The Harmonic Oscillator

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Lecture 4

Working with operators

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- Calculate and show.
- What is the value of $(D-3)$ operating on zero?

- We already know the following,

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- Notice that the same function will also solve,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = (D-3)(D-2)y = 0$$

- Similarly $c_2 e^{3x}$ solves

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- so it also is a solution of

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- Does that mean that there are two different solutions $c_1 e^{2x}$ and $c_2 e^{3x}$, for the differential equation

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what do you think?

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  what do you think?

- The equation is a second order differential equation, so solving it requires two integration and hence two constants of integration.

- The expression $y(x) = c_1 e^{2x} + c_2 e^{3x}$ is known as the general solution of the given differential equation.

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- We shall discuss similar such equations later.

- Consider Oscillatory motion of a point mass inside a potential well $V(x)$.

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- The force on the mass is given by $\vec{F} = -\frac{dV}{dx}$.
- Notice that the force is zero at the maxima and minima of the potential. Those are equilibrium of the particle

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- Hence near the minimum of a potential, the particle with a low energy exhibits oscillatory motion.
- Why does not the particle stop as it arrives at the equilibrium?

Harmonic oscillations

- The simplest expression of force near equilibrium is linear $F = -kx$, which is the force on a mass attached to a spring. (What happened to the equilibrium length?)

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- Applying Newton's second law, the equation of motion is,

$$m\frac{d^2x}{dt^2} = -kx$$

- In operator notation the equation can be written as

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- As seen in the previous example the general solution can be written as,

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

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- Rewrite the expression of $x(t)$ in terms of $\sin$ and $\cos$ functions.
- Rewrite expression as $x(t) = Csin(\omega t + \phi)$. What is the physical meaning of ϕ ?

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(2) $x(t) = A\cos(\omega t) + B\sin(\omega t)$
(3) $x(t) = Csin(\omega t + \phi)$

- Note that all the three expressions are mathematically equivalent after rewriting the coefficients appropriately.

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- Once the general solution is known one can write the solution of SHO for any set of initial values.

- Exercise
  (1) Make the small angle approximation of the pendulum equation $\sin\theta \approx \theta$ and write doen the solution.

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  (1) Make the small angle approximation of the pendulum equation $\sin\theta \approx \theta$ and write doen the solution.
  (2) A mass attached to a spring hangs vertically down with the other end of the spring held fixed. Write the corresponding differential equation and solve.

- Write the position vector of a point in uniform circular motion as a function of time.

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  Write the position vector of a point on the rim of a bicycle wheel rolling in straight line.