(1) L-37 16-4-18

— $\dot{X}\uparrow = AX\uparrow$, $X(t=0)\uparrow = X_0\uparrow$ initial value

The general solution

$$X(t)\uparrow = c_1 e_1\uparrow e^{\lambda_1 t} + c_2 e_2\uparrow e^{\lambda_2 t}$$

(1) where c_1 & c_2 are constants, to be decided by $X(t=0)\uparrow = X_0\uparrow$

(2) $e_1\uparrow$, $e_2\uparrow$ are eigenvectors of A.

(3) λ_1, λ_2 are eigenvalues of A.

If $S = \begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix}$

a change of variables.

$$Y\uparrow = S^{-1} X\uparrow$$

decouples the differential equations. $Y\uparrow$ components are called the normal coordinates.

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Solve

$$\frac{dx}{dt} = -3x + 10y$$

$$\frac{dy}{dt} = -3x + 8y$$

$x(0) = 2$ $y(0) = 3$

Find $x(t)$, $y(t)$.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 10 \\ -3 & 8 \end{bmatrix}.$$

The characteristic equation for λ is,

$$\text{Det}\begin{bmatrix} -3-\lambda & 10 \\ -3 & 8-\lambda \end{bmatrix} = 0$$

solution gives $\lambda_1 = 2$ $\lambda_2 = 3$.

$$e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3/5 \end{bmatrix}$$

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$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} e^{3t}$$

This is the general solution.

To fix c_1 and c_2, use

$x(0) = 2$

$y(0) = 3$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\left.\begin{aligned} 2c_1 + 5c_2 &= 2 \\ c_1 + 3c_2 &= 3 \end{aligned}\right\} \quad \begin{aligned} c_1 &= -9 \\ c_2 &= 4. \end{aligned}$$

The solution for initial value problem is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -9 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 4 \begin{bmatrix} 5 \\ 3 \end{bmatrix} e^{3t}$$

find $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ such that $\frac{dx_1}{dt}$ & $\frac{dy_1}{dt}$ are decoupled.

$$S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\left.\begin{aligned} x_1 &= 3x - 5y \\ y_1 &= -x + 2y. \end{aligned}\right\} \text{ Normal coordinates}$$

$\rightarrow$ use $\frac{dx}{dt}$, $\frac{dy}{dt}$

show that

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \lambda_1 x_1 \\ \frac{dx_2}{dt} &= \lambda_2 x_2 \end{aligned}\right\} \text{ Decoupled equations.}$$

$$\frac{dx_1}{dt} = \frac{d}{dt}(3x - 5y) = 3\frac{dx}{dt} - 5\frac{dy}{dt}$$

$$= 3(-3x + 10y) - 5(-3x + 8y)$$

$$= -9x + 15x + 30y - 40y$$

$$= 6x - 10y = 2(3x - 5y) = 2x_1$$

$$\boxed{\frac{dx_1}{dt} = \overset{\lambda_1}{2} x_1}$$ similarly $$\boxed{\frac{dx_2}{dt} = \overset{\lambda_2}{3} x_2}$$

Decoupled equations.

Determinants.

Given a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

the determinant of the matrix is

$ad - bc = \text{Det } A = |A|$

- What is the geometric meaning of Det A ?

- Manasi : Det A is the area of the parallelogram created by the column vectors of the matrix as edges.

- Where have you used determinants before ?

$$\iint f(x,y)\, dx\, dy = \int f(x(X,Y), y(X,Y)).\, |J|\, dX\, dY.$$

The determinant corresponding to coordinate change.

dx : infinitesimal length

$\cancel{dx = \lim_{\Delta x \to 0} \Delta x}$

tangent

dy : "

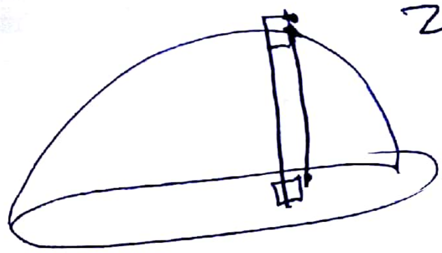
Don't need to worry

$dx\,dy$: infinitesimal area.

$z = f(x,y) \rightarrow$ surface

$z = \sqrt{1-x^2-y^2}$

z,



$\iint f(x,y)\,dx\,dy$

$\downarrow$

volume

height * area = volume

$X = 2x$

$Y = 3y$.

$dX\,dY \neq dx\,dy$.

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Similarly determinants of 3X3 matrices, correspond to volume of parallelopiped created by the column vectors as edges.

→ Similarly for higher dimensions,

→ If two columns are same the corresponding geometric quantity becomes zero.

→ In 2D two vectors along the same line don't enclose an area

→ In 3D, 3 vectors on the same plane don't enclose a volume

That is why when column vectors are linearly dependent, determinant is zero.