LU decomposition

Mitaxi Mehta: Lecture 8

- While multiplying a matrix with a number, the number gets multiplied to every element of the matrix. For example,

$$3\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 3a_{11} & 3a_{12} \\ 3a_{21} & 3a_{22} \end{pmatrix}$$

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- One may also consider the column picture of the equation and apply the same reasoning.

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- Note that interchanging of rows of a matrix is accomplished by matrix multiplication as follows,

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- Satisfy yourself that matrix products are non-commutative.

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$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

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- Often, it is of interest to decompose a matrix in upper and lower triangular form like $A = LU$

- We shall do the LU decomposition of a matrix using an example, for more details consult section 1.5 of the reference book.

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- Consider the following matrix equation,

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

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- Notice that this is essentially the identity matrix with the entry (2,1) being the multiplying factor.

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- Check that the following sequence of matrix multiplication brings the matrix in an upper traibgular form. $G\,F\,E\,A = U$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

- The inversion works as follows,

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\begin{aligned}
& GFEA = U \\
\Rightarrow \quad & FEA = G^{-1}U \\
\Rightarrow \quad & EA = F^{-1}G^{-1}U \\
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- Notice that the order of the multiplication is important.

- Convince yourself that mutiplication of two lower (upper) triangular matrices results in a lower (upper) triangular matrix.

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- Also notice that inverse of an upper triangular matrix is another upper triangular matrix.

- Exercise: find the value of x in the following equation.

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$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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- The above equation is easy to solve for $y \uparrow$
- Plugging in $y \uparrow$ in the definition $ux \uparrow = y \uparrow$ and solving for $x \uparrow$ is also easy.