On Defining Smooth Manifolds and Smooth Maps

A note

Preface

This note was written during the summer of 2014 as I taught a seminar on Lie Groups.

The note consitutes my understanding of how to define smooth manifolds and maps between them.

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Contents

1 Introduction

The notion of a smooth manifold is fundamental to modern geometry. In this note, we motivate and explore this notion along with the notion of a smooth map between manifolds. We conclude by exploring several different (and equivalent) ways of understanding the derivative of a smooth map.

2 Manifolds

We begin by recalling the notion of a manifold. Remember that the idea behind an n-dimensional manifold is that it is a topological space that looks locally like $\mathbb{R}^n$. More precisely, we have the following definition.

Definition 1. An **n-dimensional topological manifold** is a second-countable, Hausdorff space which is locally homeomorphic to the Euclidean space $\mathbb{R}^n$.

Here, locally homeomorphic to $\mathbb{R}^n$ means that every point of the manifold has an open neighborhood which is homeomorphic to some open subset of $\mathbb{R}^n$ (using the standard topology induced by the Euclidean norm). The additional requirements of being second-countable and Hausdorff simply stipulate that a manifold has "enough" open sets.

Another way of formulating the local condition in the definition of a manifold is the following. Let X be a topological space. An *n-dimensional coordinate chart* for X is a pair (U, ϕ) where U is an open subset of X and $\phi : U \to \mathbb{R}^n$ is a continuous map taking U homeomorphically onto its image $\phi(U)$ in $\mathbb{R}^n$. An *atlas of n-dimensional charts* for X is a collection $\{(U_\alpha, \phi_\alpha)\}$ of n-dimensional charts such that the open sets U_α form a cover for X, that is, such that $\cup_\alpha U_\alpha = X$. The following corollary is immediate from the definition above.

Corollary 1. *Let X be a second-countable, Hausdorff topological space. Then X is an n-dimensional manifold if and only if there is an atlas of n-dimensional charts for X.*

3 Smooth manifolds

In the setting of Euclidean vector spaces, it makes sense to talk about differentiable functions between such spaces. Moreover, the notion of differentiability is a local notion, in the sense that the derivative of a function at a point depends only on the information of the function near that point. Because of these two observations, we could hope that the notion of differentiability could be extended to the setting of manifolds, since manifolds are locally the same as Euclidean spaces. Indeed, this turns out to be the case, and this is the main motivation for introducing C^k-manifolds. These manifolds are ones with an additional structure, which allows us to talk about C^k-functions on such manifolds without ambiguity.

To understand how we can extend the notion of differentiability, let us recall what differentiability means in a setting where we understand the term, namely, the setting of finite-dimensional Euclidean spaces. There are more general settings than this one, but for our purposes, we are content to work in finite dimensions.

Recall that we say a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable at the point* $x \in \mathbb{R}^n$ if there is a linear map $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ which approximates f at the point $x \in \mathbb{R}^n$ in the sense that

$$\lim_{\substack{y \in U \\ y \to 0}} \frac{\|f(x+y) - f(x) - Df_x(y)\|}{\|y\|} = 0$$

The linear map Df_x is called *the derivative of* f *at* x. If f is differentiable at every point of its domain $U \subset \mathbb{R}^n$, then we call f a *differentiable* map.

Note that if f is differentiable at every point of its domain $U \subset \mathbb{R}^n$, then we obtain an assignment

$$\begin{aligned} Df : U &\to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \\ x &\mapsto Df_x. \end{aligned}$$

The space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is a Euclidean space in its own right. Indeed, it has dimension mn and it comes equipped with the *operator norm*, which is defined by the rule

$$\|A\| = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

for a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Thus it makes sense to talk about whether the assignment Df is continuous or differentiable. If the assignment Df is continuous, then we say that f is C^1*-differentiable*. If the assignment Df is differentiable, then we obtain another assignment

$$\begin{aligned} D^2 f : U &\to \mathrm{Hom}(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \\ x &\mapsto D(Df)_x. \end{aligned}$$

Again, the codomain is a Euclidean space (of dimension mn^2), and we can ask whether this assignment is continuous or differentiable. If the assignment $D^2 f$ is continuous, then we say that f is C^2*-differentiable*. More generally, if the assignment $D^{k-1} f$ is differentiable, we can form a map $D^k f$, and we say that f is C^k*-differentiable* if this map $D^k f$ is continuous.

There is a more explicit description of C^k maps which is also useful. Working with f and x as above, let u be a vector in U. The *directional derivative of* f *at* x *with respect to* u, written $f'(x; u)$, is the value of the limit

$$f'(x; u) = \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h}$$

provided the limit exists. When u is the standard basis vector e_i, we write $f'(x, e_i) = D_i f(x)$ and we call this vector the *i-th partial derivative of* f *at* x.

The assignment $x \mapsto D_i f(x)$ defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. If for each i satisfying $1 \leq i \leq n$, the function $D_i f$ is a continuous map, then we say that f is C^1-differentiable. More generally, for a positive integer k, we say that f is C^k-differentiable if each function of the form

$$D_{i_1} \cdots D_{i_r} f$$

exists and is continuous, where i_t are positive integers satisfying $1 \leq i_t \leq n$ and $i_1 + \cdots + i_r = k$. The set of C^k maps on a subset U of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is denoted $C^k(U, \mathbb{R}^m)$. Note that we have the inclusions

$$C^1(U, \mathbb{R}^m) \supset C^2(U, \mathbb{R}^m) \supset \cdots \supset C^k(U, \mathbb{R}^m) \supset \cdots$$

When a map is C^k-differentiable for all $k = 1, 2, \ldots$, then we call it *smooth*, and we denote the space of smooth maps on U by $C^\infty(U, \mathbb{R}^m)$. It is typically with this space of functions that we often work, so that we can take as many derivatives as we need without worry.

Now let us try to extend this notion of smoothness (resp. C^k-differentiability) to the setting of continuous maps $f : V \to \mathbb{R}^m$ from some open subset V of a manifold X. For a point x of V, there is a neighborhood U containing x, which is mapped by some homeomorphism $\phi : U \to \mathbb{R}^n$ onto some subset $\phi(U)$ of $\mathbb{R}^n$. Let y denote the value $\phi(x)$ in $\mathbb{R}^n$. Now, note that the composition

$$f \circ \phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a continuous map between two Euclidean spaces! This means that it makes sense to talk about smoothness (resp. C^k-differentiability) of this composition near y. What we would like to say is that f is smooth (resp. C^k-differentiable) at x precisely when this composition is smooth (resp. C^k-differentiable) at y. However, note that such a definition would depend upon a choice of homeomorphism ϕ. So we would be obliged to include this clause in our definition, producing a definition such as the following.

Definition 2. Let $f : V \to \mathbb{R}^m$ be a continuous function from an open subset V of an n-manifold X. We say that f is **smooth (resp. C^k-differentiable) at the point x with respect to the chart** (U, ϕ) if the composition $f \circ \phi^{-1}$ is smooth (resp. C^k-differentiable) at the point $\phi(x)$.

It would be easy enough to try to eliminate this clause from the definition of smoothness (resp. C^k-differentiability): we simply declare f to be smooth (resp. C^k-differentiable) at x if f is smooth (resp. C^k-differentiable) at x with respect to all charts containing x. In this way, we can extend the notion of smoothness (resp. C^k-differentiability)to a function $f : U \to \mathbb{R}^m$ from some subset U of a manifold X.

However, there is a minor problem in this attempted definition: it would be tedious to verify. Indeed, we would need to check smoothness (resp. C^k-differentiability) with respect to *all* such charts. It would be better if we could construct things so that we would only need to check smoothness (resp. C^k-differentiability) with respect to one chart, and somehow smoothness (resp. C^k-differentiability) with respect to other charts would follow. This motivates the notion of a smooth (resp. C^k-) manifold.

Before introducing the notion of a smooth (resp. C^k-) manifold, let us investigate how smoothness (resp. C^k-differentiability) with respect to one chart could imply smoothness (resp. C^k-differentiability) with respect to another. Suppose that $f : U \to \mathbb{R}^m$ is a continuous map that is smooth (resp. C^k-differentiable) at x with respect to the chart (U, ϕ). Let $(\widetilde{U}, \widetilde{\phi})$ be another chart containing x. Note that we have the following equality of functions

$$f \circ \widetilde{\phi}^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \widetilde{\phi}^{-1})$$

wherever makes sense, that is, at the intersection of their domains, which happens to be $\widetilde{\phi}(U \cap \widetilde{U})$. We are already given that the function $f \circ \phi^{-1}$ is smooth (resp. C^k-differentiable) at the point $\phi(x)$. Using the chain rule and the above equality of functions, we know that the assertion that the function $f \circ \widetilde{\phi}^{-1}$ is smooth (resp. C^k-differentiable) at the point $\widetilde{\phi}(x)$ is equivalent to the assertion that the function $\phi \circ \widetilde{\phi}^{-1}$ is smooth (resp. C^k-differentiable) at the point $\widetilde{\phi}(x)$.

The preceding paragraph implies the following. If we know that

- the function $f : U \to \mathbb{R}^m$ is smooth (resp. C^k-differentiable) at x with respect to the chart (U, ϕ) and

- a chart $(\widetilde{U}, \widetilde{\phi})$ containing x satisfies the property that the map $\phi \circ \widetilde{\phi}^{-1}$ is smooth (resp. C^k-differentiable),

then we also know that the function f is smooth (resp. C^k-differentiable) at x with respect to the chart $(\widetilde{U}, \widetilde{\phi})$. This motivates the following definitions.

Definition 3. Let X be an n-dimensional manifold. Two charts (U, ϕ) and (V, ψ) for X are said to be **smoothly (resp. C^k-) compatible** if either $U \cap V = \varnothing$ or the **transition map** $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a smooth (resp. C^k) map. An atlas is said to be a **smooth (resp. C^k-) atlas** if each chart in the atlas is smoothly (resp. C^k-) compatible with every other chart of the atlas.

Because of the discussion before the previous definition, we also have the following corollary.

Corollary 2. *Let $f : V \to \mathbb{R}^m$ be a continuous map from an open subset V of an n-manifold X, and let x be a point of V. Suppose X is equipped with a smooth (resp. C^k-) atlas. Then f is smooth (resp. C^k-differentiable) at x with respect to one chart in the atlas if and only if f is smooth (resp. C^k-differentiable) at x with respect to all charts in the atlas.*

This means that we can introduce a notion of smoothness (resp. C^k-differentiability) with respect to a smooth (resp. C^k-) atlas as follows.

Definition 4. Let $f : V \to \mathbb{R}^m$ be a continuous map from an open subset V of an n-manifold X, let A be a smooth (resp. C^k-) atlas for X, and let x be a point of V. We say that f is **smooth (resp. C^k-differentiable) at the point x with respect to the atlas** A if there is a chart (U, ϕ) in A such that f is smooth (resp. C^k-differentiable) with respect to the chart (U, ϕ). We say that f **is smooth (resp. C^k-differentiable) with respect to the atlas** A if f is smooth (resp. C^k-differentiable) at all points of its domain.

The discussion above shows that this definition is well-defined, meaning that to check whether a given function f is smooth (resp. C^k-differentiable) with respect to a smooth (resp. C^k-) atlas, it does not matter which chart we choose to check the smoothness (resp. C^k-differentiability) of f; f will either be smooth (resp. C^k-differentiable) in all of them simultaneously or in none of them.

Thus, we are inclined to say that a smooth (resp. C^k-) manifold is a manifold equipped with a smooth (resp. C^k-) atlas of charts. However, this definition would not be ideal, since two different smooth (resp. C^k-) atlases for a given manifold can give rise to the same collection of smooth (resp. C^k-differentiable) functions. We would like to consider two atlases to be different if and only if they determine different collections of smooth (resp. C^k-differentiable) functions on the given manifold. Therefore, we need to be a little more careful about the notion of a smooth (resp. C^k-) manifold. In particular, we will see that it will be formulated as a manifold together with an equivalence class of smooth (resp. C^k-) atlases, each of which determines the same collection of smooth (resp. C^k-differentiable) on the given manifold.

Let us first see how two atlases can given rise to the same collection of smooth (resp. C^k-differentiable) functions. Suppose we are given a smooth (resp. C^k-) atlas A for a manifold X. Now suppose that we add to this atlas another chart (U, ϕ) which is smoothly (resp. C^k-) compatible with all the charts in the given atlas A, and we call the resulting atlas A'. Note that this resulting atlas is again a smooth (resp. C^k-) atlas, since each chart in A' is smoothly (resp. C^k-) compatible with all the other charts in A'. Moreover, note that if a function is smooth (resp. C^k-differentiable) with respect to the atlas A, then the function is smooth (resp. C^k-differentiable) with respect to the atlas A', and vice-versa. Therefore, the smooth (resp. C^k-) atlases A and A' determine the same set of smooth (resp. C^k-differentiable) functions on the manifold X.

We are lead to introduce the following relation.

Definition 5. Two smooth (resp. C^k-) atlases A and A' for a manifold X are said to be **equivalent** if their union $A \cup A'$ is a smooth (resp. C^k-) atlas. One can show that this relation is indeed an equivalence relation, and we let $[A]$ denote the equivalence class of an atlas under this equivalence relation. A choice of an equivalence class is called a **smooth (resp. C^k-) structure on X.**

Moreover, one can show that equivalent atlases give rise to the same collections of smooth (resp. C^k-differentiable) functions, as the following lemma asserts (whose proof we leave as an exercise).

Lemma 1. *Let A and A' be two equivalent smooth (resp. C^k-)atlases for a manifold X and let $f : U \to \mathbb{R}^m$ be a function from an open subset U of X into the Euclidean space $\mathbb{R}^m$. Then f is smooth (resp. C^k-differentiable) with respect to the atlas A if and only if f is smooth (resp. C^k-differentiable) with respect to the atlas A'.*

Because of this lemma, we can now introduce the notion of a smooth (resp. C^k-) manifold. It will be a manifold X together with an equivalence class of smooth (resp. C^k-) atlases on X, each of which determines the same collection of smooth (resp. C^k-differentiable) functions on X.

Definition 6. A *smooth (resp. C^k-) manifold* is a pair $(M, [A])$ where X is a manifold and $[A]$ is a smooth (resp. C^k-) structure on X.

Now we are ready to introduce the notion of a differentiable function on a smooth (resp. C^k-) manifold.

Definition 7. Let X be a smooth (resp. C^k-) manifold whose smooth (resp. C^k-) structure is represented by the atlas A. A function $f : V \to \mathbb{R}^m$ from an open subset V of a smooth (resp. C^k-) manifold X into Euclidean space $\mathbb{R}^m$ is said to be **smooth (resp. C^k-differentiable) at the point x in V** if there is a chart (U, ϕ) in A containing x such that the composition $f \circ \phi^{-1}$ is smooth (resp. C^k-differentiable) at the point $\phi(x)$. The function f is said to be **smooth (resp. C^k-differentiable)** if it is smooth (resp. C^k-differentiable) at each point of its domain.

These definitions are well-defined because the previous discussion shows that for a smooth (resp. C^k-) manifold, the notion of smoothness (resp. C^k-differentiability) is independent not only of the choice of chart in the atlas but also of the choice of atlas representing the smooth (resp. C^k-) structure.

For a map $f : X \to Y$ between two smooth (resp. C^k-) manifolds (possibly of different dimension), we can also introduce a notion of smoothness (resp. C^k-differentiability). Again, because of the smooth (resp. C^k-) structure, one can show that the notion of smoothness (resp. C^k-differentiability) depends only on the behavior of f in one pair of charts (one from an atlas representing the smooth (resp. C^k-) structure for X and one from an atlas representing the smooth (resp. C^k-) structure for Y), not on the behavior of f in all such pairs. We leave this fact as an exercise for the interested reader.

Definition 8. Let X and Y be two smooth (resp. C^k-) manifolds, possibly of different dimensions with chosen atlases representing the given smooth (resp. C^k-) structures. A continuous map $f : W \to Y$ from some open subset W of X into Y is said to be **smooth (resp. C^k-differentiable) at the point $x \in W$** if there is a chart (U, ϕ) containing x and a chart (V, ψ) containing $f(x)$ such that the composition

$$\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(V) \cap U) \longrightarrow \psi(V \cap f(U))$$

is a smooth (resp. C^k-differentiable) map at $\phi(x)$. We say that f is **smooth (resp. C^k-differentiable)** if f is smooth (resp. C^k-differentiable) at all points of its domain.

This section was quite long, so let us recap what we have done. First we recalled the familiar notion of a smooth (resp. C^k-differentiable) map between Euclidean spaces. Then we used this notion to introduce a notion of smoothness (resp. C^k-differentiability) with respect to a certain chart on a manifold X. We showed that by choosing a smooth (resp. C^k-) atlas for X (that is, an atlas of charts all of whose transition functions are smooth [resp. C^k-differentiable]), we could say that a function is smooth (resp. C^k-differentiable) if it is smooth (resp. C^k-differentiable) with respect to only one chart, not all of them. We then introduced an equivalence relation on smooth (resp. C^k-) atlases, identifying as those which determine the same smooth (resp. C^k-) functions on X as equivalent. From there, we introduced the notion of a smooth (resp. C^k-) manifold, and we showed how we could extend our notion of smoothness (resp. C^k-differentiability) to smooth (resp. C^k-) manifolds in a well-defined manner.

4 Tangent spaces and the differential

Even though the previous section describes *when* a map $f : M \to N$ between smooth manifolds is differentiable, it gives no process for determining what a derivative should be! The goal of this section is to determine what is meant by a derivative in this setting of smooth manifolds.

Recall that in the finite-dimensional Euclidean setting, the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $x \in \mathbb{R}^n$ is a linear approximation $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ to f at the point x. In a similar way, the derivative of a smooth map $f : M \to N$ between smooth manifolds will be an appropriate linear approximation of the map at a point $x \in M$. In this setting, however, it is not apparent what vector spaces should be the domain and codomain of the linear approximation. In the setting of Euclidean spaces, it is easy: we simply take the domain and codomain to be the domain and codomain of the original map f. The setting of smooth manifolds requires us to introduce a new type of vector space, namely, the tangent space T_xM of a manifold M at the point x.

There are many ways to define the tangent space T_xM, and in this section we hope to cover as many of these ways as we can. However, we first discuss an example which provides a geometric picture motivating our forthcoming dicussion.

In the case where the underlying manifold M is a sphere in $\mathbb{R}^3$, each tangent space can be pictured as an affine plane in $\mathbb{R}^3$. Indeed, one can picture the tangent space T_xM as a plane attached to the sphere M at the point x, with the property that the sphere's radius is a normal vector to the plane. This picture is a good one to keep in mind, and motivates the notion of a tangent space for an arbitrary manifold M.

We now begin our discussion of the tangent space. We will discuss three different yet equivalent approaches, in order of increasing abstraction. As is often the case, the most concrete approach is intuitive, but somewhat cumbersome to use in practice. On the other hand, the most abstract approach is elegant, yet somewhat divorced from geometric intuition. Therefore, we find that it is important to understand all three approaches, since they each have their own advantages and disadvantages.

For our discussion, let us fix a smooth n-dimensional manifold M and a point $x \in M$. We will construct the tangent space T_xM to M at x. This will be a vector space of the same finite dimension as M.

Definition using curves. The idea for this approach is to construct a vector space whose elements are equivalence classes of curves passing through the point x.

Let C_x denote the set

$$C_x = \{\gamma : (-a, a) \to M \mid \gamma(0) = x,\ \gamma \text{ is smooth},\ a > 0\}$$

of smooth curves passing through the point x. We define an equivalence relation on the set C_x by the rule $\gamma_1 \sim \gamma_2$ if and only if there is a chart (U, ϕ) containing x such that

$$\frac{d}{dt}(\phi \circ \gamma_1)(t)|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2)(t)|_{t=0}.$$

It is easy to see that if the above equality holds for one chart, then it must hold for all of them. For a curve $\gamma \in C_x$, we let $\gamma'(0)$ denote the equivalence class of γ under this equivalence relation, and we let T_xM denote the set of all such equivalence classes.

We now endow T_xM with the structure of a vector space. The idea will be to use a chart to transfer the vector space structure from $\mathbb{R}^n$ to T_xM.

We define addition and we leave the definition of scalar multiplication up to the interested reader. Let (U, ϕ) denote a chart containing x. Up to composing ϕ with a translation, we may assume that $\phi(x) = 0$. For two curves γ_1 and γ_2 in C_x, let γ denote the curve defined by the rule

$$\gamma(t) = \phi^{-1}(\phi(\gamma_1(t)) + \phi(\gamma_2(t)))$$

for whichever t this makes sense. (One should check that this does make sense for at least some t and that γ belongs to C_x!) We then define

$$\gamma_1'(0) + \gamma_2'(0) := \gamma'(0).$$

To show that this addition is well-defined, we need to show that it does not depend on the particular representatives γ_1 and γ_2. We show now that it does not depend on the representative γ_1 and the proof for γ_2 will be similar.

Let $\widetilde{\gamma}_1$ be a curve equivalent to γ_1, meaning that

$$\frac{d}{dt}(\phi \circ \widetilde{\gamma}_1)(t)|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_1)(t)|_{t=0}.$$

Let $\widetilde{\gamma}$ denote the curve defined by

$$\widetilde{\gamma}(t) = \phi^{-1}(\phi(\widetilde{\gamma}_1(t)) + \phi(\gamma_2(t))).$$

Then note that

$$\begin{aligned}
\frac{d}{dt}(\phi \circ \widetilde{\gamma})(t)|_{t=0} &= \frac{d}{dt}(\phi(\widetilde{\gamma}_1(t)) + \phi(\gamma_2(t)))|_{t=0} \\
&= \frac{d}{dt}(\phi(\gamma_1(t)) + \phi(\gamma_2(t)))|_{t=0} \\
&= \frac{d}{dt}(\phi \circ \gamma)(t)|_{t=0},
\end{aligned}$$

showing that $\widetilde{\gamma}$ and γ are equivalent, as desired.

After constructing scalar multiplication in a similar fashion, the reader will be convinced that the set T_xM carries the structure of a vector space. However, note that this vector space structure seems to depend upon our choice of chart (U, ϕ). Indeed, one might be concerned that another choice of chart (V, ψ) would endow the set T_xM with a different vector space structure entirely! This would be bad, for it would mean that we could not really talk about *the* tangent space at the point x; we would need to talk about the tangent space with respect to some chart. However, as is often the case when dealing with smooth manifolds, one can show that the above vector space structure does *not* depend on choice of chart, and hence we are justified in calling T_xM *the* tangent space at x. We outline a proof of this fact now.

Let γ_1 and γ_2 be curves in C_x. For a chart (U, ϕ), construct the path γ representing the sum of $\gamma_1'(0)$ and $\gamma_2'(0)$ as above. Let (V, ψ) be another chart containing x. Again, up to composing with a translation, we may assume that $\psi(x) = 0$. Let ρ be the path in C_x defined by the rule

$$\rho(t) = \psi^{-1}(\psi(\gamma_1(t)) + \psi(\gamma_2(t))).$$

Our goal is to show that ρ is equivalent to γ. We can do this by using the chain rule. In what follows, we use the notation DF_xv to denote the derivative of F at x applied to the vector v. For a single variable map $\tau : (a, b) \to \mathbb{R}^m$, we write $\tau'(c)$ for the derivative of τ at c. Using this notation, we have

$$\begin{aligned}
\frac{d}{dt}(\phi \circ \rho)(t)|_{t=0} &= \frac{d}{dt}(\phi \circ \psi^{-1} \circ \psi \circ \rho)(t)|_{t=0} \\
&= \frac{d}{dt}((\phi \circ \psi^{-1})(\psi(\gamma_1(t)) + \psi(\gamma_2(t)))))|_{t=0} \\
&= D(\phi \circ \psi^{-1})_{\psi(\gamma_1(0))+\psi(\gamma_2(0))}((\psi \circ \gamma_1)'(0) + (\psi \circ \gamma_2)'(0)) \\
&= D(\phi \circ \psi^{-1})_0((\psi \circ \gamma_1)'(0) + (\psi \circ \gamma_2)'(0))
\end{aligned}$$

Recall that the derivative at a point is a linear map. This implies that we have

$$\begin{aligned}
&D(\phi \circ \psi^{-1})_0((\psi \circ \gamma_1)'(0) + (\psi \circ \gamma_2)'(0)) \\
&= D(\phi \circ \psi^{-1})_0(\psi \circ \gamma_1)'(0) + D(\phi \circ \psi^{-1})_0(\psi \circ \gamma_2)'(0) \\
&= (\phi \circ \psi^{-1} \circ \psi \circ \gamma_1)'(0) + (\phi \circ \psi^{-1} \circ \psi \circ \gamma_2)'(0) \\
&= (\phi \circ \gamma_1)'(0) + (\phi \circ \gamma_2)'(0) \\
&= \frac{d}{dt}(\phi(\gamma_1(t)) + \phi(\gamma_2(t)))|_{t=0} \\
&= \frac{d}{dt}(\phi \circ \gamma)(t)|_{t=0},
\end{aligned}$$

which shows that ρ and γ are equivalent, as desired.

Now it remains to find the dimension of the vector space T_xM. As expected, it will have the same dimension as the dimension of the manifold M, which, in this case, is n. The idea is to use a chart to construct a linear isomorphism $T_xM \to \mathbb{R}^n$.

Let (U, ϕ) be a chart containing x and assume that $\phi(x) = 0$. We define a map $\widetilde{d\phi}_x : C_x \to \mathbb{R}^n$ by the rule

$$\widetilde{d\phi}_x(\gamma) = \frac{d}{dt}(\phi \circ \gamma)(t)|_{t=0}.$$

This map induces a well-defined map $d\phi_x : T_xM \to \mathbb{R}^n$ defined by

$$d\phi_x(\gamma'(0)) = \frac{d}{dt}(\phi \circ \gamma)(t)|_{t=0},$$

which is a linear map with respect to the vector space structure on T_xM. We argue now that this is a bijective map. Since the map is clearly injective by the definition of T_xM, it remains only to show that the map is surjective.

Let v be a vector in $\mathbb{R}^n$. Let $\alpha_v : (-1, 1) \to \mathbb{R}^n$ denote the path defined by $\alpha(t) = tv$. There is an $\epsilon > 0$ so that the image $\alpha(-\epsilon, \epsilon)$ is contained in $\phi(U)$. Let $\gamma_v : (-\epsilon, \epsilon) \to M$ denote the path in M defined by the rule $\gamma_v(t) = (\phi^{-1} \circ \alpha)(t)$. Then it is routine to check that $d\phi_x(\gamma_v'(0)) = v$, showing that $d\phi_x$ is surjective, as desired.

This concludes our discussion of the tangent space T_xM using the approach of curves.

Definition using derivations. The idea for this approach is to construct a vector space whose elements are derivations on the set of smooth real-valued functions on the manifold M.

Let $C^\infty(M)$ denote the vector space of real-valued smooth functions $f : M \to \mathbb{R}$. A *derivation at x* is a linear map $D : C^\infty(M) \to \mathbb{R}$ such that

$$D(f \cdot g) = D(f) \cdot g(x) + f(x) \cdot D(g) \quad \text{for each } f, g \in C^\infty(M).$$

We can define addition and scalar multiplication on the set of derivations at x in the usual way, and we obtain a vector space, which we call the tangent space to M at x. For notation's sake, we denote this vector space by T_xM_{Der}, though it is standard to refer to this vector space as T_xM as well.

We outline now how to show that this approach is equivalent to the previous approach by constructing a bijective correspondence between "tangent vectors" in each approach. The idea is to send a curve to the derivation which differentiates along this curve. More precisely, we define a map $\widetilde{D} : C_x \to T_xM_{\mathrm{Der}}$ by $\gamma \mapsto \widetilde{D}_\gamma$ where $\widetilde{D}_\gamma$ is the derivation defined by the rule

$$\widetilde{D}_\gamma(f) = \frac{d}{dt}(f \circ \gamma)(t)|_{t=0}.$$

One can show that this map only depends on the equivalence class of the path γ and hence gives rise to a well-defined map $D : T_xM \to T_xM_{\mathrm{Der}}$ defined by $D_{\gamma'(0)} = \widetilde{D}_\gamma$. It is somewhat beyond the scope of these notes to show that this map is a linear isomorphism, but this fact can be shown.

Definition using the cotangent space. The idea for this approach is to construct a vector space whose elements are dual vectors to some quotient space constructed from subspaces of $C^\infty(M)$.

The motivation of this construction is based upon the following observation. The space $C^\infty(M)$ is a commutative $\mathbb{R}$-algebra under pointwise multiplication. In this algebra, the set of all functions vanishing at x forms an ideal, which we denote by I. Elements of the square I^2 of this ideal must belong to the kernel of any derivation, by the defining property of derivations. Hence any derivation $D : \mathbb{C}^\infty(M) \to \mathbb{R}$ gives rise to a well-defined linear map $\alpha_D : I/I^2 \to \mathbb{R}$. This means that to every derivation D we can associate an element α_D of the dual space $(I/I^2)^*$.

It turns out that the map $D \mapsto \alpha_D$ is actually bijective, and hence we are justified in defining the tangent space T_xM to be the dual space $(I/I^2)^*$. The vector space structure on this space is canonical and needs no explanation. However, the correspondence between this space and the set of derivations is somewhat subtle, and we let the interested reader supply further details.

This completes our discussion of the different formulations of the tangent space T_xM. Although equivalent to the others, each has its own merits, so it is left to the reader to decide which formulation is most applicable in which circumstance.

We now return to our main goal of this section: associating a type of derivative to a smooth map. As in the case of finite-dimensional Euclidean space, the derivative at a point will be a linear map. In this new setting of smooth manifolds, it will be a linear map from one tangent space to another. More precisely, if $f : M \to N$ is a smooth map between manifolds, the differential at $x \in M$ will be a linear map $df_x : T_xM \to T_{f(x)}N$. Because there are different formulations for the tangent space, there are also different formulations for the differential. We cover some of these formulations now.

Differential using curves. We first formulate the differential using curves. For a smooth map $f : M \to N$ and a point $x \in M$, the differential of f at x, denoted df_x, is a linear map $df_x : T_xM \to T_{f(x)}N$ defined by the rule

$$df_x(\gamma'(0)) = (f \circ \gamma)'(0).$$

Recall here that $\gamma'(0)$ denotes the equivalence class of the curve γ, and so $(f \circ \gamma)'(0)$ denotes the equivalence class of the curve $f \circ \gamma$ in N. It is routine to show that this map is well-defined (meaning that it is independent of the choice of representative γ) and that it is linear with respect to the vector space structure on the tangent spaces. We leave these exercises for the interested reader.

Differential using derivations. This is perhaps the most standard approach to defining the differential. In this case, we let $df_x : T_xM \to T_{f(x)}N$ denote the linear map defined by the rule $X \mapsto df_x(X)$ where $df_x(X)$ is the derivation at $f(x)$ defined by

$$df_x(X)(g) = X(f \circ g)$$

for $g \in C^\infty(N)$. Again, we leave it to the reader to check that the map df_x is a linear map.

Differential using cotangent space. This case is essentially the same as the former. In this case, we let $df_x : T_xM \to T_{f(x)}N$ denote the linear map defined by the rule $X \mapsto df_x(X)$ where we view X as an element of the dual space $(I/I^2)^*$ with I being the ideal of smooth functions on M vanishing at x. The element $df_x(X)$ is then the element of the dual space $(J/J^2)^*$ defined by the rule

$$df_x(X)(g) = X(f \circ g)$$

for $g \in J/J^2$ where J is the ideal of smooth functions on N vanishing at $f(x)$.

In summary, we have accomplished our main goal of the section: we have associated to a smooth map a type of derivative. In fact, we have do so in many ways, using various formulations for the tangent space.