Lecture 16: Area Laws I

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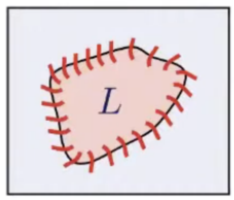
Area laws are statements about constraints on entanglement in ground states of gapped lattice Hamiltonians. Roughly, it says that the entanglement between a subsystem L and its complement $\overline{L}$ only scales with the boundary ∂L (area law). This is in contrast to generic quantum states, where the entanglement scales with the size of the entire system L (volume law).



$$S_L \propto |\partial L|$$

It is conjectured to be true for all lattice Hamiltonians with a constant spectral gap, and still remains wide open in this most general case. Here, the celebrated result is the area law of entanglement entropy in any 1D gapped system by Hastings in 2007. Today we will start looking at area laws in a specific 1D case called frustration-free systems.

1 Background

Definition 1.1. Consider a 1D Hamiltonian $H = \sum_i h_{i,i+1}$ with $h_{i,i+1} \succeq 0$. It is said to be frustration-free (FF) if there exists a state $|\Omega\rangle$ s.t. $H|\Omega\rangle = 0 \Leftrightarrow h_{i,i+1}|\Omega\rangle = 0\ \forall i$. I.e. $|\Omega\rangle$ satisfies all local terms. It turns out that if H is frustration free, we can assume wlog the local terms are projectors $h_{i,i+1} = I - \Pi_{i,i+1}$.

Recall the DL operator $\mathrm{DL} = (\prod_{i \text{ even}} \Pi_{i,i+1})(\prod_{i \text{ odd}} \Pi_{i,i+1})$ and the detectability lemma:

Lemma 1.2. *If H has spectral gap γ, then DL has gap $\Theta(\gamma)$.*

The motivation for area laws is the expectation that gapped ground states should have simpleer structure/complexity than ground states of gapless Hamiltonians. Such belief is widespread in many areas of physics, as also evidenced by the fact that numerical methods based on area laws, such as tensor networks seem to work very well with gapped systems in practice. E.g. people had known the DMRG algorithm (White 1992) worked very well in practice for these systems[1], long before Hastings' proof.

Before stating the 1D area law, we first need to define a notion of low entanglement/complexity. Consider a 1D cut $A : B$, here are some common notions:

[1] On the other hand, if the gap is $1/\mathrm{poly}(n)$, then it is known that the ground state could have high entanglement (Gottesman-Hastings 2009).

- Mutual information: $I(A:B)_{|\Omega\rangle} = S(\Omega_{AB} \| \Omega_A \otimes \Omega_B) = S(\Omega_A) + S(\Omega_B) - S(\Omega_{AB})$, where $S(\Omega_A)$ is the entanglement entropy.. This equals $2S(\Omega_A)$ for pure state $|\Omega\rangle$.
- Renyi entanglement entropies: $S_\alpha(\Omega_A) = \frac{1}{1-\alpha} \log \mathrm{Tr}(\Omega_A^\alpha)$ for $\alpha \geq 0$. Let $\Omega_A = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$
  - When $\alpha = \infty$, $S_\infty = \log(1/\lambda_{max})$ is the min entropy.
  - When $\alpha = 0$, $S_0 = \log(|\operatorname{supp}(\Omega_A)|)$ (log of number of nonzero eigenvalues).
  - When $\alpha = 1$ we recover the entanglement entropy $S = \sum_j \lambda_j \log(1/\lambda_j)$.
  - When $\alpha = 1/2$, $S_{1/2} = 2\log(\sum_j \sqrt{\lambda_j})$
  - When $\alpha = 2$, $S_{1/2} = -\log(\sum_j \lambda_j^2)$ is the purity.

Note that: $S_0 \geq S_{1/2} \geq S_1 \geq S_2 \geq S_\infty$. We will see that S_0 and $S_{1/2}$ are better for the purpose of having a simple description of quantum states.

2 Statements

Theorem 2.1 (Hastings)**.** *For any gapped 1D Hamiltonian H (not necessarily frustration-free), with unique ground state $|\Omega\rangle$ and gap γ, and 1D bipartition $A : B$, it holds that $S_{1/2}(\Omega_A) \leq \exp(O(1/\Delta))$.*

The way he proved this was via first bounding S_∞, then bootstrapping to $S_{1/2}$. Here's why $S_{1/2}$ is interesting.

Fact 2.2. $S_{1/2}$ upperbounds the smooth version of S_0, which is defined as $S_0^\varepsilon = \min_{\Omega' \approx_\varepsilon \Omega} S_0(\Omega'_A)$. It holds that $S_0^\varepsilon \leq S_{1/2} + \log(1/\varepsilon)$.

This fact allows a matrix product state approximation of the ground state with bond dimension (Schmidt rank) bounded by S_0^ε. The rough idea is according to area law, $S_0^{1/n^2}(\Omega_A) \leq O(\log n)$ for any cut $A : B$. Then one can building the MPS site-by-site.

2.1 Some history

Let us recap the improvements/simplifications following Hastings proof, all restricted to 1D systems:

- Arad-Landau-Vazirani'12: For FF systems, improved $S_{1/2}$ upperbound to $O(1/\Delta^3)$.
- Arad-Kitaev-Landau-Vazirani'13: For all systems, improved $S_{1/2}$ upperbound to $O(1/\Delta)$ – which is believed to be tight because matches correlation length.
  Here's an argument for the tightness: Consider a bipartition $A : B$, with a further partition $B = CB'$ where $C \approx 1/\Delta$ (the correlation length from previous lecture), then if we believe $I(A : B') \approx 0$, then $I(A : B) = I(A : B') + I(A : C|B') \approx I(A : C|B') \leq |C| = 1/\Delta$.
- Gottesman-Hastings'09: lowerbound $\Omega(1/\Delta^{1/4})$.
- Polynomial algorithms by Landau-Vazirani-Vidick, Arad-Landau-Vazirani-Vidick'16.

3 Proof warm-up

A first attempt one can try is to approximate the ground state by the Gibbs state ρ_β, which is known to have area law (of mutual information). This does work when the density of eigenstates is nice. Suppose that for $E \geq \gamma$, the number of eigenstates with energy E scales polynomially as n^{cE}. Such scaling arises in physics systems with quasiparticles, where excited states are produced by local excitations, leading to a scaling like $\binom{n}{E} \approx n^{cE}$.

Then

$$\left\| \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})} - |\Omega\rangle\langle\Omega| \right\|_1 \leq 2 \frac{\sum_{E\geq\gamma} n^{cE} e^{-\beta E}}{1 + \sum_{E\geq\gamma} n^{cE} e^{-\beta E}} \leq 1/\mathrm{poly}(n), \tag{1}$$

when $\beta \gg (2c \log n)/\eta$. Then using $I(A:B)_{\rho_\beta} \leq O(\beta|\partial A|)$ and continuity of entropies we obtain an area law for the ground state. This proof works regardless of the geometry of the Hamiltonian, but fails as soon as the number of eigenstates near γ scales exponentially with n (for whatever reason).

The key idea is to relax the approximation requirement to ℓ_∞ norm rather then ℓ_1. This is the notion of *approximate ground state projector* (AGSP).

Definition 3.1. A (Δ, D)-AGSP is an operator K, with Schmidt rank K across the cut A:B.

$$K = \sum_{i=1}^{D} K_i^A \otimes K_i^B. \tag{2}$$

In addition, $K|\Omega\rangle = |\Omega\rangle$ and approximates the ground state projector in ℓ_∞ norm

$$\| K - |\Omega\rangle\langle\Omega| \|_\infty \leq \Delta \tag{3}$$

A simple example is the Gibbs state itself, with $\Delta = e^{-\beta\gamma}$ (re-normalized so that $|\Omega\rangle$ is not shrunken). Another example is the detectability lemma operator in the frustration free case, the DL is a $(\Delta = 1/(1+\Theta(\gamma)), O(1))$-AGSP.

The intuition is that applying K multiple times on some good initial state we can get close the $|\Omega\rangle$, while not producing too much entanglement. This is made precise by the following theorem.

Theorem 3.2 (Arad-Landau-Vazirani'12)**.** *If there is a good AGSP, satisfying $D \cdot \Delta < 1/2$, then $S_{1/2}(\Omega_A) = O(\log D)$.*

Here the factor 1/2 is an arbitrary constant < 1.

Proof. Let $|\psi\rangle_A \otimes |\psi\rangle_B$ be the closest product state to $|\Omega\rangle$. Decompose $|\psi\rangle_A \otimes |\psi\rangle_B = \mu|\Omega\rangle + \sqrt{1-\mu}\,|\phi\rangle$. Consider $|\chi\rangle = (K(|\psi\rangle_A \otimes |\psi\rangle_B))/\|K(|\psi\rangle_A \otimes |\psi\rangle_B)\|$, then $|\chi\rangle$ has overlap $\geq \frac{\mu}{\sqrt{\mu^2+(1-\mu^2)\Delta^2}}$ with $|\Omega\rangle$. The state $|\chi\rangle = \sum_{i=1}^{D} \sqrt{\lambda_i}\,\left|\chi_A^i\right\rangle \otimes |\chi\rangle_B^i$, where each component has overlap at most μ with $|\Omega\rangle$, so by Cauchy-Schwarz $|\langle\chi|\Omega\rangle| \leq D\mu$. Combine the upper and lower bounds and $\Delta^2 D^2 \leq 1/4$ yields $\mu \geq \sqrt{3/4} \cdot 1/D$. This shows $S_\infty(\Omega) \leq \log D + O(1)$. We then apply K a few more times to get to the ground state. $\square$

This theorem thus reduces the entire question to showing the existence of a good AGSP, which we will do next lecture. Note that even in the FF case, this won't simply be the DL operator, as it has $D\Delta > 1$.

Let's make another remark. This theorem is false if $D\Delta$ is larger some fixed constant! Here's a counter example. Let $\{U_i\}_{i\in[D]}$ be an ensemble of n-qubit quantum expander unitaries with $D = O(1)$, then $K = \frac{1}{c}\sum_i U_i \otimes U_i^*$ is an AGSP of the EPR state $|\mathrm{EPR}\rangle^{\otimes n}$, as K fixes the EPR state and shrinks all other states by a factor $\Delta < 1$. But $|\mathrm{EPR}\rangle^{\otimes n}$ has volume law entanglement across the cut.