Problem Set 2: First Chern classes of pojective manifolds

The purpose of this problem set is to understand how to compute the first Chern classes of projective manifolds.

1. Verify that the transition functions g_{jk} for the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^n$ relative to the standard charts are described by

$$g_{jk}([Z_0 : \ldots : Z_n]) = \frac{Z_j}{Z_k}$$

(and that these are well-defined functions on the charts).

2. Show that the transition functions g_{jk} for the bundle $\mathcal{O}(\ell)$ are

$$g_{jk}([Z_0 : \ldots : Z_n]) = \left(\frac{Z_k}{Z_j}\right)^{\ell}$$

3. Check that the canonical bundle of $\mathbb{CP}^n$ is isomorphic to the bundle $\mathcal{O}(-n-1)$. (Hint: A local frame over the chart U_0 is $dz_1 \wedge \cdots \wedge dz_n$ with $z_j = \frac{Z_j}{Z_0}$. Using the local frame $(-1)^i dz_0 \wedge \cdots \widehat{dz_i'} \wedge \cdots \wedge dz_n'$ over U_i with the coordinates $z_j' = \frac{Z_j}{Z_i}$, argue that the transition functions correspond to those for $\mathcal{O}(-n-1)$ from Problem 2.)

4. The **curvature of a line bundle** L **with hermitian metric** h is defined to be the $(1,1)$-form on M with local expression

$$-\sqrt{-1}\partial\bar{\partial}\log \|s\|_h^2$$

for a local non-vanishing holomorphic section s of L. Check that this $(1,1)$-form is independent of the choice of holomorphic section s.

5. Restricting the standard hermitian metric on the trivial vector bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ to the subbundle $\mathcal{O}(-1)$ equips $\mathcal{O}(-1)$ with a hermitian metric. Check that the curvature of this metric is given by $-\omega_{FS}$. More generally, argue that the curvature of $\mathcal{O}(\ell)$ is given by $\ell\omega_{FS}$.

6. We say that a $(1,1)$-form α is **positive** if the 2-tensor defined by $(X,Y) \mapsto \alpha(X, JY)$ is positive definite. We say that a line bundle is positive if it admits a hermitian metric whose corresponding curvature form is positive. Check that $\mathcal{O}(\ell)$ is positive if and only if ℓ is positive.

7. The **first Chern class** $c_1(L)$ **of** L is the cohomology class determined by the $(1,1)$-form with local expression

$$\frac{-\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \|s\|_h^2$$

for some hermitian metric h on L and some local holomorphic non-vanishing section s of L. Check that this cohomology class is independent of both the choice of h and the choice of s.

8. Show that if η is a real $(1,1)$-form representing the cohomology class $c_1(L)$, then there is a metric h' on L such that $2\pi\eta$ is the curvature of h'.

9. (Adjunction formula) For a complex submanifold $V \subset M$, define the **normal bundle** N_V to be the quotient

$$0 \to T_{1,0}V \to T_{1,0}M|_V \to N_V \to 0$$

of the holomorphic tangent bundle $T_{1,0}M|_V$ restricted to V. Check that for a complex submanifold $V \subset M$ of codimension 1, the canonical bundle K_V satisfies

$$K_V = (K_M|_V) \otimes N_V.$$

10. Show that if $L \to M$ be a holomorphic line bundle with holomorphic global section s such that $V = s^{-1}(0) \subset M$ is a complex submanifold, then

$$N_V = L|_V.$$

11. Suppose that $V \subset \mathbb{CP}^n$ is a projective hypersurface of degree $d > 0$, that is, V is the zero locus of a global section of $\mathcal{O}(d)$. Show that

$$K_V \simeq \mathcal{O}(d-n-1)|_V.$$

Hence,

- If $d > n+1$, then the first Chern class of V is positive $c_1(V) < 0$.
- If $d < n+1$, then the first Chern class of V is negative $c_1(V) > 0$.
- If $d = n+1$, then the first Chern class of V is zero $c_1(V) = 0$.

12. Suppose that $V \subset \mathbb{CP}^n$ is a projective submanifold of codimension r determined by the transverse intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$. If $d = d_1 + \cdots + d_r$. Show that

$$K_V \simeq \mathcal{O}(d-n-1)|_V.$$

Hence,

- If $d > n+1$, then the first Chern class of V is positive $c_1(V) < 0$.
- If $d < n+1$, then the first Chern class of V is negative $c_1(V) > 0$.
- If $d = n+1$, then the first Chern class of V is zero $c_1(V) = 0$.

(Hint: Show that $K_V \simeq K_M|_V \otimes \bigwedge^r N_V$ and also that

$$\bigwedge^r(\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)) = \mathcal{O}(d_1 + \cdots + d_r).$$

for integers $d_1, \ldots, d_r$.)