

# Chapter 1

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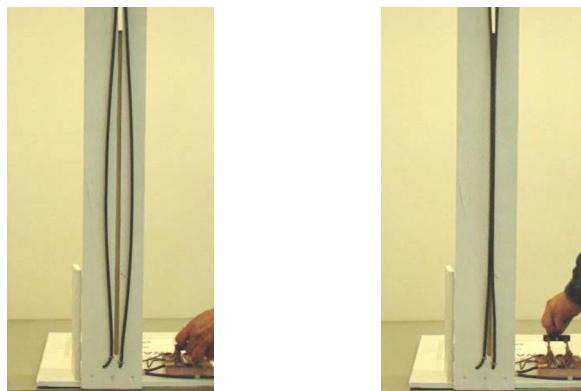
# Fields

## 1.1 Fields as the Agent of All Interactions

### 1.1.1 Action at a Distance Versus Field Theory

Classical electromagnetic field theory emerged in more or less complete form in 1873 in James Clerk Maxwell's *A Treatise on Electricity and Magnetism*. Maxwell based his theory in large part on the intuitive insights of Michael Faraday we discuss in the next section. The wide acceptance of Maxwell's theory has caused a fundamental shift in our understanding of physical reality. In this theory, electromagnetic fields are the agents of the interaction between material objects. This view differs radically from the older "action at a distance" view that preceded field theory.

What is "action at a distance?" It is a worldview in which the interaction of two material objects requires no mechanism other than the objects themselves and the empty space between them. That is, two objects exert a force on each other simply because they are present. Any mutual force between them (for example, gravitational attraction or electric repulsion) is instantaneously transmitted from one object to the other through empty space. There is no need to take into account any method or agent of transmission of that force, or any finite speed for the propagation of that agent of transmission. This is known as "action at a distance" because objects exert forces on one another ("action") with nothing but empty space ("distance") between them. No other agent or mechanism is needed.



(a) [Link to movie](#)

(b) [Link to movie](#)

**Figure 1.1.1** Current-carrying wires can repel or attract each other.

To be concrete, consider the following. Two long parallel wires carrying current in opposite directions will repel one another. This is shown in Figure 1.1.1(a). If the

current is in the same direction in both wires, the wires will attract one another, as in Figure 1.1.1(b). The movie linked to Figure 1.1.1 shows this behavior for the two directions of current. In the action at a distance model this behavior requires nothing more than the current carrying-current wires and the empty space between them. And indeed the space between the wires does appear to be empty to our normal senses. At first blush, we cannot see, hear, or touch anything that indicates that there is something in the space between the wires shown in Figure 1.1.1.

But despite the evidence of their senses, many natural philosophers objected to the action at a distance model because in our everyday experience, forces are exerted by one object on another only when the objects are in direct contact. In the field theory view, this is always true in some sense. That is, objects that are not in direct contact (objects separated by apparently empty space) must exert a force on one another *through the presence of an intervening medium or mechanism existing in the space between the objects*.

The force between the two objects is transmitted by direct “contact” from the first object to an intervening mechanism immediately surrounding that object, and then from one element of space to a neighboring element, in a continuous manner, until the force is transmitted to the region of space contiguous to the second object, and thus ultimately to the second object itself.

Although the two objects are not in direct contact with one another, they are in direct contact with a medium or mechanism that exists between them. The force between the objects is transmitted (at a finite speed) by stresses induced in the intervening space by the presence of the objects. The “field theory” view thus avoids the concept of “action at a distance” and replaces it by the concept of “action by continuous contact.” The “contact” is provided by a stress, or “field,” induced in the space between the objects by their presence.

And indeed, if we use more sophisticated instruments than our normal senses, we find that we can measure the presence of something in the space between the wires in Figure 1.1.1. That something is a continuous magnetic field created by the current in the wires that transmits forces from one wire to the other. A movie of these magnetic fields is shown in Section 9.2.1. Even though we are not aware of these fields with our normal senses, it is this invisible magnetic field that is causing the wires to move.

This is the essence of field theory, and it is the foundation of all modern approaches to understanding the world around us. Classical electromagnetism was the first field theory. It involves many concepts that are mathematically complex. As a result, even now it is difficult to appreciate. In this textbook, we will give you all the mathematics you need to understand how electromagnetic fields are created, store energy, and transmit forces. We will also provide movies and interactive simulations to try to make these difficult mathematical concepts more intuitive, following Faraday’s insights as outlined below.

### 1.1.2 Rubber Bands and Strings and the Forces Transmitted by Fields

Faraday was the first to understand the connection between the shape of electromagnetic fields and the forces they produce. The forces transmitted by electromagnetic fields can be understood conceptually as analogous to the forces transmitted by rubber bands and strings. This was Faraday's great insight, and it led him directly to the hypothesis of fields as the mediator of interactions between material objects.

To understand Faraday's insight, we must show depictions of the *total* field, that is the field due to all objects being considered. For example, to understand, as Faraday did, the reason that a point charge in a constant electric field feels a force, we must depict the total field due to both the constant electric field *and* the field of the point charge. This depiction of the total field is rare in introductory texts, but it allows profound insight into the mechanisms whereby fields transmit forces. The stresses transmitted by electromagnetic fields can be understood as analogous to the forces transmitted by rubber bands and strings, but to reach this understanding we must show a representation of the *total* field. A representation of the total magnetic field that exists between the wires in Figure 1.1.1 is shown in Figure 9.2.2, and one can understand their attraction or repulsion, exactly as Faraday did, by looking at the shape of the magnetic field lines in this Figure and in the accompanying movie.

Throughout this text we show figures and give link to movies that show the total field configuration in various circumstances. We do this because this allows you to better make the intuitive connection between the stresses transmitted by electromagnetic fields and the forces transmitted by rubber bands and strings. Thereby, you can relate unfamiliar concepts about forces in electromagnetism to your more familiar experience with the way the world works in terms of common everyday things that you can see, hear, and touch, and already understand at an intuitive level, such as strings and rubber bands.

## 1.2 Scalar Fields

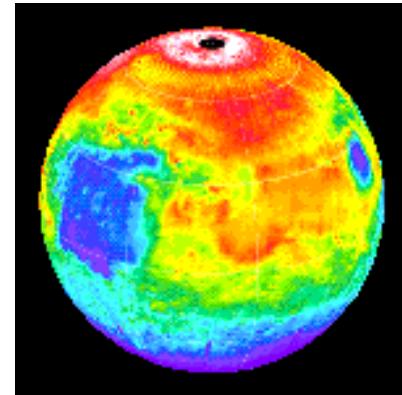
“... In order therefore to appreciate the requirements of the science [of electromagnetism], the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress ...”

James Clerk Maxwell [1855]

In this first chapter of your introduction to field theory, we discuss what a *field* is, and how we represent fields. We begin with scalar fields. A *scalar field* is a function that gives us a single value of some variable for every point in space. As an example, the image in Figure 1.2.1 shows the nighttime temperatures measured by the Thermal Emission Spectrometer instrument on the Mars Global Surveyor (MGS). The data were

acquired during the first 500 orbits of the MGS mapping mission. The coldest temperatures, shown in purple, are  $-120^{\circ}\text{C}$  while the warmest, shown in white, are  $-65^{\circ}\text{C}$ .

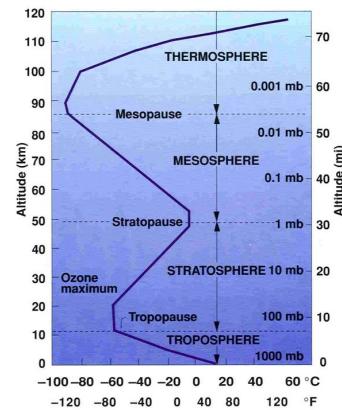
The view is centered on Isidis Planitia ( $15^{\circ}\text{N}$ ,  $270^{\circ}\text{W}$ ), which is covered with warm material, indicating a sandy and rocky surface. The small, cold (blue) circular region to the right is the area of the Elysium volcanoes, which are covered in dust that cools off rapidly at night. At this season, the north polar region is in full sunlight and is relatively warm at night. It is winter in the southern hemisphere and the temperatures are extremely low.



**Figure 1.2.1** Nighttime temperature map for Mars

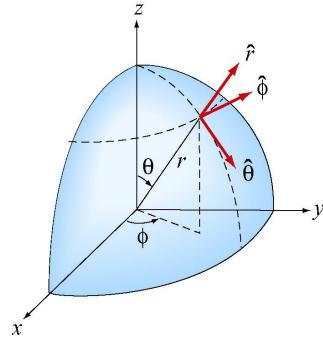
The various colors on the map represent the surface temperature. This map, however, is limited to representing only the temperature on a two-dimensional surface and thus, it does not show how temperature varies as a function of altitude. In principal, a scalar field provides values not only on a two-dimensional surface in space but for every point in space.

Figure 1.2.2 illustrates the variation of temperature as a function of height above the surface of the Earth, which is a third dimension which complements the two dimensions shown in Figure 1.2.1.



**Figure 1.2.2** Atmospheric temperature variation as a function of altitude above the Earth's surface

How do we represent three-dimensional scalar fields? In principle, one could create a three-dimensional atmospheric volume element and color it to represent the temperature variation.



**Figure 1.2.3** Spherical coordinates

Another way is to simply represent the temperature variation by a mathematical function. For the Earth we shall use spherical coordinates  $(r, \theta, \phi)$  shown in Figure 1.2.3 with the origin chosen to coincide with the center of the Earth. The temperature at any point is characterized by a function  $T(r, \theta, \phi)$ . In other words, the value of this function at the point with coordinates  $(r, \theta, \phi)$  is a temperature with given units. The temperature function  $T(r, \theta, \phi)$  is an example of a “scalar field.” The term “scalar” implies that temperature at any point is a number rather than a vector (a vector has both magnitude and direction).

### Example 1.1: Half-Frozen /Half-Baked Planet

As an example of a scalar field, consider a planet surrounded by an atmosphere. The planet rotates with the same angular speed about its axis as the planet orbits about a nearby star, *i.e.*, one hemisphere always faces the star. Let  $R$  denote the radius of the planet. Use spherical coordinates  $(r, \theta, \phi)$  with the origin at the center of the planet, and choose  $\phi = \pi / 2$  for the center of the hemisphere facing the star. A model for the temperature variation at any point is given by

$$T(r, \theta, \phi) = [T_0 + T_1 \sin^2 \theta + T_2 (1 + \sin \phi)] e^{-\alpha(r-R)} \quad (1.2.1)$$

where  $T_0$ ,  $T_1$ ,  $T_2$ , and  $\alpha$  are constants. The dependence on the variable  $r$  in the term  $e^{-\alpha(r-R)}$  indicates that the temperature decreases exponentially as we move radially away from the surface of the planet. The dependence on the variable  $\theta$  in the term  $\sin^2 \theta$  implies that the temperature decreases as we move toward the poles. Finally, the  $\phi$  dependence in the term  $(1 + \sin \phi)$  indicates that the temperature decreases as we move away from the center of the hemisphere facing the star.

A scalar field can also be used to describe other physical quantities such as the atmospheric pressure. However, a single number (magnitude) at every point in space is

not sufficient to characterize quantities such as the wind velocity since a direction at every point in space is needed as well.

### 1.2.1 Representations of a Scalar Field

A field, as stated earlier, is a function that has a different value at every point in space. A scalar field is a field for which there is a single number associated with every point in space. We have seen that the temperature of the Earth's atmosphere at the surface is an example of a scalar field. Another example is

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + (y+d)^2 + z^2}} - \frac{1/3}{\sqrt{x^2 + (y-d)^2 + z^2}} \quad (1.2.2)$$

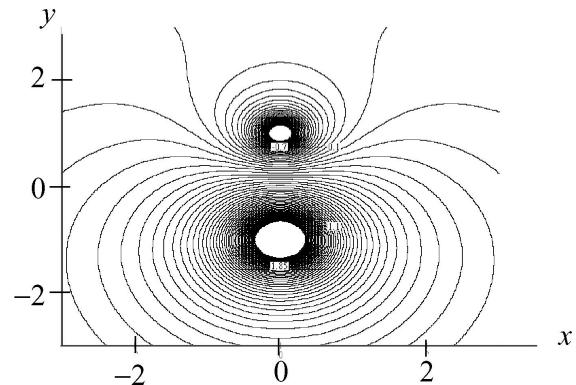
This expression defines the value of the scalar function  $\phi$  at every point  $(x, y, z)$  in space. How do we visually represent a scalar field defined by an equation such as Eq. (1.2.2)? Below we discuss three possible representations.

#### 1. Contour Maps

One way is to fix one of our independent variables ( $z$ , for example) and then show a contour map for the two remaining dimensions, in which the curves represent lines of constant values of the function  $\phi$ . A series of these maps for various (fixed) values of  $z$  then will give a feel for the properties of the scalar function. We show such a contour map in the  $xy$ -plane at  $z = 0$  for Eq. (1.2.2), namely,

$$\phi(x, y, 0) = \frac{1}{\sqrt{x^2 + (y+d)^2}} - \frac{1/3}{\sqrt{x^2 + (y-d)^2}} \quad (1.2.3)$$

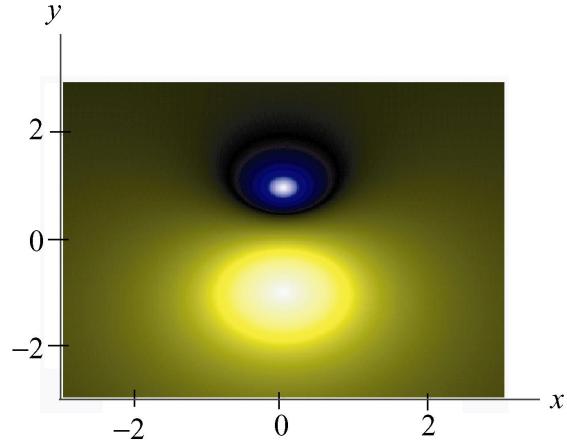
Various contour levels are shown in Figure 1.2.4, for  $d = 1$ , labeled by the value of the function at that level.



**Figure 1.2.4** A contour map in the  $xy$ -plane of the scalar field given by Eq. (1.2.3).

## 2. Color-Coding

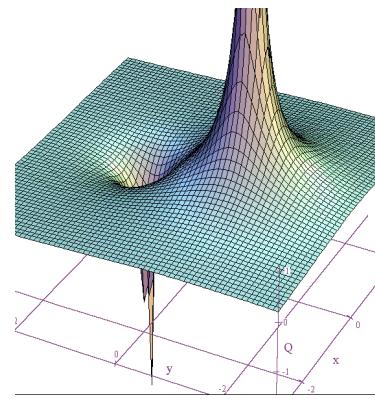
Another way we can represent the values of the scalar field is by color-coding in two dimensions for a fixed value of the third. This was the scheme used for illustrating the temperature fields in Figures 1.2.1 and 1.2.2. In Figure 1.2.5 a similar map is shown for the scalar field  $\phi(x, y, 0)$ . Different values of  $\phi(x, y, 0)$  are characterized by different colors in the map.



**Figure 1.2.5** A color-coded map in the  $xy$ -plane of the scalar field given by Eq. (1.2.3).

## 3. Relief Maps

A third way to represent a scalar field is to fix one of the dimensions, and then plot the value of the function as a height versus the remaining spatial coordinates, say  $x$  and  $y$ , that is, as a relief map. Figure 1.2.6 shows such a map for the same function  $\phi(x, y, 0)$ .

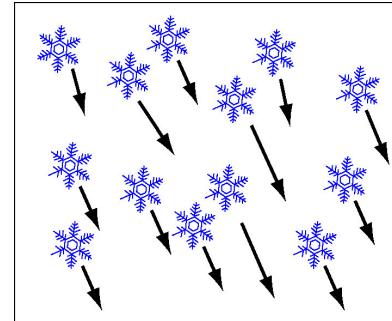


**Figure 1.2.6** A relief map of the scalar field given by Eq. (1.2.3).

## 1.3 Vector Fields

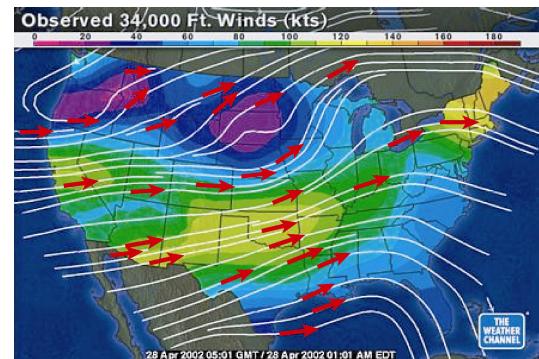
A vector is a quantity that has both a magnitude and a direction in space. Vectors are used to describe physical quantities such as velocity, momentum, acceleration and force, associated with an object. However, when we try to describe a system that consists of a large number of objects (e.g., moving water, snow, rain, ...) we need to assign a vector to each individual object.

As an example, let's consider falling snowflakes, as shown in Figure 1.3.1. As snow falls, each snowflake moves in a specific direction. The motion of the snowflakes can be analyzed by taking a series of photographs. At any instant in time, we can assign, to each snowflake, a velocity vector that characterizes its movement.



**Figure 1.3.1** Falling snow.

The falling snow is an example of a collection of discrete bodies. On the other hand, if we try to analyze the motion of continuous bodies such as fluids, a velocity vector then needs to be assigned to every point in the fluid at any instant in time. Each vector describes the direction and magnitude of the velocity at a particular point and time. The collection of all the velocity vectors is called the velocity vector field. An important distinction between a vector field and a scalar field is that the former contains information about both the direction and the magnitude at every point in space, while only a single variable is specified for the latter. An example of a system of continuous bodies is airflow. Figure 1.3.2 depicts a scenario of the variation of the jet stream, which is the wind velocity as a function of position. Note that the value of the height is “fixed” at 34,000 ft.



**Figure 1.3.2** Jet stream with arrows indicating flow velocity. The “streamlines” are formed by joining arrows from head to tail.

## 1.4 Fluid Flow

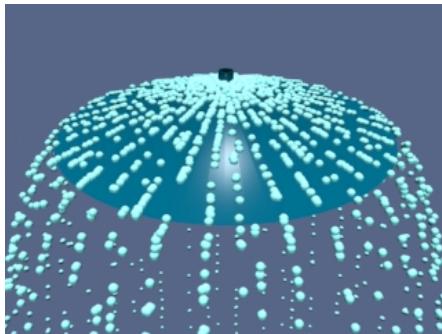
### 1.4.1 Sources and Sinks

In general, a vector field  $\vec{F}(x, y, z)$  can be written as

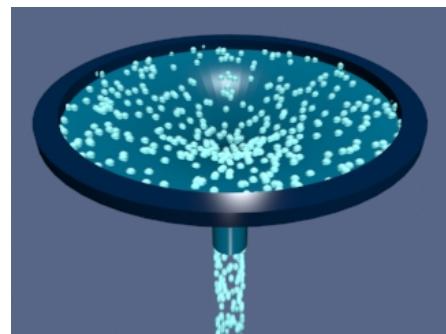
$$\vec{F}(x, y, z) = F_x(x, y, z)\hat{i} + F_y(x, y, z)\hat{j} + F_z(x, y, z)\hat{k} \quad (1.4.1)$$

where the components are scalar fields. Below we use fluids to examine the properties associated with a vector field since fluid flows are the easiest vector fields to visualize.

In Figure 1.4.1 we show physical examples of a fluid flow field, where we represent the fluid by a finite number of particles to show the structure of the flow. In Figure 1.4.1(a), particles (fluid elements) appear at the center of a cone (a “source”) and then flow downward under the effect of gravity. That is, we create particles at the origin, and they subsequently flow away from their creation point. We also call this a diverging flow, since the particles appear to “diverge” from the creation point. Figure 1.4.1(b) is the converse of this, a converging flow, or a “sink” of particles.



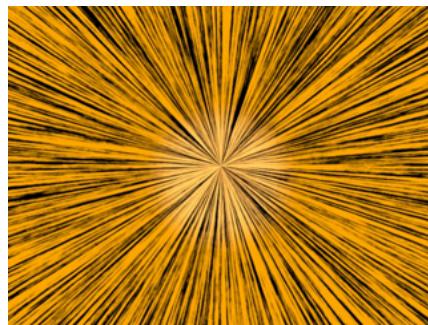
(a) [Link to movie](#)



(b) [Link to movie](#)

**Figure 1.4.1** (a) An example of a source of particles and the flow associated with a source, and (b) an example of a sink of particles and the flow associated with a sink.

A different representation of the diverging fluid flow in Figure 1.4.1(a) is depicted in Figure 1.4.2. Here the direction of the flow is represented by a texture pattern in which the direction of the field (radial) has correlations in lightness and darkness. The lightness or darkness has no physical meaning. We are simply using correlations in the lightness and darkness to indicate the direction of the vector field.

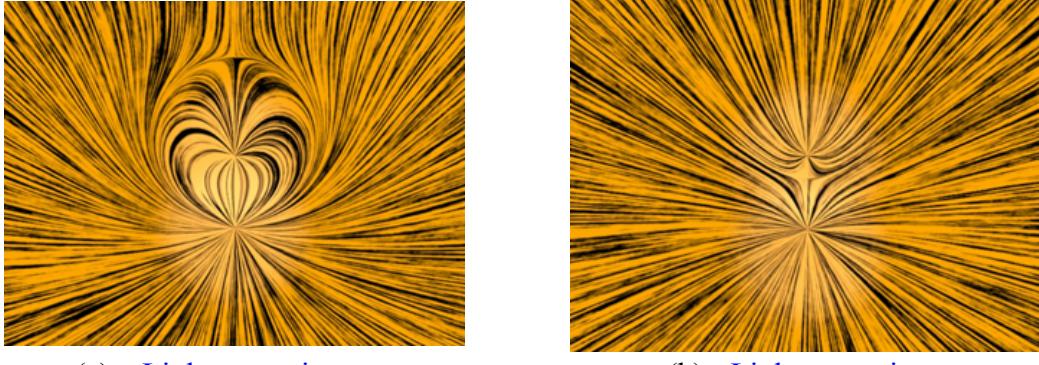


[Link to movie](#)

**Figure 1.4.2** Representing the flow field associated with a source using a texture pattern.

We can use this same technique to visualize fluid flows with multiple sources or sinks. For example, Figure 1.4.3(a) shows a source below a sink of lesser magnitude. We see

flow out of the source, some of which flows into the sink, but much of which flows to infinity. In contrast, Figure 1.4.3(b) shows two sources of unequal strength. We see flow out of both of the sources, all of which flows to infinity, with a comet-like tail streaming out from the source of lesser strength.

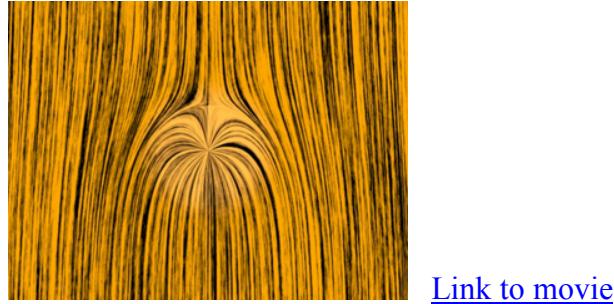


(a) [Link to movie](#)

(b) [Link to movie](#)

**Figure 1.4.3** The flow fields associated with (a) a source (lower) and a sink (upper) where the sink is smaller than the source, and (b) two sources of unequal strength.

Finally, in Figure 1.4.4, we illustrate a constant downward flow interacting with a diverging flow (source). The diverging flow is able to make some headway “upwards” against the downward constant flow, but eventually turns and flows downward, overwhelmed by the strength of the “downward” flow.



[Link to movie](#)

**Figure 1.4.4** A constant downward flow interacting with a diverging flow (source).

In the language of vector calculus, we represent the flow field of a fluid by

$$\bar{\mathbf{v}}(x, y, z) = v_x(x, y, z)\hat{\mathbf{i}} + v_y(x, y, z)\hat{\mathbf{j}} + v_z(x, y, z)\hat{\mathbf{k}} \quad (1.4.2)$$

where  $v_x(x, y, z)$ ,  $v_y(x, y, z)$ , and  $v_z(x, y, z)$  are the component functions. For an incompressible flow, the point  $(x, y, z)$  is a source of fluid if the divergence of  $\bar{\mathbf{v}}(x, y, z)$  is greater than zero. That is, if

$$\nabla \cdot \vec{v}(x, y, z) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} > 0 \quad (1.4.3)$$

where

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad (1.4.4)$$

is the del operator. On the other hand, the point  $(x, y, z)$  is a sink if the divergence of  $\vec{v}(x, y, z)$  is less than zero. When  $\nabla \cdot \vec{v}(x, y, z) = 0$ , then the point  $(x, y, z)$  is neither a source nor a sink.

### 1.4.2 Circulations

A flow field that is neither a source nor a sink may exhibit another class of behavior, which we call circulation. In Figure 1.4.5(a) we show a physical example of a circulating flow field where particles are not created or destroyed (except at the beginning of the animation in (a)), but merely move in circles. The purely circulating flow can also be represented by textures, as shown in Figure 1.4.5(b).



(a) [Link to movie](#)



(b) [Link to movie](#)

**Figure 1.4.5** (a) An example of a circulating fluid. (b) Representing a circulating flow using textures.

A flow field can have more than one system of circulation centered about different points in space. In Figure 1.4.6(a) we show a flow field with two circulations. The flows are in opposite senses, and one of the circulations is stronger than the other. In Figure 1.4.6(b) we have the same situation, except that now the two circulations are in the same sense.



(a) [Link to movie](#)



(b) [Link to movie](#)

**Figure 1.4.6** A flow with two circulation centers with (a) opposite directions of circulation, and (b) the same direction of circulation.

In Figure 1.4.7, we show a constant downward flow interacting with a counter-clockwise circulating flow. The circulating flow is able to make some headway against the downward constant flow, but eventually is overwhelmed by the strength of the “downward” flow.



[Link to movie](#)

**Figure 1.4.7** A constant downward flow interacting with a counter-clockwise circulating flow.

In the language of vector calculus, the flows shown in Figures 1.4.5 through 1.4.7 are said to have a non-zero curl, but zero divergence. In contrast, the flows shown in Figures 1.4.2 through 1.4.4 have a zero curl (they do not move in circles) and a non-zero divergence (particles are created or destroyed).

Finally, in Figure 1.4.8, we show a fluid flow field that has both a circulation and a divergence (both the divergence and the curl of the vector field are non-zero). Any vector field can be written as the sum of a curl-free part (no circulation) and a divergence-free part (no source or sink). We will find in our study of electrostatics and magnetostatics that the electrostatic fields are curl free (e.g. they look like Figures 1.4.2 through 1.4.4) and the magnetic fields are divergence free (e.g. they look like Figures 1.4.5 through 1.4.7). Only when dealing with time-varying situations will we encounter electric fields that have both a divergence and a curl. Figure 1.4.8 depicts a field whose curl and divergence are both non-vanishing. As far as we know, even in time-varying situations, magnetic fields always remain divergence-free. Therefore, magnetic fields will always look like the patterns shown in Figures 1.4.5 through 1.4.7.



[Link to movie](#)

**Figure 1.4.8** A flow field that has both a source (divergence) and a circulation (curl).

### 1.4.3 Relationship between Fluid Fields and Electromagnetic Fields

Vector fields that represent fluid flow have an immediate physical interpretation: the vector at every point in space represents a direction of motion of a fluid element, and we can construct animations of those fields, as above, which show that motion. A more general vector field, for example the electric and magnetic fields discussed below, do not have that immediate physical interpretation of a flow field. There is no “flow” of a fluid along an electric field or magnetic field.

However, even though the vectors in electromagnetism do not represent fluid flow, we carry over many of the terms we use to describe fluid flow to describe electromagnetic fields as well. For example we will speak of the flux (flow) of the electric field through a surface. If we were talking about fluid flow, “flux” would have a well-defined physical meaning, in that the flux would be the amount of fluid flowing across a given surface per unit time. There is no such meaning when we talk about the flux of the electric field through a surface, but we still use the same term for it, as if we were talking about fluid flow. Similarly we will find that magnetic vector field exhibit patterns like those shown above for circulating flows, and we will sometimes talk about the circulation of magnetic fields. But there is no fluid circulating along the magnetic field direction.

We use much of the terminology of fluid flow to describe electromagnetic fields because it helps us understand the structure of electromagnetic fields intuitively. However, we must always be aware that the analogy is limited.

## 1.5 Gravitational Field

The gravitational field of the Earth is another example of a vector field that can be used to describe the interaction between a massive object and the Earth. According to Newton’s universal law of gravitation, the gravitational force on an object of mass  $m$  due to the interaction with another object of mass  $M$  is given by

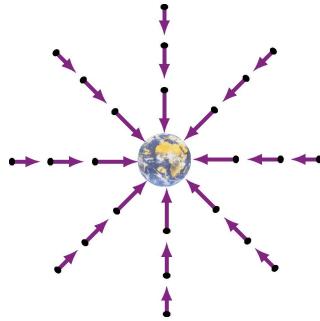
$$\vec{F}_g = -G \frac{Mm}{r^2} \hat{r} \quad (1.5.1)$$

where  $r$  is the distance between the two objects and  $\hat{r}$  is the unit vector located at the position of  $m$  that points from  $M$  towards  $m$ . The constant of proportionality is the gravitational constant  $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ . Notice that the force is always attractive, with its magnitude being proportional to the inverse square of the distance between the masses.

As an example, if  $M$  is the mass of the Earth, the gravitational field  $\vec{g}$  at a point  $P$  in space, defined as the gravitational force per unit mass, can be written as

$$\vec{g} = \lim_{m \rightarrow 0} \frac{\vec{F}_g}{m} = -G \frac{M}{r^2} \hat{r} \quad (1.5.2)$$

From the above expression, we see that the field is radial and points toward the center of the Earth, as shown in Figure 1.5.1.

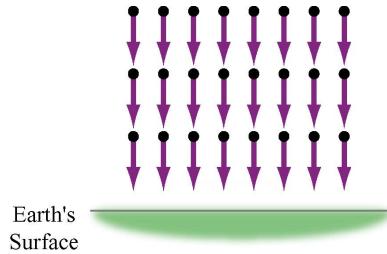


**Figure 1.5.1** Gravitational field of the Earth.

Near the Earth's surface, the gravitational field  $\vec{g}$  is approximately constant:  $\vec{g} = -g\hat{r}$ , where

$$g = G \frac{M}{R_E^2} \approx 9.8 \text{ m/s}^2 \quad (1.5.3)$$

and  $R_E$  is the radius of Earth. The gravitational field near the Earth's surface is depicted in Figure 1.5.2.



**Figure 1.5.2** Uniform gravitational field near the surface of the Earth.

Notice that an object in a constant gravitational field does not necessarily move in the direction of the field. This is true only when its initial velocity is in the same direction as the field. On the other hand, if the initial velocity has a component perpendicular to the gravitational field, the trajectory will be parabolic.

## 1.6 Electric Fields

The force between electrically charged objects at rest is called the electrostatic force. However, unlike mass in gravitational force, there are two types of electric charge: positive and negative. The electrostatic force between charged objects falls off as the inverse square of their distance of separation, and can be either attractive or repulsive. Charged objects exert forces on each other in a manner that is analogous to gravitation. Consider an object that has charge  $Q$ . A “test charge” with charge  $q$  that is placed at a point  $P$  a distance  $r$  from  $Q$  will experience a Coulomb force:

$$\vec{F}_e = k_e \frac{Qq}{r^2} \hat{r} \quad (1.6.1)$$

where  $\hat{r}$  is the unit vector that points from  $Q$  to  $q$ . The constant of proportionality  $k_e = 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$  is called the Coulomb constant. The electric field at  $P$  is defined as

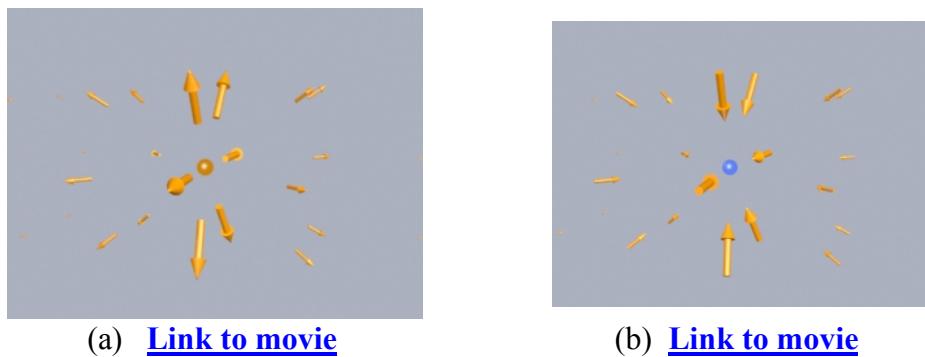
$$\vec{E} = \lim_{q \rightarrow 0} \frac{\vec{F}_e}{q} = k_e \frac{Q}{r^2} \hat{r} \quad (1.6.2)$$

The SI unit of electric field is newtons/coulomb (N/C). If  $Q$  is positive, its electric field points radially away from the charge; on the other hand, the field points radially inward if  $Q$  is negative (Figure 1.6.1). In terms of the field concept, we may say that the charge  $Q$  creates an electric field  $\vec{E}$  that exerts a force  $\vec{F}_e = q\vec{E}$  on  $q$ .



**Figure 1.6.1** Electric field for (a) positive and (b) negative charges.

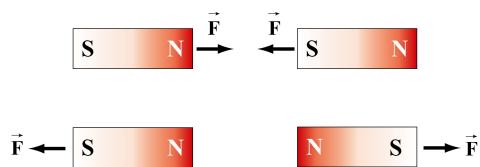
The charges in Figure 1.6.1 are stationary. We can also talk about the electric fields of moving charges. For example, Figure 1.6.2 shows one frame of movies of the electric field of a moving positive and negative point charge, assuming the speed of the charge is small compared to the speed of light.



**Figure 1.6.2** The electric fields of (a) a moving positive charge, (b) a moving negative charge when the speed of the charge is small compared to the speed of light.

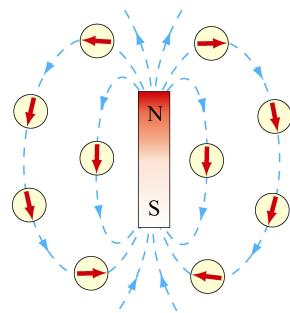
## 1.7 Magnetic Fields

The magnetic field is another example of a vector field. The most familiar source of magnetic fields is a bar magnet. One end of the bar magnet is called the north pole and the other, the south pole. Like poles repel while opposite poles attract (Figure 1.7.1).



**Figure 1.7.1** Magnets attracting and repelling.

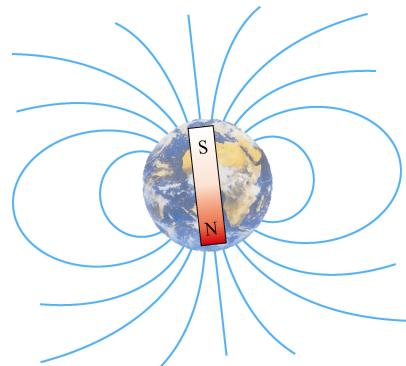
If we place some compasses near a bar magnet, the needles will align themselves along the direction of the magnetic field, as shown in Figure 1.7.2.



**Figure 1.7.2** Magnetic field of a bar magnet.

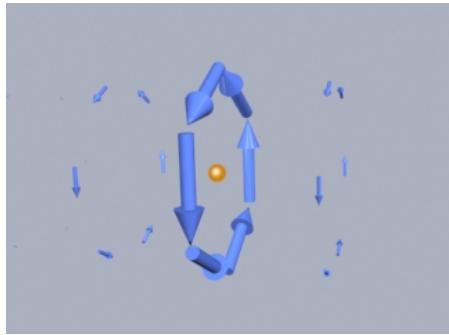
The observation can be explained as follows: a magnetic compass consists of a tiny bar magnet that can rotate freely about a pivot point passing through the center of the magnet. When a compass is placed near a bar magnet which produces an external magnetic field, it experiences a torque which tends to align the north pole of the compass with the external magnetic field.

The Earth's magnetic field behaves as if there were a bar magnet in it (Figure 1.7.3). Note that the south pole of the magnet is located in the northern hemisphere.

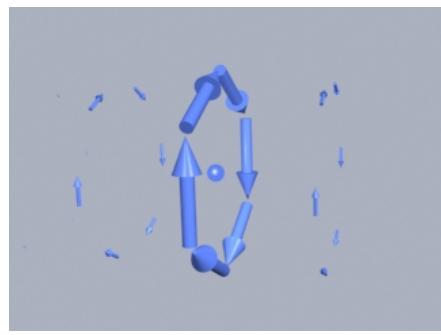


**Figure 1.7.3** Magnetic field of the Earth

As we will see in Chapter 9, moving electric charges produce magnetic fields. Thus the moving charges in Figure 1.6.2 are accompanied not only by electric fields, but by magnetic fields as well. Figure 1.7.4 shows one frame of the animations of the magnetic field of a moving positive and negative point charge, assuming the speed of the charge is small compared to the speed of light. If the charge is not moving, there is no magnetic field.



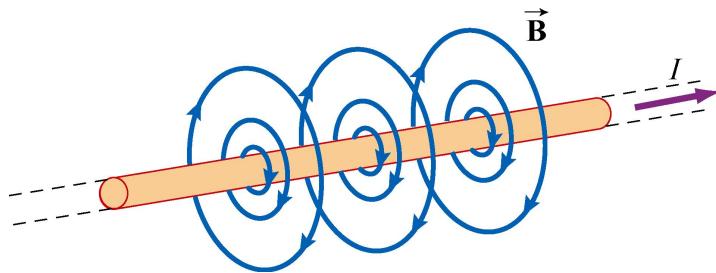
(a) [Link to movie](#)



(b) [Link to movie](#)

**Figure 1.7.4** The magnetic field of (a) a moving positive charge and of (b) a moving negative charge when the speed of the charge is small compared to  $c$ .

Another familiar source of magnetic fields is the current-carrying wire. In Figure 1.7.5, we show the magnetic field associated with an infinitely long current-carrying wire. The magnetic field is wrapped in circles about the wire, with the direction of the rotation of the circles determined by the right hand rule (if the thumb of your right hand is in the direction of the current, your fingers will curl in the direction of the magnetic field).



**Figure 1.7.5** Magnetic field lines due to an infinite wire carrying current  $I$ .

## 1.8 Representations of a Vector Field

How do we represent vector fields? Since there is much more information (magnitude and direction) in a vector field, our visualizations are correspondingly more complex when compared to the representations of scalar fields.

Let us introduce an analytic form for a vector field and discuss the various ways that we represent it. Let

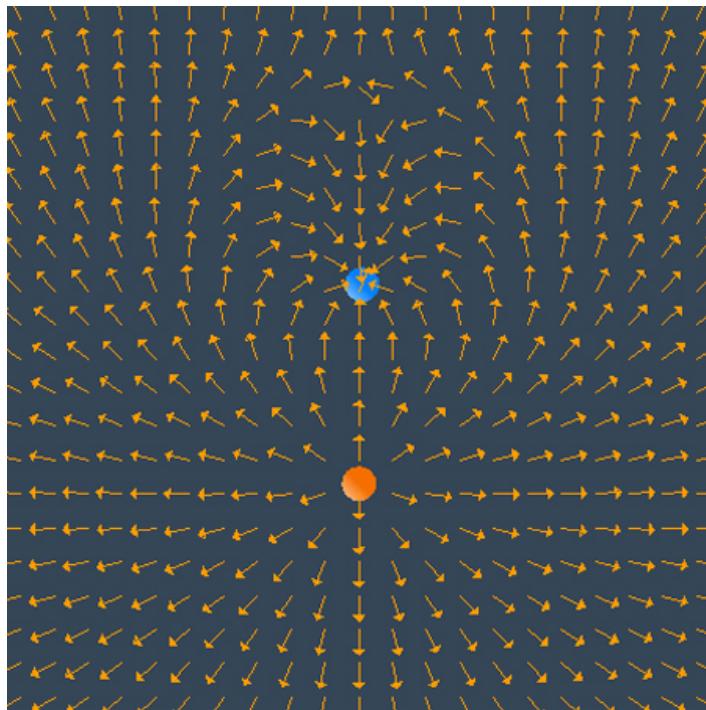
$$\vec{E}(x, y, z) = \frac{x\hat{i} + (y+d)\hat{j} + z\hat{k}}{[x^2 + (y+d)^2 + z^2]^{3/2}} - \frac{1}{3} \frac{x\hat{i} + (y-d)\hat{j} + z\hat{k}}{[x^2 + (y-d)^2 + z^2]^{3/2}} \quad (1.8.1)$$

This field is proportional to the electric field of two point charges of opposite signs, with the magnitude of the positive charge three times that of the negative charge. The positive

charge is located at  $(0, -d, 0)$  and the negative charge is located at  $(0, d, 0)$ . (We discuss how this field is calculated in Section 2.7.)

### 1.8.1 Vector Field Representation

Figure 1.8.1 is an example of a “vector field” representation of Eq. (1.8.1), in the plane where  $z = 0$ . We show the charges that would produce this field if it were an electric field, one positive (the orange charge) and one negative (the blue charge). We will always use this color scheme to represent positive and negative charges.



[Link to simulation](#)

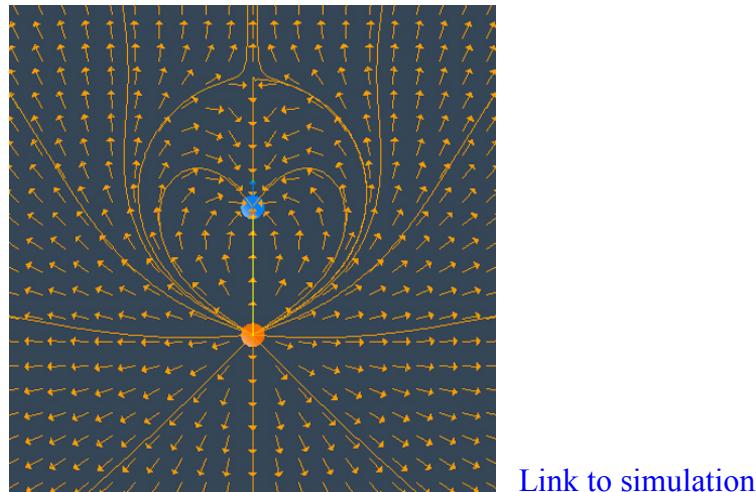
**Figure 1.8.1** A “vector field” representation of the field of two point charges, one negative and one positive, with the magnitude of the positive charge three times that of the negative charge.

In the vector field representation, we put arrows representing the field direction on a rectangular grid. The direction of the arrow at a given location represents the direction of the vector field at that point. In many cases, we also make the length of the vector proportional to the magnitude of the vector field at that point. But we also may show only the direction with the vectors (that is make all vectors the same length), and color-code the arrows according to the magnitude of the vector. Or we may not give any information about the magnitude of the field at all, but just use the arrows on the grid to indicate the direction of the field at that point.

Figure 1.8.1 is an example of the latter situation. That is, we use the arrows on the vector field grid to simply indicate the direction of the field, with no indication of the magnitude of the field, either by the length of the arrows or their color. Note that the arrows point away from the positive charge (the positive charge is a “source” for electric field) and towards the negative charge (the negative charge is a “sink” for electric field).

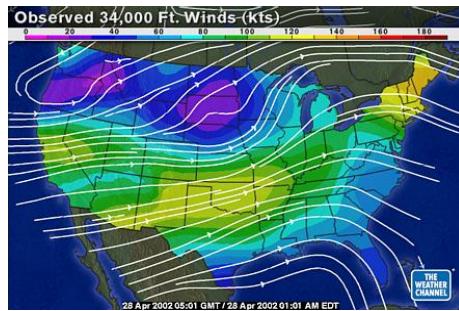
### 1.8.2 Field Line Representation

There are other ways to represent a vector field. One of the most common is to draw “field lines.” Faraday called the field lines for electric field “lines of force.” To draw a field line, start out at any point in space and move a very short distance in the direction of the local vector field, drawing a line as you do so. After that short distance, stop, find the new direction of the local vector field at the point where you stopped, and begin moving again in that new direction. Continue this process indefinitely. Thereby you construct a line in space that is everywhere tangent to the local vector field. If you do this for different starting points, you can draw a set of field lines that give a good representation of the properties of the vector field. Figure 1.8.2 below is an example of a field line representation for the same two charges we used in Figure 1.8.1.



**Figure 1.8.2** A “vector field” representation of the field of two point charges, and a “field line” representation of the same field. The field lines are everywhere tangent to the local field direction.

For the example of the velocity field of the jet stream discussed in 1.3, the field line representation is shown in Figure 1.8.3.



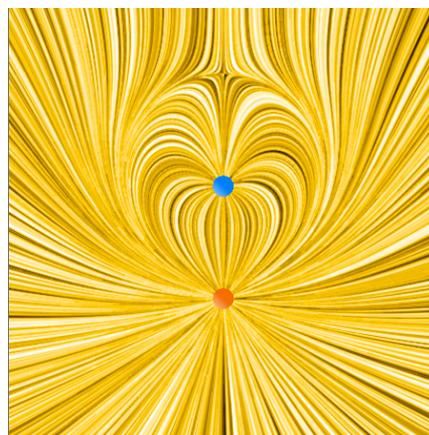
**Figure 1.8.3** Field lines of jet stream

In summary, the field lines are a representation of the collection of vectors that constitute the field, and they are drawn according to the following rules:

- (1) The direction of the field line at any point in space is tangent to the field at that point.
- (2) The field lines never cross each other, otherwise there would be two different field directions at the point of intersection.

### 1.8.3 Grass Seeds and Iron Filings Representations

The final representation of vector fields is the “grass seeds” representation or the “iron filings” representation. For an electric field, this name derives from the fact that if you scatter grass seeds in a strong electric field, they will orient themselves with the long axis of the seed parallel to the local field direction. They thus provide a dense sampling of the shape of the field. Figure 1.8.4 is a “grass seeds” representation of the electric field for the same two charges in Figures 1.8.1 and 1.8.2.



[Link to simulation](#)

**Figure 1.8.4:** A “grass seeds” representation of the electric field that we considered in Figures 1.8.1 and 1.8.2.

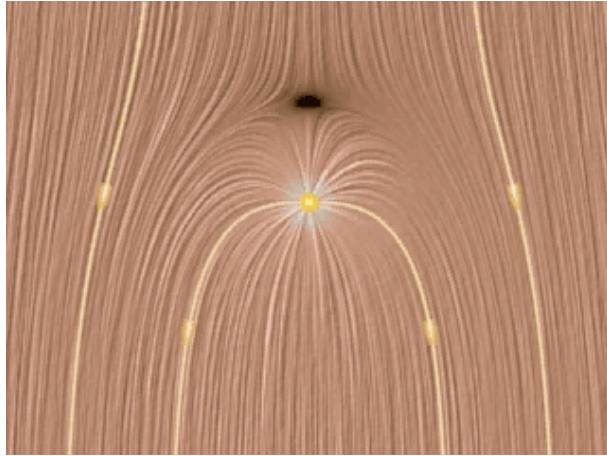
The local field direction is in the direction in which the texture pattern in this figure is correlated. This “grass seeds” representation gives by far the most information about the spatial structure of the field.

We will also use this technique to represent magnetic fields, but when used to represent magnetic fields we call it the “iron filings” representation. This name derives from the fact that if you scatter iron filings in a strong magnetic field, they will orient themselves with their long axis parallel to the local field direction. They thus provide a dense sampling of the shape of the magnetic field.

A frequent question from the student new to electromagnetism is “What is between the field lines?” Figures 1.8.2 and 1.8.4 make the answer to that question clear. There are more field lines that we have chosen not to draw between the field lines we chose to draw. The field itself is a continuous feature of the space between the charges, and the “grass seeds” representation of field makes that apparent.

#### 1.8.4 Motion of Electric and Magnetic Field Lines

In this text we will show the spatial structure of electromagnetic fields using all of the methods discussed above. In addition, for the field line, grass seeds, and iron filings representation, we will frequently show movies of the time evolution of the fields. We do this by having the field lines and the grass seed patterns or iron filings patterns move in the direction of the energy flow in the electromagnetic field at a given point in space. We will discuss the concept of energy flow in electromagnetic fields in Chapter 13. This energy flow is in the direction of  $\vec{E} \times \vec{B}$ , the cross product of the electric field  $\vec{E}$  and the magnetic field  $\vec{B}$ , and is perpendicular to both  $\vec{E}$  and  $\vec{B}$ . As an example, one frame of a movie of the electric fields of a charge moving upward in a downward constant field is shown in Figure 1.8.5, and there is a link to the movie. The flow direction in this movie, which is always perpendicular to  $E$ , is very different from that in our movies of the fluid flow fields above. In the movies in Section 1.4 above, the direction of the flow is in the same direction as the velocity field itself, where as in the movie linked to Figure 1.8.5, the flow is perpendicular to the  $E$  field.



[Link to movie](#)

**Figure 1.8.5:** One frame of a movie showing the electric field lines of a positive charge moving upward in a downward constant field. The motion of the field lines and texture patterns in this movie are in the direction of  $\vec{E} \times \vec{B}$ .

We adopt this animation technique for time-changing electromagnetic fields because these fields can both support the flow of energy and can store energy as well. We will discuss quantitatively how to compute this energy flow later, when we discuss the Poynting vector in Chapter 13. For now we simply note that when we animate the motion of the field line or grass seeds or iron filings representations, the direction of the pattern motion indicates the direction in which energy in the electromagnetic field is flowing.

In keeping with our discussion in Section 1.1.2, we also note that plotting the *total* field in Figure 1.8.5, that is the electric field of both the constant downward electric field *and* the field of the positive point charge, gives insight into why there is a downward force on the point charge. In the movie linked to Figure 1.8.5, one sees that as the charge moves upward, it is forcing itself into the pre-existing electric field, or medium, with stresses in that medium that develop due to this encroachment. These stresses resist the movement of the charge into this region. Moreover, as the charge moves upward and slows, there is a continual transfer of the kinetic energy of the charge to the electric energy stored in the electric field, thus conserving total energy. We see this in the movie because it shows the flow of electromagnetic energy away from the charge as it slows, and vice versa when he speed up again. We discuss this example in more detail in Section 2.11.2.

## 1.9 Summary

In this chapter, we have discussed the concept of fields. A scalar field  $T(x, y, z)$  is a function on all the coordinates of space. Examples of a scalar field include temperature and pressure. On the other hand, a vector field  $\vec{F}(x, y, z)$  is a vector each of whose components is a scalar field. A vector field  $\vec{F}(x, y, z)$  has both magnitude and direction at

every point  $(x, y, z)$  in space. Gravitational, electric and magnetic fields are all examples of vector fields.

We have also emphasized that the forces transmitted by electromagnetic fields can be understood conceptually as analogous to the forces transmitted by rubber bands and strings. To see this analogy, however, we must show depictions of the *total* field, that is the field due to all objects being considered. This depiction of the total field is rare in introductory texts, but we do it frequently here. As we proceed, we will always try to point out how the forces transmitted by electromagnetic fields can be understood by analogy to the more familiar forces transmitted by rubber bands and strings.

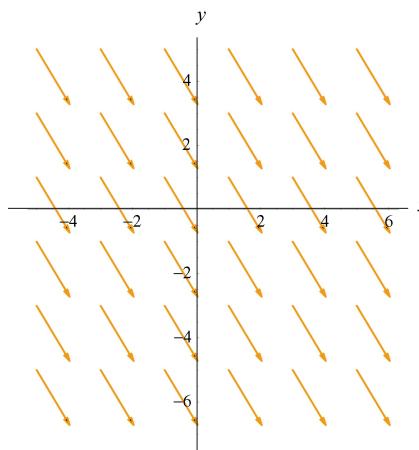
## 1.10 Solved Problems

### 1.10.1 Vector Fields

Make a plot of the following vector fields:

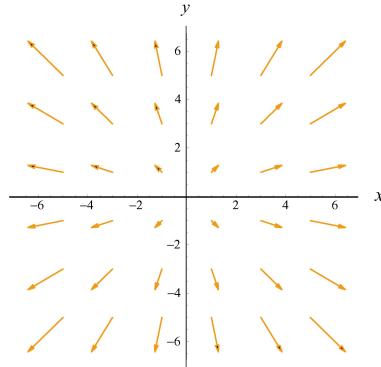
$$(a) \vec{v} = 3\hat{i} - 5\hat{j}$$

This is an example of a constant vector field in two dimensions. The plot is depicted in Figure 1.10.1:



**Figure 1.10.1**

$$(b) \vec{v} = \hat{r}$$



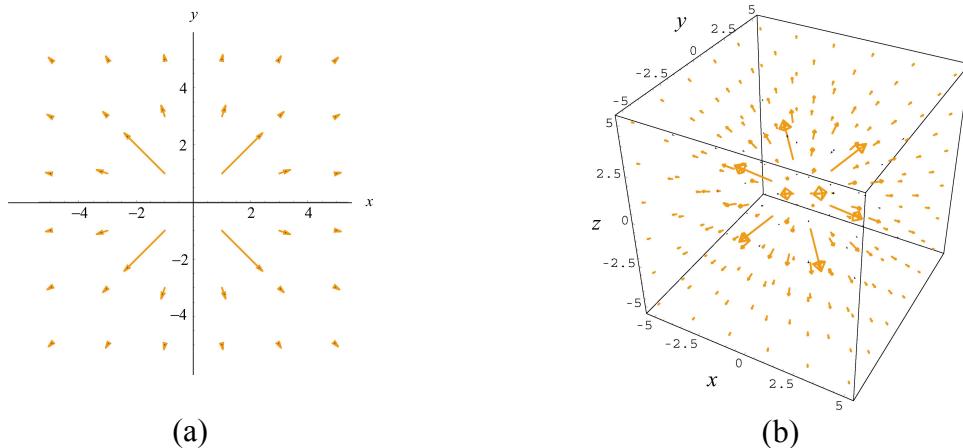
**Figure 1.10.2**

$$(c) \vec{v} = \frac{\hat{r}}{r^2}$$

In two dimensions, using the Cartesian coordinates where  $\vec{r} = x\hat{i} + y\hat{j}$ ,  $\vec{v}$  can be written as

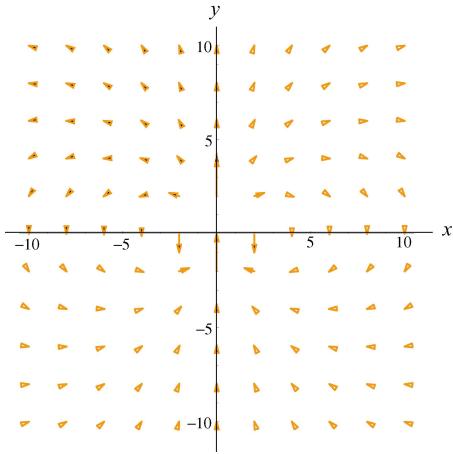
$$\vec{v} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}}$$

The plot is shown in Figure 1.10.3(a). Both the gravitational field of the Earth  $\vec{g}$  and the electric field  $\vec{E}$  due to a point charge have the same characteristic behavior as  $\vec{v}$ . In three dimensions where  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , the plot looks like that shown in Figure 1.10.3(b).



**Figure 1.10.3**

$$(d) \vec{v} = \frac{3xy}{r^5}\hat{i} + \frac{2y^2 - x^2}{r^5}\hat{j}$$



**Figure 1.10.4**

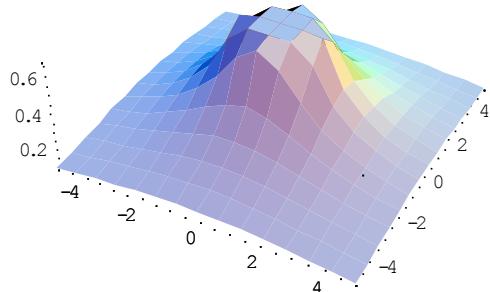
The plot is characteristic of the electric field due to a point electric dipole located at the origin.

### 1.10.2 Scalar Fields

Make a plot of the following scalar functions in two dimensions:

$$(a) f(r) = \frac{1}{r}$$

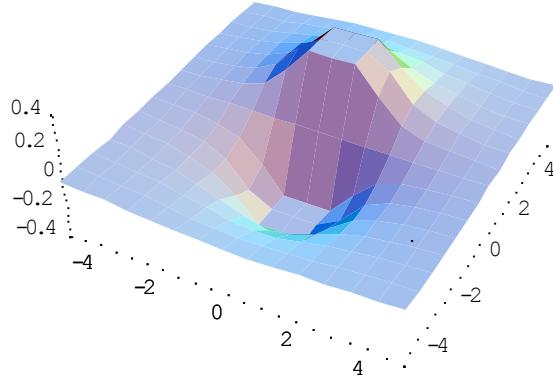
In two dimensions, we may write  $r = \sqrt{x^2 + y^2}$ .



**Figure 1.10.5**

Figure 1.10.5 can be used to represent the electric potential due to a point charge located at the origin. Notice that the mesh size has been adjusted so that the singularity at  $r = 0$  is not shown.

$$(b) f(x,y) = \frac{1}{\sqrt{x^2 + (y-1)^2}} - \frac{1}{\sqrt{x^2 + (y+1)^2}}$$



**Figure 1.10.6**

This plot represents the potential due to a dipole with the positive charge located at  $y = 1$  and the negative charge at  $y = -1$ . Again, singularities at  $(x,y) = (0,\pm 1)$  are not shown.

## 1.11 Additional Problems

### 1.11.1 Plotting Vector Fields

Plot the following vector fields:

- (a)  $y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$
- (b)  $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$
- (c)  $\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{2}}$
- (d)  $2y\hat{\mathbf{i}}$
- (e)  $x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$
- (f)  $\frac{y\hat{\mathbf{i}} - x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}}$
- (g)  $xy\hat{\mathbf{i}} - x\hat{\mathbf{j}}$
- (h)  $\cos x\hat{\mathbf{i}} + \sin y\hat{\mathbf{j}}$

### 1.11.2 Position Vector in Spherical Coordinates

In spherical coordinates (see Figure 1.2.3), show that the position vector can be written as

$$\vec{\mathbf{r}} = r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$$

### 1.11.3 Electric Field

A charge  $+Q$  is situated at the point  $(-x, 0, 0)$  and a charge  $-Q$  is situated at the point  $(x, 0, 0)$ . Find the electric field of these two charges at an arbitrary point  $(0, y, 0)$  on the  $y$ -axis.

### 1.11.4 An Object Moving in a Circle

A particle moves in a circular path of radius  $r$  in the  $xy$ -plane with a constant angular speed  $\omega = d\theta / dt$ . At some instant  $t$ , the particle is at  $P$ , as shown in Figure 1.11.1.

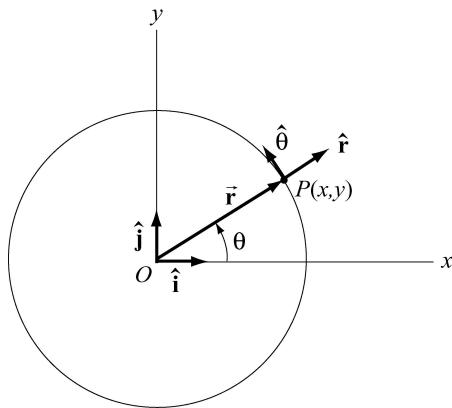


Figure 1.11.1

- Write down the position vector  $\vec{r}(t)$ .
- Calculate the velocity and acceleration of the particle at  $P$ .
- Express the unit vectors  $\hat{r}$  and  $\hat{\theta}$  in polar coordinates in terms of the unit vectors  $\hat{i}$  and  $\hat{j}$  in Cartesian coordinates.

### 1.11.5 Vector Fields

- Find a vector field in two dimensions which points in the negative radial direction and has magnitude 1.
- Find a vector field in two dimensions that makes an angle of  $45^\circ$  with the  $x$ -axis and has a magnitude  $(x + y)^2$  at any point  $(x, y)$ .
- Find a vector field in two dimensions whose direction is tangential and whose magnitude at any point  $(x, y)$  is equal to its distance from the origin.

- (d) Find a vector field in three dimensions which is in the positive radial direction and whose magnitude is 1.

### 1.11.6 Object Moving in Two Dimensions

An object moving in two dimensions has a position vector

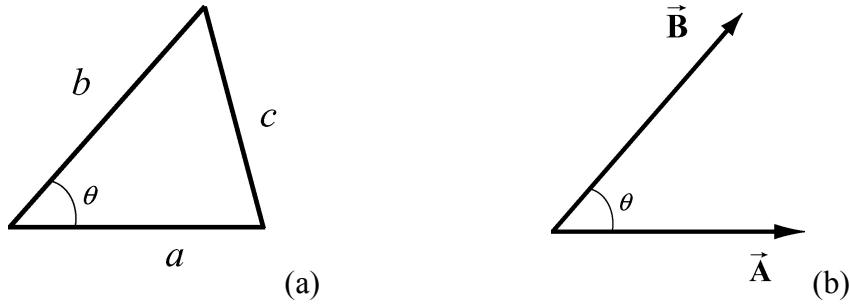
$$\vec{r}(t) = a \sin \omega t \hat{i} + b \cos \omega t \hat{j}$$

where  $a$ ,  $b$  and  $\omega$  are constants.

- (a) How far is the object from the origin at time  $t$ ?
- (b) Find the velocity and acceleration as function of time for the object.
- (c) Show that the path of the object is elliptical.

### 1.11.7 Law of Cosines

Two sides of the triangle in Figure 1.11.2(a) form an angle  $\theta$ . The sides have lengths  $a$  and  $b$ .



**Figure 1.11.2** Law of cosines

The length of the side opposite  $\theta$  is given by the relation triangle identity

$$c^2 = a^2 + b^2 - 2ab \cos \theta .$$

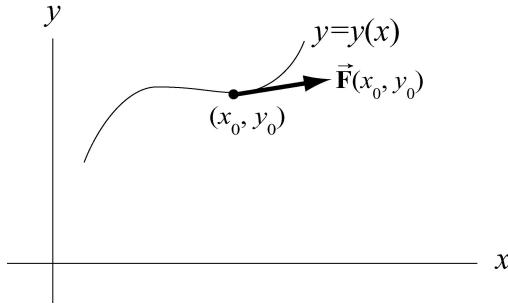
Suppose we describe the two given sides of the triangles by the vectors  $\vec{A}$  and  $\vec{B}$ , with  $|\vec{A}|=a$  and  $|\vec{B}|=b$ , as shown in Figure 1.11.2(b)

- (a) What is the geometric meaning of the vector  $\vec{C} = \vec{B} - \vec{A}$ ?

- (b) Show that the magnitude of  $\vec{C}$  is equal to the length of the opposite side of the triangle shown in Figure 1.11.2(a), that is,  $|\vec{C}| = c$ .

### 1.11.8 Field Lines

A curve  $y = y(x)$  is called a field line of the vector field  $\vec{F}(x, y)$  if at every point  $(x_0, y_0)$  on the curve,  $\vec{F}(x_0, y_0)$  is tangent to the curve (see Figure 1.11.3).



**Figure 1.11.3**

Show that the field lines  $y = y(x)$  of a vector field  $\vec{F}(x, y) = F_x(x, y)\hat{i} + F_y(x, y)\hat{j}$  represent the solutions of the differential equation

$$\frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)}$$