MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SELF-PACED STUDY GUIDE in GEOMETRY and ANALYTIC GEOMETRY

Undergraduate Academic Affairs Office, August, 1992

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How to Use the Self-Paced Review Module

The *Self-Paced-Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and Analytic Geometry; Trigonometry; and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic. A numbering system facilitates easy tracking of subject material. For example, in the topic area **Algebra,** the subtopic **Linear Equations** are numbered by **A.III.** Problems and the self-evaluations are categorized according to this numbering system.

When using the *Self-Paced Review,* it is important to differentiate between concept learning and problem solving. The review modules are oriented towards refreshing concept understanding while the problems and self-tests are designed to develop problem solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules which should be solved while working through the module.The problems should be attempted without looking out the solutions. If a problem can not be solved after at least two honest efforts, then consult the solutions. The tests should be taken only when both an understanding of the material and a problem solving ability has been achieved. The self-evaluation is a useful tool to evaluate one's mastery of the material. The previous Diagnostic Exams should provide the finishing touch.

The review modules were written by Professor A.P. French (Physics Department) and Adelaida Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department).

Produced by the Undergraduate Academic Affairs Office, August, 1992

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

GEOMETRY
&
ANALYTIC GEOMETRY
REVIEW MODULES

prepared by
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GEOMETRY REVIEW MODULE

Introduction

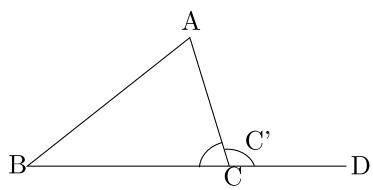
This is a short module. It is not intended to be a full review of the theorems of Euclidean geometry. Its purpose if just to remind you of some of the chief results that will be of direct use to you in your work in science and in other areas of mathematics. In particular, the plane geometry in this module leads quickly into trigonometry, which is the subject of a separate module.

I. TRIANGLES

1) You will be familiar with the fact that, in Euclidean geometry, the angles of a triangle add up to 180°. We shall take this property of triangles as given. It is worth remembering, though, that this is not a general result. It holds only for plane triangles. The angles of a triangle drawn on the surface of a sphere add up to *more* than 180°. The reason why that isn't important to us for most purposes is that the triangles we deal with usually have very small linear dimensions compared to the radius of the earth, so, for example, most surveying on the earth's surface can use Euclidean geometry with no significant error. (And then there are also other "non-Euclidean geometries" that obey different rules, which we'll certainly not deal with.) But here, this 180° property is a starting point: In a triangle ABC,

$$\angle A + \angle B + \angle C = 180°.$$

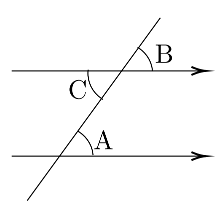
2) Another familiar property is that, if one side of a triangle is extended, as in the diagram here, the exterior angle C' is equal to the sum of the interior angles A and B. (A reminder of the proof: Since BCD is a straight line, $\angle C + \angle C' = 180°$, so $\angle C' = 180° - \angle C$. Then use the result above.)



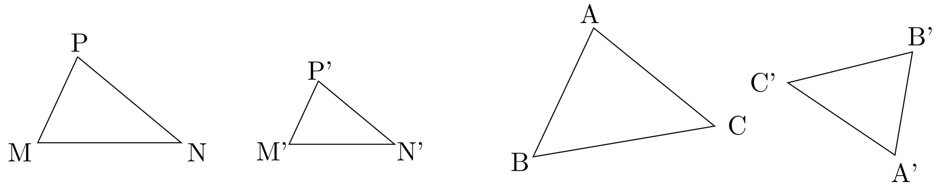
3) Angles & Similar Triangles.

Some basic results of Euclidean geometry are very useful in analyzing geometrical situations in general and triangles in particular. Where one straight line cuts across two parallel straight lines:

$\angle A = \angle B$ (because lines are parallel);
$\angle B = \angle C$ (opposite angles are equal);
$\angle A = \angle C$ (alternate interior angles are equal).



These find many applications in recognizing and relating *similar* triangles. It is easy to recognize similar triangles when they are placed side by side in the same orientation, like those below on the left. It make not be so easy if the triangles are in quite different orientations like those on the right.

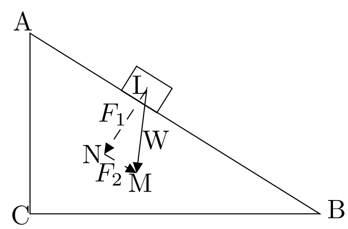


A common type of situation is the following mechanics problem.

Example:

A block sits on an inclined plane. It is acted on by the vertical force of gravity. In the solution of the problem, you want to *resolve* this gravitational force (the weight W) into components parallel and perpendicular to the plane. The triangle LMN representing the analysis of the downward force W into two mutually perpendicular parts, F_1 and F_2, is similar to the triangle ABC composed of the inclined plane and its horizontal and vertical dimensions. One can therefore put:

$$F_1 : F_2 : W = BC : AC : AB.$$



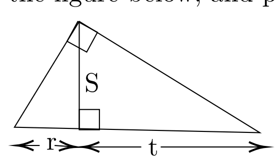
(This is more than an exercise in pure geometry, because we are comparing a geometrical triangle to a triangle of forces. But it is important and useful to know that this can be done.)

Exercise I.1.

Some artists used to use a *camera obscura* (literally just a "dark room") with a small hole in one wall to form and image of a distant scene on a screen inside he room. (Then all they had to do was trace over the image!) Imagine a situation where the screen was 2 meters from the hole, forming an image of a building 10 meters high and 50 meters away. Draw a sketch showing rays of light passing to the screen from the top and bottom of the building, and calculate the height of the image.

Exercise I.2.

Find three similar triangles in the figure below, and prove that $s^2 = rt$.



Exercise I.3.

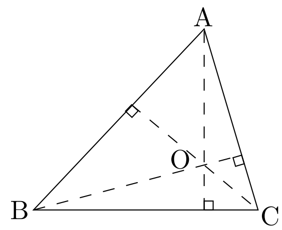
In the example of an inclined plane discussed above, find the forces F_1 and F_2 in terms of W if $\angle A = 30°$.

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that needs to be corrected, please write to us.

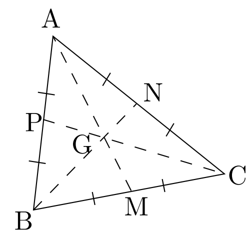
4) Orthocenters and Centroids.

In any triangle:

(a) The three perpendicular lines from the angles to the opposite sides (the altitudes) intersect at a single point. This is the orthocenter.



(b) The three lines from the angles to the mid-points of the opposite side (the medians) intersect at a single point. This is the centroid.



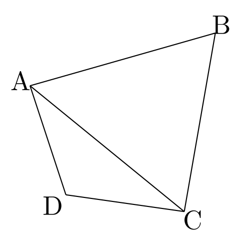
The centroid is of particular significance physically. If a triangle is made from a sheet of material of uniform thickness, then the centroid is its balance point (its center of gravity), G. This can be understood by drawing lines parallel to each of the sides in turn. Suppose the figure is suspended from the corner A. Imagine the triangle divided up into strips parallel to the opposite side BC. Then the center of each strip lies on the line AM, and the triangle will be in balance, when hung from A, with AM vertical. Similarly for the other two medians. So, if suspended from the centroid, the triangle has no tendency to take up any other particular orientation.

II. SOME OTHER PLANE FIGURES

1) Quadrilaterals.

For any quadrilateral, the sum of the angles is 360°:

$$A + B + C + D = 360^\circ$$



(Think of it as two triangles, ABC and ACD.)

Some special quadrilaterals:

(a) Trapezoid, with AB parallel to CD. Its width at half altitude is equal to $(AB + CD)/2$

(b) Parallelogram, with AB parallel to CD, and AD parallel to BC.

Other properties:

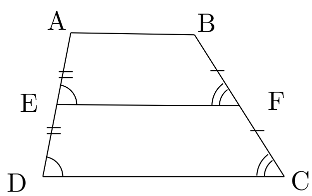
$$\angle A = \angle C; \ \angle B = \angle D.$$

$$AB = DC, \ AD = BC.$$

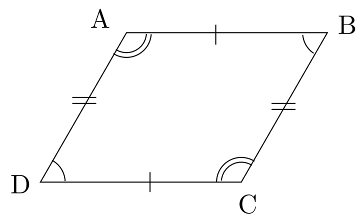
(c) Rhombus, a parallelogram with all sides equal:

$$AB = BC = CD = DA.$$

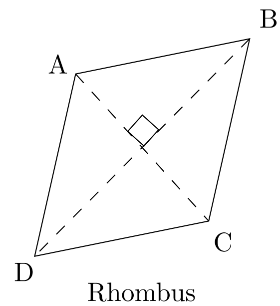
Its diagonals are perpendicular.



Trapezoid



Parallelogram



Rhombus

2) Other Polygons.

The literal meaning of "polygon" is "many-cornered," but we usually think in terms of the number of sides. Since the number of angles equals the number of sides, n, you can't go wrong. It's useful to know the names of a few polygons beyond the quadrilateral:

$n = 5$:	Pentagon;	$n = 8$:	Octagon;
$n = 6$:	Hexagon	$n = 10$:	Decagon;
$n = 7$:	Heptagon;	$n = 12$:	Dodecagon.

(The ones omitted are seldom met.)

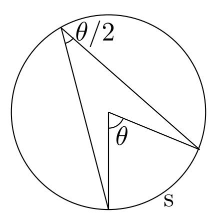
There is a general formula for the sum of all the angles of a polygon with n sides or corners:

$$\text{Angle Sum } (S) = (n - 2) \times 180°$$

III. THE ALL-MIGHTY CIRCLE

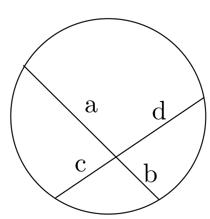
Some properties worth knowing about – even if you don't memorize them:

1) Given any arc of the circle, the angle it subtends at the center is twice the angle it subtends at the circumference. (Special case: A semicircle subtends 180° at the center and a right angle at the circumference)



2) Intersecting chords: The products of their separate parts are equal:

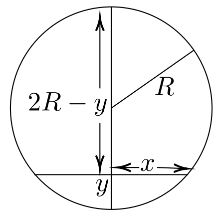
$$a \times b = c \times d.$$



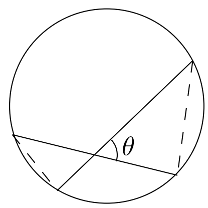
(Special case: A diameter and a chord perpendicular to it: With the figure labeled as shown,

$$y(2R - y) = x^2.$$

If $y \ll R$, then $y = x^2/2R$. This means that a small part of the circle has almost the same shape as a parabola. That is important in many mathematical and physical approximations. Admittedly, this is analytic geometry, not plane geometry!)



3) The figures formed by any two intersecting chords, not necessarily passing through the center of the circle, are geometrically similar.



IV. LENGTHS, AREAS & VOLUMES

There are many useful formulas for the perimeters, areas and volumes of geometrical figures. Some of them you should definitely memorize. But before you look at the tabulation below, consider the following statements concerning the dimensions of such quantities:

Any perimeter must have the dimension of length. That is, it must be expressed in terms of length to the first power only.

Any area must have the dimension of $(\text{length})^2$.

Any volume must have the dimension of $(\text{length})^3$.

For example, if you are trying to remember the formula for the area of a circle, consider that it couldn't possible be $2\pi r$. Whatever else there may be, it's got to have r to the 2cd power. Guessing that it's $2\pi r^2$ (as against the correct formula πr^2) is an error you shouldn't make – but it's less serious than using a formula that could only represent a length.

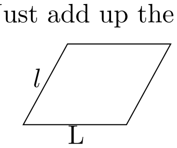
Likewise, the circumference of a circle could not possibly be given by πr^2 or $2\pi r^2$.

FORMULAS

Make a special effort to memorize the formulas marked with an asterisk (*). People very often get them wrong!

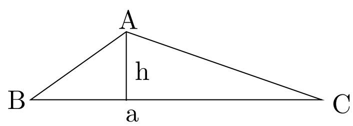
1) Perimeters

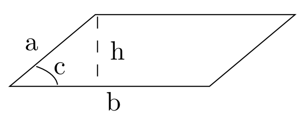
Rectangle or Parallelogram:	$2(L + l)$.
*Circle:	$2\pi r$.
Any other straight-sided figure:	Just add up the lengths.

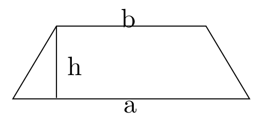


2) <u>Areas</u>

Triangle:	$\frac{1}{2}bh = \frac{1}{2}ab\sin C =$ etc. (cyclically).
Parallelogram:	$bh = ab\sin C$.
Trapezoid:	$\frac{1}{2}h(a+b)$.
*Circle:	πr^2.
*Cylinder:	$2\pi Rh$ (sides) $+2\pi R^2$ (ends).
Prism:	$P_bh + 2A_b$. ($P_b =$ perimeter of base; $A_b =$ area of base; $h =$ height.)
*Sphere	$4\pi R^2$.

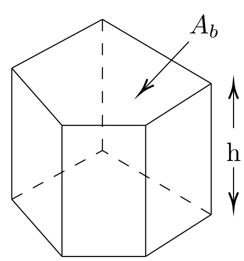






3) <u>Volumes</u>

Rectangular Parallelepiped:	$abc = A_bh$.
Pyramid or Cone:	$4\frac{1}{3}A_bh = \frac{1}{3}\pi R^2h$ (for circular cone).
*Sphere:	$\frac{4\pi}{3}R^3$.



<u>Exercise IV.1</u>.

Prove that the area of a flat circular ring or washer, with inner radius r_1 and outer radius r_2, is equal to $2\pi r_{ave.}\Delta r$, where $r_{ave.}$ is the average radius $(r_1+r_2)/2$, and Δr is the width (r_2-r_1).

<u>Ecercise IV.2</u>.

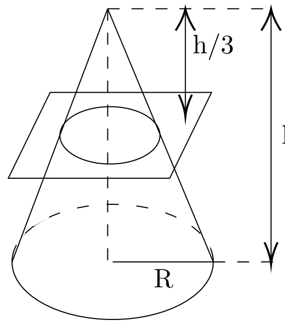
Given that a sphere has area 64π, what is its volume?

<u>Exercise IV.3</u>.

A cube of edge-length L has a sphere just fitting inside it (i.e., the diameter of the sphere is equal to L). Calculate the ratios of (a) the surface areas, and (b) the volumes of these two figures.

Exercise IV.4.

A plane, parallel to the base of a circular cone of height h and of radius R at the base, cuts across it at a distance of $h/3$ from the top. Calculate the ratio of the volumes of the small cone and the original cone.

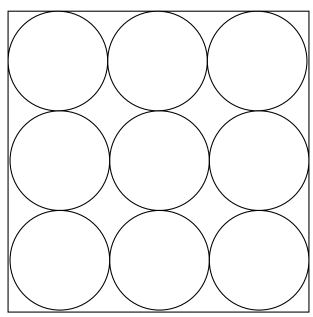


Ecercise IV.5.

A cube of side a is packed full with small spheres of diameter a/n (n = positive integer) as shown in the diagram.

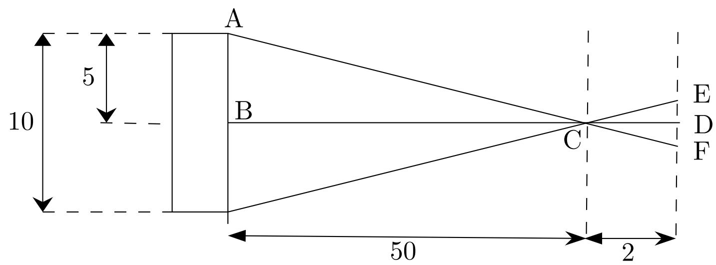
Consider the total volume of the spheres. Do you think it increases or decreases as n gets bigger?

Given two equal-size and equal-price boxes of mothballs, should you buy the one with the larger mothballs or the one with the smaller mothballs?



ANSWERS TO EXERCISES

Exercise I.1.



(not to scale)

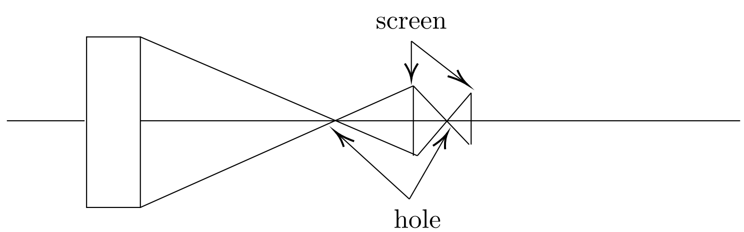
Find similar triangles (see diagram above):

$$\triangle ABC \sim \triangle CDF \rightarrow \frac{AB}{BC} = \frac{DF}{CD}$$

or, using the distances given,

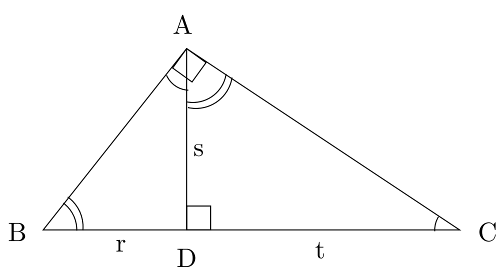
$$\frac{5}{50} = \frac{DF}{2} \rightarrow DF = \frac{1}{5}$$

Then the height FE of the image is $FE = 2 \times DF = \frac{2}{5}m = 0.4m$. Note: If the artists used only one hole in the wall, they would trace an inverted image on the screen. (The rays passing through the hole from the top and the bottom of the building end up on the screen at points F and E, respectively).



In order to get the image right side up we would need two holes and two screens (see diagram above).

Exercise I.2.



(1) $$\triangle ABC \sim \triangle BDA \rightarrow \frac{AB}{AC} = \frac{BD}{AD}$$

(2) $$\triangle ABC \sim \triangle ADC \rightarrow \frac{AB}{AC} = \frac{AD}{DC}$$

Using both (1) and (2), we have:

$$\frac{BD}{AD} = \frac{AD}{DC}, \text{ or } \frac{r}{s} = \frac{s}{t} \to s^2 = rt$$

[Solution without using similar triangles, but using Pythagoras's theorem instead:

(3) $$AB^2 = BD^2 + AD^2 \to AB^2 = r^2 + s^2$$

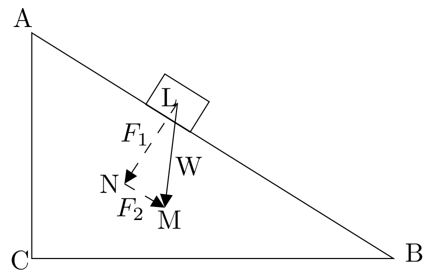
(4) $$AC^2 = AD^2 + DC^2 \to AC^2 = s^2 + t^2$$

But $BC^2 = AB^2 + AC^2$, and $BC = r + t$. By substitution (see (3) and (4)) we obtain:

$BC^2 = r^2 + s^2 + s^2 + t^2$
$(r+t)^2 = r^2 + 2s^2 + t^2$
$r^2 + 2rt + t^2 = r^2 + 2s^2 + t^2$

or $2rt = 2s^2$, which gives us the answer, $rt = s^2$.]

Exercise I.3.



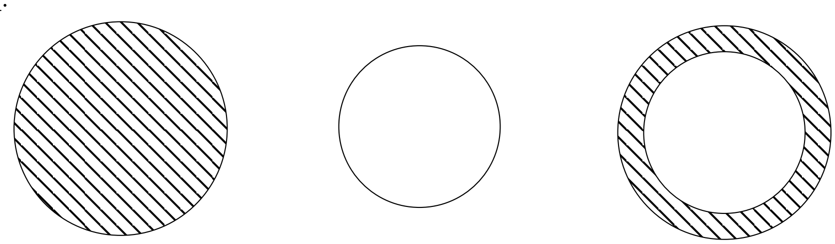
$$\triangle ABC \sim \triangle LMN \to \frac{AB}{LM} = \frac{AC}{MN} = \frac{BC}{LN}, \text{ or } \frac{AB}{W} = \frac{AC}{F_2} = \frac{BC}{F_1},$$

thus $\frac{BC}{AB} = \frac{F_1}{W}$ and $\frac{AC}{AB} = \frac{F_2}{W}$ (1)
$\angle A = 30°$. Using the known relationships in a 30°-60°-90° triangle, (mentioned in the Trigonometry Module), $\frac{BC}{AB} = \frac{1}{2}$, $\frac{AC}{AB} = \frac{\sqrt{3}}{2}$. Using the two equalities above and (1), we obtain:

$$\frac{BC}{AB} = \frac{1}{2} = \frac{F_1}{W} \to F_1 = \frac{W}{2}$$

$$\frac{AC}{AB} = \frac{\sqrt{3}}{2} = \frac{F_2}{W} \to F_2 = \frac{W\sqrt{3}}{2}.$$

Exercise IV.1.

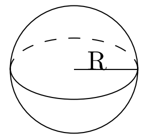


$A_{outer} = \pi r_2^2$ $\qquad$ $A_{inner} = \pi r_1^2$

The washer is what is left of the outer circle, when the inner one is taken away. Its area will be: $A_W = \pi r_2^2 - \pi r_1^2$. Using a few algebraic manipulations,

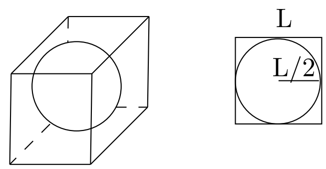
$$A_w = \pi(r_2^2 - r_1^2) = \pi(r_2 + r_1)(r_2 - r_1) = 2\pi\frac{(r_2 + r_1)}{2}(r_2 - r_1) = 2\pi r_{ave.}\Delta r.$$

Exercise IV.2.



$A_{sphere} = 4\pi R^2 = 64\pi$
$R^2 = 16,\ R = 4$
$V_{sphere} = \frac{4\pi R^3}{3} = \frac{4\pi\times 64}{3} = \frac{256\pi}{3}$.

Exercise IV.3.



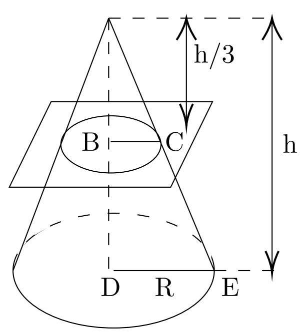
The radius of the sphere is $R = \frac{L}{2}$.
a)

$$A_{cube} = 6L^2$$
$$A_{sphere} = 4\pi R^2 = 4\pi\left(\frac{L}{2}\right)^2 = \pi L^2$$
$$\frac{A_{cube}}{A_{sphere}} = \frac{6L^2}{\pi L^2} = \frac{6}{\pi} \approx 1.91$$

b)

$$V_{cube} = L^3$$
$$V_{sphere} = \frac{4\pi R^3}{3} = \frac{4\pi}{3}\times\left(\frac{L}{2}\right)^3 = \frac{\pi L^3}{6}$$
$$\frac{V_{cube}}{V_{sphere}} = L^3/\left(\frac{\pi L^3}{6}\right) = \frac{6}{\pi} \approx 1.91$$

Exercise IV.4.



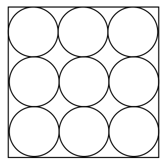
$$\triangle ABC \sim \triangle ADE \to \frac{AB}{AD} = \frac{BC}{DE},\ \text{or}\ \frac{h/3}{h} = \frac{BC}{R} \to BC = \frac{R}{3}.$$
$$V_{sm.\ cone} = \frac{1}{3}\pi(BC)^2 \times AD = \frac{1}{3}\pi R^2 h.$$
$$V_{lg.\ cone} = \frac{1}{3}\pi(DE)^2 \times AD = \frac{1}{3}\pi R^2 h.$$

$$\frac{V_{sm.\ cone}}{V_{lg.\ cone}} = \frac{\pi R^2 h}{81} \times \frac{1}{\frac{\pi R^2 h}{3}} = \frac{1}{27}$$

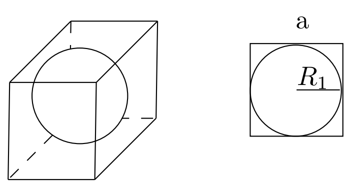
[N.B.: no units, since it is a ratio of two quantities with the same dimension, $(\text{length})^3$.]
Another Method (Dimensional Analysis): Since volume has dimension $(length)^3$, the formula for V must be $V = ch^3$ for some constant c related to the angle at the vertex. Hence,

$$\frac{V_{sm.\ cone}}{V_{lg.\ cone}} = \frac{c(h/3)^3}{ch^3} = \frac{ch^3/27}{ch^3} = \frac{1}{27}.$$

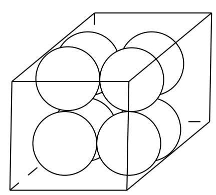
Exercise IV.5



Since the diameter of the small spheres is a/n, the radius will be $R = \frac{a}{2n}$. Considering the three-fold symmetry of the sphere and cube, it should be clear that we are dealing with n^3 small spheres packed inside the cube. Take first just one big mothball (sphere) inside the cube: $n = 1$; $r^3 = 1$; $R_1 = \frac{a}{2}$.



The next number of spheres we can pack is 8: with $n = 2$, $n^3 = 8$, $R_2 = \frac{a}{2\times 2} = \frac{a}{4}$ (check it for yourself).



and so on – you can continue with $n = 3$, packing $n^3 = 27$ spheres of radius $R_3 = \frac{a}{6}$, etc.
The volume of one sphere is: $V = \frac{4\pi R^3}{3} = \frac{4\pi}{3} \times \left(\frac{a}{2n}\right)^3 = \frac{\pi a^3}{6n^3}$. But since we have n^3 spheres, the total volume of the spheres is $V_{tot} = n^3 \times V = n^3 \times \frac{\pi a^3}{6n^3} = \frac{\pi a^3}{6}$. So the final result is independent of n; no matter which you buy, the total mass of the mothballs – assuming constant density – is the same. (This result is exactly the as in Exercise IV.3 b.) The ratio of the volume of the inscribed sphere to the volume of the cube is $\pi/6$ regardless of the sidelength L.)

This module is based in large part on an earlier module prepared by the Department of Mathematics.

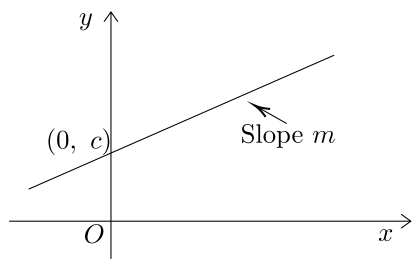
ANALYTIC GEOMETRY REVIEW MODULE

Introduction

Analytic geometry is the description and analysis of geometrical curves and shapes in terms of mathematical coordinates. The subject is concerned mainly with geometry in two or three dimensions, using various kinds of coordinate systems. The scope of the module is limited to two dimensions, using rectangular coordinates. It will deal with a few of the most basic results concerning straight lines, circles, parabolas, and other conic sections. Those are about all you will need initially in basic calculus and physics.

I. STRAIGHT LINES

First we set up our coordinate system, with an origin and the conventional x and y axes, with the x axis horizontal and the y axis perpendicular to it in the plane of the paper. (We shouldn't really call the direction of the y axis "vertical," although we often do.) The position of any point in the xy plane is then uniquely defined by the pair of numbers (x, y).



Any straight line in the xy plane can then be described by an equation of the form:

$$y = mx + c.$$

if this equation applies, we say that y is a linear function of x. (It is equally true that x is a linear function of y when $m \neq 0$.) When $x = 0$, $y = c$, and when $y = 0$, $x = -c/m$. These conditions define the intercepts of the line on the x and y axes respectively. If both m and c are positive, the general appearance of the line is as shown in the above diagram.

If (x_1, y_1) and (x_2, y_2) are any two points on the line, we have

$$y_1 = mx_1 + c;$$

$$y_2 = mx_2 + c.$$

By subtraction, we get the result:

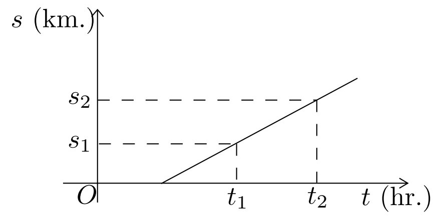
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

This ratio, the ratio of the vertical "rise" to the horizontal "run" between any two points, is what is called the slope of the line, and the equation $y = mx + c$ is called the slope-intercept form of the equation for a line.

[A Note about Slopes: If we label the angle between a straight line and the positive direction of the x axis as θ, then we have the relation:

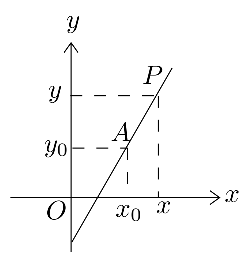
$$m = \tan\theta.$$

This *geometrical* identification of the slope of a line in terms of an angle is fine in mathematics, where x and y are pure numbers and the axes are assumed to be marked off with equal scale divisions. However, it is seldom necessary, and you should be warned that it loses its usefulness – and can in fact be misleading – when you come to graphs of the relationship between two different *kinds* of quantities, measured in different units.



For example, the graph above might describe the motion of a car traveling at a constant speed along a straight road. On the y axis we show its position s measured, say, in kilometers relative to some fixed point; the horizontal axis shows the time t, measured, say, in hours. The speed of the car, in kilometers per hour, is the slope of this graph, measured as the change in s divided by the change in t between ant two points on the line. We simply read off these values from the scales of s and t along the axes, and these scales are purely a matter of convenience, chosen to give us a graph whose slope is neither too steep or too small for easy analysis. There is no particular meaning to the geometrical slope of the line as such.

Most of the graphs we draw in science, engineering, economics, etc. are constructed in this way.]



If we have a straight line as shown, where the point A (x_0, y_0) is a particular point on the line and the point P (x, y) is *any* other point, we can put

$$\frac{y - y_0}{x - x_0} = m.$$

This can be rewritten in the form:

$$y - y_0 = m(x - x_0).$$

This is the so-called point-slope form of the equation of a straight line.

The general equation of a straight line can be written:

$$ax + by + c = 0.$$

You should be able to recognize and use any of the above three forms of the equation of a straight line. For most purposes, the form $y = mx + c$ is the easiest and the most convenient to use.

Exercise I.1.

(a) Find the intercept c in the slope-intercept equation in terms of the quantities x_0, y_0 and m of the point slope equation;

(b) Find the slope m and the intercept c of the slope-intercept equation in terms of the constants a, b, c of the general equation.

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that needs to be corrected, please write to us.

Exercise I.2.

Draw a set of xy axes, marked off in equal intervals between -5 and $+5$ for both x and y, and sketch the straight lines with the following values of m and c:
(a) $m = +2$, $c = -4$; (b) $m = -1$, $c = +4$; (c) $m = +1/2$, $c = +1$; (d) $c = -1$, $m = -1/3$.

Exercise I.3.

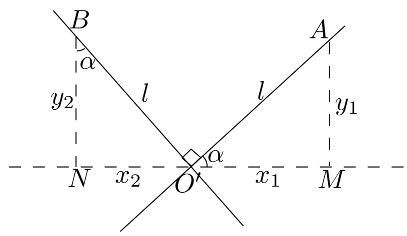
Draw another set of xy axes, marked off as in Exercise I.2, and draw he following lines, all passing through the point $(2, 1)$: (a) $m = 0$; (b) $m = +3/2$; (c) $m = \infty$; (d) $m = -2/3$.

Exercise I.4.

Draw another set of xy axes, marked off as before, and draw lines described by the following equations: (a) $x + y = 2$; (b) $2x - y + 3 = 0$; (c) $x + 2y + 4 = 0$; (d) $x - 4 = 0$.

By definition, lines with the same values of m have the same slope. Lines with the same value of m but different values of c are therefore parallel.

If two lines are perpendicular, the product of their slopes is -1. We included an example of this in Exercise I.3, parts (b) and (d). The result is easily shown. Here is one way:



Let the point of intersection of two perpendicular lines be called O'. Take O' as a new origin, and mark off a distance l along each line as shown in the diagram. If $\angle AO'M = \alpha$, then $\angle O'BN$ is also α. With respect to O', the coordinates of A are x_1, y_1, where $x_1 = l\cos\alpha$, $y_1 = l\sin\alpha$, and we have:

$$m_1 = \frac{y_1}{x_1} = \tan\alpha.$$

Also, with respect to O', the coordinates of B are x_2, y_2, where $x_2 = -l\sin\alpha$, $y_2 = l\cos\alpha$, and so we have:

$$m_2 = \frac{y_2}{x_2} = \frac{l\cos\alpha}{(-l\sin\alpha)} = -\cot\alpha.$$

Therefore: $m_1 m_2 = (\tan\alpha)(-\cot\alpha) = -1$.

Unless two lines in the xy plane are parallel, the must intersect somewhere. To find the point of intersection, we use the condition that this point must be on both lines, which gives us a pair of simultaneous equations.

Example:

Find the point of intersection of the lines $2x - y - 2 = 0$, $x - 2y + 2 = 0$. Let the point of intersection be (x_1, y_1).

$$\text{Then} \quad 2x_1 - y_1 = 2 \quad \text{and} \quad x_1 - 2y_1 = -2$$

Solving these equations gives $x_1 = 2$, $y_1 = 2$. (Check this!)
(Drawing sketched for such problems is highly recommended!)

Exercise I.5.

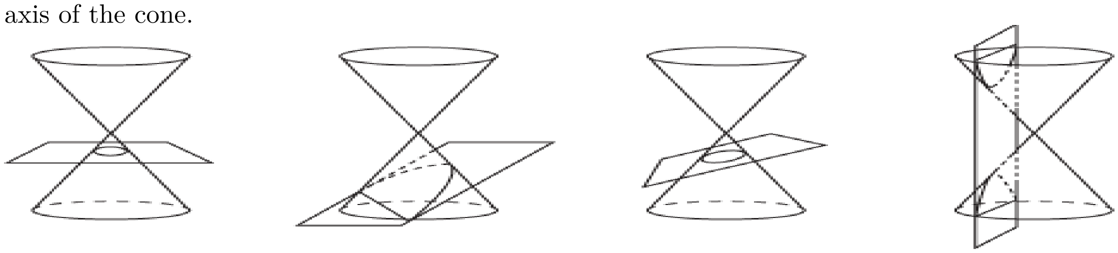
Find the point of intersection of the lines $3x - 4y - 5 = 0$ and $x + 2y + 3 = 0$.

Exercise I.6.

Which of the following lines are parallel? perpendicular? intersect?
(a) $y = 2x + 4$; (b) $2y + x - 1 = 0$; (c) $y = (-3/2)x + 1$; (d) $3x + 2y = -1$; (e) $2 = (y - 4)/(x - 0)$.

II. CONIC SECTIONS

You have probably learned that the circle, the parabola, the ellipse and the hyperbola are called "conic section." This is because they arise from the intersection of various planes with a circular cone. The diagram below shows how this comes about. Circles are formed when a plane is drawn at right angles to the axis of the cone. Parabolas are formed when a plane is drawn parallel to one of the generators (sloping sides) of the cone. Ellipses are formed by planes intermediate between these two directions. Hyperbolas are formed by planes that intersect both top and bottom parts of the complete double cone; the diagram shows the particular plane that is parallel to the axis of the cone.

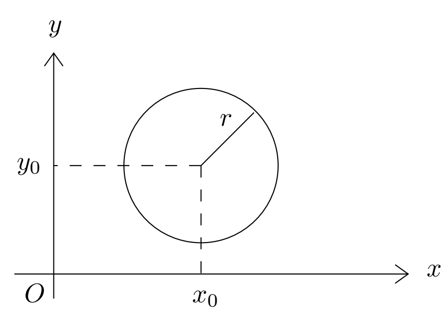


[From Hughes-Hallett, Math Workshop: Elementary Functions, W.W. Norton Co. (1980).]

We shan't make use of these pictures to obtain equations for conics. Instead we'll work from geometric definitions of how these curves are constructed in terms of a rectangular coordinate system. But it is worth seeing how they are members of a family.

III. CIRCLES

There are just two features that characterize a circle:
(i) It has a certain radius, r,
(ii) Its center is at a certain point (x_0, y_0).



To express these properties in mathematical terms, we combine them in a statement that the distance between the center of the circle and any point (x, y) on its circumference is equal to r. For any two points in the xy plane, the square of the distance between them is given by Pythagoras's Theorem:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

The equation of the circle is therefore given by:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

This is the <u>standard form</u> of the equation of a circle. If the center of the circle is at the origin this reduces to the simple form:

$$x^2 + y^2 = r^2.$$

<u>Exercise III.1</u>.

Write the equations of the following circles, and sketch them on squared paper:
(a) Center at $(3, 2)$, radius 3; (b) Center at $(0, 3)$, radius 5; (c) Center at $(-2, 2)$, passing through $(2, 5)$.

If the equation of a circle with its center at an arbitrary point is expanded out into individual terms and the contributions to the constant term are collected together, the resulting equation is of the form:

$$x^2 + y^2 + ax + by + c = 0.$$

It is <u>not</u> necessarily true, however, then *any* equation of the above form describes a circle. We can see this by completing the square for the terms in x and y. If we do this, and rearrange, we get:

$$\left(a + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - c.$$

The right-hand side can be positive, negative, or zero.

If it is positive, we can equate it to r^2; the equation does describe. circle – with its center at $(-a/2, -b/2)$.

If it is negative, there are no real values of x and y that satisfy the equation. The sum of two squares must be > 0. The equation cannot describe any real curve.

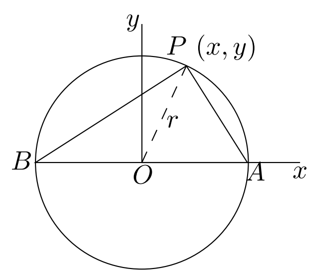
If it is zero, the equation can be satisfied only for a single point: $x = -a/2$, $y = -b/2$, since both of the squared on the left must be zero if the equation is to be satisfied. This is, if you like, a circle of zero radius with its center at $(-a/2, -b/2)$.

<u>Exercise III.2</u>.

Analyze which of these equations describe circles. For those that do, find the radius and the coordinates of the center:
(a) $x^2 + y^2 - 2x + 8 = 0$; (b) $x^2 + y^2 - 4x + 2y + 6 = 0$; (c) $x^2 + y^2 + 4x - 6y - 12 = 0$;
(d) $x^2 + y^2 - 8x + 6y + 25 = 0$.

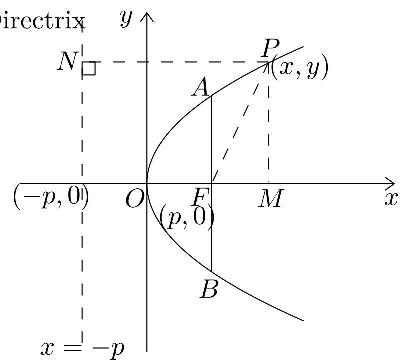
<u>Exercise III.3</u>.

Here is a nice exercise that puts together the analytic geometry of straight lines and circles. Take a circle of radius r with its center at the origin. See if you can prove, by analytic geometry, the theorem from plane geometry that says that an angle inscribed in a semicircle is a right angle. [First find the slopes of the lines AP and BP in terms of x, y, and r. To confirm the perpendicularity condition $m_1 m_2 = -1$, use $x^2 + y^2 = r^2$.]



IV. PARABOLAS

A parabola is defined geometrically as a plane curve that is the locus of points equidistant from a fixed point (the focus, F) and a fixed straight line (the directrix). The word focus can be taken literally. A mirror made in a parabolic (or, more strictly, paraboloidal) shape will bring parallel light to a focus at F. (A paraboloid is a parabola rotated about its axis of symmetry.)



Suppose that the focus is at the point $(p, 0)$, and that the directrix is the line $x = -p$. You can see right away that the origin is a point on the curve; it is halfway between the focus and the directrix. For any other point P (x, y), we have the more complicated condition that the distance FP is equal to the length of the perpendicular PN drawn from P to the directrix. Using Pythagoras's Theorem, this requires:

$$(FM)^2 + (MP)^2 = (PN)^2.$$

Putting $FM = x - p$, $MP = y$, and $PN = x + p$, this gives:

$$(x - p)^2 + y^2 = (x + p)^2.$$

Multiplying this out, we get:

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2,$$

which after cancelling and rearranging gives:

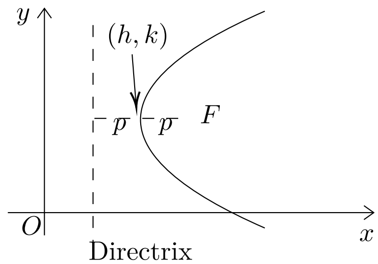
$$y^2 = 4px.$$

This equation describes a parabola with its axis along the x axis and its vertex (the rounded end) at the origin. The parabola is symmetrical about the x axis, because for any given value of x we have $y = \pm\sqrt{(4px)}$.

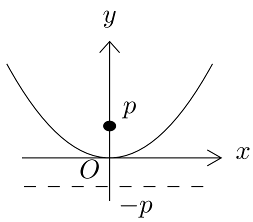
The coefficient $4p$ has a geometrical significance. If we draw the line $x = p$ through the focus, it cuts the parabola at points A and B for which $y^2 = 4p^2$ and so $y = \pm 2p$. Thus the distance between these points is equal to $4p$; it is a measure of the *width* of the parabola.

We can construct similar parabolas with different positions of the focus and different locations and directions of the directrix. For example, a parabola with the same value of p as before (i.e., the same distance between focus and directrix), with its vertex at (h, k) and with its directrix parallel to the y axis, has the equation:

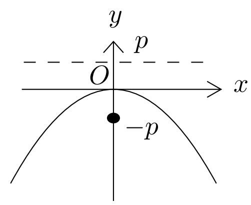
$$(y - k)^2 = 4p(x - h).$$



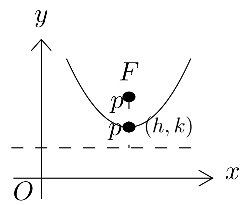
The figures below show some other examples.



$x^2 = 4py$



$x^2 = -4py$



$(x-h)^2 = 4p(y-k)$

The first of these, whose equation can be rewritten in the form $y = Cx^2$, is a particularly useful one to remember, because we often encounter relationships between two quantities where the dependent variable (y) is proportional to the square of the independent variable (x). (Example: the distance as a function of time that an object falls from rest under gravity.)

<u>Exercise IV.1</u>.

Take some squared paper and sketch the following parabolas:
(a) $y = -x^2$;   (b) $x = 2(y-1)^2$;   (c) $y = (x+2)^2$.

There are two standard forms for the equation of a parabola, depending on whether its axis is along the x direction or the y direction. Notice that if the axis is parallel to x, the equation is quadratic in y; if the axis is parallel to y, the equation is quadratic in x. (You can of course have parabolas whose axis is in some arbitrary direction, but we won't bother with those.)

If the axis is parallel to x, the equation can be written in the form:

$$x^2 + ax + by + c = 0.$$

As with the circle, we can put this into standard form, beginning by completing the square on the terms of x:

$$\left(x + \frac{a}{2}\right)^2 = -by - c + \frac{a^2}{4} = -b\left[y + \left(\frac{c}{b} - \frac{a^2}{4b}\right)\right].$$

Comparing this with $(x-h)^2 = 4p(y-k)$, we can find the values of h, k, and p.

<u>Example:</u>

Find the coordinates of the focus and the vertex of the parabola:

$$x^2 - 6x - 8y + 17 = 0.$$

Completing the square on the x terms and rearranging, we have:

$$(x-3)^2 = 8y - 17 + 9 = 8y - 8 = 8(y-1).$$

Thus the vertex is at $(3,1)$, and $p = 2$. In this parabola, the focus is a distance p above the vertex; its coordinates are therefore $(3,3)$.

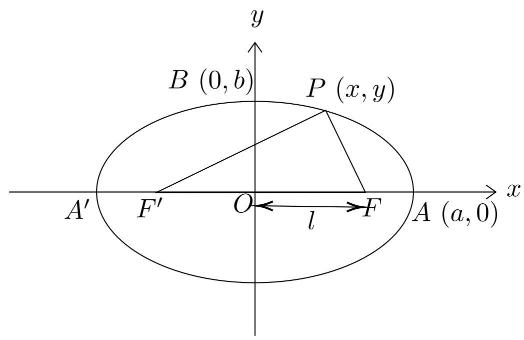
Exercise IV.2.

The following equations are in the general polynomial form of equations for parabolas. Put them into standard form, locate the vertices and the foci, and sketch the curves.:
(a) $x^2 - 4x - 4y = 0$; (b) $12x - 6y - y^2 - 33 = 0$; (c) $y^2 + 4y + 4x = 8$;
(d) $2x^2 - 4x - y + 1 = 0$.

V. ELLIPSES

You are very likely familiar with the "pins-and-string" definition of an ellipse. Take a piece of string and attach its ends to pins at two fixed points, F and F', on a piece of paper. Take a sharp pencil and hold it vertically. Move it until it just makes the string tight, and then move it around, always in contact with the string, so as to create a closed curve. This curve is an ellipse. If the length of the string is called $2a$, we have:

$$FP + FP' = 2a.$$



The points F and F' are the foci. If the ellipse were a mirror, light from point source at F would come to a focus at F', and *vice versa.* The ratio of the distance OF to the distance OA is called the eccentricity, ϵ of the ellipse:

$$\epsilon = \frac{OF}{OA}.$$

(Note that, if F and F' are both moved to the center O, the ellipse becomes a circle and by the above definition the eccentricity is *zero.*)

Take an origin at the geometrical center of the figure, with an x axis along the line $F'F$ ($A'A$), and a y axis along OB. Let the distance OF ($= OF'$) be l, and let OB be b.

First imagine that P is moved to A. Then $FA + AF = 2a$. But, by symmetry, $FA = FA'$. Therefore $AF' + FA'$ (which adds up to AA'), is equal to $2a$. So $OA = a$.

Next, imagine that P is moved to B; this gives FB ($= BF$) $= a$. Using Pythagoras's Theorem, one finds $b = \sqrt{(a^2 - l^2)}$.

Finally, let P be at some arbitrary point (x, y) on the ellipse. From the condition $FP + PF' = 2a$, and using Pythagoras's Theorem again, we have:

$$\sqrt{[(l-x)^2 + y^2]} + \sqrt{[(l+x)^2 + y^2]} = 2a.$$

If you do some algebra on this, you can verify that it leads to the following equation for the ellipse in xy coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The distances a and b are called the semi-major axis and the semi-minor axis respectively ($a > b$).

Exercise V.1.

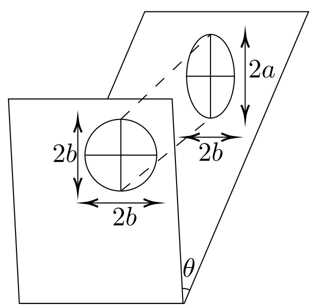
Go through thte process of deriving this last equation from the preceding one. [Begin by putting one of the radicals on the other side of the equation and squaring. Some rearrangement, followed by another squaring, will do most of the rest, but you will need the relation among a, b and l.]

If the center of the ellipse is not at the origin but at some point (h, k), one just replaces x by $(x - h)$ and y by $(y - k)$ in the equation. The resulting equation, when multiplied out, will in general have quadratic and linear terms in x and y, plus a constant. The thing that immediately distinguishes the equation of an ellipse from the equation of a circle is that, for a circle, the coefficients of the x^2 and y^2 terms are equal, whereas for an ellipse they are different.

Exercise V.2.

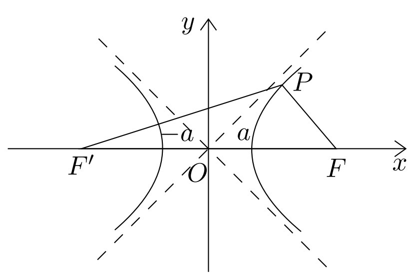
Sketch the ellipse $4x^2 + 9y^2 = 36$, locate its foci and find the eccentricity.

For $b = a$ the equation of an ellipse turns into the equation for a circle, and it is clear that the connection between the two is close. In fact, an ellipse can be thought of as the geometrical projection of a circle onto another plane. The diagram below indicates this. If the angle between the planes is θ, then a circle of radius b projects into an ellipse of semi-minor axis b and semi-major axis $a = b \sec\theta$. If the coordinates of a point P on the circle are labelled (x, y), the coordinates of the corresponding point P' on the ellipse are $(x \sec\theta, y)$. Thus every x-coordinate on the ellipse is stretched by the factor $\theta = a/b$. One can use this fact to infer that the area of an ellipse is given by πab.



VI. HYPERBOLAS

The way of constructing a hyperbola looks very similar to that for an ellipse, but is not nearly so easy to do in practice. Again one has an axis with two foci on it. However, the prescription for a hyperbola is to move a point P so that the difference of its distances from the foci is a constant.

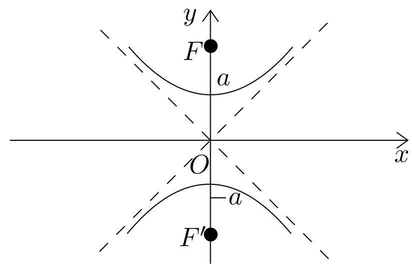


If the foci are on the x axis, the xy equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is rather like an ellipse turned inside out. The distance intercepted by the curve on the x axis is again $2a$, but this time it is a minimum distance rather than a maximum. At large distances from the origin, the two branches of the curve approach two straight lines – the asymptotes – whose equations are the two possibilities allowed by the equation $(x^2/a^2) - (y^2/b^2) = 0$:

$$y = \pm \frac{b}{a} x.$$



If the foci lie on the y axis, the branches of the hyperbola are as shown in the second diagram. The equation to the curve is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

and the asymptotes are given by: $y = \pm \frac{a}{b} x$. In other words, the whole picture is turned through $90°$ with respect to the first one.

Exercise VI.1.

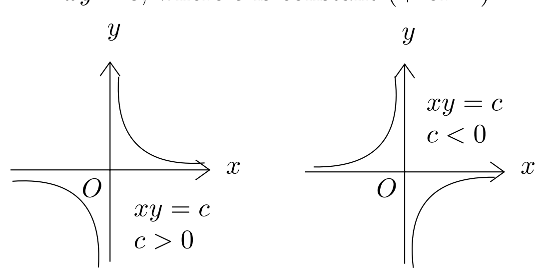
Using the condition $F'P - FP = 2a$. together with the standard equation of a hyperbola, show that the distance between the foci is $2\sqrt{(a^2 + b^2)}$.

Exercise VI.2.

Sketch the hyperbola $4x^2 - 9y^2 = 36$, and its asymptotes. Also locate the foci.

Another very important form of the equation of a hyperbola – and in some respects much the simplest, is:

$$xy = c, \text{ where } c \text{ is constant } (+ \text{ or } -)$$



This gives the two possibilities shown here. This is the rectangular hyperbola – its asymptotes are perpendicular. A good physical example of this (width $c > 0$) is the relation between pressure and volume for an ideal gas at constant temperature:

$$pV = \text{constant}$$

Exercise VI.3.

Sketch the hyperbola $xy = 9$, and locate its foci.

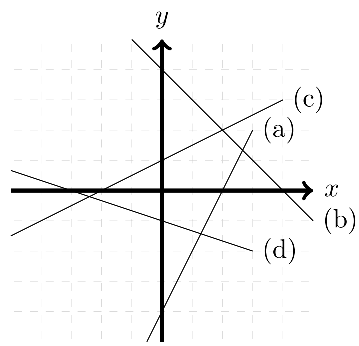
ANSWERS TO EXERCISES

Answers not given to exercises V.1. VI,1,

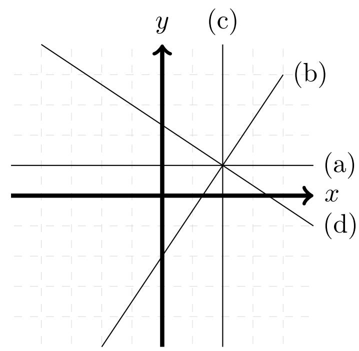
Exercise 1.1.

(a) $c = y_0 - mx_0$; (b) It is very confusing to use the same letter (c) for two different quantities in a problem. Before solving, we change the same of "c" to "d" in the general equation, $ax + by + d = 0$, and leave is as "c" in the slope-intercept equation, $y = mx + c$. The $m = -a/b$, $c = -d/b$.

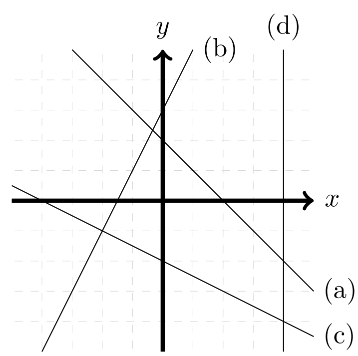
Exercise I.2.



Exercise I.3.



Exercise I.4.



Exercise I.5.

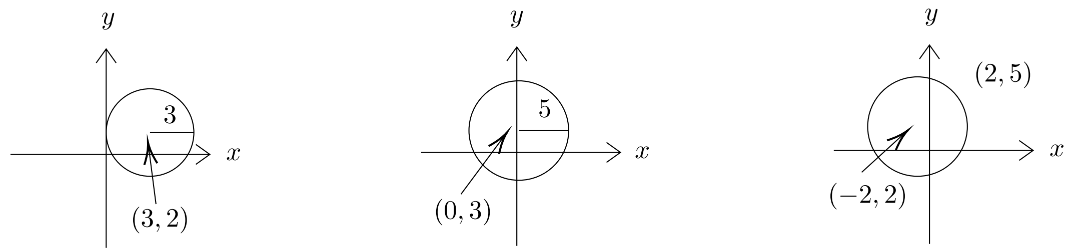
$x = -1/5$, $y = -7/5$.

Exercise I.6.

$l_c \parallel l_d$; $l_b \perp l_a$, $l_b \perp l_e$ (l_e, l_a = the same line). All non-parallel lines intersect!

Exercise III.1.

(a) $(x-3)^2 + (y-2)^2 = 9$ (b) $x^2 + (y-3)^2 = 25$ (c) $(x+2)^2 + (y-2)^2 = 25$



Exercise II.2

(a) No; (b) No; (c) Center at $(-2, 3)$, $r = 5$; (d) Single point, $(4, -3)$

Exercise III.3.

$$m_1 = \text{ slope } AP = \frac{y-0}{x-r}$$

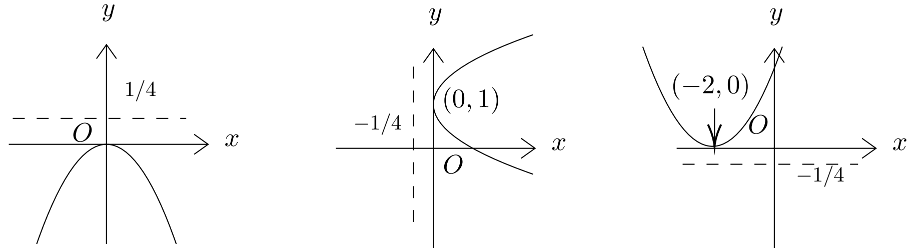
$$m_2 = \text{ slope } BP = \frac{y-0}{x-(-r)}$$

$$m_1 m_2 = \frac{y}{x-r} \times \frac{y}{x+r} = \frac{y^2}{x^2-r^2}$$

$$\text{But } x^2 + y^2 = r^2 \therefore x^2 - r^2 = -y^2 \therefore m_1 m_2 = y^2/(-y^2) = -1.$$

Exercise IV.1.

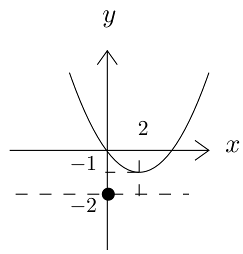
a) b) c)



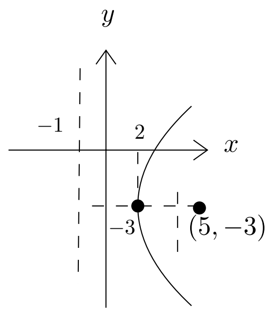
Exercise IV.2.

(a) $x-2)^2 = 4(y+1)$; (b) $(y+3)^2 = 12(x-2)$; (c) $(y+2)^2 = -4(x-3)$;
(d) $(x-1)^2 = (1/2)(y+1)$.

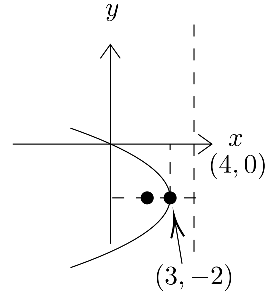
a)



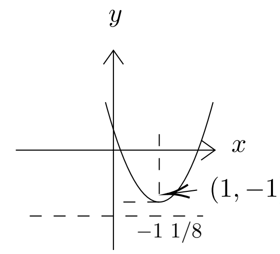
b)



c)

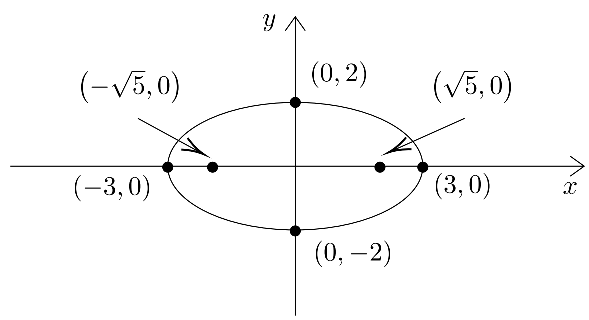


d)



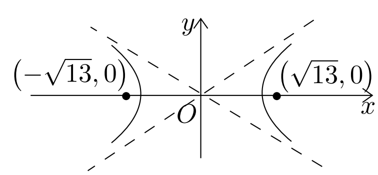
Exercise V.2.

$(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$; $\epsilon = l/a = \sqrt{5}/3$



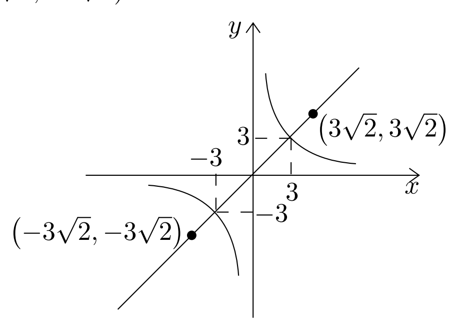
Exercise VI.2.

Foci at $(\sqrt{13}, 0)$, $(-\sqrt{13}, 0)$:



Exercise VI.3.

Foci at $(3\sqrt{2}, 3\sqrt{2})$, $(-3\sqrt{2}, -3\sqrt{2})$:



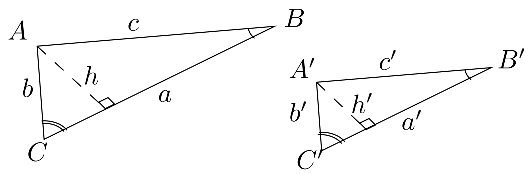
This module is based in the large part on an earlier module prepared by the Department of Mathematics.

Geometry and Analytic Geometry Review Problems

G.I Triangles, similarity of figures.

Two triangles are similar when the three angles of one are the same as the three angles of the other. (In practice, you only have to show this for two of the angles, since the third angle will then automatically be equal.)

If two triangles are similar, *corresponding sides are proportional*, and *the altitudes on these sides are proportional.* This is the essential fact that is most often used in scientific problems.

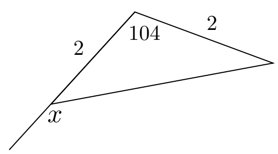


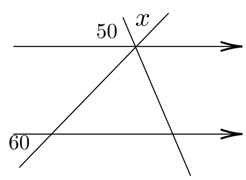
$\measuredangle A = \measuredangle A'$
$\measuredangle B = \measuredangle B'$
$\vdots$

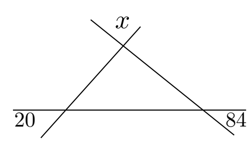
$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{h}{h'}$ *etc.*

In working the problems below, in your diagram mark the equal angles; for similar triangles, mark two of the three pairs of equal angles , as is done above. Parallel lines are indicated by arrows. Figures are not drawn to scale!

1. In each of the figures, tell how many degrees angle x has.

a) 

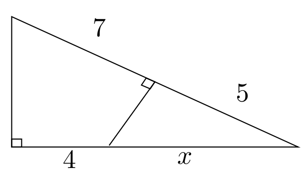
b) 

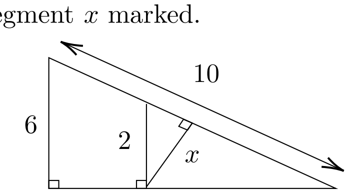
c) 

2. A line segment of length 3 joins two sides of a triangle , is parallel to the third side, and has distance 1 from that side and 2 from the opposite vertex. How long is the third side of the triangle?

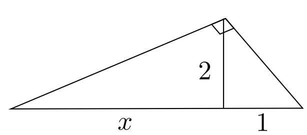
3. The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a vertical screen 2 meters away from the hole. If the image is 1.2 meters high, how tall is the building?

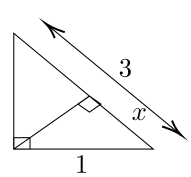
4. In each of the diagrams, find the length of the line segment x marked.

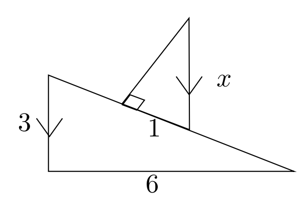
a) 

b) 

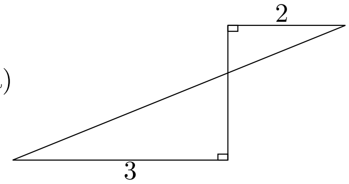
5. In each picture, find the length of the line segment x marked.

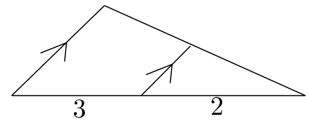
a) 

b) 

c) 

6. In the accompanying figures, the smaller triangle has area A; what is the area of the larger triangle?

a) 

b) 

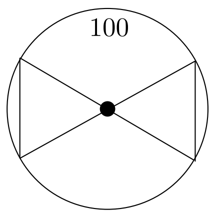
7. A streetlamp 20 feet high is 15 feet horizontally distant from a woman 5 feet high. How long is the shadow she casts?

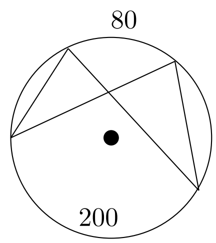
8. How long is the diagonal of (a) a cube of side 1 (b) a rectangular box of sides 1, 2, and 3?

G.II Circle Problems.

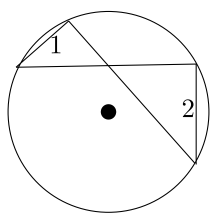
Note: in these problems the center is indicated by a heavy dot.

1. Some arcs are given. Determine the angles in the left triangle.

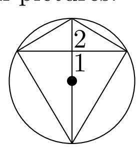
a) 

b) 

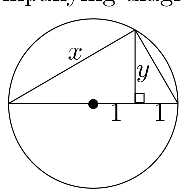
2. The smaller triangle has area A; what is the area of the larger triangle?



3. Find the area of the kite-shaped region pictures.

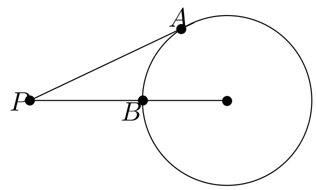


4. Find the lengths x and y in the accompanying diagram.

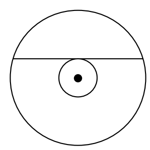


2

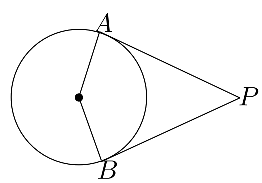
5. The circle has radius 1, the line segment PA has length 2, and is tangent to the circle. What is the length of PB?



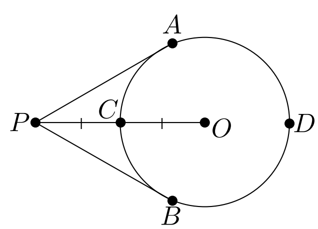
6. The two circles are concentric, with radii respectively 1 and 3. The chord is tangent to the inner circle. How long is it?



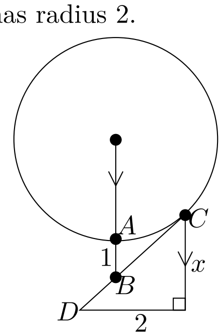
7. The circle has radius 2; PA and PB are tangents, and PO has length 4. What is the area of quadrilateral $OAPB$?



8. The two line segments marked are equal, and PA, PB are tangents. What are the degrees of the two circular arcs $\widehat{ADB}$ and $\widehat{ACB}$ having A and B as their endpoints?



9. Find the length of x in the accompanying picture. The arrows indicate parallel line segments; $AB = 1$ and CD is tangent; the circle has radius 2.

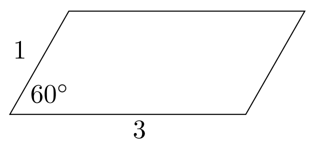


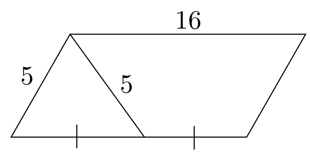
G.III Areas and Volumes.

1. A circle has radius R; what is the area and perimeter of a 72° circular sector?

2. What is the area of a triangle having
   a) all sides of length 3;
   b) two sides of length 4, with a 30° angle between them?

3. Find the areas of the two parallelograms shown:

a) 

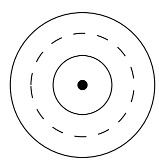
b) 

4. A trapezoid has parallel sides of lengths 5 and 9, and its other two sides both have length 3. What is its area?

5. A square and a circle have the same perimeter. What is the ratio of their respective areas?

6. The ring-shaped region between two concentric circles has inner radius a and outer radius b. Give an expression for its area A, and show that is we draw the median circle on the ring, half-way between the inner and outer boundaries, then

$$A = (\text{length of median circle}) \times (\text{thickness of ring}).$$



7. The radius of a right circular cylinder is 1/3 of its height; the volume of the cylinder is 24π. What is the radius and height?

8. These connect radius R, surface area S, and volume V of a sphere.
   a) A sphere has surface area 12π; what is its volume?
   b) Find in terms of R the ratio S/V; simplify your answer.
   c) What is the total surface area of a solid hemisphere of radius R?
   d) A sphere of radius R is centered at the origin of xyz-coordinates. What is the surface area and volume of that portion of it for which $x, y, z \geq 0$?

9. A sphere is inscribed in a cube, so that it is tangent to all six sides of the cube. What is the ration (sphere:cube) of a) their surface areas b) their volumes?

10. An upright symmetrical pyramid has a square base with sides of length 10, and a height of 12. What is a) its volume b) the area of one of its slanted faces?

11. A right circular cone has radius 2 and height 6. What is the volume of the slice cut off by a plane parallel to the base and distance 2 from it?

12. A tetrahedron has three edges of lengths 2, 3, and 5 which intersect at right angles. Sketch it and determine its volume.

G.IV Lines.

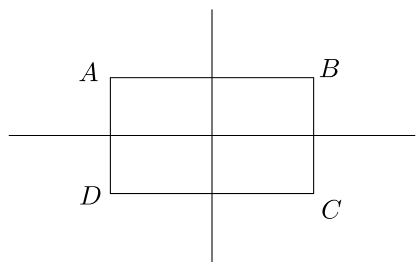
1. A line of slope 6 passes through the point $(2, -3)$. Where does it cross the x-axis?

2. At what point do the lines given by $3x + 4y = 12$ and $3x - 5y = -6$ intersect?

3. At what point does a line with slope -1 passing through the point $(2, 5)$ intersect the line having slope 2 and y-intercept -2?

4. What is the equation of a line perpendicular to the line $2x - 5y = 1$ and having y-intercept 3?

5. The rectangle $ABCD$ is symmetric about the y-axis; What is the equation of a line through C parallel to the diagonal BD? ($B = (5, 3)$, $C = (5, -1)$)



6. Find the area of the finite region bounded by the lines $x = 0$, $y = 0$, and $3x + 5y = 30$.

7. Find the equation of a line through the origin perpendicular to the line having y-intercept -2 and x-intercept 3. (Draw a sketch.)

G.V Circles.

1. Give in the form $x^2 + y^2 + ax + by + c = 0$ the equation of the circle
   a) with center at the origin and radius 3.
   b) with center at the origin and tangent to the line $x + y = 1$.
   c) with center at $(1, 1)$ and passing through the origin.
   d) with center at $(-1, 1)$ and tangent to both coordinate axes.

2. A circle centered at the origin has radius 11. Is the point $(7, 8)$ outside or inside the circle?

3. Where do the circle and line intersect in the following examples?
   a) $x^2 + y^2 = 5$ and $x + y = 3$
   b) $x^2 + y^2 = 10$ and $x + 3y = 10$
   c) circle of radius 3 centered at $(0, 0)$, line of slope 3 through $(0, 0)$

`G.VI` Conics.

1. What is the length of the semi-major and semi-minor axes of the ellipses
a) $\frac{x^2}{10} + \frac{y^2}{3} = 1$ b) $9x^2 + 4y^2 = 25$?

2. Write the equation of the ellipse with center at the origin, axes along the two coordinate axes, and whose x-intercepts are ± 3, y-intercepts are ± 2.

3. Write the equation of a parabola whose minimum point is on the y-axis at -1, and whose x-intercepts are ± 2.

4. Write in the form $y = ax(x - c)$ the equation of a parabola whose high point is at $(2, 1)$, and which goes through the origin.

5. Write the general form for the equation of a hyperbola having the lines $y = \pm 2x$ as asymptotes.

6. What kind of geometric locus do the following equation represent: circle, ellipse, parabola, hyperbola, straight lines, point, or no locus?
a) $x^2 + 2y^2 - 12 = 0$ b) $6x^2 - 5y^2 = 0$ c) $y^2 + 2x = 5$ d) $x^2 - y^2 = 0$ e) $x^2 + 2y^2 + 1 = 0$

7. Answer the same question as in 6 for
a) $4y^2 - x^2 = 2$ b) $x^2 + y^2 + 2x - 5 = 0$ c) $y^2 + 2x = 5$ d) $x^2 - y^2 = 0$ e) $x^2 + 2y^2 + 1 = 0$

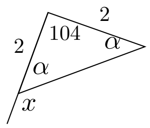
8. Answer the same question as in 6 for
a) $xy = 5$ b) $x^2 + y = 0$ c) $x^2 - y^2 = 4$ d) $2x^2 + 3y^2 = 5$ e) $xy = 0$

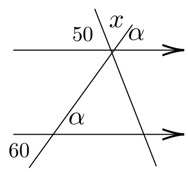
9. Answer the same question as in 6 for
a)$3x^2 + 3y^2 - 6y = 10$ b) $x^2 - 5y^2 = 10$ c) $x^2 + 3y^2 = 0$ d) $x + 3y^2 - 5 = 0$ e) $x^2 - 4y^2 = 0$

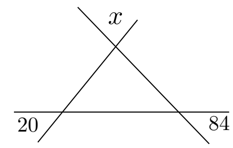
Solutions: Geometry and Analytic Geometry Review Problems

`G.I` Triangles, similarity of figures.

1.

a) 

b) 

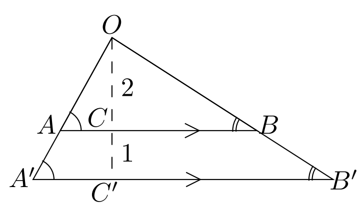
c) 

a) Since the triangle is isosceles, the base angles α are equal.
   $2\alpha + 104 = 180 \therefore \alpha = 38 \therefore x = 180 - 38 = \boxed{142}$ (or $x =$ sum $104 + \alpha = 104 + 28 = 142$)

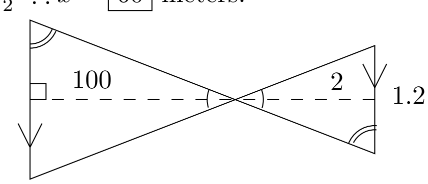
b) Two angles α are equal since since lines are parallel. $\alpha = 60$ (vertical angles are equal).
   $50 + x + 60 = 180 \therefore x = \boxed{70}$

c) Since vertical angles are equal, $x + 20 + 84 = 180 \therefore x = \boxed{76}$

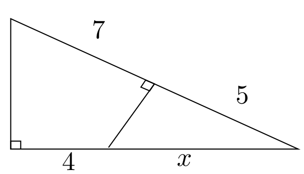
2. The two triangles are similar, so corresponding lines are proportional:
$\frac{OC}{OC'} = \frac{AB}{A'B'} \therefore \frac{2}{3} = \frac{3}{A'B'}$ $A'B' = \boxed{9/2}$. [If you prefer, do it in two steps, using similarity of the right triangles: $\frac{OC}{OC'} = \frac{OA}{OA'} = \frac{AB}{A'B'}$]

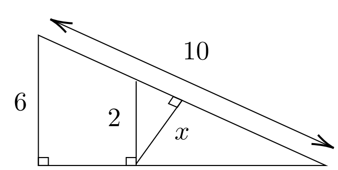


3. Triangles are similar, since bases are parallel lines. Corresponding lines are proportional (see reasoning in 2 above). $\frac{x}{100} = \frac{1.2}{2} \therefore x = \boxed{60}$ meters.



4.

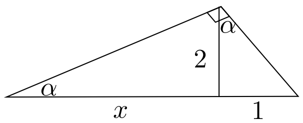
a) 

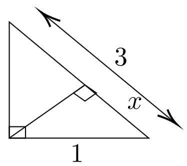
b) 

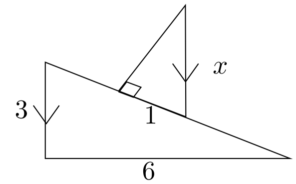
a) The triangles are similar (both have angle A and a right angle).
   $\frac{x}{5} = \frac{12}{x+4} \therefore x^2 + 4x = 60 \therefore (x+10)(x-6) = 0 \therefore x = \boxed{6}$ ($x = -10$ makes no sense).

b) $\triangle ABC$ is similar to the small triangle, (two equal angles and 2 right angles): $\frac{x}{2} = \frac{AB}{10}$,
   $AB = 8$ (Pythagorean Theorem) therefore $x = \frac{2\times 8}{10} = \boxed{8/5}$.

5.

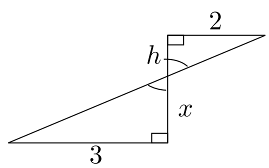
a) 

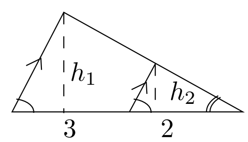
b) 

c) 

a) The two angles marked α are equal since both of them equal $90 - \beta$ therefore the left and right triangles are similar (both have θ and a right angle). Therefore $\frac{1}{2} = \frac{2}{x}$ so $x = \boxed{4}$.
b) This is the same picture as the above. The bottom and the big triangle are similar (both have angle A and a right angle), therefore $\frac{x}{1} = \frac{1}{3}$, $x = \boxed{1/3}$.
c) The triangles are similar, since two angles are equal (alternate interior angles to two parallel lines). $\frac{1}{x} = \frac{3}{\sqrt{3^2+6^2}}$, $x = \frac{\sqrt{3^2+6^2}}{3} = \frac{3\sqrt{1+2^2}}{3}$, $x = \boxed{\sqrt{5}}$.

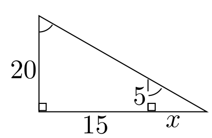
6.

a) 

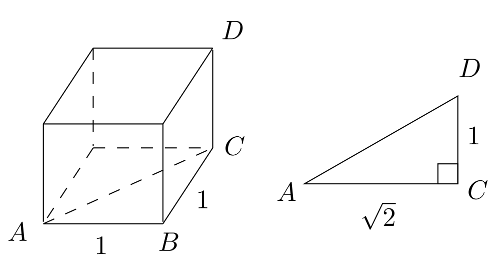
b) 

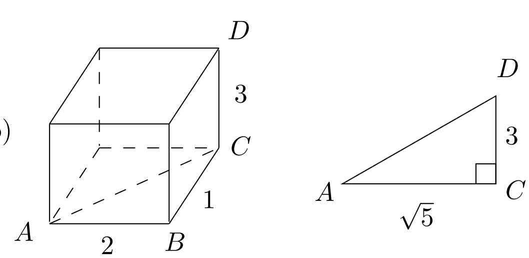
a) Triangles are similar (2 right angles and equal vertical angles). $\frac{2}{3} = \frac{h}{x}$ If $A_{top} = \frac{2h}{2}$, then $A_{bottom} = \frac{3x}{2} = 3h\frac{3}{2}$ Thus $A_{bottom} = \boxed{9/4} A_{top}$
b) The two triangles are similar, so corresponding lines are proportional thus $\frac{h_1}{h_2} = \frac{5}{2}$ so $h_1 = \frac{5}{2}h_2$. $A_1 = \frac{5h_1}{2}$, $A_2 = \frac{2h_2}{2}$, therefore $A_1 = \frac{5}{2} \times \frac{5}{2} A_2 (= A)$ so $A_1 = \boxed{\frac{25}{4}A}$.

7. Two triangles are similar. Therefore $\frac{x}{5} = \frac{x+15}{20}$, cross multiply $20x = 5x + 75$, $15x = 75$, $x = \boxed{5}$.



8.

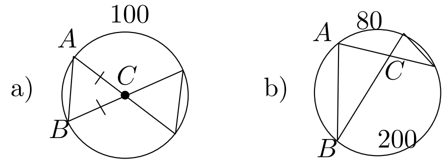
a) 

b) 

a) The diagonal is AD. We have a right triangle therefore $AD = \sqrt{3}$.
b) By the same reasoning as in part (a), $AD = \sqrt{14}$.

$\texttt{G.II}$ Circle Problems

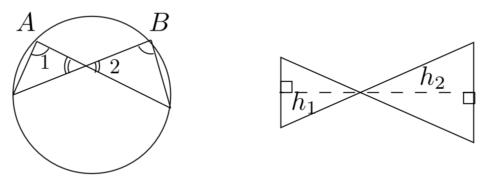
1.



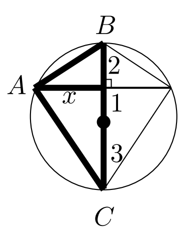
a) $\widehat{AB} = 180 - 100 = 80 \therefore \measuredangle ACB = \boxed{80^\circ}$ (central angle equals arc it subtends). $\triangle ACB$ is isosceles therefore the other two angles are both $\boxed{50^\circ}$.

b) $\measuredangle A = \boxed{100}$, (half the arc it subtends), $\measuredangle B = \boxed{40}$ therefore $\measuredangle C = \boxed{40}$

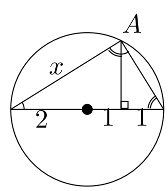
2. Angles A and B are equal, since both subtend the same arc. The two triangles are similar, thus $\frac{h_2}{h_1} = \frac{2}{1}$, so $h_2 = 2h_1$. $A_2 = \frac{2h_2}{2}$, $A_1 = \frac{1h_1}{2}$, therefore $A_2 = h_2 = 2h_1 = 4A_1$, answer: $\boxed{\text{4A}}$.



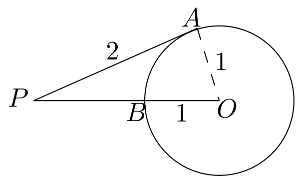
3. A is a right angle, since it subtends 180° of an arc. Thus the two triangles (outlined) are similar (the two angles are both complementary to the other angle – or they subtend equal arcs, by symmetry). Hence $\frac{2}{x} = \frac{x}{4}$. $x^2 = 8$, therefore $x = 2\sqrt{2}$, therefore Area of $\triangle ABC = \frac{2\sqrt{2}}{2}$. Area of kite is $\boxed{12\sqrt{2}}$.



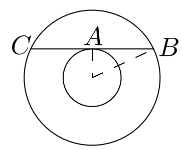
4. The angle at A is a right angle (it is inscribed in a semicircle); both smaller triangles are similar to the big one. Therefore $\frac{y}{1} = \frac{4}{y}$, $y = \boxed{2}$ and $\frac{x}{4} = \frac{3}{x}$, $x = \boxed{2\sqrt{3}}$. (check by the Pythagorean Theorem: $x^2 + y^2 = 4^2$: $(2\sqrt{3})^2 + 2^2 = 4^2$).



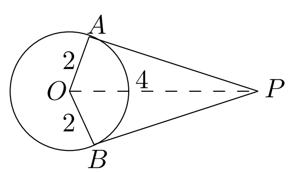
5. $OA \perp PA$ (radius is always perpendicular to tangent). By Pythagorean Theorem, $PO = \sqrt{2^2 + 1^2} = \sqrt{5} \therefore PB = \boxed{\sqrt{5} - 1}$.



6. The angle at A is a right angle since radius is perpendicular to the tangent. By Pythagorean Theorem: $AB = \sqrt{3^2 + 1^2} = 2\sqrt{2} \therefore CB = \boxed{4\sqrt{2}}$.

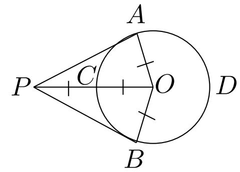


7. A and B are right angles,(radius perpendicular to tangent), therefore area $OPB = \frac{2 \times 2\sqrt{3}}{2} = 2\sqrt{3}$ By symmetry, OPB and OAP are congruent, therefore area $AOBP = \boxed{4\sqrt{3}}$.

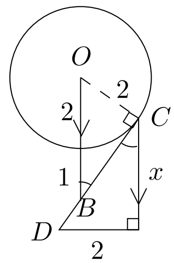


$$\begin{aligned} PB &= \sqrt{4^2 - 2^2} \\ &= 2\sqrt{3} \end{aligned}$$

8. A is a right angle (radius perpendicular to tangent) $OA = OC$ (radii of circle), therefore $OA = \frac{1}{2}OP$, so we have a $30°$-$60°$-$90°$ triangle. Therefore $\widehat{AC} = 60°$, so $\widehat{ACB} = \boxed{120°}$, $\widehat{ADB} = \boxed{240°}$.

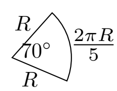


9. The two triangles are similar (the two equal angles are alternate interior angles).
$\frac{x}{2} = \frac{CB}{2} \therefore x = CB = \sqrt{3^2 - 2^2} = \boxed{\sqrt{5}}$

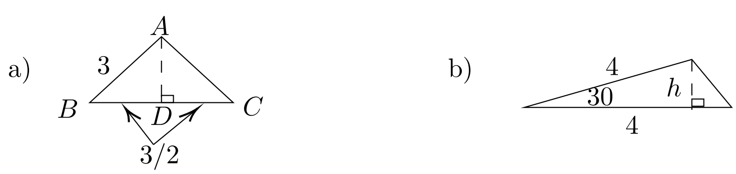


G.III Areas and Volumes.

1. $\frac{360}{72} = 5$. Therefore Area is $\frac{1}{5}$ of the Area circle so $\boxed{\frac{\pi R^2}{5}}$. Perimeter is $\boxed{2R + \frac{2\pi R}{5}}$ (see diagram).



2.



a) ABD is a 30-60-90 triangle, so $AD = \frac{3}{2}\sqrt{3}$ (or use Pythagorean Theorem: $AD = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = 3\sqrt{\frac{3}{4}} = \frac{3}{2}\sqrt{3}$.) Therefore area $ABC = \frac{1}{2} \times 3 \times \frac{3}{2}\sqrt{3} = \boxed{\frac{9}{4}\sqrt{3}}$.

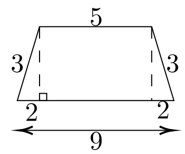
b) $h = 2$ (30-60-90 triangle), therefore area is $\frac{1}{2} \times 4 \times 2 = \boxed{4}$ (could also use trigonometry).

3.

a) Altitude is $\frac{\sqrt{3}}{2}$, since we have a 30-60-90 triangle area is $\boxed{\frac{3\sqrt{3}}{2}}$.

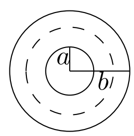
b) $h = 3$ by Pythagorean Theorem therefore area is $3 \times 16 = \boxed{48}$.

4. $h = \sqrt{3^2 - 2^2} = \sqrt{5}$ by the Pythagorean Theorem. Therefore Area is the area of the rectangle plus the two triangles so $5\sqrt{5} + 2\frac{2\sqrt{5}}{2}$ or $h \times$ average of bases $= \boxed{7\sqrt{5}}$ (formula for area of trapezoid).

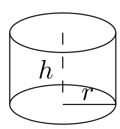


5. Let $a =$ side of square, $r =$ radius of circle; then $2\pi r = 4a \therefore r = \frac{2a}{\pi}$ so $\frac{\text{area square}}{\text{area circle}} = \frac{a^2}{\pi\frac{4a^2}{\pi^2}} = \boxed{\frac{\pi}{4}}$.

6. $A = \pi b^2 - \pi a^2 = \pi(b^2 - a^2) = \pi(b+a)(b-a) = 2\pi\left(\frac{b+a}{2}\right)(b-a)$. [$2\pi\left(\frac{b+a}{2}\right)$ is the perimeter of median circle and $(b-a)$ is the thickness of the ring.]



7. $r = \frac{h}{3}$, $V = \pi r^2 h = \frac{\pi h^2}{9}h = 24\pi$. Therefore $h^3 = 9 \times 24$, $h = \boxed{6}$, $r = \boxed{2}$.



8.

a) Surface area is $4\pi r^2 = 12\pi \therefore r = \sqrt{3}$, Volume is $\frac{4}{3}\pi r^3 = \boxed{4\pi\sqrt{3}}$.

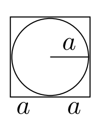
b) $\frac{S}{V} = \frac{4\pi R^2}{\frac{4}{3}\pi R^3} = \boxed{\frac{3}{R}}$.

c) Surface area hemisphere is $\frac{4\pi R^2}{2} + \pi R^2 = \boxed{3\pi R^2}$.

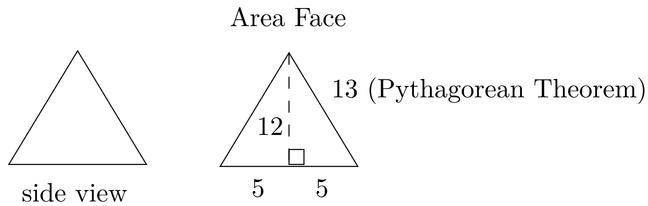
d) This is $\frac{1}{8}$ of both surface area and volume. So surface area is $\boxed{\frac{\pi R^2}{2}}$ and volume is $\boxed{\frac{\pi R^3}{6}}$.

9. Edge of cube an diameter of sphere are of the same length. $\frac{\text{area sphere}}{\text{area cube}} = \frac{4\pi a^2}{6(2a)^2} = \boxed{\frac{\pi}{6}}$.

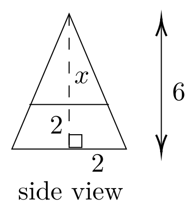
$\frac{\text{volume sphere}}{\text{volume cube}} = \frac{\frac{4}{3}\pi a^3}{(2a)^3} = \boxed{\frac{\pi}{6}}$.



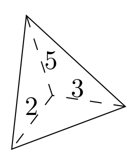
10. Volume is $\frac{1}{3}$ base times height so $\frac{1}{3} \times 100 \times 12 = \boxed{400}$. Area is $\frac{1}{2} \times 10 \times 13 = \boxed{65}$.



11. vol. slice = vol. cone − vol. top cone $= \frac{1}{3}\pi 2^2 \times 6 - \frac{1}{3}\pi\left(\frac{4}{3}\right)^2 \times 4 = \frac{\pi}{3}\left(24 - \frac{64}{9}\right) = \boxed{\dfrac{8\pi}{3} \times \dfrac{19}{9}}$.
vol. cone $= \frac{1}{3}$ base times ht. By similar triangles $\frac{x}{2} = \frac{4}{6} \therefore x = \frac{4}{3}$.



12. Volume is $\frac{1}{3}$ base times height. $\frac{1}{3} \times \frac{2\times 3}{2} \times 5 = \boxed{5}$.



G.IV Lines.

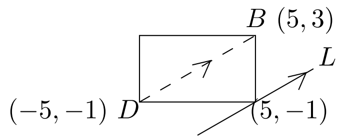
1. Point-slope equation of line is $y + 3 = 6(x - 2)$. Crosses x-axis when $y = 0$: i.e., $6(x-2) = 3$, so $x - 2 = \frac{1}{2}$. $x = \boxed{\dfrac{5}{2}}$.

2. Solve simultaneeously $3x + 4y = 12$ and $3x - 5y = -6$. Subtracting, $9y = 18$, so $y = 2$, $x = \frac{4}{3}$. So $\boxed{(4/3, 2)}$.

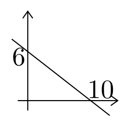
3. Slope -1, through $(2,3)$: $y - 5 = -(x - 2)$, or $y = -x + 7$. Slope 2, y-intercept -2: $y = 2x - 2$. Solving simultaneously: $2x - 2 = -x + 7 \therefore x = 3,\ y = 4$. Ans: $\boxed{(3,4)}$.

4. $2x - 5y = 1 \to y = \frac{2}{5}x - \frac{1}{5}$ therefore slope is $\frac{2}{5}$. So slope of line perpendicular is $-\frac{5}{2}$ (negative reciprocal). Since y-intercept is 3, its equation is $\boxed{y = -\dfrac{5}{2}x + 3}$.

5. Slope of $BD = \frac{3-(-1)}{5-(-5)} = \frac{4}{10} = \frac{2}{5}$, therefore L has equation $\boxed{y+1 = \frac{2}{5}(x-5)}$ (or $5y - 2x + 15 = 0$).

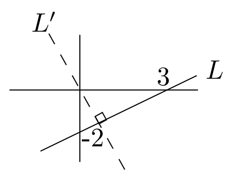


6. The x-intercept is where $y = 0$, so $3x = 30$, $x = 10$. y-intercept is where $x = 0$. so $5y = 30$, $y = 6$. Therefore area of the triangle is $\frac{1}{2} \times 6 \times 10 = \boxed{30}$.



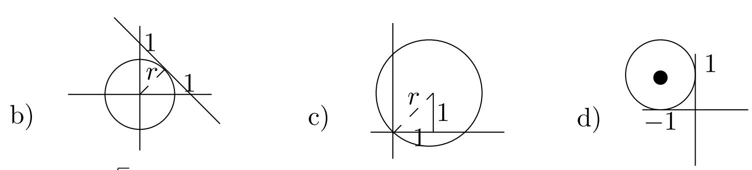
$3x + 5y = 30$

7. L has slope $\frac{2}{3}$. Therefore L' has slope $-\frac{3}{2}$. $\boxed{y = -\frac{3}{2}x}$.



G.V Circles.

1.



$r = \frac{\sqrt{2}}{2}\ \therefore r^2 = \frac{1}{2}$ $\quad$ $r = \sqrt{2}\ \therefore r^2 = 2$ $\quad$ $r = 1$

a) $x^2 + y^2 - 9 = 0$.
b) $x^2 + y^2 - \frac{1}{2} = 0$.
c) $(x-1)^2 + (y-1)^2 = 2 \therefore x^2 + y^2 - 2x - 2y = 0$.
d) $(x+1)^2 + (y+1)^2 = 1^2 \therefore x^2 + y^2 + 2x - 2y + 1 = 0$.

2. The equation is $x^2 + y^2 = 11^2$. Inside is where $x^2 + y^2 < 11^2$, outside is where $x^2 + y^2 > 11^2$ ($11^2 = 121$). Here $7^2 + 8^2 = 113 < 121$. Therefore $(7,8)$ is $\boxed{\textit{inside}}$ the circle.

3.

a) $x^2 + y^2 = 5$, $x + y = 3$. Solving simultaneously: $y = 3 - x$; substitute with equation of circle $x^2 + (3-x)^2 = 5$, $2x^2 - 6x + 4 = 0$ or $x^2 - 3x + 2 = 0$, $(x-2)(x-1) = 0$. Therefore $x = 2$ and $y = 1$ and $x = 1$ and $y = 2$ are the solutions. Answer: $\boxed{(2,1) \text{ and } (1,2)}$.

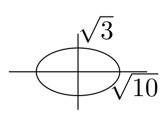
b) $x^2 + y^2 = 10$, $x + 3y = 10$ gives $(10-3y)^2 + y^2 = 10$, $100 - 60y + 10^2 = 10$ or $y^2 - 6y + 9 = 0$, $(y-3)^2 = 0$. Therefore $x = 1$, $y = 3$ is the solution. $\boxed{(1,3)}$.

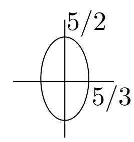
c) $x^2 + y^2 = 9$ (circle), $y = 3x$ (line). $x^2 + 9x^2 = 9$, $10x^2 = 9$, therefore $x = \pm\frac{3}{\sqrt{10}}$, $y = \pm\frac{9x}{\sqrt{10}}$.

$$\boxed{\left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right), \left(-\frac{3}{\sqrt{10}}, -\frac{9}{\sqrt{10}}\right)}.$$

G.VI Conics.

1.

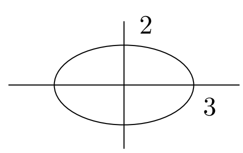
a) 

b) 

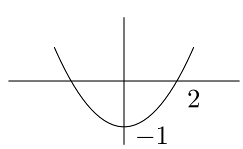
a) Intercepts are when $x = 0$, $y = \pm\sqrt{3}$ and $y = 0$, $x = \pm\sqrt{10}$. Therefore semi-major axis is $\boxed{\sqrt{10}}$, semi-minor axis $\boxed{\sqrt{3}}$.

b) Intercepts are at $x = 0$, $y = \pm\frac{5}{2}$ and $y = 0$, $x = \pm\frac{5}{3}$. Therefore semi-major axis is $\boxed{\dfrac{5}{2}}$, semi-minor axis is $\boxed{\dfrac{5}{3}}$.

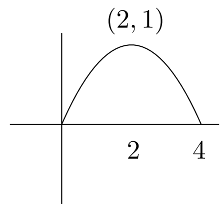
2. $\boxed{\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1}$, or $\boxed{\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1}$.



3. Equation is $y = ax^2 - 1$ (parabola $y = ax^2$ moved down 1 unit). Choose A so it goes through $(2,0)$: $0 = 4a - 1 \therefore a = \frac{1}{4}$. $\boxed{y = \dfrac{x^2}{4} - 1}$. [Another way: since roots are at ± 2, equation is $y = c(x+2)(x-2) = c(x^2 - 4)$ choose c so that $(0,-1)$ is on parabola.]



4. Since high point is at $x = 2$ and it goes through the origin, the other intercept is $x = 4$. (High point is midway between the two x-intercepts). Therefore $y = ax(x-4)$ choose a so $1 = 2a(2-4) \therefore a = -\frac{1}{4}$. $\boxed{y = -\dfrac{1}{4}x(x-4)}$.



5. The lines are $y + 2x = 0$ and $y - 2x = 0$; taken together, their equation is $(y+2x)(y-2x) = 0$ or $y^2 - 4x^2 = 0$. Equation os hyperbola is then $\boxed{y^2 - 4x^2 = c}$ (c can be positive or negative but not 0).

6. a) ellipse b) two lines c) circle d) parabola e) hyperbola

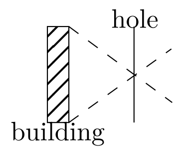
7. a) hyperbola b) circle c) parabola d) two lines e) no locus

8. a) hyperbola b) parabola c) hyperbola d) ellipse e) two lines

9. a) circle b) hyperbola c) point $(0, 0)$ d) parabola e) two lines

Geometry and Analytic Geometry Diagnostic Test #1

1. The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building?



2. An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of 100°. What are the angles of the arcs cut off from the other side?

3. The radius of a right circular cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?

4. One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?

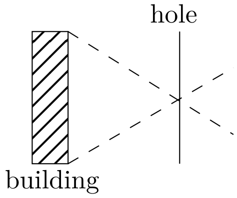
5. A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside of the circle?

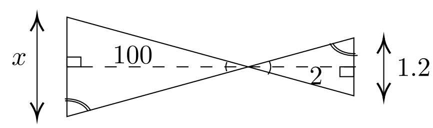
6. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

(a) $x^2 + 2y^2 - 12 = 0$.
(b) $6x^2 - 5y^2 = 0$.
(c) $15 - x^2 - y^2 = 0$.
(d) $xy = 0$.
(e) $zy = 2$.

Solutions: Geometry and Analytic Geometry Diagnostic Test #1

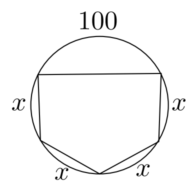
1. The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building?





Triangles are similar (indicated angles are equal) therefore corresponding sides proportional. $\frac{x}{100} = \frac{1.2}{2}$, $x = .6 \times 100 = \boxed{60}$ meters.

2. An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of 100°. What are the angles of the arcs cut off from the other side?



$100 + 4x = 360$, $x = \frac{260}{4} = \boxed{65^\circ}$

3. The radius of a right circular cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?
$r = \frac{1}{3}h$, $V = \pi r^2 h = 24\pi$, therefore $\pi\frac{h^2}{9} \times h = 24\pi$, $h^3 = 9 \times 24 = 3^2 \times (3 \times 2^3)$. So $h = \boxed{6}$, $r = \boxed{2}$.

4. One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?
First line: $y - 5 = -(x - 2)$ or $x + y = 7$. Solve: $x + y = 7$ with $2x - y = 2$. Adding: $3x = 9$, $x = 3 \therefore y = 4$. Answer: $\boxed{(3, 4)}$.

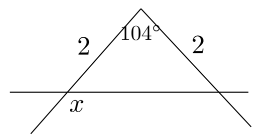
5. A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside of the circle?
Circle is: $x^2 + y^2 = 11^2$. (x, y) inside if $x^2 + y^2 < 11^2$, outside id $x^2 + y^2 > 11^2$. Hence $7^2 + 8^2 < 11^2$, since $49 + 64 = 113 < 121$. Therefore it is $\boxed{\textit{inside}}$.

6. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

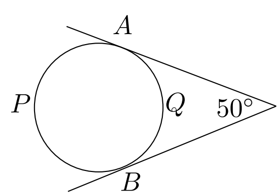
(a) $x^2 + 2y^2 - 12 = 0$. $\boxed{\textit{Ellipse}}$.
(b) $6x^2 - 5y^2 = 0$. $\boxed{\textit{Two lines}}$.
(c) $15 - x^2 - y^2 = 0$.$\boxed{\textit{Circle}}$.
(d) $xy = 0$.$\boxed{\textit{Two lines}}$.
(e) $zy = 2$.$\boxed{\textit{Hyperbola}}$.

Geometry and Analytic Geometry Diagnostic Test #2

1. What is x in degrees?

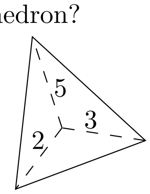


2. Two distinct lines are both tangent to a circle. They meet outside of the circle at an angle of 50°. What are the central angles subtended by the two arcs they cut off on the circle?



$\widehat{APB} = ?$
$\widehat{AQB} = ?$

3. A tetrahedron has three edges which all intersect at right angles. Those edges have length 2, 3, and 5. What is the volume of the tetrahedron?



4. A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?

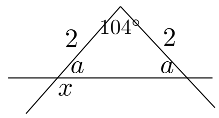
5. Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.

6. What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, point, or no locus?

(a) $4y^2 - x^2 = 2$.
(b) $x^2 + y^2 + 2x - 5 = 0$.
(c) $y^2 + 2x = 5$.
(d) $2y - 4x^2 = 3$.
(e) $x^2 + 3y^2 = 6$.

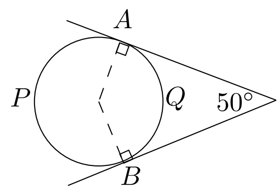
Solutions: Geometry and Analytic Geometry Diagnostic Test #2

1. What is x in degrees?



Triangle is isosceles so the two angles marked a are equal. $2a + 104 = 180 \therefore a = 38$. Then $x = 180 - a = \boxed{142°}$ (or $x = 104 + a = 142°$).

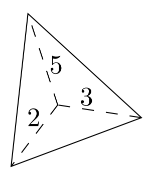
2. Two distinct lines are both tangent to a circle. They meet outside of the circle at an angle of 50°. What are the central angles subtended by the two arcs they cut off on the circle?



$\widehat{APB} = ?$
$\widehat{AQB} = ?$

Draw in dotted lines (they are perpendicular to the tangents).
$\widehat{AQB} = 360 - (90 + 90 + 50) = \boxed{130°}$. $\widehat{APB} = 360 - 130 = \boxed{230°}$

3. A tetrahedron has three edges which all intersect at right angles. Those edges have length 2, 3, and 5. What is the volume of the tetrahedron?



Volume is $\frac{1}{3}$ base times height. $\frac{1}{3} \times \frac{3\times 2}{2} \times 5 = \boxed{5}$.

4. A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?
Line is: $y + 3 = \sqrt{3}(x - 2)$ crosses x-axis where $y = 0$. Solving for x: $\sqrt{3}(x - 2) = 3 \therefore x - 2 = \sqrt{3}$, $x = \boxed{2 + \sqrt{3}}$.

5. Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.
Solve simultaneously: $x^2 + y^2 = 2$ and $x + y - 1 = 0$. Eliminate y: $y = 1 - x$ from eqn. 2. Substituting into eqn. 1: $x^2 + (1 - x)^2 = 2$ or $2x^2 - 2x - 1 = 0$. Therefore $x = \frac{2\pm\sqrt{4+8}}{4} = (1 \pm \sqrt{3})/2 \therefore y = (1 \mp \sqrt{3})/2$. So $\boxed{\left(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)}$ and $\boxed{\left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)}$.

6. What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, point, or no locus?

(a) $4y^2 - x^2 = 2$. $\boxed{Ellipse}$

(b) $x^2 + y^2 + 2x - 5 = 0$. $\boxed{Circle}$ [complete square: $(x+1)^2 + y^2 = 6$.]

(c) $y^2 + 2x = 5$. $\boxed{Parabola}$.

(d) $2y - 4x^2 = 3$. $\boxed{Parabola}$.

(e) $x^2 + 3y^2 = 6$. $\boxed{Ellipse}$.

Self-Evaluation Summary

You may want to informally evaluate your understanding of the various topic areas you have worked through in the *Self-Paced Review.* If you meet with tutors, you can show this evaluation to them and discuss whether you were accurate in your self-assessment.

For each topic which you have covered, grade yourself on a one to ten scale. One means you completely understood the topic are able to solve all the problems without any hesitation. Ten means you couldnot solve any problems easily without review.

G. Geometry and Analytic Geometry Review

Geometry

______	G.I	Triangles: Similarity of Figures
______	G.II.1	Quadrilaterals
______	G.II.2	Other Polygons
______	G.III	Circles
______	G.IV	Lengths, Areas, & Volumes

Analytic Geometry

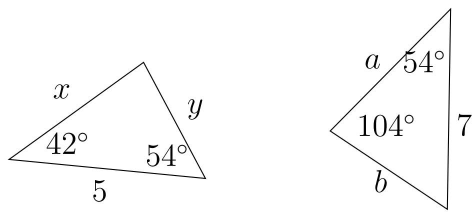
______	G.V	Straight Lines
______	G.VI.1	Conic Sections: Circles
______	G.VI.2	Conic Sections: Parabolas
______	G.VI.3	Conic Sections: Ellipses
______	G.VI.4	Conic Sections: Hyperbolas

Part I

Geometry

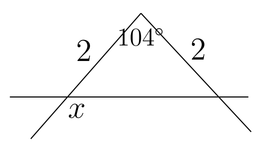
1 Triangles

1. What is a and b in terms of x and y?

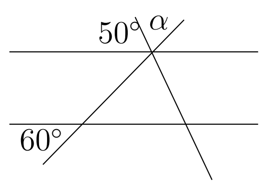


2. The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building?

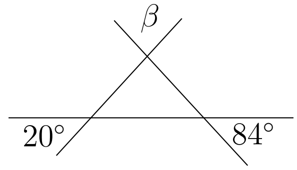
3. What is x in degrees?



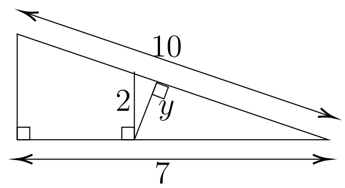
4. Two different lines cross two parallel lines as shown. What is α in degrees?



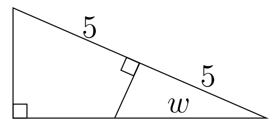
5. What is angle β in degrees?



6. Find the length of y.

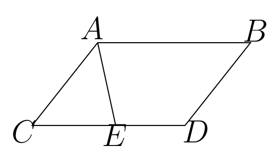


7. Find the length of w.

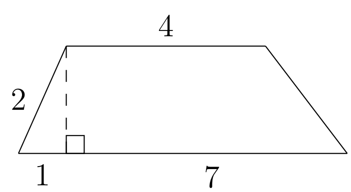


2 Some Other Plane Figures

1. Line segment AB is parallel to line segment CD, and both have length 16. Also E is the midpoint of CD, and $AC = AE = 5$. What is the area of the quadrilateral $ABCD$?



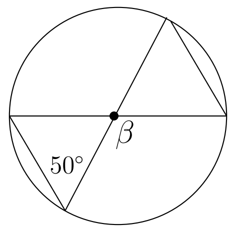
2. What is the area of the trapezoid?



3 The All-Mighty Circle

1. A circle with diameter 10, a chord of length 6 is bisected by a diameter. The chord divides that diameter into two parts. What are the lengths of those two parts?

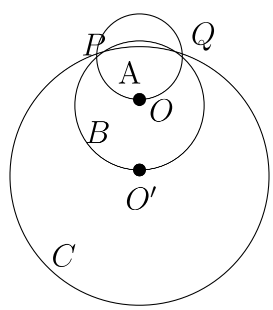
2. What is β in degrees?



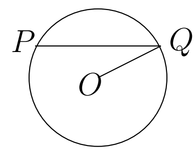
3. Two circles are tangent to each other, and a line passes through their point of intersection. The angle between that line and the line that connects the two circles' centers is 50°. The line cuts each of the circles into two arcs. What are the angles that subtend these arcs?

4. An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of 100°. What are the angles of the arcs cut off from the other side?

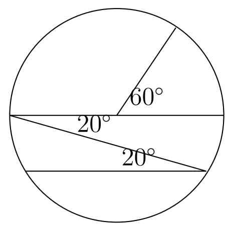
5. Circle A passes through the center O of circle B, which in turn passes through the center O' of circle C. The three circles all intersect in the same two points P and Q, as shown below. If the arc PQ in circle A subtends a central angle of 60°, what central angle does the arc PQ subtend in circle C?



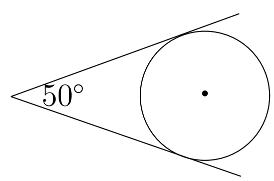
6. The points P and Q divide a circle with center O into two arcs, whose lengths are in the ratio of 1:2. How many degrees in angle PQO?



7. What is $x + y$?



8. Two distinct lines are both tangent to a circle. They meet outside of the circle at an angle of 50°. What are the central angles subtended by the two arcs they cut off on the circle?



9. A line L is tangent at point P to circle C of radius 6. Point Q on L is at distance of 8 from P. What is the distance between Q and the center of the circle?

4 Formulas

1. A square and a circle have the same perimeter. What is the ratio between their respective areas?

2. An equilateral triangle has sides of length 3. What is its area?

3. A solid sphere of radius R is cut by a plane into two equal half-spheres. What is the total surface area of one of the half-spheres?

4. A cube has the same area as a sphere. What is the ratio between E, one of the cube's edges, and r, the radius of the sphere?

5. A disk of radius r_1 has a disk of radius r_2 cut out from it. What is the area of the ring-shaped region that is left?

6. If a sphere has surface area of 12π, what is its volume?

7. Object A has length 10 centimeters and a volume of 25 cubic centimeters. On the planet Doon, length is measured in units called orbs. If A has a volume of 200 cubic orbs, what is the length of A in orbs?

8. The cost of an amuso ball is given by the formula $17r^a$ where r is its radius in meters. If the cost is proportional to the cube of the surface area, what is a?

9. Two sides of a triangle each have length 4, and their included angle is $30°$. What is the area of the triangle?

10. A trapezoid has parallel sides of lengths 5 and 9, and its other two sides both have length 3. What is the area of the trapezoid?

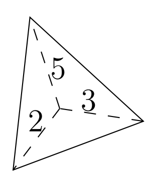
11. The radius of a cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?

12. A regular hexagon and a square have the same perimeter. What is the ratio between the square's area and the hexagon's area?

13. A pyramid has a square base with sides of length 10, and its height is 12. What is the area of one of its faces?

14. A sphere inside of a cubical box is tangent to each of the faces of the box. What fraction of the volume inside the box lies outside the sphere?

15. A tetrahedron has three edges which all intersect at right angles. Those edges have length 2, 3, and 5. What is the volume of the tetrahedron?



Part II
Analytic Geometry

1 Straight Lines

1. A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?

2. A line has the equation $(a+1)x + (a^2 - 1)y + 5 = 0$. Express its slope and y-intercept in terms of the constant a.

3. One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?

4. Line A has slope 17 and is perpendicular to line B, which is parallel to line C. What is the slope of C?

5. What is the slope of a line which is perpendicular to the line described by $2x - 5y + 3 = 0$ intersect?

7. For the following collections of lines, find the area of the finite region that they bound:

$$y = 0,\ x = 0,\ 12x + 5y = -2.$$

8. Find the area of the quadrilateral whose top and bottom are given by $y = 0$, $y = 1$ and whose sides are given by

$$y = \frac{1}{2}x,\ 2y = x - 6.$$

2 Circles

1. What is the radius of the circle defined by $4x^2 + 4y^2 - 49 = 0$?

2. A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside the circle?

3. Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.

4. Find the center and radius of the circle described by

$$4x^2 + 4y^2 + 4x - 20y - 23 = 0?$$

5. At which points does the line $2x + y + 1 = 0$ intersect the circle

$$x^2 + y^2 - 5 = 0?$$

6. Give in the form $x^2 + y^2 + ax + by + c = 0$ the equation of a circle whose center is at $(1, 1)$ and having a radius 2.

3 Conic Sections

1. What is the length of the semi-major axis of the ellipse given by

$$\frac{x^2}{10} + \frac{y^2}{3} = 1?$$

2. What is the length of the semi-minor axis of the ellipse given by

$$\frac{x^2}{24} + \frac{y^2}{50} = 1?$$

3. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

(a) $x^2 + 2y^2 - 12 = 0$.

(b) $6x^2 - 5y^2 = 0$.

(c) $15 - x^2 - y^2 = 0$.

(d) $xy = 0$.

(e) $xy = 2$.

4. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

(a) $4y^2 + x^2 = 2$.

(b) $x^2 + y^2 + 2x - 5 = 0$.

(c) $y^2 + 2x = 5$.

(d) $2y - 4x^2 = 3$.

(e) $x^2 + 3y^2 = 6$.

5. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

(a) $12x^2 + 12y^2 = 1$.

(b) $4x^2 - y^2 = 1$.

(c) $x^2 - 4y + x = 7$.

(d) $7x^2 + 4y^2 = 0$.

(e) $2x^2 + 3y^2 = 5$.

6. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

(a) $x^2 + y = 0$.

(b) $x^2 - y^2 = 4$.

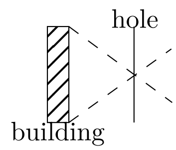
(c) $x^2 - 4y^2 = 0$.

(d) $xy = 5$.

(e) $x^2 + y^2 + 9 = 0$.

Geometry and Analytic Geometry Diagnostic Test #1

1. The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building?



2. An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of $100°$. What are the angles of the arcs cut off from the other side?

3. The radius of a right circular cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?

4. One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?

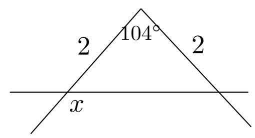
5. A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside of the circle?

6. What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

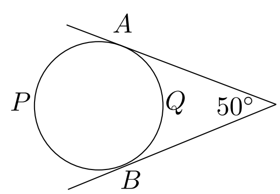
(a) $x^2 + 2y^2 - 12 = 0$.
(b) $6x^2 - 5y^2 = 0$.
(c) $15 - x^2 - y^2 = 0$.
(d) $xy = 0$.
(e) $zy = 2$.

Geometry and Analytic Geometry Diagnostic Test #2

1. What is x in degrees?

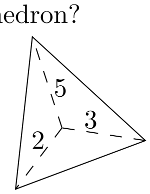


2. Two distinct lines are both tangent to a circle. They meet outside of the circle at an angle of 50°. What are the central angles subtended by the two arcs they cut off on the circle?



$\widehat{APB} = ?$
$\widehat{AQB} = ?$

3. A tetrahedron has three edges which all intersect at right angles. Those edges have length 2, 3, and 5. What is the volume of the tetrahedron?



4. A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?

5. Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.

6. What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, point, or no locus?

(a) $4y^2 - x^2 = 2$.
(b) $x^2 + y^2 + 2x - 5 = 0$.
(c) $y^2 + 2x = 5$.
(d) $2y - 4x^2 = 3$.
(e) $x^2 + 3y^2 = 6$.