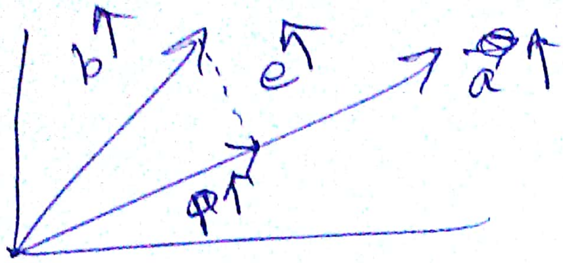
28-2-18 L-19.



$$p\uparrow = x\, a\uparrow$$

Small p vector

$$= \frac{a\uparrow \vec{a}}{\vec{a}\, a\uparrow}\, b\uparrow$$

$$= P \quad b\uparrow$$

capital P is a matrix

P is the projection matrix.

Properties of the projection matrix

(1) P is a symmetric matrix

$$P_{ij} = P_{ji}, \quad (P = P^T).$$

$$a\uparrow = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$P = \frac{a\uparrow \vec{a}}{\vec{a}\, a\uparrow} \quad \frac{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \frac{\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix}}{\|\vec{a}\|^2}$$

$$P = \frac{a\uparrow \vec{a}}{\vec{a}\, a\uparrow}$$

$$P^T = \frac{1}{\|a\uparrow\|^2} (a\uparrow \vec{a})^T$$

$$(AB)^T = B^T A^T$$

$$= \frac{1}{\|a\uparrow\|^2} (\vec{a})^T (a\uparrow)^T$$

$$= \frac{a\uparrow \vec{a}}{\|a\uparrow\|^2} \quad \left(\text{Note that } \|a\uparrow\|^2 = \|\vec{a}\|^2\right)$$

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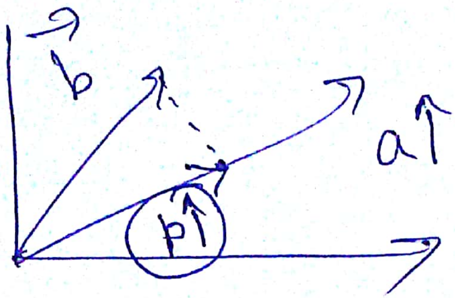
(2) $P^2 = P$ ~~identity~~

Nisarg P., Hariyali, Abutibyan GOT IT.

$$\text{L.H.S.} = P^2 = \frac{a\uparrow(\vec{a}}{\|\vec{a}\|^2} \cdot \frac{a\uparrow)\vec{a}}{\|\vec{a}\|^2}$$

$$= \frac{\|\vec{a}\|^2\, a\uparrow \vec{a}}{\|\vec{a}\|^2\, \|\vec{a}\|^2} = \frac{a\uparrow \vec{a}}{\|\vec{a}\|^2} = P$$

Sanity check for $P^2 = P$.



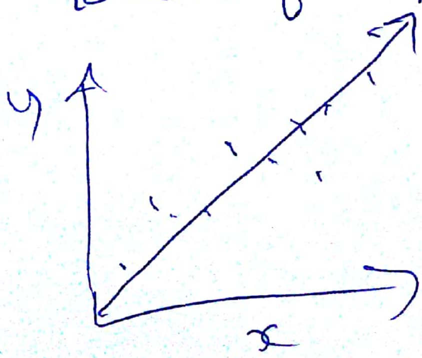
$p\uparrow = x\, a\uparrow$
$y\uparrow = m\, x\uparrow$

$$P\, b\uparrow = p\uparrow$$

$$\Rightarrow P^2 b\uparrow = P\, p\uparrow = p\uparrow$$

Applying P on P^2 has the same effect on $b\uparrow$, Hence $P^2 = P$.

Best fit problems.



Fit $y = mx$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$ → all data y values

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$ all data x values

$$y\uparrow - m\, x\uparrow = e\uparrow$$

Compare with projection problem
$b\uparrow \neq x\, a\uparrow$ because $b\uparrow$ & $a\uparrow$ are not colinear.

$\vec{p} = x\,\vec{a}$

or $\vec{y} = m\,\vec{x}$

can be done only if $\vec{y}$ is in the column space of $\vec{x}$

More general fits.

Data is $n+1$ dimensional.

$(y_1, x_1, x_2, \ldots x_n)$.

$x_i \rightarrow$ Concentration of n different chemicals

$y_1 \rightarrow$ Sediments, outcome

Fit a linear model.

$$\boxed{y_1 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.}$$

You have m data points.

$x_1^1 \quad x_2^1 \quad \ldots \quad x_n^1 \quad y_1^1$ point 1

$x_1^2 \quad x_2^2 \quad\quad x_n^2 \quad y_1^2$

$$\underbrace{\begin{bmatrix} x_1^1 & x_2^1 & \cdots & x_n^1 \\ x_1^2 & x_2^2 & & x_n^2 \\ \vdots & & & \\ x_1^m & x_2^m & & x_n^m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1^1 \\ \vdots \\ y_1^m \end{bmatrix}}_{\vec{y}}$$

The equation would be exactly true if $\begin{bmatrix} y_1^1 \\ \vdots \\ y_1^m \end{bmatrix}$ is in $C(A)$.

Remember the column picture

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

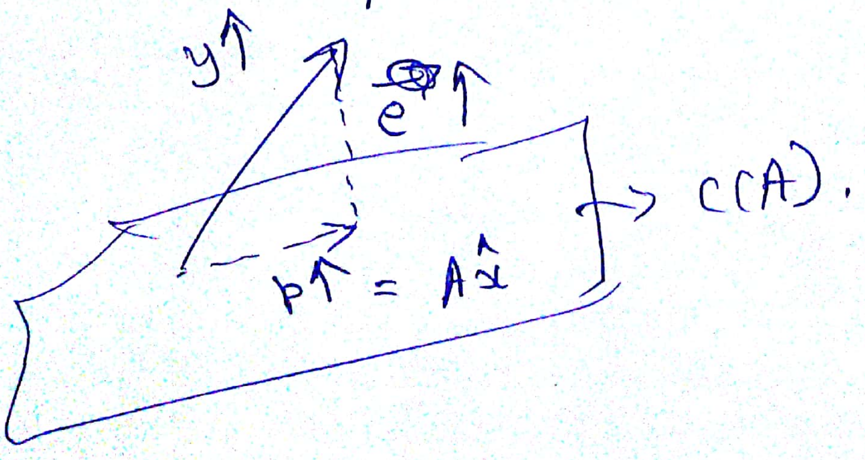
$$\Rightarrow \quad x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

We know that the data is not exact & the theory is not perfect so

$$A\vec{a} = \vec{y}$$

will not be exact.

because $\vec{y} \notin C(A)$.



$\vec{e} \perp C(A)$

$$\vec{y} - A\vec{a} = \vec{e}.$$

$\vec{e}$ is the error vector.

For best fit $\vec{a}$ should be chosen such that $\vec{e}$ is $\perp$ to $C(A)$.

Since $\vec{e} \perp C(A)$

$$\vec{e} \in N(A^T).$$

$$A^T \vec{e} = 0$$

$$\Rightarrow A^T(\vec{y} - A\vec{a}) = 0.$$

$$\Rightarrow A^T\vec{y} = A^T A \vec{a}$$

The above eqn. is true if $\vec{a}$ is chosen such that $\vec{e} \perp C(A)$, I have best fit.

Name that choice of parameters as

$\vec{a} = \hat{a}$

$$\Rightarrow \hat{a} = \underbrace{(A^T A)^{-1} A^T \vec{y}}$$ all known values.

Unknown parameters.

$$\vec{p} = A\hat{a} = A(A^T A)^{-1} A^T \vec{y}$$

Surprise quiz:

Prove that

if $\vec{y} \in C(A)$

then $\vec{p} = \vec{y}$.

in the previous discussion

$$\vec{p} = A(A^T A)^{-1} A^T \vec{y}$$

$\vec{y} \in C(A)$

$\hat{a}$ exists such that $A\hat{a} = \vec{y}$

$$\vec{p} = A(A^T A)^{-1}(A^T A)\hat{a}$$

$$= A\hat{a}$$

$$= \vec{y}$$