Chapter 8

Magnetic Field

Introduction to Magnetic Fields

8.1 Introduction

In this Chapter, we define what we mean by a magnetic field and discuss at length the effect the magnetic field has on moving electric charges. In Chapter 9, we consider the manner by which magnetic fields are produced.

We have seen that a charged object produces an electric field $\vec{\mathbf{E}}$ at all points in space. In a similar manner, a bar magnet is a source of a magnetic field $\vec{\mathbf{B}}$. This can be readily demonstrated by moving a compass near the magnet. The compass needle will line up along the direction of the magnetic field produced by the magnet, as depicted in Figure 8.1.1.

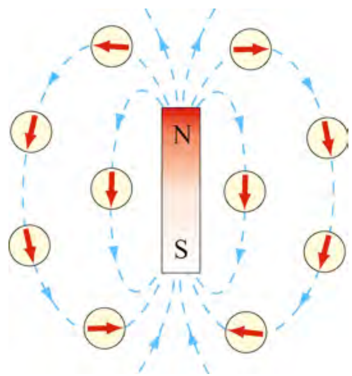


Figure 8.1.1 Magnetic field produced by a bar magnet.

A bar magnet consists of two poles, which are designated as the north (N) and the south (S). Magnetic fields are strongest at the poles. The magnetic field lines leave from the north pole and enter the south pole. When holding two bar magnets close to each other, the like poles will repel each other while the opposite poles attract (Figure 8.1.2).

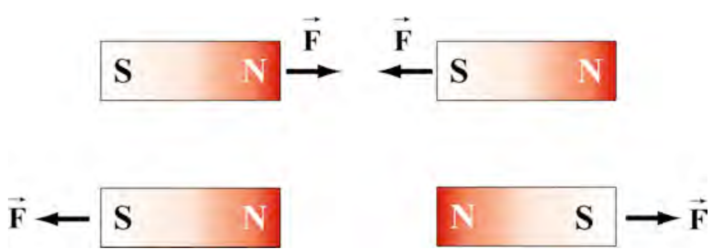


Figure 8.1.2 Magnets attracting and repelling

Unlike electric charges, which can be isolated, the two magnetic poles always come in a pair. When you break the bar magnet, two new bar magnets are obtained, each with a north pole and a south pole (Figure 8.1.3). In other words, magnetic "monopoles" do not exist in isolation, although they are of theoretical interest.

Figure 8.1.3 Magnetic monopoles do not exist in isolation

Another familiar source of magnetic fields is the current-carrying wire. In Figure 8.1.4, we show the magnetic field associated with an infinitely long current-carrying wire. The magnetic field is wrapped in circles about the wire, with the direction of the rotation of the circles determined by the right hand rule (if the thumb of your right hand is in the direction of the current, your fingers will curl in the direction of the magnetic field).

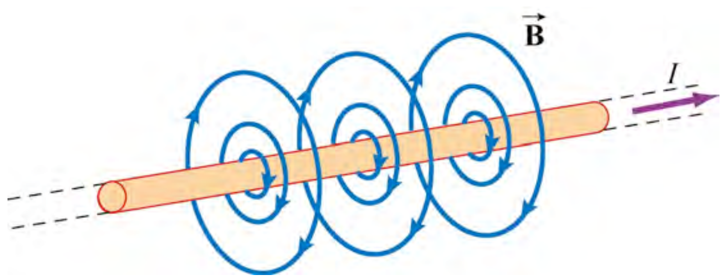


Figure 8.1.4 Magnetic field lines due to an infinite wire carrying current I.

As we have already seen in Section 1.7, moving electric charges also have magnetic fields, with a configuration similar to that shown in Figure 8.1.4, that is, circles centered on an axis defined by the vector velocity of the charge.

How do we define the magnetic field $\vec{\mathbf{B}}$? In the case of an electric field $\vec{\mathbf{E}}$, we have already seen that the field is defined as the force per unit charge:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}_e}{q}. \tag{8.1.1}$$

However, due to the absence of magnetic monopoles, $\vec{\mathbf{B}}$ must be defined in a different way.

8.2 The Definition of a Magnetic Field

To define the magnetic field at a point P, consider a particle of charge q, located at point P, moving with velocity $\vec{\mathbf{v}}$ at P.

(1) We find by experiment that the force $\vec{\mathbf{F}}_B$ at P on the charged particle at the point P is perpendicular to the direction of $\vec{\mathbf{v}}$ at P.

(2) When we vary the direction of the velocity of the charged particle through P, then for a particular direction of $\vec{\mathbf{v}}$ (and also opposite that direction), we observe that the force $\vec{\mathbf{F}}_B$ at P is zero. We define the **direction of the magnetic field at** P, $\vec{\mathbf{B}}$, to point

along that particular line formed by the direction of $\vec{\mathbf{v}}$ at P. In a moment we will establish by convention, which direction along that line the magnetic field points.

(3) We then vary the direction of $\vec{\mathbf{v}}$ at P, so that it moves perpendicular to $\vec{\mathbf{B}}$, until the magnitude of the force $\vec{\mathbf{F}}_B$ is maximal, $F_{B,\max}$. We define the **magnitude of the magnetic field at** P, B, by

$$B = \frac{F_{B,\max}}{|q||\vec{\mathbf{v}}|}. \tag{8.2.1}$$

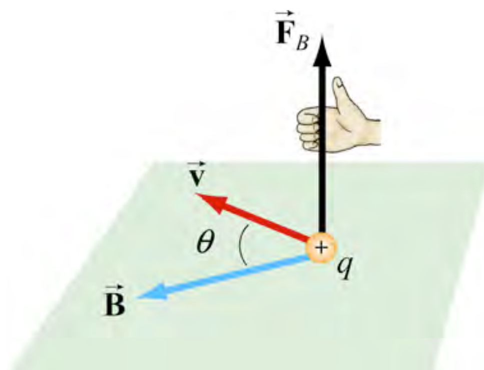


Figure 8.2.1 The definition of the magnetic force by the vector cross product

The above observations can be summarized by the following definition for the magnetic field at any point P. The field causes a force on a moving electric charge given by

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}. \tag{8.2.2}$$

In this section we simply state this force law, without trying to justify it in any way. This is the path followed by almost all introductory textbooks. But you will get a better understanding of this law at an intuitive level by reading Section 8.6.1 below, where we explain how Faraday thought of this force law, in terms of lines of force (see also Section 1.1).

When the sign of the charge of the particle is switched from positive to negative (or vice versa), we observe that the direction of the magnetic force also reverses. For a positively charged particle, we have chosen the direction of $\vec{\mathbf{B}}$ such that (Figure 8.2.1),

$$\text{direction}\,(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) = \text{direction}\,\vec{\mathbf{F}}_B\,,\ (\text{positvely charged particle}) \tag{8.2.3}$$

For the positively charged particle shown in Figure 8.2.1, the magnitude of $\vec{\mathbf{F}}_B$ is given by

$$F_B = |q|\,vB\sin\theta, \quad 0 \le \theta \le \pi. \tag{8.2.4}$$

The SI unit of magnetic field is the tesla (T),

$$1 \text{ tesla} = 1 \text{ T} = 1 \ \frac{\text{Newton}}{\text{(Coulomb)(meter/second)}} = 1 \frac{\text{N}}{\text{C} \cdot \text{m/s}} = 1 \ \frac{\text{N}}{\text{A} \cdot \text{m}} .$$

Another commonly used non-SI unit for $\vec{\mathbf{B}}$ is the *gauss* (G), where $1\text{T} = 10^4 \text{ G}$.

Note that $\vec{\mathbf{F}}_B$ is always perpendicular to $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, and cannot change the particle's speed v (and thus the kinetic energy). In other words, magnetic force cannot speed up or slow down a charged particle. Consequently, $\vec{\mathbf{F}}_B$ can do no work on the particle,

$$dW = \vec{\mathbf{F}}_B \cdot d\,\vec{\mathbf{s}} = q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{v}}\, dt = q(\vec{\mathbf{v}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{B}}\, dt = 0 . \qquad (8.2.5)$$

The direction of $\vec{\mathbf{v}}$, however, can be altered by the magnetic force, as we shall see below.

8.3 Magnetic Force on a Current-Carrying Wire

In this section we derive in the standard way the force that a current-carrying wire feels in a magnetic field. (You will get a better understanding of this law at an intuitive level by reading Section 8.6.3 below.)

8.3.1 Magnetic Force as the Sum of Forces on Charge Carriers in the Wire

We have just seen that a charged particle moving through a magnetic field experiences a magnetic force $\vec{\mathbf{F}}_B$. Because electric current consists of a collection of charged particles in motion, when placed in a magnetic field, a current-carrying wire will also experience a magnetic force.

Consider a long straight wire suspended in the region between the two magnetic poles. The magnetic field points out the page and is represented with dots (•). It can be readily demonstrated the wire is deflected to the left when the direction of the current is downward in the wire. However, when the current is upward, the deflection is rightward, as shown in Figure 8.3.1.

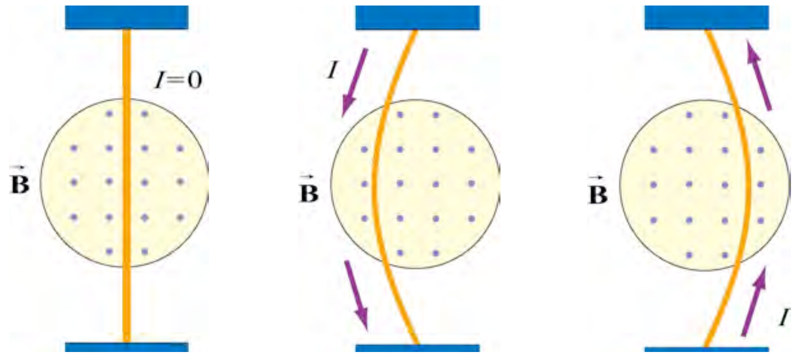


Figure 8.3.1 Deflection of current-carrying wire by magnetic field

To calculate the force exerted on the wire, consider a segment of wire of length s and cross-sectional area A, as shown in Figure 8.3.2. The magnetic field points into the page, and is represented with crosses (X).

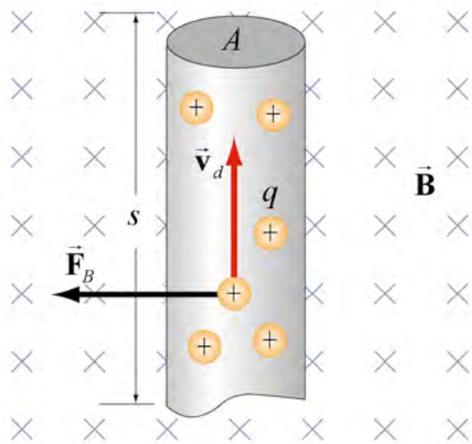


Figure 8.3.2 Magnetic force on a conducting wire

The charges move at an average drift velocity $\vec{\mathbf{v}}_d$. Because the amount of charge in this segment is $Q_{\text{tot}} = q(nAs)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

$$\vec{\mathbf{F}}_B = Q_{\text{tot}}\vec{\mathbf{v}}_d \times \vec{\mathbf{B}} = qnAs(\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}) = I(\vec{\mathbf{s}} \times \vec{\mathbf{B}}), \qquad (8.3.1)$$

where $I = nqv_dA$, and $\vec{\mathbf{s}}$ is a *length vector* with a magnitude s and directed along the direction of the electric current.

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{\mathbf{s}}$ (Figure 8.3.3).

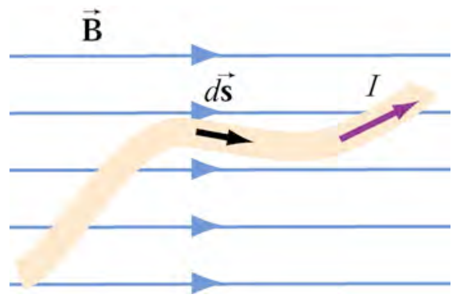


Figure 8.3.3 Current-carrying wire placed in a magnetic field

The magnetic force acting on the segment is

$$d\vec{\mathbf{F}}_B = I d\,\vec{\mathbf{s}}\times\vec{\mathbf{B}}. \tag{8.3.2}$$

Thus, the total force is

$$\boxed{\vec{\mathbf{F}}_B = I\int_a^b d\,\vec{\mathbf{s}}\times\vec{\mathbf{B}}}, \tag{8.3.3}$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$, as shown in Figure 8.3.4.

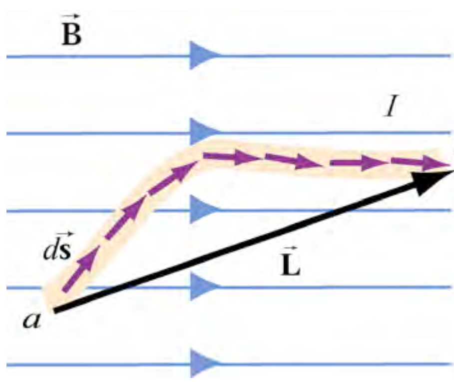


Figure 8.3.4 A curved wire carrying a current I.

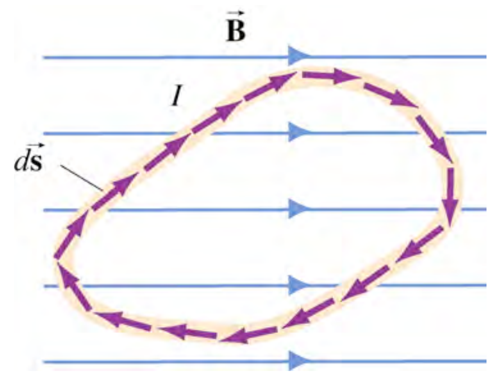


Figure 8.3.5 A closed loop carrying a current I in a uniform magnetic field.

Using Eq. (8.3.3), and the fact that the magnetic field is uniform so we can pull it out of the integral, the magnetic force on the wire is given by

$$\vec{\mathbf{F}}_B = I\left(\int_a^b d\,\vec{\mathbf{s}}\times\vec{\mathbf{B}}\right) = I\left(\int_a^b d\,\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} = I\vec{\mathbf{L}}\times\vec{\mathbf{B}}, \tag{8.3.4}$$

where $\vec{\mathbf{L}}$ is the length vector directed from a to b. However, if the wire forms a closed loop of arbitrary shape (Figure 8.3.5), then the set of differential length elements $d\,\vec{\mathbf{s}}$

form a closed polygon, and their vector sum is zero, i.e., $\oint d\vec{\mathbf{s}} = 0$. Therefore the magnetic force on a closed current loop is $\vec{\mathbf{F}}_B = \vec{0}$,

$$\vec{\mathbf{F}}_B = I\left(\oint d\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} = \vec{\mathbf{0}}, \quad (\text{uniform } \vec{\mathbf{B}}). \tag{8.3.5}$$

Example 8.1: Magnetic Force on a Semi-Circular Loop

Consider a closed semi-circular loop lying in the xy-plane carrying a current I in the counterclockwise direction, as shown in Figure 8.3.6.

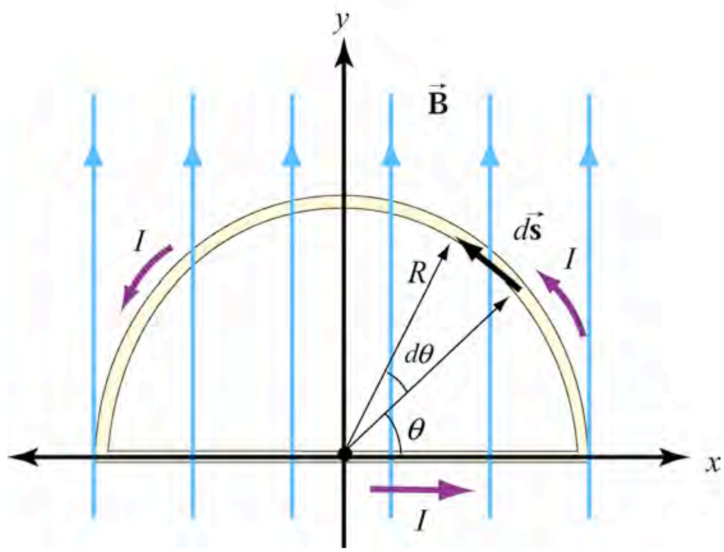


Figure 8.3.6 Semi-circular loop carrying a current I

A uniform magnetic field pointing in the $+y$ direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.

Solution: Let $\vec{\mathbf{B}} = B\hat{\mathbf{j}}$, and $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are the forces acting on the straight segment and the semicircular parts, respectively. Using Eq. (8.3.3) and noting that the length of the straight segment is $2R$, the magnetic force is

$$\vec{\mathbf{F}}_1 = I(2R\,\hat{\mathbf{i}})\times(B\,\hat{\mathbf{j}}) = 2IRB\,\hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}$ is directed out of the page. To evaluate $\vec{\mathbf{F}}_2$, we first note that the differential length element $d\vec{\mathbf{s}}$ on the semicircle can be written as

$$d\vec{\mathbf{s}} = ds\hat{\boldsymbol{\theta}} = Rd\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}).$$

The force acting on the length element $d\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{F}}_2 = Id\,\vec{\mathbf{s}}\times\vec{\mathbf{B}} = IR\,d\theta(-\sin\theta\,\hat{\mathbf{i}}+\cos\theta\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{j}}) = -IBR\sin\theta d\theta\;\hat{\mathbf{k}}\,.$$

Here we see that $d\vec{\mathbf{F}}_2$ points into the page. Integrating over the entire semi-circular arc, we have

$$\vec{\mathbf{F}}_2 = -IBR\,\hat{\mathbf{k}}\int_0^{\pi}\sin\theta d\theta = -2IBR\,\hat{\mathbf{k}}\,.$$

Thus, the net force acting on the semi-circular wire is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \vec{\mathbf{0}}\,.$$

This is consistent with Eq. (8.3.5) that the magnetic force acting on a closed current-carrying loop must be zero.

8.4 Torque on a Current Loop

In this section we derive in the standard way the torque on a current loop in a magnetic field. (You will get a better understanding of this torque at an intuitive level by reading Section 8.6.4 below.)

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{i}}$ which runs parallel to the plane of the loop, as shown in Figure 8.4.1(a)?

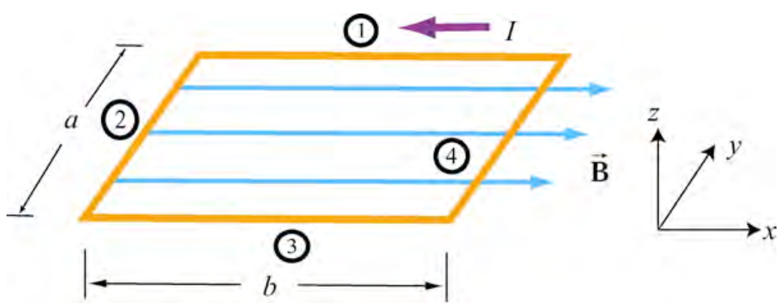


Figure 8.4.1(a) A rectangular current loop placed in a uniform magnetic field.

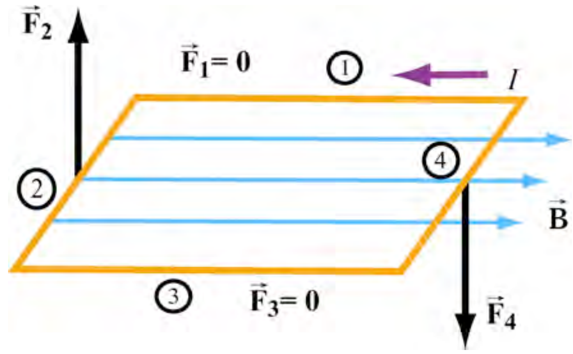


Figure 8.4.1(b) The magnetic forces acting on sides 2 and 4.

From Eq. (8.3.1), we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{\mathbf{s}}_1 = -b\,\hat{\mathbf{i}}$ and $\vec{\mathbf{s}}_3 = b\,\hat{\mathbf{i}}$ are parallel and anti-parallel to $\vec{\mathbf{B}}$ and their cross products vanish. The magnetic forces acting on segments 2 and 4 are non-vanishing,

$$\begin{cases} \vec{\mathbf{F}}_2 = I(-a\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{i}}) = IaB\,\hat{\mathbf{k}} \\ \vec{\mathbf{F}}_4 = I(a\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{i}}) = -IaB\,\hat{\mathbf{k}}. \end{cases} \tag{8.4.1}$$

Thus, the force on the rectangular loop is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}}\,, \tag{8.4.2}$$

as expected. Even though the force on the loop vanishes, the forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ will produce a torque that causes the loop to rotate about the y-axis (Figure 8.4.2). The torque about the center of the loop is

$$\begin{aligned} \vec{\boldsymbol{\tau}} &= -(b/2)\hat{\mathbf{i}}\times\vec{\mathbf{F}}_2 + (b/2)\hat{\mathbf{i}}\times\vec{\mathbf{F}}_4 = -(b/2)\hat{\mathbf{i}}\times(IaB\hat{\mathbf{k}}) + (b/2)\hat{\mathbf{i}}\times(-IaB\hat{\mathbf{k}}) \\ &= (IabB/2 + IabB/2)\hat{\mathbf{j}} = IabB\hat{\mathbf{j}} = IAB\hat{\mathbf{j}} \end{aligned} \tag{8.4.3}$$

where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y-axis.

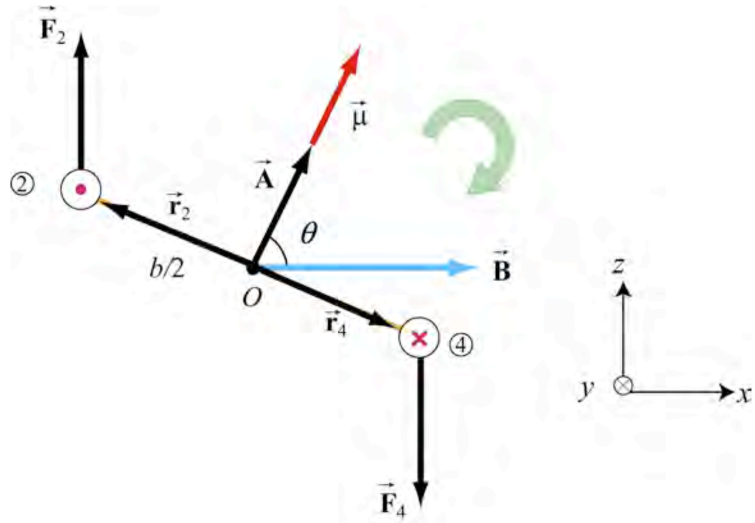


Figure 8.4.2 Rotation of a rectangular current loop in a uniform magnetic field

It is convenient to introduce the area vector $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{\mathbf{n}}$ is set by the conventional right-hand rule. In our case, we have $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$. The above expression for torque can then be rewritten as

$$\vec{\boldsymbol{\tau}} = I\vec{\mathbf{A}} \times \vec{\mathbf{B}} \tag{8.4.4}$$

The magnitude of the torque is at a maximum when $\vec{\mathbf{B}}$ is parallel to the plane of the loop (or perpendicular to $\vec{\mathbf{A}}$).

Consider now the more general situation where the loop (or the area vector $\vec{\mathbf{A}}$) makes an angle θ with respect to the magnetic field.

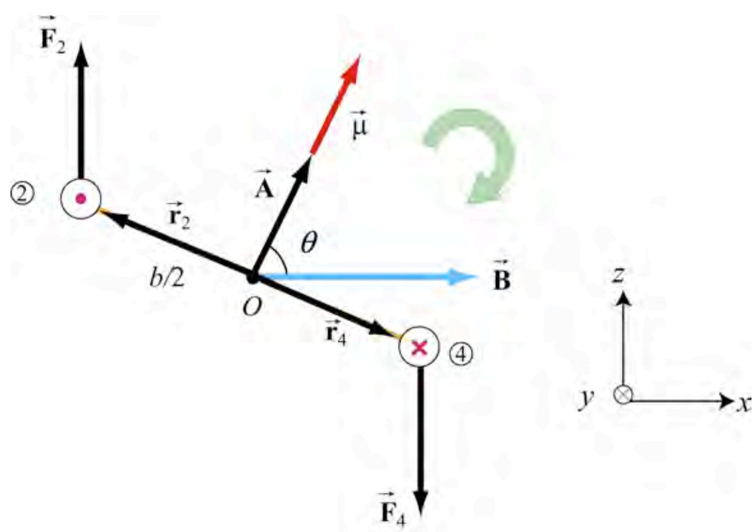


Figure 8.4.2 Rotation of a rectangular current loop

From Figure 8.4.2, the lever arms and can be expressed as

$$\vec{\mathbf{r}}_2 = \frac{b}{2}(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}) = -\vec{\mathbf{r}}_4\,. \tag{8.4.5}$$

The torque about the center of the loop becomes

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}}_2\times\vec{\mathbf{F}}_2 + \vec{\mathbf{r}}_4\times\vec{\mathbf{F}}_4 = 2\vec{\mathbf{r}}_2\times\vec{\mathbf{F}}_2 = 2\cdot(b/2)(-\sin\theta\,\hat{\mathbf{i}}+\cos\theta\,\hat{\mathbf{k}})\times(IaB\,\hat{\mathbf{k}})\\ &= IabB\sin\theta\,\hat{\mathbf{j}} = I\vec{\mathbf{A}}\times\vec{\mathbf{B}}.\end{aligned} \tag{8.4.6}$$

For a loop consisting of N turns, the magnitude of the toque is

$$\tau = NIAB\sin\theta\,. \tag{8.4.7}$$

The quantity $NI\vec{\mathbf{A}}$ is called the *magnetic dipole moment* $\vec{\mu}$,

$$\boxed{\vec{\mu} = NI\vec{\mathbf{A}}}. \tag{8.4.8}$$

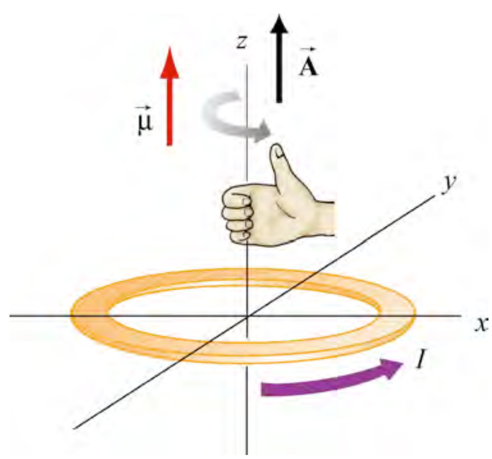


Figure 8.4.3 Right-hand rule for determining the direction of $\vec{\boldsymbol{\mu}}$

The direction of $\vec{\mu}$ is the same as the area vector $\vec{\mathbf{A}}$ (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure 8.4.3). The SI unit for the magnetic dipole moment is ampere-meter^2 $(\mathrm{A}\cdot\mathrm{m}^2)$. Using the expression for $\vec{\boldsymbol{\mu}}$, the torque exerted on a current-carrying loop can be rewritten as

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}}\times\vec{\mathbf{B}} \tag{8.4.9}$$

The above equation is analogous to $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}}\times\vec{\mathbf{E}}$ in Eq. (2.8.3), the torque exerted on an electric dipole moment $\vec{\mathbf{p}}$ in the presence of an electric field $\vec{\mathbf{E}}$. Recalling that the potential energy for an electric dipole is $U = -\vec{\mathbf{p}}\cdot\vec{\mathbf{E}}$ [see Eq. (2.8.7)], a similar form is expected for the magnetic case. In figure 8.4.2, the angular velocity is given by $\vec{\boldsymbol{\omega}} = (d\theta / dt)(-\hat{\mathbf{j}})$. The work done by the magnetic field to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$\begin{aligned} W &= \int_{\theta_0}^{\theta} \vec{\tau}\cdot\vec{\boldsymbol{\omega}}\,dt = \int_{\theta_0}^{\theta} (\mu B \sin\theta'\hat{\mathbf{j}})\cdot(d\theta'/dt)(-\hat{\mathbf{j}})\,dt \\ &= -\int_{\theta_0}^{\theta} (\mu B\sin\theta')\,d\theta' = \mu B(\cos\theta - \cos\theta_0). \end{aligned} \tag{8.4.10}$$

The result shows that a *positive* work is done by the field when $\cos\theta > \cos\theta_0$, (when $\theta < \theta_0$). The change in potential energy ΔU of the dipole is the negative of the work done by the field,

$$\Delta U = U - U_0 = -W = -\mu B(\cos\theta - \cos\theta_0)\,, \tag{8.4.11}$$

We shall choose our zero point for the potential energy when the angle between the dipole moment and the magnetic field is $\pi/2$, $U(\theta_0 = \pi/2) = 0$. Then, when the dipole moment is at an angle θ with respect to the direction of the external magnetic field, we define the potential energy function by

$$U(\theta) = -\mu B\cos\theta = -\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}}, \qquad \text{where } U(\pi/2) = 0 \tag{8.4.12}$$

The configuration is at a stable equilibrium when $\vec{\boldsymbol{\mu}}$ is aligned parallel to $\vec{\mathbf{B}}$, making U a minimum with $U_{\min} = -\mu B$. On the other hand, when $\vec{\boldsymbol{\mu}}$ and $\vec{\mathbf{B}}$ are anti-parallel, $U_{\max} = +\mu B$ is a maximum and the system is unstable.

8.4.1 Magnetic force on a dipole

As we have shown above, the force experienced by a current-carrying rectangular loop, (which we consider as a magnetic dipole) that is placed in a uniform magnetic field is zero. What happens if the magnetic field is non-uniform? In this case, there will be a force acting on the dipole.

Consider the situation where a small dipole $\vec{\boldsymbol{\mu}}$ is placed along the symmetric axis of a bar magnet, as shown in Figure 8.4.4.

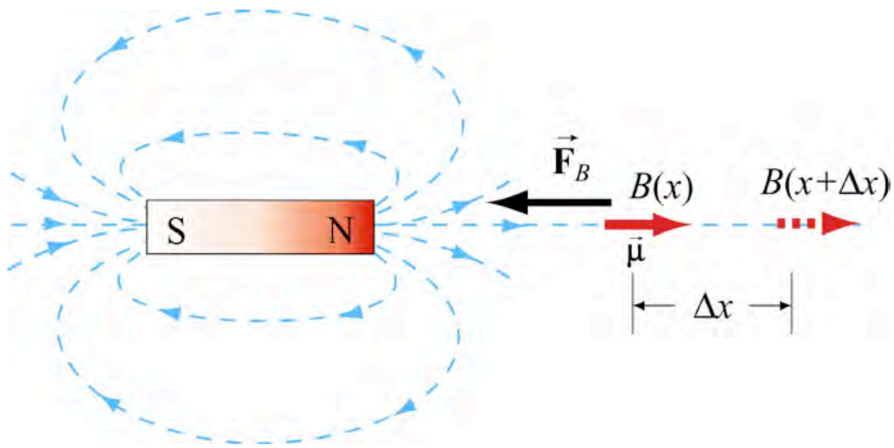


Figure 8.4.4 A magnetic dipole near a bar magnet.

The dipole experiences an attractive force by the bar magnet whose magnetic field is non-uniform in space. Thus, an external force must be applied to move the dipole to the right. The amount of force F_{ext} exerted by an external agent to move the dipole by a distance Δx is given by

$$F_{\text{ext}}\Delta x = W_{\text{ext}} = \Delta U = -\mu B(x+\Delta x) + \mu B(x) = -\mu[B(x+\Delta x) - B(x)], \quad (8.4.13)$$

where we have used Eq. (8.4.11). For small Δx, the external force may be obtained as

$$F_{\text{ext}} = -\mu \frac{[B(x+\Delta x) - B(x)]}{\Delta x} = -\mu \frac{dB}{dx}, \quad (8.4.14)$$

which is a positive quantity since $dB/dx < 0$, i.e., the magnetic field decreases with increasing x. This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have

$$F_B = \mu \frac{dB}{dx} = \frac{d}{dx}(\vec{\mu} \cdot \vec{\mathbf{B}}). \quad (8.4.15)$$

More generally, the magnetic force experienced by a dipole $\vec{\mu}$ placed in a non-uniform magnetic field $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu} \cdot \vec{\mathbf{B}}), \quad (8.4.16)$$

where

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \quad (8.4.17)$$

is the gradient operator.

8.5 Charged Particles in a Uniform Magnetic Field

In this section we discuss charge motion in a uniform magnetic field in the standard manner. (You will get a better understanding of this motion at an intuitive level by reading Section 8.6.2 below.)

If a particle of mass m moves in a circle of radius r at a constant speed v, there must be a radial force acting on the particle that always points toward the center and is perpendicular to the velocity of the particle.

In Section 8.2, we have already shown that the magnetic force $\vec{\mathbf{F}}_B$ always points in the direction perpendicular to the velocity $\vec{\mathbf{v}}$ of the charged particle and the magnetic field $\vec{\mathbf{B}}$. Since $\vec{\mathbf{F}}_B$ can do no work, it can only change the direction of $\vec{\mathbf{v}}$ but not its magnitude. What would happen if a charged particle moves through a uniform magnetic field $\vec{\mathbf{B}}$ with its initial velocity $\vec{\mathbf{v}}$ at a right angle to $\vec{\mathbf{B}}$? For simplicity, let the charge be $+q$ and the direction of $\vec{\mathbf{B}}$ be into the page. The magnitude force $\vec{\mathbf{F}}_B$ acting on the particle is a centripetal force (acting radially inward) given by

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}\times\vec{\mathbf{B}} = qvB(-\hat{\mathbf{r}})\,. \tag{8.5.1}$$

The charged particle will move in a circular path in a counterclockwise direction, as shown in Figure 8.5.1.

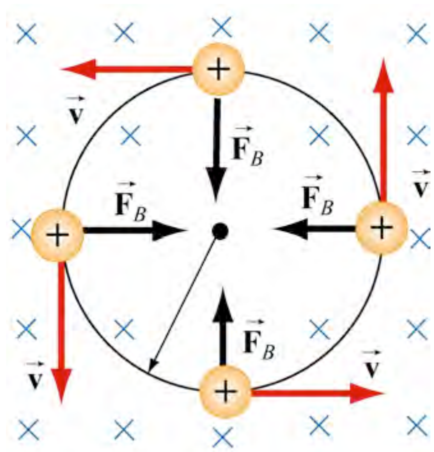


Figure 8.5.1 Path of a charge particle moving in a uniform $\vec{\mathbf{B}}$ field with velocity $\vec{\mathbf{v}}$ initially perpendicular to $\vec{\mathbf{B}}$.

Recall that a particle undergoing uniform circular motion accelerates to the center according to

$$\vec{\mathbf{a}} = \frac{v^2}{r}(-\hat{\mathbf{r}})\,. \tag{8.5.2}$$

Therefore the radially component of Newton's Second Law becomes

$$qvB = \frac{mv^2}{r}. \tag{8.5.3}$$

We cam solve Eq. (8.5.3) for the radius of the circle

$$\boxed{r = \frac{mv}{qB}}. \tag{8.5.4}$$

The period T (time required for one complete revolution) is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB}. \tag{8.5.5}$$

Similarly, the angular speed (cyclotron angular frequency) ω of the particle can be obtained as

$$\boxed{\omega = 2\pi f = \frac{v}{r} = \frac{qB}{m}}. \tag{8.5.6}$$

If the initial velocity of the charged particle has a component parallel to the magnetic field $\vec{\mathbf{B}}$, instead of a circle, the resulting trajectory will be a helical path, as shown in Figure 8.5.2:

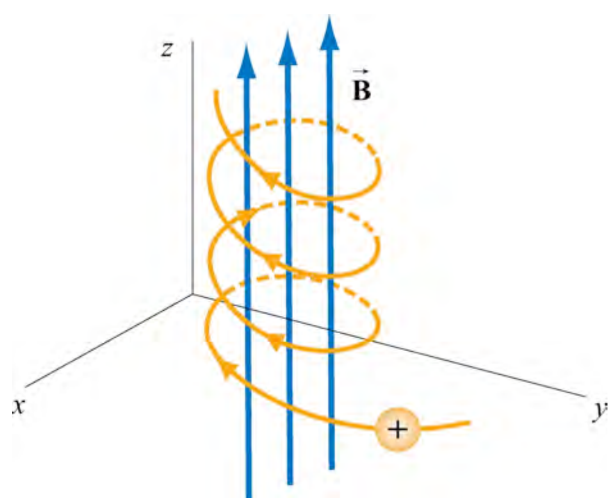


Figure 8.5.2 Helical path of a charged particle in an external magnetic field. The velocity of the particle has a non-zero component along the direction of $\vec{\mathbf{B}}$.

8.6 Rubber Bands and Strings and the Forces Transmitted by Magnetic Fields

We now return to our considerations in Section 1.1.2, where we asserted that depictions of the *total* field, that is the field due to all currents being considered, allows profound insight into the mechanisms whereby fields transmit forces. The stresses transmitted by magnetic fields can be understood as analogous to the forces transmitted by rubber bands and strings, but to reach this understanding we must show a representation of the *total* magnetic field, as we do in the four examples following. The examples below show you how Faraday, the father of field theory, understood how his "lines of force" picture explained the Lorentz Force Law at a more intuitive level than simply stating it, as we did in Eq. (8.2.2) above. The interactive simulation of the force on a current loop in the field of a magnetic dipole in Section 8.13 will give you additional insight into the cause of magnetic forces, from Faraday's point of view.

8.6.1 A Charged Particle in a Time-Varying Magnetic Field Movie

As an example of how we can understand the forces transmitted by magnetic fields if we look at the total field, following Faraday, consider a moving positive point charge at the origin in a rapidly changing time-dependent external field. This external field is uniform in space but varies in time according to the equation

$$\vec{\mathbf{B}} = -B_0 \sin^4\left(\frac{2\pi t}{T}\right)\hat{\mathbf{k}}\,. \tag{8.6.1}$$

We assume that the variation of this field is so rapid that the charge moves only a negligible distance in one period T. The magnetic field of the moving charge is given by the following expression, assumed non-relativistic motion (we will discuss this equation in Chapter 9, Section 9.1.2).

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2}\,. \tag{8.6.2}$$

Figure 8.6.1 shows two frames of an animation of the total magnetic field configuration for this situation, that is, the sum of the background field and the field of the moving charge. We note that Equation (8.2.2) would predict a magnetic force to the right in this situation. Figure 8.6.1(a) is at $t = 0$, when the vertical background magnetic field is zero, and we see only the magnetic field of the moving charge (the charge is moving out of the page, so the field circulates counterclockwise). Frame 8.6.1(b) is at a quarter period later, when the vertically downward magnetic field is at a maximum. To the left of the charge, where the field of the charge is in the same direction as the external magnetic field (downward), the magnetic field is enhanced. To the right of the charge, where the field of the charge is opposite that of the external magnetic field, the magnetic field is reduced (and is zero at one point to the right of the charge).

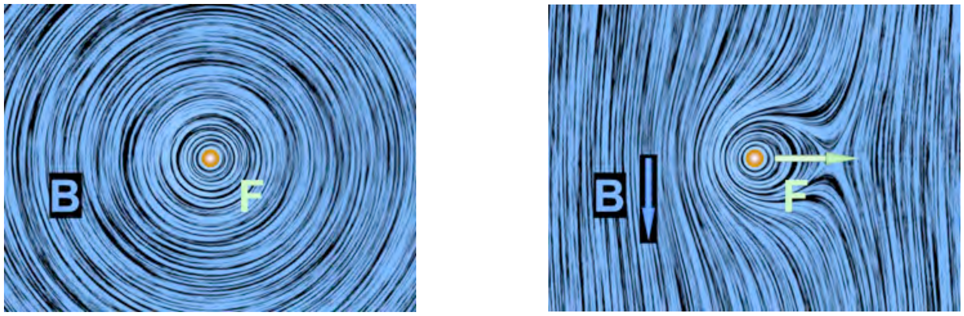


Figure 8.6.1 Two frames of an animation of the magnetic field around a positive charge moving out of the page in a time-changing magnetic field that points downward. The blue vector is the magnetic field and the green vector is the force on the point charge. http://youtu.be/zjy9b_PLvbw

Faraday would have interpreted the field configuration in Figure 8.6.1(b) as indicating a net force to the right on the moving charge, in agreement with the predictions of Equation (8.2.2). He would have said that this occurs because the pressure of the magnetic field is much higher on the left as compared to the right. Note that if the charge had been moving into the page instead of out of the page, the force would have been to the left, because the magnetic pressure would have been higher on the right. The animation of Figure 8.6.1 shows dramatically the inflow of energy into the neighborhood of the charge as the external magnetic field grows, with a resulting build-up of stress that transmits a sideways force to the moving positive charge.

We can estimate the magnitude of the force on the moving charge in Figure 8.6.1(b) as follows. At the time shown in Figure 8.6.1(b), the distance r_0 to the right of the charge at which the magnetic field of the charge is equal and opposite to the constant magnetic field is determined by

$$B_0 = \frac{\mu_0}{4\pi}\frac{qv}{r_0^2}. \tag{8.6.3}$$

The surface area of a sphere of this radius is $A = 4\pi r_0^2 = \mu_0 q v / B_0$. Suppose the pressure (force per unit area) and/or tension transmitted across the surface of this sphere surrounding the charge is of the order of $B^2/2\mu_0$. Since the magnetic field on the surface of the sphere is of the order B_0, the total force transmitted by the field is of order

$$F = PA = \frac{B_0^2}{2\mu_0}(4\pi r_0^2) = \frac{B_0^2}{2\mu_0}\cdot\frac{\mu_0 qv}{B_0} \approx qvB_0. \tag{8.6.4}$$

as we expect from Equation 8.2.2. Of course this net force to the right is a combination of a pressure pushing to the right on the left side of the sphere and a tension pulling to the right on the right side of the sphere in Figure 8.6.1(b).

The rough estimate that we have just made demonstrates that the pressures and tensions transmitted across the surface of this imaginary sphere surrounding the moving charge are plausibly of the order $B^2 / 2\mu_0$. In addition, this argument gives us some intuitive insight in to why the magnetic force on a moving charge is transverse to the velocity of the charge and to the direction of the background field. This is because of the side of the charge on which the total magnetic pressure is the highest. It is this pressure that causes the deflection of the charge.

8.6.2 Charged Particle Moving in a Uniform Magnetic Field Movie

We now use Faraday's insights to understand in a different way the circular motion of a charged particle in a magnetic field, as we discussed in the traditional manner in Section 8.5 above. Figure 8.6.2 shows a charge moving toward a region where the magnetic field is vertically upward. When the charge enters the region where the external magnetic field is non-zero, it is deflected in a direction perpendicular to that field and to its velocity as it enters the field. This causes the charge to move in an arc that is a segment of a circle, until the charge exits the region where the external magnetic field is non-zero. We show in the animation the total magnetic field, that is the sum of the external magnetic field and the magnetic field of the moving charge.

The bulging of that field on the side opposite the direction in which the particle is pushed is due to the buildup in magnetic pressure on that side. It is this pressure that causes the charge to move in a circle.

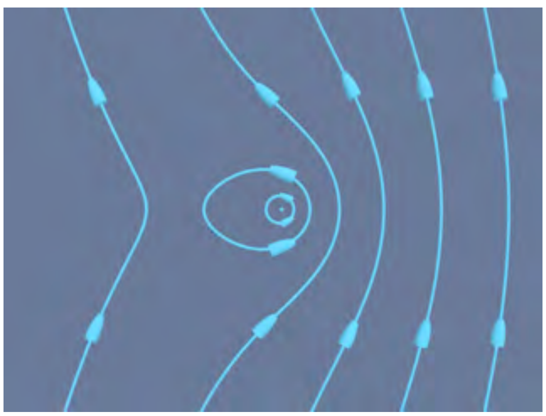
(a) http://youtu.be/eDojdmQ5sbY (b) http://youtu.be/a69-qC0S87U

Figure 8.6.2 Two views of a charged particle moving in a magnetic field that is non-zero over the pie- shaped region shown. The external field is upward.

Consider also momentum conservation. The moving charge in the movies of Figure 8.6.2 changes its direction of motion by ninety degrees over the course of the movies. How do we conserve momentum in this process? Momentum is conserved because momentum is transmitted by the field *from* the moving charge *to* the currents that are generating the constant external field. This is plausible given the field configuration shown in Figure 8.6.2. The magnetic field stress, which *pushes* the moving charge sideways, is accompanied by a tension *pulling* the current source for the external field in the opposite direction. To see this, look closely at the field stresses where the external field lines enter the region where the currents that produce them are hidden, and remember that the magnetic field acts as if it were exerting a tension parallel to itself. The momentum loss by the moving charge is transmitted to the hidden currents producing the constant field in this manner, so that the total momentum of the system is conserved.

8.6.3 Magnetic Force on a Current-Carrying Wire Movie

As another example, let us use Faraday's approach to understand intuitively why there is a force on a current-carrying wire in a magnetic field, as we discussed in the traditional manner in Section 8.3 above. A wire carrying current out of the page and free to move impinges on a region with a constant upward magnetic field. Figure 8.6.3 shows the total magnetic field, that is the field of the wire and the constant background field. We see that the force given by Eq. (8.3.1), which is to the left, is caused by the build-up of magnetic pressure to the right of the current-carrying wire. We can also see intuitively that the momentum being lost by the wire is being taken up by the sources of current that produce the constant field (not shown), because of the implied tension being transmitted to those sources of current by the total field at the top and bottom of the figure.



http://youtu.be/jIbhrRs5Q-Q

Figure 8.6.3 A wire carrying current out of the page and free to move impinges on a region with a constant upward magnetic field.

8.6.4 Torques on a Dipole in a Constant Magnetic Field Movie

> "…To understand this point, we have to consider that a [compass] needle vibrates by gathering upon itself, because of it magnetic condition and polarity, a certain amount of the lines of force, which would otherwise traverse the space about it…"
>
> Michael Faraday [1855]

Finally, let us use Faraday's approach to understand intuitively why there is a torque on a magnetic dipole in a background magnetic field, as we discussed in the traditional manner in Section 8.4 above. Consider a magnetic dipole in a constant background field. Historically, we note that Faraday understood the oscillations of a compass needle in exactly the way we describe here. We show in Figure 8.6.4 a magnetic dipole in a “dip needle” oscillating in the magnetic field of the Earth, at latitude approximately the same as that of Boston. The magnetic field of the Earth is predominantly downward and northward at these Northern latitudes, as the visualization indicates.

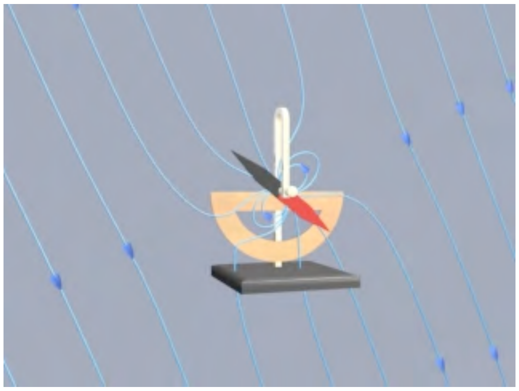
 http://youtu.be/hnCFz8N7Juk

Figure 8.6.4 A magnetic dipole in the form of a dip needle oscillates in the magnetic field of the Earth.

To explain what is going on in this visualization, suppose that the magnetic dipole vector is initially along the direction of the earth’s field and rotating clockwise. As the dipole rotates, the magnetic field lines are compressed and stretched. The tensions and pressures associated with this field line stretching and compression results in an electromagnetic torque on the dipole that slows its clockwise rotation. Eventually the dipole comes to rest. But the counterclockwise torque still exists, and the dipole then starts to rotate counterclockwise, passing back through being parallel to the Earth’s field again (where the torque goes to zero), and overshooting.

As the dipole continues to rotate counterclockwise, the magnetic field lines are now compressed and stretched in the opposite sense. The electromagnetic torque has reversed sign, now slowing the dipole in its counterclockwise rotation. Eventually the dipole will come to rest, start rotating clockwise once more, and pass back through being parallel to the field, as in the beginning. If there is no damping in the system, this motion continues indefinitely.

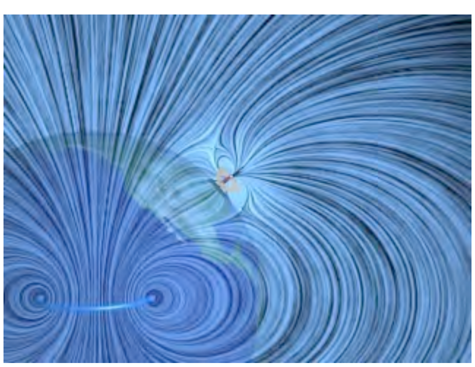
 http://youtu.be/2Ub4MMEWWn0

Figure 8.6.5 A magnetic dipole in the form of a dip needle oscillates in the magnetic field of the Earth. We show the currents that produce the earth's field in this visualization.

What about the conservation of angular momentum in this situation? Figure 8.6.5 shows a global picture of the field lines of the dip needle and the field lines of the Earth, which are generated deep in the core of the Earth. If you examine the stresses transmitted between the Earth and the dip needle in the accompanying movie, you can convince yourself that any clockwise torque on the dip needle is accompanied by a counterclockwise torque on the currents producing the earth's magnetic field. Angular momentum is conserved by the exchange of equal and opposite amounts of angular momentum between the compass and the currents in the Earth's core that generate the Earth's magnetic field. In contrast, the rotational kinetic energy of the compass when it comes to rest has been stored locally in the magnetic energy of the local magnetic field, from which it returns when the compass begins to rotate again.

8.6.5 Pressures and Tensions Transmitted by Magnetic Fields

> "…It appears therefore that the stress in the axis of a line of magnetic force is a tension, like that of a rope…"
>
> J. C. Maxwell [1861].

Let's now consider a more general case of stress (pressure or tension) transmitted by magnetic fields, as we did in Section 2.11.5 for electric fields. These two discussions are very similar. In Figure 8.6.6, we show an imaginary closed surface (a box) placed in a magnetic field. If we look at the face on the left side of this imaginary box, the field on that face is perpendicular to the outward normal to that face. Using the result illustrated in Figure 8.6.6, the field on that face transmits a pressure perpendicular to itself. In this case, this is a *push* to the *right*. Similarly, if we look at the face on the right side of this imaginary box, the field on that face is perpendicular to the outward normal to that face, the field on that face transmits a pressure perpendicular to itself. In this case, this is a *push* to the *left*.

If we want to know the total magnetic force transmitted to the interior of this imaginary box in the left-right direction, we add these two transmitted stresses. If the magnetic field is homogeneous, this total magnetic force transmitted to the interior of the box in the left-right direction is a push to the left and an equal but opposite push to the right, and the transmitted force adds up to zero.

Figure 8.6.6 An imaginary dark blue box in a magnetic field (long light blue vectors). The short gray vectors indicate the directions of stresses transmitted across the surface of the imaginary box by the field, either pressures (on the left or right faces of the box) or tensions (on the top and bottom faces of the box).

In contrast, if the right side of this imaginary box is sitting inside a long vertical solenoid, for which the magnetic field is vertical and constant, and the left side is sitting outside of that solenoid, where the magnetic field is zero, then there is a net push to the left, and we say that the magnetic field exerts a outward *pressure* on the walls of the solenoid. We can deduce this by simply looking at the magnetic field topology. At sufficiently high magnetic field, such forces will cause the walls of a solenoid to explode outward. A quantitative calculation of the pressure transmitted by magnetic fields in such a situation is presented in Section 11.11.

Similarly, if we look at the top face of the imaginary box in Figure 8.6.6, the field on that face is parallel to the outward normal to that face, and one may show that the field on that face transmits a tension along itself across that face. In this case, this is an *upward pull*, just as if we had attached a string under tension to that face, pulling upward. On the other hand, if we look at the bottom face of this imaginary box, the field on that face is anti-parallel to the outward normal to that face, and Faraday would again have said that the field on that face transmits a tension along itself. In this case, this is a *downward pull*, just as if we had attached a string to that face, pulling downward. Note that this is a *pull* parallel to the outward surface normal, whether the field is into the surface or out of the surface, since the pressures or tensions are proportional to the squares of the field magnitudes.

If we want to know the total magnetic force transmitted to the interior of this imaginary box in the up-down direction, we add these two transmitted stresses. If the magnetic field is homogeneous, this total magnetic force transmitted to the interior of the box in the up-down direction is a pull upward plus an equal and opposite pull downward, and adds to zero.

8.7 Applications

There are many applications involving charged particles moving through a uniform magnetic field.

8.7.1 Velocity Selector

In the presence of both electric field $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$, the total force on a charged particle is

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) . \tag{8.7.1}$$

This is known as the *Lorentz force law*. By combining the two fields, particles that move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons. In Figure 8.7.1 the schematic diagram of Thomson's apparatus is depicted.

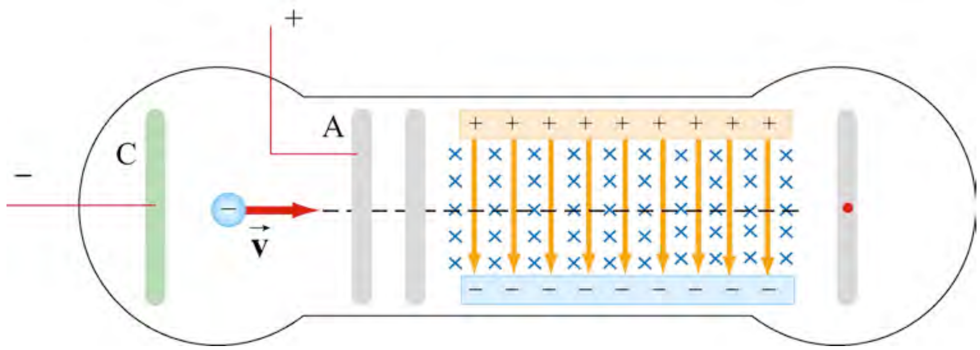


Figure 8.7.1 Thomson's apparatus

The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be $V_A - V_C = \Delta V$. The change in potential energy is equal to $\Delta U = q\Delta V = -e\Delta V$. The change in kinetic energy is $\Delta K = mv^2 / 2$. Because there is no work done by external forces, (the electric field is considered part of the system), the energy is constant and so $\Delta K = -\Delta U$. Therefore

$$mv^2 / 2 = e\Delta V \tag{8.7.2}$$

Thus, the speed of the electrons is given by

$$v = \sqrt{\frac{2e\Delta V}{m}}\,. \tag{8.7.3}$$

If the electrons then pass through a region where there exists a downward uniform electric field, the negatively charged electrons will be deflected upward. However, if in addition to the electric field, a magnetic field directed into the page is also applied, then the electrons will experience an additional downward magnetic force $-e\vec{\mathbf{v}}\times\vec{\mathbf{B}}$. When the two forces exactly cancel, the electrons will move in a straight path. From Eq. 8.7.1, we see that when the condition for the cancellation of the two forces is given by $eE = evB$, which implies

$$v = \frac{E}{B}\,. \tag{8.7.4}$$

In other words, only those particles with speed $v = E / B$ will be able to move in a straight line. Combining the two equations, we obtain

$$\boxed{\frac{e}{m} = \frac{E^2}{2(\Delta V)B^2}}\,. \tag{8.7.5}$$

By measuring E, ΔV, and B, the charge-to-mass ratio can be readily determined. The most precise measurement to date is $e / m = 1.758820174(71)\times 10^{11}$ C/kg.

8.7.2 Mass Spectrometer

Various methods can be used to measure the mass of an atom. One possibility is through the use of a mass spectrometer. The basic feature of a *Bainbridge* mass spectrometer is illustrated in Figure 8.7.2. A particle carrying a charge $+q$ is first sent through a velocity selector.

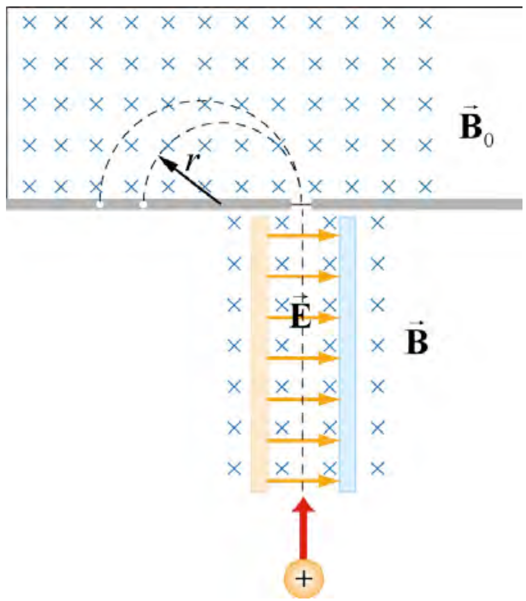


Figure 8.7.2 A Bainbridge mass spectrometer

The applied electric and magnetic fields satisfy the relation $E = vB$ so that the trajectory of the particle is a straight line. Upon entering a region where a second magnetic field $\vec{\mathbf{B}}_0$ pointing into the page has been applied, the particle will move in a circular path with radius r and eventually strike the photographic plate. Using Eq. (8.5.4), we have

$$r = \frac{mv}{qB_0}. \tag{8.7.6}$$

Because $v = E/B$, the mass of the particle can be written as

$$m = \frac{qB_0 r}{v} = \frac{qB_0 Br}{E}. \tag{8.7.7}$$

8.8 Summary

- The **magnetic force** acting on a charge q traveling at a velocity $\vec{\mathbf{v}}$ in a magnetic field $\vec{\mathbf{B}}$ is defined by expression

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}\times\vec{\mathbf{B}}\,.$$

- The magnetic force acting on a wire of length s carrying a steady current I in a magnetic field $\vec{\mathbf{B}}$, where $\vec{\mathbf{s}}$ points in the direction of the current is

$$\vec{\mathbf{F}}_B = I\vec{\mathbf{s}}\times\vec{\mathbf{B}}\,.$$

- The magnetic force $d\vec{\mathbf{F}}_B$ generated by a small portion of current I of length $d\vec{\mathbf{s}}$ in a magnetic field $\vec{\mathbf{B}}$ is

$$d\vec{\mathbf{F}}_B = I\,d\vec{\mathbf{s}}\times\vec{\mathbf{B}}$$

- The **torque** $\vec{\boldsymbol{\tau}}$ acting on a close loop of wire of area A carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\boldsymbol{\tau}} = I\vec{\mathbf{A}}\times\vec{\mathbf{B}}\,,$$

  where $\vec{\mathbf{A}}$ is a vector, which has a magnitude of A and a direction perpendicular to the loop.

- The **magnetic dipole moment** $\vec{\boldsymbol{\mu}}$ of a closed loop of wire of area A carrying a current I is given by

$$\vec{\boldsymbol{\mu}} = I\vec{\mathbf{A}}\,.$$

- The torque exerted on a magnetic dipole $\vec{\boldsymbol{\mu}}$ placed in an external magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}}\times\vec{\mathbf{B}}\,.$$

- The potential energy of a magnetic dipole placed in a magnetic field is

$$U = -\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}}\,.$$

- If a particle of charge q and mass m enters a magnetic field of magnitude B with a velocity $\vec{\mathbf{v}}$ perpendicular to the magnetic field lines, the radius of the circular path that the particle follows is given by

$$r = \frac{mv}{|q|B}\,.$$

  and the angular speed of the particle is

$$\omega = \frac{|q|B}{m}\,.$$

- As predicted in Section 1.1.2, if we look at the shape of magnetic field lines for the total magnetic field, as Faraday did, the magnetic forces transmitted by fields can be understood at an intuitive level by analogy to the more familiar forces exerted by strings and rubber bands.

8.9 Problem-Solving Tips

We have shown that in the presence of both magnetic field $\vec{\mathbf{B}}$ and the electric field $\vec{\mathbf{E}}$, the total force acting on a moving particle with charge q is $\vec{\mathbf{F}} = \vec{\mathbf{F}}_e + \vec{\mathbf{F}}_B = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$, where $\vec{\mathbf{v}}$ is the velocity of the particle. The direction of $\vec{\mathbf{F}}_B$ involves the cross product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, based on the right-hand rule. In Cartesian coordinates, the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ satisfy the following properties

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \ \ \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \ \ \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \ \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \ \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

For $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$, the cross product may be obtained as

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = (v_y B_z - v_z B_y)\hat{\mathbf{i}} + (v_z B_x - v_x B_z)\hat{\mathbf{j}} + (v_x B_y - v_y B_x)\hat{\mathbf{k}}\,.$$

If only the magnetic field is present, and $\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{B}}$, then the trajectory is a circle with a radius $r = mv / |q| B$, and an angular speed $\omega = |q| B / m$.

When dealing with a more complicated case, it is useful to work with individual force components. For example,

$$F_x = ma_x = qE_x + q(v_y B_z - v_z B_y)$$

8.10 Solved Problems

8.10.1 Rolling Rod

A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance s, as shown in the Figure 8.9.1. The rod carries a current I and rolls without slipping along rails, which are placed in a uniform magnetic field $\vec{\mathbf{B}}$ directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?

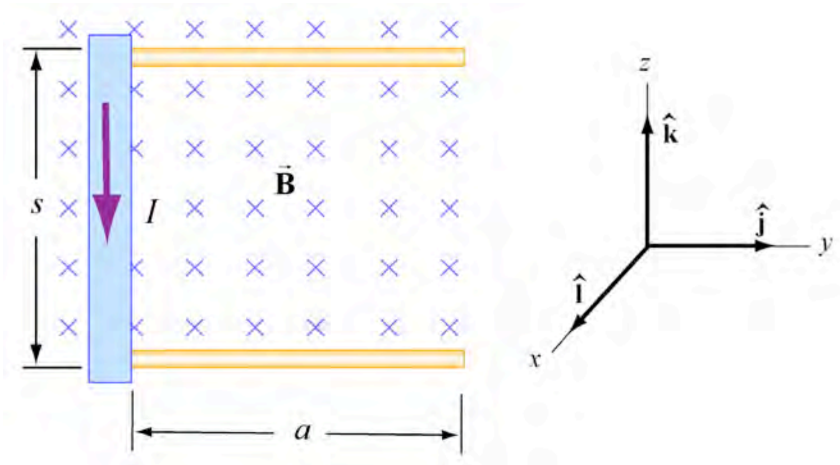


Figure 8.9.1 Rolling rod in uniform magnetic field

Solution: The total work done by the magnetic force on the rod as it moves through the region is

$$W = \int \vec{\mathbf{F}}_B \cdot d\,\vec{\mathbf{s}} = F_B a = (IsB)a\,. \tag{8.10.1}$$

By the work-energy theorem, W must be equal to the change in kinetic energy

$$\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\,, \tag{8.10.2}$$

where both translation and rolling are involved. Since the moment of inertia of the rod is given by $I = mR^2/2$, and the condition of rolling without slipping implies $\omega = v/R$, we have

$$I\ell Ba = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2\,. \tag{8.10.3}$$

Thus, the speed of the rod as it leaves the rails is

$$v = \sqrt{\frac{4I\ell Ba}{3m}}\,. \tag{8.10.4}$$

But wait! Didn't we say that magnetic forces do no work because they are always perpendicular to the velocity of the moving charges? What's going on here?

Let's first write the current in the rod as $I = \lambda s$, where λ is the charge density in the wire and $\vec{\mathbf{u}} = -u\,\hat{\mathbf{k}}$ is the velocity of the charges. Using the coordinate system shown in

Figure 8.9.1, the magnetic force acting on a small segment of charge dq moving with speed $\vec{\mathbf{u}} = -u\hat{\mathbf{k}}$ is given by

$$d\vec{\mathbf{F}}_{B,1} = dq\vec{\mathbf{u}} \times \vec{\mathbf{B}} = dqu(-\hat{\mathbf{k}}) \times (-B\hat{\mathbf{i}}) = dquB\hat{\mathbf{j}}. \qquad (8.10.5)$$

This is not the total force because once the rod is moving in the horizontal direction with speed $\vec{\mathbf{w}} = w\hat{\mathbf{j}}$, the magnetic field exerts an additional force on the charge dq in the rod given by

$$d\vec{\mathbf{F}}_{B,2} = dq\vec{\mathbf{w}} \times \vec{\mathbf{B}} = dq(w\hat{\mathbf{j}}) \times (-B\hat{\mathbf{i}}) = dqwB\hat{\mathbf{k}}. \qquad (8.10.6)$$

The force $d\vec{\mathbf{F}}_{B,2}$ is in the direction opposes the flow of charge, so there must be some source of emf that does work on the moving charges to keep the current steady. Note that the sum of these forces

$$d\vec{\mathbf{F}}_{B} = d\vec{\mathbf{F}}_{B,1} + d\vec{\mathbf{F}}_{B,2} = dqwB\hat{\mathbf{k}} + dquB\hat{\mathbf{j}} = dqB(w\hat{\mathbf{k}} + u\hat{\mathbf{j}}),$$

is perpendicular to the velocity $\vec{\mathbf{v}} = w\hat{\mathbf{j}} + u\hat{\mathbf{k}}$,

$$d\vec{\mathbf{F}}_{B} \cdot \vec{\mathbf{v}} = dqB(w\hat{\mathbf{k}} + u\hat{\mathbf{j}}) \cdot (w\hat{\mathbf{j}} + u\hat{\mathbf{k}}) = 0.$$

The magnetic force does no work. The kinetic changes so what agent is doing the work? If we integrate the vertical force over the wire, we find that

$$\vec{\mathbf{F}}_{B,2} = \int_{\text{charge}} d\vec{\mathbf{F}}_{B,2} = \int_{\text{charge}} dqwB\hat{\mathbf{k}} = qwB\hat{\mathbf{k}} = \lambda swB\hat{\mathbf{k}}, \qquad (8.10.7)$$

where $q = \lambda s$ is the total charge in the length s. The work done by the emf must oppose this vertical force in order to keep a steady current. In a time dt, the charges moves downward a vertical distance $d\vec{\mathbf{s}} = -udt\hat{\mathbf{k}}$, hence the electromotive force must do work

$$\varepsilon = \int(-\vec{\mathbf{F}}_{B,2}) \cdot d\vec{\mathbf{s}} = \int(-\lambda swB\hat{\mathbf{k}}) \cdot (-udt\hat{\mathbf{k}}) = \lambda suB\int wdt = \lambda suBa = IsBa, \quad (8.10.8)$$

where $a = \int wdt$ is the distance the rod rolled and $I = \lambda s$. So the work done by the emf force is exactly what we at first thought was the work done by the magnetic force.

8.10.2 Suspended Conducting Rod

A conducting rod having a linear mass density λ (mass per unit length) is suspended by two flexible wires in a uniform magnetic field $\vec{\mathbf{B}}$ which points out of the page, as shown in Figure 8.9.2.

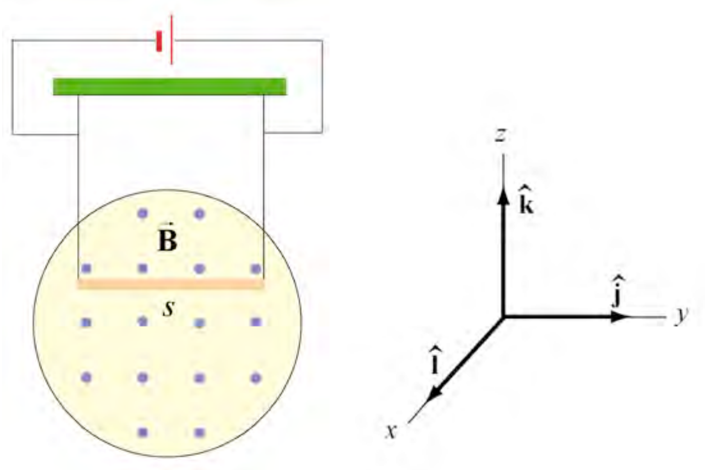


Figure 8.9.2 Suspended conducting rod in uniform magnetic field

If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

Solution: In order that the tension in the wires is zero, the magnetic force $\vec{\mathbf{F}}_B = I\vec{\mathbf{s}} \times \vec{\mathbf{B}}$ acting on the conductor must exactly cancel the downward gravitational force $\vec{\mathbf{F}}_g = -mg\hat{\mathbf{k}}$. For $\vec{\mathbf{F}}_B$ to point in the positive z-direction, we must have $\vec{\mathbf{s}} = -s\hat{\mathbf{j}}$, i.e., the current flows to the left, so that

$$\vec{\mathbf{F}}_B = I\vec{\mathbf{s}} \times \vec{\mathbf{B}} = I(-s\,\hat{\mathbf{j}}) \times (B\,\hat{\mathbf{i}}) = -IsB(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = +I\ell B\,\hat{\mathbf{k}} \,. \tag{8.10.9}$$

The magnitude of the current can be obtain from

$$IsB = mg \,. \tag{8.10.10}$$

Therefore the current in the wire is

$$I = \frac{mg}{Bs} = \frac{\lambda g}{B}\,, \tag{8.10.11}$$

where $m = \lambda s$.

8.10.3 Charged Particles in Magnetic Field

Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B, are accelerated from rest by a potential difference ΔV, and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution: The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V\,, \tag{8.10.12}$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}}\,. \tag{8.10.13}$$

The charges move in semicircles, since the magnetic force points radially inward and therefore by Newton's Second Law,

$$\frac{mv^2}{r} = qvB\,. \tag{8.10.14}$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B}\sqrt{\frac{2m\Delta V}{q}}\,, \tag{8.10.15}$$

Therefore r is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \frac{(m_A/q_A)^{1/2}}{(m_B/q_B)^{1/2}} \quad \Rightarrow \quad \frac{R}{2R} = \frac{(m_A/q)^{1/2}}{(m_B/2q)^{1/2}}\,, \tag{8.10.16}$$

therefore

$$\frac{m_A}{m_B} = \frac{1}{8}\,. \tag{8.10.17}$$

8.10.4 Ring of Current in the B Field of a Dipole (see also Section 8.13)

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current I, as shown in the Figure 8.9.3. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

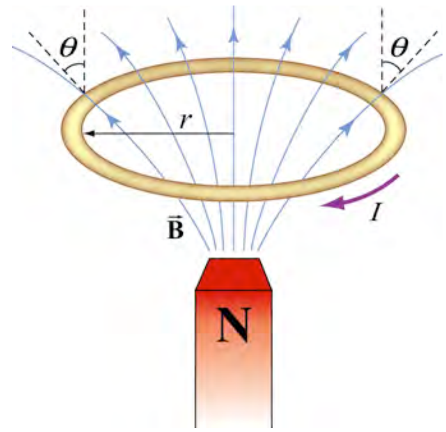


Figure 8.9.3 A bar magnet approaching a conducting ring

Solution: The magnetic force acting on a small differential current-carrying element $I\,d\vec{\mathbf{s}}$ on the ring is given by $d\vec{\mathbf{F}}_B = Id\,\vec{\mathbf{s}}\times\vec{\mathbf{B}}$, where $\vec{\mathbf{B}}$ is the magnetic field due to the bar magnet. Using cylindrical coordinates $(\hat{\mathbf{r}},\hat{\boldsymbol{\phi}},\hat{\mathbf{z}})$ as shown in Figure 8.9.4, we have

$$d\vec{\mathbf{F}}_B = I(-ds\,\hat{\boldsymbol{\phi}})\times(B\sin\boldsymbol{\theta}\,\hat{\mathbf{r}} + B\cos\boldsymbol{\theta}\,\hat{\mathbf{z}}) = (IBds)\sin\boldsymbol{\theta}\,\hat{\mathbf{z}} - (IBds)\cos\boldsymbol{\theta}\,\hat{\mathbf{r}} \quad . \tag{8.10.18}$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the z-component.

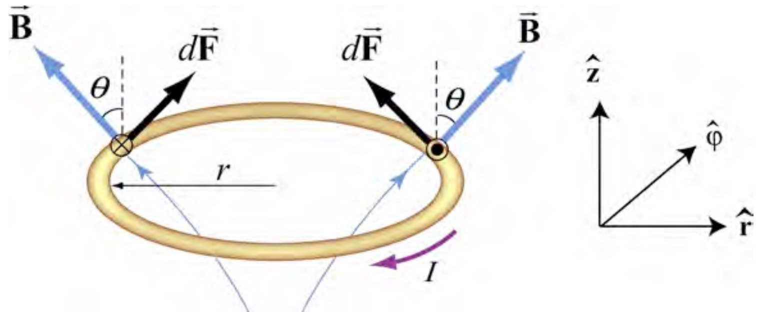


Figure 8.9.4 Magnetic force acting on the conducting ring

The total force acting on the ring then becomes

$$\vec{\mathbf{F}}_B = (IB\sin\theta)\hat{\mathbf{z}}\oint ds = (2\pi rIB\sin\theta)\,\hat{\mathbf{z}}\,. \tag{8.10.19}$$

The force points in the $+z$ direction and therefore is repulsive.

8.11 Conceptual Questions

1. Can a charged particle move through a uniform magnetic field without experiencing any force? Explain.

2. If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?

3. Suppose a charged particle is moving under the influence of both electric and magnetic fields. How can the effect of the two fields on the motion of the particle be distinguished?

4. What type of magnetic field can exert a force on a magnetic dipole? Is the force repulsive or attractive?

5. If a compass needle is placed in a uniform magnetic field, is there a magnetic force acting on the needle? Is there a torque?

8.12 Additional Problems

8.12.1 Force Exerted by a Magnetic Field

The electrons in the beam of television tube have an energy of 12 keV where $1\text{ eV} = 1.6\times10^{-19}\text{ J}$. The tube is oriented so that the electrons move horizontally from south to north. At MIT, the Earth's magnetic field points roughly vertically down (i.e. neglect the component that is directed toward magnetic north) and has magnitude $B \sim 5\times10^{-5}$ T.

(a) In what direction will the beam deflect?

(b) What is the acceleration of a given electron associated with this deflection? [Ans. $\sim10^{-15}$ m/s^2.]

(c) How far will the beam deflect in moving 0.20 m through the television tube?

8.12.2 Magnetic Force on a Current Carrying Wire

A square loop of wire, of length $s = 0.1\text{ m}$ on each side, has a mass of 50 g and pivots about an axis AA' that corresponds to a horizontal side of the square, as shown in Figure 8.11.1. A magnetic field of 500 G, directed vertically downward, uniformly fills the region in the vicinity of the loop. The loop carries a current *I* so that it is in equilibrium at $\theta = 20°$.

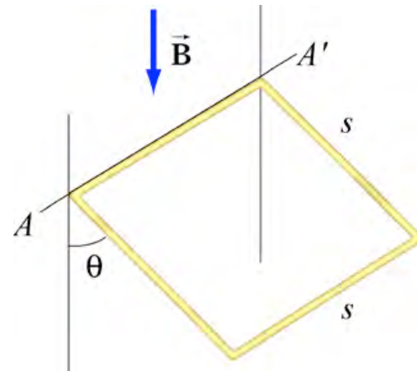


Figure 8.11.1 Magnetic force on a current-carrying square loop.

(a) Consider the force on each segment separately and find the direction of the current that flows in the loop to maintain the $20°$ angle.

(b) Calculate the torque about the axis due to these forces.

(c) Find the current in the loop by requiring the sum of all torques (about the axis) to be zero. (Hint: Consider the effect of gravity on each of the 4 segments of the wire separately.) [Ans. $I \sim 20$ A.]

(d) Determine the magnitude and direction of the force exerted on the axis by the pivots.

(e) Repeat part (b) by now using the definition of a magnetic dipole to calculate the torque exerted on such a loop due to the presence of a magnetic field.

8.12.3 Sliding Bar

A conducting bar of length s is placed on a frictionless inclined plane that is tilted at an angle θ from the horizontal, as shown in Figure 8.11.2.

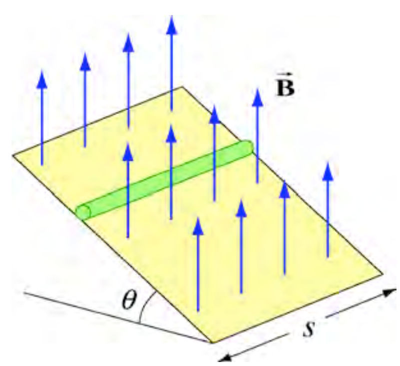


Figure 8.11.2 Magnetic force on a conducting bar

A uniform magnetic field is applied in the vertical direction. To prevent the bar from sliding down, a voltage source is connected to the ends of the bar with current flowing through. Determine the magnitude and the direction of the current such that the bar will remain stationary.

8.12.4 Particle Trajectory

A particle of charge $-q$ is moving with a velocity $\vec{\mathbf{v}}$. It then enters midway between two plates where there exists a uniform magnetic field pointing into the page, as shown in Figure 8.11.3.

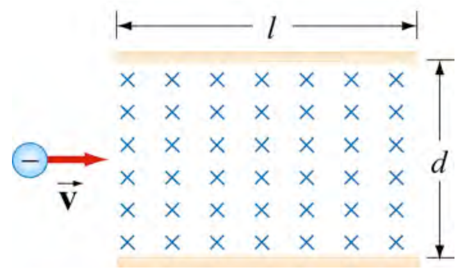


Figure 8.11.3 Charged particle moving under the influence of a magnetic field

(a) Is the trajectory of the particle deflected upward or downward?

(b) Compute the distance between the left end of the plate and where the particle strikes.

8.12.5 Particle Orbits in a Magnetic Field

Suppose the entire xy-plane to the right of the origin O is filled with a uniform magnetic field $\vec{\mathbf{B}}$ pointing out of the page, as shown in Figure 8.11.4.

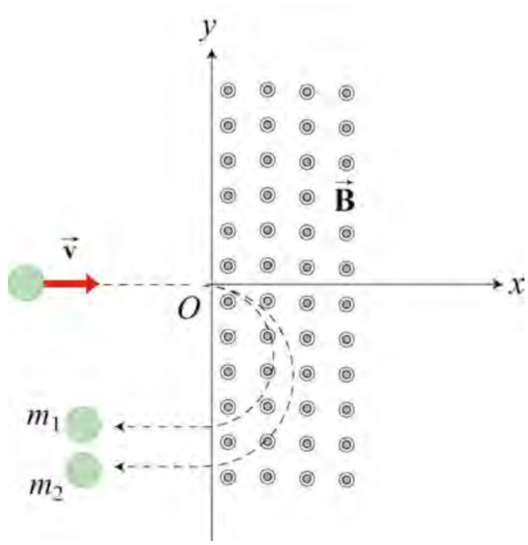


Figure 8.11.4

Two charged particles, starting in the region $x < 0$, travel along the x-axis in the positive x direction, each with speed v, and enter the magnetic field at the origin O. The two particles have the same charge q, but have different masses, m_1 and m_2. When in the magnetic field, their trajectories both curve in the same direction, but describe semi-circles with different radii. The radius of the semi-circle traced out by particle 2 is exactly *twice* as big as the radius of the semi-circle traced out by particle 1.

(a) Is the charge q of these particles such that $q > 0$, or is $q < 0$?

(b) Derive an expression for the radius R_1 of the semi-circle traced out by particle 1, in terms of q, v, B, and m_1.

(c) What is the ratio m_2 / m_1?

(d) Is it possible to apply an electric field $\vec{\mathbf{E}}$ in the region $x > 0$ only, which will cause both particles to continue to move in a straight line after they enter the region $x > 0$? If so, indicate the magnitude and direction of that electric field, in terms of the quantities given. If not, why not?

8.12.6 Force and Torque on a Current Loop

A current loop consists of a semicircle of radius R and two straight segments of length s with an angle θ between them. The loop is then placed in a uniform magnetic field pointing to the right, as shown in Figure 8.11.5.

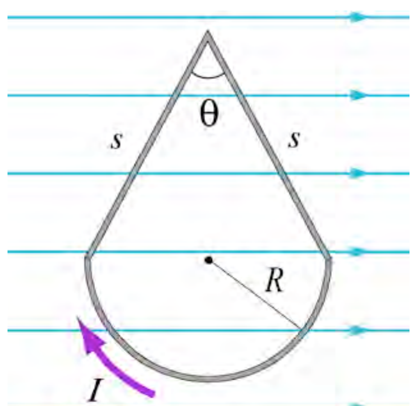


Figure 8.11.5 Current loop placed in a uniform magnetic field

(a) Find the net force on the current loop.

(b) Find the net torque on the current loop.

8.12.7 Force on a Wire

A straight wire of length 0.2 m carries a 7.0 A current. It is immersed in a uniform magnetic field of 0.1 T whose direction lies 20 degrees from the direction of the current.

(a) What is the direction of the force on the wire? Make a sketch to show your answer.

(b) What is the magnitude of the force? [Ans. ~0.05 N]

(c) How could you maximize the force without changing the field or current?

8.12.8 Levitating Wire

A copper wire of diameter d carries a current density $\vec{\mathbf{J}}$ at the Earth's equator where the Earth's magnetic field is horizontal, points north, and has magnitude $B = 0.5\times10^{-4}\,\mathrm{T}$. The wire lies in a plane that is parallel to the surface of the Earth and is oriented in the east-west direction. The density and resistivity of copper are $\rho_m = 8.9\times10^{3}\,\mathrm{kg/m^3}$ and $\rho = 1.7\times10^{-8}\,\Omega\cdot\mathrm{m}$, respectively.

(a) How large must $\vec{\mathbf{J}}$ be, and which direction must it flow in order to levitate the wire? Use $g = 9.8\ \mathrm{m/s^2}$

(b) When the wire is floating how much power will be dissipated per cubic centimeter?

8.13 Force Between a Magnetic Dipole and a Ring of Current Simulation

In this section we explore the meaning of Ampere's Law using a 3D simulation that creates imaginary, moveable Amperian loops (closed paths) in the presence of real, moveable line currents. This simulation illustrates Ampere's Law for a circular or rectangular Amperian loop, in the presence of current carrying wires current currents both into and out of the page.

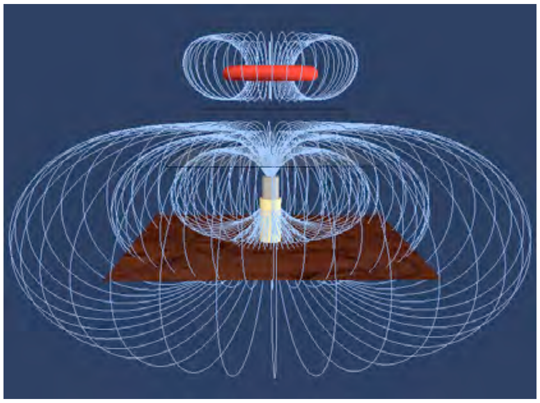
 http://peter-edx.99k.org/FloatingCoil.html

Figure 9.14.1 Screen Shot of Ampere's Law Simulation

You begin with one a wire carrying current out of the page and one carrying current into the page in the scene. You can add additional line currents, or delete all line currents present and start again. Left clicking and dragging on the line current can move the line currents. You can choose whether your imaginary Amperian loop is a circle or a rectangle, and you can move that loop. You will see tangents to the Amperian loops at many points on the loop. At those same points you will see the local magnetic field (blue vectors) on the loop due to all the line currents in the scene. If you left click and drag in the view, your perspective will change so that you can see the field vector and tangent orientation better. If you want to return to the original view you can "Reset Camera."

Use the simulation to verify the following properties of Ampere's Law. For the Amperian loop, you may choose either the circle or the rectangle.

(1) If line currents do not carry current through an Amperian loop, the line integral of the magnetic field around the loop flux through is zero.

(2) If line currents do carry current through an Amperian loop, the line integral of the magnetic field around the loop is positive or negative depending on the direction of the total current penetrating the surface of the loop.

Then use the simulation to answer the two following questions. Consider two line currents. Place one of the charged line currents ***inside*** your Amperian loop and the other ***outside***.

(1) Is the magnetic field at any point on the loop due only to the line currents that that are inside that loop?

(2) Is the dot product of the magnetic field with the local tangent at any point on the loop due only to the line currents that are inside the loop?

(3) Is the *total* line integral of the magnetic field around the entire Amperian loop due only to the line currents that are inside the loop?