Indefinite Integrals

A.1 Antiderivative and Indefinite Integral: 2

A.1.1 Example Antiderivative of a Constant Function: 2

A.1.2 Example Antiderivative of $f(x) = \cos x$ 2

A.1.3 Example Antiderivative of $f(x) = e^x$ 2

A.1.3 Example Antiderivative of $f(x) = \ln x$ 2

A.1.4 Integration Table (Antiderivative Table) 3

A.2 Change of Variable Formula 4

A.2.1 Example: $\int \sin(3x)\, dx$ 4

A.2.1 Example: $\int xe^{-x^2}\, dx$. 4

A.2.1 Example: $\int \frac{dx}{a^2 + b^2x^2}$ 4

A.3 Integration by Parts 4

A.3.1 Example: $\int x \sin x\, dx$ 5

A.3.2 Example: $\int xe^x\, dx$ 5

A.4 Method of Partial Fractions 5

A.4.1 Example: $y(x) = \frac{1}{1-x^2}$ 5

A.1 Antiderivative and Indefinite Integral:

Integration is defined as the inverse operation of differentiation. When a function $F(x)$ is differentiated to give $f(x) = dF/dx$, then $F(x)$ is called an *antiderivative* of $f(x)$, (or also called the *primitive*) . A constant can always be added to a function without changing its derivative. If $F(x)$ is an antiderivative of $f(x)$, then all the antiderivatives of $f(x)$ are given by

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant. The expression $\int f(x)\,dx$ is also called the *indefinite integral* of $f(x)$. The symbol $\int ...dt$ means the 'integral, with respect to t, of …", and is thought of as the inverse of the symbol $\frac{d}{dt}$..... In terms of a differentials: $dF = f(x)dx$; then the indefinite integral is written as

$$\int dF(x) = F(x) + c.$$

Therefore for every differentiation formula there is a corresponding integration formula.

A.1.1 Example Antiderivative of a Constant Function:

Let $f(x) = 1$, then $F(x) = x$ and $\int dx = x + c$. (Note that $F'(x) = \frac{d}{dx}(x) = 1 = f(x).$)

A.1.2 Example Antiderivative of $f(x) = \cos x$

Because $\frac{d}{dx}\sin x = \cos x,$ therefore $\int \cos x\,dx = \sin x + c.$

A.1.3 Example Antiderivative of $f(x) = e^x$

Because $\frac{d}{dx}e^x = e^x$, therefore $\int e^x dx = e^x + c.$

A.1.3 Example Antiderivative of $f(x) = \ln x$

An antiderivative of $f(x) = \ln x$ is $F(x) + c = x\ln x - x$. To verify this note that

$$\frac{dF}{dx} = \frac{d}{dx}(x\ln x - x + c) = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x = f(x).$$

A.1.4 Integration Table (Antiderivative Table)

An *integral table* is a list of antiderivatives. The following table consists the indefinite integrals of some common functions, where the arbitrary integration constants are omitted; and a and n are constants. More complete Integral Tables can be found online.

Integral Table

1. $\int a\, dx = ax$
2. $\int af(x)\, dx = a\int f(x)\, dx$
3. $\int (u+v)\, dx = \int u\, dx + \int v\, dx$
4. $\int x^n\, dx, = \dfrac{x^{n+1}}{n+1} \quad n \neq -1$
5. $\int \dfrac{dx}{x} = \ln|x|$
6. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\ln\left|a+bx\right|$
7. $\int e^x\, dx = e^x$
8. $\int e^{ax} dx = \dfrac{e^{ax}}{a}$
9. $\int b^{ax}\, dx = \dfrac{b^{ax}}{a\ln b}$
10. $\int \ln|x|\, dx = x\ln|x| - x$
11. $\int \sin x\, dx = -\cos x$
12. $\int \cos x\, dx = \sin x$
13. $\int \tan x\, dx = -\ln|\cos x|$
14. $\int \cot x\, dx = \ln|\sin x|$
15. $\int \sec x\, dx = \ln|\sec x + \tan x|$
16. $\int \sin x \cos x\, dx = \dfrac{1}{2}\sin^2 x$
17. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
18. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$
19. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x - \sqrt{x^2 \pm a^2}\right|$
20. $\int w(u)\, dx = \int w(u)\dfrac{dx}{du}\, du$
21. $\int u\, dv = uv - \int v\, du$

A.2 Change of Variable Formula

An integral with respect to the variable x can be converted into another integral, depending on the variable $u(x)$, using the rule:

$$\int w(x)\, dx = \int \left[w(u) \frac{dx}{du} \right] du .$$

A.2.1 Example: $\int \sin(3x)\, dx$

Calculate $\int \sin(3x)\, dx$. Because $w(x) = \sin(3x)$, let $u = 3x$, then $dx / du = 1/3$. Thus

$$\int \sin(3x)\, dx = \frac{1}{3}\int \sin u\, du = \frac{1}{3}(-\cos u + c) = \frac{1}{3}(-\cos(3x) + c).$$

A.2.1 Example: $\int x e^{-x^2}\, dx.$

Consider the integral $\int x e^{-x^2}\, dx$. Let $u = x^2$, or $x = \sqrt{u}$, hence $dx = \dfrac{1}{2\sqrt{u}} du$ and $w(u) = \sqrt{u} e^{-u}$.

Using the rule for change of variable, the integral becomes

$$\int x e^{-x^2}\, dx = \int \sqrt{u} e^{-u} \frac{1}{2\sqrt{u}} du = \frac{1}{2}\int e^{-u}\, du = -\frac{1}{2} e^{-u} + c = -\frac{1}{2} e^{-x^2} + c.$$

To check that this result is correct, note that

$$\frac{d}{dx}\left(-\frac{1}{2} e^{-x^2} + c \right) = x e^{-x^2}.$$

A.2.1 Example: $\int \dfrac{dx}{a^2 + b^2 x^2}$

Evaluate $\int \dfrac{dx}{a^2 + b^2 x^2}$, where *a* and *b* are constants. Let $u = bx$, then $dx = du / b$ and using the integral formula 17 in the Integral Table, yields

$$\int \frac{dx}{a^2 + b^2 x^2} = \frac{1}{b}\int \frac{du}{a^2 + u^2} = \frac{1}{ab}\left(\tan^{-1} \frac{u}{a} \right) + c = \frac{1}{ab}\left(\tan^{-1} \frac{bx}{a} \right) + c.$$

A.3 Integration by Parts

Let u and v be any two functions of x. Then the rule for *integration by parts* is $\int u\, dv = uv - \int v\, du$. To justify this, start with the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate both sides of this equation with respect to x.

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx,$$

which can be written as

$$\int d(uv) = \int u\,dv + \int v\,du.$$

Because $\int d(uv) = uv$, therefore

$$\int u\,dv = uv - \int v\,du.$$

A.3.1 Example: $\int x\sin x\,dx$

Find $\int x\sin x\,dx$. Let $u = x$, $dv = \sin x dx$. Then $du = dx$, and $v = -\cos x = \int \sin x\,dx$, . Thus

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + c.$$

A.3.2 Example: $\int xe^x\,dx$

Find $\int xe^x\,dx$. Because $e^x = \int e^x\,dx$, let $u = x$, $dv = e^x dx$, so that $du = dx$, $v = e^x$. Then,

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + c = (x-1)e^x + c.$$

A.4 Method of Partial Fractions

A useful manipulation from elementary algebra is to combine two simple fractions into one function. The *method of partial fractions* involves reversing this process in which you split a function into a sum of fractions with simpler denominators.

A.4.1 Example: $y(x) = \dfrac{1}{1-x^2}$

Consider the function $y(x) = 1/(1-x^2)$. Because $1-x^2 = (1-x)(1+x)$, write

$$\frac{1}{1-x^2} = \frac{a}{1-x} + \frac{b}{1+x},$$

where a and b, which are yet to be defined, are called *undetermined coefficients*. Combining terms yields

$$\frac{1}{1-x^2} = \frac{a}{1-x} + \frac{b}{1+x} = \frac{a(1+x)+b(1-x)}{1-x^2} = \frac{(a+b)+(a-b)x}{1-x^2}.$$

Equate coefficient of like powers, (this is comparable to solving a system of linear equations). Therefore $a+b=1$ and $a-b=0$. Thus $a=b=1/2$. Now integrate $1/(1-x^2)$ in terms of the simpler integrals:

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}(-\ln|1-x| + \ln|1+x|) + c = \ln\left(\left|\frac{1+x}{1-x}\right|\right) + c.$$

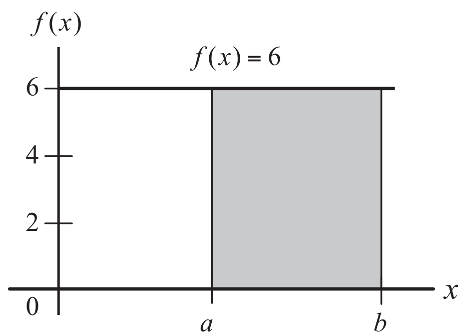
Definite Integrals

I.1 Definite Integrals and Area 2
I.1.1 Example Area Under a Curve: 2
I.1.2 Sign Convention: 2
I.1.3 Definite Integral: 2
I.2. Fundamental Theorems of Calculus 4
I.2.1 Mean Value Theorem 4
I.2.2 Mean Value Theorem for Definite Integrals: 5
I.2.3 Are Function 5
I.2.3 First Fundamental Theorem of Calculus: 6
I.2.4 Example Constant Function 6
I.2.5 Example Linear Function 7
I.2.6 Second Fundamental Theorem of Calculus: 7
I.2.6 Example Quadratic Function 8
I.2.7 Exchanging Limits of Integral 8
I.2.8 Example Integrating $\sin\theta$ 8
I.3 Some Applications of Integration 9
I.3.1 Position, Velocity, and Acceleration 9
I.3.2 Example Non-constant Acceleration 9
I.3.3 Example Exponentially Decreasing Velocity 10
I.3.3 The Method of Slices: Volume of a Right Circular Cone. 11
I.3.3 Example: Volume of a Right Circular Cone. 11
I.3.3 Example: Volume of a Symmetric Object 12
I.4 Double Integrals 13
I.4.1 Example: Double Integral 15
I.4.2 Example: Moment of Inertia of a Sheet 16

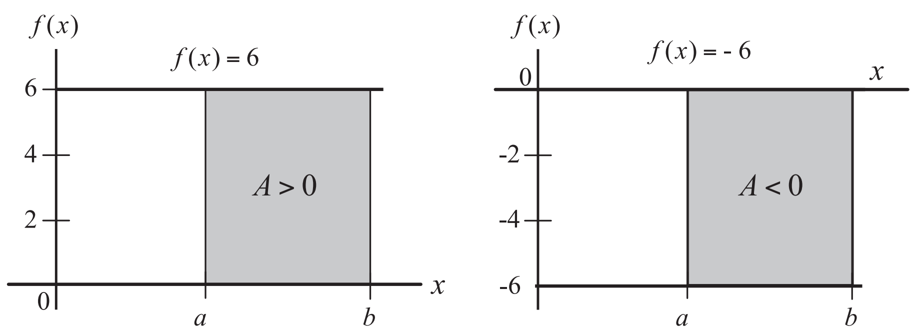
I.1 Definite Integrals and Area

I.1.1 Example Area Under a Curve:

Consider a graph of a straight line given by $f(x) = 6$.



The area in the rectangle is the product of the base, $(b-a)$, and the height, 6. Thus the area is $6(b-a)$.

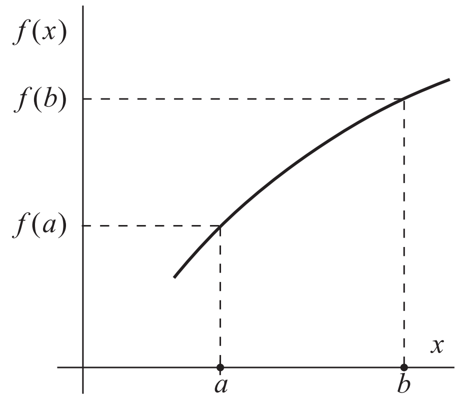


I.1.2 Sign Convention:

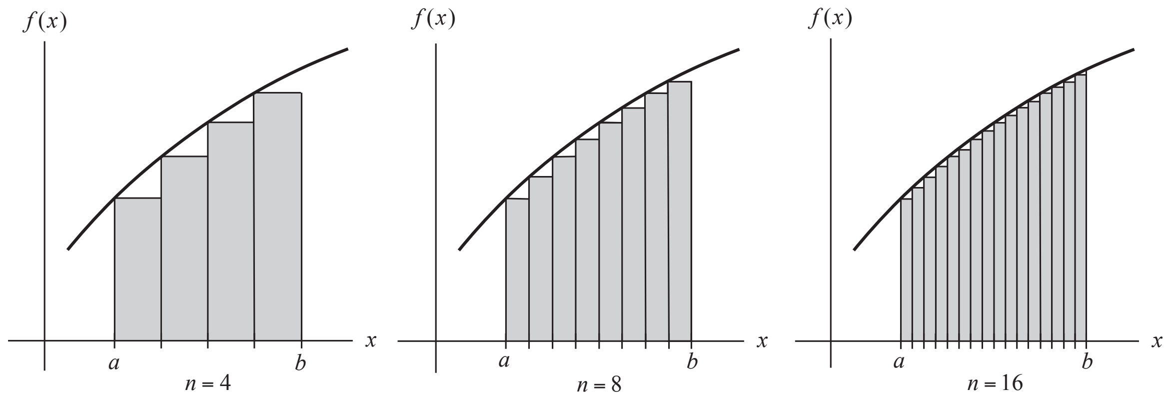
There is an important convention in the sign of the area. In the drawing at the above left, because $f(x)$ is positive, the area is positive. However, the area under the graph at the above right is negative because the height is $f(x)$, which is negative. Thus, areas can be positive or negative.

I.1.3 Definite Integral:

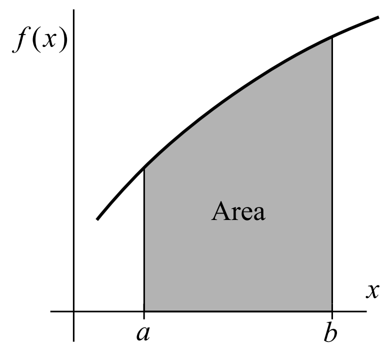
Consider the graph of an arbitrary function $f(x)$ bordered by the x-axis between $x = a$ and $x = b$, and $f(a)$ and $f(b)$, shown in the figure below.



In order to first approximate the area under the curve, divide the area into n strips of equal widths by drawing lines parallel to the vertical axis. The width of each strip is $\Delta x = (b-a)/n$. In the figures below $n=4$, $n=8$, and $n=16$.



The height of the first rectangular shape is $f(x_1)$, where $x_1 = a$ is the value of x at the beginning of the first strip. Similarly, the height of the second rectangular shape is $f(x_2)$ where $x_2 = x_1 + \Delta x$. The height of the *i*th strip is $f(x_i)$. The area of the rectangular strips is then $A \approx \sum_{i=1}^{n} f(x_i)\,\Delta x$. As the number of strips gets larger, the approximation to the area under the curve by the rectangular strips gets better and better.



In the limit where $\Delta x \to 0$, the approximation becomes an equality. Thus,

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\, \Delta x.$$

This limit is called the *definite integral* and is written $\int_a^b f(x)\,dx$. The function $f(x)$ is called the *integrand*, x is called the *integration variable*, and the points a and b are called the *limits of the integral*.

The indefinite and definite integral both employ the integral symbol $\int$, and so they can easily be confused. They are entirely different: the definite integral is a *number* and is equal to the area under the curve between limits *a* and *b*. In contrast the indefinite integral is a *function* — the antiderivative of the integrand.

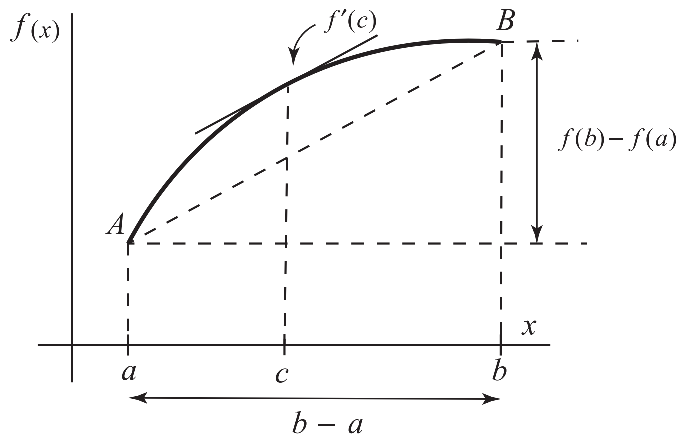
I.2. Fundamental Theorems of Calculus

Finding the limit of an infinite sum requires a detailed analysis of the limit. Fortunately, the value of a definite integral can be determined by a far simpler method using the techniques of integration for indefinite integrals. In order to show this, we need a few theorems.

I.2.1 Mean Value Theorem

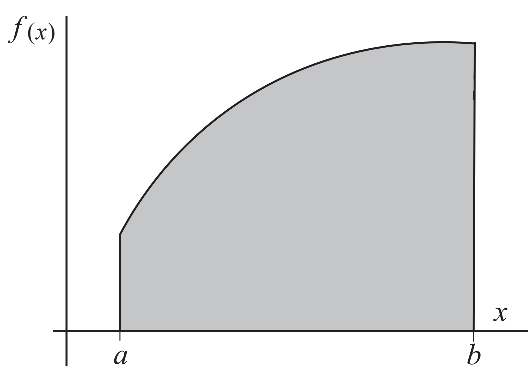
Let $f(x)$ be a continuous positive function on the interval $[a,b]$, and has a derivative at each value of x, for $a < x < b$. Then there is a least one real number c, with $a < c < b$ such that $f(b) - f(a) = f'(c)(b-a)$.

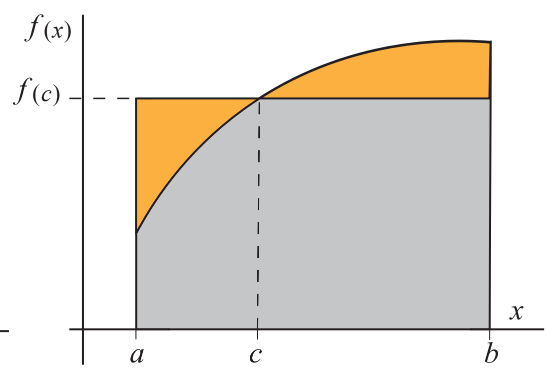
In the figure below, $f'(c)$ is the slope at the point c and is equal to the slope of the chord connecting the points A and B, thus $f'(c) = (f(b) - f(a))/(b-a)$.



I.2.2 Mean Value Theorem for Definite Integrals:

In the figure below left the definite integral $\int_a^b f(x)\,dx$ corresponds to the shaded area under the curve $y = f(x)$ between the points $x = a$ and $x = b$. There exists a point c between $x = a$ and $x = b$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$. The figure below right illustrates this area equality.





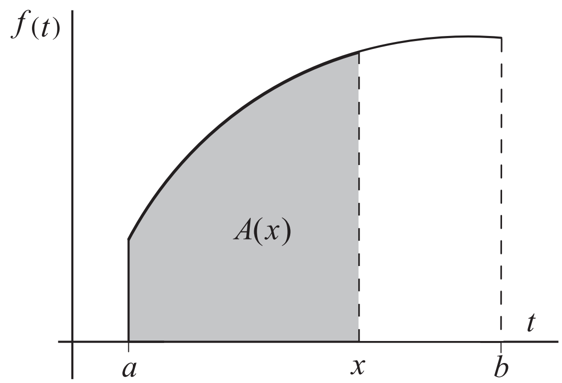
I.2.3 Are Function

Let $f(x)$ be a continuous positive function on the interval $[a,b]$. The definite integral corresponding to the area under the curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by the limit of the summation

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx .$$

Define the *area function* by letting the variable x replace the upper limit b of the definite integral and using the symbol t to represent the integration variable:

$$A(x) = \int_a^x f(t)\,dt; \quad \text{if } a \le x \le b.$$



I.2.3 First Fundamental Theorem of Calculus:

The area function $A(x)$ is the antiderivative of the function $f(x)$, i.e. $A'(x) = f(x)$. To understand why this is true let $A(x+\Delta x) = \int_a^{x+\Delta x} f(t)\,dt$. Then

$$A(x+\Delta x) - A(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt.$$

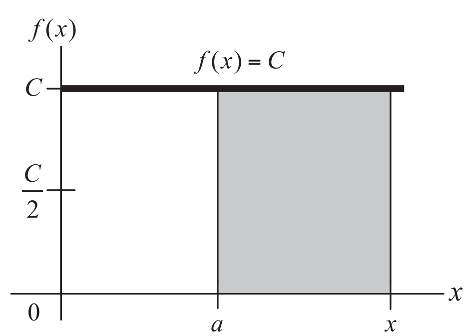
By the mean-value theorem for definite integrals $\int_x^{x+\Delta x} f(t)\,dt = f(c)\Delta x$. Then $(A(x+\Delta x) - A(x)) / \Delta x = f(c)$. Because $f(x)$ is continuous

$$\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{c \to x} f(c) = f(x).$$

Thus $A'(x) = f(x)$.

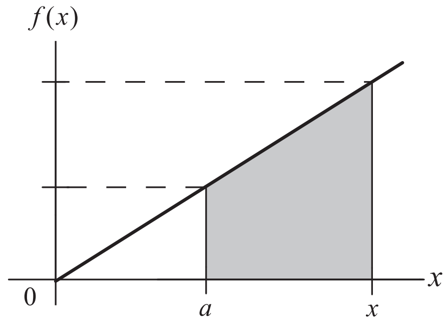
I.2.4 Example Constant Function

To illustrate that $A'(x) = f(x)$, consider the area under the curve $f(x) = C$, where C is a constant, is $A(x) = C(x-a)$. Differentiating, $A'(x) = C = f(x)$.



I.2.5 Example Linear Function

Find the area $A(x)$ under $f(x) = Cx$ between a and x.



The area above is the difference of the area of two right triangles. Using area = ½ base × height: $A(x) = \frac{1}{2}xf(x) - \frac{1}{2}af(a) = \frac{1}{2}Cx^2 - \frac{1}{2}Ca^2$. Therefore the derivative of the area function is $A'(x) = \frac{d}{dx}\left(\frac{1}{2}Cx^2 - \frac{1}{2}Ca^2\right) = Cx = f(x)$.

I.2.6 Second Fundamental Theorem of Calculus:

Let $F(x)$ be any indefinite integral of $f(x)$ i.e. $F'(x) = f(x)$. Then the definite integral of $f(x)$ is

$$A = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Idea of the proof: The area function is equal to $A(x) = \int_a^x f(t)\,dt = F(x) + c$. Set $x = a$. Then $0 = A(x) = \int_a^a f(t)\,dt = F(a) + c$. Therefore $c = -F(a)$. Hence the area under the curve $f(x)$ is equal to $A(x) = \int_a^x f(t)\,dt = F(x) - F(a)$.

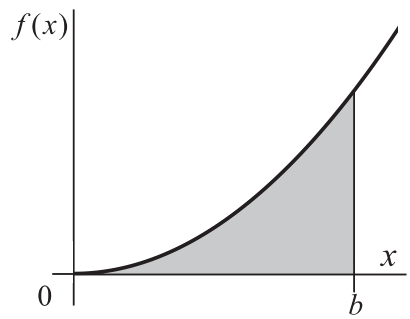
Now set $x = b$. Thus the definite integral corresponding to the area under the curve of the function $f(x)$ for $a \le x \le b$ can be determined by calculating the value of any indefinite integral F of f at $x = b$ and subtracting the value of F at $x = a$. The definite integral is then

$$A = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

The significance of the Second Fundamental Theorem of Calculus is that instead of determining the limit of an infinite sum in an indefinite integral, a definite integral can be determined by evaluating an antiderivative at the endpoints of the interval. In other words, calculating a definite integral is the inverse operation of differentiation.

I.2.6 Example Quadratic Function

Find the area under the curve $y = x^2$ between $x = 0$ and $x = b$.



An antiderivative is $F(x) = x^3/3$ because $F'(x) = x^2$. Therefore

$$A = \int_0^b x^2\,dx = F(b) - F(0) = \frac{1}{3}b^3 - \frac{1}{3}0^3 = \frac{1}{3}b^3.$$

Note that the definite integral yields a number. Furthermore, we no longer need to introduce an undetermined constant c whenever we evaluate an expression such as $F(b) - F(0)$.

I.2.7 Exchanging Limits of Integral

The proof that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ is as follows:

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where} \quad F(x) = \int f(x)\,dx.$$

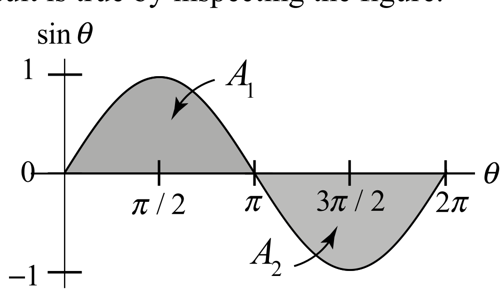
Reversing the limits of the integral yields

$$\int_b^a f(x)\,dx = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f(x)\,dx.$$

I.2.8 Example Integrating $\sin\theta$

$$\int_0^{2\pi} \sin\theta\,d\theta = -\cos\theta\Big|_0^{2\pi} = -(1-1) = 0.$$

It is easy to see why this result is true by inspecting the figure.



The integral yields the total area under the curve, from 0 to 2π, which is the sum of the area A_1 between 0 to π, and A_2 between π and 2π. But A_2 is negative, because $\sin\theta$ is negative in that region. By symmetry, the two areas just add to 0. For example $A_1 = \int \sin\theta \, d\theta = -\cos\theta \,|_0^{\pi} = -[-1-(+1)] = 2.$ In evaluating $\cos\theta$ at the limits, $\cos\pi = -1$, and $\cos 0 = 1$.

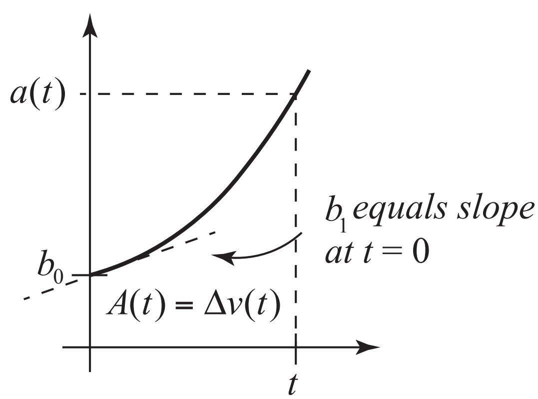
I.3 Some Applications of Integration

I.3.1 Position, Velocity, and Acceleration

For one dimensional motion, the velocity $v(t)$ of an object moving the derivative of the position function with respect to time, $v = dx/dt$. the acceleration $a(t)$ of an object moving the derivative of the velocity with respect to time, $a = dv/dt$. (For two or three dimensional motion, position, velocity, and acceleration are vectors so the relationship between them are vector derivatives, $\vec{\mathbf{v}} = d\vec{\mathbf{x}}/dt$, and $\vec{\mathbf{a}} = d\vec{\mathbf{v}}/dt$. For simplicity the following discussion will just consider one dimensional motion). Now reverse the procedure and find the velocity from acceleration by integration using definite integrals, and then the position from velocity also by integration.

I.3.2 Example Non-constant Acceleration

Suppose an object starts out with an initial velocity v_0 at time t_0 and travels in a straight line with a given acceleration $a(t) = b_0 + b_1 t + b_2 t^2$. The graph of the acceleration *vs*. time is shown in the figure below.



The change in velocity $\Delta v(t) = v(t) - v_0$ during the interval $[0,t]$ is equal to the area $A(t)$, and hence the integral

$$A(t) = \Delta v(t) = v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')\,dt' = \int_{t'=0}^{t'=t} (b_0 + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}.$$

The velocity as a function in time is then

$$v(t) = v_0 + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}.$$

There may be some confusion in the notation because the symbol t, which is the dependent variable of the function appears both as an endpoint of the integral and as the integration variable in the integrand $v(t)$. To avoid this confusion, the integration variable, called a *dummy variable,* was replaced by t'.

Denote the initial position at time $t = 0$ by x_0. The change in position called the *displacement* as a function of time is given by the integral

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt' = \int_{t'=0}^{t'=t} \left(v_0 + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt'$$

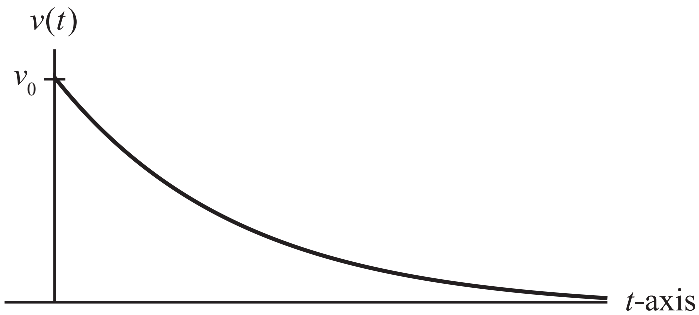
$$= v_0 t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}.$$

The position at time t is then

$$x(t) = x_0 + v_0 t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}.$$

I.3.3 Example Exponentially Decreasing Velocity

Suppose an object moves with a velocity that continually decreases according to $v(t) = v_0 e^{-bt}$, (v_0 and b are positive constants).



At $t = 0$ the object is at the origin; $x(0) = 0$. The distance, $x(t)$, the object will have moved after an infinite time is given by the integral

$$x(t) - x(0) = \int_0^t v(t)\,dt = \int_0^t v_0\, e^{-bt}\,dt = -\frac{v_0}{b} e^{-bt}\Big|_0^t = -\frac{v_0}{b}(e^{-bt} - 1).$$

We are interested in the $\lim_{t\to\infty} x(t)$. Because $e^{-bt} \to 0$ as $t \to \infty$, we have

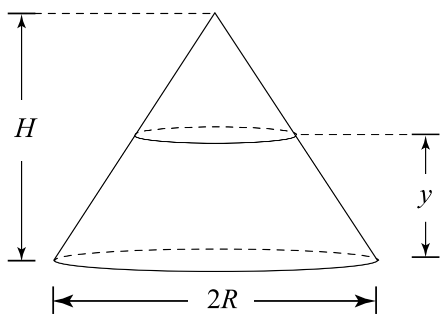
$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(-\frac{v_0}{b}(e^{-bt}-1)\right) = -\lim_{t\to\infty}\frac{v_0}{b}e^{-bt} + \lim_{t\to\infty}\frac{v_0}{b} = \frac{v_0}{b}.$$

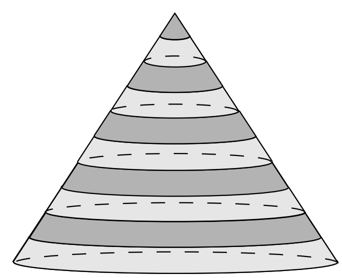
Although the object never comes completely to rest, its velocity gets so small that the total distance traveled is finite.

I.3.3 The Method of Slices: Volume of a Right Circular Cone.

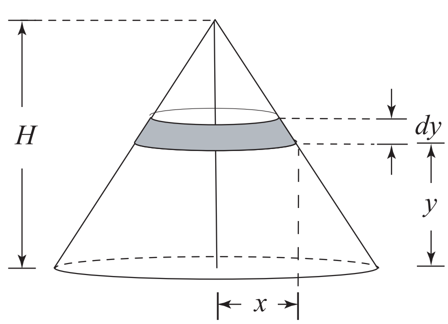
Integration can be used to find the volumes of solids of known geometry, in particular the volume of symmetric solids.

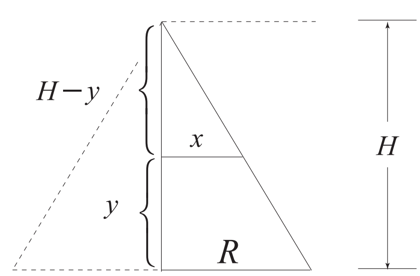
I.3.3 Example: Volume of a Right Circular Cone.





The height of a right circular cone is H, and the radius of the base is R. Let y represent distance vertically from the base. Slice the cone into a number of discs whose volume is approximately that of the cone in the figure (the cone has been approximated by ten circular discs). Then $V \approx \sum_{i=1}^{10} \Delta V_i$, where ΔV_i is the volume of one of the discs. In the limit where the height of each disc (and hence the volume) goes to 0, $V = \int dV$.





A section of the cone is shown in the figure above left, which is represented by a disc. The radius of the disc is x and its height is dy. The volume of this disc is the product of the area and height. Thus, $dV = \pi x^2 dy$. The radius x is a function of the height y. The figure on the right shows a cross section of the cone. Because x and R are corresponding edges of similar triangles, therefore $\frac{x}{R} = \frac{H-y}{H}$, or $x = R\left(1-\frac{y}{H}\right)$.

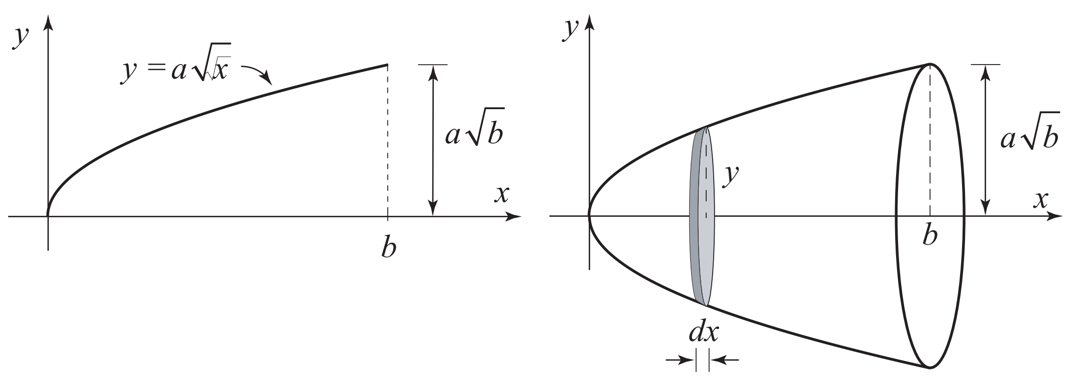
Thus, $dV = \pi x^2 dy = \pi R^2\left(1-\frac{y}{H}\right)^2 dy$. The integral for V is now

$$V = \int_0^H \pi R^2\left(1-\frac{y}{H}\right)^2 dy = \pi R^2 \int_0^H \left(1-\frac{2y}{H}+\frac{y^2}{H^2}\right) dy$$

$$= \pi R^2\left(y-\frac{y^2}{H}+\frac{1}{3}\frac{y^3}{H^2}\right)\Bigg|_0^H = \pi R^2\left(H-H+\frac{1}{3}H\right) = \frac{1}{3}\pi R^2 H.$$

I.3.3 Example: Volume of a Symmetric Object

Generate the volume (figure below right) by rotating the area under the curve $y = a\sqrt{x}$ (figure below left) about the x-axis, for the range $x=0$ to $x=b$?



Consider a disk of thickness dx and radius y, located at a distance x from the origin as shown in the figure above right. The differential volume dV for the disk is

$$dV = \pi y^2 dx = \frac{\pi a^2}{h^2} x^2 dx\,.$$

The limits of the variable x are $x=0$ to $x=h$. The volume integral is then

$$V = \int_0^b \pi a^2 x \, dx = \pi a^2 \left. \frac{x^2}{2} \right|_0^b = \frac{\pi a^2 b^2}{2} .$$

I.4 Double Integrals

The subject of this section—multiple integrals—introduces some new concepts and enables us to apply calculus to a world of problems that involve *multiple variable calculus*, in contrast to *single variable calculus*.

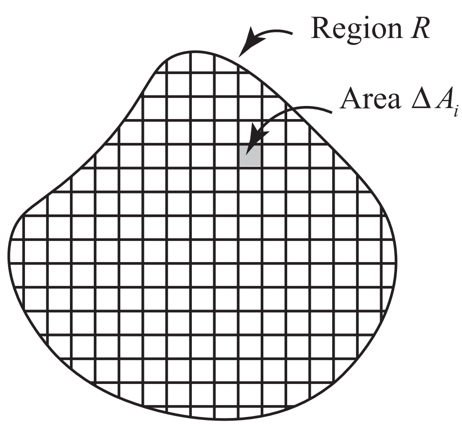
Double integrals are similarly defined for two independent variables, x and y, $z = f(x,y)$ will be the dependent variable. Thus, z is a function of two variables. (In general, multiple integrals are defined for an arbitrary number of independent variables.)

The definite integral of $f(x)$ between a and b was defined by

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\, \Delta x.$$

The double integral is similarly defined, but with two independent variables. There are, however, some important differences. For a single definite integral the integration takes place over a closed interval between a and b on the x-axis. In contrast, the integration of $z = f(x,y)$ takes place over a closed region R in the $x-y$ plane.

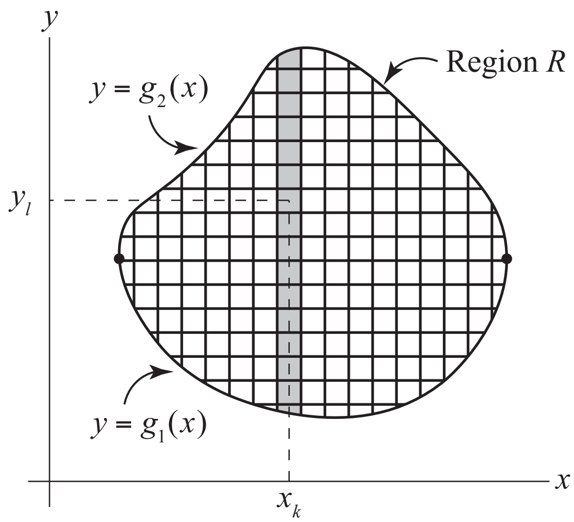
To define the double integral, divide the region R into N smaller regions each of area ΔA_i.



Let (x_i, y_i) be an arbitrary point inside the region ΔA_i. Then in analogy to the integral of a single variable, the double integral is defined as

$$\iint f(x,y)\, dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i .$$

The double integral is often evaluated by taking ΔA_i to be a small rectangle with sides parallel to the x and y axes. The procedure is first evaluate the sum and limit along one direction and then along the other. Consider the upper portion of the region R in the $x-y$ plane to be bounded by the curve $y = g_2(x)$, while the lower portion is bounded by $y = g_1(x)$, as in the diagram.



Let $\Delta A_i = \Delta x_k \Delta y_l$, then

$$\iint\limits_R f(x,y)\, dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i = \lim_{\Delta x_k \to 0} \lim_{\Delta y_l \to 0} \sum_{k=1}^{p} \sum_{l=1}^{q} f(x_k, y_l)\, \Delta y_l\, \Delta x_k .$$

This can be simplified by carrying it out in two separate steps.

$$\iint\limits_R f(x,y)\, dA = \lim_{\Delta x_k \to 0} \sum_{k=1}^{p} \left[\lim_{\Delta y_l \to 0} \sum_{l=1}^{q} f(x_k, y_l) \Delta y_l \right] \Delta x_k .$$

The first step is to carry out the operation within the brackets. Note that x_k is not altered as when summing over l in the brackets. This corresponds to summing over the crosshatched strip in the diagram with x_k treated as approximately a constant. The quantity in square brackets is then merely a definite integral of the variable y, with x treated as a constant. Although the limits of integration, $g_1(x)$ and $g_2(x)$, are constants for a particular value of x, they are in general non-constant functions of x. The quantity in square brackets can then be written as

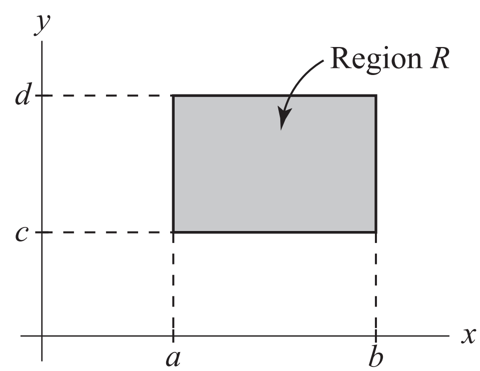
$$\int_{g_1(x_k)}^{g_2(x_k)} f(x_k, y)\, dy.$$

This quantity will no longer depend on y, but it will depend on x_k both through the integrand $f(x_k, y)$ and the limits $g_1(x_k)$, $g_2(x_k)$. Consequently,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x_k \to 0} \sum_{k=1}^{p} \left[\int_{g_1(x_k)}^{g_2(x_k)} f(x_k, y)\,dy \right] \Delta x_k = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

In calculations it is essential that one first evaluate the integral in the square brackets while treating x as a constant. The result is some function, which depends only on x. The next step is to calculate the integral of this function with respect to x, treating x now as a variable.

Multiple integrals are most easily evaluated if the region R is a rectangle whose sides are parallel to the x and y coordinate axes, as shown in the drawing.



The double integral is

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx.$$

If the integration over x is carried over before the integration over y, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

This can be found merely by interchanging the y and x operations in evaluating the double integral. (Note that the integration limits must also be interchanged.)

I.4.1 Example: Double Integral

Evaluate the double integral of $f(x,y) = 3x^2 + 2y$ over the rectangle in the $x-y$ plane bounded by the lines $x = 0$, $x = 3$, $y = 2$, and $y = 4$. The double integral is equal to the iterated integral.

$$\iint_R \left(\frac{1}{3}x^2 + y \right) dA = \int_0^3 \left[\int_2^4 (3x^2 + 2y)\,dy \right] dx.$$

Integrating first over y and then over x yields

$$\int_0^3\left[\int_2^4 (3x^2+2y)\,dy\right]dx = \int_0^3 (3x^2y+y^2)\Big|_2^4\,dx = \int_0^3 [3x^2(4-2)+(16-4)]\,dx$$

$$= \int_0^3 (6x^2+12)\,dx = \left(6\frac{x^3}{3}+12x\right)\Bigg|_0^3 = 54+36=90.$$

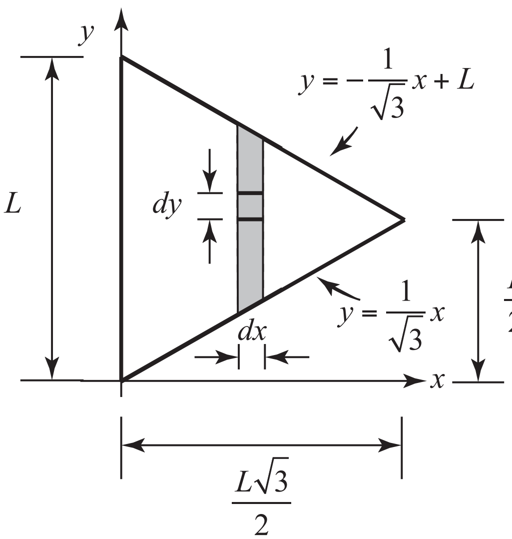
Alternatively, integrating first over x and then over y yields

$$\int_2^4\left[\int_0^3 (3x^2+2y)\,dx\right]dy = \int_2^4 (x^3+2yx)\Big|_0^3\,dy = \int_2^4 (27+6y)\,dy = (27y+3y^2)\Big|_2^4$$

$$=108+48-(54+12)=90.$$

I.4.2 Example: Moment of Inertia of a Sheet

Find the moment of inertia of a very thin sheet in the shape of an equilateral triangle about the y-axis. The moment of inertia about the y-axis is defined to be the double integral, $I = \iint\limits_{sheet} x^2\,dm$. The length of each side is L and a total mass m.



The area of the sheet is $A = (L^2\sqrt{3})/4$. Consider a small rectangular element of the sheet of area $da = dxdy$ and a distance x from the y-axis. The mass contained in this small area is $dm = \sigma\,dxdy = \frac{m}{A}dxdy = \frac{4m}{L^2\sqrt{3}}dxdy$.

Integrate first over y from $y = (1/\sqrt{3})x$ to $y = -(1/\sqrt{3})x + L$. Then integrate over x from $x = 0$ to $x = L\sqrt{3}/2$. Thus

$$I = \iint_{sheet} x^2 \; dm = \frac{4m}{L^2\sqrt{3}} \int_0^{x=L\sqrt{3}/2} \left(\int_{y=Lx/\sqrt{3}}^{y=-Lx/\sqrt{3}+L} x^2 \, dy \right) dx$$

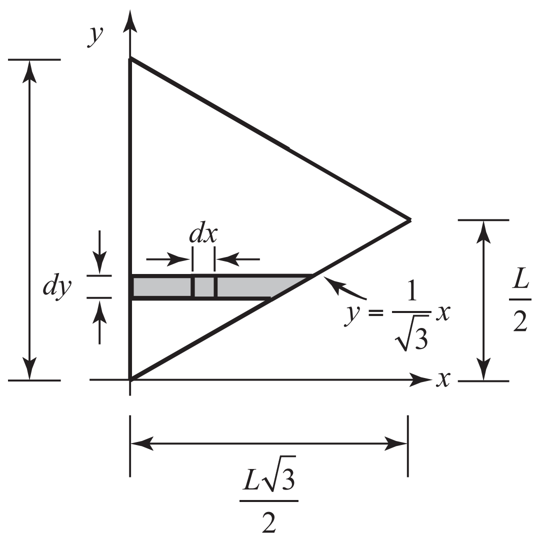
$$= \frac{4m}{L^2\sqrt{3}} \int_{x=00}^{x=L\sqrt{3}/2} x^2 \left(y\Big|_{y=x/\sqrt{3}}^{y=-x/\sqrt{3}+L} \right) dx = \frac{4m}{L^2\sqrt{3}} \int_0^{x=L\sqrt{3}/2} x^2 \left(L - \frac{2x}{\sqrt{3}} \right) dx$$

The x integral is then

$$I = \frac{4m}{L^2\sqrt{3}} \int_0^{x=L\sqrt{3}/2} x^2 \left(L - \frac{2x}{\sqrt{3}} \right) dx = \frac{4m}{L^2\sqrt{3}} \left(\frac{x^3 L}{3} - \frac{x^4}{2\sqrt{3}} \right) \Bigg|_0^{x=L\sqrt{3}/2}$$

$$\frac{4m}{L^2\sqrt{3}} \left(\frac{L^4 3\sqrt{3}}{24} - \frac{L^4 9}{32\sqrt{3}} \right) = \frac{1}{8} m L^2$$

Alternately, first integrate over x from $x = 0$ to $x = \sqrt{3}y$. Then integrate over y from $y = 0$ to $y = L/2$, and multiply by a factor of 2.

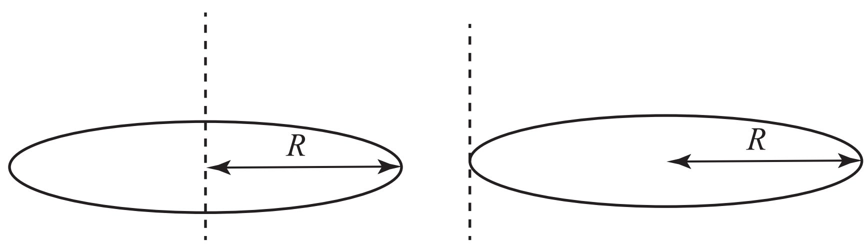


Thus

$$I = \iint\limits_{sheet} x^2\, dm = 2\frac{4m}{L^2\sqrt{3}} \int_0^{L/2} \left(\int_{x=0}^{x=\sqrt{3}y} x^2\, dx \right) dy =$$

$$\frac{8m}{L^2\sqrt{3}} \int_0^{L/2} \left(\frac{x^3}{3}\bigg|_{x=0}^{x=\sqrt{3}y} \right) dy = \frac{8m}{L^2} \int_0^{L/2} y^3\, dy = \frac{8m}{L^2}\frac{x^4}{4}\bigg|_{x=0}^{x=L/2} = \frac{1}{8} mL^2.$$

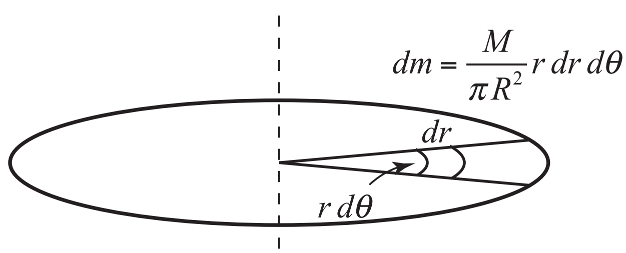
Problem: A thin uniform disc of mass M and radius R is mounted on an axis passing through the center of the disk, perpendicular to the plane of the disc. Determine the moment of inertia about two different parallel axes that are perpendicular to the plane of the disk, (i) an axis passing through the center of mass of the disc and (ii) an axis passing through a point on the rim of the disc a distance R from the center.



(i) (ii)

Answers:

i) As a starting point, consider the contribution to the moment of inertia from the mass element dm show in the figure below.



Take the point S to be the center of mass of the disc. The axis of rotation passes through the center of the disc, perpendicular to the plane of the disc. Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element $da = r\,dr\,d\theta$ is the product of arc length $r\,d\theta$ and the radial width dr. Because the disc is uniform, the mass per unit area is a constant, and is equal to $M / \pi R^2$. Therefore the mass in the infinitesimal area element a distance r from the axis of rotation, is given by $dm = (M / \pi R^2) r\,dr\,d\theta$. The mass element is a distance r from the central axis. The moment of inertia integral over the body is an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus

$$I = \int_{\text{body}} r^2\, dm = \frac{M}{\pi R^2} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr \,.$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2} \int_{r=0}^{r=R} \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) r^3\, dr = \frac{M}{\pi R^2} \int_{r=0}^{r=R} 2\pi r^3\, dr = \frac{2M}{R^2} \int_{r=0}^{r=R} r^3\, dr \,,$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2} \int_{r=0}^{r=R} r^3\, dr = \frac{2M}{R^2} \frac{r^4}{4} \Bigg|_{r=0}^{r=R} = \frac{2M}{R^2} \frac{R^4}{4} = \frac{1}{2} MR^2 \,.$$

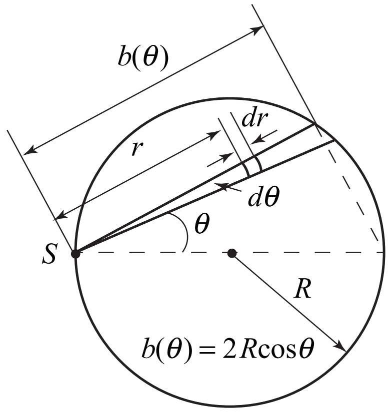
As an alternative approach, instead of taking the area element as a small patch $da = r\, dr\, d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by $da = \pi(r + dr)^2 - \pi r^2 = 2\pi r\, dr + \pi(dr)^2$. In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This is equivalent to first integrating the $d\theta$ variable

$$da = r\, dr \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) = 2\pi r\, dr \,.$$

Then the mass element is $dm = \sigma da = \dfrac{M}{\pi R^2} 2\pi r\, dr$. The moment of inertia integral is then just an integral in the variable r,

$$I = \int_{\text{body}} r^2\, dm = \frac{2\pi M}{\pi R^2} \int_{r=0}^{r=R} r^3\, dr = \frac{1}{2} MR^2 \,.$$

(ii) In order to find the moment of inertia about the point S on the rim, choose coordinates shown in the figure below.



The mass element is still $dm = \sigma\, r\, dr\, d\theta = \dfrac{M}{\pi R^2} r\, dr\, d\theta$, but the variables r and θ are not the same as in the previous part. The integral for the moment of inertia becomes

$$I = \int_{\text{body}} r^2\, dm = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=b(\theta)} r^3\, dr\, d\theta \,.$$

First integrate in the radial direction, yielding

$$I = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{b^4}{4}\, d\theta \,.$$

However, the variable b is a non-constant function of θ. The triangle formed by the diameter of the circle and two chords is a right triangle; thus $b(\theta) = 2R\cos\theta$,

and the integral becomes

$$I = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{(2R\cos\theta)^4}{4}\, d\theta \,.$$

Using the fact that the integral (try to show this)

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{8} \,,$$

the moment of inertia about the point S is

$$I = \frac{4\,MR^4}{\pi R^2}\int_{\theta=-\pi/2}^{\theta=\pi/2}\cos^4\theta\,d\theta = \frac{4\,MR^4}{\pi R^2}\frac{3\pi}{8} = \frac{3}{2}\,MR^2\,.$$

Problems for Indefinite and Definite Integrals

A.1.1 Antiderivatives and Indefinite Integrals 2

Problem A.1.1: $f(x) = \sin x \cos x$ 2

Problem A.1.2: $\int \sin x \, dx$? 2

Problem A.1.3: $\int x^n \, dx$, $n \neq -1$ 2

A.2 Change of Variable Formula 2

Problem A.2.1 $\int \sin x \cos x \, dx$ 2

Problem A.2.2 $\int \sin(x/2)\cos(x/2) \, dx$. 2

Problem A.2.2 $\int \frac{x \, dx}{x^2+4}$. 3

Problem A.2.3 $\int x\sqrt{1+x^2} \, dx$. 3

Problem A.3.1 $\int (\cos 5x + b) \, dx$ 3

Problem A.3.2 $\int x \ln x^2 \, dx$ 3

I.2.6 Second Fundamental Theorem of Calculus: 4

Problem I.2.1: $y = 2x^2$ 4

Problem I.2.2: $y = 4x^3$ 5

Problem I.2.3: $y = x^3 + 2$ 5

Problem I.2.4: $f(x) = 1 - e^{-x}$ 5

Problem I.2.5: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ 6

I.3 Some Applications of Integration 6

Problem I.2.6: Position, Velocity, and Acceleration 6

Problem I.2.7: Position and Velocity 9

I.3 The Method of Slices: 9

Problem I.3.1 Volume of a Sphere 9

I.4 Integral Table 10

A.1.1 Antiderivatives and Indefinite Integrals

Problem A.1.1: $f(x) = \sin x \cos x$

Verify that when $f(x) = \sin x \cos x$, an antiderivative is $F(x) = \frac{1}{2}\sin^2 x$.

Answer:

$$\frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right) = \frac{1}{2}(2\sin x)\left(\frac{d}{dx}\sin x\right) = \sin x \cos x.$$

Problem A.1.2: $\int \sin x \, dx$?

What is the integral $\int \sin x \, dx$?

Answer: Because $\frac{d}{dx}\cos x = -\sin x$, $\int \sin x \, dx = -\cos x + c$.

Problem A.1.3: $\int x^n \, dx$, $n \neq -1$

What is the indefinite integral $\int x^n \, dx$, $n \neq -1$, equal to?

Answer:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{dx^{n+1}}{dx} = \frac{1}{n+1}(n+1)\,x^n = x^n$$

Hence,

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1.$$

A.2 Change of Variable Formula

Problem A.2.1 $\int \sin x \cos x \, dx$

Evaluate $\int \sin x \cos x \, dx$.

Answer: Let $u = \sin x$. Then $\sin x \cos x = u \cos x$ and $\frac{du}{dx} = \cos x$. Hence $dx = \frac{1}{\cos x} du$ and the integral becomes

$$\int \sin x \cos x \, dx = \int u \cos x \frac{1}{\cos x} \, du = \int u \, du = \frac{1}{2}u^2 + c = \frac{1}{2}\sin^2 x + c.$$

Problem A.2.2 $\int \sin(x/2)\cos(x/2) \, dx.$

Evaluate $\int \sin(x/2)\cos(x/2) \, dx.$

Answer: Let $u = x/2$, then $dx = 2du$ and $\int \sin(x/2)\cos(x/2)\,dx = 2\int \sin u \cos u\,du$.
Then using the result from the previous problem,

$$\int \sin(x/2)\cos(x/2)\,dx = 2\int \sin u \cos u\,du = \sin^2 u + 2c = \sin^2(x/2) + 2c.$$

Check this result: By the chain rule

$$\frac{d}{dx}\left(\sin^2(x/2) + 2c\right) = 2\left(\sin(x/2)\cos(x/2)\right)\left(\frac{1}{2}\right) = \sin(x/2)\cos(x/2).$$

Problem A.2.2 $\int \frac{x\,dx}{x^2+4}$.

Evaluate $\int \frac{x\,dx}{x^2+4}$.

Answer: Suppose we let $u^2 = x^2 + 4$. Then $2udu = 2xdx$, and

$$\int \frac{x\,dx}{x^2+4} = \int \frac{u\,du}{u^2} = \int \frac{du}{u} = \ln|u| + c = \ln\left|\sqrt{x^2+4}\right| + c.$$

Problem A.2.3 $\int x\sqrt{1+x^2}\,dx$.

Evaluate the integral: $\int x\sqrt{1+x^2}\,dx$.

Answer: Taking $u^2 = 1 + x^2$, then $2udu = 2xdx$ and

$$\int x\sqrt{1+x^2}\,dx = \int u(u\,du) = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}(1+x^2)^{3/2} + c.$$

Problem A.2.4 $\int (\cos 5x + b)\,dx$

Evaluate $\int (\cos 5x + b)\,dx$, where b is a constant.

Answer: $\int (\cos 5x + b)\,dx = \int \cos 5x\,dx + \int b\,dx = \frac{1}{5}\int \cos 5x\,d(5x) + \int b\,dx = \frac{1}{5}\sin 5x + bx + c.$

Problem A.2.5 $\int x \ln x^2\,dx$

Evaluate $\int x \ln x^2\,dx$

Answer: let $u = x^2$, $du = 2xdx$. Then

$$\int x \ln x^2\,dx = \frac{1}{2}\int \ln u\,du = \frac{1}{2}(u \ln u - u + c).$$

Therefore,

$$\int x \ln x^2\,dx = \frac{1}{2}(x^2 \ln x^2 - x^2 + c).$$

A.3.1 Problem: $\int x\cos x\,dx$.

Find $\int x\cos x\,dx$. Let $u = x$ and $dv = \cos x\,dx$, and integrate by parts. Thus $du = dx$, $v = \sin x$, and the integral is

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c.$$

A.4.1 Problem: $\int \frac{3x-4}{(x^2-2x-3)}dx$

Use the method of partial fractions to evaluate the integral $\int \frac{3x-4}{(x^2-2x-3)}dx$.

Answer: Because $x^2 - 2x - 3 = (x+1)(x-3)$, write

$$\frac{3x-4}{(x^2-2x-3)} = \frac{3x-4}{(x+1)(x-3)} = \frac{a}{x+1} + \frac{b}{x-3} = \frac{(-3a+b)+(a+b)x}{(x+1)(x-3)}.$$

Next, compare coefficients: $-3a + b = -4$ and $a + b = 3$. Solve these two equations and with the result that $a = 7/4, b = 5/4$. Therefore

$$\frac{3x-4}{(x^2-2x-3)} = \frac{7}{4(x+1)} + \frac{5}{4(x-3)}.$$

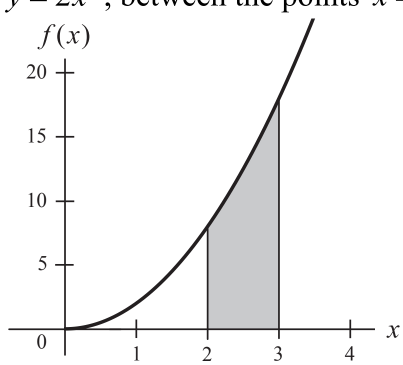
Hence the integral is

$$\int \frac{3x-4}{(x^2-2x-3)}dx = \frac{7}{4}\int \frac{dx}{x+1} + \frac{5}{4}\int \frac{dx}{x-3} = \frac{7}{4}\ln|1+x| + \frac{5}{4}\ln|x-3|) + c.$$

I.2.6 Second Fundamental Theorem of Calculus:

Problem I.2.1: $y = 2x^2$

What is the area under the curve $y = 2x^2$, between the points $x = 2$ and $x = 3$?



Answer: An antiderivative for the function $f(x) = 2x^2$ is $F(x) = (2/3)x^3$. Therefore

$$A = \int_2^3 2x^2\,dx = F(3) - F(2) = (2/3)x^3\Big|_2^3 = \frac{2}{3}(3^3 - 2^3) = \frac{38}{3}.$$

Problem I.2.2: $y = 4x^3$

Find the area under the curve $y = 4x^3$ between $x = -2$ and $x = 1$.

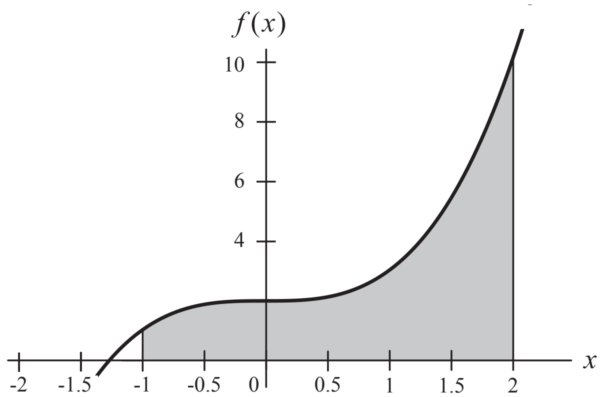
Answer: An antiderivative for the function $f(x) = 4x^3$ is $F(x) = x^4$. Therefore

$$A = \int_{-2}^{1} 4x^3 \, dx = x^4 \Big|_{-2}^{1} = 1^4 - (-2)^4 = 1 - 16 = -15$$

Note that this area is negative.

Problem I.2.3: $y = x^3 + 2$

The graph shows a plot of $y = x^3 + 2$. Find the area between the curve and the x-axis from $x = -1$ and $x = 2$.

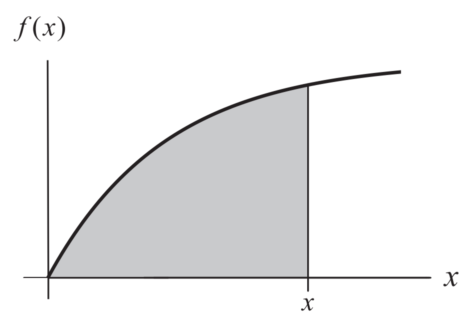


Answer: one antiderivative is $F(x) = (x^4 / 4) + 2x$. Therefore

$$A = F(2) - F(-1) = F(x)\big|_{-1}^{2} = \left(\frac{1}{4}x^4 + 2x\right)\Bigg|_{-1}^{2} = \left(\frac{16}{4} + 4\right) - \left(\frac{1}{4} - 2\right) = \frac{39}{4}.$$

Problem I.2.4: $f(x) = 1 - e^{-x}$

A graph of the function $f(x) = 1 - e^{-x}$ is shown in the figure below. What is the shaded area under the curve between the origin and x?



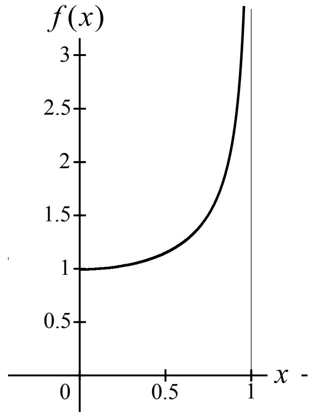
Answer:

$$A(x) = \int_0^x f(x) \, dx = \int_0^x (1 - e^{-x}) \, dx = \int_0^x dx - \int_0^x e^{-x} \, dx$$

$$= [x - (-e^{-x})]\Big|_0^x = [x + e^{-x}]\Big|_0^x = x + e^{-x} - 1,$$

Problem I.2.5: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Find $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. A graph of $f(x)=\frac{1}{\sqrt{1-x^2}}$ is shown below. Although the function is discontinuous at x = 1, the area under the curve is well defined.



Answer: Note that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$. Therefore,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\Big|_0^1 = \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Because $\sin\frac{\pi}{2} = 1$, hence $\sin^{-1}1 = \frac{\pi}{2}$. Similarly, $\sin^{-1}$ 0 = 0.

I.3 Some Applications of Integration

Problem I.2.6: Position, Velocity, and Acceleration

At $t=0$, a sports car starting at rest at $x=0$ accelerates with an acceleration given by

$$a(t) = \begin{cases} \alpha t - \beta t^3, & 0 \le t \le (\alpha/\beta)^{1/2} \\ 0, & (\alpha/\beta)^{1/2} < t \le t_f \end{cases}.$$

The constants $\alpha, \beta > 0$.

(i) Find expressions for the velocity and position vectors of the sports car as a function of time for $t \ge 0$.

(ii) Sketch graphs of the x-component of the position, velocity and acceleration as a function of time for $t > 0$. Use for the parameters $\alpha = 10\ \text{m}\cdot\text{s}^{-3}$ and $\beta = 0.9\ \text{m}\cdot\text{s}^{-5}$.

Solution:

(i) For the interval $0 < t < (\alpha/\beta)^{1/2}$, the change in the velocity as a function of time is given by

$$v(t)-v_0=\int_{t'=0}^{t'=t} a(t')\,dt'=\int_{t'=0}^{t'=t}(\alpha t'-\beta t'^3)\,dt'=\frac{\alpha}{2}t^2-\frac{\beta}{4}t^4.$$

Because the sports car starts from rest at $t=0$, $v_0=0$, the velocity as a function of time is given by

$$v(t)=\frac{\alpha}{2}t^2-\frac{\beta}{4}t^4,\quad 0\le t\le(\alpha/\beta)^{1/2}.$$

At $t=(\alpha/\beta)^{1/2}$, the velocity is

$$v(t=\alpha/\beta)=\frac{\alpha^2/\beta}{4}.$$

Because the acceleration is zero afterwards the car maintains this constant speed. Hence

$$v(t)=\begin{cases}\dfrac{\alpha}{2}t^2-\dfrac{\beta}{4}t^4, & 0\le t\le(\alpha/\beta)^{1/2}\\ \dfrac{\alpha^2/\beta}{4}, & (\alpha/\beta)^{1/2}\le t\le t_1\end{cases}.$$

For the interval $0\le t\le(\alpha/\beta)^{1/2}$, the displacement is

$$x(t)-x_0=\int_{t'=0}^{t'=t}\left(\frac{\alpha}{2}t'^2-\frac{\beta}{4}t'^4\right)dt'=\frac{\alpha}{6}t^3-\frac{\beta}{20}t^5,\quad 0\le t\le(\alpha/\beta)^{1/2}.$$

The initial position of the sports car is given by $x_0=0.$ Thus the position as a function of time is given by

$$x(t)=\frac{\alpha}{6}t^3-\frac{\beta}{20}t^5,\quad 0\le t\le(\alpha/\beta)^{1/2}.$$

At $t=(\alpha/\beta)^{1/2}$, the position is

$$x(t=\alpha/\beta)=\frac{7\alpha^{5/2}\beta^{-3/2}}{60}.$$

For the interval $(\alpha/\beta)^{1/2}\le t\le t_1$, the position is given by

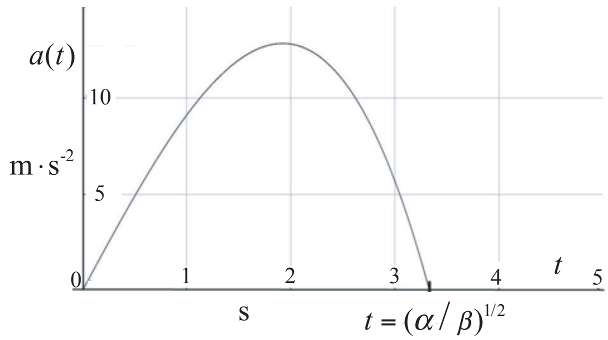
$$x(t)=x(t=\alpha/\beta)+\int_{t'=(\alpha/\beta)^{1/2}}^{t'=t} v(t')\,dt'=\frac{7\alpha^{5/2}\beta^{-3/2}}{60}+\int_{t'=(\alpha/\beta)^{1/2}}^{t'=t}\frac{\alpha^2/\beta}{4}dt'$$

$$x(t)\ =\frac{7\alpha^{5/2}\beta^{-3/2}}{60}+\frac{\alpha^2/\beta}{4}(t-(\alpha/\beta)^{1/2}),\qquad (\alpha/\beta)^{1/2}\le t\le t_1.$$

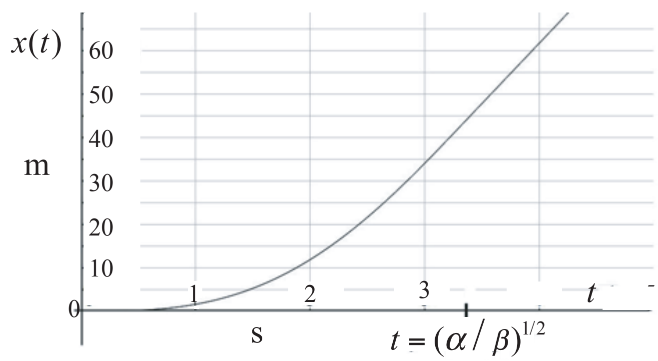
In summary

$$x(t)=\begin{cases}\dfrac{\alpha t^3}{6}-\dfrac{\beta t^5}{20}, & 0\le t\le(\alpha/\beta)^{1/2}\\ \dfrac{7\alpha^{5/2}\beta^{-3/2}}{60}+\dfrac{\alpha^2/\beta}{4}(t-(\alpha/\beta)^{1/2}), & 0\le t\le(\alpha/\beta)^{1/2}t\ge(\alpha/\beta)^{1/2}\end{cases}.$$

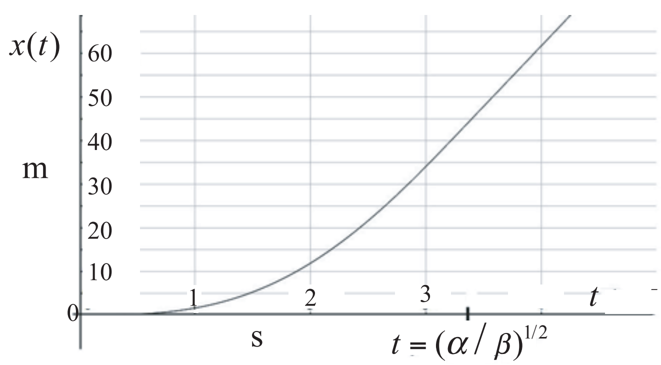
(ii) The following graphs use the parameters $\alpha = 10\ \mathrm{m\cdot s^{-3}}$ and $\beta = 0.9\ \mathrm{m\cdot s^{-5}}$. Then the car accelerates for a time interval $(\alpha/\beta)^{1/2} = 3.3\ \mathrm{s}$ and when immediately finished accelerating reaches a speed $v(t = (\alpha/\beta)^{1/2}) = \dfrac{\alpha^2/\beta}{4} = 27.8\ \mathrm{m\cdot s^{-1}}$. At that instant the car has traveled a distance $x(t = (\alpha/\beta)^{1/2}) = \dfrac{7\alpha^{5/2}\beta^{-3/2}}{60} = 43.2\ \mathrm{m}$. The graph of the acceleration as a function of time for $t > 0$ is shown below.



The graph of the velocity as a function of time for $t \geq 0$ is shown below.



The graph of the position as a function of time for $t \geq 0$ is shown below.



Problem I.2.7: Position and Velocity

A particle starts from the origin at $t = 0$ with a velocity $v(t) = v_0 / (b + t)$, where v_0 and b are constants. How far does it travel as $t \to \infty$?

Answer:

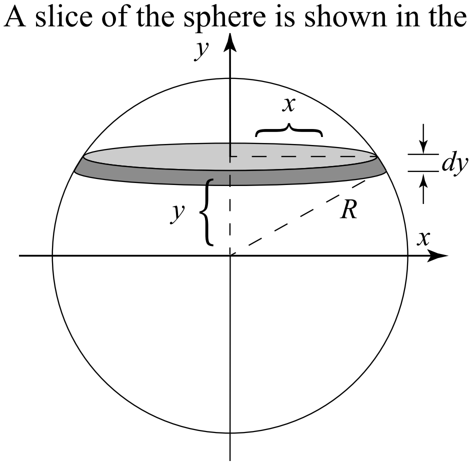
$$x(t) - 0 = \int_0^t v_0 \frac{dt}{b+t} = v_0 \ln(b+t)\Big|_0^t = v_0[\ln(b+t) - \ln b] = v_0 \ln\left(1 + \frac{t}{b}\right).$$

Because $\ln(1 + (t/b)) \to \infty$ as $t \to \infty$, we see that $x(t) \to \infty$ as $t \to \infty$. In this case, the particle is always moving fast enough so that its motion is unlimited. Or, alternatively, the area under the curve $v(t) = v_0 / (b + t)$ increases without limit as $t \to \infty$.

I.3 The Method of Slices*:*

Problem I.3.1 Volume of a Sphere

Find the volume of a Sphere. A slice of the sphere is shown in the figure below.



Solution:

In order to calculate the volume of the sphere, consider a disk of thickness dy and radius x, located at a distance y from the center of the sphere as shown in the drawing. The differential volume dV of the disk between y and $y + dy$ is $dV = \pi x^2 dy$. By the Pythagorean theorem $x = \sqrt{R^2 - y^2}$, therefore $dV = \pi(R^2 - y^2)dy$. The limits of the variable y are $y = -R$ to $y = R$. The volume integral is

$$V = \int_{-R}^{R} \pi(R^2 - y^2)\,dy = 2\int_{0}^{R} \pi(R^2 - y^2)\,dy = (2\pi R^2 y - 2\pi y^3/3)\Big|_0^R.$$

$$V = 2\pi R^3 - 2\pi R^3/3 = 4\pi R^3/3.$$

I.4 Integral Table

1. $\int a\,dx = ax$
2. $\int af(x)\,dx = a\int f(x)\,dx$
3. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$
4. $\int x^n\,dx, = \dfrac{x^{n+1}}{n+1} \quad n \neq -1$
5. $\int \dfrac{dx}{x} = \ln|x|$
6. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\ln\left|a+bx\right|$
7. $\int e^x\,dx = e^x$
8. $\int e^{ax}dx = \dfrac{e^{ax}}{a}$
9. $\int b^{ax}\,dx = \dfrac{b^{ax}}{a\ln b}$
10. $\int \ln|x|\,dx = x\ln|x| - x$
11. $\int \sin x\,dx = -\cos x$
12. $\int \cos x\,dx = \sin x$
13. $\int \tan x\,dx = -\ln|\cos x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sec x\,dx = \ln|\sec x + \tan x|$
16. $\int \sin x \cos x\,dx = \dfrac{1}{2}\sin^2 x$
17. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
18. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$
19. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x - \sqrt{x^2 \pm a^2}\right|$
20. $\int w(u)\,dx = \int w(u)\dfrac{dx}{du}\,du$
21. $\int u\,dv = uv - \int v\,du$