1 Experiment

	R_1	R_2	C_1	C_2	output: V_{out}
values	x_1^1	x_2^1	x_3^1	x_4^1	V_1
	x_1^2	x_2^2	x_3^2	x_4^2	V_2
	$\vdots$				
	x_1^m	x_2^m	$\dots$	x_4^m	V_m

↳ m sets of observation

A linear model

$$V_{out} = a_1 R_1 + a_2 R_2 + a_3 C_1 + a_4 C_2.$$

data is expected to fit like

$$\underbrace{V_1}_{\text{observatio}} = a_1 \underbrace{x_1^1}_{\text{observation}} + a_2 x_2^1 + a_3 x_3^1 + a_4 x_4^1$$

~~orbs~~ observation ... → unkown

→ m such equations.

Can be written in terms of a matrix equation

$$\begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ \vdots & & & \\ x_1^m & x_2^m & & x_4^m \end{bmatrix} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\text{unknowns}} = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix}$$

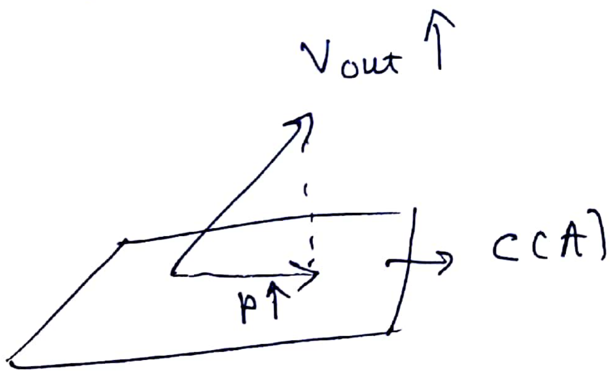
Reminder

The equation

$$A\, a\uparrow = V_{out}\uparrow$$

- can be exact only if $V_{out}\uparrow$ is in the column space of A
- Because of experimental errors.
  - * Sloppy observer
  - * Limited resolution (least count)
  - * Systematic errors.
  - * environmental factors.
- Theoretical approximations.

In general $V_{out}\uparrow \notin C(A)$.



The best you can do is take a vector in $C(A)$ which is closest to $V_{out}\uparrow$

i.e. take projection of $V_{out}\uparrow$ on $C(A)$

if $V_{out}\uparrow$ is replaced by $p\uparrow$ an exact solution is possible.

Solve the following problem in two different ways. least square, projection.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

find best fit for

$$A \hat{a} = \hat{b}$$

This is a toy exaple.

? What is the data here ?

x_1 Inputs	x_2	output y
1	0	1
0	1	1
1	1	0

? What is the model ?

Equation $y = a_1 x_1 + a_2 x_2$.

a_1, a_2 are unknowns to be found.

For each data point there is an equation

$$\left.\begin{aligned} 1 &= a_1 + 0 a_2 \\ 1 &= 0 a_1 + a_2 \\ 0 &= a_1 + a_2 \end{aligned}\right\} \text{~~In~~ inconsistent, can't be a fit}$$

calculat errors (difference betwen L.H.S. & R.H.S.)

$$\|E\|^2 = (a_1-1)^2 + (a_2-1)^2 + (a_1+a_2)^2$$

minimize $\|E\|^2$

$$\frac{\partial \|E\|^2}{\partial a_1} = 0$$

two unknowns &

$$\frac{\partial \|E\|^2}{\partial a_2} = 0$$

two equations

Solve.

$$\frac{\partial \|E\|^2}{\partial a_1} = 2(a_1-1) + 2(a_1+a_2) = 0$$

$$\frac{\partial \|E\|^2}{\partial a_2} = 2(a_2-1) + 2(a_1+a_2) = 0.$$

$\Rightarrow \quad a_1 = a_2$

$6a_1 - 2 = 0$

$\Rightarrow \quad a_1 = 1/3$

similarly $a_2 = 1/3$

Solve the same problem geomethically.

The matrix equation is

$A\,\vec{a} = \vec{b} \rightarrow$ Not solvable

$A^T A\,\hat{\vec{a}} = A^T \vec{b}$ (The normal eqn.)

$\downarrow$ solvable for the best fit parameters $\hat{\vec{a}}$

Write the normal eqn. & solve

$$A^T A \hat{a} \uparrow = A^T b \uparrow$$

- A is not square matrix in general. can't be inverted.
- What are the dimensions of $A^T A$?

→ A is $m \times n$ matrix

m: Number of observations

n: " " parameters.

→ A^T is $n \times m$ "

→ $A^T A$ is $n \times n$, a square matrix [invertable if it is not singular].

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \hat{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Best fit parameters

Solve $A^T A \hat{a} \uparrow = A^T b$.

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_1 = a_2 = \frac{1}{3}$$

Matrix transformations

- Define a linear transformation that rotates a vector.

- Any linear transformation is completely defined by its action on unit vectors.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The action of A on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

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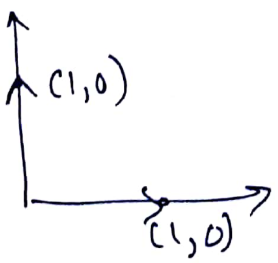
rotation by $\pi/2$



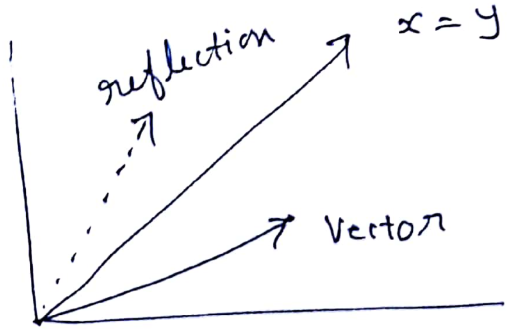
$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

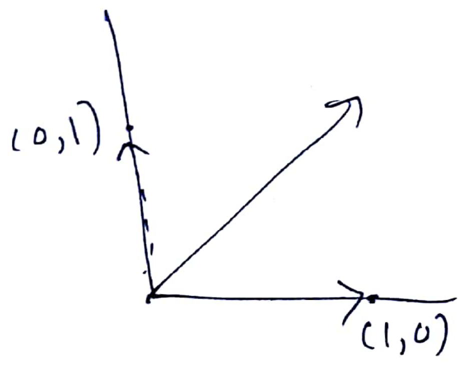
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ $\rightarrow$ rotation of any vector in 2D by $\theta = \pi/2$.

Scanned by CamScanner

Find the linear transformation that reflects any vector about the line $x = y$.





Assume $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

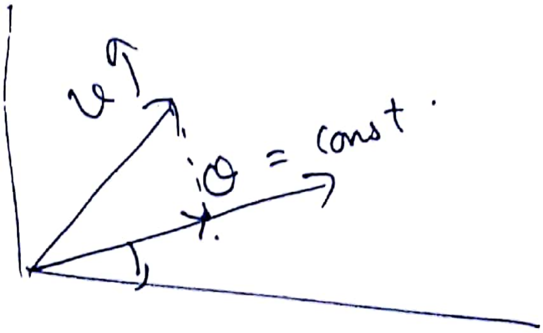
$$\Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Also $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Projection along $\theta = \text{const.}$



The projection is along the vector $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$P = \frac{a^{\uparrow}\, \vec{a}}{\vec{a}\, a^{\uparrow}}$$

$$= \frac{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}}{\cos^2\theta + \sin^2\theta}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \sin\theta \cos\theta & \sin^2\theta \end{bmatrix}$$