Chapter 11: Momentum, System of Particles, and Conservation of Momentum

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Chapter 11

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Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both [1]

Isaac Newton *Principia*

11.1 Introduction

When we apply a force to an object, through pushing, pulling, hitting or otherwise, we are applying that force over a discrete interval of time, Δt. During this time interval, the applied force may be constant, or it may vary in magnitude or direction. Forces may also be applied continuously without interruption, such as the gravitational interaction between Earth and the Moon. In this chapter we will investigate the relationship between forces and the time during which they are applied, and in the process learn about the quantity of momentum, the principle of conservation of momentum, and its use in

[1]Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 13.

solving a new set of problems involving systems of particles.

11.2 Momentum (Quantity of Motion) and Average Impulse

Consider a point-like object (particle) of mass m, that is moving with velocity $\vec{\mathbf{v}}$ with respect to some fixed reference frame. The quantity of motion or the momentum, $\vec{\mathbf{p}}$, of the object is defined to be the product of the mass and velocity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \tag{11.1}$$

Momentum is a reference frame dependent vector quantity. The direction of momentum is the same as the direction of the velocity. The magnitude of the momentum is the product of the mass with the instantaneous speed.

Units: In the SI system of units, momentum has units of $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$ There is no special name for this combination of units.

Consider the following. During a time interval Δt, a force $\vec{\mathbf{F}}$ is applied to a point-like particle. Newton's Second Law is then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{v}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt}. \tag{11.2}$$

Because we are assuming that the mass of the point-like object does not change, the Second Law can be written as

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{p}}}{dt}. \tag{11.3}$$

The *impulse* of the force $\vec{\mathbf{F}}$ acting on a particle during a time interval Δt is defined as the definite integral of the force from t to $t + |deltat$,

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')dt'. \tag{11.4}$$

The SI units of impulse are $[\mathrm{N} \cdot \mathrm{s}] = [\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}]$ which are the same units as the units of momentum. Apply Newton's Second Law in Equation 11.4:

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')dt' = \int_{t'=t}^{t'=t+\Delta t} \frac{d\vec{\mathbf{p}}'}{dt'}dt' = \int_{\vec{\mathbf{p}}'=\vec{\mathbf{p}}'(t)}^{\vec{\mathbf{p}}'=\vec{\mathbf{p}}'(t+\Delta t)} d\vec{\mathbf{p}}' = \vec{\mathbf{p}}(t+\Delta t) - \vec{\mathbf{p}}(t) = \Delta\vec{\mathbf{p}}. \tag{11.5}$$

Equation 11.5 represents the integral version of Newton's Second Law: the impulse applied by a force during the time interval $[t, t+\Delta t]$ is equal to the change in momentum of the particle during that time interval. The average value of that force during that time interval is given by the integral expression

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{1}{\Delta t} \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')dt'. \tag{11.6}$$

The product of the average force acting on an object and the time interval over which it is applied is called the *average impulse*,

$$\vec{\mathbf{I}}_{ave} = \vec{\mathbf{F}}_{ave}\,\Delta t. \tag{11.7}$$

Multiply each side of Equation 11.6 by Δt with the result that the average impulse applied to a particle during the time interval $[t, t + \Delta t]$ is equal to the change in momentum of the particle during that time interval,

$$\vec{\mathbf{I}}_{ave} = \Delta\vec{\mathbf{p}}. \tag{11.8}$$

11.2.1 Example: impulse for a non-constant force

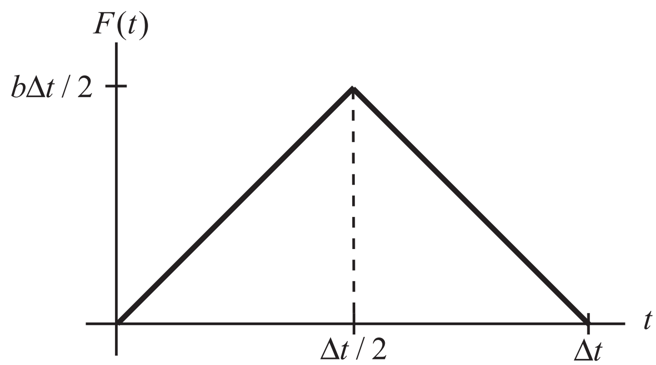

Figure 11.1: Graph of force vs. time.

Suppose you push an object for a time $\Delta t = 1.0$ s in the $= x$ -direction. For the first half of the interval, you push with a force that increases linearly with time according to

$$\vec{\mathbf{F}}(t) = bt\hat{i}, \quad 0 \le t \le 0.5, \tag{11.9}$$

where $b = 2.0 \times 10^1\ \text{N} \cdot \text{s}^{-1}$. Then for the second half of the interval, you push with a linearly decreasing force,

$$\vec{\mathbf{F}}(t) = (d - bt)\hat{\mathbf{i}}, \quad 0.5\,\text{s} \le t \le 1.0\,\text{s}, \tag{11.10}$$

where $d = 2.0 \times 10^1$ N. The force vs. time graph is shown in Figure 11.1. What is the impulse applied to the object?

Answer

We can find the impulse by calculating the area under the force vs. time curve. Since the force vs. time graph consists of two triangles, the area under the curve is easy to calculate and is given by

$$\begin{aligned}\vec{\mathbf{I}} &= \left[\tfrac{1}{2}(b\,\Delta t/2)(\Delta t/2) + \tfrac{1}{2}(b\,\Delta t/2)(\Delta t/2)\right]\hat{\mathbf{i}} \\ &= \tfrac{1}{4}b\,(\Delta t)^2\,\hat{\mathbf{i}} = \tfrac{1}{4}(2.0\times 10^1\ \mathrm{N\cdot s^{-1}})(1.0\,\mathrm{s})^2\,\hat{\mathbf{i}} = (5.0\,\mathrm{N\cdot s})\hat{\mathbf{i}}.\end{aligned} \tag{11.11}$$

11.3 External and Internal Forces and the Change in Momentum of a System

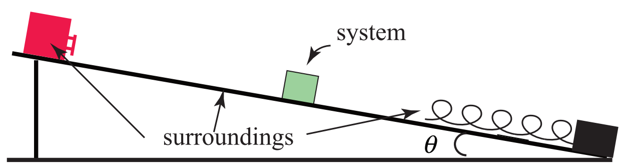

Figure 11.2: Diagram of a block as a system and its surroundings.

So far we have restricted ourselves to considering how the momentum of an object changes under the action of a force. For example, if we analyze in detail the forces acting on the block sliding down an inclined plane (Figure 11.2), we determine that while the block is sliding but before it hits the spring, there are two forces acting on the block: the contact force between the inclined plane and the block, $\vec{\mathbf{C}}_{p,b}$ and the gravitational force $m\vec{\mathbf{g}}$. When the block collides with the spring there is an additional force that the spring applies to the block, $\vec{\mathbf{F}}_{s,b}$. If we choose the block as our system, then everything else acts as the surroundings. We illustrate this division of system and surroundings in Figure 11.2. When the block is in contact with the spring, the forces acting on the cart are *external forces*. The vector sum of these external forces that are applied to the system (the block) is the *total external force*,

$$\vec{\mathbf{F}}_b^{ext} = \vec{\mathbf{F}}_{s,b} + \vec{\mathbf{F}}^g_{e,b} + \vec{\mathbf{C}}_{p,b}. \tag{11.12}$$

Then Newton's Second Law applied to the cart, in terms of impulse, is

$$\Delta\vec{\mathbf{p}}_{sys} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{ext}\,dt \;\equiv\; \vec{\mathbf{I}}_{sys}. \tag{11.13}$$

Let's extend our system to two interacting objects, for example the block and the spring. The forces between the spring and block are now *internal forces*. Both objects, the block and the spring, experience these internal forces, which by Newton's Third Law are equal in magnitude and applied in opposite directions. When we sum up the internal forces for the whole system, they cancel in pairs. Thus the sum of all the internal forces is always zero,

$$\sum_i \vec{\mathbf{F}}_i^{int} = \vec{\mathbf{0}}. \tag{11.14}$$

External forces are still acting on our system; the gravitational force, the contact force between the inclined plane and the block and the force of the support on the spring.. The total force acting on the system is the sum of the internal and the external forces. However, as we have shown, the internal forces cancel, so we have that

$$\vec{\mathbf{F}}_{sys}^{total} = \vec{\mathbf{F}}_{sys}^{ext} + \vec{\mathbf{F}}_{sys}^{int} = \vec{\mathbf{F}}_{sys}^{ext}. \tag{11.15}$$

11.4 System of Particles

Suppose we have a system of N particles labeled by the index i. The force on the i^{th} particle is

$$\vec{\mathbf{F}}_i = \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,\,i}. \tag{11.16}$$

In this expression $\vec{\mathbf{F}}_{j,i}$ is the force on the i^{th} particle due to the interaction between the j^{th} and i^{th} particles. We sum over all j particles with $j \neq i$ because a particle cannot exert a force on itself (equivalently, we could define $\vec{\mathbf{F}}_{i,i} = \vec{\mathbf{0}}$). The the total internal force acting on the i^{th} particle is

$$\vec{\mathbf{F}}_i^{int} = \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,\,i}. \tag{11.17}$$

The force acting on the system is the sum over all the i particles, of the force acting on each particle,

$$\vec{\mathbf{F}}_{sys}^{total} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i^{ext} + \sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{F}}_{sys}^{ext}. \tag{11.18}$$

Note that the double sum vanishes,

$$\sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{0}}, \tag{11.19}$$

because all internal forces cancel in pairs,

$$\vec{\mathbf{F}}_{j,i} + \vec{\mathbf{F}}_{i,\,j} = \vec{\mathbf{0}}. \tag{11.20}$$

The force on the i^{th} particle is equal to the rate of change in momentum of the i^{th} particle,

$$\vec{\mathbf{F}}_i = \frac{d\,\vec{\mathbf{p}}_i}{dt}. \tag{11.21}$$

When can now substitute Equation 11.21 into Equation 11.18 and determine that that the external force is equal to the sum over all particles of the momentum change of each particle,

$$\vec{\mathbf{F}}_{sys}^{ext} = \sum_{i=1}^{i=N} \frac{d\,\vec{\mathbf{p}}_i}{dt}. \tag{11.22}$$

The momentum of the system is given by the sum

$$\vec{\mathbf{p}}_{sys} = \sum_{i=1}^{i=N} \vec{\mathbf{p}}_i. \tag{11.23}$$

Momentum Law

We conclude that the external force causes the momentum of the system to change, and we thus restate and generalize Newton's Second Law for a system of objects as

$$\vec{\mathbf{F}}_{sys}^{ext} = \frac{d\,\vec{\mathbf{p}}_{\mathrm{sys}}}{dt}. \tag{11.24}$$

In terms of impulse, the statement is

$$\Delta\vec{\mathbf{p}}_{\mathrm{sys}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\mathrm{ext}} dt \equiv \vec{\mathbf{I}}. \tag{11.25}$$

11.5 Center of Mass

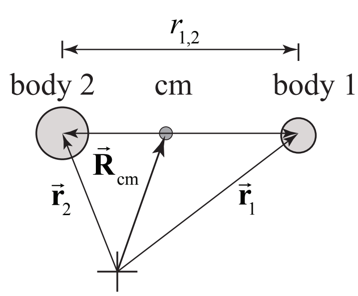

Figure 11.3: Center of mass coordinate system.

Consider two point-like particles with masses m_1 and m_2. Choose a coordinate system such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 11.3). The *center of mass vector*, $\vec{\mathbf{R}}_{cm}$, of the two-body system is defined as

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\,\vec{\mathbf{r}}_1 + m_2\,\vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{11.26}$$

We shall now extend the concept of the center of mass to more general systems. Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. Choose a coordinate system and denote the position of the i^{th} particle as $\vec{\mathbf{r}}_i$. The mass of the system is given by the sum

$$m_{sys} = \sum_{i=1}^{i=N} m_i. \tag{11.27}$$

Define the position of the center of mass of the system of particles is by

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_{sys}} \sum_{i=1}^{i=N} m_i \, \vec{\mathbf{r}}_i. \tag{11.28}$$

For a continuous rigid body, divide the body into infinitesimal point-like particles where each point-like particle has mass dm and is located at the position $\vec{\mathbf{r}}_{dm}$. The center of mass is then defined as an integral over the body,

$$\vec{\mathbf{R}}_{cm} = \frac{\displaystyle\int_{body} dm \, \vec{\mathbf{r}}_{dm}}{\displaystyle\int_{body} dm}. \tag{11.29}$$

11.5.1 Center of mass of the Earth-Moon system

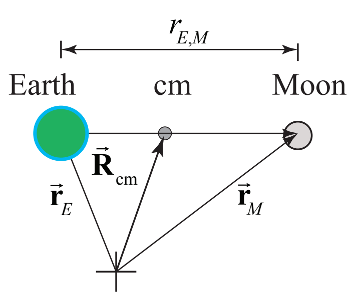

Figure 11.4: Center of mass of Earth-Moon system.

The mean distance from the center of Earth to the center of the Moon is $r_{E,M} = 3.84 \times 10^8 \text{ m}$. The mass of Earth is $m_E = 5.98 \times 10^{24} \text{ kg}$ and the mass of the Moon is $m_M = 7.34 \times 10^{22} \text{ kg}$. The mean radius of Earth is $r_E = 6.37 \times 10^6 \text{ m}$. The mean radius of the Moon is $r_M = 1.74 \times 10^6 \text{ m}$. Where is the location of the center of mass of the Earth-Moon system? Is it inside Earth's radius or outside?

Answer

Denote Earth as body 1 and the Moon as body 2. The center of mass of the Earth-Moon system is

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_{sys}} \sum_{i=1}^{i=2} m_i \vec{\mathbf{r}}_i = \frac{1}{m_E + m_M} (m_E \vec{\mathbf{r}}_E + m_M \vec{\mathbf{r}}_M). \tag{11.30}$$

Choose an origin at the center of Earth and a unit vector $\hat{i}$ pointing towards the moon,

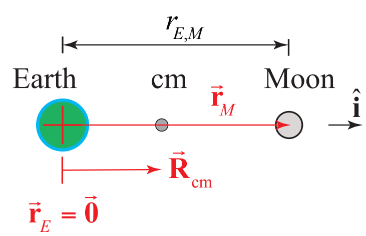

Figure 11.5: Coordinate system with origin at center of Earth.

then $\vec{\mathbf{r}}_E = \vec{\mathbf{0}}$. The center of mass of the Earth-Moon system is then

$$\begin{aligned} \vec{\mathbf{R}}_{cm} &= \frac{m_M \mathbf{r}_M}{m_E + m_M} = \frac{m_M r_{E,M}}{m_E + m_M} \hat{\mathbf{i}} \\ &= \frac{(7.34 \times 10^{22}\ \mathrm{kg})(3.84 \times 10^8\ \mathrm{m})}{(5.98 \times 10^{24}\ \mathrm{kg} + 7.34 \times 10^{22}\ \mathrm{kg})} \hat{\mathbf{i}} = 4.66 \times 10^6\ \mathrm{m}\ \hat{\mathbf{i}} \end{aligned} \tag{11.31}$$

The earth's mean radius is $r_E = 6.37 \times 10^6\ \mathrm{m}$ so the center of mass of the Earth-Moon system lies within the earth.

11.5.2 Example: center of mass of a rod

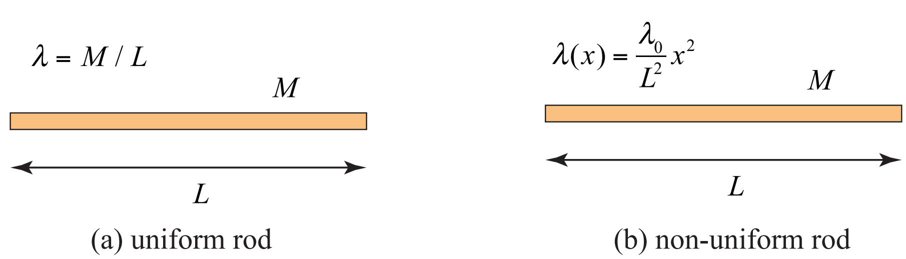

Figure 11.6: (a) Uniform rod; (b) non-uniform rod.

A thin rod has length L and mass M.

(a) Suppose the rod is uniform (Figure 11.6(a)) with mass per unit length λ. Find the position of the center of mass with respect to the left end of the rod.

(b) Now suppose the rod is not uniform (Figure 11.6(b)) with a linear mass density that varies with the distance from the left end according to

$$\lambda(x) = \frac{\lambda_0}{L^2}x^2. \tag{11.32}$$

where λ_0 is a constant and has SI units $[\mathrm{kg}\cdot\mathrm{m}^{-1}$. Find λ_0 and the position of the center of mass with respect to the left end of the rod.

Answer

(a) Choose a coordinate system with the rod aligned along the x-axis and the origin located at the left end of the rod. The center of mass of the rod can be found using the definition given in Equation 11.29. In that expression dm is an infinitesimal mass element and $\vec{\mathbf{r}}_{dm}$ is the vector from the origin to the mass element dm (Figure The vector from the origin to the mass element is $\vec{\mathbf{r}}_{dm} = x'\hat{i}$. The center of mass is found by integration (Equation 11.7)). Choose the infinitesimal mass element dm located a

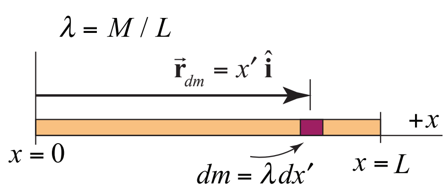

Figure 11.7: Infinitesimal mass element for rod

distance x' from the origin. In this problem x' will be the integration variable. Let the length of the mass element be dx'. The mass per unit length $\lambda = M/L$. Then

$$dm = \lambda dx' = (M/L)dx'. \tag{11.33}$$

The center of mass is then

$$\vec{\mathbf{R}}_{cm} = \frac{1}{M}\int_{rod} \vec{\mathbf{r}}_{dm}\, dm \;= \frac{1}{L}\int_{x'=0}^{x} x'dx'\,\hat{\mathbf{i}} = \frac{1}{2L}\,x'^2\Big|_{x'=0}^{x'=L}\,\hat{\mathbf{i}} = \frac{1}{2L}(L^2-0)\,\hat{\mathbf{i}} = \frac{L}{2}\,\hat{\mathbf{i}}. \tag{11.34}$$

(b) For the non-uniform rod (Figure 11.8), the infinitesimal mass element is

$$dm = \lambda(x')dx' = \lambda = \frac{\lambda_0}{L^2}x'^2dx'. \tag{11.35}$$

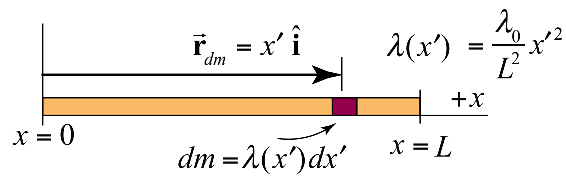

Figure 11.8: Non-uniform rod

The mass is found by integrating the mass element over the length of the rod

$$\begin{aligned} M &= \int_{rod} dm = \int_{x'=0}^{x=L} \lambda(x')dx' = \frac{\lambda_0}{L^2}\int_{x'=0}^{x=L} x'^2 dx' \\ &= \frac{\lambda_0}{3L^2}\, x'^3\Big|_{x'=0}^{x'=L} = \frac{\lambda_0}{3L^2}(L^3 - 0) = \frac{\lambda_0}{3}L \end{aligned} \tag{11.36}$$

Therefore

$$\lambda_0 = \frac{3M}{L}. \tag{11.37}$$

The center of mass is again found by integration

$$\begin{aligned} \vec{\mathbf{R}}_{cm} &= \frac{1}{M}\int_{rod} \mathbf{r}\; dm = \frac{3}{\lambda_0 L}\int_{x'=0}^{x} \lambda(x')x'dx'\;\hat{\mathbf{i}} = \frac{3}{L^3}\int_{x'=0}^{x} x'^3 dx'\;\hat{\mathbf{i}} \\ &= \frac{3}{4L^3}\, x'^4\Big|_{x'=0}^{x'=L}\;\hat{\mathbf{i}} = \frac{3}{4L^3}(L^4 - 0)\;\hat{\mathbf{i}} = \frac{3}{4}L\;\hat{\mathbf{i}}. \end{aligned} \tag{11.38}$$

11.6 Translational Motion of the Center of Mass

The velocity of the center of mass is found by differentiation,

$$\vec{\mathbf{V}}_{cm} = \frac{1}{m_{sys}}\sum_{i=1}^{i=N} m_i\vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}_{sys}}{m_{sys}}. \tag{11.39}$$

The momentum is then expressed in terms of the velocity of the center of mass by

$$\vec{\mathbf{p}}_{sys} = m_{sys}\,\vec{\mathbf{V}}_{cm}. \tag{11.40}$$

We have already determined that the external force is equal to the change of the momentum of the system (Equation 11.24). If we now substitute Equation 11.40 into Equation 11.24, and continue with our assumption of constant masses m_i, we have that

$$\vec{\mathbf{F}}^{ext} = \frac{d\,\vec{\mathbf{p}}_{sys}}{dt} = m_{sys}\frac{d\,\vec{\mathbf{V}}_{cm}}{dt} = m_{sys}\vec{\mathbf{A}}_{cm}. \tag{11.41}$$

where $\vec{\mathbf{A}}_{cm}$, the derivative with respect to time of $\vec{\mathbf{V}}_{cm}$, is the acceleration of the center of mass. From Equation 11.41 we can conclude that in considering the linear motion of the center of mass, the sum of the external forces may be regarded as acting at the center of mass.

11.6.1 Example: forces on a baseball bat

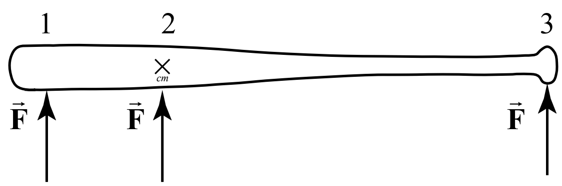

Figure 11.9: Forces acting on a baseball bat.

Suppose you push a baseball bat lying on a nearly frictionless table at the center of mass, position 2, with a force $\vec{\mathbf{F}}$ (Figure 11.9). Will the acceleration of the center of mass be greater than, equal to, or less than if you push the bat with the same force at either end, positions 1 and 3.

Answer

The acceleration of the center of mass will be equal in the three cases. From our

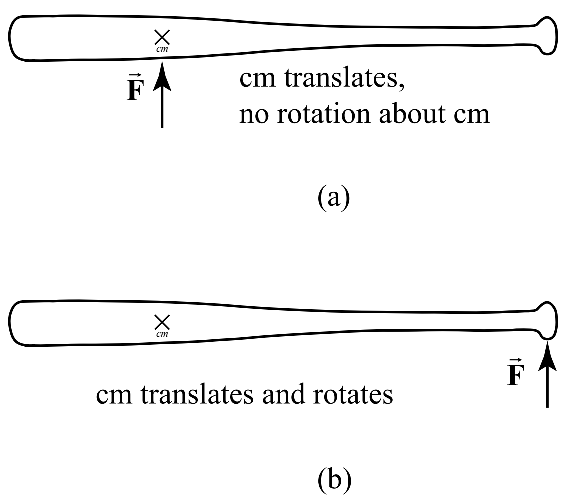

Figure 11.10: (a) Force applied at center of mass; (b)Force applied at end of bat.

previous discussion, (Equation 11.41), the acceleration of the center of mass is independent of where the force is applied. However, the bat undergoes a very different motion if we apply the force at one end or at the center of mass. When we apply the force at the center of mass all the particles in the baseball bat will undergo linear motion (Figure 11.10(a)). When we push the bat at one end, the particles that make up the baseball bat will no longer undergo a linear motion even though the center of mass undergoes linear motion. In fact, each particle will rotate about the center of mass of the bat while the center of mass of the bat accelerates in the direction of the applied force (Figure 11.10(b)).

11.7 Constancy of Momentum and Isolated Systems

Suppose we now completely isolate our system from the surroundings so that there are no external forces acting on the system. Then the system is called an *isolated system*. For an isolated system, the change in the momentum of the system is zero,

$$\Delta\vec{\mathbf{p}}_{sys} = \mathbf{0} \qquad \text{(isolated system)}, \tag{11.42}$$

therefore the momentum of an isolated system is constant. The initial momentum of the system is the sum of the initial momentum of the individual particles,

$$\vec{\mathbf{p}}_{sys,\,i} = m_1\vec{\mathbf{v}}_{1,\,i} + m_2\vec{\mathbf{v}}_{2,\,i} + \cdots. \tag{11.43}$$

The final momentum is the sum of the final momentum of the individual particles,

$$\vec{\mathbf{p}}_{sys,\,f} = m_1\vec{\mathbf{v}}_{1,\,f} + m_2\vec{\mathbf{v}}_{2,\,f} + \cdots. \tag{11.44}$$

Note that the right-hand-sides of Equations. 11.43 and 11.44 are vector sums. When the external force on a system is zero, then the initial momentum of the system equals the final momentum of the system,

$$\vec{\mathbf{p}}_{sys,\,i} = \vec{\mathbf{p}}_{sys,\,f} \tag{11.45}$$

11.8 Momentum Changes and Non-isolated Systems

Suppose the external force acting on the system is not zero, and hence the system is not isolated. By Newton's Third Law, the sum of the forces on the surroundings is equal in magnitude but opposite in direction to the sum of the external force acting on the system,

$$\vec{\mathbf{F}}^{sur} \equiv \sum_i \vec{\mathbf{F}}_i^{sur} = -\sum_i \vec{\mathbf{F}}_i^{ext} \equiv -\vec{\mathbf{F}}^{ext}. \tag{11.46}$$

All the internal forces in the surroundings sum to zero. Thus the sum of the external force acting on the system and the force acting on the surroundings is zero,

$$\vec{\mathbf{F}}^{sur} + \vec{\mathbf{F}}^{ext} = \vec{\mathbf{0}}. \tag{11.47}$$

We have already found (Equation 11.24 that the external force $\vec{\mathbf{F}}^{ext}$ acting on a system is equal to the rate of change of the momentum of the system. Similarly, the force on the surrounding is equal to the rate of change of the momentum of the surroundings. Therefore the momentum of both the system and surroundings is always conserved.

$$\Delta\vec{\mathbf{p}}_{sur} + \Delta\vec{\mathbf{p}}_{sys} = \vec{\mathbf{0}}. \tag{11.48}$$

Principle of Conservation of Momentum

> *For a system and all of the surroundings that undergo any change of state, the change in the momentum of the system and its surroundings is zero.*

11.9 Momentum Problem Solving Strategies

11.9.1 Understand – get a conceptual grasp of the problem

The first question you should ask is whether or not momentum is constant in some system that is changing its state after undergoing an interaction. First you must identify the objects that compose the system and how they are changing their state due to the interaction. As a guide, try to determine which objects change their momentum in the course of interaction. You must keep track of the momentum of these objects before and after any interaction. Second, momentum is a vector quantity so the question of whether momentum is constant or not must be answered in each relevant direction. In order to determine this, there are two important considerations. You should identify any external forces acting on the system. The *Momentum Law* is that an external force is equal to the change in momentum of the system, thus generalizing Newton's Second Law to a system of objects:

$$\vec{\mathbf{F}}^{ext}_{sys} = \frac{d\,\vec{\mathbf{p}}_{\text{sys}}}{dt}. \tag{11.49}$$

Equation 11.49 is a vector equation; if the external force in some direction is zero, then the change of momentum in that direction is zero. In some cases, external forces may act but the time interval during which the interaction takes place is so small that the impulse is small in magnitude compared to the momentum and might be negligible. Recall that the average external impulse changes the momentum of the system

$$\vec{\mathbf{I}}_{ave} = \vec{\mathbf{F}}^{\text{ext}}\Delta t_{\text{int}} = \Delta\vec{\mathbf{p}}_{\text{sys}}. \tag{11.50}$$

If the interaction time is small enough, the momentum of the system is constant, because $\lim\limits_{\Delta t \to 0} \Delta\vec{\mathbf{p}} = \vec{\mathbf{0}}$. If the momentum is not constant then you must apply either Equation 11.49 or Equation 11.50 .If the momentum of the system is constant, then

$$\vec{\mathbf{p}}_{sys,i} = \vec{\mathbf{p}}_{sys,f}\,. \tag{11.51}$$

If there is no net external force in some direction, for example the x-direction, the component of momentum is constant in that direction, and you must apply

$$\vec{\mathbf{p}}_{sys,x,i} = \vec{\mathbf{p}}_{sys,x,f}\,. \tag{11.52}$$

11.9.2 Devise a Plan - set up a procedure to obtain the desired solution

(a) Clearly identify which objects form your system by identifying the objects that change momentum during the interaction.

(b) Draw a diagram that illustrates two states, the state *iommediately before* the interaction, a state called the *initial state*, and the state *immediately after* the interaction, a state called the *final state*|. Sometimes the initial state is called the *before state* and the final state is called the *after state*. This type of diagram is called a *momentum flow diagram*. The diagram should contain all the objects that change momentum between the initial and final states.

(c) In your diagram, identify a set of positive directions and associated unit vectors for the diagram. For each object clearly indicate the momentum in each state with an arrow and a symbol. Choose symbols to identify the mass and velocity of each object in the system. Choose your symbols to reference the state and the partticular object, (this facilitates an easy interpretation). Decide whether you are using components or magnitude s and directions for your velocity symbols. You can choose either approach.

11.9.3 Carry out your Plan

As an example consider a system consisting of two objects of known masses m_1 and m_2 that are initially moving in the $x-y$- plane, and after the interaction they remian in the $x-y$ plane. Assume that there are no external forces acting on the system.

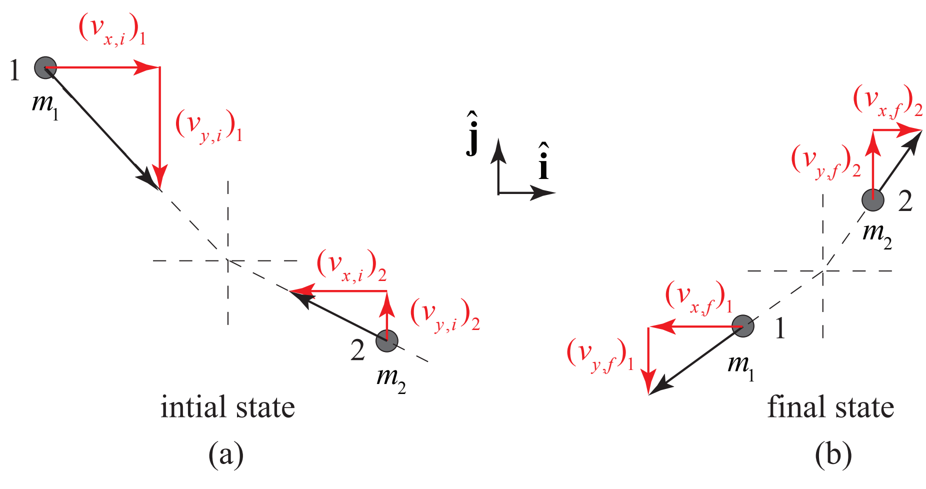

Figure 11.11: (a) Initial state; (b) Final state.

Component approach: For the component approach decompose each velocity vector into its x- and y-components. For example, $(v_{x,i})_1$ represents the x-component of the velocity of object 1 in the initial state and $(v_{x,f})_1$ represents the x-component of the velocity of object 1 in the final state). The y-components of object 1 in the initial and final states are $(v_{y,i})_1$ and $(v_{y,f})_1$. Similar expressions hold for object 2.

The initial x-component of the momentum of the system is

$$p_{\text{sys},\,x,\,i} = m_1(v_{x,\,i})_1 + m_2(v_{x,\,i})_2. \tag{11.53}$$

The final x-component of the momentum is

$$p_{\text{sys},\,x,\,f} = m_1(v_{x,\,f})_1 + m_2(v_{x,\,f})_2. \tag{11.54}$$

The initial y-component of the momentum of the system is

$$p_{\text{sys},\,y,\,i} = m_1(v_{y,\,i})_1 + m_2(v_{y,\,i})_2. \tag{11.55}$$

The final y-component of the momentum is

$$p_{\text{sys},\,y,\,f} = m_1(v_{y,\,f})_1 + m_2(v_{y,\,f})_2. \tag{11.56}$$

Because there are no external forces the components of the momentum are constant. The x-components satisfy the equation

$$m_1(v_{x,\,i})_1 + m_2(v_{x,\,i})_2 = m_1(v_{x,\,f})_1 + m_2(v_{x,\,f})_2. \tag{11.57}$$

The y-components satisfy the equation

$$m_1(v_{y,\,i})_1 + m_2(v_{y,\,i})_2. = m_1(v_{x,\,f})_1 + m_2(v_{x,\,f})_2. \tag{11.58}$$

In the example illustrated by Figure 11.11, suppose you known the initial components of the velocities. Equations 11.57 and 11.58 are two linearly independent equations with four unknown quantities, the final components of the velocities of the two objects, $(v_{x,\,f})_1$, $(v_{y,\,f})_1$, $(v_{x,\,f})_2$ and $(v_{y,\,f})_2$. In order to solve this system of equations you will need two additional conditions.

Suppose you know the final components of the velocities of object 2, $(v_{x,\,f})_2$ and $(v_{y,\,f})_2$. Then Equations 11.57 and 11.58 imply that $(v_{x,\,f})_1$ and $(v_{y,\,f})_1$ are given by

$$\begin{aligned} (v_{x,\,f})_1 &= (v_{x,\,i})_1 + \frac{m_2}{m_1}[(v_{x,\,i})_2 - (v_{x,\,f})_2], \\ (v_{y,\,f})_1 &= (v_{y,\,i})_1 + \frac{m_2}{m_1}[(v_{y,\,i})_2 - (v_{y,\,f})_2]. \end{aligned} \tag{11.59}$$

If the final components of the velocity of object 1 are such that $(v_{x,f})_1 < 0$ and $(v_{y,f})_1 < 0$, then after the interaction, object 1 is moving in the lower left quadrant as in Figure 11.11..

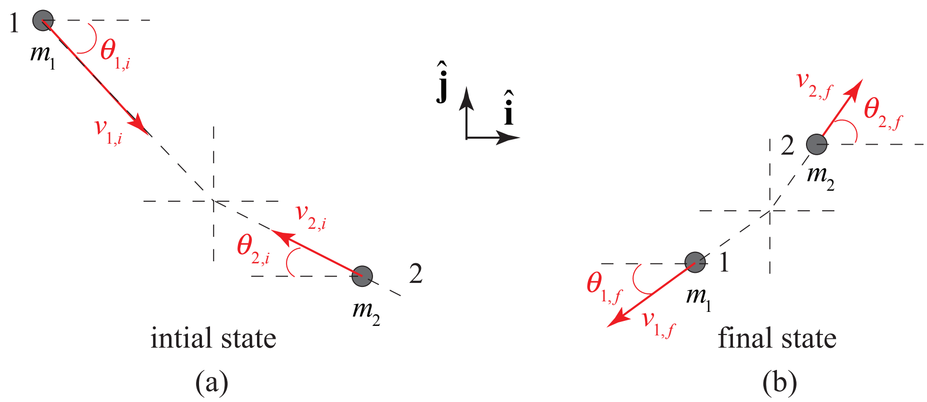

Figure 11.12: (a) Initial state; (b) Final state.

Magnitude and direction approach: In this approach, you represent the direction of velocity vector of each object by arrows and choose the symbol v for the magnitude and θ with respect to the x-axis for the angle. The angle is chosen such that $0 \leq \theta \leq \pi/2$. Depending on your choicefor the direction of the arrow, the component of the momentum can be positive or negative. For example in Figure 11.12, the equation for the constancy of the momentum in the positive $\hat{\mathbf{i}}$-direction is

$$m_1 v_{1,i} \cos(\theta_{1,i}) - m_2 v_{2,i} \cos(\theta_{2,i}) = -m_1 v_{1,f} \cos(\theta_{1,f}) + m_2 v_{2,f} \cos(\theta_{2,f}). \quad (11.60)$$

the equation for the constancy of the momentum in the positive $\hat{\mathbf{j}}$-direction is

$$-m_1 v_{1,i} \sin(\theta_{1,i}) + m_2 v_{2,i} \sin(\theta_{2,i}) = -m_1 v_{1,f} \sin(\theta_{1,f}) + m_2 v_{2,f} \sin(\theta_{2,f}). \quad (11.61)$$

Depending on the given conditions and which quantities you are trying to solve for, you may need to interpret the sign of your answers. For example, suppose the quantity $v_{1,f} < 0$. This means that the final direction of object 1 is opposite your choice of direction indicated by your choice of arrow in your momentum flow diagram..

11.9.4 Look Back – check your solution and method of solution

After you carry out your plan, check your solution, especially dimensions or units and any relevant vector directions. Substitute your calculated quantities into the momentum equations to make sure that you did not make any algebraic errors.

11.10 Worked Examples

11.10.1 Example: Exploding Projectile

An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_i. The projectile breaks into two pieces that fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2 To the surprise of the scientist in charge, the smaller piece returns to Earth at the launching station. Neglect air resistance and effects due to Earth's curvature. How far away, $x_{3,f}$, from the original launching point does the larger piece land?

Answer

We can solve this problem two different ways. The easiest approach utilizes the

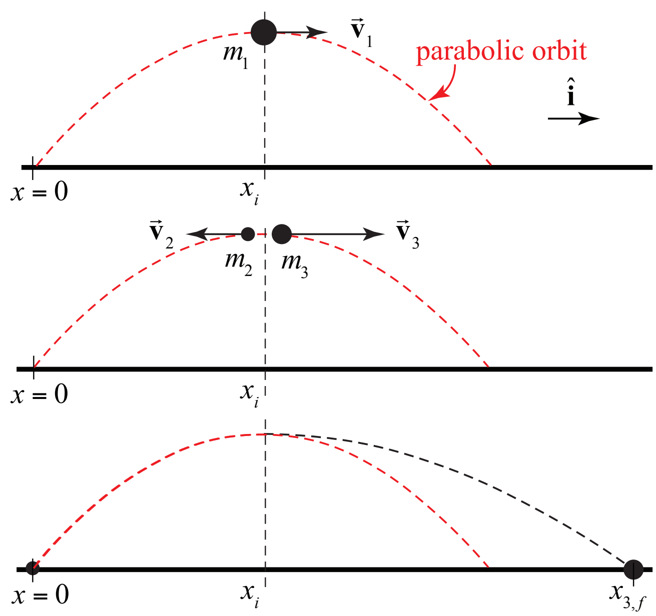

Figure 11.13: Exploding projectile trajectories.

fact that the external force is the gravitational force and therefore the center of mass of the system follows a parabolic trajectory. From the information given in the problem $m_2 = m_1/4$ and $m_3 = 3m_1/4$. Thus when the two objects return to the ground the center of mass of the system has traveled a distance $\vec{\mathbf{R}}_{cm} = 2x_i\hat{\mathbf{i}}$. We now use the definition of center of mass to find where the object with the greater mass hits the ground. Choose an origin at the starting point. The center of mass of the system is given by

$$\vec{\mathbf{R}}_{cm} = \frac{m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_2 + m_3} \,. \tag{11.62}$$

When the objects hit the ground the center of mass is located at $R_{\mathrm{cm}} = 2x_i$, the object with the smaller mass returns to the origin, $\vec{\mathbf{r}}_2 = \vec{\mathbf{0}}$, and the position vector of the other object is $\vec{\mathbf{r}}_3 = x_{3,f}\,\hat{\mathbf{i}}$. Thus the center of mass is

$$2x_i\,\hat{\mathbf{i}} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1/4+3m_1/4} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1} = \frac{3}{4}x_{3,f}\,\hat{\mathbf{i}}\,. \tag{11.63}$$

Therefore

$$x_{3,f} = \frac{8}{3}x_i. \tag{11.64}$$

Neither the vertical height above the ground nor the gravitational acceleration g entered into our answer.

Alternatively, we can use conservation of momentum and kinematics to find the distance traveled. Because the smaller piece returns to the starting point after the collision, the velocity of the smaller piece immediately after the explosion is equal to the negative of the velocity of original object immediately before the explosion. Because the collision is instantaneous, the horizontal component of the momentum is constant during the collision. We can use this to determine the speed of the larger piece after the collision. The larger piece takes the same amount of time to return to the ground as the projectile originally takes to reach the top of the flight. We can therefore determine how far the larger piece traveled horizontally.

We begin by identifying various states in the problem. State 0at time $t_0 = 0$: the projectile is launched.

State 1 at time t_1: the projectile is at the top of its flight trajectory immediately before the explosion. The mass is m_1 and the velocity of the projectile is $\vec{\mathbf{v}}_1 = v_1\hat{\mathbf{i}}$.

State 2 at time $t_2 \approx t_1$: immediately after the explosion, the projectile has broken into two pieces, one of mass m_2 moving backwards (in the negative x -direction) with velocity $\vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_1 = -v_1\hat{\mathbf{i}}$. The other piece of mass m_3 is moving in the positive x-direction with velocity $\vec{\mathbf{v}}_3 = v_3\hat{\mathbf{i}}$.

State 3: the two pieces strike the ground at time $t_3 = 2t_1$, one at the original launch site and the other at a distance $x_{3,f}$ from the launch site. The pieces take the same amount of time to reach the ground $(t_3 - t_2) = t_1$ because both pieces are falling from the same height as the original projectile reached at time t_1, and each has no component of velocity in the vertical direction immediately after the explosion.

The momentum flow diagram with state 1 as the initial state and state 2 as the final state are shown in the upper two diagrams in Figure 11.13.

The initial momentum at time t_1 immediately before the explosion is

$$\vec{\mathbf{p}}_{sys}(t_1) = m_1\,\vec{\mathbf{v}}_1 = m_1 v_1\hat{\mathbf{i}}. \tag{11.65}$$

The momentum at time t_2 immediately after the explosion is

$$\vec{\mathbf{p}}_{sys}(t_2) = m_2\,\vec{\mathbf{v}}_2 + m_3\,\vec{\mathbf{v}}_3 = -\frac{1}{4}\,m_1\vec{\mathbf{v}}_1 + \frac{3}{4}m_1\vec{\mathbf{v}}_3 = -\frac{1}{4}\,m_1 v_1\hat{\mathbf{i}} + \frac{3}{4}m_1 v_3\hat{\mathbf{i}}. \tag{11.66}$$

During the duration of the instantaneous explosion, impulse due to the external gravitational force may be neglected and therefore the momentum of the system is constant. Therefore the constancy of momentum in the horizontal direction is

$$m_1 \, v_1 \hat{\mathbf{i}} = -\frac{1}{4} \, m_1 v_1 \hat{\mathbf{i}} + \frac{3}{4} m_1 v_3 \hat{\mathbf{i}}, \tag{11.67}$$

which can now be solved for the speed of the larger piece immediately after the collision,

$$v_3 = (5/3) v_1 . \tag{11.68}$$

The larger piece travels a distance

$$x_{3,f} = v_3 \, t_1 = \frac{5}{3} v_1 \, t_1 = \frac{5}{3} x_i . \tag{11.69}$$

Therefore the total distance the larger piece traveled from the launching station is

$$x_f = x_i + \frac{5}{3} x_i = \frac{8}{3} x_i , \tag{11.70}$$

in agreement with our approach based on the motion of the center of mass. This is an example of using additional physics concepts to make a calculation easier.