Linear Independence

Mitaxi Mehta: Lectures 16

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- Another example,

$$\begin{aligned} 2x + 3y + 2 &= 0 \\ x - y + 2 &= 5 \end{aligned}$$

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- Aarya: The RHS is in the column space, so there is at-least one solution. Drashti: There are infinnite number of solutions since z is a free variable.

- Linear independence: A set of vectors $\{v_1 \uparrow, v_2 \uparrow, ..., v_n \uparrow\}$ is linearly independent if

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- Hence the two vectors are linearly independent.

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- Shrey, does not agree. in that case show that only $c_1 = c_2 = 0$ case is possible.

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$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 + c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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- Hence $c_1 = c_2 = 0$.

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- Nonzero c_1, c_2, c_3 exist such that $c_1 v_1 \uparrow + c_2 v_2 \uparrow + c_3 v_3 \uparrow = 0$, $c_1 = c_2 = 1, c_3 = -1$. Thus one of the vectors can be written in terms of the rest two.

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- Express $(3, 2)$ as a linear combination of $(1, 0)$ and $(1, 1)$.

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- $(1,0),(0,1)$ is a basis set of R^2 so is, $(1,0),(1,1)$.
- Express $(3,2)$ as a linear combination of $(1,0)$ and $(1,1)$. Ans: $(3,2)=(1,0)+2(1,1)$.

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- The standard notation for norm is $||\vec{a}||^2 = a^T a$. Note that in the dot product the order is not important $\vec{a} b \uparrow = \vec{b} a \uparrow$.

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- If $a^T b = 0$ what can you conclude? Give the four possible conclusions.
- Either the two vectors are orthogonal or one of them is zero or both are zero.