# Differentiation

*D.1 Differentiation*

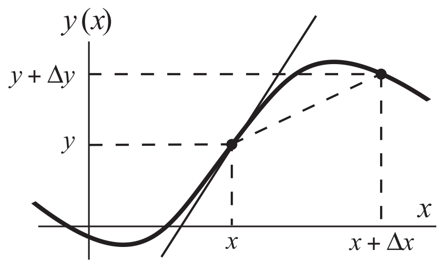
*D.1.1 Definition of a Derivative*

Consider any continuous function defined by $y = f(x)$ where $y$ is the dependent variable, and $x$ is the independent variable. When the independent variable $x$ changes by $\Delta x$, there is a corresponding change $\Delta y$ in the dependent variable $y$. The limit as $\Delta x \to 0$ of the ratio $\Delta y / \Delta x$, is called the *derivative of $y$ with respect to $x$*, (provided it exists) and is denoted by the symbol

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

This limit is also written as $y'(x)$, or just $y'$.

Geometrically, $dy/dx$ can be found by drawing a straight line through the point $(x, y)$ and the point $(x + \Delta x, y + \Delta y)$ as shown.




The slope of that line is given by $\Delta y / \Delta x$, and $dy/dx$ is the slope of the tangent line to the curve at $(x, y)$.

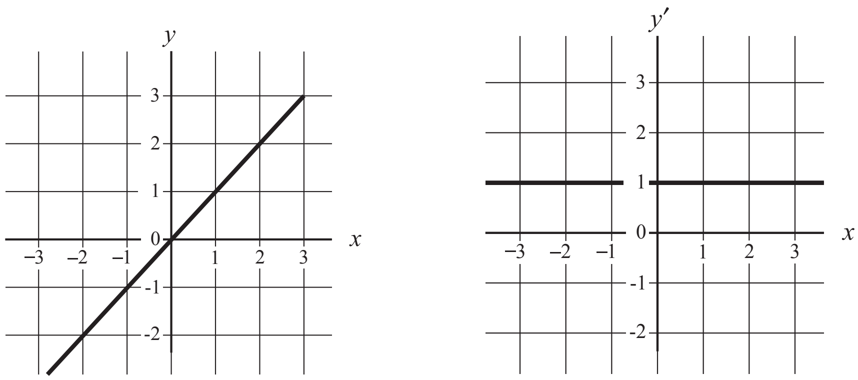
*D.1.2 Derivative of a Function*

When the dependent variable $y$ is given by a function $f(x)$, the derivative can be written as $df / dx$, or more simply as $f'$. The symbol $\dfrac{d}{dx}$ is a derivative operator, operating on the function $f$. Thus $\dfrac{d}{dx}(\quad)$ means differentiate with respect to $x$ whatever function $f(x)$ happens to be in the parentheses. In order to calculate $f'$, obtain an expression for $\Delta f = f(x + \Delta x) - f(x)$, and then evaluate the limit

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

*D.1.3 Graphs of Functions and Their Derivatives*

The graphs of the function $y = x$ and its derivative $dy/dx$ are plotted in the figures below. Because the slope of $y$ is positive and equal to one, $y' = 1$.

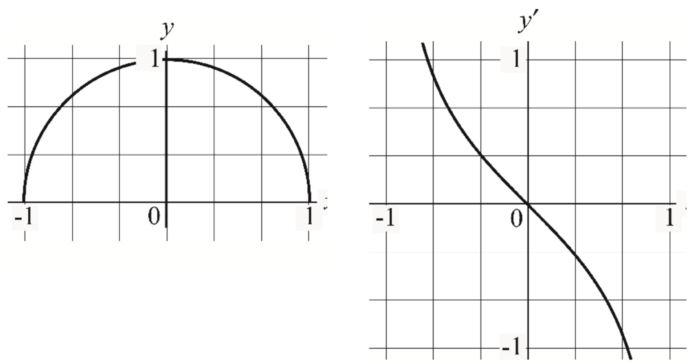



To prove that $\frac{d}{dx}x = 1$, let $y(x) = x$. Then $\Delta y = y(x + \Delta x) - y(x) = x + \Delta x - x = \Delta x$. Hence,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

*D.1.4 Example Semicircle*

Consider the semicircle. The plots of $y$ and $y'$ are shown below.




The slope of the semicircle does not behave nicely at the extreme values of $x$. The tangent line to the curve at $x = 0$, is parallel to the $x$-axis, with slope equal to zero. Thus, $y' = 0$ at $x = 0$. For $x > 0$, a line tangent to the curve has negative slope, so $y' < 0$. As $x$ approaches 1, the tangent becomes increasingly steep, and $y'$ becomes increasingly negative. In fact, as $x \to 1$, $y' \to -\infty$. For $x < 0$, as $x$ approaches $-1$, the tangent becomes increasingly steep, and $y'$ becomes increasingly positive and as $x \to -1$, $y' \to \infty$.

*D.1.5 Derivatives of Polynomials*

*D.1.5.1 Example Derivative of $y = a$*

Consider the constant function $y = a$. To find $y'$, use the fact that $y(x+\Delta x) = a$ in the definition of the derivative

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a-a}{\Delta x} = 0 .$$

*D.1.5.2 Example Derivatives of $y = ax$*

The derivative of a linear function $y = ax$ is found as follows: Because $y(x+\Delta x) - y(x) = a(x+\Delta x) - ax = (ax + a\Delta x) - ax = a\Delta x$, therefore

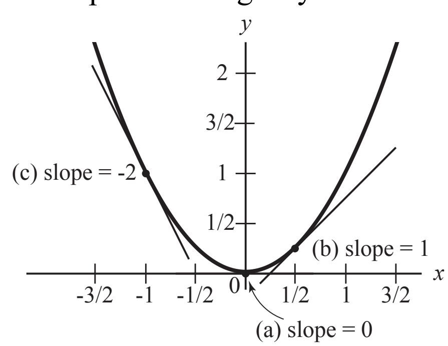
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{a\,\Delta x}{\Delta x} = a.$$

*D.1.5.3 Example Derivatives of $f(x) = x^2$*

The derivative of a quadratic function $f(x) = x^2$ is found as follows. The difference $f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2$, so

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x} = 2x .$$

To illustrate this, a graph of $y(x) = x^2$ is drawn in the figure. Because the slope of the curve at a point is simply the derivative at that point, each of the straight lines tangent to the curve has a slope equal to the derivative evaluated at the point of tangency.




The tangent through the origin has a slope of (2)(0) = 0. Line (*b*) passes through the point $x = ½$, and has slope (2)(½) = 1. Line (*c*) passes through the point $x = -1$, and has slope (2)(−1) = −2.

*D.1.5.4 General Polynomial Case*

The derivative of $x^n$, where $n$ is any real number, is given by

$$\frac{d}{dx}x^n = nx^{n-1}.$$

*D.1.5.5 Example* $y = 1/x^2$

Find $dy/dx$ for $y = 1/x^2$. Use the general rule: $\frac{d}{dx}x^n = nx^{n-1}$. Thus $y(x) = 1/x^2 = x^{-2}$; here $n = -2$, therefore $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$.

*D.1.5.6 Example* $y = x^{2/3}$

Find $dy/dx$ for $y = x^{2/3}$. For $y = x^{2/3}$; $n = 2/3$, hence $\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$.

*D.2 Differentiation Rules*

Let $u(x)$ and $v(x)$ stand for any two functions that depend on $x$ in the following sets of rules for differnentiation.

*D.2.1.1 Sum Rule*

Our first rule will let us evaluate the derivative of the sum of $u(x)$ and $v(x)$ in terms of their derivatives. We will derive the rule here. Let $y(x) = u(x) + v(x)$. Then

$$\begin{aligned}\frac{d}{dx}(u+v) &= \lim_{\Delta x \to 0}\frac{[u(x+\Delta x)+v(x+\Delta x)-u(x)-v(x)]}{\Delta x} \\ &= \lim_{\Delta x \to 0}\frac{[u(x+\Delta x)-u(x)]}{\Delta x} + \lim_{\Delta x \to 0}\frac{[v(x+\Delta x)-v(x)]}{\Delta x}. \\ &= \frac{du}{dx}+\frac{dv}{dx}.\end{aligned}$$

*D.2.1.1.1 Example Differentiate* $y = x^4 + 8x^3$

In order to find the derivative of the function $y = x^4 + 8x^3$, let $u(x) = x^4$, $v(x) = 8x^3$. Then

$$\frac{d}{dx}(u+v) = \frac{d}{dx}(x^4+8x^3) = \frac{d}{dx}x^4 + \frac{d(8x^3)}{dx} = 4x^3 + 24x^2.$$

*D.2.1.2 Product Rule*

The rule for the differentiation of the product of two functions, $f(x) = u(x)v(x)$, is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'.$$

*D.2.1.2.1 Example Differentiate* $y(x) = (x^5 + 7)(x^3 + 17x)$.

In order to find the derivative of the function $y(x) = (x^5 + 7)(x^3 + 17x)$, let $u(x) = x^5 + 7$ and $v(x) = x^3 + 17x$, then $y(x) = u(x)v(x)$. The product rule is then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^5 + 7)\frac{d}{dx}(x^3 + 17x) + (x^3 + 17x)\frac{d}{dx}(x^5 + 7).$$

Because $du/dx = 5x^4$ and $dv/dx = 3x^2 + 17$, the result is

$$\frac{dy}{dx} = (x^5 + 7)(3x^2 + 17) + (x^3 + 17x)(5x^4).$$

*D.2.1.3 Quotient Rule*

The rule for differentiating the quotient of two functions, $u(x)/v(x)$, is

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2} = \frac{vu' - uv'}{v^2}.$$

*D.2.1.3.1 Example Differentiate* $y(x) = (1 + x)/x^2$.

Find the derivative of the function $y(x) = (1 + x)/x^2$. Let $u(x) = 1 + x$, $v(x) = x^2$. Then $du/dx = 1$, $dv/dx = 2x$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} = \frac{x^2 - (1 + x)(2x)}{x^4} = \frac{x^2 - 2x - 2x^2}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2}.$$

*D.2.1.4 Chain Rule*

Suppose $f(u)$ is a function that depends on $u$, and $u(x)$ in turn depends on $x$. Then $f(u(x))$ also depends on $x$. The rule for calculating the derivative of $f(u(x))$, called the *chain rule*, is

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

*D.2.1.4.1 Example Differentiate* $f(x) = (x + x^2)^2$.

In order to differentiate $f(x) = (x + x^2)^2$, let $u(x) = (x + x^2)$, in which case $f = u^2$. Then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{du^2}{du}\frac{du}{dx} = 2u\frac{du}{dx}.$$

Because $\frac{du}{dx} = 1 + 2x$, the derivative is then

$$\frac{df}{dx} = 2(x+x^2)(1+2x).$$

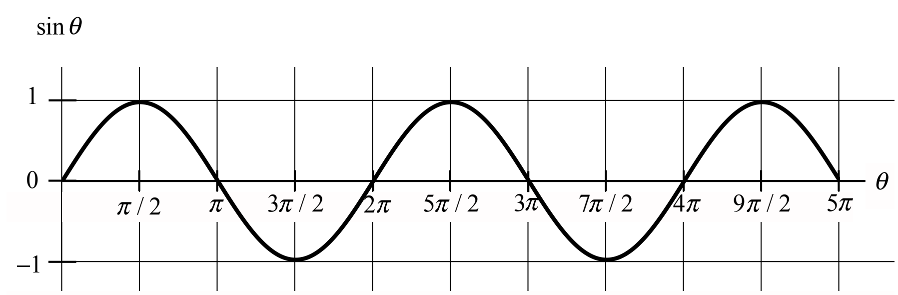
*D.2.1.4.2 Example Differentiate* $f(v(x)) = \dfrac{1}{v(x)}$.

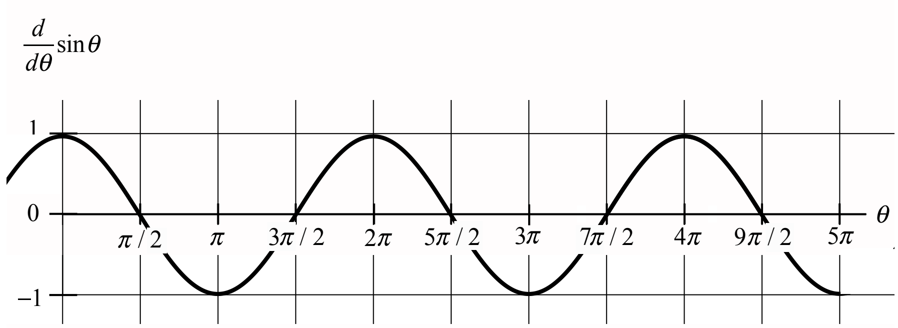
Use the chain rule to derive $\dfrac{d}{dx}\left(\dfrac{1}{v}\right)$ in terms of $v$ and $\dfrac{dv}{dx}$, where $v(x)$ depends on $x$. To find $\dfrac{d}{dx}\left(\dfrac{1}{v}\right)$, apply the chain rule as follows. Let $f = \dfrac{1}{v} = v^{-1}$ with $\dfrac{df}{dx} = \dfrac{df}{dv}\dfrac{dv}{dx}$, where $\dfrac{df}{dv} = \dfrac{d}{dv}v^{-1} = -\dfrac{1}{v^2}$. Thus $\dfrac{d}{dx}\left(\dfrac{1}{v}\right) = -\dfrac{1}{v^2}\dfrac{dv}{dx}$.

*D.3 Differentiating Trigonometric Functions*

Trigonometric functions occur in so many applications that it is useful to know their derivatives; for example $\dfrac{d}{d\theta}\sin\theta$. By definition,

$$\frac{d}{d\theta}\sin\theta = \lim_{\Delta\theta \to 0}\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\Delta\theta}.$$



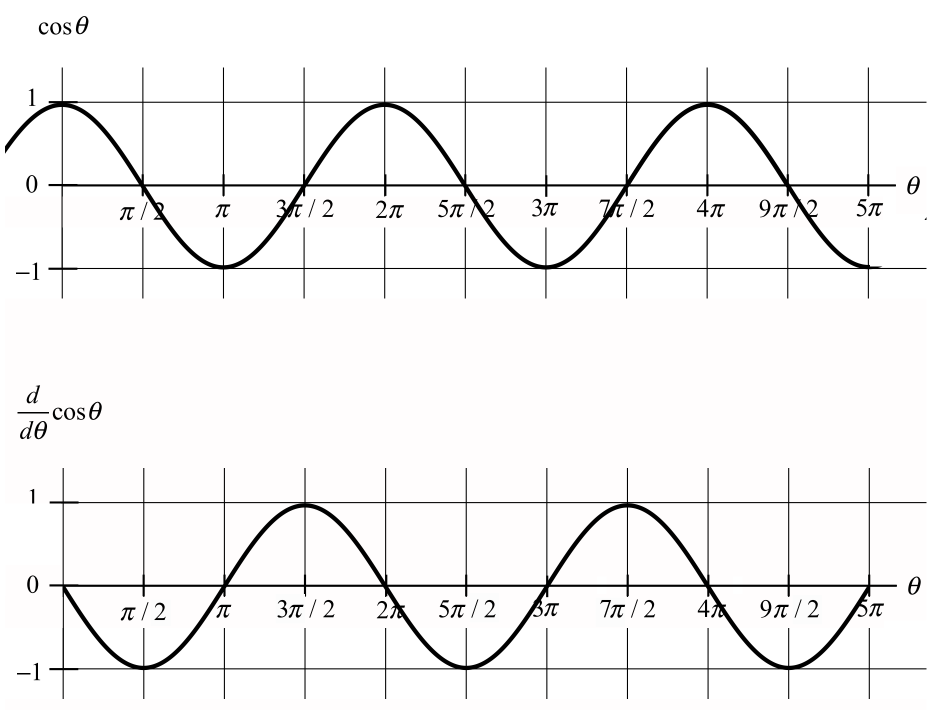



The plots of $\sin\theta$ and $\frac{d}{d\theta}\sin\theta$ over the interval $0 \le \theta \le 5\pi$ are shown in the figure above, ($\theta$ is measured in radians). Note that where the slope of $\sin\theta$ is greatest, at 0 and $2\pi$, $\frac{d}{d\theta}\sin\theta$ has its greatest value, and that where the slope is 0, at $\theta = \pi/2$ and $\theta = 3\pi/2$, $\frac{d}{d\theta}\sin\theta$ is 0. By looking at the graphs, the derivative is

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

This relation is true *only* when the angle is measured in radians—this is why the radian is such a useful unit.

The plots of $\cos\theta$ and $\frac{d}{d\theta}\cos\theta$, shown in the figure below illustrate the result that $\frac{d}{d\theta}\cos\theta = -\sin\theta$,




To summarize:

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

*D.3.1 Example Differentiate* $\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right)$.

Use the quotient rule:

$$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta\frac{d(\sin\theta)}{d\theta} - \sin\theta\frac{d(\cos\theta)}{d\theta}}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

*D.3.2 Example Differentiate* $\frac{d}{d\theta}\sec\theta$.

Use the definition $\sec\theta = \frac{1}{\cos\theta}$. Therefore

$$\frac{d}{d\theta}\sec\theta = \frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right) = -\frac{1}{\cos^2\theta}\frac{d}{d\theta}\cos\theta = \frac{1}{\cos^2\theta}\sin\theta = \frac{\tan\theta}{\cos\theta} = \sec\theta\tan\theta.$$

*D.3.3 Example Differentiate* $\frac{d}{d\theta}(\sin\theta)^2$.

Let $u(\theta) = \sin\theta$. Then $\frac{du}{d\theta} = \cos\theta$, and

$$\frac{d}{d\theta}(\sin\theta)^2 = \frac{d}{d\theta}(u^2) = \frac{d}{du}(u^2)\frac{du}{d\theta} = 2u\frac{du}{d\theta} = 2\sin\theta\cos\theta.$$

## *D.4 Differentiating Logarithms and Exponentials*

### *D.4.1 Natural Logarithms*

Natural logarithms, $\ln x = \log_e x$, with $x > 0$, use the base $e = 2.71828...$ The rules for manipulating logarithms are summarized below. These rules apply to logarithms to any base, including the base $e$.

From the definition of $\log x$, $a = 10^{\log a}$ and $b = 10^{\log b}$. Consequently, from the properties of exponentials,

(a) $ab = (10^{\log a})(10^{\log b}) = 10^{\log a + \log b}$.

Taking the log of both sides, and again using $\log 10^x = x$ gives

(b) $\log(ab) = \log 10^{\log a + \log b} = \log a + \log b$.

Similarly, $a/b = 10^{\log a}10^{-\log b} = 10^{\log a - \log b}$. Therefore

(c) $\log(a/b) = \log a - \log b$

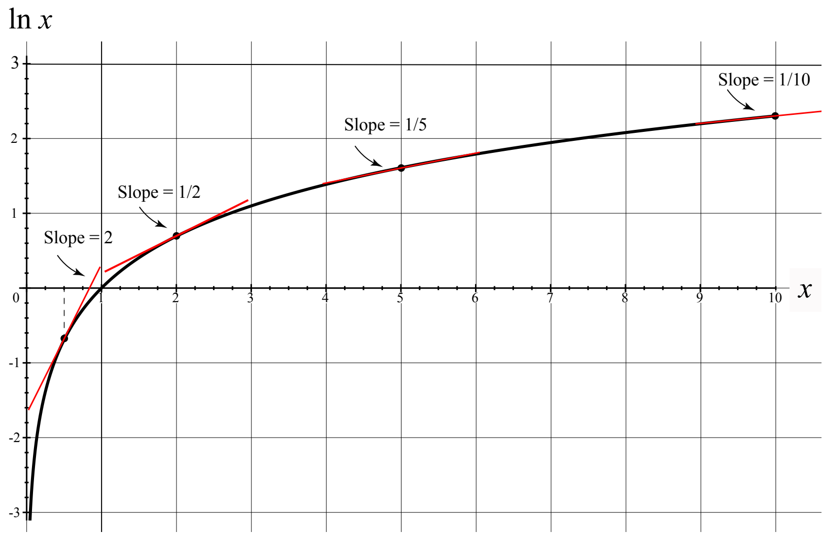
Likewise, $a^n = (10^{\log a})^n = 10^{n \log a}$, so that

(f) $\log(a^n) = n \log a$.

These rules apply to logarithms to any bases, including the base $e$.

*D.4.1 Differentiating Natural Logarithms*

The qualitative features of the slope at various points in a plot of $\ln x$ vs. $x$, are shown in the figure below.




For small values of $x$ the derivative $\dfrac{d}{dx}\ln x$ is large, and for large values of $x$ the derivative is small. In the figure above tangents are shown at a few points.

Let $y = \ln x$, hence $y(x+\Delta x) = y + \Delta y = \ln(x+\Delta x)$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\ln(x+\Delta x) - \ln x}{\Delta x}\frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right) = \frac{1}{x}\frac{x}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right).$$

Using the property that $\ln(a)^b = b \ln a$, then

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x} = \frac{1}{x}\ln(1+c)^{1/c},$$

where $c = \Delta x / x$. Note that as $\Delta x \to 0$, $c \to 0$. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[ \frac{1}{x} \ln(1+c)^{1/c} \right] = \frac{1}{x} \ln\left[ \lim_{c \to 0} (1+c)^{1/c} \right] = \frac{1}{x} \ln e = \frac{1}{x},$$

where $e = \lim_{c \to 0} (1+c)^{1/c}$ is the Euler number and the fact that $\ln e = 1$. Therefore

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

*D.4.1.1 Example Differentiate* $\frac{d}{dx} \ln(x^2)$.

Use the chain rule: $\frac{d}{dx} (\ln x)^2 = (2 \ln x) \left( \frac{d}{dx} \ln x \right) = \frac{2 \ln x}{x}$.

*D.4.2 Exponential Derivatives*

*D.4.2.1 Example Differentiate* $y(x) = a^x$

Consider the function $y(x) = a^x$, where $a$ is a positive constant and $x$ is the variable. In order to find the derivative, first take the natural logarithm: $\ln y = \ln(a^x) = x \ln a$. The derivative of $\ln y(x)$ with respect to $x$ is

$$\frac{d}{dx} \ln y = \frac{d}{dx} (x \ln a) = \ln a.$$

Now apply the chain rule to $\frac{d}{dx} \ln y$

$$\frac{d}{dx} \ln y = \frac{d}{dy} \ln y \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}.$$

Equate the two expressions for derivatives obtaining $\frac{1}{y} \frac{dy}{dx} = \ln a$. Solve for the derivative $\frac{dy}{dx} = (\ln a) y$ and substitute $y = a^x$ yielding

$$\frac{d}{dx} a^x = a^x \ln a.$$

*D.4.2.2 Example Differentiate* $y(x) = e^x$

An important special case occurs when $a = e$. Because $\ln e = 1$,

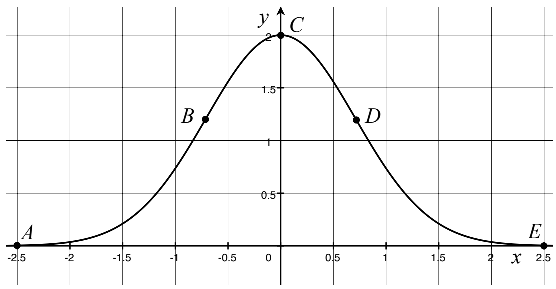
$$\frac{d}{dx} e^x = e^x.$$

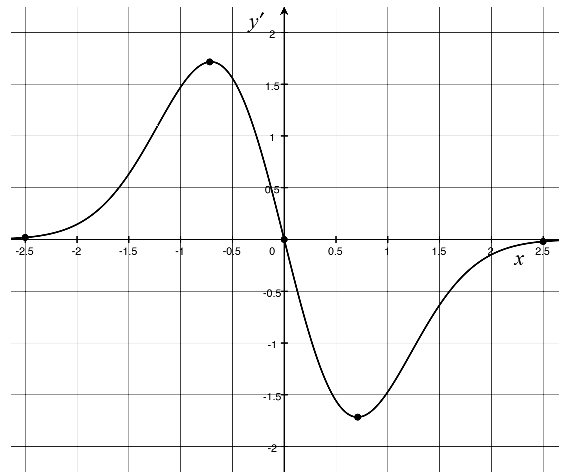
*D.4.2.3 Example Differentiate* $f(x) = 2e^{-x^2}$

Let $f(x) = 2e^{-x^2}$. Set $u(x) = -x^2$ with $du/dx = -2x$. Then use the chain rule:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = (2e^{u})(-2x) = -4xe^{-x^2}.$$

The graphs of the function $f(x) = 2e^{-x^2}$, and its derivative $f'(x) = -4xe^{-x^2}$ are plotted in the figures below.




When $x < 0$, $y$ increases with $x$ so that $y'$ is positive. The slope of $y$ is greatest near point *B*, but it must abruptly decrease beyond *B* because it vanishes at $C(x = 0)$. At *D*, $y$ is decreasing rapidly, so $y'$ is negative. At the points, *A* and *E*, the slope of $y$ is small and $y'$ is close to zero.

*D.4.3 Logarithmic Derivation*

Consider a function $f(x)$ for which $f(x) \neq 0$ for a range of values of $x$. Then the derivative is

$$\frac{df}{dx} = \left(\frac{d}{dx}\ln f\right) f.$$

This technique is useful for certain functions $f(x)$ in which $\ln f(x)$ can be simplified as much as possible using properties of the natural logarithm and if the derivative $\frac{d}{dx}\ln f$ is fairly straightforward to calculate.

*D.4.3.1 Example Differentiate* $f(x) = \left(\frac{x+1}{x-1}\right)^{1/3}$.

If $y = \left(\frac{x+1}{x-1}\right)^{1/3}$, for $x > 1$, what is $\frac{dy}{dx}$? Set $f(x) = \left(\frac{x+1}{x-1}\right)^{1/3}$. Then

$$\ln f(x) = \ln\left(\frac{x+1}{x-1}\right)^{1/3} = \frac{1}{3}(\ln(x+1) - \ln(x-1)).$$

Thus

$$\frac{d}{dx}\ln f = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) = -\frac{2}{3}\left(\frac{1}{(x+1)(x-1)}\right).$$

Therefore

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{x+1}{x-1}\right)^{1/3} = \left(\frac{d}{dx}\ln f\right)f = -\frac{2}{3}\left(\frac{1}{(x+1)(x-1)}\right)\left(\frac{x+1}{x-1}\right)^{1/3}$$

$$= -\frac{2}{3}\frac{1}{((x+1)^2(x-1)^4)^{1/3}} = -\frac{2}{3}\frac{1}{(((x+1)(x-1))^2(x-1)^2)^{1/3}} = -\frac{2}{3}\frac{1}{((x^2-1)^2(x-1)^2)^{1/3}}.$$

$$= -\frac{2}{3}((x^2-1)(x-1))^{-2/3}.$$

*D.5 Higher Order Derivatives*

The *nth derivative* of $f$ with respect to $x$, is written as $f^{(n)} = \dfrac{d^n f}{dx^n}$, where $n$ is a positive integer.

*D.5.1 Example Second Order Derivative*

Find the second derivative $\dfrac{d^2 f}{dx^2}$ of $f = 2x^3$. The first derivative is $\dfrac{df}{dx} = 6x^2$, and the second derivative is then $\dfrac{d^2 f}{dx^2} = \dfrac{d}{dx}\left(\dfrac{df}{dx}\right) = \dfrac{d}{dx}(6x^2) = 12x.$

*D.5.2 Acceleration*

Velocity is the rate of change of position with respect to time, $v = \dfrac{dx}{dt}$. Acceleration is the rate of change of velocity with respect to time $a = \dfrac{dv}{dt}$. Therefore $a = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2 x}{dt^2}$.

*D.5.2.1 Example Oscillatory Motion*

Let the position of a object be given by $x(t) = A\sin(\omega t)$, where $A$ and $\omega$ are constants. Find the acceleration. The velocity is $v = \dfrac{dx}{dt} = \dfrac{d}{dt}A\sin(\omega t) = A\omega\cos(\omega t)$. The acceleration is

$$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt} A\omega\cos(\omega t) = -A\omega^2\sin(\omega t).$$

*D.5.2.2 Example Differentiate a Quartic Function* $f(x) = x^4$

Let $f(x) = x^4$, find $f^{(4)} = \dfrac{d^4 f}{dx^4}$. The four derivatives are

$$f^{(4)} = \frac{d^4 f}{dx^4} = \frac{d^4}{dx^4}(x^4) = \frac{d}{dx}\left(\frac{d}{dx}\left\{\frac{d}{dx}\left[\frac{d}{dx}(x^4)\right]\right\}\right)$$
$$= (4)\frac{d^3}{dx^3}(x^3) = (4)(3)\frac{d^2}{dx^2}(x^2) = (4)(3)(2)\frac{d}{dx}x = (4)(3)(2)(1).$$

*D.5.2.2 Generalization: nth Derivative of a Polynomial* $f(x) = x^n$

The nth derivative of $f(x) = x^n$ is $\dfrac{d^n}{dx^n}x^n = (n)(n-1)(n-2)\cdots(1) = n!$

($n!$ is called $n$ *factorial* and is $(n)(n-1)(n-2)\cdots(1)$. By definition $0! = 1$.)

*D.6 Maxima and Minima*

If a function $f(x)$ has a maximum or a minimum for some value of $x$ within a given interval, then its derivative $f'$ is zero for that $x$. Wherever $f' = 0$, $f(x)$ has a maximum value if $f'' < 0$, and $f(x)$ has a minimum value if $f'' > 0$. (If $f'' = 0$, this test is not useful.)

*D.6.1 Example Values of Maxima or Minima of* $f(x) = 8x + \dfrac{2}{x}$

For which the value(s) of $x$ does the function $f(x) = 8x + \dfrac{2}{x}$ have a maximum or minimum?

Problem:

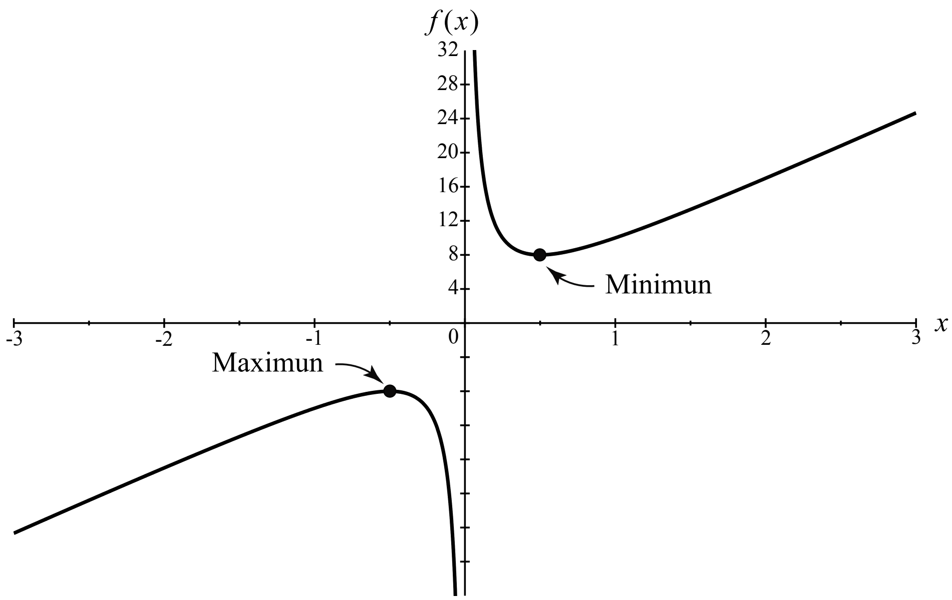
For which the value(s) of $x$ do the following functions have a maximum or minimum.

(a) $f(x) = x^2 + 6x$, (b) $f(x) = 8x + \dfrac{2}{x}$, (c) $f(x) = e^{-x^2}$.

Solution:

(a) The maximum or minimum occurs where $x$ satisfies $f' = 2x + 6 = 0$. Thus the maximum or minimum occurs at $x = -3$. The second derivative $f'' = 2 > 0$, so the function has a minimum at $x = -3$.

(b) The desired points are solutions of the equation $f' = 8 - \frac{2}{x^2} = 0$, which are at $x = +1/2$ and $x = -1/2$. The second derivative is $f'' = \frac{4}{x^3}$. At $x = -1/2$, $f''(-1/2) = -32 < 0$, so at that point the function has a maximum. At $x = +1/2$, $f''(1/2) = 32 > 0$, hence that point is a minimum. A plot of $f(x)$ is shown in the figure below.

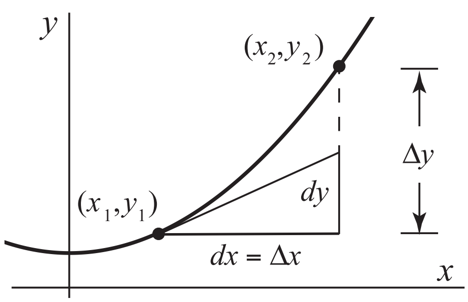


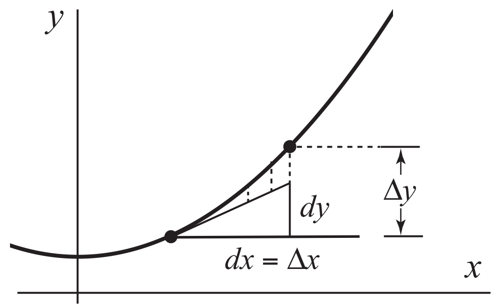

*D.7 Differentials*

Suppose that $x$ is an independent variable, and that $y = f(x)$. Then the *differential* $dx$ of $x$ is defined as equal to any *increment*, $dx = \Delta x = x_2 - x_1$, where $x_1$ is the point of interest. The differential $dx$ can be positive or negative, large or small. The differential $dx$ can also be regarded as an independent variable. The *differential* $dy$ for the interval $dx = x_2 - x_1$ is defined by the following rule:

$$dy = y'(x_1)dx = \left.\frac{dy}{dx}\right|_{x=x_1} (x_2 - x_1).$$

where $y'(x_1) = \left.\frac{dy}{dx}\right|_{x=x_1}$ indicates that the derivative has been evaluated at the point $x_1$. The differential $dy = y'(x_1)dx$ is not the same as the difference $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$. The figure below shows that $dy$ and $\Delta y$ are different quantities.




Although $dy$ and $\Delta y$ are different, in the limit where $dx \to 0$, $dy$ may be substituted for $\Delta y$.

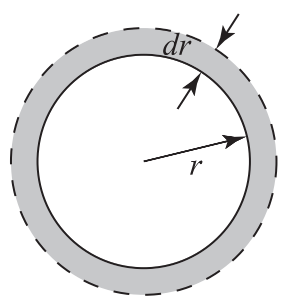
The differential of a function $f(x)$ is $d(f(x)) = \dfrac{df}{dx} dx$.

*D.7.1.1 Example Differential of $f(x) = x^n$*

Let $f(x) = x^n, df(x) = d(x^n) = \dfrac{d}{dx}(x^n)dx = nx^{n-1}\, dx.$

*D.7.1.2 Differential Area of a Disc*

The diagram shows the surface of a disc to which a thin rim has been added.




In order to approximate the change in area $\Delta A$ which occurs when the radius is increased from $r$ to $r + dr$. the differential increase in area is given by

$$dA = \left(\frac{dA}{dr}\right) dr = \frac{d}{dr}(\pi r^2)\, dr = 2\pi r\, dr.$$

The difference of the two areas is $\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r\, \Delta r + \pi\, \Delta r^2$. When $\Delta r$ is small compared with r, the last term can be neglected and $\Delta A \approx 2\pi r\, \Delta r$. Set $\Delta r = dr$, and take the limit as $dr \to 0$, then $\Delta A \to dA = 2\pi r\, dr$. As a check, because the rim is thin, its area $dA$ is the approximate length, $2\pi r$, multiplied by its width, $dr$. Hence, $dA = 2\pi r\, dr$.

*D.8 Appendix*

*D.8.1 Definition of a Limit*

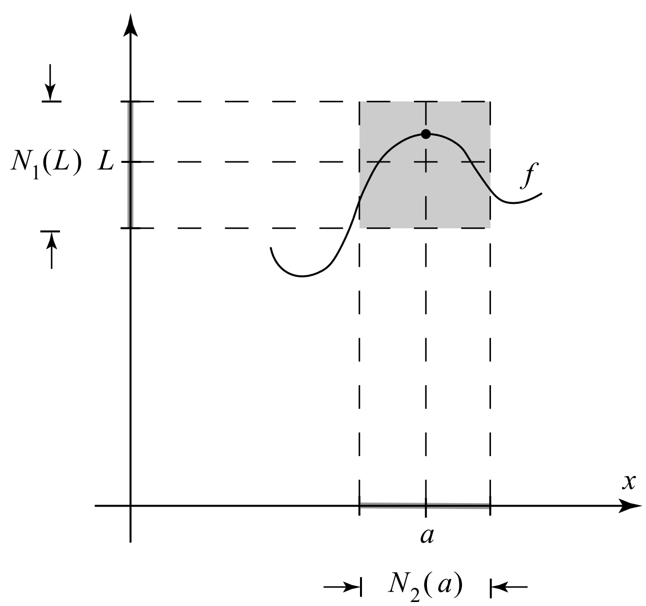
Let $f(x)$ be defined for all $x$ in an interval about $x=a$, but not necessarily at $x=a$. If there is a number $L$ such that to each positive number $\varepsilon$ there corresponds a positive number $\delta$ such that

$$|f(x)-L|<\epsilon \text{ provided } 0<|x-a|<\delta,$$

we say that $L$ is the *limit* of $f(x)$ as $x$ approaches $a$ and write

$$\lim_{x\to a} f(x)=L.$$

A *neighborhood of a point* $a$, $N(a)$ is defined to be an open interval such that $a$ lies at its midpoint. The limit of $f(x)$, as $x$ approaches $a$, is equal to $L$, means that for every neighborhood $N_1(L)$ there is some neighborhood $N_2(a)$ such that $f(x)\in N_1(L)$ whenever $x\in N_2(a)$ and $x\neq a$. Consider a neighborhood $N_1(L)$. A neighborhood $N_2(a)$, corresponding to $N_1(L)$ is shown in the figure below. The entire graph of $f(x)$ above the interval $N_2(a)$ lies within the shaded rectangle except possibly for $f(a)$.




*D.8.2 Definition of Continuity of a Function*

A $f(x)$ function is continuous at a point $a$ if

$a)$ $f(a)$ is defined at the point $a$,

$b)$ $\lim_{x\to a} f(x)=f(a)$.

In terms of neighborhoods: A function $f(x)$ is continuous at a point $a$ if for every neighborhood $N_1(f(a))$ there is some neighborhood $N_2(a)$ such that $f(x) \in N_1(f(a))$ whenever $x \in N_2(a)$.

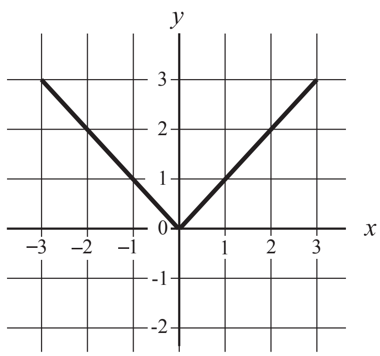
In terms of epsilons and deltas: A function $f(x)$ is continuous at a point $a$ if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta$.

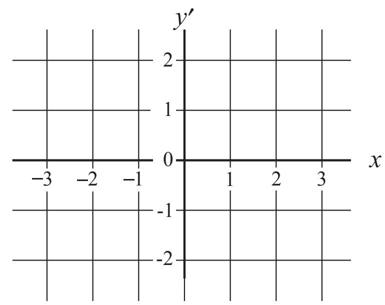
Problems for Differentiation

*D.1.3 Graphs of Functions and Their Derivatives*

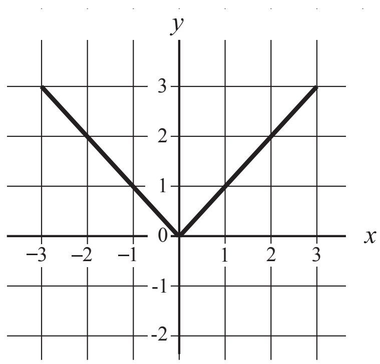
*D.1.3.1 Problem* $y = |x|$

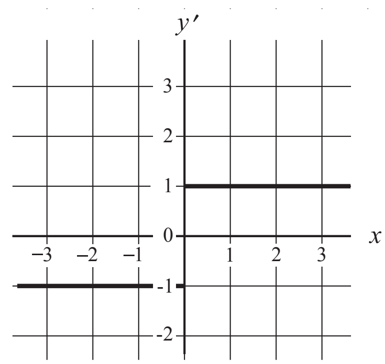
Consider the function $y = |x|$. On the coordinates below, sketch $y'$.




Answer: Here are sketches of $y = |x|$ and $y'$.

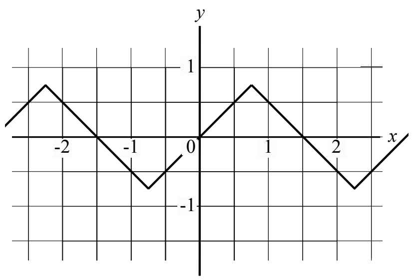



As you can see from the graph, $y = |x| = x$ for $x > 0$. So for $x > 0$ the slope is one, hence $y' = 1$. However, for $x < 0$, the slope of $|x|$ is negative and is easily seen to be $-1$. At $x = 0$, the slope is undefined, for it has the value +1 if we approach 0 along the positive $x$-axis and has the value $-1$ if we approach 0 along the negative $x$-axis. Therefore, $\frac{d}{dx}|x|$ is discontinuous at $x = 0$. (The function $|x|$ is continuous at this point, but the break in its slope at $x = 0$ causes a discontinuity in the derivative.)

*D.1.3.2 Problem Sketching Derivatives of Functions*

Try sketching the derivatives for the fours function shown in the figures below.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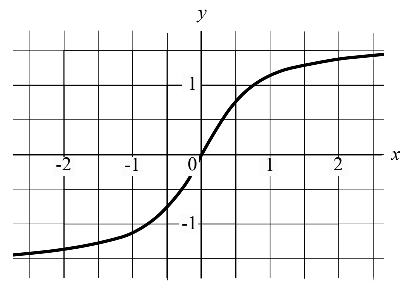
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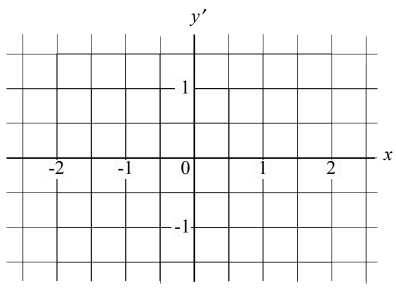
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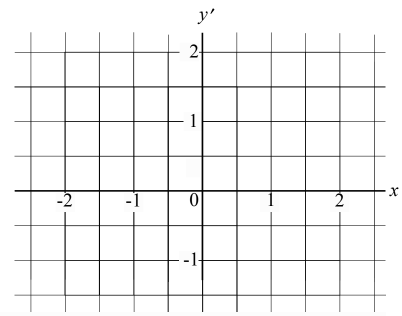
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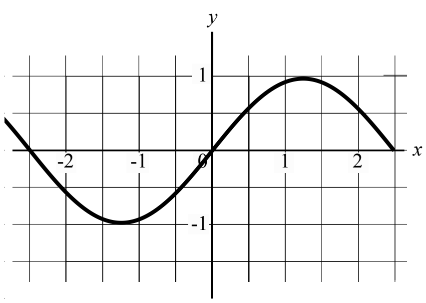
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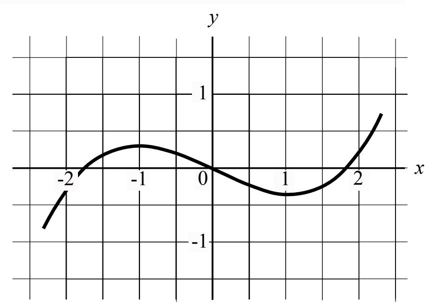
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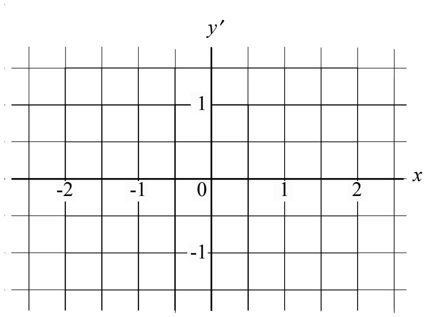
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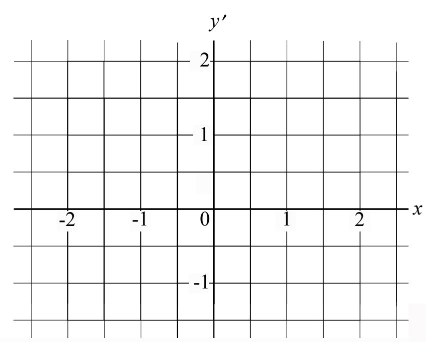
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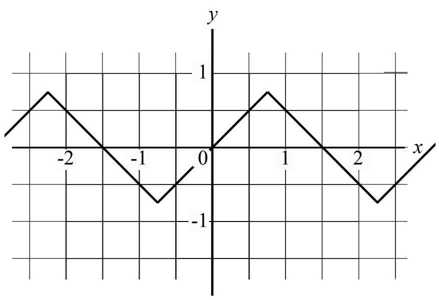
Solutions: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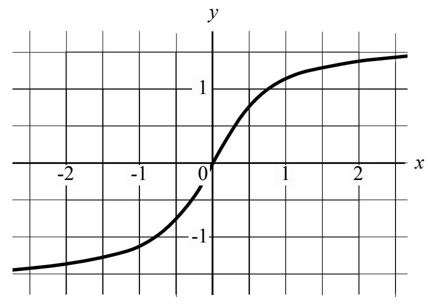
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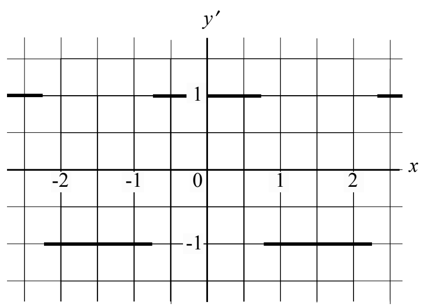
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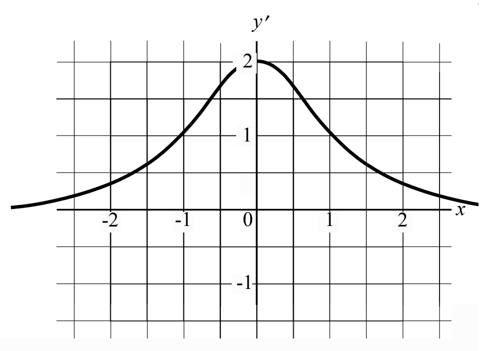
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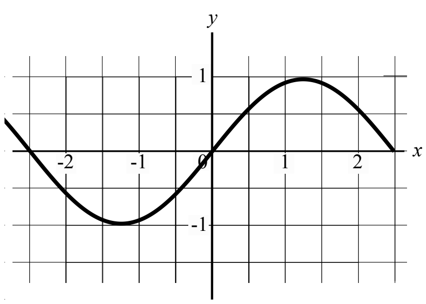
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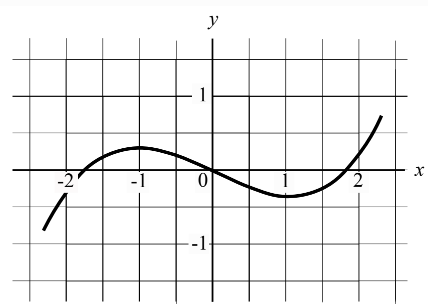
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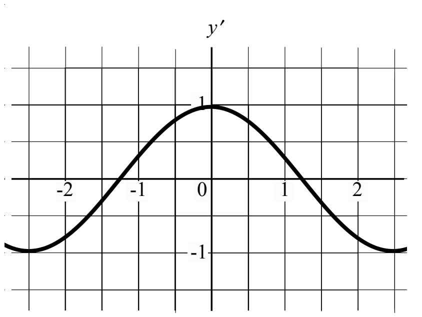
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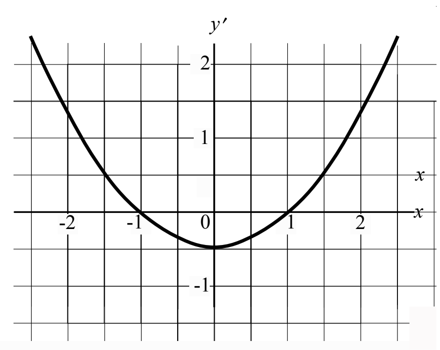
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*D.1.5.2 Problem Derivative of $f(x)=3x^2+7x+2$*

If $f(x)=3x^2+7x+2$, find $f'$.

Solution: Start with $f(x+\Delta x)=3[x^2+2x\Delta x+(\Delta x)^2]+7(x+\Delta x)+2,$ therefore

$$\Delta f = f(x+\Delta x)-f(x)=6x\ \Delta x+3\,(\Delta x)^2+7\ \Delta x.$$

The derivative is then

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\left(\frac{6x\ \Delta x+3\,(\Delta x)^2+7\ \Delta x}{\Delta x}\right)=\lim_{\Delta x\to 0}(6x+3\,\Delta x+7)=6x+7\,.$$

*D.1.5.3 Problem Derivatives of Polynomial Functions*

Problem: Find $\dfrac{dy}{dx}$ for each of the following functions:

(a) $y=x^{-7}$, (b) $y(x)=x^3$, (c) $y=\dfrac{1}{x}$, (d) $y=\dfrac{-1}{3}x^{-3}$, and (e) $y=x^{1/2}$.

Solution: Use the general rule: $\dfrac{d}{dx}x^n=nx^{n-1}$.

(a) $y(x)=x^{-7}$; hence $\dfrac{d}{dx}x^{-7}=-7x^{-7-1}=-7x^{-8}$.

(b) $y(x)=x^3$; hence $\dfrac{d}{dx}x^3=3x^{3-1}=3x^2$.

(c) $y=\dfrac{1}{x}$; here $n=-1$, therefore $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)=-x^{-2}$.

(d) $y=-\dfrac{1}{3}x^{-3}$; thus $-\dfrac{1}{3}\dfrac{d}{dx}\left(\dfrac{1}{x^3}\right)=-\dfrac{1}{3}(-3x^{-4})=x^{-4}$.

(e) $y=x^{1/2}$; the rule $\dfrac{d}{dx}x^n=nx^{n-1}$ is true for any value of $n$. In this case, $n=1/2$, $\dfrac{d}{dx}x^{1/2}=\dfrac{1}{2}x^{-1/2}$.

*D.2.1.2 Product Rule*

*D.2.1.2.1 Problem Derivative of $y(x)=(3x+7)(4x^2+6x)$*

Use the product rule to find the derivative of $y(x)=(3x+7)(4x^2+6x)$.
Solution:

Let $u(x) = 3x + 7$ and $v(x) = 4x^2 + 6x$. Then $u' = 3$ and $v' = 8x + 6$. Hence

$$\frac{d}{dx}(uv) = uv' + vu' = (3x+7)(8x+6) + (4x^2 + 6x)(3).$$

*D.2.1.2.2 Problem Derivative of* $y(x) = (2x+3)(x^5)$

Use the product rule to find the derivative of $y(x) = (2x+3)(x^5)$.

Solution:
Let $u(x) = 2x + 3$ and $v(x) = x^5$. Then $u' = 2$ and $v' = 5x^4$.

$$\frac{d}{dx}\left[(2x+3)(x^5)\right] = (2x+3)(5x^4) + (x^5)(2).$$

*D.2.1.4.1 Problem Differentiate* $f(x) = \sqrt{1+x^2}$.

Solution: Let $u(x) = 1 + x^2$, then $f(u) = \sqrt{u}$. Hence

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{x}{\sqrt{1+x^2}}.$$

*D.2.1.4.2 Problem Differentiate* $f(x) = \left(x^3 + x^{-1}\right)^{-3}$.

Solution: Let $u(x) = \left(x^3 + x^{-1}\right)$, then $f(u) = u^{-3}$. The chain rule is then

$$\frac{df}{dx} = -3u^{-4}\frac{du}{dx} = -3u^{-4}\left(3x^2 - x^{-2}\right) = -3\left(x^3 + x^{-1}\right)^{-4}\left(3x^2 - x^{-2}\right).$$

*D.2.1.4.3 Problem Differentiate* $y(x) = \left(1 + x^{-1}\right)^2$.

Solution: Let $u = 1 + x^{-1}$. Then $f(u) = u^2$, and

$$\frac{df}{dx} = 2u\frac{du}{dx} = 2\left(1 + x^{-1}\right)\left(-x^{-2}\right).$$

*D.2.1.4.4 Problem Differentiate* $y(x) = (2x + 7x^2)^{-2}$.

Solution: Let $u(x) = 2x + 7x^2$, then $\frac{du}{dx} = 2 + 14x$, and $f(u) = u^{-2}$. Therefore

$$\frac{df}{dx} = -2u^{-3}\frac{du}{dx} = -2(2x + 7x^2)^{-3}(2 + 14x).$$

*D.2.1.4.5 Problem Differentiate* $y(x) = 12(x^2 + 4)^4 + 7(x^2 + 4)$.

Solution: Let $u(x) = x^2 + 4$, then $\frac{du}{dx} = 2x$ and $f(u) = 12u^4 + 7u$ with $\frac{df}{du} = 48u^3 + 7$. Therefore

$$\frac{df}{dx} = (48u^3 + 7)(2u) = [48(x^2 + 4)^3 + 7](2x).$$

*D.2.1.4.6 Problem Derive the Quotient Rule.*

Derive the quotient rule for the derivative of the quotient of two functions. Find $\frac{d}{dx}\left(\frac{u}{v}\right)$ in terms of $u$, $v$, $\frac{du}{dx}$, $\frac{dv}{dx}$.

Solution: Let $w = \frac{1}{v}$, then first use the product rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx}$. Apply the result from the last example that $\frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$, therefore

$$\frac{d}{dx}\left(\frac{u}{v}\right) = -\frac{u}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx} = \frac{v(du/dx) - u(dv/dx)}{v^2}.$$

*D.3.1.1 Problem Differentiate* $\frac{d}{d\theta}\cos(\theta^3)$.

Differentiate $\frac{d}{d\theta}\cos(\theta^3)$.

Solution. Let $u = \theta^3$, then $f(u) = \cos u$. Then

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = -\sin u 3\theta^2 = -\sin(\theta^3)3\theta^2.$$

*D.3.1.2 Problem Differentiate* $\frac{d}{dt}\sin(\omega t)$.

Differentiate $\frac{d}{dt}\sin(\omega t)$, where $\omega$ is a constant.

Solution. Let $u = \omega t$, then $f(u) = \sin u$. Hence

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \cos u \frac{d(\omega t)}{dt} = \omega \cos(\omega t).$$

*D.4.1.1 Problem Differentiate* $\frac{d}{dx}\ln(5x)$.

Solution. Because $\ln(x^2) = 2\ln x$, $\frac{d}{dx}\ln(x^2) = \frac{d}{dx}(2\ln x) = \frac{2}{x}$.

*D.4.1.2 Problem Differentiate* $\frac{d}{dx}(\ln x)^2$.

Solution: To find $\frac{d}{dx}\ln(5x)$ recall that $\ln(5x) = \ln 5 + \ln x$. Hence,

$$\frac{d}{dx}\ln(5x) = \frac{d}{dx}\ln 5 + \frac{d}{dx}\ln x = 0 + \frac{1}{x} = \frac{1}{x}.$$

*D.4.1.3 Problem Differentiate* $\frac{d}{dx}\frac{1}{\ln x}$.

Solution. Use the chain rule. Let $u(x) = \ln x$. Then

$$\frac{d}{dx}\left(\frac{1}{\ln x}\right) = \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx} = -\frac{1}{u^2}\frac{1}{x} = -\frac{1}{(\ln x)^2 x}.$$

*D.4.2.1 Problem Differentiate* $\frac{d}{dx}e^{cx}$.

Solution. Let $u(x) = cx$. Thus $\frac{d}{dx}e^{cx} = \frac{d}{du}e^u\frac{du}{dx} = e^u c = ce^{cx}$.

*D.4.2.2 Problem Differentiate* $\frac{d}{dx}e^{cx}$.

Solution. Set $c = -1$, then $\frac{d}{dx}e^{-x} = -e^{-x}$.

*D.4.3.1 Problem Differentiate* $f(x) = x^x$

Solution. Let $f(x) = x^x$, what is $\frac{df}{dx}$? To find the derivative of $f(x) = x^x$, consider $\ln f(x) = x \ln x$. We can differentiate this with respect to $x$,

$$\frac{df}{dx} = \frac{d}{dx}x^x = \left(\frac{d}{dx}\ln f\right)f = \left(\frac{d}{dx}\ln x^x\right)x^x = \frac{d}{dx}(x\ln x)x^x = (\ln x + 1)x^x.$$

*D.5.1.1 Problem Second Order Derivative* $f(x) = x + x^{-1}$

Find the second derivative $\frac{d^2 f}{dx^2}$ of $f(x) = x + x^{-1}$.

Solution: Let $f(x) = x + x^{-1}$, $\frac{df}{dx} = 1 - x^{-2}$, $\frac{d^2 f}{dx^2} = 0 - 1\left(-2x^{-3}\right) = 2x^{-3}$.

*D.6.1.1 Problem Values of Maxima or Minima of* $f(x) = 8x + \frac{2}{x}$

For which the value(s) of $x$ does the function $f(x) = 8x + \frac{2}{x}$ have a maximum or minimum?

Solution: The maximum or minimum occurs where $x$ satisfies $f' = 2x + 6 = 0$. Thus the maximum or minimum occurs at $x = -3$. The second derivative $f'' = 2 > 0$, so the function has a minimum at $x = -3$.

*D.6.1.2 Problem Values of Maxima or Minima of* $f(x) = e^{-x^2}$

For which the value(s) of $x$ does the function $f(x) = e^{-x^2}$ have a maximum or minimum?

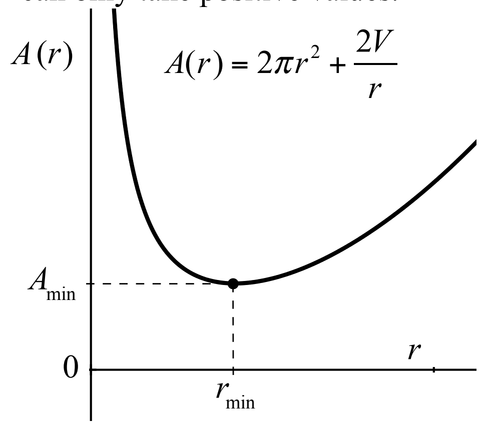
Solution: In order to calculate the first derivative, set $u = x^2$, then $f(u) = e^{-u}$, and the chain rule yields $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = -e^{-u}2x = -2xe^{-x^2}$. The first derivative $-2xe^{-x^2} = 0$ at $x = 0$. Use the product rule to calculate the second derivative $f'' = -2e^{-x^2} + 4x^2 e^{-x^2} = (-2 + 4x^2)e^{-x^2}$. At $x = 0$, $f'' = (-2 + (4)(0))e^{-0} = -2 < 0$, therefore $f(0) = 1$ is the maximum value.

*D.6.1.3 Minimal Surface Area of Cylinder*

Consider a cylinder of radius $r$ and height $h$. What ratio of the radius to the height to radius, $h/r$ minimizes the surface area for a given fixed volume $V$?

Solution:

The total surface area is the sum of the surface area of the cylinder, plus the area of the two end-caps, each of area $\pi r^2$, hence $A = 2\pi r^2 + 2\pi rh$. Because the height and volume are related by $h = V / \pi r^2$, the total area can be expressed as a function of the radius $r$ and the constant volume $V$ according to $A = 2\pi r^2 + 2V / r$. In the figure below, the area is plotted as a function of $r$. Note the radius $r$ can only take positive values.




At the minimum, the first derivative is zero, $0 = dA / dr = 4\pi r - 2V / r^2$. Thus the radius that minimizes the surface area is $r = \left(V / 2\pi\right)^{1/3}$. Because $V = \pi r^2 h$, $r = \left(r^2 h / 2\right)^{1/3}$. Therefore, the ratio of the radius to the height $h / r = 2$.

*D.7.1.1 Differential of* $f(x) = \sin x$

Solution: $d(\sin x) = \left(\dfrac{d(\sin x)}{dx}\right) dx = \cos x \, dx$.

*D.7.1.2 Differential of* $f(x) = x^{-1}$

Solution: $d\left(x^{-1}\right) = \left[\dfrac{d}{dx}\left(x^{-1}\right)\right] dx = -x^{-2} dx$.

*D.7.1.2 Differential of* $f(x) = e^x$

Solution: $de^x = \left(\dfrac{de^x}{dx}\right) dx = e^x dx.$

*D.8 Additional Problems*

*D.8.1 Velocity for Oscillatory Motion*

The position of a particle along a straight line is given by the following expression:

$x(t) = A\sin(\omega t) + B\cos(2\omega t)$ where $A$, $B$ and $\omega$ (omega) are constants. Find the velocity of the particle.

Solution:

$$v(t) = \frac{dS}{dt} = \frac{d}{dt}(A\sin(\omega t)\cos(\omega t)) = A\sin(\omega t)\left(\frac{d}{dt}\cos(\omega t)\right) + \left(\frac{d}{dt}A\sin(\omega t)\right)\cos(\omega t)$$
$$= \omega A(\cos^2(\omega t) - \sin^2(\omega t))$$

Equivalently use the trigonometric identity that $\sin(\omega t)\cos(\omega t) = (1/2)\sin(2\omega t)$. Then,

$$v(t) = \frac{d}{dt}\left(\frac{A}{2}\sin(2\omega t)\right) = \omega A\cos(2\omega t) = \omega A(\cos^2(\omega t) - \sin^2(\omega t)).$$

*D.8.2 Velocity for Free Fall*

Suppose the height of a ball above the ground is given by $y(t) = y_0 + v_0 t - (g/2)t^2$. Find the velocity in the $y$ - direction.

Solution: $v(t) = \frac{dy}{dt} = \frac{d}{dt}(y_0 + v_0 t - (g/2)t^2) = v_0 - gt$.

*D.8.3 Velocity for the Position Function* $x(t) = \frac{a}{(t+c)^2} + bt$

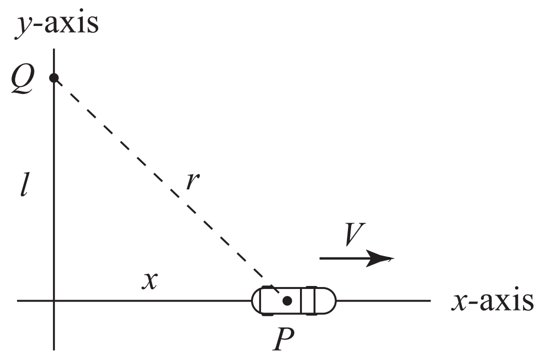
The position of an objects is given by $x(t) = \frac{a}{(t+c)^2} + bt$, where $a$, $b$, and $c$ are constant, and $t \geq 0$. Find the velocity.

Solution:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{a}{(t+c)^2} + bt\right) = -\frac{2a}{(t+c)^3} + b.$$

*D.8.4 Rate of Change of Distance Between Man and Car*

A car $P$ moves along a road in the $x$ direction with a constant velocity $v$. The problem is to find how fast it is moving away from a man standing at point $Q$, distance $l$ away from the road, as shown in the figure below. Let $r$ be the distance between $Q$ and $P$, find $dr/dt$.




Solution:

From the figure $r = (x^2 + l^2)^{1/2}$. Use the chain rule:

$$\frac{dr}{dt} = \frac{dr}{dx}\frac{dx}{dt} = \frac{d}{dx}(x^2 + l^2)^{1/2}\frac{dx}{dt} = \frac{1}{2}\frac{2x}{(x^2 + l^2)^{1/2}}\frac{dx}{dt} = v\frac{x}{(x^2 + l^2)^{1/2}}.$$