Chapter 2 Units, Dimensional Analysis, Problem Solving, and Estimation

Peter Dourmashkin

September 6, 2023

Contents

Chapter 2

Units, Dimensional Analysis, Problem Solving, and Estimation

> But we must not forget that all things in the world are connected with one another and depend on one another, and that we ourselves and all our thoughts are also a part of nature. It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the change of things; made because we are not restricted to any one definite measure, all being interconnected. A motion is termed uniform in which equal increments of space described correspond to equal increments of space described by some motion with which we form a comparison, as the rotation of the earth. A motion may, with respect to another motion, be uniform. But the question whether a motion is in itself uniform, is senseless. With just as little justice, also, may we speak of an "absolute time" ¬— of a time independent of change. This absolute time can be measured by comparison with no motion; it has therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception.
>
> Ernst Mach[1]

2.1 International System of Units

The system of units most commonly used throughout science and technology today is the Système International (SI). The seven base quantities and their corresponding base units, are shown in Table 2.1.

[1]E. Mach, *The Science of Mechanics*, translated by Thomas J. McCormack, Open Court

Table 2.1: Base Units

Base Quantity	**Base Unit**
Time	second (s)
Length	length (m)
Mass	kilogram (kg)
Electric Current	ampere (A)
Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

Many physical quantities are then derived from the base quantities by a set of algebraic relations defining the physical relation between these quantities. Mechanics is based on just the first three of these quantities, the second, the meter, and the kilogram, MKS or meter-kilogram-second system. An alternative metric system, still widely used, is the CGS system (centimeter-gram-second).

Example 2.1: Derived units for velocity, acceleration, force, energy, and power

Velocity is defined to be the rate of change of position with respect to time and therefore has the derived SI unit $m \cdot s^{-1}$. Acceleration is defined to be the rate of velocity with respect to time and therefore has the derived SI unit $m \cdot s^{-2}$. The *newton*, symbol N, is the derived SI unit for force which is equal to the product of mass with acceleration. Therefore force has the derived SI unit $kg \cdot m \cdot s^{-1}$. The *joule*, symbol J, is the derived SI unit for energy which is equal to the product of force with distance. Therefore the derived SI unit for energy is $kg \cdot m^2 \cdot s^{-1}$. The *watt*, symbol W, is the derived SI unit for power which is equal to the rate of change of energy with respect to time. Therefore the derived SI unit for power is $kg \cdot m^2 \cdot s^{-2}$.

The base quantities were originally determined by experiment with uncertainties in their values. Gradually the base units were no longer defined by physical prototypes like the standard kilogram or standard meter bar but were defined by asset of constants. As of May 20, 2019, all the base SI units are now defined in terms of seven $constants$ shown in Table 2.2.

Table 2.2: Defining Constants

Quantity	**Symbol**	**Numerical Value**	**SI Units**
Cesium hyperfine frequency	$\Delta\nu_{Cs}$	9192631770	s^{-1}
Speed of light in vacuum	c	299792458	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Planck constant	h	$6.62607015 \times 10^{-34}$	$\mathrm{J} \cdot \mathrm{s}$
Elementary charge	e	$1.602176634 \times 10^{-19}$	$\mathrm{A} \cdot \mathrm{s}$
Boltzmann constant	k	1.380649×10^{-23}	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{K}^{-1}$
Avogadro constant	K_A	$6.02214076 \times 10^{23}$	mol
Luminous efficacy of a defined visible radiation	Kcd	638	$\mathrm{cd} \cdot \mathrm{sr} \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^3$

The International Committee for Weights and Measures describes practical methods *mises en practique* for "realizing" a unit.[2] This means a method for the establishment of the value and associated uncertainty of a quantity of the same kind as the unit that is consistent with the definition of the unit. Any method that is traceable to the seven constants could be used.

The future definition of a unit does not imply any particular experiment for its practical realization. Any method capable of deriving an amount of base quantity value traceable to the set of seven reference constants could, in principle, be used. A *primary method* for realizing each unit is one that achieves the best precision and lowest uncertainty.

In what follows, the exact language for the definitions of the constants and SI base units is used. The definitions specify the exact numerical value of each constant when its value is expressed in the corresponding SI unit. By fixing the exact numerical value the unit becomes defined, since the product of the ***numerical value*** and the ***unit*** has to equal the ***value*** of the constant, which is postulated to be invariant. The seven constants are chosen in such a way that any unit of the SI can be written either through a defining constant itself or through products or quotients of defining constants.[3]

2.1.1 Definition of the second

Isaac Newton, in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), distinguished between time as duration and an absolute concept of time.

> *"Absolute true and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.* "[4]

Before 1960, the second was defined as the fraction $\frac{1}{86400}$ of the mean solar day, which varied due to slight changes in the earth rotation and so this was not a good definition. The development of clocks based on atomic oscillations allowed measures of timing with accuracy on the order of 1 part in 10^{14}, corresponding to errors of less than one microsecond (one millionth of a second) per year. Given the incredible accuracy of this measurement, and clear evidence that the best available timekeepers were atomic in nature, the *second* , SI unit, s, was defined in 1967 by the International Committee on Weights and Measures as a certain number of cycles of electromagnetic radiation emitted by cesium (or caesium) atoms as they make transitions between two designated quantum states:

---

[2]https://www.bipm.org/en/publications/mises-en-pratique/

[3]https://www.bipm.org/en/measurement-units/base-units.html

[4]Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 6.

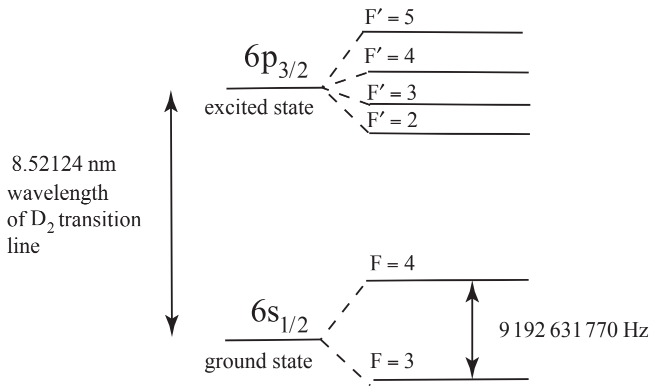

Figure 2.1: Energy levels of Cesium atom, showing the hyperfine transition between the two hyperfine levels F = 4 and F = 3 of the ground state $6\mathrm{s}_{1/2}$ and the wavelength of D_2 line corresponding to the fine structure doublet transition $6\mathrm{s}_{1/2} \longrightarrow 6\mathrm{p}_{3/2}$.
2.1

> *"The second, symbol* s*, is the SI unit of time. It is defined by taking the fixed numerical value of the cesium frequency* $\Delta\nu_{Cs}$*, the unperturbed ground-state hyperfine transition frequency of the cesium 133 atom to be 9192631770 when expressed in the unit Hz, which is equal to* s^{-1}".[5]

The second is equal to the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels F = 4 and F = 3 of the ground state $6\mathrm{s}_{1/2}$ of the cesium 133 atom.

The hyperfine frequency is defined exactly to be

$$\Delta\nu_{Cs} = 9192631770\mathrm{s}^{-1}. \tag{2.1}$$

Therefore the unit second is equal to

$$1\mathrm{s} = \frac{9192631770}{\Delta\nu_{Cs}} \tag{2.2}$$

A *primary method* to "realize" a second, the unit of time, with the highest accuracy involves certain national metrology laboratories that design *primary frequency standards*. The primary frequency standards aim at exactly realizing the SI second using the transition between the two hyperfine levels of the ground state of the cesium 133 atom, and for which corrections with respect to all known systematic shifts due to a variety of physical phenomena (gravitation is especially problematic in correcting for its effect) have been applied to best knowledge. The accuracy order of the best primary frequency standards that define the SI second approaches 1 part in a 10^7.

[5]https://www.bipm.org/utils/en/pdf/si-mep/SI-App2-second.pdf

The invention of the optical frequency comb made it possible to increase the “tic rate” of an atomic clock by a factor of one thousand or more, opening the way to replacing the cesium frequency standard by an optical frequency standard with uncertainty at the level of 1 part in 10^{18}, using an atom such as strontium or ytterbium, or an ion such as ytterbium or aluminum.[6]

2.1.2 Definition of the meter

The meter was originally defined as 1/10,000,000 of the arc from the Equator to the North Pole along the meridian passing through Paris. In 1889 the meter was redefined in terms of the international prototype meter by the 1st Conférence Générale des Poids et Mèsures (CGPM). The prototype meter was a platinum bar with an etched length scale to aid in calibration and ease of comparison, preserved near Paris. The accuracy of the bar was limited to one part in 10^{16}. Effects of temperature and pressure needed to be precisely calibrated as well as the mounting in order to ensure that the bar was straight.

In 1960 the CGPM introduced a microscopic reference by defining the meter as the length equal to 1650763.73 wavelengths in vacuum of the radiation corresponding to the transition between the levels $2\mathrm{p}_{10}$ and $5\mathrm{d}_5$ of the **krypton 86** atom. This definition of the meter depended on a particular radiation and cannot be realized to better than 4 parts in 10^9.

In 1983 the 17th CGPM defined the speed of light as a constant and the meter was redefined in terms of the distance that light traveled in $\frac{1}{299792458}$ of a second.

> *“The meter, symbol* m*, is the SI unit of length. It is defined by taking the fixed numerical speed of light in vacuum c to be 299792458 when expressed in the unit* $m \cdot s^{-1}$*, where the second is defined in terms of the cesium frequency* $\Delta\nu_{Cs}$*."*[7]

The definition of the speed of light states that

$$\mathrm{c} = 299792458\ \mathrm{m} \cdot \mathrm{s}^{-1} \tag{2.3}$$

The meter is therefore defined by

$$1\mathrm{m} = \frac{c}{299792458} \cdot 1\mathrm{s} \tag{2.4}$$

Because the second is defined by Eq. 2.2, therefore Eq. 2.4 becomes

$$1\mathrm{m} = \frac{\mathrm{c}}{299792458} \cdot \frac{9192631770}{\Delta\nu_{Cs}} = 30.6633149 \frac{\mathrm{c}}{\Delta\nu_{Cs}}. \tag{2.5}$$

This new definition, opens the way to major improvements in the precision with which the meter can be realized using laser wavelength and frequency measurement techniques. It is worth noting that the new definition has only become practicable with

[6] Andrew D. Ludlow, Martin M. Boyd,Jun Ye, E. Peik, and P.O. Schmidt, Rev. Mod. Phys. 87, 637
[7] https://www.bipm.org/utils/en/pdf/si-mep/SI-App2-metre.pdf

the development of techniques for the measurement of frequencies of electromagnetic radiations in the visible and near infrared. These can now be measured directly in terms of the frequency of the cesium standard which is used in the definition of the second.[8]

2.1.3 Example: Krypton Frequency Transition

The experimental value for frequency of the transition between the levels $2\mathrm{p}_{10}$ and $5\mathrm{d}_5$ of the $^{86}\mathrm{Kr}$ (krypton 86 atom) is equal to $f = 494886516.5$ MHz. Using the definition of the speed of light, Eq. 2.3, (i) calculate the wavelength corresponding to this transition, (ii) How does this frequency compare to the frequency derived from the wavelength $\lambda_{Kr} = \frac{1}{1650763.73}$ m in the 1960 definition of the meter?

Answer: (i)

$$\lambda = \frac{c}{f} = \frac{299792458 \mathrm{~m} \cdot \mathrm{s}^{-1}}{494886516.5 \mathrm{~MHz}} = 605780210.2 \mathrm{~fm}.$$

(ii) In the 1960 definition, the wavelength was equal to $\lambda_{Kr} = \frac{1}{1650763.73}$ m. Using $\mathrm{c} = 299792458 \mathrm{~m} \cdot \mathrm{s}^{-1}$, the frequency is then

$$f_{Kr} = \frac{c}{\lambda_{Kr}} = \frac{299792458 \mathrm{~m} \cdot \mathrm{s}^{-1}}{605780210.2 \mathrm{~fm}} = 494886516.2 \mathrm{~MHz}.$$

2.1.4 Definition of the kilogram

The kilogram was the last base unit in the International System of Units (SI) that was replaced in Nov 18, 2018 by a new definition in terms of the constants. The old kilogram was a physical artifact, known as the "International Prototype of the Standard Kilogram." George Matthey (of Johnson Matthey) made the prototype in 1879 in the form of a cylinder, 39 mm high and 39 mm in diameter, consisting of an alloy of 90% platinum and 10% iridium. The international prototype is kept in the Bureau International des Poids et Mèsures (BIPM) at Sevres, France, under conditions specified by the 1st Conférence Générale des Poids et Mèsures (CGPM) in 1889 when it sanctioned the prototype and declared "This prototype shall henceforth be considered to be the unit of mass." It is stored at atmospheric pressure in a specially designed triple bell-jar. The prototype is kept in a vault with six official copies.

The 3rd Conférence Générale des Poids et Mèsures CGPM (1901), in a declaration intended to end the ambiguity in popular usage concerning the word "weight" confirmed that: the kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram. As of Nov 18, 2018, the kilogram is now defined as follows:

> *The kilogram, symbol* kg*, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be* $6.62607015 \times 10^{-34}$ *when expressed in the unit* $\mathrm{J} \cdot \mathrm{s}$ *which is equal to* $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$*, where the meter and second are defined in terms of c and* $\Delta\nu_{Cs}$*. Thus the Planck*

[8]"Documents concerning the New Definition of the Metre", Metrologia 19 (1984) 163. DOI: 10.1088/0026-1394/19/4/004

constant h is exactly $\mathrm{h} = 6.62607015 \times 10^{-34}\,\mathrm{J} \cdot \mathrm{s}$.*This numerical value of h defines the unit joule second in the SI and, in combination with the SI second and meter, defines the kilogram. The numerical value of h given in the definition of the kilogram has ensured the continuity of the unit of mass with the previous definition of the kilogram.*[9]

2.1.5 Example: Definition of the kilogram in terms of the SI constants

Find an expression for the unit kilogram in terms of the defining constants h, c, and $\Delta\nu_{\mathrm{Cs}}$.

Answer: The Planck constant is exactly defined as $\mathrm{h} = 6.62607015{\times}10^{-34}\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$. Therefore the kilogram is defined to be

$$1\mathrm{kg} = \frac{h}{6.62607015 \times 10^{-34}}\ \mathrm{m}^{-2} \cdot \mathrm{s}^{1} \tag{2.6}$$

Using the definitions of the second, Eq. 2.2, and meter, Eq. 2.5, Eq. 2.6 becomes

$$1\mathrm{kg} = \frac{h}{6.62607015 \times 10^{-34}}\ \mathrm{m}^{-2} \cdot \mathrm{s}^{1} \cdot (30.6633149\frac{\mathrm{c}}{\Delta\nu_{Cs}})^2 \cdot \frac{\Delta\nu_{Cs}}{9192631770} \tag{2.7}$$

Therefore

$$1\mathrm{kg} = 1.4755214 \times 10^{40}\frac{h\Delta\nu_{Cs}}{c^2} \tag{2.8}$$

Realizing the kilogram requires practical methods that are describes in an article in Physics Today by Wolfgang Ketterle (MIT) and Alan Jamison (Institute for Quantum Computing at the University of Waterloo).[10]They begin by using the above definition to count photons and then describe a series of more practical methods. The internal energy E of an object is proportional to the rest mass according to

$$E = mc^2, \tag{2.9}$$

where c is the speed of the light. The energy of a light particle (photon) is proportional to the frequency of oscillation of the monochromatic classical wave associated to the photon,

$$E = hf = \frac{hc}{\lambda}, \tag{2.10}$$

yielding an effective mass

$$m = \frac{hc}{c^2}, \tag{2.11}$$

The mass equivalence of a photon emitted due to the hyperfine transition between two hyperfine ground states of cesium-133 atoms with frequency of $f = \Delta\nu_{Cs}$ is then

$$m = \frac{hc}{c^2} = \frac{(6.62607015 \times 10^{-34}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})(9192631770\ \mathrm{s}^{-1})}{(299792458\ \mathrm{m} \cdot \mathrm{s}^{-1})^2} = 6.77726531{\times}10^{-41}\,\mathrm{kg}. \tag{2.12}$$

[9]https://www.bipm.org/utils/en/pdf/si-mep/SI-App2-kilogram.pdf
[10]https://physicstoday.scitation.org/doi/10.1063/PT.3.4472

The number N of photons, emitted due to the hyperfine transition between two hyperfine ground states of cesium-133 atoms, needed in order for their total mass to equal one kilogram is then

$$N = \frac{1}{6.77726531 \times 10^{-41}} = 1.4755214 \times 10^{40}. \tag{2.13}$$

One kilogram is now equal to the mass of 1.4755214×10^{40} photons at the cesium hyperfine frequency.

$$1 \text{ kg} = 1.4755214 \times 10^{40} \frac{\Delta\nu_{Cs}}{c^2} \tag{2.14}$$

2.1.6 Definition of ampere

> *The ampere, symbol* A*, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge* e *to be* $1.602176634 \times 10^{-19}$ *when expressed in the unit Couloimb, symbol* C*, which is equal to* $\mathrm{A} \cdot \mathrm{s}$ *where the second is defined in terms of* $\frac{\Delta\nu_{\mathrm{Cs}}}{c^2}$.[11]

2.1.7 Example: Definition of ampere in terms of SI constants

Find an expression for the unit ampere in terms of the defining e and $\Delta\nu_{\mathrm{Cs}}$.

Answer: The exact definition of the elementary charge is

$$e = 1.602176634 \times 10^{-19} \text{ C} = 1.602176634 \times 10^{-19} \text{ A} \cdot \text{s}. \tag{2.15}$$

Therefore the unit ampere is defined to be

$$1\text{A} = \frac{e}{1.602176634 \times 10^{-19}} \cdot (1 \text{ s}^{-1}) = \frac{e}{1.602176634 \times 10^{-19}} \cdot \frac{\Delta\nu_{\mathrm{Cs}}}{9192631770}. \tag{2.16}$$

Hence

$$1\text{A} = 6.789687 \times 10^{8} e\Delta\nu_{\mathrm{Cs}} \tag{2.17}$$

2.1.8 Definition of the kelvin

> *The kelvin, symbol mathrmK, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k to be* 1.380649×10^{-23} *when expressed in the unit,* $\mathrm{J} \cdot \mathrm{K}^{-1}$*, which is equal to* $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{K}^{-1}$*, where the kilogram, meter and second are defined in terms of h, c, and* $\Delta\nu_{\mathrm{Cs}}$.
>
> *This definition implies the exact relation* $k = 1.380649 \times 10^{-23} \mathrm{J} \cdot \mathrm{K}^{-1}$*. Its effect is that one kelvin is equal to the change of thermodynamic temperature T that results in a change of thermal energy KT by* $k = 1.380649 \times 10^{-23} \mathrm{J}$.[12]

[11] https://www.bipm.org/en/measurement-units/base-units.html
[12] https://www.bipm.org/en/measurement-units/base-units.html

2.1.9 Example: Definition of kelvin in terms of SI constants

Find an expression for the unit kelvin in terms of the defining constants k, h, c, and $\Delta\nu_{\mathrm{Cs}}$.

Answer: The Boltzmann constant is exactly

$$k = 1.380649 \times 10^{-23}\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{K}^{-1}. \tag{2.18}$$

Therefore the unit kelvin is defined as

$$1\ \mathrm{K} = \frac{1.380649 \times 10^{-23}}{k} \cdot \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}. \tag{2.19}$$

Using our above definitions for the kilogram, meter and second, Eq. 2.19 becomes

$$\begin{aligned} &1\ \mathrm{K} = \left(\tfrac{1.380649\times10^{-23}}{k}\right) \cdot \left(1.4755214 \times 10^{40}\ \tfrac{h\Delta\nu_{\mathrm{Cs}}}{c^2}\right) \\ &\cdot\left(30.663\ 314\ 9\ \tfrac{c}{\Delta\nu_{\mathrm{Cs}}}\right)^2 \cdot \left(\tfrac{9\ 192\ 631\ 770}{\Delta\nu_{\mathrm{Cs}}}\right)^{-2} \end{aligned} \tag{2.20}$$

One kelvin is then equal to

$$1\ \mathrm{K} = 2.266\ 665\ 265\ \frac{h\Delta\nu_{\mathrm{Cs}}}{k} \tag{2.21}$$

2.1.10 Definition of the mole

> *The mole, symbol* mol*, is the SI unit of of amount of substance. One mole contains exactly* $6.02214076 \times 10^{23}$ *elementary entities. This number is the fixed numerical value of the Avogadro constant,* $\mathrm{N_A}$*, when expressed in the unit* $rmmol^{-1}$ *and is called the* **Avogadro number**.
>
> *The amount of substance, symbol n, of a system is a measure of the number of specified elementary entities. An elementary entity may be an atom, a molecule, an ion, an electron, any other particle or specified group of particles.*[13]

2.1.11 Example: Definition of mole in terms of SI constants

Find an expression for the unit mole in terms of the defining constant $\mathrm{N_A}$.

Answer: The Avogadro number is exactly defined as

$$N_A = 6.02214076 \times 10^{23}\ \mathrm{mol}^{-1}. \tag{2.22}$$

Therefore, 1 mole is given by the expression:

$$1\ \mathrm{mol} = 6.02214076 \times 10^{23} \frac{1}{N_A}. \tag{2.23}$$

[13] https://www.bipm.org/en/measurement-units/base-units.html

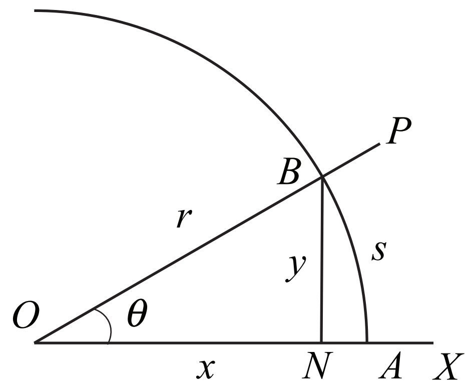

Figure 2.2: Trigonometric relations

2.1.12 Radians

Consider the triangle drawn in Figure 2.2. The basic trigonometric functions of an angle θ in a right triangle ONB are $\sin(\theta) = y/r$, $\cos(\theta) = x/r$, and $\cos(\theta) = y/x$.

It is very important to become familiar with using the measure of the angle θ itself as expressed in *radians* [rad]. Let θ be the angle between two straight lines OX and OP. Draw a circle of radius r centered at O. The lines OP and OX cut the circle at the points A and B where $OA = OB = r$. Denote the length of the arc AB by s, then the radian measure of θ is given by

$$\theta = s/r, \tag{2.24}$$

and the ratio is the same for circles of any radii centered at O just as the ratios y/r and y/x are the same for all right triangles with the angle θ at O. As θ approaches $360°$, s approaches the complete circumference $2\pi r$ of the circle, so that $360° = 2\pi r$.

Let's compare the behavior of $\sin(\theta)$, $\tan(\theta)$, and θ itself for small angles. One can see from Figure 2.2 that $s/r > y/r$. It is less obvious that $y/x > \theta$. It is instructive to plot $\sin(\theta)$, $\tan(\theta)$, and θ as functions of θ measured in rad between 0 and $\pi/2$ on the same graph (see Figure 2.3). For small θ, the values of all three functions are almost equal. But how small is "small"? An acceptable condition is for $\theta << 1$ measured in radians.

We can show this with a few examples. Recall that $360° = 2\pi r$, $57.3° = 1\mathrm{r}$, so an angle $6° \cong (6°)(2\pi\ \mathrm{rad}/360^o) \cong 0.1\ \ \mathrm{rad}$ when expressed in radians. In Table 2.3 we compare the value of θ (measured in radians) with $\sin(\theta)$, $\tan(\theta)$, $\theta - \sin(\theta)$, and $\theta - \tan(\theta)$, for $\theta = 0.1\mathrm{rad}.\ 0.2\mathrm{rad}$, $0.5\mathrm{rad}$ and $1.0\mathrm{rad}$. The values for $\theta - \sin(\theta)$, and $\theta - \tan(\theta)$ for $\theta = 0.2\mathrm{rad}$ are less than $\pm 1.4\%$. Provided that θ is not too large, the approximation that

$$\sin(\theta) \approx \tan(\theta) \approx \theta, \tag{2.25}$$

called the *small angle approximation*, can be used almost interchangeably, within some small percentage error. This is the basis of many useful approximations in physics calculations.

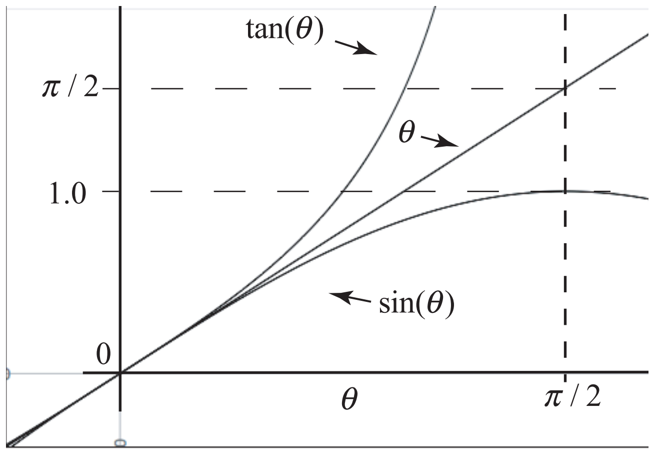

Figure 2.3: Radians compared to trigonometric functions

Table 2.3: Small Angle Approximation

$\theta[\mathrm{rad}]$	$\theta[\mathrm{deg}]$	$\sin(\theta)$	$\tan(\theta)$	$\theta - \sin(\theta)$	$\theta - \tan(\theta)$
0.1	5.72958	0.09983	0.10033	0.00167	-0.00335
0.2	11.45916	0.19867	0.20271	0.00665	-0.01355
0.5	28.64789	0.47943	0.54630	0.04115	-0.09260
1.0	57.29578	0.84147	1.55741	0.15853	-0.55741

2.1.13 Example: Parsec

A standard astronomical unit is the **parsec**. Consider two objects that are separated by a distance of one astronomical unit, $1\mathrm{AU} = 1.50 \times 10^{11}\mathrm{m}$, which is the mean distance between the earth and sun. (One astronomical unit is roughly equivalent to eight light minutes, $1\mathrm{AU} = 8.3\mathrm{light-minutes}$. **One parsec** is the distance at which one astronomical unit subtends an angle $\theta = 1\mathrm{arc-second} = (1/3600)\ \mathrm{deg}$.

Suppose is a spacecraft is located in a space a distance 1 parsec from the Sun as shown in Figure 2.4. How far is the spacecraft in terms of light years and meters?

Answer: Because one arc second corresponds to a very small angle, one parsec is therefore equal to distance divided by angle, hence

$$
\begin{aligned}
&1\ \mathrm{pc} = \tfrac{(1\ \mathrm{AU})}{(1/3600)} = (2.06 \times 10^5\ \mathrm{AU}) \left(\tfrac{1.50\times10^{11}\ \mathrm{m}}{1\ \mathrm{AU}}\right) = 3.09 \times 10^{16}\ \mathrm{m} \\
&= (3.09 \times 10^{16}\ \mathrm{m}) \left(\tfrac{1\ \mathrm{ly}}{9.46\times10^{15}\ \mathrm{m}}\right) = 3.26\ \mathrm{ly}.
\end{aligned} \tag{2.26}
$$

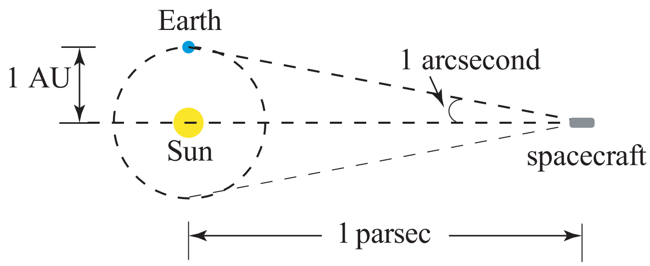

Figure 2.4: One parsec

2.1.14 Steradians

The *steradian* [sr] is the unit of solid angle that, having its vertex in the center of a sphere, cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere. The conventional symbol for steradian measure is Ω, the uppercase Greek letter "Omega." The total solid angle Ω_{sph} of a sphere is then found by dividing the surface area of the sphere by the square of the radius,

$$\Omega_{\text{sph}} = 4\pi r^2/r^2 = 4\pi. \tag{2.27}$$

This result is independent of the radius of the sphere.

2.1.15 Definition of the candela

> *The candela, symbol* cd, *is the SI unit of luminous intensity in a given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency* $540\times10^{12}\text{Hz}$, Kcd, *to be 683 when expressed in the unit* $\text{m}\cdot\text{W}^{-1}$ *which is equal to* $\text{cd}\cdot\text{sr}\cdot\text{W}^{-1}$ *or* $\text{cd}\cdot\text{sr}\cdot\text{kg}^{-1}\cdot\text{m}^{-2}\cdot\text{s}^{3}$ *where the kilogram, meter and second are defined in terms of h, c and* $\Delta\nu_{\text{Cs}}$.[14]

2.1.16 Example: Definition of candela in terms of SI constants

Find an expression for the unit candela in terms of the defining constants K_{cd}, h and $\Delta\nu_{\text{Cs}}$.

Answer: One candela expressed in terms of the defining constants by inverting K_{cd}, the luminous efficacy of monochromatic radiation of frequency $540\times10^{12}\text{Hz}$. Starting with

$$K_{\text{cd}} = 683\ \text{cd}\cdot\text{sr}\cdot\text{kg}^{-1}\cdot\text{m}^{-2}\cdot\text{s}^{3} \tag{2.28}$$

we have that

$$1\ \text{cd} = \frac{\text{K}_{\text{CD}}}{683}\text{sr}^{-1}\cdot\text{kg}\cdot\text{m}^{-2}\cdot\text{s}^{-3} \tag{2.29}$$

[14] https://www.bipm.org/en/measurement-units/base-units.html

substituting the units kg, m and s by their corresponding expressions in terms of the defining constants:

$$\begin{aligned}
1\ \mathrm{cd} &= \tfrac{K_{\mathrm{cd}}}{683}\mathrm{sr}^{-1}\cdot\mathrm{kg}\cdot\mathrm{m}^{-2}\cdot\mathrm{s}^{-3}\\
1\ \mathrm{cd} &= \tfrac{K_{\mathrm{cd}}}{683}\left(1.4755214\times10^{40}\ \tfrac{h\Delta\nu_{\mathrm{Cs}}}{c^2}\right)\left(30.6633149\tfrac{c}{\Delta\nu_{\mathrm{Cs}}}\right)^2\cdot\left(\tfrac{9\ 192\ 631\ 770}{\Delta\nu_{\mathrm{Cs}}}\right)^{-3}\\
1\ \mathrm{cd} &= \tfrac{(1.4755214\times10^{40})(30.6633149)^2(9\ 192\ 631\ 770)^{-3}}{683}K_{\mathrm{cd}}h\Delta\nu_{\mathrm{Cs}}{}^2\\
1\ \mathrm{cd} &= 2.614830\times10^{10}\ K_{\mathrm{cd}}h\Delta\nu_{\mathrm{Cs}}{}^2
\end{aligned}\tag{2.30}$$

The effect of this definition is that one candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540\times10^{12}\mathrm{Hz}$ and has a radiant intensity in that direction of $(1/683)$
$rmW\cdot\mathrm{sr}^{-1}$.[15]

2.2 Dimensions of Commonly Encountered Quantities

Introduction Many physical quantities are derived from the base quantities by a set of algebraic relations defining the physical relation between these quantities. We shall refer to the *dimension* of the base quantity by the quantity itself, for example

$$\dim\,(\mathrm{length})\equiv\mathrm{length}\equiv\mathrm{L},\ \dim\,(\mathrm{mass})\equiv\mathrm{mass}\equiv\mathrm{M},\ \dim\,(\mathrm{time})\equiv\mathrm{time}\equiv\mathrm{T}.$$

The dimension of a derived quantity is written as a power of the dimensions of the base quantities. For example velocity is a derived quantity and the dimension is given by the relationship

$$\dim\mathrm{velocity}=(\mathrm{length})/(\mathrm{time})=\mathrm{L}\cdot\mathrm{T}^{-1},$$

where $\mathrm{L}=\mathrm{length}$ and $\mathrm{T}=\mathrm{time}$. Force is also a derived quantity and has dimension

$$\dim\mathrm{force}=\frac{(\mathrm{mass})(\mathrm{dimvelocity})}{(\mathrm{time})}$$

where $\mathrm{M}=\mathrm{mass}$. We can also express force in terms of mass, length, and time by the relationship

$$\dim\mathrm{force}=\frac{(\mathrm{mass})(\mathrm{length})}{(\mathrm{time})^2}\ =\ \mathrm{M}\cdot\mathrm{L}\cdot\mathrm{T}^{-2}.$$

The derived dimension of kinetic energy is

$$\dim\mathrm{kinetic\,energy}=(\mathrm{mass})(\mathrm{dimvelocity})^2,$$

which in terms of mass, length, and time is

[15] https://www.bipm.org/en/measurement-units/base-units.html

$$\text{dim kinetic energy} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = \mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}.$$

The derived dimension of work is

$$\text{dim work} = (\text{dimforce})(\text{length}).$$

which in terms of our fundamental dimensions is

$$\text{dimwork} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = \mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}.$$

So work and kinetic energy have the same dimensions. Power is defined to be the rate of change in time of work so the dimensions are

$$\text{dimpower} = \frac{\text{dimwork}}{\text{time}} = \frac{(\text{dimforce})(\text{length})}{\text{time}} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^3} = \mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-3}.$$

In Table 2.4 we include the derived dimensions of some common mechanical quantities in terms of mass, length, and time.

Table 2.4: Dimensions of Some Common Mechanical Quantities M = mass, L = length and T = time

Quantity	**Dimension**	**MKS unit**
Angle	dimensionless	Dimensionless = radian
Solid Angle	dimensionless	Dimensionless = sterradian
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	s^{-1} = Hz
Velocity	$\mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Acceleration	$\mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{m} \cdot \mathrm{s}^{-2}$
Angular Velocity	T^{-1}	$\mathrm{rad} \cdot \mathrm{s}^{-1}$
Angular Acceleration	T^{-2}	$\mathrm{rad} \cdot \mathrm{s}^{-2}$
Mass Density	$\mathrm{M} \cdot \mathrm{L}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^{-3}$
Momentum	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$
Angular Momentum	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$
Force	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ = newton = N
Work, Energy	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$ = joule = J
Torque	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$ = watt = W
Pressure	$\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}$ = pascal = Pa

2.2.1 Dimensions of Fundamental Constants

A number of fundamental constants appear in the fundamental laws of physics. The values of these constants depend on the choice of units. We denote the dimension of a quantity D by square brackets [D]: or example the dimension of frequency is denoted by $[f]$ and is equal to

$$[f] = \mathrm{T}^{-1}$$

In what follows all constants will be given in SI units.

2.2.2 Example: Dimension of the Universal Gravitation Constant G

Newton's Universal Law of Gravitation describes the gravitational force between two bodies with masses, m_1 and m_2. This force points along the line connecting the bodies, is attractive, and its magnitude is proportional to the inverse square of the distance, r_{12}, between the bodies:

$$|\mathbf{F}| = G\frac{m_1 m_2}{{r_{12}}^2},$$

where $G = 6.6742(10) \times 10^{-11}\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$. The dimensions of the Universal Gravitation Constant G are then

$$[G] = \frac{\mathrm{L}^3}{\mathrm{M} \cdot \mathrm{T}^2}$$

.

2.2.3 Example: Dimensions of Coulomb's constant $\frac{1}{4\pi\epsilon_0}$

Coulomb's Law describes the electric force between two charged bodies q_1 and q_2, separated by a distance r_{12}. The fundamental charge has magnitude $e = 1.602176634 \times 10^{-19}$ C. The magnitude of the force exerted on an electron with charge $q_1 = e$ is given by

$$|\mathbf{F}_{12}| = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{{r_{12}}^2}.$$

The vacuum permittivity is defined as

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}$$

. where the speed of light $c = 299792458\ \mathrm{m} \cdot \mathrm{s}^{-1}$, and the vacuum permeability is experimentally determined with a value $\mu_0 = 1.25663706212(19) \times 10^{-6}\ \mathrm{N} \cdot \mathrm{C}^{-2} \cdot \mathrm{s}^2$. (Note that before the new definitions of units, $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N} \cdot \mathrm{C}^{-2} \cdot \mathrm{s}^2$. Therefore

$$\varepsilon_0 = 8.8541878128(13) \times 10^{-12}\mathrm{N}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{C}^2$$

Note that $1/4\pi\varepsilon_0 = 8.9875517923 \times 10^9 \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-2}$. Therefore its dimensions are

$$[1/4\pi\varepsilon_0] = \frac{(force)(length)^2}{(charge)^2} = \frac{(mass))length)}{(time)^2}\frac{(length)^2}{(charge)^2} = \frac{\mathrm{M} \cdot \mathrm{L}^3}{\mathrm{T}^2 \cdot \mathrm{C}^2}$$

.

2.2.4 Example: Dimension of the Planck Constant h

The internal energy of an object is proportional to the rest mass according to

$$E = mc^2$$

The energy of a light particle (photon) is proportional to the frequency of oscillation of the monochromatic classical wave associated to the photon,

$$E = hf = \frac{hc}{\lambda},$$

where Planck's constant is defined to be exactly $\mathrm{h} = 6.62607015 \times 10^{-34} \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$. The dimensions of Planck's constant can be determined from the relation

$$h = E/f$$

. The dimension of energy can be determined from the relation $E = mc^2$:

$$[E] = \frac{(mass)(length)^2}{(time)^2} = \frac{\mathrm{M} \cdot \mathrm{L}^2}{\mathrm{T}^2}.$$

The dimension of Planck's constant is therefore

$$[h] = (energy)\,(time) = \frac{(mass)\,(length)^2}{(time)} = \frac{\mathrm{M} \cdot \mathrm{L}^2}{\mathrm{T}}.$$

2.3 Dimensional Analysis

There are many phenomena in nature that can be explained by simple relationships between the observed phenomena.

2.3.1 Example: Period of a Pendulum

Consider a simple pendulum consisting of a massive bob suspended from a fixed point by a string. Let Tdenote the time interval (period of the pendulum) that it takes the bob to complete one cycle of oscillation. How does the period of the simple pendulum depend on the quantities that define the pendulum and the quantities that determine the motion?

Answer: What possible quantities are involved? The length of the pendulum l, the mass of the pendulum bob m, the gravitational acceleration g and the angular amplitude of the bob θ_0 are all possible quantities that may enter into a relationship for the period of the swing. Have we included every possible quantity? We can never be sure but let's first work with this set and if we need more than we will have to think harder! Our problem is then to find a function f such that

$$T = f(l, m, g, \theta_0)$$

. We first make a list of the dimensions of our quantities as shown in Table 2.5. Our

Table 2.5: Dimensions of Quantities Relevant to the Period of Pendulum

Quantity	**Symbol**	**Dimensional Formula**
Time of swing	t	T
Length of pendulum	l	L
Mass of pendulum bob	m	M
Gravitational acceleration	g	$\mathrm{L} \cdot \mathrm{T}^{-2}$
Angular amplitude of swing	θ_0	dimensionless

first observation is that the mass of the bob cannot enter into our relationship, as our final quantity has no dimensions of mass and no other quantity has dimensions of mass. Let's focus on the length of the string and the gravitational acceleration. In order to eliminate length, these quantities must divide each other when appearing in some functional relation for the period T. If we choose the combination l/g, the dimensions are

$$\dim[l/g] = \frac{\text{length}}{\text{length}/(\text{time})^2} = \mathrm{T}^2$$

. It appears that the time of swing may proportional to the square root of this ratio. Thus we have a candidate formula

$$T \sim \left(\frac{l}{g}\right)^{1/2}.$$

(in the above expression, the symbol $\propto$ represents a proportionality, not an approximation). Because the angular amplitude θ_0 is dimensionless, it may or may not appear. We can account for this by introducing some function $y(\theta_0)$ into our relationship, which is beyond the limits of this type of analysis. The period is then

$$T = y(\theta_0)\left(\frac{l}{g}\right)^{1/2}.$$

We shall discover later on that $y(\theta_0)$ is nearly independent of the angular amplitude θ_0 and for very small amplitudes $y(\theta_0) = 2\pi$. Then for small angles of oscillation

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}.$$

2.3.2 Example: Planck Length

Suppose the Universal Gravitational Constant is fixed at $G = 6.6742(10)\times10^{-11}\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$ and the Cesium hyperfine frequency is removed as a fundamental constant. The set of defining constants are now shown in Table 2.6

Table 2.6: Dimensions of Quantities Relevant to the Planck Length

Quantity	**Symbol**	**Numerical Value**	**SI Units**
Gravitational constant	G	$6.6742(10)\times10^{-11}$	$\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$
Speed of light in vacuum	c	299792458	$\mathrm{m}\cdot\mathrm{s}^{-1}$
Planck constant	g	6.62607015×10^{-34}	$\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$

Physicists often use $\hbar = \frac{h}{2\pi} = 1.05457182\times10^{-34}\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$ instead of the Planck constant.

(a) Using dimensional analysis, find a combination of the defining constants $\hbar$, c and G, that has the dimensions of length, called the *Planck length*, l_P.

(b) What is the value of the *Planck length*, l_P?

(c) Find an expression for the unit meter in terms of the defining constants $\hbar$, c and G.

Answer: (a) Let $l_p = c^{a_1}\hbar^{a_2}G^{a_3}$ where $[l_P] = \mathrm{L}$ has the dimensions of length. Therefore in order for the dimensions to match

$$\begin{aligned}\mathrm{L} &= (\mathrm{L}\cdot\mathrm{T}^{-1})^{a_1}(\mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{T}^{-1})^{a_2}(\mathrm{L}^3\cdot\mathrm{T}^{-2}\cdot\mathrm{M}^{-1})^{a_3}\\ &= \mathrm{L}^{a_1+2a_2+3a_3}\cdot\mathrm{M}^{a_2-a_3}\cdot\mathrm{T}^{-a_1-a_2-2a_3}.\end{aligned}$$

Therefore we have three conditions

$$\begin{aligned}1 &= a_1+2a_2+3a_3\\ 0 &= a_2-a_3\\ 0 &= -a_1-a_2-2a_3\end{aligned}$$

We can solve these equations and find that

$$\begin{aligned}a_2 &= a_3 = 1/2\\ a_1 &= -3a_3 = -3/2\end{aligned}$$

Thus the combination

$$l_p = c^{-3/2}\hbar^{1/2}G^{1/2} = \sqrt{\frac{G\hbar}{c^3}}$$

has the dimensions of length.

(b) The magnitude of the Planck length unit in SI units is

$$\begin{aligned}l_p &= \left(\frac{\hbar G}{c^3}\right)^{1/2} = \left(\frac{(1.05457182\times10^{-34}\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1})(6.6742\times10^{-11}\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2})}{(2.99792458\ \times10^8\mathrm{m}\cdot\mathrm{s}^{-1})^3}\right)^{1/2}\\ &= 1.61624\times10^{-35}\mathrm{m}\end{aligned}$$

(c) Because $l_P = 1.61624 \times 10^{-35}\mathrm{m}$. Therefore

$$1\ \mathrm{m} = \frac{l_p}{1.61624 \times 10^{-35}\mathrm{m}} = 6.18719 \times 10^{34} \left(\frac{\hbar G}{c^3}\right)^{1/2}$$

2.3.3 Example: Fine Structure Constant

(a) What combination of constants h, c, e and ϵ_0 gives a dimensionless quantity? Hint: Let $\gamma = h^{a_1} c^{a_2} e^{a_3} {\varepsilon_0}^{a_4}$ and find a_1, a_2, a_3 and a_4, such that $[\gamma]$ is dimensionless.

Answer: Let $\gamma = h^{a_1} c^{a_2} e^{a_3} {\varepsilon_0}^{a_4}$ such that $[\gamma]$ is dimensionless. This means that

$$\gamma = (\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})^{a_1} (\mathrm{m} \cdot \mathrm{s}^{-1})^{a_2} (\mathrm{C})^{a_3} (\mathrm{kg}^{-1} \cdot \mathrm{s}^2 \cdot \mathrm{m}^{-3} \cdot \mathrm{C}^2)^{a_4}$$

must has no units. Therefore

$$\begin{aligned}
&a_1 - a_4 = 0 \\
&2a_1 + a_2 - 3a_4 = 0 \\
&-a_1 - a_2 + 2a_4 = 0 \\
&a_3 + 2a_4 = 0
\end{aligned}$$

We can solve these equations and find that

$$\begin{aligned}
a_1 &= a_2 = a_4 \\
a_3 &= -2a_1
\end{aligned}$$

The smallest integer solution for a positive value of a_1 is given by

$$\begin{aligned}
a_1 &= a_2 = a_4 = 1 \\
a_3 &= -2
\end{aligned}$$

Therefore

$$\gamma = \frac{hc\varepsilon_0}{e^2}.$$

(b) Fine structure constant: Define $\beta = (h/2\pi)c(4\pi\varepsilon_0)/e^2$. Find the value of the fine structure constant α given by

$$\alpha = \frac{1}{\beta} = \frac{e^2}{2\varepsilon_0 ch} = \frac{\mu_0 ce^2}{2h}.$$

Answer:

$$\alpha \simeq \frac{1}{137}.$$

2.3.4 Example: Rydberg constant

The *Rydberg constant* is named after a Swedish physicist Johannes Rydberg. It corresponds to the wavenumber (inverse wavelength) of the lowest-energy photon that can ionize an atom from its ground state. It's expressed as R_H for the hydrogen atom and as R_∞ for any atom whose nucleus is infinitely heavier than a single orbiting electron.

According to the CODATA[16] the value of the Rydberg constant for heavy atoms is:

$$R_\infty = \frac{2\pi^2 m_e\, e^4}{(4\pi\varepsilon_0)^2 h^3 c} = 10\ 973\ 731.568\ 160\ \mathrm{m}^{-1}$$

where $m_e = 9.1093837015{\times}10^{-31}$ kg is the rest mass of the electron, e is the elementary charge, ϵ_0 is the permittivity of free space, h is the Planck constant, and c is the speed of light in vacuum. In atomic physics, the *Rydberg unit of energy*, R_y describes the energy of the photon with a wavenumber equal to R_∞ thus corresponding to the ionization energy of the hydrogen atom. It's expressed as follows:

$$R_y = hcR_\infty = \frac{2\pi^2 m_e\, e^4}{(4\pi\varepsilon_0)^2 h^2} = \frac{1}{2}(m_e c^2)\alpha^2.$$

Find the numerical value of the Rydberg unit of energy. Answer: The exact value, provided by the CODATA, is:

$$R_y = \tfrac{m_e\, e^4}{8{\varepsilon_0}^2 h^2} = 2.179\ 872\ 361\ 1035 \times 10^{-18}\mathrm{J}$$
$$R_y = 13.605\ 693\ 122994\ \mathrm{eV}$$

2.4 Order of Magnitude Estimates - Fermi Problems

Counting is the first mathematical skill we learn. We came to use this skill by distinguishing elements into groups of similar objects, but counting becomes problematic when our desired objects are not easily identified, or there are too many to count. Rather than spending a huge amount of effort to attempt an exact count, we can try to estimate the number of objects. For example, we can try to estimate the total number of grains of sand contained in a bucket of sand. Because we can see individual grains of sand, we expect the number to be very large but finite. Sometimes we can try to estimate a number, which we are fairly sure but not certain is finite, such as the number of particles in the universe.

We can also assign numbers to quantities that carry dimensions, such as mass, length, time, or charge, which may be difficult to measure exactly. We may be interested in estimating the mass of the air inside a room, or the length of telephone wire in the United States, or the amount of time that we have slept in our lives. We choose some set of units, such as kilograms, miles, hours, and coulombs, and then we can attempt to estimate the number with respect to our standard quantity.

Often we are interested in estimating quantities such as speed, force, energy, or power. We may want to estimate our natural walking speed, or the force of wind acting against a bicycle rider, or the total energy consumption of a country, or the electrical power necessary to operate a university. All of these quantities have no exact, well-defined value; they instead lie within some range of values.

When we make these types of estimates, we should be satisfied if our estimate is reasonably close to the middle of the range of possible values. But what does "reasonably close" mean? Once again, this depends on what quantities we are estimating. If we are

[16]https://physics.nist.gov/cgi-bin/cuu/Value?ryd

describing a quantity that has a very large number associated with it, then an estimate within an order of magnitude should be satisfactory. The number of molecules in a breath of air is close to 10^{22}; an estimate anywhere between 10^{21} and 10^{23} molecules is close enough. If we are trying to win a contest by estimating the number of marbles in a glass container, we cannot be so imprecise; we must hope that our estimate is within 1% of the real quantity. These types of estimations are called *Fermi problems*. The technique is named after the physicist Enrico Fermi, who was famous for making these sorts of "back of the envelope" calculations.

Methodology for Estimation Problems

Estimating is a skill that improves with practice. Here are two guiding principles that may help you get started.

(1) You must identify a set of quantities that can be estimated or calculated.

(2) You must establish an approximate or exact relationship between these quantities and the quantity to be estimated in the problem.

Estimations may be characterized by a precise relationship between an estimated quantity and the quantity of interest in the problem. When we estimate, we are drawing upon what we know. But different people are more familiar with certain things than others. If you are basing your estimate on a fact that you already know, the accuracy of your estimate will depend on the accuracy of your previous knowledge. When there is no precise relationship between estimated quantities and the quantity to be estimated in the problem, then the accuracy of the result will depend on the type of relationships you decide upon. There are often many approaches to an estimation problem leading to a reasonably accurate estimate. So use your creativity and imagination!

2.4.1 Example: Fermi's Problem Piano Tuners

A famous type of estimation problem is named after the physicist Enrico Fermi. One of his favorite examples was estimating the number of piano tuners in Chicago with the only given information the population of Chicago which at the time was approximately 3 million people.

Solution: Our estimate will be based on how many individual living units there are in Chicago. Assume that on average 5 people live together. Then there are 600,000 living units. Assume that one in ten living units has a piano, which amounts to 60,000 pianos. Suppose a piano tuner can tune three pianos a day. If a tuner works a 50 week year, five days a week, then each tuner can tune 750 pianos per year. Therefore there is a need for $60,000/750 = 80$, so we estimate there are 100 piano tuners in Chicago.

2.4.2 Example: One Kilometer Line of Pennies

In this example our goal is to estimate the number of pennies needed to mark off 1 kilometer.

Solution: The first step is to consider what type of quantity is being estimated. In this example we are estimating a dimensionless scalar quantity, the number of pennies. We can now give a precise relationship for the number of pennies needed to mark off 1 kilometer

$$\# \text{ of pennies} = \frac{\text{total distance}}{\text{diameter of penny}}.$$

We can estimate a penny to be approximately 2 centimeters wide. Therefore the number of pennies is

$$\begin{aligned}\# \text{ of pennies} &= \tfrac{\text{total distance}}{\text{length of a penny}} = \tfrac{(1\text{ km})}{(2\text{ cm})(1\text{ km}/10^5\text{ cm})} \\ &= 50,000 \text{ pennies} = 5\times 10^4 \text{ pennies}.\end{aligned}$$

When applying numbers to relationships we must be careful to convert units whenever necessary. How accurate is this estimation? If you measure the size of a penny, you will find out that the width is 1.9 cm, so our estimate was accurate to within 5%. This accuracy was fortuitous. Suppose we estimated the length of a penny to be 1 cm. Then our estimate for the total number of pennies would be within a factor of 2, a margin of error we can live with for this type of problem.

2.4.3 Example: Estimation of Mass of Water on Surface of Earth

Solution: In this example we are estimating mass, a quantity that is a fundamental in SI units, and is measured in kg. We start by approximating that the amount of water on Earth is approximately equal to the amount of water in all the oceans. Initially we will try to estimate two quantities: the density of water and the volume of water contained in the oceans. Then the relationship we want is

$$\text{mass} = (\text{density})(\text{volume}).$$

One of the hardest aspects of estimation problems is to decide which relationship applies. One way to check your work is to check dimensions. Density has dimensions of mass/volume, so our relationship is correct dimensionally. The density of fresh water is $\rho_{\text{w}} = 1.0\text{g}\cdot\text{cm}^{-3}$; the density of seawater is slightly higher, but the difference won't matter for this estimate. You could estimate this density by estimating how much mass is contained in a one-liter bottle of water. (The density of water is a point of reference for all density problems. Suppose we need to estimate the density of iron. If we compare iron to water, we might estimate that iron is 5 to 10 times denser than water. The actual density of iron is $\rho_{\text{iron}} = 7.8\text{g}\cdot\text{cm}^{-3}$.) Because there is no precise relationship, estimating the volume of water in the oceans is much harder. Let's model the volume occupied by the oceans as if the water completely covers the earth, forming a spherical shell of radius R_E and thickness d (Figure 2.5, which is decidedly not to scale), where R_E is the radius of the earth and d is the average depth of the ocean. The volume of that spherical shell is

$$\text{volume} \approx 4\pi R_{\text{earth}}^2 d.$$

We also estimate that the oceans cover about 75% of the surface of the earth. So we can refine our estimate that the volume of the oceans is

$$\text{volume} \approx (0.75)(4\pi {R_E}^2 d).$$

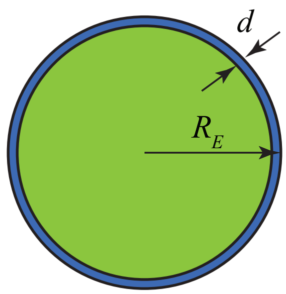

Figure 2.5: A model for estimating the mass of the water on the surface of the Earth.

We therefore have two more quantities to estimate, the average depth of the ocean, which we can estimate as $d = 1\ \mathrm{km}$, and the radius of the earth, which is approximately $R_E \approx 6\times10^6\mathrm{m}$. (The quantity that you may remember is the circumference of the earth, which is about $25,000\mathrm{miles}$. Historically the circumference of the earth was defined to be $4\times10^6\mathrm{m}$.) The radius R_E and the circumference s are related by $R_E = 2\pi\mathrm{s}$. Thus $R_E = \frac{s}{2\pi} = \frac{(2.5\times10^4\ \mathrm{mi})(1.6\ \mathrm{km\cdot mi^{-1}})}{2\pi} = 6.4\times10^3\ \mathrm{km}$ We will use $R_E \approx 6\times10^6\mathrm{m}$; additional accuracy is not necessary for this problem, since the ocean depth estimate is clearly less accurate. In fact, the factor of75% is not needed, but included more or less from habit. Altogether, our estimate for the mass of the oceans is

$$\mathrm{mass} = (\mathrm{density})(\mathrm{volume}) \simeq \rho(0.75)(4\pi {R_E}^2 d)$$

$$\mathrm{mass} \simeq \left(\frac{1\mathrm{g}}{\mathrm{cm}^3}\right)\left(\frac{1\ \mathrm{kg}}{10^3\ \mathrm{g}}\right)\left(\frac{(10^5\ \mathrm{cm})^3}{(1\ \mathrm{km})^3}\right)(0.75)(4\pi)(6\times10^3\ \mathrm{km})^2(1\mathrm{km})$$

$$\mathrm{mass} \simeq 3\times10^{20}\ \mathrm{kg} \simeq 10^{20}\ \mathrm{kg}.$$

An interesting question: what is the origin of this water?

2.4.4 Example: Lao-Tzu's Last Breath

What is the probability that none of the molecules in your next breath were in the last breath of Lao-Tzu (b. 551 B.C.E.), author of the *Tao Te Ching*, who expired in 471 B.C.E.?

Answer: We begin by making a few notational definitions.

Let N be the number of molecules in a breath.
Let p be the probability that one molecule in your next breath was contained in Lao-Tzu's last breath.
Let $q = 1 - p$ be the probability that one molecule in your next breath was not contained in Lao-Tzu's last breath.

The probability P that none of the molecules in your next breath were contained in Lao-Tzu's last breath is therefore

$$P = q^N = (1-p)^N.$$

We would like to estimate the number of molecules, N, in a breath, and the probability p that one molecule in your next breath was also contained in Lao-Tzu's last breath. We shall estimate the number of molecules in one breath by first estimating the volume $V\mathrm{breath}$ of one breath, and then converting this to number of moles, n_m, at STP (Standard Temperature and Pressure). I estimate that the volume of one breath is approximately 1 L. At STP, one mole of an ideal gas (which is a good approximation for the actual properties of air) occupies 22.4 L, so the number of moles of air in one breath is

$$N = N_A n_m \simeq \left(6 \times 10^{23} \mathrm{molecules/mole}\right)\left(4 \times 10^{-2} \mathrm{mole}\right) \cong 3 \times 10^{22} \mathrm{molecules}$$

where $N_A \approx 6 \times 10^{-23} \mathrm{molecules/mole}$ is the Avogadro constant.

We shall estimate the probability p as the ratio of the volume of a breath to the volume of the atmosphere:

$$p \simeq \frac{\mathrm{volume\ of\ one\ breath}}{\mathrm{volume\ of\ atmosphere}} = \frac{V_{\mathrm{breath}}}{V_{\mathrm{atm}}}.$$

In order to estimate the volume, V_{atm} of the earth's atmosphere, let's assume that the atmosphere is a uniform spherical shell of thickness t and radius r; this is the same model used for the volume of the oceans in another example. Then the volume of the shell is approximately $V_{\mathrm{atm}} \simeq 4\pi r^2 t$. We need to make two estimations. Let's suppose: 1) the thickness of the shell is approximately $t = 10\ \mathrm{km} = 10^4\ \mathrm{m}$; 2) the radius $r = R_E$, where we estimate the radius of the Earth as $R_E \approx 6 \times 10^6 \mathrm{m}$. Note that $t << R_E$, a necessary condition for using the spherical shell approximation as given above. Then our estimate for the volume of the earth's atmosphere is

$$V_{\mathrm{atm}} \cong 4\pi\, r^2 t = 4\pi\left(6 \times 10^6\ \mathrm{m}\right)^2 \left(10^4\ \mathrm{m}\right) = 4 \times 10^{18}\ \mathrm{m}^3.$$

We estimated the volume of one breath as 1 L; let's convert this to cubic meters:

$$V_{\mathrm{breath}} \approx 1\ \mathrm{L} = \left(\frac{1\ \mathrm{L}}{1 \times 10^3\ \mathrm{cm}^3}\right)\left(\frac{\left(1 \times 10^{-2}\ \mathrm{m}\right)^3}{1\ \mathrm{cm}^3}\right) = 1 \times 10^{-3}\ \mathrm{m}^3$$

So the probability that one specific molecule in your next breath was contained in Lao-Tzu's last breath is estimated as

$$p \cong \frac{\mathrm{volume\ of\ one\ breath}}{\mathrm{volume\ of\ atmosphere}} = \frac{V_{\mathrm{breath}}}{V_{\mathrm{atm}}} \cong \frac{1 \times 10^{-3}\ \mathrm{m}^3}{4 \times 10^{18}\ \mathrm{m}^3} \cong 1 \times 10^{-21}.$$

The probability, $q = 1 - p$, that one specific molecule in your next breath was not contained in Lao-Tzu's last breath, is essentially one: $q = 1 - p \cong 1 - 10^{-21}$. The total probability P that none of the molecules in your next breath were contained in Lao-Tzu's last breath is then

$$P = q^N = (1 - p)^N \cong \left(1 - 10^{-21}\right)^{3 \times 10^{22}},$$

or alternately,

$$P = q^N = (1 - p)^N \cong \left(1 - 10^{-21}\right)^{3 \times 10^{22}} = \left(1 - \frac{3 \times 10^1}{3 \times 10^{22}}\right)^{3 \times 10^{22}}.$$

To simplify the calculation, we can now use a representation for the exponential function (see Appendix 2.A for a proof of this result),

$$e^{-x} = \lim_{N\to\infty}\left(1 - \frac{x}{N}\right)^N$$

Thus

$$P \simeq \left(1 - \frac{3\times 10^1}{3\times 10^{22}}\right)^{3\times 10^{22}} \simeq e^{-3\times 10^1} \simeq 1\times 10^{-13} \simeq 0$$

So the probability P is nearly zero that none of the molecules in your next breath were contained in Lao-Tzu's last breath. From this, we see that it is certain that at least one molecule of air you inhale was in Lao-Tzu's last gasp.

Note one key assumption that we make in this problem: in 2500 years, the atmosphere has re-circulated to the point that Lao-Tzu's final exhalation is equally distributed throughout the world. Recently, a $3\mathrm{km}$ ice core sample had been drilled from the Antarctic ice sheet.

Challenge Question for reader: what is the probability that an air bubble in the ice at the bottom of the sample contains an atom in Lao-Tzu's final exhalation?

2.A Representation of the exponential function

Show that

$$\lim_{N\to\infty}\left(1 - \frac{a}{N}\right)^N = e^{-a}.$$

Proof: Let's begin by proving that

$$\lim_{y\to 0}\frac{\ln(1-ay)}{y} = -a.$$

Apply the Taylor formula to the numerator

$$\ln(1-ay) = -ay - \frac{1}{2}(ay)^2 + O(y^3).$$

Then

$$\lim_{y\to 0}\frac{\ln(1-ay)}{y} = \lim_{y\to 0}\frac{-ay - \frac{1}{2}(ay)^2 + O(y^3)}{y} = -a.$$

Define b^x for all $b > 0$ raised to any real number x by $b^x = e^{x\ln b}$. Let $b = \ln(1-ay)$ and $\ln(1-ay)^{1/y} = e^{(1/y)\ln(1-ay)}$. Thus

$$\lim_{y\to 0}\ln(1-ay)^{1/y} = \exp(\lim_{x\to 0}((1/y)\ln(1-ay)) = e^{-a}.$$

Let $y = 1/N$. Then

$$\lim_{N\to\infty}\ln(1-\frac{a}{N})^N = e^{-a}$$