MATH S1202: Calculus IV
Quiz 2 Solutions
May 31, 2018

1.

(a) Write the standard Cartesian coordinates (x, y, z) in terms of cylindrical coordinates (r, θ, z) about the z-axis.

(b) In terms of cylindrical coordinates (r, θ, z), give a description of the solid region E lying below the plane $z = 4$ and above the cone $z^2 = x^2 + y^2$.

(c) Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Write an expression for the triple integral $\iiint_E f(x, y, z)dV$ in terms of iterated integrals in cylindrical coordinates (r, θ, z). You only need to write an expression; you don't need to compute an exact value.

Solution. (a) We have

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z. \end{aligned}$$

(b) The solid region E is given by

$$E = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 4 \\ r \leqslant z \leqslant 4 \end{cases}.$$

(c) We have

$$\iiint_E f(x, y, z)\, dV = \int_0^{2\pi} \int_0^4 \int_r^4 r\sqrt{r^2 + z^2}\, dz\, dr\, d\theta.$$

2.

(a) Write the standard Cartesian coordinates (x, y, z) in terms of spherical coordinates (ρ, θ, φ).

(b) In terms of spherical coordinates (ρ, θ, φ), give a description of the ball B of radius 1 centered about the origin.

(c) Compute the triple integral of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the ball B. This time I want you to compute.

Solution. (a) We have

$$\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}$$

(b) The solid region B is given by

$$B = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant \varphi \leqslant \pi \\ 0 \leqslant \rho \leqslant 1 \end{cases}.$$

(c) We have

$$
\begin{aligned}
\iiint_B f(x,y,z)\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{1} \rho^3 d\rho \\
&= 2\pi\cdot 2\cdot\frac{1}{4} \\
&= \pi.
\end{aligned}
$$

3. Find the *surface area* of the part of the plane $4x+2y+z=8$ that lies in the first octant.

Solution. The surface lies above the region D in the xy-plane in the first quadrant below the line $4x+2y=8$. Above D, the surface is the graph of $f(x,y)=8-4x+2y$. The surface area is given by the formula

$$
\begin{aligned}
\text{Area} &= \iint_D \sqrt{1+f_x^2+f_y^2}\,dA \\
&= \iint_D \sqrt{1+(-4)^2+(-2)^2}\,dA \\
&= \sqrt{21}\iint_D dA \\
&= \sqrt{21}\cdot\text{Area}(D).
\end{aligned}
$$

The intercepts of the line $4x+2y=8$ are $(2,0)$ and $(4,0)$, and hence D is right triangle with legs of length 2 and 4. It follows that the surface area is

$$\text{Area} = 4\sqrt{21}$$