L-22-20 ① 9 March 18

# The Gram-Schmidt orthogonalization

Definition: Orthonormal vectors.
A set of vectors $q_1, \ldots, q_n$ is $\perp$ set.
~~Two vectors~~ $q_i^{\uparrow}$ and $q_j^{\uparrow}$ are ortho normal if

$$\vec{q_i}\, q_j^{\uparrow} = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

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Example the basis set of $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

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Are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ ortho normal ?

These are orthogonal vectors. ( $\perp$ to each other) but not orthonormal. (magnitude of the vectors ~~is not~~ $\neq 1$ ).

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$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$ orthonormal.

The task at hand

$\boxed{\text{Given a set of } n \text{ vectors } a, b, c, \ldots \text{ Write a set of } n \text{ orthonormal vectors } e_1, e_2, e_3 \ldots}$

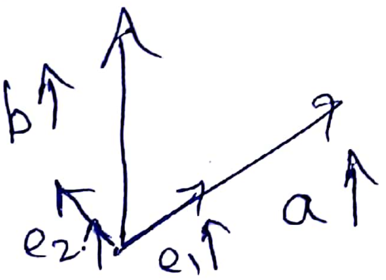
[which span the same vector space].

* Extension to integration, differentiation, transformations will follow. Elaborate.

The Gram Schmidt method of orthogonalization.

$$e_1\uparrow = \frac{a\uparrow}{\|\vec{a}\|}.$$

The geometric picture




Given two vectors $a\uparrow$ and $b\uparrow$ create two orthogonal and unit length vectors out of them

$e_1\uparrow \perp e_2\uparrow$ & $\|e_1\uparrow\| = \|e_2\uparrow\| = 1$

Consider
(1, 0, 0) and (2, 3, 0).

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These two vectors span a plane
why?
Both the vectors are in the x, y plane
Abu Tibyan.

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Any two linearly independent vectors in 3 dimensions, span a plane..

Jainam K: Two lines define a plane.
Shubham: Two free variables, $c_1$ & $c_2$ define a 2D geometric object, & the object is linear because $c_1$ & $c_2$ appear linearly, Hence it is a plane
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valid also in n-dimensions.

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Challenge Q: Mathematically derive eqn. of a plane for the 3D case.

$\vec{a} = \cancel{(1,0,0)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b} = \cancel{(2,3,0)}$, $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$$\vec{e_1} = \frac{\vec{a}}{\|\vec{a}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

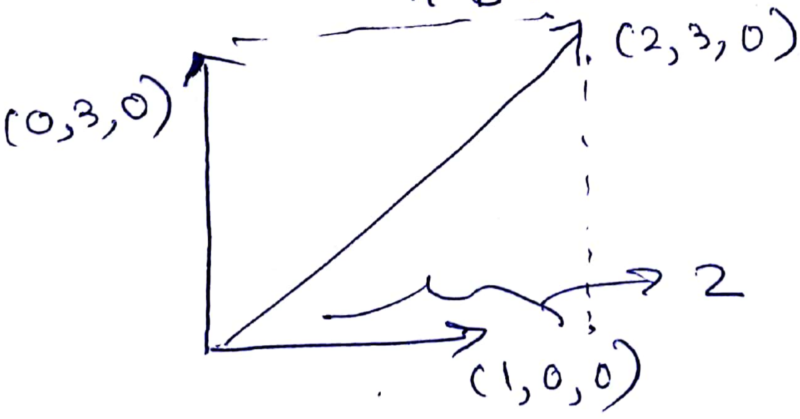
$$\vec{b} \cdot \vec{e_1} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$\vec{B} = \vec{b} - (\vec{b} \cdot \vec{e_1})\, \vec{e_1}$$

$$\vec{B} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - (2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$\vec{e_2} = \frac{\vec{B}}{\|\vec{B}\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$




$\xleftarrow{\quad 2 \quad}\!\!\rightarrow \; (\vec{b} \cdot \vec{e_1})\, \vec{e_1}$

In general.

$a\uparrow$, $b\uparrow$, $c\uparrow$, $d\uparrow$, $e\uparrow$ are given vectors.

define $e_1\uparrow$, $e_2\uparrow$, $e_3\uparrow$, ... as orthonormal vectors.

(1) $$e_1\uparrow = \frac{a\uparrow}{\|\vec{a}\|}$$

(2) $$B\uparrow = b\uparrow - (b\uparrow \cdot e_1\uparrow)\, e_1\uparrow$$

$$e_2\uparrow = \frac{B\uparrow}{\|\vec{B}\|}.$$