MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

8.02

Driven RLC Circuits: Impedance and Admittance with Worked Examples

Section 1: Representation of Voltage and Current as Complex Numbers:

Consider the RLC circuit with ac voltage source shown in Figure 1.

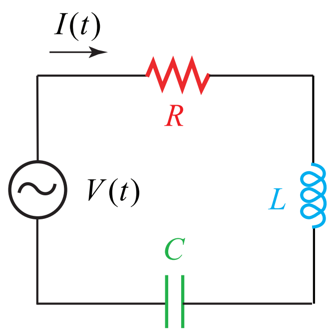

Figure 1: RLC circuit with ac voltage source

Consider an alternating voltage source (ac) voltage source given by

$$V(t) = V_0 \cos(\omega t) \ , \tag{1}$$

where V_0 is the amplitude and ωt is a time dependent phase. The alternating current in the circuit is given by

$$I(t) = I_0 \cos(\omega t - \phi) \ , \tag{2}$$

where I_0 is the amplitude and $\omega t - \phi$ is the time dependent phase. The quantity ϕ is called the *phase constant* and is equal to the difference between the phases of the voltage and the current $\phi = \omega t - (\omega t - \phi)$. For that reason the phase constant is also called the *phase shift*.

We shall represent voltage and current as complex numbers, basing our representation on the Euler formula,

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \ , \tag{3}$$

which equates the complex number $e^{i\omega t}$ to the sum of a real number $\cos(\omega t)$ with an imaginary number $i\sin(\omega t)$. We define two projection operators. The first one takes the complex number $e^{i\omega t}$ and gives its real part,

$$\text{Re}(e^{i\omega t}) = \cos(\omega t) \ . \tag{4}$$

The second operator takes the complex number $e^{i\omega t}$ and gives its imaginary part, which is the real number

$$\mathrm{Im}(e^{i\omega t}) = \sin(\omega t) \tag{5}$$

We introduce a complex valued function for the ac voltage

$$V_{\mathbb{C}}(t) = V_0 e^{i\omega t} \ . \tag{6}$$

The actual ac voltage is the real part of this complex number

$$V(t) = \mathrm{Re}(V_{\mathbb{C}}(t)) = \mathrm{Re}(V_0 e^{i\omega t}) = V_0 \cos(\omega t) \ . \tag{7}$$

We also introduce a complex valued function for the current

$$I_{\mathbb{C}}(t) = I_0 e^{i(\omega t - \phi)} \ . \tag{8}$$

The actual current is the real part of the complex current

$$I(t) = \mathrm{Re}(I_{\mathbb{C}}(t)) = \mathrm{Re}(I_0 e^{i(\omega t - \phi)}) = I_0 \cos(\omega t - \phi) \tag{9}$$

In what follows we shall just refer to the complex valued functions for voltage and current as $V(t)$ and $I(t)$ respectively.

Section 2: Impedance

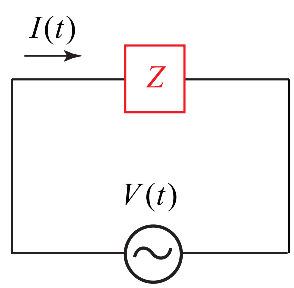

Figure 2: Representation of Impedance of a Circuit Element

The *impedance* Z is a complex number that generalizes the concept of resistance to any circuit element (Figure 2), as the proportionality constant between voltage and current for any circuit element,

$$V(t) = Z\, I(t) \ . \tag{1}$$

Because impedance is a complex number $Z = x + iy$, it can be expressed in terms of a modulus and phase factor as

$$Z = |Z| e^{i\delta} \ , \tag{2}$$

where the amplitude is

$$|Z| = (x^2 + y^2)^{1/2} \ , \tag{3}$$

and the phase factor is

$$\delta = \tan^{-1}(y/x) \ . \tag{4}$$

The voltage across the branch and the current in the branch are related by

$$V(t) = ZI(t) = |Z| e^{i\delta} I(t) \ . \tag{5}$$

Using our complex valued functions for voltage and current, Eq. (5) can be rewritten as

$$V_0 e^{i\omega t} = |Z| e^{i\delta} I_0 e^{i\omega t} e^{-i\phi} \ , \tag{6}$$

which after dividing through by the factor $e^{i\omega t}$ becomes

$$V_0 = |Z| e^{i\delta} I_0 e^{-i\phi} \ . \tag{7}$$

Because the left hand side of Eq. (7) is a real number, the two phase factors on the right hand side must satisfy

$$e^{i\delta} e^{-i\phi} = 1 \ . \tag{8}$$

Therefore the current phase constant is equal to the phase constant of the impedance

$$\phi = \delta \ . \tag{9}$$

Eq. (7) the simplifies to

$$V_0 = |Z| I_0 \ . \tag{10}$$

The amplitude of the current is then equal to

$$I_0 = V_0 / |Z| \ . \tag{11}$$

Thus knowledge of the modulus $|Z|$ and phase δ of the impedance of a circuit element determines the amplitude I_0 and phase constant ϕ of the current through that element.

Section 3: Circuit Elements in Series:

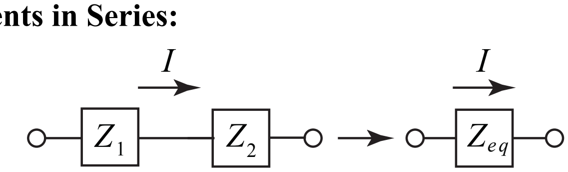

Figure 3: Equivalent impedance of circuit elements in series

For the two circuit elements connected in series shown in the Figure 3, the potential difference adds

$$V = V_1 + V_2 \ . \tag{12}$$

Define the equivalence impedance by $V = Z_{eq} I$. The Eq. (12) becomes

$$Z_{eq} I = Z_1 I + Z_2 I \ . \tag{13}$$

Hence impedance adds for circuit elements connected in series

$$Z_{eq} = Z_1 + Z_2 \ . \tag{14}$$

By writing the equivalence impedance in terms of a modulus and phase factor

$$Z_{eq} = \left| Z_{eq} \right| e^{i\delta} \ , \tag{15}$$

we have that the current phase constant satisfies

$$\phi = \delta \ , \tag{16}$$

and the amplitude of the current satisfies

$$I_0 = V_0 / \left| Z_{eq} \right| \ . \tag{17}$$

Section 4: Circuit Elements in Parallel:

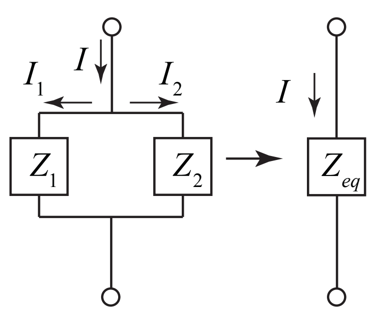

Figure 4: Equivalent impedance of circuit elements in parallel

When two circuit elements are connected in parallel (Figure 4), the potential difference V across each element is the same and the current into a junction satisfies

$$I = I_1 + I_2 \tag{18}$$

Define the equivalence impedance by $V = Z_{eq} I$. The Eq. (18) can be rewritten as

$$\frac{1}{Z_{eq}}V = \frac{1}{Z_1}V + \frac{1}{Z_2}V \quad . \tag{19}$$

Hence the impedance adds inversely for circuit elements connected in parallel

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \quad . \tag{20}$$

If the complex number associated to the inverse of the impedance is written as

$$\frac{1}{Z_{eq}} = u + iv \tag{21}$$

then the impedance is

$$Z_{eq} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{(u^2+v^2)} \quad . \tag{22}$$

The equivalent impedance can be then written in terms of modulus and phase factor $Z_{eq} = \left|Z_{eq}\right| e^{i\delta}$, where

$$\left|Z_{eq}\right| = \frac{1}{(u^2+v^2)^{1/2}} \quad . \tag{23}$$

$$\delta = \tan^{-1}(-v/u) \quad . \tag{24}$$

Hence the current $I(t) = I_0 e^{i(\omega t - \phi)}$ has a phase constant equal to

$$\phi = \delta \quad , \tag{25}$$

and an amplitude equal to

$$I_0 = V_0 / \left|Z_{eq}\right| \quad . \tag{26}$$

Worked Example: Circuit Elements in Parallel and Series

Consider an ac circuit with elements that are in parallel and series as shown in Figure 5. We shall reduce this circuit by the same methods that we used for dc circuits. We first find the equivalent impedance for the two parallel elements. Then we find the total impedance for the two series elements.

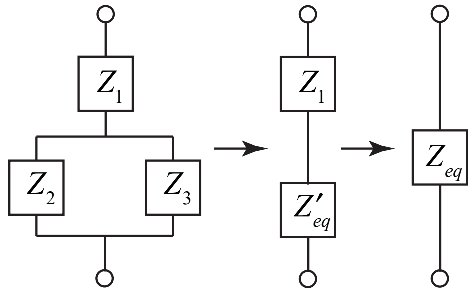

Figure 5: Network of circuit elements in series and parallel

The equivalent impedance for the two parallel elements is

$$\frac{1}{Z'_{eq}} = \frac{1}{Z_2} + \frac{1}{Z_3} \ . \tag{27}$$

We now have reduced the network to two series elements with equivalent impedance

$$Z_{eq} = Z_1 + Z'_{eq} = \left|Z_{eq}\right| e^{i\delta} \ . \tag{28}$$

As we showed in Section 2, Eqs. (11) and (9), the amplitude of the current is $I_0 = V_0 / \left|Z_{eq}\right|$, and the phase constant for the current is $\phi = \delta$.

Section 5: Impedance for Resistor, Capacitors, and Inductors

The complex values for the impedance of a resistor, inductor, and capacitor are determined in the following three worked examples.

Worked Example 3: Driven R Circuit (pure resistive, $C = \infty$, and $L = 0$)

Let's consider the purely resistive circuit with ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 e^{i(\omega t - \phi)}$. The circuit diagram is shown in Figure 6.

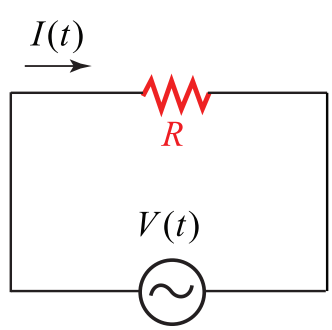

Figure 6: ac circuit with resistor as circuit element

The circuit equation is

$$V(t) = RI(t) \tag{29}$$

The phase shift $\phi = 0$ is zero, and the amplitude is $I_0 = V_0 / R$. The impedance Z_R satisfies

$$V = Z_R I \, . \tag{30}$$

Therefore the impedance for a resistor is the real number

$$Z_R = R \ . \tag{31}$$

A plot of voltage and current vs. time is shown in Figure 7. The phase shift $\phi = 0$ means that the voltage is peaking at the same time as the current. We say that the 'voltage is in phase with the current'.

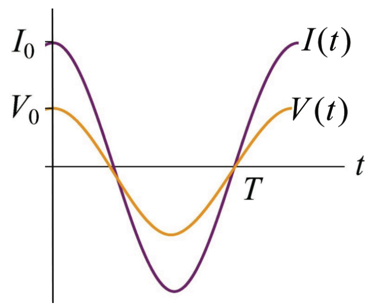

Figure 7: Voltage-current plot for ac circuit with resistor as circuit element

Worked Example 4: Driven L Circuit (pure inductive $C = \infty$, and $R = 0$)

Let's consider the purely inductive circuit (Figure 8) with ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 e^{i(\omega t - \phi)}$.

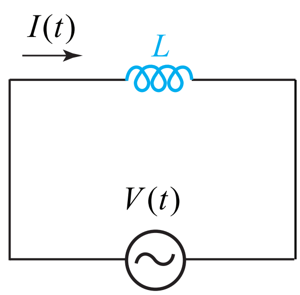

Figure 8: ac circuit with inductor as circuit element

We shall first determine the impedance Z_L of the inductor and then calculate the amplitude I_0 and phase constant ϕ of the current. There are two emf's in this circuit, the ac voltage supply $V(t)$ and the back emf $-LdI / dt$ arising from the changing current. Because there is no resistance, the two emf's sum to zero

$$V(t) - LdI/dt = 0 \ . \tag{32}$$

Introduce complex valued functions for the voltage $V(t) = V_0 e^{i\omega t}$ and the current $I(t) = I_0 e^{i\omega t} e^{-i\phi}$. Then Eq. (32) becomes

$$V_0 e^{i\omega t} - i\omega L I_0 e^{i\omega t} e^{-i\phi} = 0 \ . \tag{33}$$

Eq. (33) simplifies to

$$V(t) = i\omega L I(t) \ . \tag{34}$$

Recall that we define impedance for this inductive circuit element by the relation $V(t) = Z_L I(t)$, and hence

$$Z_L = i\omega L \, . \tag{35}$$

Because $i = e^{i\pi/2}$, we can write the impedance for the inductor as

$$Z_L = \omega L e^{i\pi/2} = |Z_L| e^{i\delta} \, . \tag{36}$$

The modulus is $|Z_L| = \omega L$ and the phase is $\delta = \pi/2$. Therefore using Eqs. (16) and (17), the amplitude of the current is

$$I_0 = \frac{V_0}{|Z_L|} = \frac{V_0}{\omega L} \ , \tag{37}$$

and the phase constant is

$$\phi = \delta = \pi/2 \ . \tag{38}$$

A plot of voltage and current vs. time is shown in Figure 9. The phase shift $\phi = \pi/2$ means that the voltage is peaking earlier in time than the current. We say that the 'voltage is leading the current' or equivalently, the 'current is lagging the voltage'.

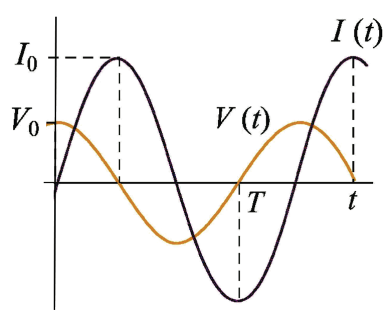

Figure 9: Voltage-current plot for ac circuit with inductor as circuit element

Worked Example 5: Driven C Circuit (pure capacitive, $R = 0$, and $L = 0$)

Let's consider the purely capacitive circuit with ac voltage supply given by $V(t) = V_0 e^{i\omega t}$ and with current $I(t) = I_0 e^{i(\omega t - \phi)}$. The circuit diagram is shown in Figure 10.

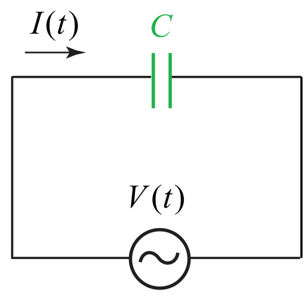

Figure 10: ac circuit with capacitor as circuit element

The circuit equation is given by

$$V(t) - \frac{Q}{C} = 0 \ . \tag{39}$$

We can differentiate this equation yielding

$$\frac{dV}{dt} - \frac{I}{C} = 0 \ . \tag{40}$$

We shall first determine the impedance Z_C of the capacitor and then calculate the amplitude I_0 and phase constant ϕ of the current. Introduce complex valued functions for the voltage $V(t) = V_0 e^{i\omega t}$ and the current $I(t) = I_0 e^{i\omega t} e^{-i\phi}$. Then Eq. (40) becomes

$$i\omega V_0 e^{i\omega t} = \frac{I_0 e^{i\omega t} e^{-i\phi}}{C} \ , \tag{41}$$

which we can rewrite as

$$V(t) = \frac{1}{i\omega C} I(t) = \frac{-i}{\omega C} I(t) = Z_C I(t) \ . \tag{42}$$

Therefore the impedance is

$$Z_C = \frac{-i}{\omega C} . \tag{43}$$

Using the fact that $-i = e^{-i\pi/2}$, we write the impedance for the capacitor as

$$Z_C = \frac{-i}{\omega C} = \frac{1}{\omega C} e^{-i\pi/2} = |Z_C| e^{i\delta} . \tag{44}$$

The modulus is $|Z_C| = \dfrac{1}{\omega C}$ and the phase is $\delta = -\pi / 2$. Therefore using Eqs. (16) and (17), the amplitude of the current is

$$I_0 = \frac{V_0}{|Z_C|} = \omega C V_0 \ , \tag{45}$$

and the phase constant is

$$\phi = \delta = -\pi / 2 \ . \tag{46}$$

A plot of voltage and current vs. time is shown in Figure 11. The phase shift $\phi = -\pi / 2$ means that the voltage is peaking later in time than the current. We say that the 'voltage is lagging the current' or equivalently, the 'current is leading the voltage'.

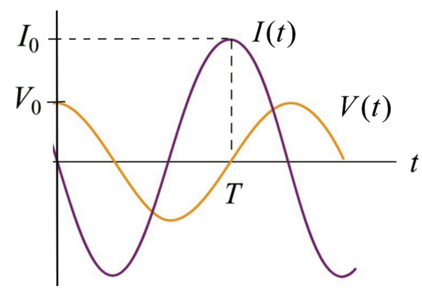

Figure 11: Voltage-current plot for ac circuit with capacitor as circuit element

Summary Table:

The complex values for the impedance for the resistor, capacitor and inductor are shown in Table 1. We shall derive these results in the W13D1 class.

Table 1: Impedance for Circuit Elements

Circuit element	impedance
inductor	$i\omega L$
resistor	R
capacitor	$-i / \omega C$

Section 6: Worked Examples of Circuit Elements in Series and Parallel

Worked Example 6: Driven LR circuit

Let's consider a series circuit with inductance L and resistance R and an ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 \cos(\omega t - \phi)$. The circuit diagram is shown Figure 12.

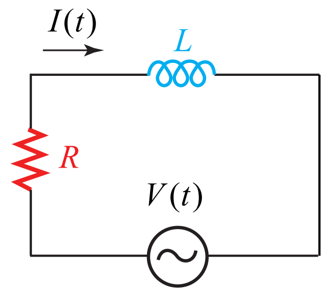

Figure 12: ac circuit with inductor and resistor in series

For this series circuit, we use our definitions for the impedance across the resistor $Z_R = R$, and the inductor $Z_L = i\omega L$, to find the equivalent impedance

$$Z_{eq} = (Z_R + Z_L) = (R + iL\omega) \ . \tag{47}$$

We write the impedance in terms of a modulus and phase factor

$$Z_{eq} = |Z_T| e^{i\delta} \ , \tag{48}$$

where the modulus is

$$|Z_{eq}| = (R^2 + (\omega L))^{1/2} \tag{49}$$

and the phase is

$$\delta = \tan^{-1}(L\omega/R) \tag{50}$$

Therefore using Eqs. (16) and (17), the amplitude of the current is

$$I_0 = \frac{V_0}{|Z_{eq}|} = \frac{V_0}{(R^2 + (\omega L))^{1/2}} \ , \tag{51}$$

and the phase constant is

$$\phi = \delta = \tan^{-1}(L\omega/R) \ . \tag{52}$$

The phase constant lies within the range $0 < \phi < \pi/2$.

Worked Example 7: Driven RC circuit

Let's consider a series circuit with capacitance C and resistance R and an ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 \cos(\omega t - \phi)$. The circuit diagram is shown in Figure 13.

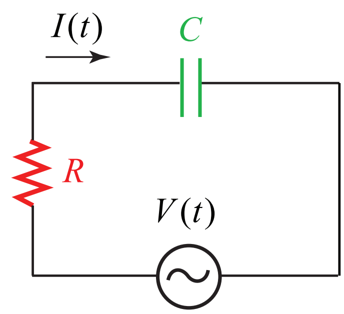

Figure 13: ac circuit with inductor and capacitor in series

For this series circuit, we use our definitions for the impedance across the resistor $Z_R = R$, and the inductor $Z_C = -i/\omega C$, to find the equivalent impedance

$$Z_{eq} = (Z_R + Z_C) = (R - \frac{i}{\omega C}) \quad . \tag{53}$$

We write the impedance in terms of a modulus and phase factor

$$Z_{eq} = \left| Z_T \right| e^{i\delta} \quad , \tag{54}$$

where the modulus is

$$\left| Z_{eq} \right| = (R^2 + (1/\omega C)^2)^{1/2} \tag{55}$$

and the phase is

$$\delta = \tan^{-1}(-1/\omega RC) \tag{56}$$

Therefore using Eqs. (16) and (17), the amplitude of the current is

$$I_0 = \frac{V_0}{\left| Z_{eq} \right|} = \frac{V_0}{(R^2 + (1/\omega C)^2)^{1/2}} \quad , \tag{57}$$

and the phase constant is

$$\phi = \delta = \tan^{-1}(-1/\omega RC) \quad . \tag{58}$$

The phase constant lies within the range $-\pi/2 < \phi < 0$.

Worked Example 8: Driven RLC circuit (series case)

Let's consider a series circuit with inductance L, capacitance C, and resistance R and an ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 \cos(\omega t - \phi)$. The circuit diagram is shown in Figure 14.

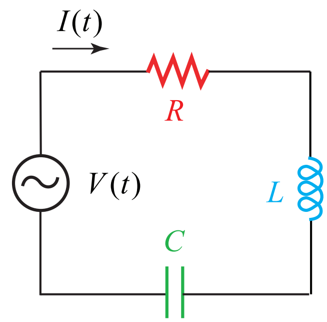

Figure 14: ac circuit with resistor, inductor and capacitor in series

Based on our experience from the last two cases, we just need the modulus and phase of the equivalent impedance,

$$Z_{eq} = Z_R + Z_C + Z_L = R + i\left(\omega L - \frac{1}{\omega C}\right) = |Z_T| e^{i\delta} \ . \tag{59}$$

The modulus is

$$|Z_{eq}| = \left(R^2 + (\omega L - 1/\omega C)^2\right)^{1/2} \tag{60}$$

The phase angle is given by

$$\phi = \delta = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) . \tag{61}$$

Therefore using Eqs. (16) and (17), the amplitude of the current is

$$I_0 = \frac{V_0}{|Z_{eq}|} = \frac{V_0}{(R^2 + (\omega L - 1/\omega C)^2)^{1/2}} \ , \tag{62}$$

and the phase constant is

$$\phi = \delta = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) . \tag{63}$$

The phase constant can vary between $-\pi/2 < \phi < \pi/2$.

Worked Example 9: Driven RLC circuit (parallel case)

Let's consider a circuit with inductance L, capacitance C, and resistance R and an ac voltage supply given by $V(t) = V_0 \cos(\omega t)$ and with current $I(t) = I_0 \cos(\omega t - \phi)$. The resistor, capacitor, and inductor are connected in parallel to the ac voltage source as shown in Figure 15.

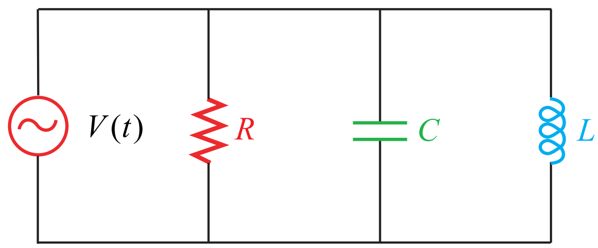

Figure 15: ac circuit with resistor, inductor and capacitor in parallel

The equivalent impedance is

$$\frac{1}{Z_{eq}} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + \frac{1}{-i\omega C} + \frac{1}{i\omega L} = \frac{1}{R} + i(\omega C - 1/\omega L) \ . \tag{64}$$

Setting

$$\frac{1}{Z_{eq}} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + \frac{1}{-i\omega C} + \frac{1}{i\omega L} = \frac{1}{R} + i(\omega C - 1/\omega L) = u + iv \ , \tag{65}$$

we can use Eqs. (23) and (24) to determine the modulus and phase of the equivalent impedance

$$\left|Z_{eq}\right| = \frac{1}{((1/R)^2 + (\omega C - 1/\omega L)^2)^{1/2}} \ , \tag{66}$$

$$\delta = -\tan^{-1}(R(\omega C - 1/\omega L)) \tag{67}$$

Using Eqs. (11) and (9) the amplitude of the current is given by

$$I_0 = V_0 / \left|Z_{eq}\right| = ((1/R)^2 + (\omega C - 1/\omega L)^2)^{1/2} V_0 ,$$

and the phase constant is given by

$$\phi = \delta = -\tan^{-1}\left(R(\omega C - 1/\omega L)\right) . \tag{68}$$