Self-Paced Study Guide in Geometry and Analytic Geometry

MIT INSTITUTE OF TECHNOLOGY

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Contents

1 Credits

The review modules were written by Professor A. P. French (Physics Department) and Adeliada Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department). This document was originally produced by the Undergraduate Academic Affairs Office, August, 1992, and edited and transcribed to LaTeX by Tea Dorminy (MIT Class of 2013) in August, 2010.

2 How to Use the Self-Paced Review Module

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in the topic area Algebra, the subtopic Linear Equations is numbered by 2.3. Problems and the self-evaluations are categorized according to this numbering system.

When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problem solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules and should be solved while working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. The tests should be taken only when both an understanding of the material and a problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate one's mastery of the material. The previous Diagnostic Exams should provide the finishing touch.

3 Geometry Review Module

This is a short module. It is not intended to be a full review of the theorems of Euclidean geometry. Its purpose is just to remind you of some of the chief results that will be of direct use to you in your work in science and other areas of mathematics. In particular, thelane geometry in this module leads quickly into trigonomoetry, which is the subject of a sepparate module.

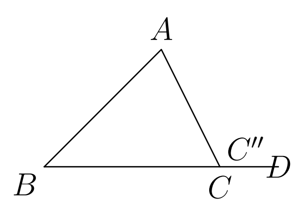
3.1 Triangles

You will be familiar with the fact that, in Eucilidean geometry, the angles of a triangle add up to 180°. We shall take this property of triangles as a given. Itis worth remembering, though, that this is not a general result. It holds only for *[plane* triangles. The angles of a triangle drawn on the surface of a sphere add up to *more* than 180°. The reason whi that isn't important to us for most purposes is that the triangles we deal with usually have very small linear dimensions relative to the radius of the Earth, so, for example, most usrveying on the earth's surface can use Euclidean geometry with no significant error. (And then there are also other "non-Euclidean geometries" that obey different rules, whicfh we'll certainly not deal with.) But here, this 180° property is a starting point:

In a triangle ABC,

$$\angle A + \angle B + \angle C = 180^\circ$$

Another familiar property is that, if one side of a triangle is extended, as in the diagram here, the exterior angle C'' is equal to the sum of the interior angles A and B. (A reminder of the proof: Since BCD is a straight line, $\angle C + \angle C'' = 180^\circ$, so $\angle C'' = 180^\circ - \angle C$. Then use the result above.)



3.1.1 Angles and Similar Triangles

ADD FIGURE:

Some basic results of Euclidean geometry are very useful in analyzing geometrical situations in general and triangles in particular. When one straight line cuts across two parallel straight lines:

- $\angle A = \angle B$ (because lines are parallel)
- $\angle B = \angle C$ (opposite angles are equal)
- $\angle A = \angle C$ (alternate interior angles are equal)

These find many applications in recognizing and relating *similar* tiranlges. It is easy to recognize similar triangles when they are placed side by side in the same orientation, like those on the right. It may not be so easy if the triangles are in quite different orientations, like those on the right.

ADD FIGURE:

A common type of situation is the following mechanics problem:

A block sits on an inclined plane. It is acted on by the vertical force of gravity. In the solution fo the problem, you want to resolve this gravitational force (the weight W) into components parallel and perpindicular to the inclined plane. The triangle LMN representing this analysis of the downard force W into two mutually perpendicular parts, F_1 and F_2, is similar to the tirangle ABC composed of the inclined plane and its horizontal and vertical dimensions. One can therefore put:

$$F_1 : F_2 : W = BC : AC : AB$$

(This is more than an exercise in pure geometry, because we are comparing a geometrical triangle to a triangle of forces. BUt it is important and useful to know that this can be done.)

ADD FIGURE:

Exercise 3.1.1:
Some artists used to use a *camera obscura* (literally just a "dark room") with a small hole in one wall to form an image of a distance scene on a screen inside the room. (Then all they had to do was trace over the image!) Imagine a situation when the screen was 2 meters from the hole, forming an image of a building 10 meters high and 50 meters away. Draw a sketch showing rays of light passing to the screen, through the point-hole, from the top and bottom of the building, and calculate the height of the image.

Exercise 3.1.2:
Find three similar triangles in the figure opposite, and prove that $s^2 = rt$.
ADD FIGURE:

Exercise 3.1.3:
In the example of an inclined plane idscussed above, find the forces F_1 and F_2 in terms of W if $\angle A = 30°$.

Note: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

3.1.2 Orthocenters and Centroids

In any triangle:

The three perpindicular lines from the angles to the opposite sides (the altitudes) intersect at a common point. This is the *orthocenter*.

ADD FIGURE:

The tree lines from the angles to the midpoints of the opposite sides (the medians) intersect at a single point. This is the *centroid*.

ADD FIGURE:

The centroid is of particular significance physically. If a triangle is made from a sheet of material of uniform thickness, then the centroid is its balance point (its center of gravity), G. This can be understood by drawing lines parallel to each of the sides in turn. SUppose the figure is supsened from the corner A. Then the center of each strip lies on the line AM, and the triangle will be in balance, when hung from A, with AM vertical. Similarly for the other two medians. So, if suspended from the centroid, the triangle has no tendency to take up any particular orientation.

3.2 Some Other Plane Figures

3.2.1 Quadrilaterals

ADD FIGURE:

For any quadrilateral, the sum of the angles is 360°:

$$A + B + C + D = 360^\circ$$

(Think of it as two triangles, ABC and BCD.)

There are some special quadrilaterals:

ADD FIGURE:

Trapezoids have AB parallel to CD. Its width at half altitude is equal to the average of AB and CD.

ADD FIGURE:

Parallelograms have AB parellel to CD, and AD parellel to BC. Other properties: $\begin{array}{ll} \angle A = \angle C & \angle B = \angle D \\ AB = DC & AD = BC \end{array}$

ADD FIGURE:

Rhombus: a parallelogram with all sides equal:

$$AB = BC = CD = AD$$

Its diagonals are perpindicular.

3.2.2 Other Polygons

The literal meaning of 'polygon' is "many-cornered", but we usually think in terms of the number of sides. Since the number of angles equals the number of sides, In, you can't go wrong. It's useful to know the names of a few polygons beyond the quadrilateral:

$n = 5$	Pentagon
$n = 6$	Hexagon
$n = 7$	Heptagon
$n = 8$	Octagon
$n = 10$	Decagon
$n = 12$	Dodecagon

(The ones omitted are seldom met.)

There is a general formula for the sum of all the interior angles of a polygon with n sides or corners: $(n - 2) \times 180^\circ$. (Note that you can obtain this by dividing the polygon up into triangles in any fashion, and noting that the sum of the interior angles of the polygon is the sum of all the interior angles of all the triangles that the polygon is divided into.)

3.3 The Almighty Circle

Some properties worth knowing about, even if you don't memorize them:

ADD FIGURE:

Given any arc of the circle, the angle it subtends at the center is twice the angle it subtends at the circumference. (Special case: A semicircle subtends 180° at the center and a right angle at the circumference.)

ADD FIGURE:

Intersecting chords: The products of their separate parts are equal:

$$ab = cd$$

This is also known as "Power of a Point." (Special case: A diameter and a chord perpindicular to it, with the figure labeled as shown, $y(2R - y) = x^2$.

ADD FIGURE:

If $y \ll R$*, then* $y \approx x^2/2R$. This means that *a small part of a circle has almost the exact same chape as a parabola.* That is important in many mathematical and physical approximations. Admittedly, this is analytic geometry, not plain geometry!)

The triangles formed by any two intersecting chords, not necessarily passing through the center of the circle, are similar.

ADD FIGURE:

3.4 Lengths, Areas, & Volumes

There are many useful formulas for the perimeters, areas, and volumes of geometrical figures. Some of them you should definitely know. But before you look at the tabulation of important ones, consider the following statements concerning the *dimensions* of such quantities:

- Any *perimeter* must have the dimensions of *length*. That is, it must be expressed in terms of units of length to the first power only.
- Any *area* must have dimensions of *(length)*^{2}.
- Any *volume* must have dimensions of *(length)*^{3}.

For example, if you are trying to remember the formula for the *area* of a *circle*, consider that it couldn't possibly be $2\pi r$. Whatever else there may be, it must have r to the 2nd power. Guessing that it's $2\pi r^2$ (as opposed to the correct formula πr^2) is an error you shouldn't make — but it's less serious than using a formula that could only represent a length.

Likewise, the circumference of a circle could not possibly be given by πr^2 or $2\pi r^2$.

ADD FIGURE:formula table

Exercise 3.4.1:
Prove that the area of a flat circular ring or washer, with inner radius r_1 and outer radius r_2, is equal to $2\pi r_{avg}\Delta r$, where r_{ave} is the average radius $\frac{r_1+r_2}{2}$ and Δr is the width $r_2 - r_2$.

Exercise 3.4.2:
Given that a sphere has surface area 64π, what is its volume?

Exercise 3.4.3:
A cube of edge-length L has a sphere just fitting inside it (i.e. the diameter of the sphere is equal to L). Calculate the ratios of the surface areas and of the volumes of these two figures.

Exercise 3.4.4:
A plane, parallel to the base of a circular cone of height h and of radius R at the base, cuts across it at a distance of $h/3$ from the top. Calculate the ratios of the volumes of the small cone and the original cone.

Exercise 3.4.5:
A cube of side a is packed full with small spheres of diameter a/n (n being a positive integer), as shown in the cross-section diagram at right. Consider the total volume of the spheres. Does it increase or decrease as n gets bigger?
Given two equal-size and equal-price boxes of mothballs, should you buy the one with the larger mothballs or the smaller mothballs?
What if mothballs' effectiveness is determined by their surface area, not their volume?
ADD FIGURE:

4 Answers to Exercises

Exercise **??**:

ADD FIGURE:

Diagram not to scale.

Find similar traingles (see diagram above):

$$\triangle ABC \sim \triangle CDF \rightarrow \frac{AB}{BC} = \frac{DF}{CD}$$

or, using the distacnes given, $\frac{5}{50} = \frac{DF}{2} \rightarrow DF = \frac{1}{5}$. Then the height FE of the image is $FE = 2 \cdot DF = \frac{2}{5}m = 0.4m$.

Note: If the artsits used only one hole in the wall, they would trace a *inverted* image on the screen. (The rays passing through the hole form the top and the bottom of the building end up on the screen at points F and E, respectively.)

ADD FIGURE:

In order to get the image right side up we would need two holes and two screens (see diagram at left).

Exercise **??**: ADD FIGURE:

$$\triangle ABC \sim \triangle BDA \rightarrow \frac{AB}{AC} = \frac{BD}{AD} \quad (1)$$

$$\triangle ABC \sim \triangle ADC \rightarrow \frac{AB}{AC} = \frac{AD}{DC} \quad (2)$$

Using both (1) and (2), we have

$$\frac{BD}{AD} = \frac{AD}{DC} \rightarrow \frac{r}{s} = \frac{s}{t} \rightarrow s^2 = rt$$

ADD FIGURE:

Solution without using similar triangles, but using the Pythagorean theorem instead:

$$AB^2 = BD^2 + AD^2 \rightarrow AB^2 = r^2 + s^2 \quad (3)$$

$$AC^2 = AD^2 + DC^2 \rightarrow AC^2 = s^2 + t^2 \quad (4)$$

But $BC^2 = AB^2 + AC^2$, and $BC = r + t$; by substitution into above, we obtain:

$$\begin{aligned} BC^2 &= r^2 + s62 + s^2 + t^2 \\ (r+t)^2 &= r^2 + 2s^2 + t^2 \\ r^2 + 2rt + t^2 &= r^2 + 2s^2 + t^2 \end{aligned}$$

or $2rt = s^2$, which gives us the answer $rt = s^2$.

Exercise **??**: ADD FIGURE:

$\triangle ABC \sim \triangle LMN \to \frac{AB}{LM} = \frac{AC}{MN} = \frac{BC}{LN}$, or $\frac{AB}{W} = \frac{AC}{F_2} = \frac{BC}{F_1}$, thus $\frac{BC}{AB} = \frac{F_1}{W}$ and $\frac{AC}{AB} = \frac{F_2}{W}$.

We have $\angle A = 30°$. Using the known relationships in a 30-60-90 triangle, mentioned in the Trigonometry Module, $\frac{BC}{AB} = \frac{1}{2}$, $\frac{AC}{AB} = \frac{\sqrt{3}}{2}$

Using the equalities above, we obtain:

$$\frac{BC}{AB} = \frac{1}{2} = \frac{F_1}{W} \to F_1 = \frac{W}{2}$$
$$\frac{AC}{AB} = \frac{\sqrt{3}}{2} = \frac{F_2}{W} \to F_2 = \frac{W\sqrt{3}}{2}$$

Exercise **??**:

ADD FIGURE:diagrams

$A_{outer} = \pi r_2^2$, and $A_{inner} = \pi r_1^2$.

The washer is what is left of the outer circle, when the inner one is taken away.

Its area will be:

$A_w = \pi r_2^2 - \pi r_1^2$

Using a few algebraic manipulations,

$A_w = \pi(r_2^2 - r_1^2) = \pi(r_2 + r_1)(r_2 - r_1) = 2\pi\frac{r_2+r_1}{2}(r_2 - r_1) = 2\pi r_{avg}\Delta r$

Exercise **??**:

ADD FIGURE:

$A_{sphere} = 4\pi R^2 = 64\pi$

$R^2 = 16, R = 4$

$V_{sphere} = \frac{4\pi R^3}{3} = \frac{4\pi \cdot 64}{3} = \frac{256\pi}{3}$

Exercise **??**:

ADD FIGURE:

The radius of the sphere is $R = \frac{L}{2}$.

For the first part, the area of the cube is $6 \cdot L^2$, and the surface area of the sphere is $4\pi R^2 = 4\pi \left(\frac{L}{2}\right)^2 = \pi L^2$. Thus the ratio is

$$\frac{A_{cube}}{A_{sphere}} = \frac{6L^2}{\pi L^2} = \frac{6}{\pi} \approx 1.91$$

For the second part, $V_{cube} = L^3$. $V_{sphere} = \frac{4\pi R^3}{3} = \frac{4\pi}{3}\left(\frac{L}{2}\right)^3 = \frac{\pi L^3}{6}$. Thus the ratio is

$$\frac{V_{cube}}{V_{sphere}} = \frac{L^2}{\frac{\pi L^3}{6}} = \frac{6}{\pi} \approx 1.91$$

Exercise **??**:
ADD FIGURE:

$$\triangle ABC \sim \triangle ADE \rightarrow \frac{AB}{AD} = \frac{BC}{DE}$$

or $\frac{h/3}{h} = \frac{BC}{R} \rightarrow BC = R/3$.
$V_{sm} = \frac{1}{3}\pi \cdot (BC)^2 \cdot AB = \frac{1}{3}\pi \left(\frac{R}{3}\right)^2 \cdot \left(\frac{h}{3}\right) = \frac{\pi R^2 h}{81}$
$V_{lg} = \frac{1}{3}\pi (DE)^2 \cdot AD = \frac{1}{3}\pi R^2 h$.

$$\frac{V_{sm}}{V_{lg}} = \frac{\pi R^2 h}{81} \cdot \frac{1}{\frac{\pi R^2 h}{3}} = \frac{1}{27}$$

N. B.: no units, since it is a *ratio* of two quantities with the same dimension, (length)^3.

Another method, dimensional analysis: since volume has dimension (length)^3, the formula for V must be ch^3 for some constant c related to the angle at the vertex. Thus, in the ratio, the constant cancels out, leaving $\left(\frac{h/3}{h}\right)^3 = \frac{1}{27}$.

Exercise **??**: ADD FIGURE:

Since the diameter of the small spheres is a/n, the radius will be

$$R = \frac{a}{2n}$$

Considering the threefold symmetry of the sphere and cube, it should be clear that we are dealing with n^3 small spheres packed inside the cube.

Take firs just one big mothball (sphere) inside the cube: $n = 1, r^3 = 1$. Then $R_1 = \frac{a}{2}$

ADD FIGURE:

The next number of spheres we can pack is 8, with $n = 2$, we get $R_2 = \frac{a}{4}$, and so on: you can continue with $n = 3$, packing $n^3 = 27$ spheres of radius $R_3 = \frac{a}{6}$, etc.

The volume of one sphere is $V = \frac{4\pi R^3}{3} = \frac{4\pi}{3}\left(\frac{a}{2n}\right)^3 = \frac{\pi a^3}{6n^3}$

But since we have n^3 spheres, the total volume of the spheres is $V_tot = n^3 \cdot V = n^3 \cdot \frac{\pi a^3}{6n^3} = \frac{\pi a^3}{6}$

So the final result is independent of n: no matter which you buy, the total mass of the mothballs – assuming constant density – is the same. This result is exactly the same as in Exercise **??**: the ratio of the volume of the inscribed sphere to the volume of the cube is $\pi/6$ no matter the sidelength L.

If mothballs' efficacy relates to their surface area, however, the surface area of one mothball is $4\pi R^2 = 4\pi\left(\frac{a}{n}\right)^2 = \frac{4\pi a^2}{n^2}$; so n^3 mothballs have surface area of $4\pi a^2 n$. This means the surface area goes up with n going up, so optimally the mothballs would be as powdered as possible. (But with sufficiently small mothballs, air no longer flows over all the surface area, but essentially only the top of the pile, negating the advantage. For our approximation of efficiency, though, finer is better.)

5 Analytic Geometry Review Module

Analytic geometry is the description and analysis of geometric curves and shapes in terms of mathematical coordinates. The subject is concerned mainly with geometry in two or three dimensions, using various kinds of coordinate systems. The scoep of tis module is limited to two dimension, using rectangular coordinates. It will deal with a few of the most basic results concerning straight lines,circles, parabolas, and other conic sections. Thos are about all you will need initially in basic calculus and physics.

5.1 Straight Lines

First we set up our coordinate system, with n origin and the conventional x and y axes, with the x axis horizontal and the y axis perpendicular to it on the plane of the paper. (We shouldn't really call the direction of the y axis "vertical", although we often do.) THe position of any point in the xy plane is the uniquely defined by the pair of numbers (x, y).

ADD FIGURE:diagram

Any straight line in the xy plane can then be described by an equation of the form

$$y = mx + c$$

If this equation applies, we say that y is a *linear function* of x. It is equally true that x is a linear function of y when $m \neq 0$. When $x = 0, y = c$, and when $y = 0$, $x = -c/m$. These conditions define the *intercepts* of the line on the x and y axes, respectively. If both m and c are positive, the general appearance of the line is as shown in the above graph.

If $(x_1, y_1$ and (x_2, y_2) are any two points on the line, we have

$$y_1 = mx_1 + c$$
$$y_2 = mx_2 + c$$

By subtraction, we get the result:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This ratio, the ratio of the vertical "rise" to the horizontal "run" between any two points, is called the *slope* of the line, and the equation $y = mx + c$ is called the *slope-intercept* form of the equation for a line. (You might remember the slope as "up over the over".)

A note about slopes: If we label the angle between a straight line and the positive direction of the x axis as θ, then we have the relation $m = \tan\theta$. This *geometrical* identification of the slope of a line in terms of an angle is fine in mathematics, where x and y are pure numbers and the axes are assumed to be marked off with equal scale divisions. However, it is seldom necessary, and you should be warned that it loses its usefulness – and can in fact be misleading – when you come to graphs of the relationship between two different kinds of quantities, measured in different units.

ADD FIGURE:diagram

For example, the graph opposite might describe the motion of a car traveling at a constant speed along a straight road. On the y axis we show its position s measured, say, in kilometers relative to its start position; the horizontal axis shows the time t measured, say, in hours. The speed of the car, in kilometers per hour, is the slope of this graph, measured as the change in s divided by the change in t between any two points on the line. We simply read off these values from the scales of s and t along the axes, and these scales are purely a matter of convenience. There is no particular meaning to the geometrical slope of the line as such.

In fact, here are two graphs of a rocket's distance from the Earth until it impacts the Sun:

ADD FIGURE:graphs

Most of the graphs we draw in science, engineering, economics, etc. are constructed with scales purely for convenience.

ADD FIGURE:graph

If we have a straight line as shown, where the point A (x_0, y_0) is a particular point on the line and the point P (x, y) is any *other* point, we can put

$$\frac{y - y_0}{x - x_0} = m$$

This can be rewritten in the form

$$y - y_0 = m(x - x_0$$

This is the so-called *point-slope* form of the equation of a line.

The *general* equation of a straight line may be written:

$$ax + by + c = 0$$

You should be able to recognize and use any of the above three forms of the equation of a straight line. For most purposes, the form $y = mx + c$ is easiest and the most convenient to use, but be comfortable with all of them.

Exercise 5.1.1:
Find the intercept c in the slope-intercept equation in terms of the quantities x_0, y_0, and m of the point-slope equation.
Find the slope m and the intercept c of the slope-intercept equation in terms of the constants a, b, c of the general equation.

Note: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

Exercise 5.1.2:
Draw a set of xy axes, marked off in equal intervals between -5 and $+5$ for both x and y, and sketch straight lines with the following values of m and c:

1. $m = 2, c = -4$
2. $m = -1, c = 4$
3. $m = \frac{1}{2}, c = 1$
4. $c = -1, m = -\frac{1}{3}$

Exercise 5.1.3:
Draw another set of xy axes, marked off as before, and draw the following lines, all passing through the point $(2, 1)$:

1. $m = 0$
2. $m = \frac{3}{2}$
3. $m = \infty$
4. $m = -\frac{2}{3}$

Exercise 5.1.4:
Draw yet another set of xy axes, marked off as before, and draw lines described by the following equations:

1. $x + y = 2$
2. $2x - y + 3 = 0$
3. $x + 2y + 4 = 0$
4. $x - 4 = 0$

By definition, *lines with the same value of m have the same slope. Lines with the same value of m but different values of c are therefore parallel.*

If two lines are perpindicular, the product of their slopes is 1.

We included an example of this in Exercise **??**. The result is easily shown: here is one way.

Let the point of intersection of two perpindicular lines be called O'. Take O' as a new origin, and mark off a distance l along each line as shown in the diagram. If $\angle AO'M = \alpha$, then $\angle O'BN$ is also α. With respect to O', the coordinates of A are (x_1, y_1), where $x_1 = l\cos\alpha$, $y_1 = l\sin\alpha$, and we then have

$$m_1 = \frac{y_1}{x_1} = \tan\alpha$$

Also, with respect to O', the coordinates of B are (x_2, y_2), where $x_2 = -l; \sin\alpha$,y$y_2 = l; \cos\alpha$, and so we have:

$$m_2 = \frac{y_1 2}{x_2} = \frac{l\cos\alpha}{-l\sin\alpha} = -\cot\alpha$$

Therefore, $m_1 m_2 = (\tan\alpha)(-\cot\alpha) = -1$

Unless two planes in the xy plane are parallel, they must intersect somewhere. To find the point of intersection, we use the condition that this point must be onboth lines, which gives us a pair of simultaneous equations.

For instance, find the point of intersection of the lines $2x - y - 2 = 0$, $x - 2y + 2 - 0$.

Let the point of intersection be (x_1, y_1). Then:

$$\begin{aligned} 2x_1 = y_1 &= 2 \\ x_1 - 2y_1 &= -2 \end{aligned}$$

Solving these equations gives $x_1 = 2<$ $y_1 = 2$. (Check this!)

Drawing sketches for such problems is highly recommended!

Exercise 5.1.5:
Find the point of intersection of the lines $3x - 4y - 6 = 0-$ and $x + 2y + 3 = 0-$.

Exercise 5.1.6:
Which of the following lines are parallel? perpendicular? intersect?

1. $y = 2x + 4$
2. $2y + x - 1 = 0$
3. $y = \frac{-3}{2}x + 1$
4. $3x + 2y = -1$
5. $2 = (y - 4)/x$

5.2 Conic Sections

You have probably learned that the circle, the parabola, the ellipse, and the hyperbola are called "conic sections." This is because they arise from the intesection of various planes with a circular cone. The diagram below shos how this comes about. Circles are formed when a plane is drawn at right angles to the axis of the cone. Parabolas are formed when a plane is drawn parallel to one of the generators (sloping sides) of the cone. Ellipses are formed by planes intermediate between these two directions. Hyperboolas are formed by planes that intersect both top and bottom parts of the copmlete double cone: the diagram shows the articular plane that is parallel to the axis of the cone.

ADD FIGURE:diagram

From Hughes-Hallett, *Math Workshop: Elementary Functions*, W. W. Norton Co. (1980).

We will not make sue of these pictures to obtain equations for conics: that would be partciularly ugly. Instaead, we',ll work from geometric definitions of how these curves are constructed in terms of a rectangular coordinate system. But it is worth seeing how they are members of a family.

5.2.1 Circles

There are just two features that characterize a circle: it has a certain radius r that is constant; and its center is at a certain point (x_0, y_0). To expressa these properties in mathematical terms, we combine them in a statement that the distacne between the center of the circle and any pouitn (x, y) on its circumference is equal to r. For any two points on the xy plane, the square of the distance between them is given by the Pythagorean Theorem. The equation of the circle is therefore given by:

$$(x - x_0)^2 + (y = y_0)^2 = r^2$$

This is the *standard form* of the eqyuation of a circle.

If the center of the circle is at the origin, this reduces to the simple form:

$$x^2 + y^2 = r^2$$

Exercise 5.2.1:
Write the equations of the following circles, and sketch them on graph paper:

1. Center at (3,2), radius 3;
2. Center at (0,3), radius 5;
3. Center at (-2,2), passing through (2,5)

If the equation of a circle with its center at an earbitrary point is expanded out into individual terms and the contributions to the constant term are collected together, the resulting equation is of the form

$$x^2 + y^2 + ax + by + c = 0$$

It is not necessarily true, however, that any equation of the abocve form descibes a circle. We can see this by completing the square for both terms. If we do this, and rearrange, we get:

$$\left(\right)x + \frac{a}{2}\Big)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - c$$

The right-hand side can be positive, negative, or zero.

If it is positive, we can equate it to r^2, the equation does describe a circle – with its center at $(-\frac{a}{2}, -\frac{b}{2})$

If it is negative, there are no real values of x and y that satisfy the equation. The sum of two squares must be greater than 0, since any one square is greater than 0. The equation cannot describe any real curve or point.

If it is zero, the equation can be satisfied only for a sinhgle point: $x = -a/2$, $y - -b/2$, since both of the squares on the left myust be xero if the equation is to be satisfied. This is, if you like, a circle of zero radius, againh with its center at $(-a/2, -b/2)$.

Exercise 5.2.2:
Analyze which of these equations describ circles. For those that do, find the radius and the coordinates of the center:

1. $x^2 + y^2 - 2x + 89 = 0$
2. $x^2 + y^2 - 4x + 2y + 6 = 0$
3. $x^2 + y^2 + 4x - 6y - 12 = 0$
4. $x^2 + y^2 - 8x + 6y + 26 = 0$

Exercise 5.2.3:
Here is a nice exercise that puts together the analytic geometry of straight lines and circles. Take a circle of radius r with its center at the origin. See if you can prove, by analytic geometry, the theorem from plane geometry that ssays that an angle inscribed ina semicircle is a right angle. [First find the slopes of the lines AP and BP in terms of x, y, and r. To confirm the perpendicularity condition $m_1 m_2 = -1$; use $x^2 + y^2 = r^2$.]
ADD FIGURE:diagram

5.2.2 Parabolas

A parabola is defined geometrically as a plane curve that is the locus of points equidistant from a fixed point (the *focus*,l F) and a fixed straight line (the *directrix*). The word focus can be taken literally: a mirror made into a parabolic (or, more strictly, a paraboloidal) shape will bribring parallel light to a focus at RF. (A paraboloid is a parabola rotated about its axis of symmetry.)

Suppose that the foucs is at the point $(p, 0()$, and that the directrix is the line $x = -p$. Youc an see right away that the oribgin is a point on the curve; it is halfway between the focus and the directix. For any other point $P = (x, y)$, we have the more complicated condition that the distacne FP is equal to the length of the perpendicular PN drawn from P to the directrix.

Using the Pythagroean Theeam, this requires

$$(FM)^2 + (MP)^2 = (PN)^2$$

Putting $FM = x - p$, $MP = y$, and $PN = x + p$, this gives:

$$(x - p)^2 + y^2 = (x + p)^2$$

Multiplying this out, we get

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$

which after canceling and rearraanging gives

$$hy^2 = 4px$$

This equation describes a parabola with its axis along the x axis and its *vertex* (the rounded end) at the orign. The parabola is symmetrical about the x axis, because for any given value of x we have $y = \pm\sqrt{4px}$.

The coefficient $4p$ has a geometrical significance. If we draw the line $x = p$ through the focus, it uts the parabola at points A and B for which $ty^2 = 4p^2$ and so $y = + \pm 2p$. Thus the distance between these points is equal to $4p$; it is a measure of of the width of the parabola.

We can construct similar parabolas with different positions of the focus and different locations and directions of the directrix. For example, a aparabola with the same value of p as before (i.e., the same fdistance between focus and directrix), with its vertex at $(h, hk0$ and its directrix parallel to the y axis has the equation

$$(y - k)^2 = 4p(x - h)$$

ADD FIGURE:diagram

The figures below show some other examples.

ADD FIGURE:figures of $x^2 = 4py$, $x^2 = 0 - 4py$, and $(x - h)^2 = 4p(y - k)$

The first of these, whose equation can be rewritten in the form $y = Cx^2$, iis a particularly useful one to rememvber, because we often encounter relationships between two quantities where the dependent variable y is porportional to the square of the independent variable (x). For instance, the distance y as a function of time x that an object falls from rest under gravity.

Exercise 5.2.4:
Take some graph paper and sketch the following parabolas:
$y = -x^2$
$x = 2(y-1)^2$
$y = (x+2)^2$

There are two standard forms for teh equation of a parabola,. depending on whether its axis is along the x direction or the y direction. Notice that if the axis is parallel to x, the equation is quadratic in y; if the axis is parallel to yk, the equation is quadratic in x. You can of course have parabolas whose axis is in some arbitrary direction, but we won't bother with those . Try to figure out what their equations would look like, though!

If the axis is parallel to x, the equation can be written in the form

$$x^2 + ax + by + c = -$$

As with the circle, we can put this into standard form, beginning by completing the square on the left side:

$$\left(x + \frac{a}{2}\right)^2 = -by - c + \frac{a^2}{4} = -b\left(y + \left(\frac{c}{b} - \frac{a^2}{4b}\right)\right)$$

Comparing this with $(x-h)^2 = 4p(y-k)$, we can find the values of h, k, and p.

For example, we can find the coordinates of the focus and the vertex of the parabola $x^2 - 6x - 8y + 17 - 0$.

Completing the square on the x terms and rearranging, we have $(x-3)^2 = 8y - 17 + 9 = 8y - 8 = 8(y-1)$. Thus the vertex is at $(3,1)$, and $p = 2$. In this parabola, the focus ais a distance p above the vertex; its coordinates are therefore (3,3).

Exercise 5.2.5:
The following equations are in the general polynomial form of tequations for parabolas. Put them into standard form, locate the vertices and foci, and sketch the curves:

1. $x^2 - 4x - 4y = 0$
2. $12x - 6y - y^2 - 33 = 0$
3. $y^2 + 4y + 4x = 8$
4. $2x^2 - 4x - y + 1 = 0$

5.2.3 Ellipses

ADD FIGURE:diagram of pin/string

You are very likely familiar with the "pins-and-string" definition of an ellipse. Take a piece of string and attach its ends to pins at two fixed points, F and F', on a piece of paper. Take a sharp pencil and hold it vertically. Move it until it just makes the string tight, and then move it around, always in contact with the string, so as to create a closed curve. This curve is an ellipse. If the length of the string is called $2a$, we have:

$$FP + F'P = 2a$$

The points F and F' are the *foci*. If the ellipse were a mirror, light from a point source at F would come to a focus at F', and vice versa. The ration of the distance OF to the distance OA is called the *eccentricity*, ϵ of the ellipse:

$$\epsilon = \frac{OF}{OA}$$

(Note that, if F and F' are both moved to the center O, the ellipse becomes a circle and by the above definition the eccentricity is zero.)

Take an origin at the geometrical center of the figure, with an x axis along the line $F'F$ (AA'), and a y axis along OB. Let the distance OF $(= OF')$ be L, and let OB be b.

ADD FIGURE:diagram repeated.

First, imagine that P is moved to A. Then $FA + AF' = 2a$> But, by symmetry, $FA = F'A'$. Therefore $AF' + F'A'$ (which adds up to AA') is equal to $2a$. So $OA = a$.

Next, imagine P is moved to B; this gives FB $(= BF')$ $= a$. Using Pythagoras' Theorem, one finds $b = \sqrt{a^2 - l^2}$.

Finally, let P be at some arbitrary point (x, y) on the ellipse. From the condition $FP + PF' = 2a$, and using Pythagoras' Theorem again, we have:

$$\sqrt{(l-x)^2 + y^2} + \sqrt{(l-x)^2 + y^2} = 2a$$

If you do some algebra on this, you can verify that it leads to the following equation for the ellipse in xy coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The distances a and b are called the *semi-major axis* and the *semi-minor axis*, respectively $(a > b)$.

Exercise 5.2.6:
Go through the process of deriving this last equation from the preceding one. [Begin by putting one of the radicals on the other side of the equation and squaring. Some rearrangement, followed by another squaring, will do most of the rest, but you will need the relation between a, b, and l.]

If the center of the ellipse is not at the origin but at some point (h, k), one just replaces x by $(x - h)$ and y by $(y - k)$ in the equation. The resulting equation, when multiplied out, will in general have quadratic and linear terms in x and y, plus a constant. The thing that immediately distinguishes the equation of an elliose from the equation of a circle is that, for a circle, the coefficients of the x^2 and y^2 terms are equal, whereas for an ellipse they are different.

Exercise 5.2.7:
Sketch the ellipse $4x^2 + 9y^2 = 36$, locate its foci and find the eccentricity.

For $b = a$ the equation of an ellipse turns into the equation for a circle, and it is clear that the connection between the two is close. In fact, an ellipse can be though of as the geometrical projection of a circle onto another plane. The diagram here indicates this. If the angle between the planes is θ, then a circle of radius b projects into an ellipse of semi-minor axis b and semi-major axis $a = b\sec\theta$. If the coordinates of a point P on the circle are labeled (x, y), the coordinates of the corresponding point P' on the ellipse are $(x\sec\theta, y)$. Thus every x-coordinate on the ellipse is stretched by the factor $\sec\theta = a/b$. One can use this fact to infer that the *area* of an ellipse is given by $\boxed{\pi ab}$.

ADD FIGURE:diagram of projection.

5.2.4 Hyperbolas

ADD FIGURE:generic hyperbola x-axis.

The way of constructing a hyperbola looks very similar to that for an ellipse, but is not nearly so easy to do in practice. Again one has an axis with two foci on it. However, the prescription for a hyperbola is to move a point P so that the *difference* of its distances from the foci is constant.

If the foci are the x axis, the xy equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is rather like an ellipse turned inside out. The distance intercepted by the curve on the x axis is again $2a$, but this time it is a minimum distance rather than a maximum. At large distances from the origin, the two branches of the curve approach two straight lines — the *asymptotes* — whose equations are the two possibilities allowed by the equation $(x^2/a^2) - (y^2/b^2) = 0$:

$$y = \pm\frac{b}{a}x$$

ADD FIGURE:generic y-axis hyperbola.

If the foci lie on the y axis, the branches of the hyperbola are as shown in the second diagram. The equation to the curve is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

and the asymptotes are given by

$$y = \pm\frac{a}{b}x$$

In other words, the whole picture is turned through 90° with respect to the first one.

Exercise 5.2.8:
Using the condition $FP - F'P = 2a$, together with the standard equation of a hyperbola, show that the distance between the foci is $2\sqrt{a^2 + b^2}$.

ANother very important form of the equation of a hyperbola — and in some respects much the simplest, is

$$xy = c$$

where c is a constant.

ADD FIGURE:asymptote=axis hyperbolas.

This gives the two possibilities shown here. This is the *rectangula hyperbola* — its asymptotes are perpindicular. A good physical example of this (with $c > 0$) is the relation between pressure and volume for an ideal gas at constant temperature:

$$pV = \text{constant}$$

Exercise 5.2.9:
Sketch the hyperbola $xy = 9$, and locate its foci.

6 Answers to Exercises

Answers not given to exercises **??**, **??**

Exercise **??**: (a) $c = y_0 - mx_0$; (b) It is very confusing to use the same letter (c) for two different quantities in a problem. Before solving, we change the name of c to d in the general equation, $ax + by + d = 0$, and leave it as c in the slope-intercept equation, $y = mx + c$. Then $m = -a/b$, $c = -d/b$.

Exercise **??**: ADD FIGURE:sketch

Exercise **??**: ADD FIGURE:sketch

Exercise **??**: ADD FIGURE:sketch

Exercise **??**: $x = -1/5, y = -7/5$.

Exercise **??**: $\ell_c \parallel \ell_d$, $\ell_b \perp \ell_a$, $\ell_b \perp \ell_e$. (ℓ_e, ℓ_a are the same line). All non-parallel lines intersect!

Exercise **??**: (a) $(x-3)^2 + (y-2)^2 = 9$ ADD FIGURE:diag

(b) $x^2 + (y-3)^2 = 25$ ADD FIGURE:diag

(c) $(x+2)^2 + (y-2)^2 = 25$ ADD FIGURE:diag

Exercise **??**: (a) No; (b) No; (c) Center at $(-2, 3)$, $r = 5$; (d) Single point, $(4, -3)$.

Exercise **??**:

$$m_1 = \text{slope } AP = \frac{y-0}{x-r} \tag{5}$$

$$m_2 = \text{slope } BP = \frac{y-0}{x-(-r)} \tag{6}$$

$$m_1 m_2 = \frac{y}{x-r} \cdot \frac{y}{x+r} = \frac{y^2}{x^2-r^2} \tag{7}$$

But $x^2 + y^2 = r^2$ so $x^2 - r^2 = -y^2$ so $m_1 m_2 = y^2/(-y^2) = -1$.

Exercise **??**: (a) ADD FIGURE:diag

(b) ADD FIGURE:diag

(c) ADD FIGURE:diag

Exercise **??**: (a) $(x-2)^2 = 4(y+1)$

(b) $(y+3)^2 = 12(x-2)$

(c) $(y+2)^2 = -4(x-3)$

(d) $(x-1)^2 = \frac{1}{2}(y+1)$

ADD FIGURE:diag x4

Exercise **??**: $(\sqrt{5}, 0), (-\sqrt{5}, 0)$; $\epsilon = \ell/a = \frac{\sqrt{5}}{3}$ ADD FIGURE:diag

Exercise **??**: Foci at $(\sqrt{13}, 0)$, $(-\sqrt{13}, 0)$ ADD FIGURE:diag
Exercise **??**: Foci at $(3\sqrt{2}, 3\sqrt{2})$, $(-3\sqrt{2}, -3\sqrt{2})$ ADD FIGURE:diag

This module is based largely on an earlier module prepared by the MIT Mathematics Department.

7 Geometry and Analytic Geometry Review Problems

7.1 Triangles, similarity of figures

Two triangles are similar when the three angles of one are the same as the three angles of the other. (In practice, you only have to show this for two angles, since the third angles will then automatically be equal.)

If two triangles are similar, *corresponding sides are proportional, and the altitudes on those sides are proportional*. This is the essential fact that is most often used in scientific problems.

ADD FIGURE:diagrams of similarity

In working the problems below, in your diagram mark the equal angles, for similar triangles, mark two of the three pairs of equal angles, as is done above. Parallel lines are indicated by arrows. Figures are not drawn to scale!

Problem 1: In each of the figures, tell how many degrees angle x has.

ADD FIGURE:3 figures

Problem 2: A line segment of length 3 joins two sides of a triangle, is parallel to the third side, and has distance 1 from that side and 2 from the opposite vertex. How long is the third side of the triangle?

Problem 3: The light from a building goes through a tiny hole 100 meters away. THe image of the building is upside down on a vertical screen 2 meters away from the hole. If the image is 1.2 meters high, how tall is the building?

Problem 4: In each of the diagrams, find the length of the line segment x marked.

ADD FIGURE:2 figures

Problem 5: In each picture, find the length of the line segment x marked.

ADD FIGURE:3 figures.

Problem 6: In the accompanying figures, the smaller triangle has area A. What is the area of the larger triangle?
ADD FIGURE:2 figures

Problem 7: A streetlamp 20 feet high is 15 feet horizontally distant from a woman 5 feet high. How long is the shadow she casts?

Problem 8: How long is the diagonal of

1. a cube of side 1
2. a rectangular box of sides 1, 2, and 3?

7.2 Circle Problems

Problem 9: Some arcs are given. Determine the angles in the left triangle.
ADD FIGURE:2 figures

Problem 10: The smaller triangle has area A; what is the area of the larger triangle?
ADD FIGURE:

Problem 11: Find the area of the kite-shaped region pictured.
ADD FIGURE:

Problem 12: Find the lengths x and y in the accompanying diagram.
ADD FIGURE:

Problem 13: The circle has radius 1, the line segment PA has length 2, and is tangent to the circle. What is the length of PB?
ADD FIGURE:

Problem 14: The two circles are concentric, with radii respectively 1 and 3. The chord is tangent to the inner circle. How long is it?
ADD FIGURE:

Problem 15: The circle has radius 2; PA and PB are tangents, and PO has length 4. What is the area of quadrilateral $OAPB$?
ADD FIGURE:

Problem 16: The two line segments marked are equal, and PA, PB are tangents. What are the degrees of the two circular arcs 6.0pt24.88pt$\widehat{\text{ADB}}$ and 6.0pt24.88pt$\widehat{\text{ACB}}$ having A and B as their endpoints?
ADD FIGURE:

Problem 17: Find the length of x in the accompanying picture. The arrows indicate parallel line segments; $AB = 1$ and CD is tangent; the circle has radius 2.
ADD FIGURE:

7.3 Areas and Volumes

Problem 18: A circle has radius R; what is the area and perimeter of a $72°$ circular sector?

Problem 19: What is the area of a triangle having

1. all sides of length 3;
2. two sides of length 4, with a $30°$ angle between them?

Problem 20: Find the areas of the two parallelograms shown:
ADD FIGURE:2 diagrams

Problem 21: A trapezoid has parallel sides of lengths 5 and 9, and its other two sides both have length 3. What is its area?

Problem 22: A square and a circle have the same perimeter. WHat is the ratio of their respective areas?

Problem 23: The ring-shaped region between two concentric circles has inner radius a and outer radius b. Give an expression for its area A, and show that if we draw the median circle on the ring, halfway between the inner and outer boundaries, then

$$A = (\text{length of median circle})(\text{thickness of ring})$$

ADD FIGURE:target diag.

Problem 24: The radius of a riht circular cylinder is $1/3$ of its height; the volume of the cylinder is 24π. What is the radius and height?

Problem 25: These connect radius R, surface area S, and volume V of a sphere:

1. A sphere has surface area 12π; what is its volume?
2. Find in terms of R the ratio S/V; simplify your answer.
3. What is the total surface area of a solid hemisphere of radius R?
4. A sphere of radius R is centered at the origin of xyz-coordinates. What is the surface area and volume of that portion of it for which $x, y, z \geq 0$?

Problem 26: A sphere is inscribed in a cube, so that it is tangent to all six sides of the cube. What is the ratio (sphere:cube) of their surface areas and volumes?

Problem 27: An upright symmetrical pyramid has a square base with sides of length 10, and a height of 12. What is its volume, and the area of one of its slanted faces?

Problem 28: A right circular cone has radius 2 and height 6. What is the volume of the slice cut off by a plane parallel to the base and distance 2 from it?

Problem 29: A tetrahedron has three edges of lengths 2, 3, and 5 which intersect at right angles. Sketch it and determine its volume.

7.4 Lines

Problem 30: A line of slope 6 passes through the point $(2, -3)$. Where does it cross the x-axis?

Problem 31: At what point do the lines given by $3x + 4y = 12$ and $3x - 5y = -6$ intersect?

Problem 32: At what point does a line with slope -1 passing through the point $(2, 5)$ intersect the line having slope 2 and y-intercept -2?

Problem 33: What is the equation of a line perpindicular to the line $2x - 5y = 1$ and having y-intercept 3?

Problem 34: The rectangle $ABCD$ is symmetric about the y-axis. What is the equation of a line through C parallel to the diagonal BD? $B = (5, 3), C = (5, -1)$.

ADD FIGURE:diagram

Problem 35: Find the area of the finite region bounded by the lines

$$x = 0; \qquad y = 0, \qquad 3x + 5y = 30$$

Problem 36: Find the equation of a line through the origin perpendicular to the line having y-intercept -2 and x-intercept 3. (Draw a sketch.)

7.5 Circles

Problem 37: Give in the form $x^2 + y^2 + ax + by + c = 0$ the equation of the circle

1. with center at the origin and radius 3
2. with center at the origin and tangent to the line $x + y = 1$.
3. with center at $(1, 1)$ and passing through the origin
4. with center at $(-1, 1)$ and tangent to both coordinate axes.

Problem 38: A circle centered at the origin has radius 11. Is the point $(7, 8)$ outside or inside the circle?

Problem 39: Where do the circle and line intersect in the following examples?

1. $x^2 + y^2 = 5$, $x + y = 3$
2. $x^2 + y^2 = 10$, $x + 3y = 10$
3. circle of radius 3 centered at $(0, 0)$, line of slope 3 through $(0, 0)$

7.6 Conics

Problem 40: What is the length of the semi-major and semi-minor axes of the ellipses

$$\frac{x^2}{10} + \frac{y^2}{3} = 1$$
$$9x^2 + 4y^2 = 25$$

Problem 41: Write the equation of the ellipse with center at the origin, axes along the two coordinate axes, and whose x-intercepts are ± 3, y-intercepts are ± 2.

Problem 42: Write the equation of a parabola whose minimum point is on the y-axis at -1, and whose x-intercepts are ± 2.

Problem 43: Write in the form $y = ax(x - c)$ the equation of a parabola whose high point is at $(2, 1)$, and which goes through the origin.

Problem 44: Write the general form for the equation of a hyperbola having the lines $y = \pm 2x$ as asymptotes.

Problem 45: What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, straight lines, point, or none?

1. $x^2 + 2y^2 - 12 = 0$

2. $6x^2 - 5y^2 = 0$
3. $15 - x^2 - y^2 = 0$
4. $x^2 + 2y - 3 = 0$
5. $xy = 2$
6. $4y^2 - x^2 = 2$
7. $x^2 + y^2 + 2x - 5 = 0$
8. $y^2 + 2x = 5$
9. $x^2 - y^2 = 0$
10. $x^2 + 2y^2 + 1 = 0$
11. $xy = 5$
12. $x^2 + y = 0$
13. $x^2 - y^2 = 4$
14. $2x^2 + 3y^2 = 5$
15. $xy = 0$
16. $3x^2 + 3y^2 - 6y = 10$
17. $x^2 - 5y^2 = 10$
18. $x^2 + 3y^2 = 0$
19. $x + 3y^2 - 5 = 0$
20. $x^2 - 4y^2 = 0$

8 Solutions to Geometry Review Problems

Solution 1: a: ADD FIGURE:

Since the triangle is isosceles, the base angles α are equal.

$$2\alpha + 104 = 180 \rightarrow \alpha = 38$$

$$x = 180 - 38 = 142$$

(or $x = 104 + \alpha = 104 + 38 = \boxed{142}$)

b) ADD FIGURE:

Two angles α are equal since lines are parallel. $\alpha = 60$ since vertical angles are equal.

$$50 + x + 60 = 180 \rightarrow \boxed{x = 70}$$

c) ADD FIGURE:

Since vertical angles are equal, picture is

ADD FIGURE:

$$x + 20 + 84 = 180 \rightarrow \boxed{x = 76}$$

Solution 2: ADD FIGURE:

The two triangles are similar, so corresponding lines are proportional:

$$\frac{OC}{OC'} = \frac{AB}{A'B'} \rightarrow \frac{2}{3} = \frac{3}{A'B'}$$

If you prefer, do it in two steps, using similarity of the right triangles:

$$\frac{OC}{OC'} = \frac{OA}{OA'} = \frac{AB}{A'B'}$$

In either method, we find $\boxed{A'B' = \frac{9}{2}}$.

Solution 3: ADD FIGURE:

Triangles are similar, since bases are parallel lines. Corresponding lines are proportional (c.f. reasoning in previous problem). So

$$\frac{x}{100} = \frac{1.2}{2} \rightarrow \boxed{x = 60 \text{ meters}}$$

Solution 4: a: ADD FIGURE:

The triangles are similar (both have angle A and a right angle) so

$$\frac{x}{5} = \frac{12}{x+4} \to x^2 + 4x = 60 \to (x+10)(x-6) = 0$$

so $\boxed{x = 6}$ ($x = -10$ makes no sense.)

b: ADD FIGURE:

$\triangle ABC$ is similar to the small triangle: two equal angles and two right angles. Thus, $\frac{x}{2} = \frac{AB}{10}$, and $AB = 8$ by the Pythagorean Theorem, so $x = \frac{2 \cdot 8}{10} = \boxed{\frac{8}{5}}$.

Solution 5: a: ADD FIGURE:

The two angles marked $|alpha$ are equal since both of them equal $90 - \beta$. Thus, the left and right triangles are similar, as both have α and a right angle. Thus, $\frac{1}{2} = \frac{2}{x} \to \boxed{x = 4}$.

b: ADD FIGURE:

This is the same picture as the above. The bottom and the big triangle are similar (both have angle A and a right angle. Thus, $\frac{x}{1} = \frac{1}{3} \to \boxed{x = \frac{1}{3}}$.

c: ADD FIGURE:

Triangles are similar, since two angles are equal (alternate interior angles to two parallel lines). Thus,

$$\frac{1}{x} = \frac{3}{\sqrt{3^2+6^2}} \to x = \frac{\sqrt{3^2+6^2}}{3} = \frac{3 \cdot \sqrt{1+2^2}}{3}$$

so $\boxed{x = \sqrt{5}}$.

Solution 6: a: ADD FIGURE:

Triangles are similar (2 right angles and equal vertical angles). Thus, $\frac{2}{3} = \frac{h}{x} \to x = \frac{3}{2}h$. Now, let $A_{top} = \frac{2h}{2}$. Then $A_{bottom} = \frac{3x}{2} = \frac{3}{2} \cdot \frac{3}{2}h$ thus $A_{bottom} = \frac{9}{4}A_{top}$.

b: ADD FIGURE:

The two triangles are similar, so corresponding lines are proportional. Thus

$$\frac{h_1}{h_2} = \frac{5}{2} \to h_1 = \frac{5}{2}h_2$$

$$A_1 = \frac{5h_1}{2} \qquad A_2 = \frac{2h_2}{2} \rightarrow A_1 = \frac{5}{2} \cdot \frac{5}{2} A_2 = \boxed{\frac{25}{4} A}$$

Solution 7: ADD FIGURE:

Two triangles are similar.

$$\frac{x}{5} = \frac{x+15}{20}$$

Crossmultiply: $20x = 5x + 75 \rightarrow x = \boxed{5}$.

Solution 8: ADD FIGURE:

The diagonal is AD. We have a right triangle

ADD FIGURE:

so $AD = \sqrt{3}$.

ADD FIGURE:

By same reasoning, $AD = \sqrt{14}$.

8.1 Circle problems

Solution 9:

a: ADD FIGURE:

6.0pt$\overset{\frown}{\text{AB}} = 180 - 100 = 80 \rightarrow \angle ACB = 80\,\text{deg}$ (central angle is equal to the arc it subtends). Then, $\triangle ACB$ is isosceles so the other two angles are both 50 deg.

b: ADD FIGURE:

$\angle A = 100$ (since half the arc it subtends)

$\angle B = 40$ (same reason)

so $\angle C = 40$

Solution 10: ADD FIGURE:

Angles A and B are equal, since both subtend the same arc. The two triangles are similar, thus $\frac{h_2}{h_1} = \frac{2}{1}$ so $h_2 = 2h_1$.

ADD FIGURE:

$A_2 = \frac{2h_2}{2}, A_1 = \frac{1h_1}{2} \rightarrow A_2 = h_2 = 2h_1 = 4A_1$, so $\boxed{4A}$.

Solution 11: A is a right angle, since it subtends 180 deg of arc. Thus the triangles

ADD FIGURE: and ADD FIGURE:

are similar (the two angles marked are both complementary to $\angle A$, or they subtend equal arcs, by symmetry). Hence, $\frac{2}{x} = \frac{x}{4}$, $x^2 = 8 \to x = 2\sqrt{2}$. Thus, the area of $\triangle ABC$ is $\frac{2\sqrt{2}\cdot 3}{2} = 6\sqrt{2}$, and the area of the kite is $\boxed{12\sqrt{2}}$.

ADD FIGURE:

Solution 12: ADD FIGURE:

The angle at A is a right angle (it is inscribed in a semicircle): both smaller triangles are similar to the big one. So, $\frac{y}{1} = \frac{4}{y} \to y = \boxed{2}$ and $\frac{x}{4} = \frac{3}{x} \to x = \boxed{2\sqrt{3}}$.

Check by the Pythagorean Theorem: $x^2 + y^2 = 4^2 = (2\sqrt{3})^2 + 2^2 = 4^2$.

Solution 13: ADD FIGURE:

$OA \perp PA$ (radius is always perpindicular to tangent) so by Pythagoras, $PO = \sqrt{2^2 + 1^2} = \sqrt{5}$, so $PB = \boxed{\sqrt{5} - 1}$.

Solution 14: ADD FIGURE:

The angle at A is a right angle, since radius is perpindicular to tangent. By Pythagorean Theorem, ADD FIGURE: $AB = \sqrt{3^2 - 1^2} = 2\sqrt{2}$ so $CB = \boxed{4\sqrt{2}}$.

Solution 15: ADD FIGURE:

$PB = \sqrt{4^2 - 2^2} = 2\sqrt{3}$.

A and B are right angles (radius perpindicular to tangent). So area of OPB is $\frac{2\cdot 2\sqrt{3}}{2} = 2\sqrt{3}$.

By symmetry, OBP and OAP are congruent, so area $AOBP = \boxed{4\sqrt{3}}$.

Solution 16: ADD FIGURE:

A is a right angle (radius perpendicular to tangent). $OA = OC$ since both are radii; so $OA = \frac{1}{2}OP$ so we have a $30\,\deg - 60\,\deg - 90\,\deg$ triangle. Thus, 6.0pt$\overset{\frown}{\text{AC}} = 90\,\deg$, so 6.0pt24.88pt$\overset{\frown}{\text{ACB}} = 120\,\deg$ and 6.0pt24.88pt$\overset{\frown}{\text{ADB}} = 240\,\deg$.

Solution 17: ADD FIGURE:

The two triangles are similar (the two equal angles marked are alternate interior angles). So $\frac{x}{2} = \frac{CB}{2} \rightarrow x = CB = \sqrt{3^2 - 2^2} = \boxed{\sqrt{5}}$.

8.2 Areas and Volumes

Solution 18: $\frac{360}{72} = 5$ so area is $\frac{1}{5}$ so area of the circle is $\pi R^2/5$, perimeter is $2R + 2\pi R/5$ (see diagram)

Solution 19: ADD FIGURE:

a: ADD FIGURE:

ABD is a 30-60-90 triangle, so $AD = \frac{3}{2}\sqrt{3}$ (or use Pythagoras: $AD = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = 3\sqrt{\frac{3}{4}} = \frac{3}{2}\sqrt{3}$.)

Thus, area $ABC = \frac{1}{2} \cdot 3 \cdot \frac{3}{2}\sqrt{3} = \boxed{\frac{9}{4}\sqrt{3}}$.

b: ADD FIGURE:

$h = 2$ (30-60-90 triangle). Thus, area is $\frac{1}{2} \cdot 4 \cdot 2 = \boxed{4}$. (Could also use trigonometry.)

Solution 20: a: ADD FIGURE:

Altitude is $\frac{\sqrt{3}}{2}$ since we have a 30-60-90 triangle. Thus, the area is $\boxed{3 \cdot \frac{\sqrt{3}}{2}}$.

b: ADD FIGURE:

$h = 3$ by Pythagorean Theorem, so area$=3 \cdot 16 = \boxed{48}$.

Solution 21: ADD FIGURE:

$h = \sqrt{3^2 - 2^2} = \sqrt{5}$ by Pythagorean Theorem.

Area is rectangle plus 2 triangles, or $5\sqrt{5} + 2 \cdot \frac{2\sqrt{5}}{2}$ (equal to the formula for area of trapezoid)

OR, area is h times the average of the two bases, giving, again, $7\sqrt{5}$.

Solution 22: Let a be the side of square, r the radius of the circle; then $2\pi r = 4a \rightarrow r = \frac{2a}{\pi}$. The ratio of the areas is then

$$\frac{a^2}{\pi \cdot \frac{4a^2}{\pi^2}} = \boxed{\frac{\pi}{4}}$$

Solution 23: ADD FIGURE:

$A = \pi b^2 - \pi a^2 = \pi \left(b^2 - a^2\right) = \pi(b+a)(b-a) = 2\pi\left(\frac{b+a}{2}\right)(b-a).$

But $2\pi\left(\frac{b+a}{2}\right)$ is the perimeter of the median circle, and $b-a$ is the thickness of the ring, as desired.

Solution 24: ADD FIGURE:

$r = \frac{h}{3}$; $V = \pi r^2 h = \pi \frac{h^2}{9} \cdot h = 24\pi$. Thus, $h^3 = 27 \cdot 8 \to \boxed{h = 6}, \boxed{r = 2}$.

Solution 25:

1. Surface area is $4\pi r^2 = 12\pi \to r = \sqrt{3}$. Volume is $\frac{4}{3}\pi r^3 = \boxed{4\pi\sqrt{3}}$.

2. $$\frac{s}{v} = \frac{4\pi R^2}{\frac{4}{3}\pi R^3} = \boxed{\frac{3}{R}}$$

3. Surface area of hemisphere is half the surface area of the sphere plus the flat side:
$$\frac{4\pi R^2}{2} + \pi R^2 = 3\pi R^2$$

4. ADD FIGURE:

   This is $\frac{1}{8}$ of both surface area and volume, so surface area is $\boxed{\frac{\pi R^2}{2}}$ and volume is $\boxed{\frac{\pi R^3}{6}}$.

Solution 26: ADD FIGURE:

Side view: edge of cube and diameter of sphere are of same length. Thus, the area ratio is $\frac{4\pi a^2}{6 \cdot (2a)^2} = \boxed{\frac{\pi}{6}}$ and the ratio of the volumes is $\frac{\frac{4}{3}\pi a^3}{(2a)^3} = \boxed{\frac{\pi}{6}}$.

Solution 27: ADD FIGURE:diag side view

volume is $\frac{1}{3}$ base $\cdot$ height, or $\frac{1}{3} \cdot 100 \cdot 12 = \boxed{400}$.

ADD FIGURE:area face diag.

area is $\frac{1}{2} \cdot 10 \cdot 13 = \boxed{65}$.

Solution 28: ADD FIGURE:diag side view

Vol. slice is equal to vol. cone less vol. top cone, or $\frac{1}{3}\pi \cdot 2^2 \cdot 6 - \frac{1}{3}\pi\left(\frac{4}{3}\right)^2 \cdot 4 = \frac{\pi}{3}\left(24 - \frac{64}{9}\right) = \frac{8\pi}{3} \cdot \frac{19}{9}$.

(Vol cone is $\frac{1}{3}\cdot$ base $\cdot$ height; by similar triangles, $\frac{x}{2} = \frac{4}{6} \to x = \frac{4}{3}$.)

Solution 29: ADD FIGURE:

Volume is $\frac{1}{3} \cdot$ base $\cdot$ height, or $\frac{1}{3} \cdot \frac{2 \cdot 3}{2} \cdot 5 = \boxed{5}$.

8.3 Lines

Solution 30: Pt.-slope equation of line is $y + 3 = 6(x - 2)$. Crosses x-axis when $y = 0$, i.e. $6(x - 2) = 3$, so $x - 2 = \frac{1}{2}$, $x = \boxed{\dfrac{5}{2}}$

Solution 31: Solve simultaneously $3x + 4y = 12$ and $3x - 5y = -6$. Subtracting, $9y = 18$, so $y = 2, x = \frac{4}{3}$; ans: $\boxed{(\dfrac{4}{3}, 2)}$

Solution 32: Slope -1, through $(2, 5)$: $y - 5 = -(x - 2)$ or $y = -x + 7$.

Slope 2, y-intercept -2: $y = 2x - 2$

Solving simultaneously, $2x - 2 = -x + 7 \to x = 3, y = 4$. Ans: $\boxed{(3, 4)}$

Solution 33: $2x - 5y = 1 \to y = \frac{2}{5}x - \frac{1}{5}$, slope is $\frac{2}{5}$. Thus, slope of perpindicular line is $-\frac{5}{2}$ (negative reciprocal). Since y-intercept is 3, its equation is $\boxed{y = -\dfrac{5}{2}x + 3}$.

Solution 34: ADD FIGURE:diagram

slope of BD is $\frac{3-(-1)}{5-(-5)} = \frac{4}{10} = \frac{2}{5}$; thus L has equation $\boxed{y + 1 = \dfrac{2}{5}(x - 5)}$ (or $5y - 2x + 15 = 0$).

Solution 35: ADD FIGURE:diagram

The x-intercept is where $y = 0$, so $3x = 30$, $x = 10$

y-intercept is where $x = 0$, so $5y = 30$, $y = 6$. Thus the area of the triangle is $\frac{1}{2} \cdot 6 \cdot 10 = \boxed{30}$.

Solution 36: ADD FIGURE:diagram

L has slope $\frac{2}{3}$ so L' has slope $-\frac{3}{2}$. $\boxed{y = -\dfrac{3}{2}x}$.

8.4 Circles

Solution 37: ADD FIGURE:3 diagrams

1. $x^2 + y^2 - 9 = 0$
2. $x^2 + y^2 - \frac{1}{2} = 0$
3. $(x-1)^2 + (y-1)^2 = 2 \to x^2 + y^2 - 2x - 2y = 0$
4. $(x+1)^2 + (y-1)^2 = 1^2 \to x^2 + y^2 + 2x - 2y + 1 = 0$

Solution 38: The equation is $x^2 + y^2 = 11^2$. *Inside* is where $x^2 + y^2 \leq 11^2$, *outside* is where $x^2 + y^2 > 11^2$. Here $7^2 + 8^2 = 113 < 11^2 = 121$. Thus $(7, 8)$ is *inside* the circle.

Solution 39:

1. $x^2 + y^2 = 5$, $x + y = 3$. Solving simultaneously, $y = 3 - x$; substitute into eqn of circle: $x^2 + (3-x)^2 = 6 \to 2x^2 - 6x + 4 = 0$, or $x^2 - 3x + 2 = 0 \to (x-2)(x-1) = 0$. Thus, the solutions are $x = 2, y = 1$ and $x = 1, y = 2$.

2. $x^2 + y^2 = 10$, $x + 3y = 10$ gives $(10-3y)^2 + y^2 = 10 \to 100 - 60y + 10y^2 = 10 \to y^2 - 6y + 9 = 0 \to (y-3)^2 = 0$. Thus, $x = 1, y = 3$ is the solution. (Note that there is only one solution because the line is *tangent* to the circle.)

3. $x^2 + y^2 = 9$, $y = 3x$ gives $x^2 + 9x^2 = 9 \to 10x^2 = 9 \to x = \pm\frac{3}{\sqrt{10}}, y = \pm\frac{9}{\sqrt{10}}$. Answers: $(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}), (-\frac{3}{\sqrt{10}}, -\frac{9}{\sqrt{10}})$.

8.5 Conics

Solution 40: a: Intercepts are when $x = 0$, $y = \pm\sqrt{3}$ and $y = 0$, $x = \pm\sqrt{10}$. Thus, semimajor axis is $\sqrt{10}$, semiminor axis $\sqrt{3}$. ADD FIGURE:diag

b: Intercepts are at $x = 0$, $y = \pm\frac{5}{2}$ and $y = 0$, $x = \pm\frac{5}{3}$. Thus, semimajor axis is $\frac{5}{2}$, semiminor axis is $\frac{5}{3}$. ADD FIGURE:diag

Solution 41: ADD FIGURE:diag

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \text{ or } \frac{x^2}{9} + \frac{y^2}{4} = 1$$

Solution 42: ADD FIGURE:diag

Equation is $y = ax^2 - 1$ (parabola $y = ax^2$ moved down by one unit). Choose a so that it goes through $(2, 0)$: $0 = a \cdot 4 - 1 \to a = \frac{1}{4}$. Thus, we have $\boxed{y = \dfrac{x^2}{4} - 1}$.

[Another way: since roots are at ± 2, equation is $y = c(x+2)(x-2) = c(x^2 - 4)$, and choose c so that $(0, -1)$ is on parabola.]

Solution 43: ADD FIGURE:diag

Since high point is at $x = 2$ and it goes through origin, other intercept is $x = 4$. (High point is midway between the two x-intercepts.) Thus, $y = ax(x-4)$; choose a so that $1 = a \cdot 2(2-4) \to a = -\frac{1}{4}$; $\boxed{y = -\dfrac{1}{4}x(x-4)}$ is our final answer.

Solution 44: The lines are $y + 2x = 0$ and $y - 2x = 0$; taken together, their equation is $(y + 2x)(y - 2x) = 0$ or $y^2 - 4x^2 = 0$. Eqn of hyperbola is then $y^2 - 4x^2 = c$ (c can be positive or negative but not 0).

Solution 45:

1. ellipse
2. two lines
3. circle

4. parabola
5. hyperbola
6. hyperbola
7. circle
8. parabola
9. two lines
10. none
11. hyperbola
12. parabola
13. hyperbola
14. ellipse
15. two lines
16. circle
17. hyperbola
18. point $(0,0)$
19. parabola
20. two lines

9 Geometry and Analytic Geometry Diagnostic Test 1

Problem 46: The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building? ADD FIGURE:diag

Problem 47: An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of 100 deg. What are the angles of the arcs cut off from the other side?

Problem 48: The radius of a right circular cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?

Problem 49: One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?

Problem 50: A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside of the circle?

Problem 51: What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

1. $x^2 + 2y^2 - 12 = 0$
2. $6x^2 - 5y^2 = 0$
3. $15 - x^2 - y^2 = 0$
4. $xy = 0$
5. $xy = 2$

10 Geometry and Analytic Geometry Diagnostic Test 1 Solutions

The light from a building goes through a tiny hole 100 meters away. The image of the building is upside down on a screen 2 meters away from the hole. If the image is 1.2 meters tall, how tall is the building? ADD FIGURE:diag

Solution 46: Solution needed

An irregular pentagon is inscribed in a circle. Four of the sides have the same length, and the other side cuts off an arc on the circle of 100 deg. What are the angles of the arcs cut off from the other sides?

Solution 47: ADD FIGURE:

$100 + 4x = 360$, so $x = \frac{260}{4} = \boxed{65\,\text{deg}}$

The radius of a right circular cylinder is $\frac{1}{3}$ as long as its height and its volume is 24π. What are the dimensions of the cylinder?

Solution 48: Solution needed

One line has a slope of -1 and passes through the point $(2, 5)$. Another line is given by the equation $2x - y - 2 = 0$. At which point do they intersect?

Solution 49: First line: $y - 5 = -(x - 2)$ or $x + y = 7$; now solve $x + y = 7$ and $2x - y = 2$, to obtain $x = 3, y = 4$. Answer: $\boxed{(3, 4)}$.

A circle has radius 11 and the origin as its center. Is the point $(7, 8)$ inside or outside of the circle?

Solution 50: Solution needed

What kind of geometric locus do the following equations represent: circle, ellipse, two straight lines, hyperbola, point, or no locus?

1. $x^2 + 2y^2 - 12 = 0$
2. $6x^2 - 5y^2 = 0$
3. $15 - x^2 - y^2 = 0$
4. $xy = 0$

5. $xy = 2$

Solution 51: Solution needed

11 Geometry and Analytic Diagnostic Test 2

Problem 52: What is x in degrees
ADD FIGURE:

Problem 53: Two distinct lines are both tangent ot a circle. They meet outside the circle at an angle of 50 deg. What are the central angles subtended by the two arcs they cut off on the circle?
ADD FIGURE:
6.0pt24.88pt$\widehat{\mathrm{APB}} = ?$
6.0pt24.88pt$\widehat{\mathrm{AQB}} = ?$

Problem 54: A tetrahedron has three edges which all intersect at right angles. Those edges have lengths 2, 3, and 5. What is the volume of the tetrahedron?
ADD FIGURE:

Problem 55: A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?

Problem 56: Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.

Problem 57: What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, point, or no locus?

1. $4y^2 - x^2 = 2$
2. $x^2 + y^2 + 2x - 5 = 0$
3. $y^2 + 2x = 5$
4. $2y - 4x^2 = 3$
5. $x^2 + 3y^2 = 6$

12 Geometry and Analytic Diagnostic Test 2 Solutions

What is x in degrees

ADD FIGURE:

Solution 52: Solution needed

Two distinct lines are both tangent ot a circle. They meet outside the circle at an angle of 50 deg. What are the central angles subtended by the two arcs they cut off on the circle?

ADD FIGURE:

6.0pt24.88pt$\widehat{APB}$ =?

6.0pt24.88pt$\widehat{AQB}$ =?

A tetrahedron has three edges which all intersect at right angles. Those edges have lengths 2, 3, and 5. What is the volume of the tetrahedron?

ADD FIGURE:

Solution 53: Solution needed

A line with a slope of $\sqrt{3}$ passes through the point $(2, -3)$. Where does it cross the x-axis?

Solution 54: Line is $y + 3 = \sqrt(3)(x - 2)$

Crosses x-axis where $y = 0$; solving for x, we have $\sqrt{3}(x - 2) = 3 \rightarrow x - 2 = \sqrt{3} \rightarrow \boxed{x = 2 + \sqrt{3}}$

Circle A is defined by $x^2 + y^2 = 2$, and line B is defined by $x + y - 1 = 0$. Find the points where A intersects B.

Solution 55: Solve simultaneously: $x^2 + y^2 = 2$ and $x + y - 1 = 0$. We eliminate y to get $y = 1 - x$ from the second equation, and substitute:

$x^2 + (1 - x)^2 = 2 \rightarrow 2x^2 - 2x - 1 = 0$. This has solutions $x = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$, $y = \frac{1 \mp \sqrt{3}}{2}$; so the intersection points are $\boxed{(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}), (\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2})}$.

What kind of geometric locus do the following equations represent: circle, ellipse, parabola, hyperbola, point, or no locus?

1. $4y^2 - x^2 = 2$

2. $x^2 + y^2 + 2x - 5 = 0$
3. $y^2 + 2x = 5$
4. $2y - 4x^2 = 3$
5. $x^2 + 3y^2 = 6$

Solution 56: Solution needed