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→ Prove that ~~if~~ since the sum of each column of a ~~xxx~~ Markov matrix is 1 the sum of entries in the vectors X_n does not change

→ Take a 2×2 matrix or a general Markov matrix

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

prove $x_{n+1} + y_{n+1} = x_n + y_n$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} x_n + a_{12} y_n \\ a_{21} x_n + a_{22} y_n \end{bmatrix}$$

$$\Rightarrow x_{n+1} + y_{n+1} = a_{11} x_n + a_{12} y_n + a_{21} x_n + a_{22} y_n$$

$$= x_n (a_{11} + a_{21}) + y_n (a_{12} + a_{22})$$

$$= x_n + y_n \quad (\because \text{columns add to } 1)$$

Prove that if $\vec{e_1}$ is an eigenvector with eigenvalue λ_1, $a_1 \vec{e_1}$ where a_1 is a constant is also an eigenvector with eigenvalue λ_1

$A \vec{e_1} = \lambda_1 \vec{e_1}$. Definition of $\vec{e_1}$ as an e-vector

$A(a\vec{e_1}) = aA\vec{e_1}$ [a scalar can multiply before or after the matrix with the same result]

$\Rightarrow \quad = a\lambda_1 \vec{e_1}$

$= \lambda_1 (a\vec{e_1})$

=

Given a matrix how will you find eigenvectors & eigen values.

$$A\, e_i \uparrow = \lambda_i\, e_i \uparrow$$

$$\Rightarrow \quad A\, e_i \uparrow - \lambda_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cdot \\ 0 & 0 & 1 \end{bmatrix} e_i \uparrow = 0 \uparrow$$

$$\Rightarrow \quad (A - \lambda_i I)\, e_i \uparrow = 0 \uparrow .$$

I is an identity matrix.

If the multiplication is zero then either $e_i \uparrow$ is a zero vector or

Mayank:

If $(A - \lambda_i I)$ is an invertible matrix then

$$e_i \uparrow = (A - \lambda_i I)^{-1}\, 0 \uparrow$$

$$= 0 \uparrow$$

To have a non-zero eigenvector $(A - \lambda I)$ Should not be invertible.

$(A - \lambda_i I)$ is a singular matrix.

$\Rightarrow$ $\mathrm{Det}(A - \lambda_i I) = 0$. This equation defines eigen values

Example:

Find the eigen-values of the matrix $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

$$\text{Det}(A - \lambda I) = 0$$

$$\Rightarrow \text{Det}\left(\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \text{Det}\begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 = 16$$

$$\Rightarrow (\lambda-1) = \pm 4.$$

$$\Rightarrow \lambda = 5, -3.$$

tangent

Use of eigenvectors & eigen values.

→ Atomic physics: Eigen values → energy levels of an atom (also molecules).
eigenvectors :→ Atomic state.

→ Mechanical engineering
eigenvalues :→ the rate at which the System evolves. eigenvectors → state of the system

Eigen values of

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$ are 5 and -3

use this information to find eigen-vectors:

Eigenvalue equation is

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

Two eqn.s of two unknowns. solve

$$x + 4y = 5x \qquad \Rightarrow \quad x = y$$
$$4x + y = 5y$$

eigenvector for $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or any multiple of it).

similarly.

eigenvector for $\lambda = -3$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ "

Find eigenvalues & eigen vectors

of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\lambda_1 = e^{i\theta}$ $\quad\quad$ $\lambda_2 = e^{-i\theta}$

Find the eigen-vectors.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = e^{i\theta} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = e^{i\theta}x$$
$$-x\sin\theta + y\cos\theta = e^{i\theta}y.$$

Short-cut: Eigen vectors always have one free variable. The variable could be x or y.

Take $x = 1$

$$\cos\theta + y\sin\theta = e^{i\theta} \Rightarrow y = i$$

for $\lambda = e^{i\theta}$ eigenvector is $\begin{bmatrix} 1 \\ i \end{bmatrix}$. [or any multiple].

→ Given the properties of a Markov Matrix

$a_{ij} > 0$ $|a_{ij}| \leq 1$

$\sum_j a_{ij} = 1$ for all i.

→ Can you see that the highest value of λ can be 1

$X_{n+1}\uparrow = \lambda X_n\uparrow = A X_n\uparrow$ if X_n is eigen-vector.

→ Doing iteration 10 times

$\boxed{X_{10}\uparrow = \lambda^{10} X_0\uparrow}$ if X_0 is an eigenvector

$\Rightarrow |X_{10}\rangle = \lambda^{10} |X_0\uparrow\rangle$

$\Rightarrow \sum_i X_{(10)i} = \lambda^{10} \sum_i X_{0i}$

Since total population does not change

$\sum X_{10i} = \sum_i X_{0i}$ so $\lambda \leq 1$.

Prove that one eigen-value of a Markov matrix is 1.

$$\text{Det}\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & a_{22}-\lambda & \cdots \\ \cdots & & a_{nn}-\lambda \end{bmatrix} = 0.$$

For the above matrix sum of all rows will add up to $(1-\lambda)$.

If λ is selected to be 1 then the Det is gaurenteed to be zero.

$\Rightarrow$ ~~λ is~~ . $\lambda = 1$ is one of the eigenvalues.

$$\Rightarrow \text{Det}\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & a_{22}-\lambda & \\ 1-\lambda & 1-\lambda & 1-\lambda \end{bmatrix} = 0.$$

$\Rightarrow \lambda = 1$ is a solution.

If you know eigenvalues & eigen-vector, Long time prediction of a population becomes simple.

→ Start with any arbitrary vector

$\begin{bmatrix} x \\ y \end{bmatrix}$ $x \rightarrow$ people in Ahd

$y \rightarrow$ people outside Ahd.

→ Rewrite in terms of eigen-vectors.

$$\begin{bmatrix} x \\ y \end{bmatrix} = a\, \vec{e_1} + b\, \vec{e_2}.$$

(assuming $\vec{e_1}$ & $\vec{e_2}$ are linearly independent and form a basis).

$$A \begin{bmatrix} x \\ y \end{bmatrix} = a\, A\, \vec{e_1} + b\, A\, \vec{e_2},$$

$$= a\, \lambda_1 \vec{e_1} + b\, \lambda_2 \vec{e_2}$$

$$A^2 \begin{bmatrix} x \\ y \end{bmatrix} = a\, \lambda_1^2 \vec{e_1} + b\, \lambda_2^2 \vec{e_2}$$

$$A^n \begin{bmatrix} x \\ y \end{bmatrix} = a\, \lambda_1^n \vec{e_1} + b\, \lambda_2^n \vec{e_2}$$

if $\lambda_1 = 1$ if $\lambda_2 < 1$

$\lambda_2^n \rightarrow 0$

$$= a\, \vec{e_1}$$

For any eigen-vector corresponding to the highest eigenvalue λ_1

$$A^n \begin{bmatrix} x \\ y \end{bmatrix} = a \lambda_1^n e_1 \uparrow + b \lambda_2^n e_2 \uparrow$$

The Highest contribution

An easy way of finding the eigen vector corresponding to the highest eigen value is

- → Select any vector
- → Apply A Large number of times.
- → Resulting vector will be close to $e_1 \uparrow$

if $\lambda_1 >> \lambda_2$

2-32

Discrete evo. systems -

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}.$$

If $A e_1 = \lambda e_1$

$\rightarrow \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \lambda^n e_1$

If e_1 & e_2 both are known then.

~~If~~ are lin. ind.

$\cancel{e_1 \neq} e_2$

$A x = \lambda x$

$A \underline{ax} = \lambda \underline{ax}$

$A \underline{ax}$ = power

$\Downarrow$ evec. 1 family line

$v \hat{} = a e_1 \hat{} + b e_2 \hat{}$

$A^n v \hat{} = a \lambda_1^n e_1 \hat{} + b \lambda_2^n e_2 \hat{}$

evolution becomes easy

solving for

Get eval & e vectors

$$\begin{bmatrix} 1 & \cancel{4} \\ 4 & 1 \end{bmatrix}$$

$(\lambda - 1)^2 = \cancel{1}6$

$\cancel{\lambda}$ $\lambda - 1 = \pm 4$

$\lambda = 1+4 \ , 1-4$

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 + 4x_2 = 5x_1$

$4x_1 + x_2 = 5x_2$

$4x_2 = 4$

$x_1 = 1$

$x_2 = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$x_1 + 4x_2 = -3x_1$

$4x_1 + x_2 = -3x_2$

get eval & evec.

for $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$(\cos\theta - \lambda)^2 = -\sin^2\theta$

$\lambda - \cos\theta = \pm i \sin\theta$

$\lambda = e^{i\theta}, e^{-i\theta}$

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = e^{i\theta} \begin{bmatrix} x \\ y \end{bmatrix}$

$x \cos\theta + y \sin\theta = e^{i\theta} x$

$-x \sin\theta + y \cos\theta = e^{i\theta} y$

$\cos\theta + y \sin\theta = e^{i\theta}$

$x = 1$

$y = i$

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