L-17

dot product of two vectors. 1

$a^T b = \vec{a}^T b \uparrow = (\vec{a} \cdot \vec{b}) \rightarrow$ old notation.

How (where) does one use a dot product?

1. To find angle between two vectors.

Find the angle between the vector $a\uparrow = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the x, y, z axis.

$\vec{x}\uparrow = [1 \; 0 \; 0]$

Dhruval

$\vec{x} \cdot a\uparrow = |x||a| \cos\theta$

$\therefore [1 \; 0 \; 0] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\sqrt{(1)^2 + 0^2 + 0^2}\right)\left(\sqrt{(1)^2 + (1)^2 + (1)^2}\right) \cos\theta.$

$\therefore 1 = (1)\sqrt{3} \cos\theta \qquad \therefore \cos\theta = 1/\sqrt{3}$

$\theta = \cos^{-1}(1/\sqrt{3})$

- Check angle of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $45°$

$\Rightarrow$ what is angle of the n dimensional vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ with coordinate axis?

Kunvar: $\cos\theta = \frac{1}{\sqrt{n}} \qquad \theta = \cos^{-1} \frac{1}{\sqrt{n}}$

Other uses of the dot product

- Akash Prasad : work, energy etc. for physical quantities.

$$\vec{F} \cdot \vec{dx} = dE$$

→ To find the length of a vector

$$a\uparrow = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Nisarg: Take the dot product of a vector with itself

$$\boxed{\begin{aligned} \| a\uparrow \|^2 &= \vec{a} \cdot a\uparrow \\ &= a_1^2 + a_2^2 + \dots + a_n^2 \end{aligned}}$$

→ Distance between two points.

Ex: (2, 3, 5) and (1, 2, 3).

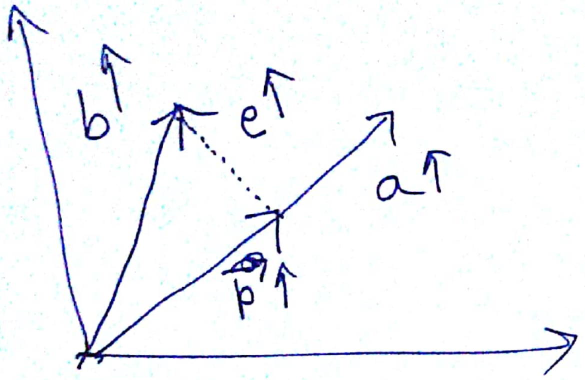
$$a = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$c\uparrow = a\uparrow - b\uparrow = \begin{bmatrix} 2-1 \\ 3-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{distance}^2 = \vec{c} \cdot c\uparrow = (1^2 + 1^2 + 2^2) = 6$$

$$\text{distance} = \sqrt{6}$$

Finding Projection of a Vector on another vector



How are $b\uparrow$, $p\uparrow$ and $e\uparrow$ related?

Abutibyan : $b\uparrow = p\uparrow + e\uparrow$

$e\uparrow = b\uparrow - p\uparrow$

Since $p\uparrow$ and $a\uparrow$ are colinear

$p\uparrow = x\, a\uparrow$ (x is a scaling factor).

From geometry, we know that

$e\uparrow \perp a\uparrow$

$\vec{a}\, e\uparrow = 0$ (in the text notation $a^T e = 0$

$\Rightarrow \vec{a}\,(b\uparrow - p\uparrow) = 0$

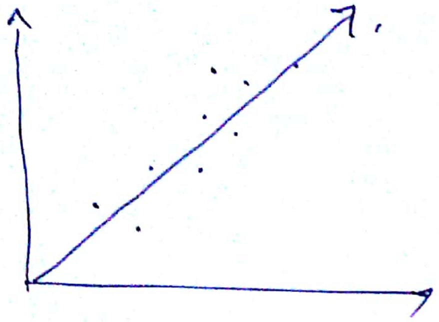
$\Rightarrow \vec{a}\,(b\uparrow - x\, a\uparrow) = 0$

$\Rightarrow \boxed{x = \dfrac{\vec{a}\, b\uparrow}{\vec{a}\, a\uparrow}}$ The projected vector $p\uparrow = x\, a\uparrow$

You are given data points

$(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots$

Fit a line passing through the data points.



Due to limitations of experiment (and theory) the data points never fall on a line.

One needs to find the best fit.

Find the slope m such that the fit is optimum.

$$y = mx$$ [the equation to be fit]

$$y_1 = m x_1$$
$$y_2 = m x_2$$
$$\vdots$$
$$y_l = m x_l.$$

Consider $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_l \end{bmatrix}$

$$\vec{y} - m\vec{x} = 0.$$

In reality the above is not zero.

Describe the error vector

$\vec{e} = \vec{y} - m\vec{x}$

where $\vec{e}$ is the error of the fit.

→ aim is to minimize $\|\vec{e}\|$

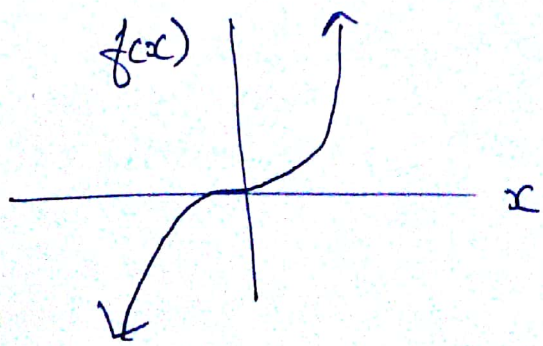
to minimize $\|\vec{e}\|$ by varying m.

$$\frac{d}{dm}\|\vec{e}\|^2 = 0.$$

Arya: Not necessarily minimum could also be a maximum.

Shrey: The max. could be at infinity.

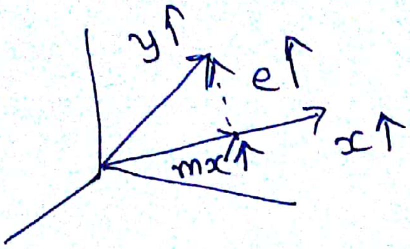
Anjali: It could be a point of inflection. → Consider x^3



$\frac{df}{dx}$ at $x=0$?

$\|\vec{e}\|^2$ is quadratic (2nd power in m). and the max has to be at infinity.

Different way of fitting a line.



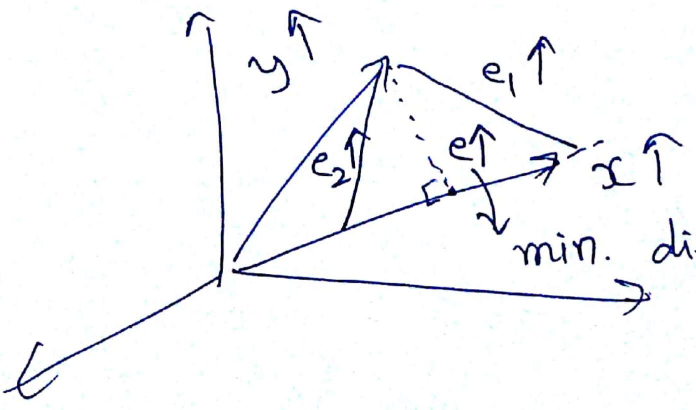
If $\vec{y}$ and $\vec{x}$ are colinear then

$\vec{y} = m\vec{x}$ is exactly solvable.

Use the same calculation as before in

the projection

$$\boxed{m = \frac{\vec{x}^{T}\,\vec{y}}{\vec{x}^{T}\,\vec{x}}} \qquad \left[\; m = \frac{x^T y}{x^T x} \text{ as per text book} \;\right]$$



min. distance from the point to the line.

given 3 points $(1, 2)$, $(0, 3)$, $(2, -1)$.

~~find~~ fit $y = mx$ to the data.

Find the best fit value of m.

fit $(2, 5)$, $(1, 3)$, $(3, 10)$.

to $y = mx$

$$m = \frac{43}{14}$$

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 5 \\ 3 \\ 10 \end{bmatrix}$$

$$m = \frac{x^T y}{x^T x} = \frac{\vec{x}^T \vec{y}}{\vec{x}^T \vec{x}} = \frac{43}{14}.$$