MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Experimental Study Group

8.022

Complex Numbers

Introduction:

Over time, mathematicians have found it useful to generalize their concept of numbers. The counting numbers 1, 2, 3, … were generalized to the integers …,-3, -2, -1, 0, 1, 2, 3… so that any two numbers could be subtracted (giving an answer that was still a valid number). Then the integers were generalized to rational numbers (fractions) so that numbers could be divided (except by zero). Then the rational numbers were generalized to real numbers so equations such as $x^2 = 2$ had solutions. Finally, real numbers were generalized to complex numbers, ensuring that every polynomial $P(x)$ of degree n has exactly n roots $x_1, x_2, x_3, ..., x_n$ where $P(x_i) = 0$ for all $x_i, i = 1\, to\, n$. Remarkably, one can prove that no further generalizations of the number concept are possible without giving up on basic properties such as commutativity of multiplication, $xy = yx$. Although mathematicians invented complex number for their perceived elegance and beauty, the have proven remarkably useful not merely in mathematics, but also in physics, for example in solving differential equations that describe electric circuits.

Rational numbers a/b can be described by a pair of integers (a,b) with $b \neq 0$, called the numerator and the denominator, and rules for how to add, subtract, multiply and divide them. For example, given a/b and c/d, the product is e/f with $e = ac$ and $f = bd$, while the sum is e/f with $e = ad + bc$ and $f = bd$. In the same way, complex numbers can be described by a pair of real numbers (x, y), where x is called the *real part* and y is called the *imaginary part*, with rules for how to add, subtract, multiply and divide them as described below. A complex number with $y = 0$ behaves like the real number x, while the complex number with $x = 0$, $y = 1$ is given the name i and has the property that its square equals 1. This means that a complex number z can be written as a sum of a real number x and a purely imaginary number iy where $i = +\sqrt{-1}$,

Section 1: Complex Numbers

The complex number z can be represented as a point in the xy - plane,

$$z = x + iy, \tag{1.1}$$

as shown in Figure 1. The complex conjugate $\overline{z}$ of a complex number z is defined to be

$$\overline{z} = x - iy \ . \tag{1.2}$$

(See Figure 1).

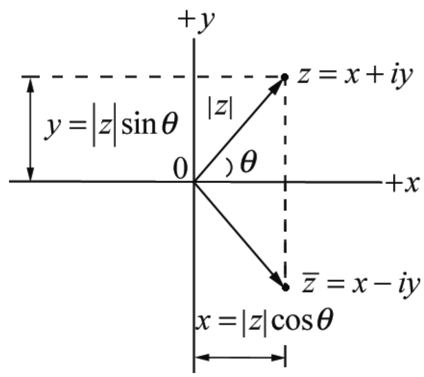


Figure 1: Representation of a complex number and its complex conjuagate

The modulus $|z|$ of a complex number z is defined to be

$$|z| = (z\bar{z})^{1/2} = ((x+iy)(x-iy))^{1/2} = (x^2+y^2)^{1/2} \tag{1.3}$$

where we used the fact that $i^2 = -1$. The inverse of a complex number is then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x-iy}{(x^2+y^2)} \quad . \tag{1.4}$$

The modulus of the inverse is the inverse of the modulus,

$$\left|\frac{1}{z}\right| = \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{|z|} \quad . \tag{1.5}$$

From Figure 1, we see that the modulus of a complex number is the length of the line connecting the origin to the point z in the xy- plane. Let θ be the angle this line makes with the positive x-axis, a quantity called the argument of z and denoted by $\arg(z)$. Then from Figure 1, we see that

$$\begin{aligned} x &= |z|\cos\theta \\ y &= |z|\sin\theta. \end{aligned} \tag{1.6}$$

The sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is

$$z_3 = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = x_3 + iy_3 \tag{1.7}$$

We can represent this by the 'vector sum' as shown in Figure 2.

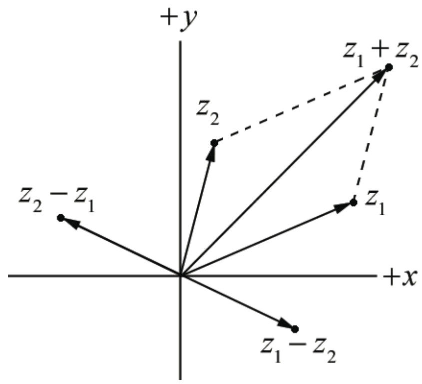


Figure 2: Addition and subtraction of two complex numbers

This means that the representation of the sum $z_3 = x_3 + iy_3$ satisfies

$$x_3 = x_1 + x_2\,, \quad y_3 = y_1 + y_2\,.$$

This is not true for the product of complex numbers,

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_3 + iy_3 \tag{1.8}$$

because

$$\begin{aligned} x_3 &= x_1 x_2 - y_1 y_2 \neq x_1 x_2 \\ y_3 &= x_1 y_2 + x_2 y_1 \neq y_1 y_2. \end{aligned} \tag{1.9}$$

Section 2: Phase and Amplitude and the Euler Formula

One of the most important identities in mathematics is the Euler formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.10}$$

This identity follows from the power series representations for the exponential, sine, and cosine functions,

$$e^{i\theta} = \sum_{n=0}^{n=\infty} \frac{1}{n!}(i\theta)^n = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \ldots \tag{1.11}$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \ldots \tag{1.12}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \tag{1.13}$$

We define two projection operators. The first one takes the complex number $e^{i\theta}$ and gives its real part,

$$\mathrm{Re}\, e^{i\theta} = \cos\theta .$$

The second operator takes the complex number $e^{i\theta}$ and gives its imaginary part, which is the real number

$$\mathrm{Im}\, e^{i\phi} = \sin\phi \tag{1.14}$$

So a complex number can also be represented as the product of a modulus $|z|$ and a phase factor $e^{i\theta}$,

$$z = |z| e^{i\theta} \tag{1.15}$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{1}{|z| e^{i\theta}} = \frac{1}{|z|} e^{-i\theta} \ , \tag{1.16}$$

where we used the fact that

$$\frac{1}{e^{i\theta}} = e^{-i\theta} \ . \tag{1.17}$$

In terms of modulus and phase, the sum of two complex numbers, $z_1 = |z_1| e^{i\theta_1}$ and $z_2 = |z_2| e^{i\theta_2}$, is

$$z_1 + z_2 = |z_1| e^{i\theta_1} + |z_2| e^{i\theta_2} . \tag{1.18}$$

A special case of this result is when the phase angles are equal, $\theta_1 = \theta_2$, then

$$z_1 + z_2 = |z_1| e^{i\theta_1} + |z_2| e^{i\theta_1} = \left(|z_1| + |z_2|\right) e^{i\theta_1} , \tag{1.19}$$

where the sum $z_1 + z_2$ has the same phase factor $e^{i\theta_1}$ as z_1 and z_2.

The product of two complex numbers, $z_1 = |z_1| e^{i\theta_1}$, and $z_2 = |z_2| e^{i\theta_2}$ is

$$z_1 z_2 = |z_1| e^{i\theta_1} |z_2| e^{i\theta_2} = |z_1||z_2| e^{i(\theta_1 + \theta_2)} \tag{1.20}$$

When the phases are equal, the product is

$$z_1 z_2 = |z_1| e^{i\phi_1} |z_2| e^{i\phi_1} = |z_1||z_2| e^{i2\phi_1} \tag{1.21}$$

which does not have the same factor as either z_1 or z_2.

Exercises:

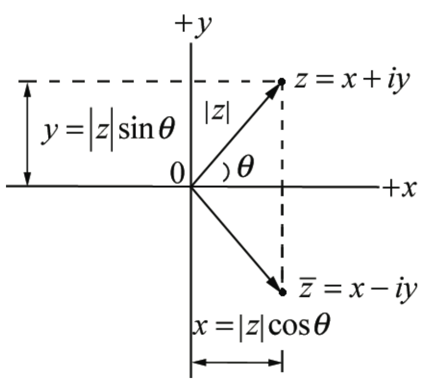
A complex number z is an ordered pair of real numbers (x, y), and can be expressed as $z = x + iy$. The ordered pair must satisfy the definitions for equality, addition, and multiplication of pairs given by

Equality: $z_1 = z_2 \Rightarrow (x_1 = x_2)$ and $(y_1 = y_2)$

Addition: $z_3 = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = x_3 + iy_3$

Multiplication: $z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = x_3 + iy_3$

The complex conjugate $\overline{z}$ of a complex number z is defined to be $\overline{z} = x - iy$.
We can also represent every non-zero complex number $z \neq 0$ by a modulus and phase according to $z = |z| e^{i\theta}$, where the modulus is given by $|z| = (z\overline{z})^{1/2} = ((x + iy)(x - iy))^{1/2} = (x^2 + y^2)^{1/2}$, and an angle θ. Note the angle θ is multivalued given by the angle shown in the figure below called the argument of z, $\arg(z)$, plus any integer multiple of 2π,
$\theta = \arg(z) + 2\pi n,\ n$ being any integer .



The inverse of a complex number z is given by $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{(x^2 + y^2)}$. Equivalently, using the modulus and argument representation, then inverse can be expressed as $\frac{1}{z} = \frac{1}{|z| e^{i\theta}} = \frac{1}{|z|} e^{-i\theta}$.

1. Express the following complex numbers in terms of modulus $|z|$ and argument of z, $\arg(z)$,

   a) $z = +i$

   b) $z = -i$

c) $z = \frac{1}{i}$

d) $z = \frac{1}{-i}$

e) $z = R + i\omega L$

f) $z = R - \frac{i}{\omega C}$

g) $z = \frac{V_0}{R + i\omega L}$

h) $z = \frac{V_0}{R - \frac{i}{\omega C}}$