ENGN2228

Assignment 2

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Q1

From graph: $T=3$, $\therefore \omega_0 = \frac{2\pi}{3}$

The function can be written as

$$x(t) = \begin{cases} t+2, & -2 \le t \le 0 \\ -2t+2, & 0 \le t \le 1 \end{cases}$$

for interval $-2 \le t \le 1$

To determine the fourier series, we must determine a_k

Firstly $a_0 = \frac{1}{T}\int_T x(t)$

$$= \frac{1}{3}\int_{-2}^{0} t+2\,dt + \frac{1}{3}\int_0^1 -2t+2\,dt$$

$$= \frac{1}{3}\left[\left|\frac{t^2}{2}+2t\right|_{-2}^{0} + \left|\frac{-2t^2}{2}+2t\right|_0^1\right]$$

$$= \frac{1}{3}[3]$$

$a_0 = 1$

now for $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$

$$= \frac{1}{3}\left[\int_{-2}^{0} te^{-jk\omega_0 t}\,dt + 2\int_{-2}^{0} e^{-jk\omega_0 t}\,dt + 2\int_0^1 e^{-jk\omega_0 t}\,dt - 2\int_0^1 te^{-jk\omega_0 t}\,dt\right]$$

Integrating, expanding & simplifying, we get

$$a_k = \frac{3j}{2\pi^2 k^2}\left[2e^{-j\frac{\pi}{3}k} \cdot \sin\left[\frac{\pi}{3}k\right] - e^{j\frac{2\pi}{3}k} \cdot \sin\left[\frac{2\pi}{3}k\right]\right], \quad k \ne 0, \quad \omega_0 = \frac{2\pi}{3}$$

$\therefore$ The fourier series representation is then

$$x(t) = \sum_{-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

Q 2

$$a_k = \begin{cases} jk & |k|<3 \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = \sum a_k \cdot e^{jk\omega t} = \sum_{-2}^{2} jk e^{jk\omega t}$$

$$= -2je^{-2j\omega t} - je^{-j\omega t} + je^{j\omega t} + 2je^{2j\omega t}$$

$$= 2\left(je^{2j\omega t} - je^{-2j\omega t}\right) + \left(je^{j\omega t} - je^{-j\omega t}\right)$$

$$= -4\sin(2\omega t) - 2\sin(\omega t)$$ from euler's identity.

Q3

a)

i) $h[n] = \delta[n+1] + \delta[n] + \delta[n-1]$
$= u[n+1] - u[n-2]$

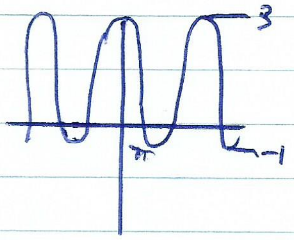
ii) $H(e^{j\omega}) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n}$

$= e^{-j\omega} + 1 + e^{j\omega} = 1 + 2\cos(\omega)$

iii) H is even

iv) $Gd = \frac{-d\phi}{d\omega}$
$= -1$
$\therefore$ linear & causal

v)



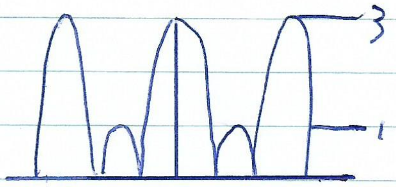
b) i) $h[n] = \delta n + \delta[n-1] + \delta[n-2]$
$= u[n] - u[n-3]$

ii) $H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega}$

iii) H is even

iv) $Gd = 1$, linear & causal

v)

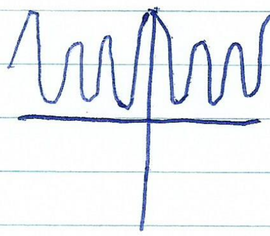


c) i) $h[n] = \delta[n-2] + \delta[n-3] + \delta[n-4]$
$= u[n-2] - u[n-5]$

ii) $H(e^{jw}) = e^{-2jw} + e^{-3jw} + e^{-4jw}$

iii) H is even

iv) $gd = -3w$, linear & causal

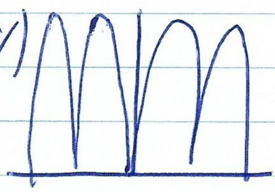
v)


d) i) $h[n] = \delta[n-1] - \delta[n+1]$

ii) $H(e^{jw}) = e^{-jw} - e^{jw} = -2j\sin(w)$

iii) H is odd

iv) phase zero; $Gg = -1$

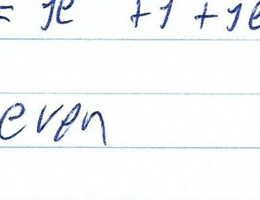
V)


e) i) $h[n] = j\delta[n+1] + j\delta[n] + j\delta[n-1]$

ii) $= je^{jw} + j + je^{-jw} = j + 2j\cos(w)$

iii) even

iv) $Gd = 0$

v)


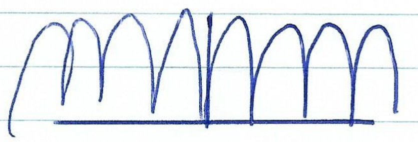
f) i) $h[n] = j\delta[n-1] - j\delta[n+1]$

ii) $= -je^{j\omega} + je^{-j\omega} = 2j\sin(-\omega)\cdot j = -2\sin(-\omega)$

iii) odd

iv) gd = -1, linear & causal

v)



g) i) $h[n] = \delta[n+2] + \delta[n] + \delta[n-2]$

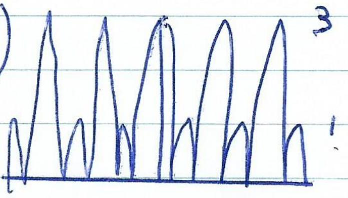
ii) $H(e^{j\omega}) = \cancel{1 + e^{-2j\omega} + \dots e}$

$1 + e^{-2j\omega} + e^{2j\omega} = 1 + 2\cos(2\omega)$

iii) even

iv) gd = -2, linear & causal

v)



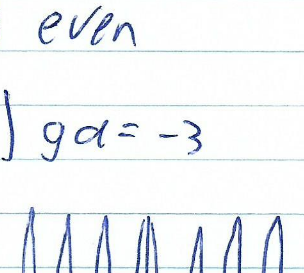
h) i) $h[n] = \delta[n+3] + \delta[n] + \delta[n-3]$

ii) $H(e^{j\omega}) = 1 + e^{-3j\omega} + e^{3j\omega} = 1 + 2\cos(3\omega)$

iii) even

iv) gd = -3

v)



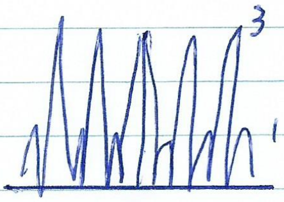
1) i) $h[n] = \delta[n+4] + \delta[n] + \delta[n-4]$

ii) $H(e^{jw}) = 1 + e^{-4jw} + e^{4jw} = 1 + 2\cos(4w)$

iii) even

iv) $gd = -4$

v)



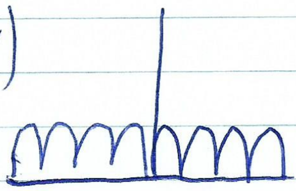
2) i) $h_n = \frac{j}{2}\delta[n+1] + \delta[n-1]$

ii) $= \frac{j}{2}e^{jw} + e^{-jw} = j\cos(w)$

iii) even

iv) $gd = 0$

v)



Q4

a) Period is 4 from graph

$$\therefore \omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$a_0 = \frac{1}{T}\int_T x(t)$$

$$= \frac{1}{4} \cdot 2$$

$$= \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega t k}$$

$$= \frac{1}{4}\int_{-2}^{2} 2e^{-j\omega t k}\, dt$$

$$= \frac{1}{4}\left[\frac{2j e^{-j\omega t k}}{k\omega}\right]_{-2}^{2}$$

$$= \frac{\sin(k\omega)}{k\pi}$$

$$\therefore a_1 = \frac{1}{\pi},\ = a_{-1}$$

$$a_2 = 0$$

$$a_3 = \frac{1}{3\pi} \ \ldots$$

$\vdots$

b) $T = 2$ from graph

$\therefore \omega_0 = \pi$

$$b_0 = \frac{1}{2}\left[\int_0^2 \cos(\pi t)\right] = \frac{1}{2}\left[\frac{\sin \pi t}{\pi}\right]_0^2$$
$$= \frac{1}{2}[0]$$
$$= 0$$

$$b_k = \frac{1}{2}\left[\int_0^2 \cos(\pi t) e^{-jn\omega_0 t}\right]$$
$$= \frac{1}{2}\left[\frac{t}{2} + \frac{je^{-2j\pi t}}{4\pi}\right]_0^2$$
$$= \frac{1}{2}$$

$b_{-1} = b_1 = \frac{1}{2}$

$\therefore$ the cooefficients for $y(t)$ are $b_0 = 0$, $b_{\pm 1} = \frac{1}{2}$

c) Fundamental frequency is 4, from graph

$\therefore \omega_0 = \frac{\pi}{2}$

d)

using the Fourier series multiplication property,

$$x(t)\,y(t) \xleftrightarrow{\mathcal{F}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

then h_k is given by

$$h_k = a_{-1} \cdot b_{k+1} + a_0 \cdot b_k + a_1 \cdot b_{k-1}$$

$$\therefore h_{-2} = a_{-1} b_{-1} + a_0 b_{-2} + a_1 \cdot b_{-3}$$

$$= \frac{1}{2} \cdot \frac{1}{\pi} = \frac{1}{2\pi}$$

~~$h_{-1} = a_{-1} \cdot b_0 + a_0 b_{-1} + a_1 b_{-2}$~~

by repeating for several h_k, it is found that

$$h_k = \begin{cases} \frac{1}{2\pi}, & |k| = 2 \\ \frac{1}{2}, & |k| = 1 \\ \frac{1}{2\pi} + \frac{1}{2}, & k = 0 \end{cases}$$

Q5

a) $x[n] = \left(\frac{3}{4}\right)^n u[n]$, $h[n] = \left[\frac{1}{2}\right]^n u[n]$

$$y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{-\infty}^{\infty} \left(\frac{3}{4}\right)^k u k \cdot \left(\frac{1}{2}\right)^{n-k} u[n-k]$$

$$= \sum_{0}^{\infty} \left(\frac{3}{4}\right)^k \cdot \left[\frac{1}{2}\right]^{n-k}$$

$$= \left[\frac{1}{2}\right]^n \sum_{0}^{\infty} \left(\frac{3}{4}\right)^k \cdot 2^k$$

$$= \frac{1}{2}^n \left[\frac{\left(\frac{3}{4}\right)^{n+1} \cdot 2^{(n+1)} - 1}{0.5}\right]$$ using $\sum \alpha^n \beta^n = \frac{\alpha^{n+1}\beta^{n+1} - 1}{\alpha\beta - 1}$

$$y[n] = \left(\frac{1}{2}\right)^n \cdot 2\left[\left(\frac{3}{4}\right)^{n+1} \cdot 2^{n+1} - 1\right]$$

b) $x[n] = \left(\frac{3}{4}\right)^n u[n]$, $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Fourier property $\alpha^n u[n] \rightarrow \frac{1}{1-\alpha e^{-j\omega}}$

$\therefore X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$, $\alpha = 3/4$, $H(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$, $\beta = \frac{1}{2}$

Fourier convolution property is $Y(z) = X(z)H(z)$

$\therefore Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$ (i)

Partial fraction expansion to find $y[n]$

$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$ (ii)

Setting equations (i) & (ii) equal, & solving for A & B yields:

$A = \frac{\alpha}{\alpha - \beta}$, $B = \frac{-\beta}{\alpha - \beta}$, $\therefore A = 3$, $B = -2$

$\therefore Y(e^{j\omega}) = \frac{3}{1-\frac{3}{4}e^{-j\omega}} \frac{-2}{1-\frac{1}{2}e^{-j\omega}}$,

using linearity property of F. transform, $y[n] = 3\left(\frac{3}{4}\right)^n u[n] - 2\left[\frac{1}{2}\right]^n u[n]$

(checked using matlab)

5 ~~d~~) c) $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$ $h[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}, \quad He^{j\omega} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{C}{1-\frac{1}{2}e^{-j\omega}}$$

Solving for A, B, & C

$A = -2$

$B = -1$

$C = 4$

$$Y(e^{j\omega}) = \frac{-2}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{4}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$y[n] = -2\left(\frac{1}{4}\right)^n u[n] - \left[\frac{1}{4}\right]^n (n+1) \cdot u[n] + 4\left(\frac{1}{2}\right)^n u[n]$$

d)

$$t e^{-\alpha t} \leftrightarrow \frac{1}{(\alpha + j\omega)^2}, \quad e^{-\alpha t} \leftrightarrow \frac{1}{\alpha + j\omega}$$

$$\therefore x(t) = t e^{-2t} u(t) \rightarrow X(j\omega) = \frac{1}{(2+j\omega)^2}$$

$$h t = e^{-4t} u(t) \rightarrow H(j\omega) = \left(\frac{1}{4+j\omega}\right)$$

$$Y(j\omega) = \frac{1}{(2+j\omega)^2} \cdot \frac{1}{(4+j\omega)}$$

Partial fractions

$$Y(j\omega) = \frac{A}{2+j\omega} + \frac{B}{(2+j\omega)^2} + \frac{C}{4+j\omega}, \quad \text{let } V = j\omega$$

$$C = \left[(V+4)Y(V)\right]_{V=-4} = \frac{1}{(2+4)^2} = \frac{1}{4}$$

$$B = \left[(2+V)^2 Y(V)\right]_{V=-2} = \frac{1}{4+V} = \frac{1}{2}$$

$$A = \frac{d}{dV}\left[(2+j\omega)^2 Y(V)\right]_{-2} = \frac{d}{dV}\left(\frac{1}{4+V}\right) = \frac{-1}{(V+4)^2} = -\frac{1}{4}$$

$$\therefore Y(j\omega) = \frac{-\frac{1}{4}}{2+j\omega} + \frac{\frac{1}{2}}{(2+j\omega)^2} + \frac{\frac{1}{4}}{4+j\omega}$$

linearity

$$\therefore h(t) = -\frac{1}{4} \cdot e^{-2t} + \frac{1}{2} t e^{-2t} + \frac{1}{4} e^{-4t}$$

e) $x(t) = te^{-2t}u(t) \qquad h(t) = te^{-4t}$

$$X(j\omega) = \frac{1}{(2+j\omega)^2} \qquad H(j\omega) = \frac{1}{(4+j\omega)^2}$$

$$Y(j\omega) = \frac{1}{(2+j\omega)^2} \cdot \frac{1}{(4+j\omega)^2}$$

$$= \frac{A}{(2+j\omega)} + \frac{B}{(2+j\omega)^2} + \frac{C}{4+j\omega} + \frac{D}{(4+j\omega)^2}$$

Let $v = j\omega$

$$D = \left[(4+v)^2 Y(v)\right]_{v=-4} = \frac{1}{(2+v)^2} = \frac{1}{4}$$

$$B = \left[(2+v)^2 Y(v)\right]_{v=-2} = \frac{1}{(4+v)^2} = \frac{1}{4}$$

$$A = \frac{d}{dv}\left[(2+v)^2 Y(v)\right]_{v=-2} = \frac{-1}{(v+4)^2} = \frac{-1}{4}$$

$$C = \frac{d}{dv}\left[(4+v)^2 Y(v)\right]_{v=-4} = \frac{-1}{(v+2)^2} = \frac{-1}{4}$$

$$Y(j\omega) = \frac{-1/4}{2+j\omega} + \frac{1/4}{(2+j\omega)^2} - \frac{1/4}{4+j\omega} + \frac{1/4}{(4+j\omega)^2}$$

$$y(t) = \frac{-1}{4}e^{-2t} + \frac{1}{4}te^{-2t} - \frac{1}{4}e^{-4t} + \frac{1}{4}te^{-4t}$$

Q6 a)

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \frac{j\omega + 4}{6 + (j\omega)^2 + 5j\omega}$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 + (j\omega)^2 + 5j\omega}$$

cross multiply

$$6Y(j\omega) + (j\omega)^2 Y(j\omega) + 5j\omega Y(j\omega) = j\omega X(j\omega) + 4X(j\omega)$$

Apply the property $\frac{d^k z(t)}{dt^k} \leftrightarrow (j\omega)^k Z(j\omega) \quad k \in \mathbb{Z}$

$$\therefore 6y(t) + \frac{d^2 y(t)}{dt^2} + 5\frac{d y(t)}{dt} = \frac{dx(t)}{dt} + 4x(t)$$

$$\therefore \text{diff. eqn} = \frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

b) $H(j\omega) = \frac{j\omega + 4}{6 + (j\omega)^2 + 5j\omega}$

let $v = j\omega$

$$H(v) = \frac{v+4}{6 + v^2 + 5v} = \frac{v+4}{(v+3)(v+2)}$$

$$H(v) = \frac{A}{v+3} + \frac{B}{v+2}$$

$$B = \left[(v+2)H(v)\right]_{v=-2} = \frac{4+v}{3+v} = \frac{2}{1} = 2$$

$$A = \left[(v+3)H(v)\right]_{v=-3} = \frac{v+4}{v+2} = \frac{1}{-1} = -1$$

$$\therefore H(j\omega) = \frac{-1}{j\omega+3} + \frac{2}{j\omega+2}$$

c) $e^{-\alpha t} \overset{F}{\leftrightarrow} \frac{1}{\alpha + j\omega}$

and note $H(j\omega) \overset{F}{\longleftrightarrow} h(t)$

then $H(j\omega) = \frac{-1}{3+j\omega} + \frac{2}{2+j\omega}$

$\frac{-1}{3+j\omega} \rightarrow e^{3t} u(t)$

$\frac{2}{2+j\omega} \rightarrow 2e^{-2t} u(t)$

$\therefore$ ~~h(t)~~ $= H(j\omega) = \frac{-1}{3+j\omega} + \frac{2}{2+j\omega} \xrightarrow{F} h(t) = e^{3t} u(t) + 2e^{-2t} u(t)$

7 a) $h[n] = \delta n - 0.5\,\delta[n-1]$

let $z = e^{j\omega}$

$Y(z) = X(z) - 0.5\,X(z-1)$

$Y(z) = X(z)\left(1 - \frac{1}{2}z^{-1}\right)$

$H(z) = \frac{Y(z)}{X(z)} = 1 - \frac{1}{2}e^{-j\omega} = H(e^{j\omega})$

b) $h[n] = \delta[n] + \frac{1}{6}h[n-1] + \frac{1}{6}h[n-2]$

let $z = e^{j\omega}$

$Y(z) = X(z) + \frac{1}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z)$

$Y(z)\left(1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right) = X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{6}{6 - e^{-j\omega} - e^{-2j\omega}} = H(e^{j\omega})$

c) $h[n] = 3\delta n - \delta[n-1] + 0.3\,h[n-1] - 0.1[n-2]$

let $z = e^{j\omega}$

$Y(z)\left(1 - 0.3z^{-1} + 0.1z^{-2}\right) = X(z)\left(3 - z^{-1}\right)$

$\frac{Y(z)}{X(z)} = \frac{3 - z^{-1}}{1 - 0.3z^{-1} + 0.1z^{-2}} = \frac{3 - e^{-j\omega}}{1 - 0.3e^{-j\omega} + 0.1e^{-2j\omega}} = H(e^{j\omega})$

d) $He^{j\omega} = \dfrac{-\frac{1}{9}e^{-j2\omega}}{1-\frac{1}{3}e^{-j\omega}}$

$$y[n] = x[n] + \tfrac{1}{3}x[n-1]$$

From the question

$H(e^{j\omega}) = F(e^{j\omega}) + Ge^{j\omega}$, $F(e^{j\omega})$ is F transform of $f[n] = \delta[n] + \frac{1}{3}\delta[n-1]$

From the fourier properties (time shift, linearity)

$F(e^{j\omega}) = 1 + \frac{1}{3}e^{-j\omega}$

So

$\therefore G(e^{j\omega}) = H(e^{j\omega}) - F(e^{j\omega})$

$$= \frac{-\frac{1}{9}e^{-j2\omega}}{1-\frac{1}{3}e^{-j\omega}} - \left(1 + \tfrac{1}{3}e^{-j\omega}\right)$$

Simplifying

$G(e^{j\omega}) = \dfrac{-1}{1-\frac{1}{3}e^{-j\omega}}$

$\therefore g[n] = -\left[\frac{1}{3}\right]^n u[n]$