Solving Linear Equations

Mitaxi Mehta: Lectures 13and14

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- The transformation A takes the row space vector $x_p \uparrow$ to the column space vector $b \uparrow$ and maps any null space vector x_n to the zero vector.

Eucledean vectors

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- In Euclidean geometry location of a vector is unimportant. The unit vectors are independent of position.

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- To define a vector space, one needs two objects. A collection of vectors V, with $v_j \uparrow \in V$ and a collection of numbers A with members $a, b, ... \in A$ and a number of properties satisfied by the elements of V and A.
- Please note that different properties apply to numbers and vectors. In this course $A = R$, the real line, unless stated otherwise.

- Question: Consider $V = Z^2$ that contains all two dimensional vectors with integer entries and $A = R$, the set of real numbers, do they define a vector space ?

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- No, because a member of V, when multiplied by a fraction, ceases to be a member of V.

- Consider V to be a set of all 2×2 matrices,

$$\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$$

with $a, b, c, d \in R$.

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- Is V a vector space ?

- Yes, because all the vector space properties are satisfied.

- For a given matrix A, consider the row space $R(A)$, the column space $C(A)$ and the null space $N(A)$.

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- To prove that each of the above spaces is a vector space, show that any linear combination of two vectors of a space also belongs to that space.

- Consider the plane $2x + 3y + z = 10$ and the set of vectors with components (x, y, z) that satisfy the above equation. Is that a vector space ?

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- Shubham Pujara: No because zero vector does not belong to the space, i.e., $x = y = z = 0$ does not satisfy the plane equation.
- Negative vectors also don't belong to the set, for example $(0, 0, 10)$ satisfies the plane equation but $(0, 0, -10)$ does not.

- Exercise: What is the null space of,

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- For the given matrix, $N(A)$ contains only one vector $(0, 0)$, it is a zero dimensional space.

Echelon form of a matrix

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- In this form, in every row, the first leading entry (nonzero entry) is such that all elements in the column below it are zero (An upper diagonal form).

Echelon form: example

- Calculate the Echelon form of the following matrix.

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$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}$$

Echelon form: example

- The row operation $R_3 = R_3 + R_1$ brings the matrix in the form

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

- Proceeding in a manner similar to Gauss elimination using $R_2 = R_2 - (-2)R_1$ and $R_3 = R_3 + (-2)R_2$, the matrix comes into the following form,

$$B = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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- The number of pivots in the Echelon form gives the rank of the matrix.
- Rank of the matrix also equals the dimensions of $R(A)$ and $C(A)$.
- The calculation of $N(A)$ becomes easy using the Echelon form of A.

Row operations as Matrix multiplication

- Each row operation is equivalent to multiplying the given matrix on the right with another matrix.

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- Each row operation is equivalent to multiplying the given matrix on the right with another matrix.
- For example, row exchange is equivalent to the following matrix multiplication,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ g\,a + c & g\,b + d \end{pmatrix}$$

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$$BAx \uparrow = 0 \uparrow .$$

- Finally when the matrix is brought to the Echelon form,

$$A_{echelon} x \uparrow = 0 \uparrow$$

- If A is taken to be matrix of our previous Echelon example,

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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- The pivot variables can be solved for in terms of the free variables.
- The free variables can be assigned any value in R.

- Writing out the two equations in

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

  and solving for the pivot variables gives,

- $x_3 = -x_4$ and $x_1 = -3x_2 + x_4$

- A general null vector can thus be rewritten as,

$$x \uparrow = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} \text{-3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ \text{-1} \\ 1 \end{pmatrix}$$

- The previous equation describes the general form of vectors in the Null space of A. The dimension of the Null space is 2.

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- The dimension of $N(A) =$ Number of columns in A - Rank of A.