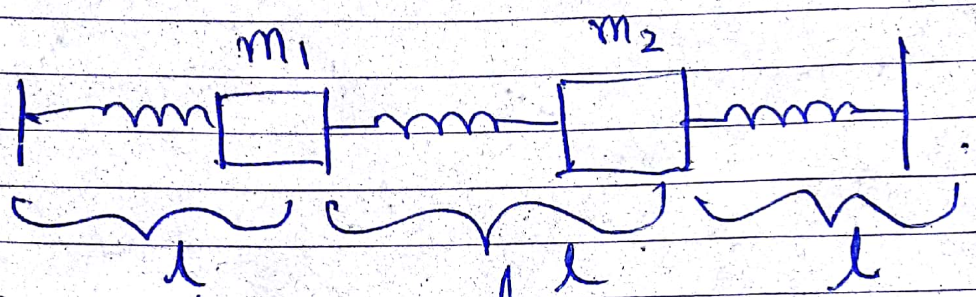
11-4-18

2-35.

Using eigen values & eigenvectors to solve linear ordinary differential equations (Very Important).



One dimensional motion $\longrightarrow \hat{x}$

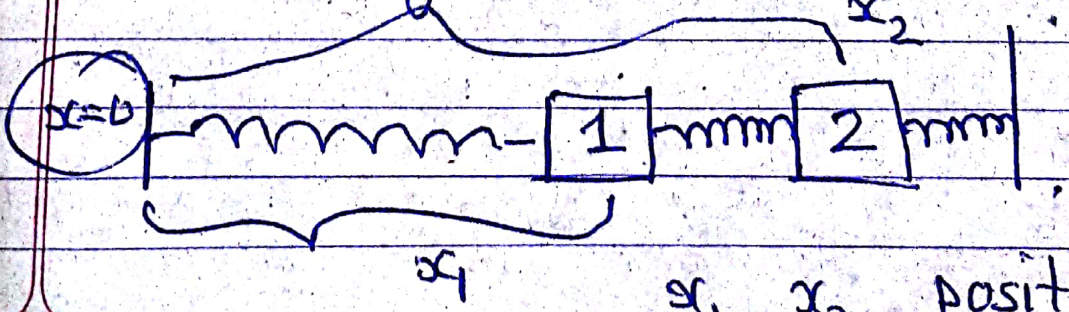
l : Equilibrium lengths of each spring.

$m_1 = m_2 = 1$

Spring constants are

$k_1 = k_2 = k_3 = k$.

Out of equilibrium



x_1, x_2 positions.

Use Newton's II law

to write

$m_1 \ddot{x}_1 = F_1$ total force on mass 1

$m_2 \ddot{x}_2 = F_2$ " " " 2.

On mass 1 there two forces:

(1) From left spring.

(2) From the right spring.

If $x_1 = l$: the first spring is at equilibrium. No force.

(1) If $x_1 > l$.

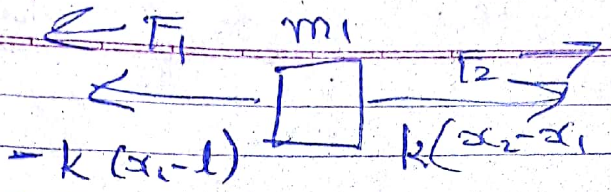
$F_1 = -k(x_1 - l)$. Negative force (inwards in $-\hat{x}$) if $x_1 > l$.

(2) The force from the second spring.

The length of the second spring is $x_2 - x_1$

The extension is $(x_2 - x_1) - l$.

If $(x_2 - x_1) - l > 0$ there is +ve force on m_1.



$k(x_2-x_1)-l > 0$

$x_1 > l.$

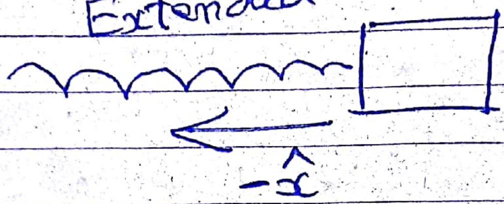
$m_1 = 1$

$m_1 \ddot{x}_1 = F_1 + F_2$

$\Rightarrow \ddot{x}_1 = -k(x_1 - l). + k(x_2 - x_1 - l).$

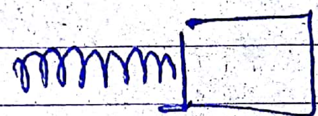
$\Rightarrow$ ~~$\ddot{x}_2$~~

if Extended spring $x_2 - x_1 - l > 0$



-ve force

$x_2 - x_1 - l < 0$



$\vec{x}$ +ve force

$\ddot{x}_2 = -k(x_2 - x_1 - l). + F_2$

F_1

what is the length of the third spring ?

length $= 3l - x_2.$

Extension $= (3l - x_2) - l = 2l - x_2$

if $2l - x_2 > 0$ positive force

$F_2 = k(2l - x_2).$

The complete system is described by

$$\begin{cases} \ddot{x}_1 = -k(x_1 - l) + k(x_2 - x_1 - l) = -2kx_1 + kx_2 \\ \ddot{x}_2 = -k(x_2 - x_1 - l) + k(2l - x_2) = -2kx_2 + kx_1 + 3kl. \end{cases}$$

Writing in matrix form

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3kl \end{bmatrix}$$

inhomogeneous

→ Coupled differential equation

change in x_1 depends on x_2 too

" x_2 " " x_1 too

We are interested only in the homogeneous part for the time being.

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$$\ddot{X}\uparrow = A X\uparrow \quad \text{--- (1)}$$

$$X\uparrow = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = k\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Change variables from $X\uparrow$ to $Y\uparrow$

$Y\uparrow = B X\uparrow$ B has number entries.

~~$A\ddot{X}\uparrow$~~

$$B\ddot{X}\uparrow = B A X\uparrow \quad \text{--- (2)}$$

$$= B A \underbrace{B^{-1} B}_{I} X\uparrow$$

$$\Rightarrow \ddot{Y}\uparrow = (B A B^{-1}) Y\uparrow \quad (3)$$

Choose B so that

$B A B^{-1} = D$ a diagonal matrix

$$\Rightarrow \ddot{Y}\uparrow = D Y\uparrow$$

$$\Rightarrow \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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$$\ddot{y}_1 = \lambda_1 y_1 \quad (1)$$
$$\ddot{y}_2 = \lambda_2 y_2 . \quad (2)$$

$$(D^2 - \lambda_1)\, y_1 = 0$$

tangent The roots of characteristic eqn. are $\pm \sqrt{\lambda_1}$

The general solution is

$$y_1(t) = C_1 e^{\sqrt{\lambda_1} t} + C_2 e^{-\sqrt{\lambda_1} t}$$

You will not do this in 2A.

The equation we want to solve is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = k \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find B such that

$$B A B^{-1} = D .$$

What matrix is B ?

Recollect that

$$A\begin{bmatrix} \uparrow & \uparrow \\ e_1 & e_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ e_1 & e_2 \\ \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

e_1 & e_2 are eigenvectors of A.

Write $S = \begin{bmatrix} \uparrow & \uparrow \\ e_1 & e_2 \\ \downarrow & \downarrow \end{bmatrix}$

$$\Rightarrow \quad AS = SD$$

$$\Rightarrow \quad S^{-1}AS = D.$$

Comparing with $BAB^{-1} = D$.

$$B = S^{-1} \quad \text{and} \quad B^{-1} = S$$

1. Find S i.e. eigen values & eigen vectors of A.

Show that $S^{-1}AS = D$.

The eigenvalues are $-1, -3$.

The eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The S matrix is

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Check that

$$S^{-1} A S = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}.$$

$B = S^{-1}$ is the matrix that gives the coordinate transformation.