Chapter 31 Non-inertial Reference Frames

Chapter 31 Rotating Reference Frames

31.1 Newton's Second Law in a Non-inertial Reference Frame

Consider an isolated object. This means that there are no physical interactions between the object and the surroundings. According to Newton's First Law an isolated object will undergo uniform motion. Choose a coordinate system such that the isolated body is at rest or is moving with a constant velocity. That coordinate system is called an ***inertial reference frame***. Do such coordinate systems exist? Newton's First Law states that it is always possible to find such a coordinate system. Newton's Second Law $\vec{\mathbf{F}}_{\text{physical}} = m\vec{\mathbf{a}}$ holds only in inertial reference frames, where $\vec{\mathbf{F}}_{\text{physical}}$ are the forces that arise from the interactions of objects.

Concept Question 1: Inertial or Non-inertial Reference Frame

You are in a spaceship with the engines turned off in a zero gravitational field. You are standing on a frictionless floor at rest. Suppose you start to slide backwards. Which of the following statements is true immediately after you start to slide backwards.

1. The spaceship is still an inertial reference frame and has not changed its speed.
2. The spaceship is accelerating backwards.
3. The spaceship is accelerating forwards.

Answer 3. Initially the spaceship defined an inertial reference frame because you, as an isolated, body, remained at rest. Once you start to slide backwards, you conclude that a fictitious force is acting on you in the direction you are moving and hence the spaceship is accelerating in the opposite (forward) direction.

Consider an inertial reference frame O and a second reference frame O' that is accelerating with a **linear acceleration** $\vec{\mathbf{A}}$ with respect to the inertial frame O (Figure 1).

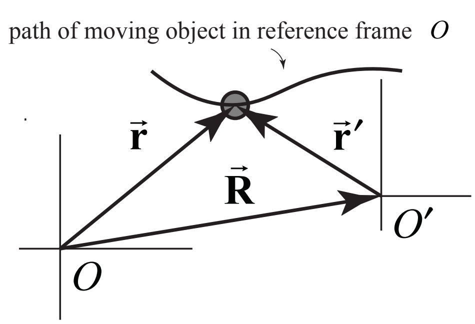


Figure 31.1

At $t=0$, the origins of the two reference frames coincide. Let $\vec{\mathbf{R}}(t)$ denote the position vector of the origin in O' as seen by an observer located at O. Then $\vec{\mathbf{V}}(t) = d\vec{\mathbf{R}}(t)/dt$ and $\vec{\mathbf{A}}(t) = d\vec{\mathbf{V}}(t)/dt$ are the velocity and acceleration of reference frame O' with respect to O. Consider a particle that has acceleration $\vec{\mathbf{a}}(t)$ in O. The path of the moving particle in reference frame O is shown in Figure 1. The position vector $\vec{\mathbf{r}}(t)$ of the object in O is related to the position vector $\vec{\mathbf{r}}'(t)$ of the object in O' by

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}'(t) + \vec{\mathbf{R}}(t) \ . \tag{1}$$

Differentiating Eq. (1) yields the relationship between the velocities of the object in the two frames:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\mathbf{V}}(t) \ . \tag{2}$$

Eq. (2) is called the *Law of Addition of Velocities.* Differentiating Eq. (2) yields the relationship between the accelerations of the object in the two frames:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{a}}'(t) + \vec{\mathbf{A}} \ . \tag{3}$$

Newton's Second Law in the inertial reference frame O, $\vec{\mathbf{F}}_{\text{physical}} = m\vec{\mathbf{a}}$. In the non-inertial frame O', Newton's Second Law needs to be modified, because

$$m\vec{\mathbf{a}}' = m\vec{\mathbf{a}} - m\vec{\mathbf{A}} = \vec{\mathbf{F}}_{\text{physical}} - m\vec{\mathbf{A}} \ . \tag{4}$$

Define a "fictitious force" by

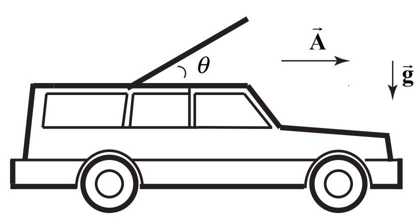
$$\vec{\mathbf{F}}_{\text{fictitious}} = -m\vec{\mathbf{A}} \ . \tag{5}$$

Then the modified Newton's Second Law in the non-inertial reference frame O' becomes

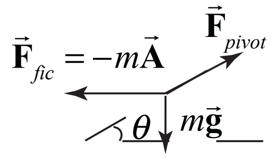
$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}' \ . \tag{6}$$

Example 31.1 Accelerating Car with Hinged Roof

A uniform thin rod of length L and mass m is pivoted at one end. The pivot is attached to the top of a car accelerating at rate $\vec{\mathbf{A}}$. What is the equilibrium value of the angle θ between the rod and the top of the car?



Solution: The free body force diagram on the hinged roof in the accelerating reference frame is shown in the figure below,



where we have added a fictitious force $\vec{\mathbf{F}}_{fic} = -m\vec{\mathbf{A}}$. Because the rod is at rest in the accelerating reference frame, Newton's Second Law becomes

$$m\vec{\mathbf{g}} - m\vec{\mathbf{A}} + \vec{\mathbf{F}}_{pivot} = \vec{\mathbf{0}}.$$

Therefore the pivot force must satisfy $m\vec{\mathbf{g}} - m\vec{\mathbf{A}} + \vec{\mathbf{F}}_{pivot} = -m(\vec{\mathbf{g}} - \vec{\mathbf{A}})$. Note that $\vec{\mathbf{g}}' = \vec{\mathbf{g}} - \vec{\mathbf{A}}$ acts like an effective gravitational field point in the direction given by

$$\theta = \tan^{-1}(g / A)$$

which is the direction that the hinged roof is angled.

31.2 Non-inertial Rotating Reference Frame

Now suppose that O is an inertial reference frame and O' is a reference frame that is rotating with an angular velocity $\vec{\omega}$ with respect to O. We shall consider two types of rotating reference frames. The first example is a reference frame fixed to a platform that is rotating with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$ (Figure 31.2) with respect to an inertial frame O.

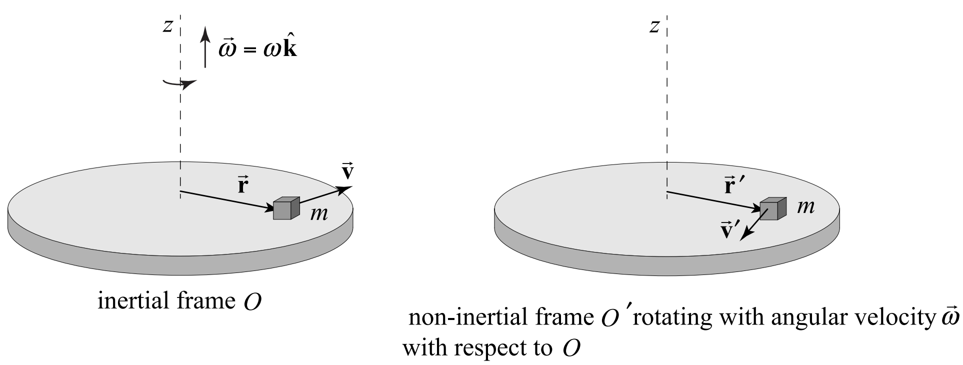


Figure 31.2: Non-inertial reference frame fixed to a rotating platform

The second example is the earth rotating with an angular velocity $\vec{\omega}$ with respect to an inertial frame at rest with respect to the distant stars (Figure 31.3).

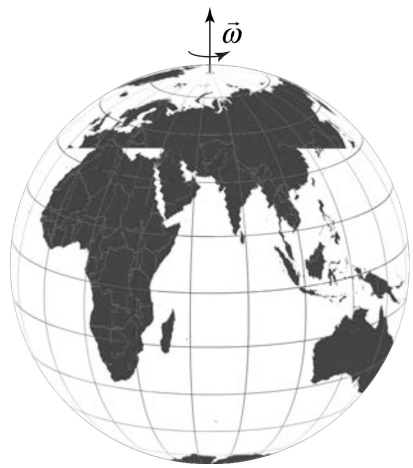


Figure 31.3: Non-inertial reference frame fixed to the earth

Let begin by considering a rotating platform with respect to a reference frame O fixed to the ground, with cylindrical coordinates shown in Figure 2. The platform is rotating about the z-axis with angular velocity $\vec{\boldsymbol{\omega}}$ given by

$$\vec{\boldsymbol{\omega}} = \omega\,\hat{\mathbf{k}}\,. \tag{7}$$

Choose a second reference frame O' that is rotating with angular frequency $\vec{\boldsymbol{\omega}}$ such that the origins of O and O' coincide.

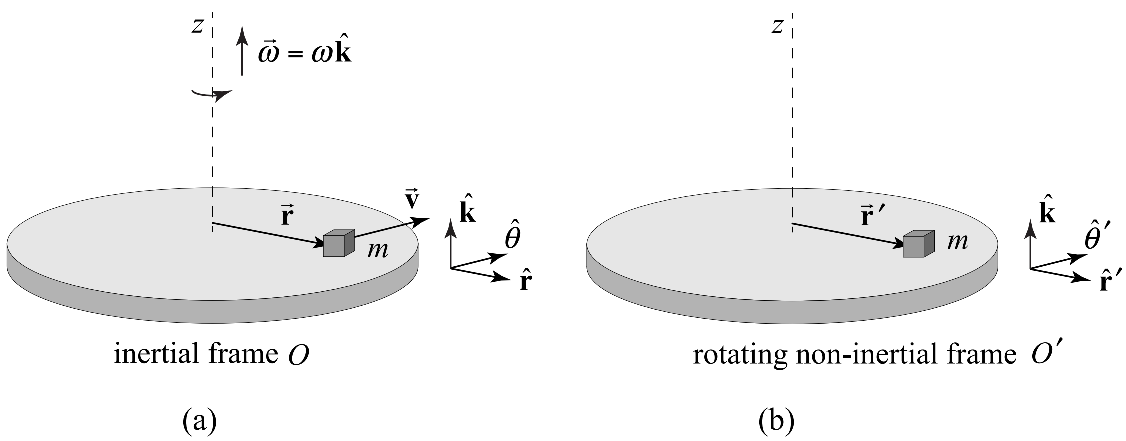


Figure 31.3 (a) Platform in reference frame O fixed to ground; (b) Reference frame O' rotating with angular velocity $\vec{\boldsymbol{\omega}}$ with respect to reference frame O

Now place an object a distance r from the center of the platform, that is at rest with respect to the rotating platform (Figure 2b), $\vec{\mathbf{v}}' = \vec{\mathbf{0}}$. In reference frame O, the object is moving in a circle with velocity $\vec{\mathbf{v}} = r\omega\hat{\theta}$ with respect to reference frame O (Figure 2a). We can write the velocity of the object with respect to reference frame O as

$$\vec{\mathbf{v}} = \omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\omega\hat{\theta}\,. \tag{8}$$

(In appendix 31.A, we provide a more general proof of this result).

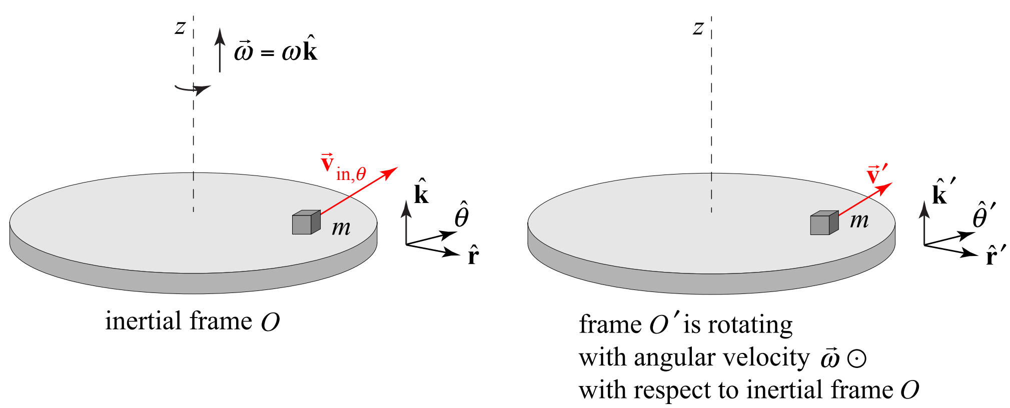
In the following examples we will compare the velocity of an object as seen by an observer in an inertial frame O and a rotating frame O'. In all of the examples, these velocities are related by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}\,. \tag{9}$$

where the velocity of the object in the inertial frame O, $\vec{\mathbf{v}}(t) = (d\vec{\mathbf{r}}\,/\,dt)_{in}$, is the derivative of the position vector $\vec{\mathbf{r}}(t)$ in O, and the velocity of the object in the inertial frame O', $\vec{\mathbf{v}}'(t) = (d\vec{\mathbf{r}}\,/\,dt)_{rot}$, is the derivative of the position vector $\vec{\mathbf{r}}'(t)$ in O'. (In appendix 31.B, we provide a more general proof of this result).

Example 31.2 Moving tangentially on a rotating platform

Consider a platform that is rotating about the z-axis with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform.



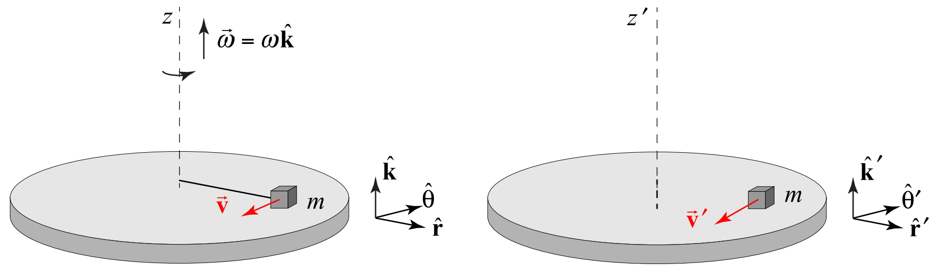
Consider an object of mass m that is moving in a circle of radius r on the platform with a constant tangential velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$ in the inertial frame O, such that $v > r\omega$. What is the velocity of the object $\vec{\mathbf{v}}'$ in the reference frame O'?

Solution: In the instant shown in the figure, the unit vectors in the two frames are equal, and therefore Eq. **(44)** can be written as

$$\vec{\mathbf{v}}'(t) = \vec{\mathbf{v}}(t) - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = v\hat{\boldsymbol{\theta}}' - (\omega\,\hat{\mathbf{k}} \times r\hat{\mathbf{r}}') = (v - r\omega)\hat{\boldsymbol{\theta}}'\,.$$

Note that if $v = r\omega$, then the object is at rest in O'.

Example 31.3 Consider a platform that is rotating about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform.



inertial frame O

frame O' is rotating
with angular velocity $\vec{\omega}$
with respect to inertial frame O

Consider an object of mass m that is moving in a circle of radius r on the platform with a constant tangential velocity $\vec{\mathbf{v}}' = -v'\hat{\boldsymbol{\theta}}'$ in the rotating frame O'. What is the velocity of the object $\vec{\mathbf{v}}$ in the reference frame O?

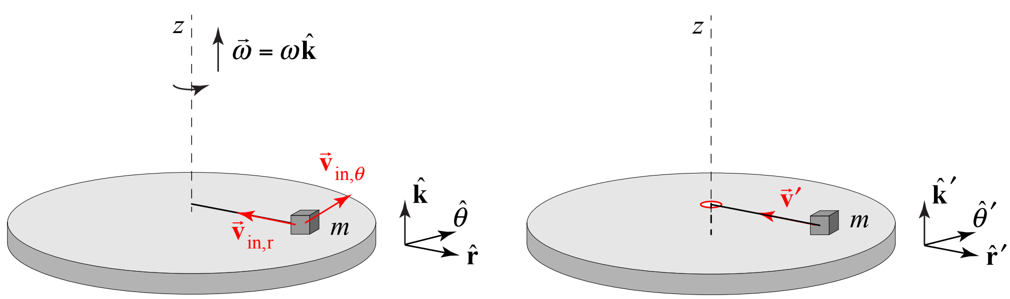
Solution: $\vec{\mathbf{v}}' = -v'\hat{\boldsymbol{\theta}}$. The velocity $\vec{\mathbf{v}}$ in the reference frame O is given by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = -v'\hat{\boldsymbol{\theta}} + (\omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}}) = (-v' + r\omega)\hat{\boldsymbol{\theta}}.$$

Note that if $v' = r\omega$, then the object is at rest in O.

Example 31.4 Moving radially inward on a rotating platform

Consider a platform that is rotating about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform.



inertial frame O

frame O' is rotating
with angular velocity $\vec{\omega}\odot$
with respect to inertial frame O

Consider an object of mass m connected to a string that is pulled radially inward along the surface of the platform at a constant speed v' in O'. At the instant shown in the figure above, the object is at a distance $r = r'$ from the center of the platform. What is the velocity of the object $\vec{\mathbf{v}}$ in the reference frame O?

Solution: $\vec{\mathbf{v}}' = -v'\hat{\mathbf{r}}$. The velocity $\vec{\mathbf{v}}$ in the reference frame O is given by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}(t) = -v'\hat{\mathbf{r}} + (\omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}}) = -v\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}} .$$

31.3 Newton's Second Law for an Object at Rest in a Rotating Reference Frame.

Consider an object of mass m at rest a distance r from the center of a rotating platform (Figure 31.3b) rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ in the reference frame O. There is a static physical friction force, $\vec{\mathbf{F}}_{\text{physical}} = \vec{\mathbf{f}}_s$, acting on the object. The object is undergoing circular motion in O moving with speed $v = r\omega$ and so it is accelerating in the radially inward direction,

$$\vec{\mathbf{a}} = -r\boldsymbol{\omega}^2\hat{\mathbf{r}} = -(v^2 / r)\hat{\mathbf{r}} . \tag{10}$$

Newton's Second Law in O is then

$$\vec{\mathbf{f}}_s = -m(v^2 / r)\hat{\mathbf{r}} . \tag{11}$$

In the non-inertial reference frame O', the object has zero acceleration $\vec{\mathbf{a}}' = \vec{\mathbf{0}}$, but the reference frame O' is accelerating with respect to O, so Newton's Second Law in the non-inertial frame O' (Eq. (6)) is then

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}' , \tag{12}$$

where $\vec{\mathbf{F}}_{\text{physical}} = \vec{\mathbf{f}}_s$, and $\vec{\mathbf{a}}' = \vec{\mathbf{0}}$. Therefore

$$\vec{\mathbf{f}}_s + \vec{\mathbf{F}}_{\text{fictitious}} = \vec{\mathbf{0}} . \tag{13}$$

Substituting Eq. (11) into Eq. (13) yields

$$\vec{\mathbf{F}}_{\text{fictitious}} = mr\boldsymbol{\omega}^2\hat{\mathbf{r}} = (mv^2 / r)\hat{\mathbf{r}} . \tag{14}$$

This fictitious force points radially outward and is called the ***centrifugal fictitious force.*** We can write this force as follows:

$$\vec{\mathbf{F}}_{\text{centrifugal}} \equiv \vec{\mathbf{F}}_{cf} = -m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) \tag{15}$$

Note that

$$\vec{\mathbf{F}}_{cf} = -m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) = -m(\boldsymbol{\omega}\hat{\mathbf{k}} \times (\boldsymbol{\omega}\hat{\mathbf{k}} \times r\hat{\mathbf{r}})) = -m(\boldsymbol{\omega}\hat{\mathbf{k}} \times r\boldsymbol{\omega}\hat{\boldsymbol{\theta}}) = mr\boldsymbol{\omega}^2\hat{\mathbf{r}} = (mv^2 / r)\hat{\mathbf{r}} \quad .(16)$$

Example 31.5 Rotating Water Bucket

In an inertial reference frame O, consider a water bucket that is rotating about the vertical z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. The rotating motion of the bucket is transformed to the fluid contained within, and after a period of time the fluid is rotating with the same angular velocity as the bucket. The surface of the fluid takes on a concave shape (Figure 31.4).

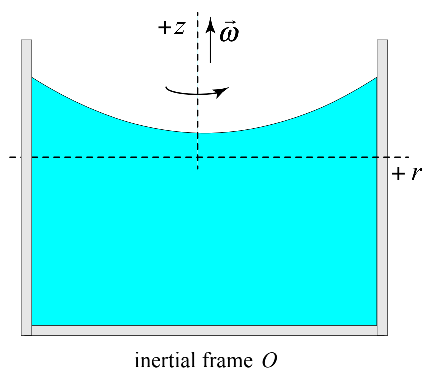


Figure 31.4

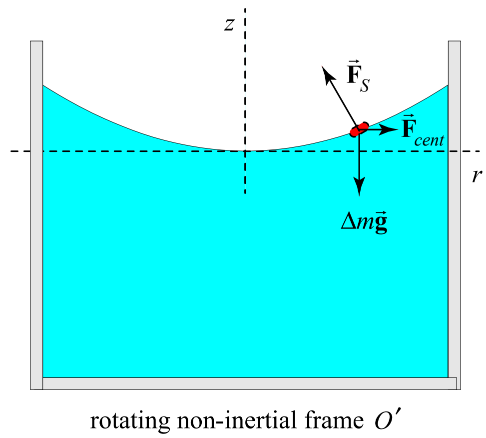


Figure 31.5

In a reference frame O' rotating with the bucket, the water is in static equilibrium. The forces acting on a small surface element of mass Δm, located at the point (r,z), are the gravitational force $\Delta m\vec{\mathbf{g}}$, the fictitious centrifugal force $\vec{\mathbf{F}}_{cf}$, and a hydrostatic force $\vec{\mathbf{F}}_S$ that the rest of the fluid exerts on the fluid element (Figure 31.5). Choose a cylindrical coordinate system with unit vectors $(\hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{k}})$ as shown in Figure 31.10.

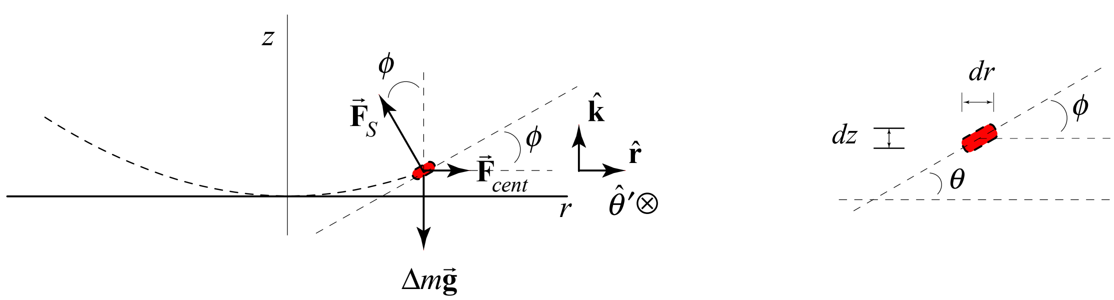


Figure 31.10 Force diagram on fluid element

The tangent line to the surface element makes an angle ϕ with respect to the horizontal axis such that the slope is given by

$$\frac{dz}{dr} = \tan\phi \tag{17}$$

The centrifugal force is given by

$$\vec{\mathbf{F}}_{cf} = -m\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}) = -m(\omega\,\hat{\mathbf{k}}\times(\omega\,\hat{\mathbf{k}}\times(r\,\hat{\mathbf{r}}+z\hat{\mathbf{k}})) = -m(\omega\,\hat{\mathbf{k}}\times r\omega\,\hat{\theta}) = mr\omega^2\hat{\mathbf{r}} .$$

Newton's modified Second Law is the $\hat{\mathbf{r}}$-direction is given by

$$-F_s\sin\phi + mr\omega^2 = 0 \ , \tag{18}$$

and in the $\hat{\mathbf{k}}$-direction is given by

$$F_s\cos\phi - mg = 0 \ . \tag{19}$$

Eqs. (18) and (19) can be solved for $\tan\phi$:

$$\tan\phi = \frac{r\omega^2}{g} \ . \tag{20}$$

Therefore the slope of the surface at the point (r,z) is given by

$$\frac{dz}{dr} = \frac{r\omega^2}{g} \ . \tag{21}$$

Eq. (21) can be integrated

$$\int_{z=0}^{z} dz = \frac{\omega^2}{g}\int_{r=0}^{r} rdr \tag{22}$$

yielding the equation for the surface of the fluid

$$z = \frac{1}{2}\frac{\omega^2}{g}r^2 \ . \tag{23}$$

31.4 Newton's Second Law for an Object Moving with Velocity $\vec{\mathbf{v}}'$ in a Rotating Reference Frame

When an object is moving with velocity $\vec{\mathbf{v}}'$ in the rotating reference frame O', there is an additional fictitious force acting on the object called the ***Coriolis fictitious force*** given by

$$\vec{\mathbf{F}}_{\text{coriolis}} \equiv \vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}' \ . \tag{24}$$

Then the modified Newton's Second law in the rotating frame becomes

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{coriolis}} + \vec{\mathbf{F}}_{\text{centrifugal}} = m\vec{\mathbf{a}}' . \tag{25}$$

See Appendix 31C for a derivation of the two fictitious forces in Eq. (25). Equation (25) will be the starting point for analyzing the motion of particles in a rotating reference frame.

The centrifugal force $\vec{\mathbf{F}}_{\text{centrifugal}}$ is perpendicular to both terms in the cross product $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\omega}} \times \hat{\mathbf{r}}$. Therefore it is perpendicular to the axis of rotation. It is a simple exercise to show that it is also pointing in the radially outward direction from the axis of rotation.

The Coriolis force $\vec{\mathbf{F}}_{\text{coriolis}}$ is perpendicular to $\vec{\boldsymbol{\omega}}$ and the velocity $\vec{\mathbf{v}}'$ of the particle in the rotating frame.

31.5 Motion on the Earth

Introduction: In an inertial reference frame O fixed with respect to the distant stars, the earth is rotating with a period of 23 hours, 53 minutes and 4 seconds corresponding to an angular speed $\omega = \dfrac{2\pi \text{ rad}}{85984 \text{ s}} = 7.307 \times 10^{-5} \text{ rad/sec}$. Choose the positive z-direction to point in the direction of the angular velocity $\vec{\omega}$. In a non-inertial reference frame O' that is rotating with the earth, consider a point located on the surface of the earth at latitude λ. Choose a spherical coordinate system with coordinates (r,θ,ϕ) with associated unit vectors, $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$, as shown in Figure 31.11, with $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.

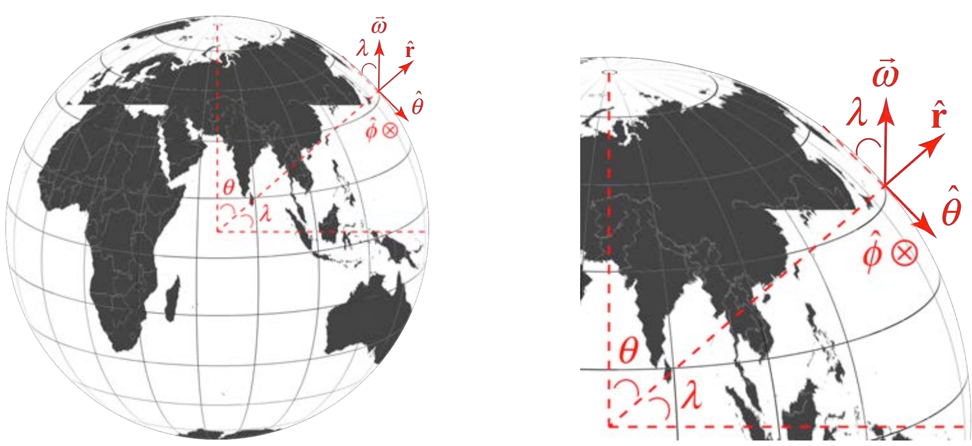


Figure 31.11

At the latitude λ, the angular velocity vector can be written as the vector sum (Figure 31.12):

$$\vec{\boldsymbol{\omega}} = \omega \sin\lambda\, \hat{\mathbf{r}} - \omega\cos\lambda\hat{\theta} = \vec{\boldsymbol{\omega}}_{\perp} + \vec{\boldsymbol{\omega}}_{\|} \;. \qquad (26)$$

where $\vec{\boldsymbol{\omega}}_{\perp} = \omega\sin\lambda\,\hat{\mathbf{r}}$ is the component of the angular velocity perpendicular to the surface of the earth and $\vec{\boldsymbol{\omega}}_{\|} = -\omega\cos\lambda\hat{\theta}$ is the component of the angular velocity tangent to the surface of the earth.

In the rotating reference frame the centrifugal force points radially away from the axis of rotation.

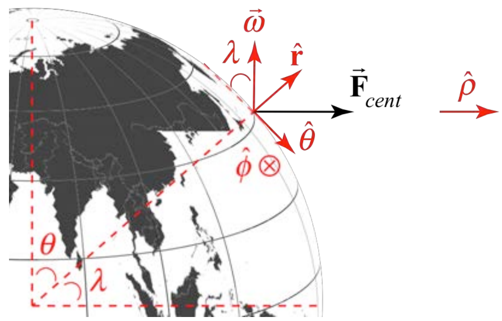


Figure 31.12

Example 31.6 The Centrifugal Force and Corrections to g

(a) Show that $\vec{\mathbf{F}}_{centrifugal} = mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\rho}}$, where $\hat{\boldsymbol{\rho}} = \cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}}$ is the unit vector pointing radially away from the rotation axis (Figure 31.12).

(b) The acceleration due to gravity measured in an earthbound rotating coordinate system is denoted by $\vec{\mathbf{g}}$. However, because of the earth's rotation, $\vec{\mathbf{g}}$ is different from the true acceleration due to gravity $\vec{\mathbf{g}}_0$, where $g_0 = \left|\vec{\mathbf{g}}_0\right|$, and $g_0 = GM_E / R_E^{\;2}$. Assuming that the earth is perfectly round, with radius R_e and angular velocity $\boldsymbol{\Omega}_e$, find $g = \left|\vec{\mathbf{g}}\right|$ as a function of latitude λ. (Assuming the earth to be round is actually not justified; the contributions to the variation of g due to the polar flattening is comparable to the effect calculated here.)

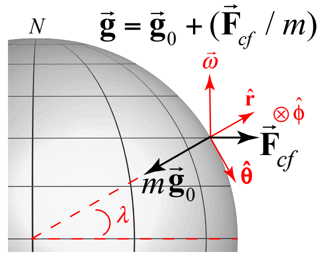
Solution:

(a) Choose coordinates in the rotating frame as shown in Figure 31.12. At the latitude λ, the angular velocity vector is given by Eq. (26): $\vec{\boldsymbol{\omega}} = \omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\theta}$. The position vector is $\vec{\mathbf{r}} = R_E\,\hat{\mathbf{r}}$. Note that $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$. Therefore the centrifugal fictitious force is given by

$$\begin{aligned}
\vec{\mathbf{F}}_{centrifugal} &\equiv \vec{\mathbf{F}}_{cf} = -m(\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}))\\
&= -m((\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times((\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times R_E\,\hat{\mathbf{r}}))\\
&= -m(\boldsymbol{\omega}\sin\boldsymbol{\lambda}\,\hat{\mathbf{r}} - \boldsymbol{\omega}\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\times\boldsymbol{\omega}\cos\boldsymbol{\lambda}R_E\,\hat{\boldsymbol{\phi}})\\
&= m(\boldsymbol{\omega}^2\sin\boldsymbol{\lambda}\cos\boldsymbol{\lambda}R_E\,\hat{\boldsymbol{\theta}} + \boldsymbol{\omega}^2\cos^2\boldsymbol{\lambda}R_E\,\hat{\mathbf{r}})\\
&= mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}(\cos\boldsymbol{\lambda}\,\hat{\mathbf{r}} + \sin\boldsymbol{\lambda}\,\hat{\boldsymbol{\theta}})\\
&= mR_E\boldsymbol{\omega}^2\cos\boldsymbol{\lambda}\,\hat{\boldsymbol{\rho}}
\end{aligned}$$

The centrifugal force points radially away from the axis of rotation as we expect.

(b) The force diagram on the object in the rotating frame is shown in the figure below.



The acceleration due to gravity measured in an earthbound rotating coordinate system is denoted by $\vec{\mathbf{g}} = \vec{\mathbf{g}}_0 + (\vec{\mathbf{F}}_{cf}\,/\,m) = (R_E\omega^2\cos^2\lambda - g_0)\hat{\mathbf{r}} + R_E\omega^2\cos\lambda\sin\lambda\,\hat{\theta}$. The magnitude of $\vec{\mathbf{g}}$ is then

$$\begin{aligned}
g = \left|\vec{\mathbf{g}}\right| &= ((R_E\omega^2\cos^2\lambda - g_0)^2 + (R_E\omega^2\cos\lambda\sin\lambda)^2)^{1/2}\\
&= ((R_E\omega^2\cos^2\lambda)^2 - 2g_0R_E\omega^2\cos^2\lambda + g_0{}^2 + (R_E\omega^2\cos\lambda\sin\lambda)^2)^{1/2}\\
&= g_0\left[\left(\frac{R_E\omega^2\cos^2\lambda}{g_0}\right)^2 - 2\frac{R_E\omega^2}{g_0}\cos^2\lambda + 1 + \left(\frac{R_E\omega^2\cos\lambda\sin\lambda}{g_0}\right)^2\right]^{1/2}
\end{aligned}$$

To simplify the calculation, let $y = R_E\omega^2/g_0$. (Note that $y = R_E\omega^2/g_0 = R_E{}^3\omega^2/GM_E$). Then

$$\begin{aligned}
g &= g_0\left[\left(y\cos^2\lambda\right)^2 - 2y\cos^2\lambda + 1 + \left(y\cos\lambda\sin\lambda\right)^2\right]^{1/2}\\
&= g_0\left[1 - (2y - y^2)\cos^2\lambda\right]^{1/2}
\end{aligned}$$

At the latitude of MIT, $\lambda = 42.36^{\circ}\mathrm{N}$. The mean radius of the earth is $R_E = 6.371\times 10^{6}\,\mathrm{m}$, the angular speed $\omega = 7.307\times 10^{-5}\ \mathrm{rad}\cdot\mathrm{s}^{-1}$, the mass of the earth $M_E = 5.972\times 10^{24}\,\mathrm{kg}$ and the universal gravitational constant is $G = 6.674\times 10^{-11}\ \mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2}$. Then

$$g_0 = (6.674\times 10^{-11}\ \mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2})(5.972\times 10^{24}\,\mathrm{kg})/(6.371\times 10^{6})^2 = 9.82\ \mathrm{m}\cdot\mathrm{s}^{-2}$$

and

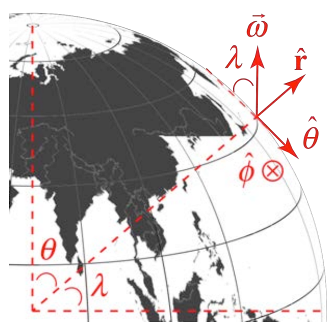
$$\begin{aligned} y &= (6.371\times 10^{6}\,\mathrm{m})^3(7.307\times 10^{-5}\ \mathrm{rad}\cdot\mathrm{s}^{-1})^2/(6.674\times 10^{-11}\ \mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2})(5.972\times 10^{24}\,\mathrm{kg}) \\ &= 3.461\times 10^{-3}. \end{aligned}$$

Then

$$\begin{aligned} g &= g_0\left[\left(y\cos^2\lambda\right)^2 - 2y\cos^2\lambda + 1 + \left(y\cos\lambda\sin\lambda\right)^2\right]^{1/2}. \\ &= g_0\left[1-(2y-y^2)\cos^2\lambda\right]^{1/2} = 9.801\,\mathrm{m}\cdot\mathrm{s}^{-2}. \end{aligned}$$

The actual value of the acceleration due to gravitation at the latitude of MIT based on the International Gravity Formula IGF) 1980 from the parameters of the Geodetic Reference System 1980 (GRS80), which determines the gravity from the position of latitude, is given by $g = 9.80381\,\mathrm{m}\cdot\mathrm{s}^{-2}$.

Example 31.7 Coriolis Fictitious Force Acting on a Moving Particle in Northern Hemisphere



Consider a particle traveling in the northern hemisphere tangent to the surface of the earth with velocity (in the rotating reference frame) $\vec{\mathbf{v}} = v_\theta\hat{\theta} + v_\phi\hat{\phi}$, where $(\hat{\mathbf{r}},\hat{\theta},\hat{\phi})$ are unit vectors in the rotating frame, (v_θ, v_ϕ) are the components of the velocity with speed $v = ({v_\theta}^2 + {v_\phi}^2)^{1/2}$. The Coriolis force is given by

$$\vec{\mathbf{F}}_{\mathrm{cor}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(\vec{\boldsymbol{\omega}}_\perp + \vec{\boldsymbol{\omega}}_{||})\times\vec{\mathbf{v}} = -2m\vec{\boldsymbol{\omega}}_\perp\times\vec{\mathbf{v}} - 2m\vec{\boldsymbol{\omega}}_{||}\times\vec{\mathbf{v}} \tag{27}$$

The contribution from the term $-2m\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}}$ is tangent to the surface of the earth, perpendicular to the velocity, and has magnitude $2m\omega_{\perp} v = 2m\omega\left|\sin\lambda\right|$.

The contribution from the term $-2m\vec{\boldsymbol{\omega}}_{\|} \times \vec{\mathbf{v}}$ is perpendicular to the surface of the earth, and has magnitude $2m\omega\left|\cos\lambda\right|$. This term is quite small compared to the gravitational force and we shall usually ignore its contribution of the motion of particles that are moving tangential to the surface of the earth.

The full vector expression for the Coriolis force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{coriolis} &= -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} = -2m(\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\theta}) \times (v_{\theta}\hat{\theta} + v_{\phi}\hat{\phi}) \\ &= 2m\omega\sin\lambda(-v_{\theta}\hat{\phi} + v_{\phi}\hat{\theta}) + 2m\omega\cos\lambda v_{\phi}\hat{\mathbf{r}}\end{aligned} \tag{28}$$

The component of the Coriolis force tangential to the surface of the earth is given by

$$\vec{\mathbf{F}}_{\text{cor},\|} = 2m\omega\sin\lambda(-v_{\theta}\hat{\phi} + v_{\phi}\hat{\theta}) \tag{29}$$

with magnitude

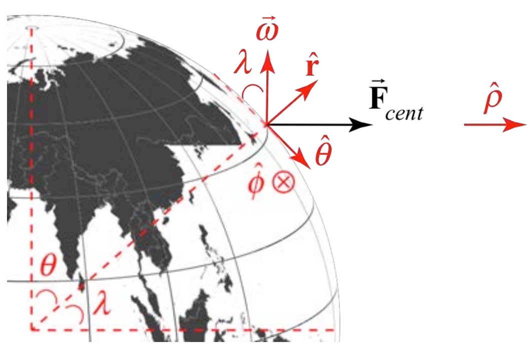
$$\left|\vec{\mathbf{F}}_{\text{cor},\|}\right| = 2m\omega\sin\lambda(v_{\lambda}{}^{2} + v_{\phi}{}^{2})^{1/2} = 2m\omega\sin\lambda v \tag{30}$$

in agreement with our discussion above. The component perpendicular to the surface of the earth is given by

$$\vec{\mathbf{F}}_{\text{cor},\perp} = 2m\omega\cos\lambda\hat{\mathbf{r}}\,. \tag{31}$$

Example 31.8 Direction of Coriolis Force in Northern Hemisphere

Consider a particle moving in the northern hemisphere at north latitude λ. Note that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.



a) If the particle is moving along a longitude line towards the North Pole with velocity $\vec{\mathbf{v}} = -v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

b) If the particle is moving along a longitude line away from the North Pole with velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

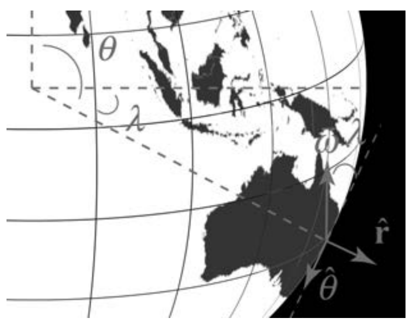
Solution:

a) $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}} = -2m(\omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}) \times (-v_{\theta}\hat{\boldsymbol{\theta}}) = 2m\omega \sin\lambda v_{\theta}\,\hat{\boldsymbol{\phi}}$. It points in the positive $\hat{\phi}$-direction, which is east.

b) $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}} = -2m(\omega \sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}) \times (v_{\theta}\hat{\boldsymbol{\theta}}) = -2m\omega \sin\lambda v_{\theta}\,\hat{\boldsymbol{\phi}}$. It points due west, in the negative $\hat{\phi}$-direction.

Example 31.9 Direction of Coriolis Force in Southern Hemisphere

Consider a particle moving in the southern hemisphere at south latitude λ. Note that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.



a) If the particle is moving along a longitude line away from the South Pole with velocity $\vec{\mathbf{v}} = -v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

b) If the particle is moving along a longitude line towards the South Pole with velocity $\vec{\mathbf{v}} = v\hat{\boldsymbol{\theta}}$, where v is the speed of the particle, find a vector expression for the Coriolis force? Does it point east, west, or some other direction?

Solution:

a) $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(-\omega\sin\lambda\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}})\times(-v\hat{\boldsymbol{\theta}}) = -2m\omega v\sin\lambda\hat{\boldsymbol{\phi}}$. It points in the negative $\hat{\phi}$-direction, which is west.

b) $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(-\omega\sin\lambda\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}})\times(v\hat{\boldsymbol{\theta}}) = +2m\omega v\sin\lambda\hat{\boldsymbol{\phi}}$. Due east in the positive $\hat{\phi}$-direction.

31.6 Trajectories of a Particle in an Inertial and Rotating Frame

Consider an object that is moving at constant velocity in an inertial reference frame O. The trajectory of that object is a straight line. Now consider a platform that is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ that lies beneath that object such that the object passes over the center of the platform. Let O' denote the non-inertial reference frame fixed to the platform i.e. O' is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to O. Choose cylindrical coordinates (r,θ,z) in O'. Let $\vec{\mathbf{v}}' = (v_\theta\hat{\theta} + v_r\hat{\mathbf{r}})$ denote the velocity of the object along the trajectory in O'. (We are dropping the primes for coordinates and component functions in O' to simplify the notation). Note that when the object is moving inward $v_r < 0$ and $v_\theta < 0$, and when the object is moving outward, $v_r > 0$ and $v_\theta < 0$. The Coriolis force is given by

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}' = -2m\omega\,\hat{\mathbf{k}}\times(v_\theta\,\hat{\boldsymbol{\theta}} + v_r\,\hat{\mathbf{r}}) = 2m\omega v_\theta\hat{\mathbf{r}} - 2m\omega v_r\,\hat{\boldsymbol{\theta}} = F_{\mathrm{cor},r}\hat{\mathbf{r}} + F_{\mathrm{cor},\theta}\hat{\boldsymbol{\theta}} \quad (32)$$

Thus when the object is moving inward, the $\hat{\boldsymbol{\theta}}$-component of the Coriolis force is positive, $F_{\mathrm{cor},\theta} > 0$, and the radial component of the Coriolis force is negative $F_{\mathrm{cor},r} < 0$. When the object is moving outward, the $\hat{\boldsymbol{\theta}}$-component of the Coriolis force is negative, $F_{\mathrm{cor},\theta} < 0$ and the radial component of the Coriolis force remains negative $F_{\mathrm{cor},r} < 0$, gradually increasing in magnitude as $|v_\theta|$ gradually increases.

There is also a centrifugal force in the radial direction

$$\vec{\mathbf{F}}_{cf} = -m(\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}})) = -m(\omega\,\hat{\mathbf{k}}\times(\omega\,\hat{\mathbf{k}}\times r\hat{\mathbf{r}})) = -m(\omega\,\hat{\mathbf{k}}\times r\omega\,\hat{\boldsymbol{\theta}}) = mr\omega^2\hat{\mathbf{r}}\,. \qquad (33)$$

In the inertial frame, O, the object is moving with a constant velocity, therefore $\vec{\mathbf{F}}^{total}_{\mathrm{phy}} = \vec{\mathbf{0}}$. Newton's Second Law, $\vec{\mathbf{F}}^{total}_{phy} + \vec{\mathbf{F}}_{cor} + \vec{\mathbf{F}}_{cf} = m\vec{\mathbf{a}}'$, applied to the object in the rotating frame O' is then

$$(2m\omega v_\theta + mr\omega^2)\hat{\mathbf{r}} - 2m\omega v_r\,\hat{\boldsymbol{\theta}} = m\vec{\mathbf{a}}' \qquad (34)$$

Recall that in polar coordinates, the expression for the acceleration of an object is

$$\vec{\mathbf{a}}' = (dv_r / dt - r(d\theta / dt)^2)\hat{\mathbf{r}} + (2v_r(d\theta / dt) + r(d^2\theta / dt^2))\hat{\boldsymbol{\theta}} \ , \tag{35}$$

where $v_r = dr / dt$ and $dv_r / dt = d^2 r / dt^2$. (See Appendix 31.E for a derivation). The equations of notion in the rotating frame are:
]
in the radial direction:

$$2\omega v_\theta + r\omega^2 = dv_r / dt - r(d\theta / dt)^2 \tag{36}$$

and in the tangential direction:

$$-2\omega v_r = 2v_r (d\theta / dt) + r(d^2\theta / dt^2) \ . \tag{37}$$

Let's consider the case in which the initial conditions are given by $(d\theta / dt)_0 = -\omega$ and $v_{r,0} = (dr / dt)_0 = v_{in}$. Then there is a unique solution to Eqs. (36) and (37) given by

$$d\theta / dt = -\omega \ . \tag{38}$$

Using that result in Eq. (36), implies $dv_r / dt = 0$: the radial component of the velocity in O' is constant. This is the condition that the radially component of the Coriolis force and the centrifugal force are equal to the centripetal acceleration. In Figure 31.13, we show the orbit in the two frames under these special conditions.

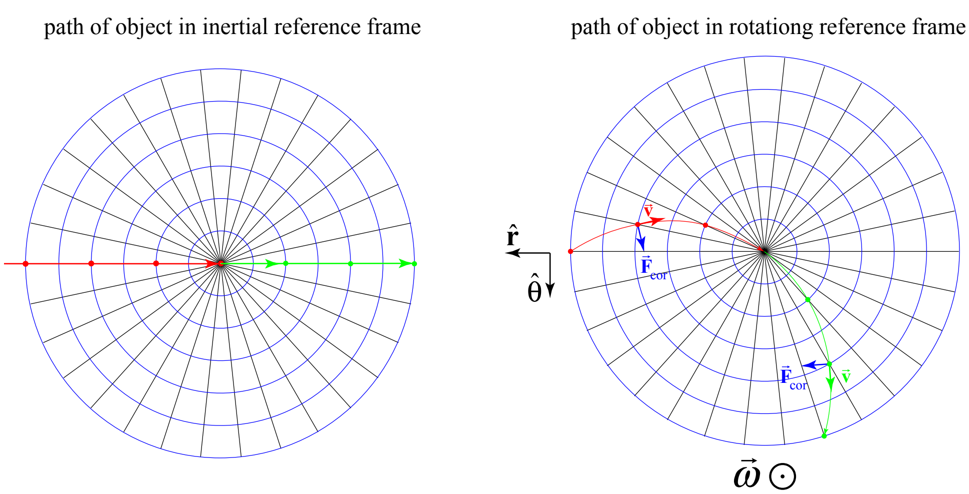


Figure 31.13

The object is moving across a rotating platform at a constant speed v_{in}. The object traverses the platform in time $T_{transit} = 2R / v_{in}$. The platform is rotating with angular speed $\omega = 2\pi / 5T_{transit}$ hence with period $T_{rot} = 2\pi / \omega = 5T_{transit}$. In the inertial reference frame, as the object travels $\Delta s = (1/3)R$ (the distance between two adjacent circles), the platform rotates $\Delta\theta_{platform} = 12^{\circ}$. During each of the these intervals, $\Delta t = (1/3)R / v_{in}$, in the reference frame rotating with the platform, the object appears to decrease it's angular position by $\Delta\theta_{object} = -12^{\circ}$.

The velocity $\vec{\mathbf{v}}'$ of the object O' is no longer constant. The tangent line at any point on trajectory in O' (red line moving inward, green line moving outward) indicates the direction of the velocity $\vec{\mathbf{v}}'$. The direction of $\vec{\mathbf{v}}'$ at various points along the trajectory in O' is shown in Figure 13. Initially, in the frame O the object is moving radially inward. Because the platform is rotating, an observer on the platform also observes that the particle is moving in the negative $\hat{\boldsymbol{\theta}}$-direction. Hence the velocity $\vec{\mathbf{v}}'$ at the initial position in O' has component inward and also in the negative $\hat{\boldsymbol{\theta}}$-direction. As the object moves inward in O', the $\hat{\boldsymbol{\theta}}$-component of the velocity becomes less negative indicating that there is a positive angular acceleration in the $\hat{\boldsymbol{\theta}}$-direction. The observer in O' attributes this angular acceleration to the Coriolis force $\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}'$, which is perpendicular to the velocity $\vec{\mathbf{v}}'$. As the object moves outward, the $\hat{\boldsymbol{\theta}}$-component of the velocity decreases (becomes more negative) indicating that there is a negative $\hat{\boldsymbol{\theta}}$-component to the acceleration.

31.7 Pendulum on a Rotating Platform

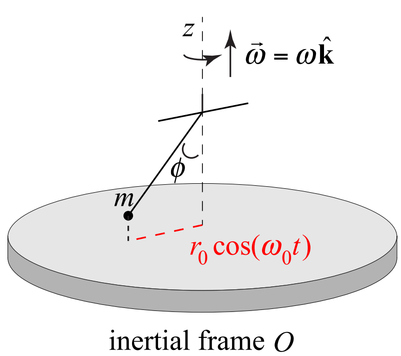


Figure 12a

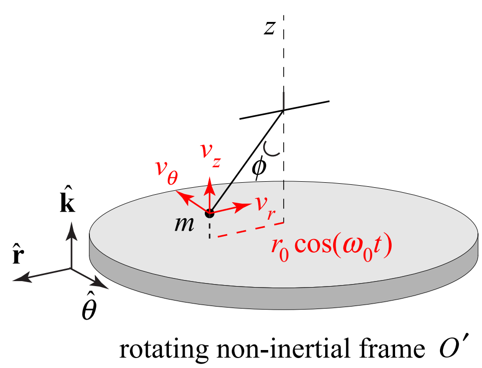


Figure 112b

Now let's consider a pendulum consisting of a bob of mass ϕ at the end of a string of length l. Choose polar coordinates on the rotating platform. Suppose the bob is released from rest at a small angle ϕ_0 with respect to the vertical axis In the frame O, the bob undergoes linear simple harmonic motion with distance from the center varying in time according to

$\vec{\mathbf{r}}(t) = r(t)\hat{\mathbf{r}} + z(t)\hat{\mathbf{k}}$, where $\omega_0 = \sqrt{g/l}$, $r(t) = l\sin(\phi_0)\cos(\phi(t))$, $z(t) = l(1-\cos\phi(t))$, and $\phi(t) = \phi_0\cos(\omega_0 t)$. In a frame O' rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ with respect to O, the motion of the bob is no longer in the radial direction because the platform is rotating underneath the bob. The velocity of the bob in O' has both a radial and tangential component

$$\vec{\mathbf{v}}'(t) = (dr/dt)\hat{\mathbf{r}} + (rd\theta/dt)\hat{\boldsymbol{\theta}} + (dz/dt)\hat{\mathbf{k}}.$$

The acceleration in the rotating cylindrical coordinates is given by

$$\vec{\mathbf{a}}(t) = (d^2r/dt^2 - r(d\theta/dt)^2)\hat{\mathbf{r}} + (2(dr/dt)(d\theta/dt) + r(d^2\theta/dt^2))\hat{\boldsymbol{\theta}}$$
$$+ (d^2z/dt^2)\hat{\mathbf{k}}.$$

Note for simplicity we have dropped the primes on all coordinates and unit vectors in the rotating frame). There is a nonzero Coriolis force given by

$$\vec{\mathbf{F}}_{cor} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}} = -2m(\omega\hat{\mathbf{k}}\times((dr/dt)\hat{\mathbf{r}} + r(d\theta/dt)\hat{\boldsymbol{\theta}} + (dz/dt)\hat{\mathbf{k}}))$$
$$= -2m\omega((dr/dt)\hat{\boldsymbol{\theta}} - r(d\theta'/dt))\hat{\mathbf{r}}.$$

There is also a centrifugal force given by

$$\vec{\mathbf{F}}_{cf} = -m(\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}) = -m(\omega\,\hat{\mathbf{k}}\times(\omega\,\hat{\mathbf{k}}\times r\,\hat{\mathbf{r}})) = -m(\omega\,\hat{\mathbf{k}}\times r\omega\,\hat{\boldsymbol{\theta}}) = mr\omega^2\hat{\mathbf{r}}.$$

The physical forces are given by

$$\vec{\mathbf{F}}_{phy} = (T\cos\phi - mg)\hat{\mathbf{k}} - T\sin\phi\,\hat{\mathbf{r}}.$$

The Coriolis force is in the $\hat{\boldsymbol{\theta}}$-direction, so Newton's Second Law in the $\hat{\boldsymbol{\theta}}$-direction is

$$-2m\omega(dr/dt)\hat{\boldsymbol{\theta}} = (2m(dr/dt)(d\theta/dt) + r(d^2\theta/dt^2))\hat{\boldsymbol{\theta}}.$$

This is a complicated equation but if we make the assumption that $d^2\theta/dt^2 \simeq 0$, then we can solve this equation $d\theta/dt$:

$$d\theta'/dt = -\omega.$$

In the frame O', the bob is precessing in the clockwise direction with angular speed ω. This should not be surprising because in the frame O, the bob is undergoing linear simple harmonic motion and the platform is rotating beneath the bob in the counterclockwise direction with angular speed ω.

Appendix 31A: Relationship between Velocity and Angular Velocity

Now consider a rigid body at time t that is instantaneously rotating about an axis, with unit normal $\hat{\mathbf{n}}$ and angle θ as shown in Figure 31A.1a and angular velocity $\vec{\boldsymbol{\omega}} = (d\theta / dt)\hat{\mathbf{n}}$.

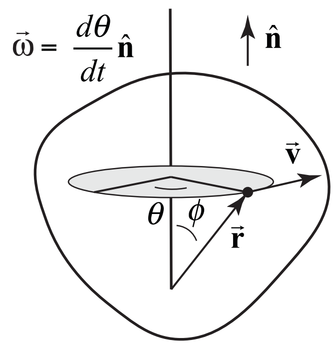


Figure 31A.1: Rigid body undergoing rotation about the instantaneous axis of rotation

The velocity of the rigid body and the angular velocity are related as follows. Every particle in the rigid body is instantaneously undergoing circular motion about the instantaneous axis of rotation (Figure 31A.1), $\vec{\boldsymbol{\omega}} = (d\theta / dt)\,\hat{\mathbf{n}}$. Recall that the position vector $\vec{\mathbf{r}}$ of the particle is constant in length and hence the velocity is given by the derivative

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \left|\vec{\mathbf{r}}\right| \sin\phi \frac{d\theta}{dt}\,\hat{\theta},$$

where $\hat{\theta}$ is a unit vector tangent to the circular path. Note also that

$$\hat{\mathbf{n}} \times \vec{\mathbf{r}} = \left|\vec{\mathbf{r}}\right| \sin\phi\,\hat{\theta}.$$

Therefore the velocity is given by the vector product

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}\ . \tag{39}$$

(Note that $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \frac{d\theta}{dt}\,\hat{\mathbf{n}} \times \vec{\mathbf{r}} = \left|\vec{\mathbf{r}}\right| \sin\phi \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$.)

Appendix 31B: Law of Addition of Velocities for a Rotating Reference Frame

Let O denote an inertial reference frame. Let O' denote a reference frame that is rotating with an angular velocity $\vec{\omega}$ respect to O. Choose a Cartesian coordinate systems for O, with coordinates (x, y, z), and O', with coordinates (x', y', z'), such that the origins of O and O' coincide at time t, and the axis of rotation of O' passes through the origin in the positive $\hat{\mathbf{k}}$-direction, therefore $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. During the time interval $[t, t+\Delta t]$, the x'- and y'-axes have rotated by the angle $\Delta\theta = \omega\Delta t$ as shown in the Figure 31.B1.

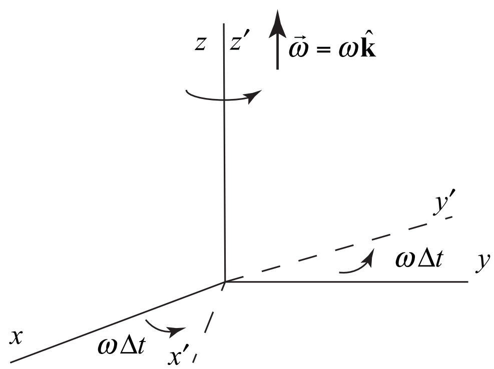
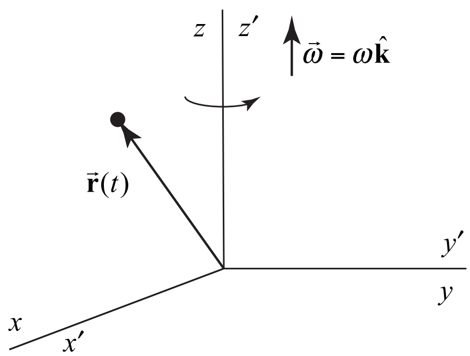


Figure 31.B1: Instantaneous rotation about z- and z'-axes

Consider the motion of a particle as seen by an observer in reference frame O. Suppose at time t, the position of the particle is located in the (x,z) plane. Denote the position vector by $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ (Figure 31.B2a). During the time interval Δt, the particle has moved to the position $\vec{\mathbf{r}}(t+\Delta t)$, with displacement $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)$ (Figure 31.B2b).



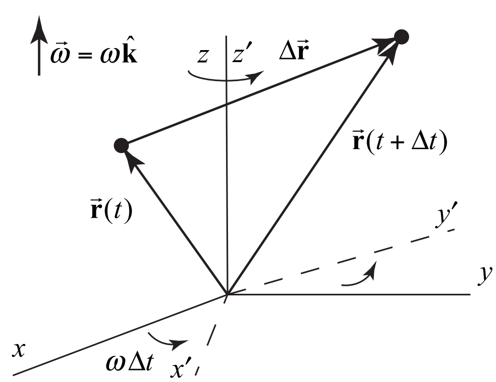


Figure 31.B2a: position at time t

Figure 31.B2b: position at time $t+\Delta t$

In the reference frame O', the position of the particle at time t is given by $\vec{\mathbf{r}}'(t) = x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}$. Because the axes of the two coordinate systems overlap at time t, $x(t) = x'(t)$ and $y(t) = y'(t)$. At time $t + \Delta t$, the position of the particle is given by $\vec{\mathbf{r}}'(t+\Delta t)$. The displacement of the particle in O' is given by $\Delta\vec{\mathbf{r}}' = \vec{\mathbf{r}}'(t+\Delta t) - \vec{\mathbf{r}}'(t)$. This displacement $\Delta\vec{\mathbf{r}}'$ is not equal to the displacement $\Delta\vec{\mathbf{r}}$ in O because the x' and y' axes have rotated by an angle $\Delta\theta = \omega\Delta t$. The initial position vector $\vec{\mathbf{r}}'(t)$ still lies in the (x',z') plane in O' but at time $t + \Delta t$, this vector has rotated with respect to the position $\vec{\mathbf{r}}(t)$ as seen by an observer in O (Figure 31.B3). The lengths of the two vectors $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$ are equal, $\left|\vec{\mathbf{r}}(t)\right| = \left|\vec{\mathbf{r}}'(t)\right|$.

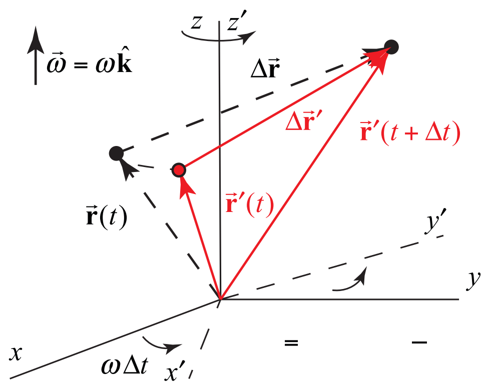


Figure 31.B3: Displacement vectors in O and O'

The difference between the displacement vectors satisfies the vector equality (Figure 31.B4a):

$$\Delta\vec{\mathbf{r}} - \Delta\vec{\mathbf{r}}' = \vec{\mathbf{r}}(t) - \vec{\mathbf{r}}'(t) \tag{40}$$

The vector in Eq. (40) is perpendicular to the axes of rotation (Figure 31.B4b).

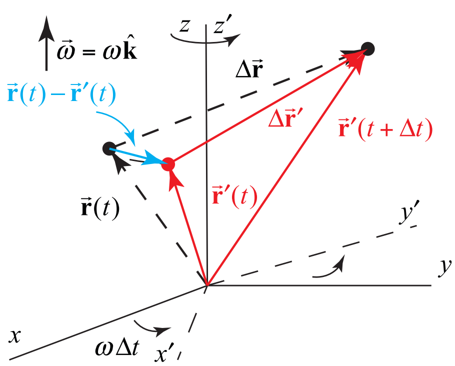


Figure 31.B4a

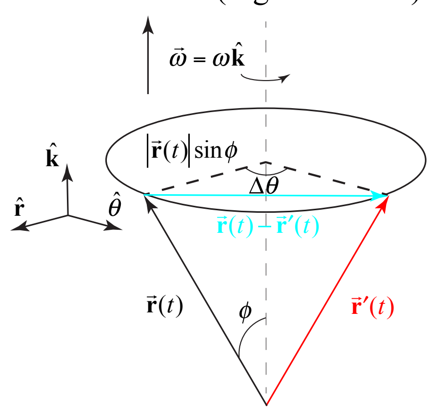


Figure 31.B4b

The magnitude of $\vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t)$ is given by

$$\left|\vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t)\right| = 2\left|\vec{\mathbf{r}}(t)\right|\sin(\phi)\sin(\Delta\theta/2) .$$

In the limit as $\Delta\theta \to 0$, $\sin(\Delta\theta/2)\to\Delta\theta/2$, and thus in the limit the magnitude is given by

$$\left|\vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t)\right| = \left|\vec{\mathbf{r}}(t)\right|\sin(\phi)\Delta\theta . \tag{41}$$

Introduce a set of unit vectors $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\mathbf{k}})$ as shown in Figure 4b. The vector $\vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t)$ is in the $\hat{\boldsymbol{\theta}}-$direction, hence

$$\vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t) = \left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\Delta\boldsymbol{\theta}\,\hat{\boldsymbol{\theta}} .$$

Therefore the difference in displacement vectors is given by

$$\Delta\vec{\mathbf{r}}-\Delta\vec{\mathbf{r}}' = \vec{\mathbf{r}}(t)-\vec{\mathbf{r}}'(t) = \left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\Delta\boldsymbol{\theta}\,\hat{\boldsymbol{\theta}} .$$

Dividing both sides by Δt and taking the limit as $\Delta t \to 0$ yields

$$\lim_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{r}}}{\Delta t} - \lim_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{r}}'}{\Delta t} = \lim_{\Delta t\to 0}\left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\frac{\Delta\boldsymbol{\theta}}{\Delta t}\hat{\boldsymbol{\theta}} .$$

Thus

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \left|\vec{\mathbf{r}}(t)\right|\sin(\boldsymbol{\phi})\boldsymbol{\omega}\,\hat{\boldsymbol{\theta}} \ , \tag{42}$$

where $\omega = d\theta/dt$. In cylindrical coordinates, the vector $\vec{\mathbf{r}}(t) = \left|\vec{\mathbf{r}}(t)\right|\sin\phi\,\hat{\mathbf{r}} + \left|\vec{\mathbf{r}}(t)\right|\cos\phi\,\hat{\mathbf{k}}$. The vector cross product is

$$\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}(t) = \boldsymbol{\omega}\,\hat{\mathbf{k}}\times(\left|\vec{\mathbf{r}}(t)\right|\sin\boldsymbol{\phi}\,\hat{\mathbf{r}} + \left|\vec{\mathbf{r}}(t)\right|\cos\boldsymbol{\phi}\,\hat{\mathbf{k}}) = \boldsymbol{\omega}\left|\vec{\mathbf{r}}(t)\right|\sin\boldsymbol{\phi}\,\hat{\boldsymbol{\theta}} \ . \tag{43}$$

Substituting Eq. (43) into Eq. (42) yields

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}'(t) + \vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}(t) \ , \tag{44}$$

where $\vec{\mathbf{v}} = (d\vec{\mathbf{r}}/dt)_{in}$ is the derivative of the position vector $\vec{\mathbf{r}}(t)$ in the inertial frame and $\vec{\mathbf{v}}' = (d\vec{\mathbf{r}}'/dt)_{rot}$ is the derivative of the position vector $\vec{\mathbf{r}}'(t)$. Eq. (44) is the rotational version of Eq. (2).

Appendix 31.C: Algebraic Derivation of Time Derivative of Vector in Rotating Reference Frame

The components of a vector $\vec{\mathbf{C}}(t)$ can be expressed in any coordinate system, even a rotating coordinate system. However the time derivative of a vector will differ in inertial and rotating coordinate systems. Consider an inertial reference frame and a reference frame O' such that the origins and z and z' axes of O and O' coincide, and O' is rotating with angular frequency $\vec{\boldsymbol{\omega}} = (d\theta / dt)\hat{\mathbf{k}}$ with respect to an inertial frame O.

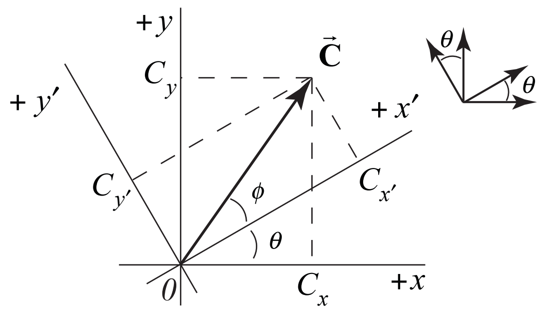


Figure 31.C.1

Let $\vec{\mathbf{C}}(t)$ be a non-constant vector lying in the $x-y$ plane (also lying in the $x'-y'$ plane). The vector expression for $\vec{\mathbf{C}}(t)$ in O is given by

$$\vec{\mathbf{C}}(t) = C_x(t)\hat{\mathbf{i}} + C_y(t)\hat{\mathbf{j}} \ , \tag{45}$$

and in O' by

$$\vec{\mathbf{C}}(t) = C_{x'}(t)\hat{\mathbf{i}}' + C_{y'}(t)\hat{\mathbf{j}}' \ . \tag{46}$$

The component functions in Eqs. (45) and (46) are related by

$$\begin{aligned} C_x &= \left|\vec{\mathbf{C}}\right|\cos(\theta+\phi) = \left|\vec{\mathbf{C}}\right|\cos(\theta)\cos(\phi) - \left|\vec{\mathbf{C}}\right|\sin(\theta)\sin(\phi) \\ &= C_{x'}\cos(\theta) - C_{y'}\sin(\theta) \end{aligned} \ , \tag{47}$$

and

$$\begin{aligned} C_y &= \left|\vec{\mathbf{C}}\right|\sin(\theta+\phi) = \left|\vec{\mathbf{C}}\right|\sin(\theta)\cos(\phi) - + \left|\vec{\mathbf{C}}\right|\cos(\theta)\sin(\phi) \\ &= C_{x'}\sin(\theta) + C_{y'}\cos(\theta) \end{aligned} \ , \tag{48}$$

The corresponding inverse transformations are

$$C_{x'} = C_x\cos(\theta) + C_y\sin(\theta) \ , \tag{49}$$

and

$$C_{y'} = -C_x \sin(\theta) + C_y \cos(\theta) \ , \tag{50}$$

Recall the vector decomposition expression for $\hat{\mathbf{i}}'(t)$ and $\hat{\mathbf{j}}'(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are given by

$$\hat{\mathbf{i}}'(t) = \cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}} \ , \tag{51}$$

$$\hat{\mathbf{j}}'(t) = -\sin(\theta)\hat{\mathbf{i}} + \cos(\theta)\hat{\mathbf{j}} \tag{52}$$

Then

$$\begin{aligned}
&\vec{\mathbf{C}}(t) = C_{x'}(t)\hat{\mathbf{i}}' + C_{y'}(t)\hat{\mathbf{j}}' \\
&= (C_x \cos(\theta) + C_y \sin(\theta))(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}) \\
&+(-C_x \sin(\theta) + C_y \cos(\theta))(\hat{\mathbf{j}}'(t) = -\sin(\theta)\hat{\mathbf{i}} + \cos(\theta)\hat{\mathbf{j}}) \\
&= C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}
\end{aligned} \ . \tag{53}$$

Let's now compare the time derivatives of the vector $\vec{\mathbf{C}}$ in the inertial and rotating reference frames. The time derivative of the vector $\vec{\mathbf{C}}$ in the inertial reference frame O is

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{in} = \frac{d}{dt}(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}) = \frac{dC_x}{dt}\hat{\mathbf{i}} + \frac{dC_y}{dt}\hat{\mathbf{j}} \ , \tag{54}$$

because the unit vectors are constant in time . The time derivative of the vector $\vec{\mathbf{C}}$ in the rotating reference frame O' is

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{rot} = \frac{dC_{x'}}{dt}\hat{\mathbf{i}}' + \frac{dC_{y'}}{dt}\hat{\mathbf{j}}' + C_{x'}\frac{d\hat{\mathbf{i}}'}{dt} + C_{y'}\frac{d\hat{\mathbf{j}}'}{dt} \ . \tag{55}$$

Note that exactly like the case with the component functions

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{in} = \frac{dC_x}{dt}\hat{\mathbf{i}} + \frac{dC_{y'}}{dt}\hat{\mathbf{j}} = \frac{dC_{x'}}{dt}\hat{\mathbf{i}}' + \frac{dC_{y'}}{dt}\hat{\mathbf{j}}' \tag{56}$$

Recall that

$$\frac{d\hat{\mathbf{i}}'}{dt} = -\sin(\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos(\theta)\frac{d\theta}{dt}\hat{\mathbf{j}} = \frac{d\theta}{dt}\hat{\mathbf{j}}' \ , \tag{57}$$

and

$$\frac{d\hat{\mathbf{j}}'}{dt} = -\cos(\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} - \sin(\theta)\frac{d\theta}{dt}\hat{\mathbf{j}} = -\frac{d\theta}{dt}\hat{\mathbf{i}}' \tag{58}$$

The time derivative in the rotating frame then becomes

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{rot} = \left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{in} - \frac{d\theta}{dt}(C_{y'}\hat{\mathbf{i}}' - C_{x'}\hat{\mathbf{j}}') \ . \tag{59}$$

The second term in Eq. (59) is just $-\vec{\boldsymbol{\omega}} \times \vec{\mathbf{C}} = -\frac{d\theta}{dt}\hat{\mathbf{k}} \times (C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}') = -\frac{d\theta}{dt}(C_{x'}\hat{\mathbf{j}}' - C_{y'}\hat{\mathbf{i}})$. Therefore Eq. (59) becomes

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{rot} = \left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{in} - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{C}} \ . \tag{60}$$

Eq. (60) is then the general result for the time derivative of a vector in a rotating reference frame.

Example 31.D.1: Let $\vec{\mathbf{r}}(t)$ be the position vector of an object, let $\vec{\mathbf{v}}(t) = (d\vec{\mathbf{r}} / dt)_{in}$ denote the velocity of the object in the inertial frame O, and let $\vec{\mathbf{v}}'(t) = (d\vec{\mathbf{r}} / dt)_{rot}$ denote the velocity of the object in the rotating frame O', Then using Eq. (60), the two velocities are related by

$$\vec{\mathbf{v}}' = \mathbf{v} - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \ , \tag{61}$$

in agreement with Eq. (44).

Appendix 31.E Acceleration in Polar Coordinates

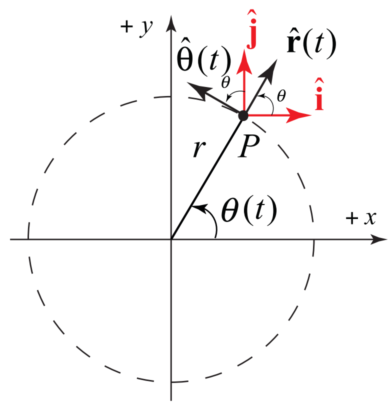


Figure 31.E1

Let's now consider central motion in a plane that is non-circular. In polar coordinates, the key point is that the time derivative dr/dt of the position function r is no longer zero. The second derivative d^2r/dt^2 also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is given by

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}\,. \tag{62}$$

Because $dr/dt \neq 0$, when we differentiate Eq. (62), we need to use the product rule

$$\vec{\mathbf{v}} = \frac{d\,\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\,\hat{\mathbf{r}}}{dt}\,. \tag{63}$$

At the point P, consider two sets of unit vectors $(\hat{\mathbf{r}}(t), \hat{\boldsymbol{\theta}}(t))$ and $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$, as shown in the figure above. The vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\,, \tag{64}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\,. \tag{65}$$

The time derivative of the unit vectors are given by

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\,) = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\,. \tag{66}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}(\cos\theta(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}}\,) = -\frac{d\theta}{dt}\hat{\mathbf{r}}\,. \tag{67}$$

Substituting Eq. (66) into Eq. (63) yields

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}\,. \tag{68}$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}\,. \tag{69}$$

In order to determine the acceleration, we now differentiate Eq. (68), again using the product rule, which is now a little more involved:

$$\vec{\mathbf{a}} = \frac{d\,\vec{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\,\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\,\hat{\boldsymbol{\theta}}}{dt}\,. \tag{70}$$

Now substitute Eqs. (66) and (67) for the time derivatives of the unit vectors in Eq. (70), and after collecting terms yields

$$\begin{aligned}\vec{\mathbf{a}} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} \\ &= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}\end{aligned}\,. \tag{71}$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of dr/dt and d^2r/dt^2. The radial component is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\,. \tag{72}$$

and the tangential component is

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\,. \tag{73}$$

The second term in the radial component of acceleration is called the centripetal acceleration. The first term in the tangential component of the acceleration, $2(dr/dt)(d\theta/dt)$ has a special name, the *coriolis acceleration*,

$$a_{cor} = 2\frac{dr}{dt}\frac{d\theta}{dt}\,. \tag{74}$$