Frontiers of quantum complexity theory

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Upcoming section

Lecture 0: Course introduction

Lecture 1: Quantum Merlin-Arthur

Quantum complexity theory

Classification of problems according to whether they are easy or hard or really hard for quantum computers. "Hardness" can be captured in multiple ways - time to solve a problem, time to verify a solution, etc.

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Classification of problems according to whether they are easy or hard or really hard for quantum computers. "Hardness" can be captured in multiple ways - time to solve a problem, time to verify a solution, etc.

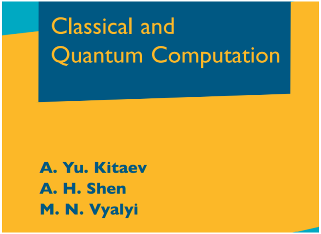
A vast topic with multiple sub-fields: quntum query complexity, quantum communication complexity, Hamiltonian complexity, multi-prover proof systems, etc.

Remarkable achievements of quantum complexity theory

- Quantum circuits to local Hamiltonians (Feynman-Kitaev clock construction).

Simulating Physics with Computers

Richard P. Feynman



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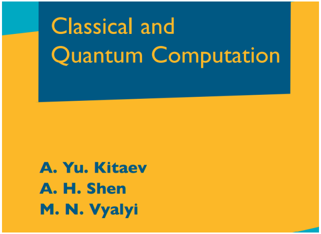

- The bridge between quantum computer science and quantum many-body physics.
- Hardness of computing ground energies of physical models, universality of adiabatic evolution, undecidability of spectral gap, etc.

Remarkable achievements of quantum complexity theory

- Connes Embedding conjecture (1970-2020).
- Study of quantum proof systems played the most essential role in its solution.

MIP*=RE

Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, Henry Yuen

Remarkable achievements of quantum complexity theory

- DMRG is a powerful heuristic method to solve quantum many-body systems.
- First rigorous DMRG-like algorithm using complexity-theoretic tools.

An area law for one-dimensional quantum systems

M B Hastings

Published 20 August 2007 • IOP Publishing Ltd

Journal of Statistical Mechanics: Theory and Experiment, Volume 2007, August 2007

A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians

Zeph Landau, Umesh Vazirani & Thomas Vidick

Nature Physics **11**, 566–569 (2015) | Cite this article

Basis for the course

Core principles:

- The future of quantum complexity lies in deeper connections with other areas, especially physics.

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Some key topics:

- Quantum Merlin-Arthur and the Quantum Cook-Levin theorem.
- Quantum PCP conjecture.
- Quantum Gibbs states and Gibbs samplers.
- 1D gapped systems: area laws and algorithms.
- QMA vs QCMA
- State synthesis vs unitary synthesis
- SYK model, Bose-Hubbard model

Our best guess of potential achievements

- Taming quantum systems beyond 1D (Area law conjecture and representing 2D systems with tensor networks).
- Robust simulation of quantum computation in quantum many-body systems (closely related to quantum PCP conjecture).
- Better approximation algorithms to quantum chemistry and quantum many-body problems.
- Better understanding of average case models (such as SYK).
- Confluence of physics and quantum cryptography!

Logistics

- 3 assignments in total.
- Scribe notes - at least one per person. Scribe notes will be evolved into lecture notes/handbook and shared with the community.
- Final projects (in groups of 2-3): various possible project topics will be suggested mid-semester.
    - Report to be submitted by the end of the semester.
    - Final presentations (at least one member per group must be present).

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Lecture 0: Course introduction

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Computational problem

- A computational problem involves a set of inputs $\mathcal{I}$, a set of outputs $\mathcal{O}$ and a function $f : \mathcal{I} \to \mathcal{O}$.
- The problem is a decision problem if $\mathcal{O}$ has size 2 (such as $\mathcal{O} = \{0, 1\}$) and for each x, $f(x)$ has size 1.
- Some problems are promise problems - the input $\mathcal{I}$ belongs to a larger domain and an algorithm or a protocol need not worry about inputs not in $\mathcal{I}$.

Class NP

- A computational problem is in the class *NP*, if it admits a protocol of the form depicted in the picture below.

- Verifier V is polynomial time.
- If $f(x) = 1$, there is a proof y (with $|y| = poly(|x|)$) such that $V(x, y) = 1$.
- If $f(x) = 0$, for all y, $V(x, y) = 0$.

Class MA

- A computational problem is in the class *MA*, if it admits a protocol of the form depicted in the picture below.

- Verifier V is polynomial time randomized on inputs x, y and randomness r.
- If $f(x) = 1$, there is a proof y (with $|y| = poly(|x|)$) such that $\text{Prob}_r[V(x, y, r) = 1] \geq 2/3$.
- If $f(x) = 0$, for all y, $\text{Prob}_r[V(x, y, r) = 1] \leq 1/3$.

Class QMA

- A computational problem is in the class *QMA*, if it admits a protocol of the form depicted in the picture below.

$\boxed{x \in \mathcal{I}}$

$\boxed{\text{Quantum Verifier}} \xleftarrow{\text{Quantum Proof } |\psi\rangle} \boxed{\text{Prover}}$

- Verifier runs a polynomial size quantum circuit Q (following by measurement of a qubit) on inputs $x, |\psi\rangle$.
- If $f(x) = 1$, there is a quantum proof $|\psi\rangle$ such that probability of 1 is at least $2/3$.
- If $f(x) = 0$, for all $|\psi\rangle$, the probability of output 1 is at most $1/3$.

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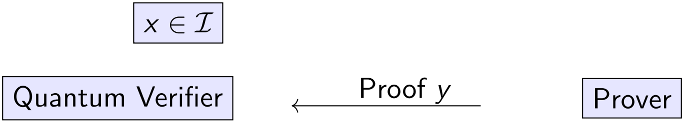

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Example 1

Kepler wants to convince Cassini that not all planets move in circular orbits. Cassini strongly believes in Copernican circular model. Given observed positions of planets at certain times, can Kepler show Cassini that there exists at least one planet whose observed positions cannot be explained under Copernican model?

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- $\mathcal{I}$ = Set of all planets
- $\mathcal{O} = \{1, 0\}$.
- Protocol: Given a planet, Kepler sends the trajectory of planets to Cassini, and Cassini geometrically checks if they lie on a circle.

This protocol puts the problem in NP.

Example 2

Kamerlingh Onnes, in 1911, claims that certain materials exhibit superconductivity. His skeptical colleague wants proof. Onnes provides a classical description of an experimental setup (for example: 'cool this sample to 4 K and check Meisner effect'). The colleague can run the quantum mechanical experiment on their own.

Example 2

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- $\mathcal{I}$ = Set of materials
- $\mathcal{O} = \{1, 0\}$.
- Protocol: given a material, Onnes sends the experimental configuration that would cool that given material to low enough temperature. The colleague cools the material themselves and then does a standard superconductivity test (Meisner effect, for example).

This protocol puts the problem in QCMA.

Example 3

The latest run of Large Hadron Collider (LHC) observed new particle decays that violates fundamental symmetries of the Standard Model. Some skeptic scientists propose new decay phenomena - based on a slight variation to the Standard model and want to test it using a new set of mini experiments.

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- $\mathcal{I}$ = Set of possible particle decays.
- $\mathcal{O} = \{1, 0\}$.
- Protocol: Skeptic scientists go to LHC and perform the experiment on the newly discovered particles themselves.

This protocol would put the problem in QMA.

Summary

- Most traditional scientific experiments could be captured within the class NP.
- However, QCMA is the right framework to capture most 20th century experiments.
- QMA is the most general framework, and is turning out to be the right framework given large scale experiments in recent decades.