Taylor Series

Mitaxi Mehta
Lecture 6

- In the last class we saw that a function can be approximated by a straight line in the neighborhood of a point.

- In the last class we saw that a function can be approximated by a straight line in the neighborhood of a point.
- This straight line is the tangent to that point to the curve of the function.

- This approximation is not very good as one moves away from the point.

- This approximation is not very good as one moves away from the point.
- Can we make a better approximation by using the information of the second derivative at the point?

- This approximation is not very good as one moves away from the point.
- Can we make a better approximation by using the information of the second derivative at the point?
- For most of the functions, the answer is yes.

- The method of Taylor approximation assumes that the given function can be approximated by a polynomial of degree n.

Taylor approximation

- The method of Taylor approximation assumes that the given function can be approximated by a polynomial of degree n.
- Provided a given function $f(x)$ is known at a specific point $x = a$ and all the derivatives of $f(x)$ up to nth derivative are known at that point.

Taylor approximation

- The method of Taylor approximation assumes that the given function can be approximated by a polynomial of degree n.
- Provided a given function $f(x)$ is known at a specific point $x = a$ and all the derivatives of $f(x)$ up to nth derivative are known at that point.
- We shall assume that the function can be approximated by a polynomial of degree n,

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n,$$

- What happens when $n = 1$?

- What happens when $n = 1$?
- The function is $f(x)$ we know its value at $f(x = a) = f(a)$ and its first derivative at $x = a$, $f'(a)$.

- What happens when $n = 1$?
- The function is $f(x)$ we know its value at $f(x = a) = f(a)$ and its first derivative at $x = a$, $f'(a)$.
- That means we may be able to approximate it by

$$P_1(x) = c_0 + c_1(x - a)$$

- If $P_1(a) = f(a)$ and $P_1'(a) = f'(a)$ can you find the values of c_0 and c_1?

- Insisting that $P_1(a) = f(a)$ gives,

$$P_1(x = a) = c_0 + c_1(a - a) = f(a)$$

Hence $c_0 = f(a)$.

- Insisting that $P_1(a) = f(a)$ gives,

$$P_1(x = a) = c_0 + c_1(a - a) = f(a)$$

  Hence $c_0 = f(a)$.

- Using $P_1'(x)|_{x=a} = f'(a)$, yields $c_1 = f'(a)$

- $c_0 = f(a)$ and $c_1 = f'(a)$ thus

$$P_1(x) = f(a) + f'(a)(x - a)$$

- $c_0 = f(a)$ and $c_1 = f'(a)$ thus

$$P_1(x) = f(a) + f'(a)(x - a)$$

- This is the same as the function approximation we talked about in the last class, provided we write $\Delta x = (x - a)$.

- $c_0 = f(a)$ and $c_1 = f'(a)$ thus
$$P_1(x) = f(a) + f'(a)(x - a)$$
- This is the same as the function approximation we talked about in the last class, provided we write $\Delta x = (x - a)$.
- Note that this is also the equation of the tangent at point $x = a$.

- Suppose we want to approximate a function $f(x)$.

- Suppose we want to approximate a function $f(x)$.
- We know $f(a)$, $f'(a)$ and $f''(a)$.

- Suppose we want to approximate a function $f(x)$.
- We know $f(a)$, $f'(a)$ and $f''(a)$.
- Assuming a quadratic polynomial of the form

$$P_2(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

find the values of c_0, c_1 and c_2

- Hence the second degree Taylor approximation about $x = a$ is,

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

- What would an n^{th} degree Taylor approximation look like ?

- the k^{th} coefficient has the form,

$$c_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad 0 \leq k \leq n.$$

- the k^{th} coefficient has the form,

$$c_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad 0 \leq k \leq n.$$

  Note: $f^{(k)}(a)$ is kth derivative of f evaluated at a, $f^k(a)$ is $(f(a))^k$.

- The polynomial has the form

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Find the third degree approximation of $\sin x$ about $x = 0$.

- Find the third degree approximation of $\sin x$ about $x = 0$.
- Find the fourth degree approximation of $\cos x$ about $x = 0$

- Find the third degree approximation of $\sin x$ about $x = 0$.
- Find the fourth degree approximation of $\cos x$ about $x = 0$
- Find the third degree approximation of $\sin x$ about $x = \pi/2$.