected fuel is $u - v_r$, (Figure **??**a). Figure **??**b shows the recoil of the rocket in the reference frame of the rocket. The rocket recoils forward with speed Δv_r. We must carefully identify the momentum of the object and the material transferred at

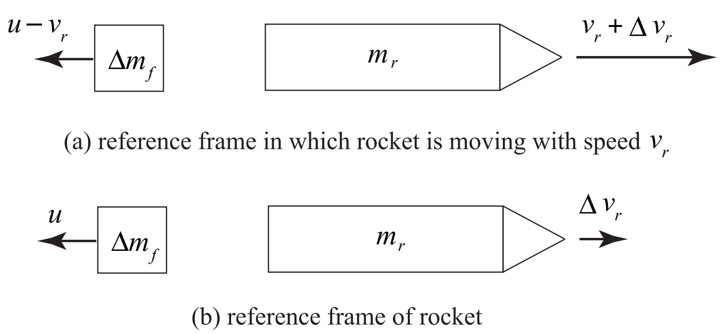

Figure 12.4: Transfer of mass out of rocket provides impulse on rocket in (a) reference frame in which rocket moves with speed v_r, (b) reference frame of rocket.

time t in order to determine $\vec{\mathbf{p}}_{\text{system}}(t)$. We must also identify the momentum of the object and the material transferred at time $t + \Delta t$ in order to determine $\vec{\mathbf{p}}_{\text{system}}(t + \Delta t)$ as well. Recall that when we defined the momentum of a system, we assumed that the mass of the system remain constant. Therefore we cannot ignore the momentum of the transferred material at time $t + \Delta t$ even though it may have left the object; it is still part of our system (or at time t even though it has not flowed into the object yet).

12.3 Worked Examples

12.3.1 Filling a Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F. At the same time coal begins to run into the car at a constant rate b from a coal hopper at rest along the track (Figure **??**). Find the speed when a mass m_c of coal has been transferred.

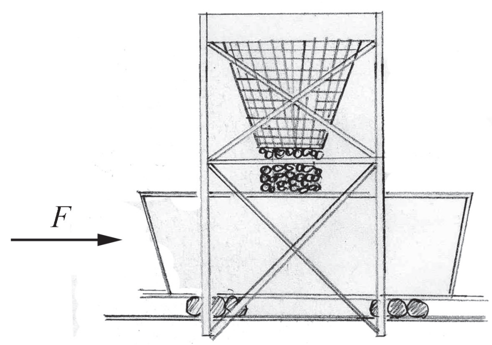

Figure 12.5: Filling a coal car.

Solution: We shall analyze the momentum changes in the horizontal direction, which we call the x-direction. Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x-direction to the coal car. So we shall take as our system the empty coal car and a mass m_c of coal that has been transferred. Our initial state at $t = 0$ is when the coal car is empty and at rest before any coal has been transferred. The x-component of the momentum of this initial state is zero,

$$p_x(0) = 0. \tag{12.3}$$

Our final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f. The x-component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f\,. \tag{12.4}$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x-direction is

$$\int_0^{t_f} F dt = \Delta p_x = p_x(t_f) - p_x(0). \tag{12.5}$$

Because the force is constant, the integral is simple and the momentum principle becomes

$$F t_f = (m_0 + b t_f) v_f . \tag{12.6}$$

The final speed is

$$v_f = \frac{F t_f}{(m_0 + b t_f)}. \tag{12.7}$$

12.3.2 Emptying a Freight Car

A freight car of mass m_c contains sand of mass m_s. At $t = 0$, the freight car is at rest and a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s/dt$. Find the speed of the freight car when all the sand is gone (Figure **??**).

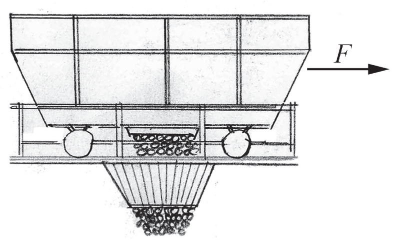

Figure 12.6: Emptying a freight car.

Solution: Choose the positive x-direction to point in the direction that the car is moving. Choose for the system the amount of sand in the fright car at time t, $m_c(t)$. At time t, the car is moving with velocity $\vec{\mathbf{v}}_c(t) = v_c(t)\hat{\mathbf{i}}$. The momentum diagram for the system at time t is shown in the diagram on the left in Figure **??**. The momentum of the system at time t is given by

$$\vec{\mathbf{p}}_{sys}(t) = m_c(t)\vec{\mathbf{v}}_c(t). \tag{12.8}$$

During the time interval $t_+\Delta t$, an amount of sand of mass Δm_s leaves the freight car and the mass of the freight car changes by $m_c(t+\Delta t) = m_c(t) + \Delta m_c$, where $\Delta m_c = -\Delta m_s$. At the end of the interval the car is moving with velocity $\vec{\mathbf{v}}_c(t+\Delta t) = \vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c =$

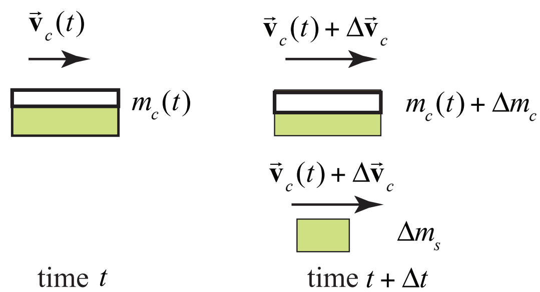

Figure 12.7: Momentum diagrams at time t and at time $t_{+}\Delta t$.

$(v_c(t)+\Delta v_c)\hat{\mathbf{i}}$. The momentum diagram for the system at time is shown in the diagram on the right in Figure **??**. The momentum of the system at time $t_{+}\Delta t$ is given by

$$\vec{\mathbf{p}}_{sys}(t+\Delta t) = (\Delta m_s + m_c(t) + \Delta m_c)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c). \quad (12.9)$$

Note that the sand that leaves the car is shown with velocity $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$. This implies that all the sand leaves the car with the velocity of the car at the end of the interval. This is an approximation. Because the sand leaves continuous, the velocity will vary from $\vec{\mathbf{v}}_c(t)$ to $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$ but so does the change in mass of the car and these two contributions to the system's moment exactly cancel. The change in momentum of the system is then

$$\Delta\vec{\mathbf{p}}_{sys} = \vec{\mathbf{p}}_{sys}(t+\Delta t) - \vec{\mathbf{p}}_{sys}(t) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) - m_c(t)\vec{\mathbf{v}}_c(t) = m_c(t)\Delta\vec{\mathbf{v}}_c. \quad (12.10)$$

Throughout the interval a constant force $\vec{\mathbf{F}} = F\hat{\mathbf{i}}$ is applied to the system so the momentum principle becomes

$$\vec{\mathbf{F}} = \lim_{\Delta t\to 0}\frac{\vec{\mathbf{p}}_{sys}(t+\Delta t) - \vec{\mathbf{p}}_{sys}(t)}{\Delta t} = \lim_{\Delta t\to 0} m_c(t)\frac{\Delta\vec{\mathbf{v}}_c}{\Delta t} = m_c(t)\frac{d\vec{\mathbf{v}}_c}{dt}. \quad (12.11)$$

Because the motion is one-dimensional, Equation **??** written in terms of x-components becomes

$$F = m_c(t)\frac{dv_c}{dt}. \quad (12.12)$$

Denote the initial mass of the car by $m_{c,0} = m_c + m_s$ where m_c is the mass of the car and m_s is the mass of the sand in the car at time $t = 0$. The mass of the sand that has left the car at time t is given by

$$m_s(t) = \int_0^t \frac{dm_s}{dt}dt = \int_0^t b\,dt = bt. \quad (12.13)$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt. \quad (12.14)$$

Therefore Equation **??** becomes

$$F = (m_c + m_s - bt)\frac{dv_c}{dt}. \tag{12.15}$$

This equation can be solved for the x-component of the velocity at time t, $v_c(t)$, (which in this case is the speed) by the method of separation of variables. Rewrite Equation **??** as

$$dv_c = \frac{F dt}{(m_c + m_s - bt)}. \tag{12.16}$$

Then integrate both sides

$$\int_{v'_c=0}^{v'_c=v_c(t)} dv'_c = \int_{t'=0}^{t'=t} \frac{F dt'}{m_c + m_s - bt'}. \tag{12.17}$$

Note the agreement of limits on the two definite integrals. Integration yields the speed of the car as a function of time

$$\begin{aligned} v_c(t) &= -\frac{F}{b}\ \ln(m_c + m_s - bt')|_{t'=0}^{t'=t} \\ &= -\frac{F}{b}\ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right) = \frac{F}{b}\ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right), \end{aligned} \tag{12.18}$$

where we used the property that $ln(a) - ln(b) = ln(a/b)$ and therefore $ln(a/b) = -ln(b/a)$. Note that $m_c + m_s \geq m_c + m_s - bt$, so the term $\ln\left(\frac{m_c+m_s}{m_c+m_s-bt}\right) \geq 0$, and the speed of the car increases as we expect.

12.3.3 Filling a Freight Car

Grain is blown into car A from car B at a rate of b kilograms per second. The grain leaves the chute vertically downward, so that it has the same horizontal velocity, u as car b, (Figure **??**). Car A is initially at rest before any grain is transferred in and has mass $m_{A,0}$. At the moment of interest, car A has mass m_A and speed v. Determine an expression for the speed car Aas a function of time t.

Solution: Choose positive x-direction to the right in the direction the cars are moving. Define the system at time t to be the car and grain that is already in it, which together has mass $m_A(t)$, and the small amount of material of mass Δm_g that is blown into car A during the time interval $[t, t + \Delta t]$. At time t, car A is moving with velocity $\vec{\mathbf{v}}_A(t) = v_A(t)\hat{\mathbf{i}}$, and the material blown into car is moving with velocity $\vec{\mathbf{u}} = u\hat{\mathbf{i}}$. At time $t + \Delta t$, car A is moving with velocity $\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A = (v_A(t) + \Delta v_A)\hat{\mathbf{i}}$, and the mass of car A is $m_A(t+\Delta t) = m_A(t) + \Delta m_A$, where $\Delta m_A = \Delta m_g$. The momentum diagrams for times t and $t + \Delta t$ are shown in Figure **??**. The momentum at time t is

$$\vec{\mathbf{p}}_{sys}(t) = m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}}. \tag{12.19}$$

The momentum at time $t + \Delta t$ is

$$\vec{\mathbf{p}}_{sys}(t + \Delta t) = (m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A). \tag{12.20}$$

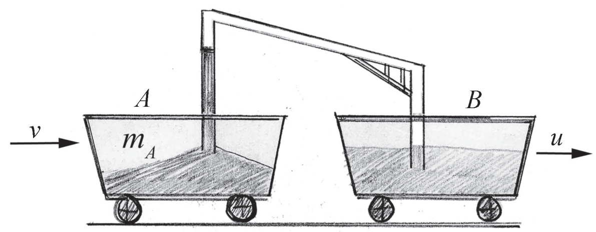

Figure 12.8: Filling a freight car.

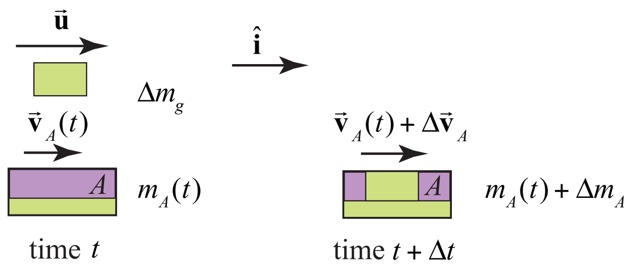

Figure 12.9: Momentum diagrams at times t and $t + \Delta t$.

There are no external forces acting on the system in the x-direction and the external forces acting on the system perpendicular to the motion sum to zero, so the momentum principle becomes

$$\mathbf{0} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{sys}(t + \Delta t) - \vec{\mathbf{p}}_{sys}(t)}{\Delta t}. \tag{12.21}$$

Using the momentum results above, the momentum principle becomes

$$\mathbf{0} = \lim_{\Delta t \to 0} \frac{(m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A) - (m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}})}{\Delta t}. \tag{12.22}$$

Apply the condition that $\Delta m_A = \Delta m_g$ and some rearrangement of terms:

$$\mathbf{0} = \lim_{\Delta t \to 0} \frac{m_A(t)\Delta\vec{\mathbf{v}}_A}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}})}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A \Delta\vec{\mathbf{v}}_A}{\Delta t}. \tag{12.23}$$

In the limit as $\Delta t \longrightarrow 0$, the product $\Delta m_A \Delta\vec{\mathbf{v}}_A$ is a second order differential (the product of two first order differentials) and the term $\Delta m_A \Delta\vec{\mathbf{v}}_A / \Delta t$ approaches zero, therefore the momentum principle yields the differential equation

$$\mathbf{0} = m_A(t)\frac{d\vec{\mathbf{v}}_A}{dt} + \frac{dm_A}{dt}(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}}). \tag{12.24}$$

The x-component of Equation **??** is then

$$0 = m_A(t)\frac{dv_A}{dt} + \frac{dm_A}{dt}(v_A(t) - u). \tag{12.25}$$

Rearranging terms and using the fact that the material is blown into car A at a constant rate $b = dm_A/dt$, we have that the rate of change of the x-component of the velocity of car A is given by

$$\frac{dv_A(t)}{dt} = \frac{b(u - v_A(t))}{m_A(t)}. \tag{12.26}$$

We cannot directly integrate Equation **??** with respect to dt because the mass of the car A is a function of time. In order to find the x-component of the velocity of car A we need to know the relationship between the mass of car A and the x -component of the velocity of the car A. There are two approaches. In the first approach we separate variables in **??** where we have suppressed the dependence on t in the expressions for m_A and v_A yielding

$$\frac{dv_A}{u - v_A} = \frac{dm_A}{m_A}, \tag{12.27}$$

which becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{m'_A=m_{A,0}}^{m'_A=m_A(t)} \frac{dm'_A}{m'_A}. \tag{12.28}$$

where $m_{A,0}$ is the mass of the car before any material has been blown in. After integration we have that

$$\ln\frac{u}{u - v_A(t)} = \ln\frac{m_A(t)}{m_{A,0}}. \tag{12.29}$$

Exponentiate both side yields

$$\frac{u}{u - v_A(t)} = \frac{m_A(t)}{m_{A,0}}. \tag{12.30}$$

We can solve this equation for the x-component of the velocity of the car

$$v_A(t) = \frac{m_A(t) - m_{A,0}}{m_A(t)}u. \tag{12.31}$$

Because the material is blown into the car at a constant rate $b = dm_A/dt$, the mass of the car as a function of time is given by

$$m_A(t) = m_{A,0} + bt. \tag{12.32}$$

Therefore substituting Equation **??** into Equation **??** yields the x-component of the velocity of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt}u. \tag{12.33}$$

In a second approach, we substitute Equation **??** into Equation **??** yielding

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_{A,0} + bt}. \tag{12.34}$$

Separate variables in Equation **??**:

$$\frac{dv_A}{u - v_A} = \frac{bdt}{m_{A,0} + bt}. \tag{12.35}$$

which then becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{t'=0}^{t'=t} \frac{dt'}{m_{A,0} + bt'}. \tag{12.36}$$

Integration yields

$$\ln \frac{u}{u - v_A(t)} = \ln \frac{m_{A,0} + bt}{m_{A,0}}. \tag{12.37}$$

Exponentiate both sides resulting in

$$\frac{u}{u - v_A(t)} = \frac{m_{A,0} + bt}{m_{A,0}}. \tag{12.38}$$

After some algebraic manipulation we can find the speed of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u. \tag{12.39}$$

in agreement with Equation **??**.

Check result

We can rewrite Equation **??** as

$$(m_{A,0} + bt)v_A(t) = btu, \tag{12.40}$$

which illustrates the point that the momentum of the system at time t is equal to the momentum of the grain that has been transferred to the system during the interval $[0, t]$.

12.4 Rocket Propulsion

A rocket at time t is moving with velocity $\vec{\mathbf{v}}_{r,i}$ with respect to a fixed reference frame. During the time interval $[t_i, t_f]$ the rocket continuously burns fuel that is continuously ejected backwards with velocity $\vec{\mathbf{u}}$ relative to the rocket. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction accelerating the rocket forward. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t)$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r/dt. Let $\vec{\mathbf{F}}_{\text{ext}}$ denote the total external force acting on the rocket. We shall use the momentum principle, to determine a differential equation that relates $d\vec{\mathbf{v}}_r/dt$, dm_r/dt, $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}_r(t)$ and $\vec{\mathbf{F}}_{\text{ext}}$, an equation known as the *rocket equation*.

We shall apply the momentum principle during the time interval $[t, t+\Delta t]$ with Δt taken to be a small interval with $t_i < t < t_f$. We shall eventually consider the limit that$\Delta t \to 0$. During this interval, choose as our system the mass of the rocket at time t,

$$m_{sys} = m_r(t) = m_{r,d} + m_f(t). \tag{12.41}$$

where $m_{r,d}$ is the dry mass of the rocket and $m_r(t)$ is the mass of the fuel in the rocket at time t. During the time interval $[t, t+\Delta t]$, a small amount of fuel of mass Δm_f is ejected backwards with velocity $\vec{\mathbf{u}}$ to the rocket. Before the fuel is ejected, it is traveling at the velocity of the rocket and so during the time interval $[t, t+\Delta t]$, the ejected fuel undergoes a change in momentum and the rocket recoils forward. At time $t+\Delta t$ the rocket has velocity $\vec{\mathbf{v}}_r(t+\Delta t)$. Although the ejected fuel continually changes its velocity, we shall assume that the fuel is all ejected at the instant $t+\Delta t$ and then consider the limit as $\Delta t \to 0$. Therefore the velocity of the ejected fuel with respect to the fixed reference frame is the vector sum of the relative velocity of the fuel with respect to the rocket and the velocity of the rocket, $\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t)$. Figure **??** represents momentum diagrams for our system at time t and $t+\Delta t$ relative to a fixed inertial reference frame in which velocity of the rocket at time t is $\vec{\mathbf{v}}_r(t)$. The momentum of the system at time

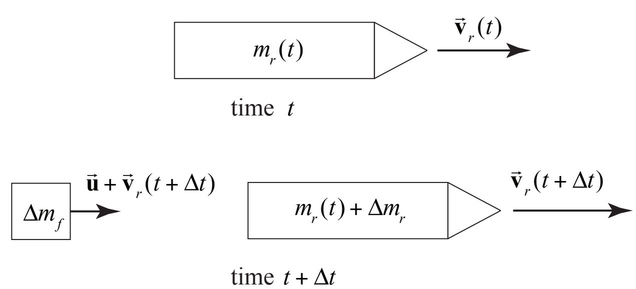

Figure 12.10: Momentum diagrams at times t and $t+\Delta t$.

t is

$$\vec{\mathbf{p}}_{\text{sys}}(t) = m_r(t)\,\vec{\mathbf{v}}_r(t). \tag{12.42}$$

Note that the mass of the system at time t is

$$m_{\text{sys}} = m_r(t). \tag{12.43}$$

The momentum of the system at time $t+\Delta t$ is

$$\vec{\mathbf{p}}_{\text{sys}}(t+\Delta t) = m_r(t+\Delta t)\vec{\mathbf{v}}_r(t+\Delta t) + \Delta m_f\,(\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t)). \tag{12.44}$$

where $m_r(t+\Delta t) = m_r(t) + \Delta m_r$. With this notation the mass of the system at time $t + \Delta t$ is given by

$$m_{\text{sys}} = m_r(t+\Delta t) + \Delta m_f = m_r(t) + \Delta m_r + \Delta m_f\,. \tag{12.45}$$

Because the mass of the system is constant, setting Equation **??** equal to Equation **??** requires that

$$\Delta m_r = -\Delta m_f\,. \tag{12.46}$$

The momentum of the system at time $t + \Delta t$ (Equation **??**) can be rewritten as

$$\begin{aligned}\vec{\mathbf{p}}_{\text{sys}}(t+\Delta t) &= (m_r(t) + \Delta m_r)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r(\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t))s \\ &= m_r(t)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r\vec{\mathbf{u}}\end{aligned}\,. \tag{12.47}$$

We can now apply Newton's Second Law in the form of the momentum principle,

$$\begin{aligned}\vec{\mathbf{F}}_{\text{ext}} &= \lim_{\Delta t\to 0}\frac{(m_r(t)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r\vec{\mathbf{u}}) - m_r(t)\vec{\mathbf{v}}_r(t)}{\Delta t} \\ &= m_r(t)\lim_{\Delta t\to 0}\frac{\vec{\mathbf{v}}_r(t+\Delta t) - \vec{\mathbf{v}}_r(t)}{\Delta t} - \lim_{\Delta t\to 0}\frac{\Delta m_r}{\Delta t}\vec{\mathbf{u}}\end{aligned}\,. \tag{12.48}$$

We now take the limit as

$$\vec{\mathbf{F}}_{\text{ext}} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt} - \frac{dm_r}{dt}\vec{\mathbf{u}}. \tag{12.49}$$

Equation **??** is known as the *rocket equation*.

Suppose the rocket is moving in the positive x-direction with an external force given by $\vec{\mathbf{F}}_{\text{ext}} = F_x\hat{\mathbf{i}}$. Then $\vec{\mathbf{u}} = -u\hat{\mathbf{i}}$, where $u > 0$ is the relative speed of the fuel and it is moving in the negative x-direction. The rocket has velocity $\vec{\mathbf{v}}_r = v_{r,x}\hat{\mathbf{i}}$. Then Equation **??** becomes

$$F_x = m_r(t)\frac{dv_{r,x}}{dt} + \frac{dm_r}{dt}u. \tag{12.50}$$

Note that the rate of decrease of the mass of the rocket, dm_r/dt, is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt}. \tag{12.51}$$

We can rewrite Equation **??** as

$$F_x - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \tag{12.52}$$

The second term on the left-hand-side of Equation **??** is called the *thrust*

$$F_{\text{thrust},x} = -\frac{dm_r}{dt}u = \frac{dm_f}{dt}u. \tag{12.53}$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f/dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x-direction.

12.4.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Equation **??** becomes

$$-\frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \tag{12.54}$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$, then multiply both sides by dt yielding

$$dv_{r,x} = -u\frac{dm_r}{m_r(t)}. \tag{12.55}$$

We now integrate both sides of Equation **??** with limits corresponding to the values of the x-component of the velocity and mass of the rocket at times t_i when the ejection of the burned fuel began and the time t_f when the burn ended,

$$\int\limits_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = -\int\limits_{m'_r=m_{r,i}}^{m'_r=m_{r,f}} \frac{u}{m'_r}dm'_r. \tag{12.56}$$

Performing the integration and substituting in the values at the endpoints yields

$$v_{r,x,f} - v_{r,x,i} = -u\ln\left(\frac{m_{r,f}}{m_{r,i}}\right). \tag{12.57}$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Equation **??** as

$$v_{r,x,f} - v_{r,x,i} = u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{12.58}$$

Note that $\ln(m_{r,i}/m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect. After a slight rearrangement of Equation **??**, we have an expression for the x-component of the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{12.59}$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} = m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \tag{12.60}$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u\ln R. \tag{12.61}$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed then if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt}. \tag{12.62}$$

Then we can integrate Equation **??**

$$\int_{m'_r=m_{r,i}}^{m'_r=m_r(t)} dm'_r = -b \int_{t'=t_i}^{t'=t} dt'. \tag{12.63}$$

yielding an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i). \tag{12.64}$$

For this special case, if we set $t_f = t$ in Equation **??**, then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u \ln\left(\frac{m_{r,i}}{m_{r,i} - bt}\right). \tag{12.65}$$

12.4.2 Example: Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81 \times 10^7\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46 \times 10^7\text{kg}$. The fuel is burned at a constant rate with total burn time is 510 s and ejected at a speed $u = 3000\ \text{m} \cdot \text{s}^{-1}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution:

The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35 \times 10^7\text{kg}$, hence $R = m_{r,i}/m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_r = u \ln R = 6250\ \text{m/s}. \tag{12.66}$$

12.4.3 Example: Two-Stage Rocket

Now suppose that the same rocket in Example **??** burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03 \times 10^7\text{kg}$ with burn time 150 s. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4 \times 10^6\text{kg}$. All the remaining fuel with mass is burned during the second stage with burn time of 360 s. What is the final speed of the rocket after all the fuel has been burned?

Solution:

The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78 \times 10^7\text{kg}$ and $R_1 = m_{r,1,i}/m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u \ln R_1 = 3840 \text{ m/s}. \tag{12.67}$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1 \times 10^6\text{kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3 \times 10^6\text{kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4 \times 10^6 kg$. Then $R_2 = m_{r,2,i}/m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u \ln R_2 = 3340 \text{ m/s}. \tag{12.68}$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u \ln R_1 + u \ln R_2 = u \ln(R_1 R_2) = 7190 \text{ m/s}, \tag{12.69}$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure **??**.

12.4.4 Rocket in a Constant Gravitational Field

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field. Neglect air resistance then the external force is

$$\vec{\mathbf{F}}_{\text{ext}} = m_r \mathbf{g}. \tag{12.70}$$

Choose the positive z-axis in the upward direction then $F_{\text{ext},z}(t) = -m_r(t)g$. Then the rocket equation (Equation **??**) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,z}}{dt}. \tag{12.71}$$

Multiply both sides of Equation **??** by dt, and divide both sides by $m_r(t)$, then

$$dv_{r,z} = -gdt - \frac{dm_r}{m_r(t)}u. \tag{12.72}$$

We now integrate both sides

$$\int_{v'_{r,x}=0}^{v'_{r,x}=v_{r,x}(t)} dv'_{r,x} = -u \int_{m'_r=m_{r,i}}^{m'_r=m_r(t)} \frac{dm'_r}{m'_r} - g\int_{t'=0}^{t'=t} dt'. \tag{12.73}$$

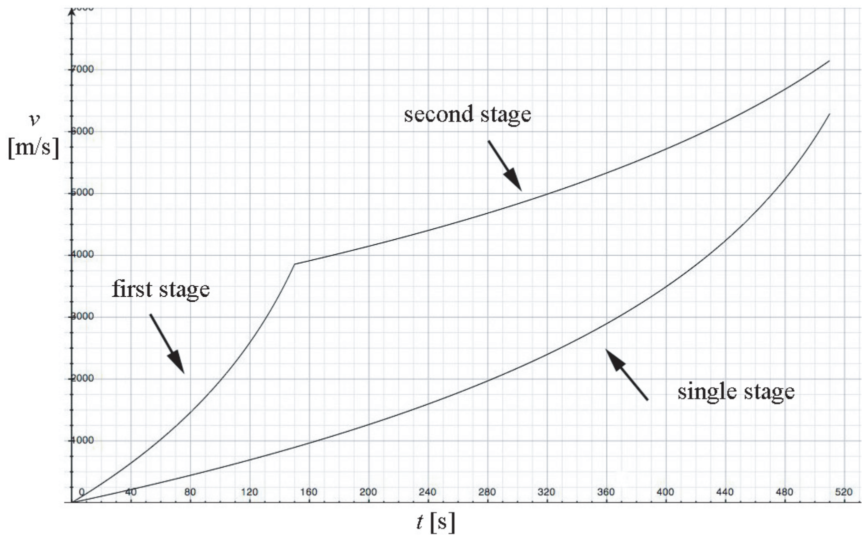

Figure 12.11: Plots of speed of rocket for both one-stage burn and two-stage burn.

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$\int_{v'_{r,x}=0}^{v'_{r,x}=v_{r,x}(t)} dv'_{r,x} = -u \int_{m'_r=m_{r,i}}^{m'_r=m_r(t)} \frac{dm'_r}{m'_r} - g \int_{t'=0}^{t'=t} dt'. \tag{12.74}$$

After all the fuel is burned at $t- = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u \ln R - g t_f . \tag{12.75}$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third turn, and hence the rocket ends up with a larger final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.

If we include a fourth coordinate indicated the time that an event happens at any spatial point, this specifes a *space-time event*.

In particular, the position of a moving body in time can be described by space-time events, with a set of space-time coordinates. You can place an *observer* at the origin of coordinate system. The coordinate system with your observer acts as a *reference frame* for describing the position, velocity, and acceleration of bodies. The position vector of the body depends on the choice of origin (location of your observer) but the displacement, velocity, and acceleration vectors are independent of the location of the observer.