Problem Set 9
More GIT
Spring 2021

1. The group $SL(n+1,\mathbb{C})$ acts on $\mathbb{C}^{n+1}$ in the usual way via the fundamental representation. Suppose that a group G acts on $\mathbb{C}^{n+1}$ through a group morphism

$$G \to SL(n+1,\mathbb{C}).$$

(a) Show that G also acts on $\mathbb{C}^{n+1} \setminus 0$.

(b) Show that the action of G "descends" to one on $\mathbb{P}^n$, when viewed as the quotient of $\mathbb{C}^{n+1} \setminus 0$ in the usual way.

(c) Because we have started with an action "upstairs," that is, a linearization of the action, we already have an action on sections of $\mathcal{O}(1)$ because we know how G acts on the variables $z_0, \ldots, z_n$. Indeed, if $f(z_0, \ldots, z_n)$ denotes a homogeneous polynomial of degree 1, then define

$$(g \cdot f)(z_0, \ldots, z_n) := f(g^{-1} \cdot (z_0, \ldots, z_n)).$$

Show that this is indeed a (left) action of G on the set of homogeneous polynomials of degree 1.

2. Suppose that $G = \mathbb{C}^*$ acts on $\mathbb{C}^3$ via

$$\lambda \cdot (z_0, z_1, z_2) = (\lambda z_0, \lambda^{-1} z_1, z_2).$$

(a) Show that this action factors through a group morphism

$$G \to SL(3,\mathbb{C})$$

and deduce that the action of G descends to one on $\mathbb{P}^2$.

(b) Show that z_2 is an invariant section of $\mathcal{O}(1)$ and $z_0 z_1$ is an invariant section of $\mathcal{O}(2)$.

(c) As a result, show that the GIT quotient is

$$\mathbb{P}^2/G \simeq \operatorname{Proj} \mathbb{C}[z_0 z_1, z_2].$$

(d) Show that the GIT quotient may be identified with weighted projective space $\mathbb{P}[2,1]$.

3. More generally, let X be a projective variety. The selection of an embedding of X into $\mathbb{P}^n$ is equivalent to the selection of a cone $\tilde{X} \subset \mathbb{C}^{n+1}$ over X, which is in turn equivalent to the selection of a line bundle $\mathcal{O}_X(1)$ over X (namely, the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ via the inclusion map $X \hookrightarrow \mathbb{P}^n$). Suppose a group G acts on $\mathbb{C}^{n+1}$ through a group morphism

$$G \to SL(n+1,\mathbb{C}).$$

Suppose further that the action of G restricts to one on the cone $\tilde{X}$.

(a) Show that G also acts on $\tilde{X} \setminus 0$.

(b) Show that the action of G "descends" to one on X, when X is viewed as the quotient of $\tilde{X} \setminus 0$ in the usual way.

(c) Because we have started with an action "upstairs," that is, a linearization of the action on $\tilde{X}$, we already have an action on sections of $\mathcal{O}_X(1)$ because we know how G acts on the variables $z_0, \ldots, z_n$. More precisely, suppose that $\tilde{X}$ is cut out by homogeneous polynomials $p_1, \ldots, p_k$ in the variables $z_0, \ldots, z_n$. Then any homogeneous polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ determines an element of the quotient ring

$$i^* f \in \frac{\mathbb{C}[z_0, \ldots, z_n]}{(p_1, \ldots, p_k)}$$

and, conversely, any homogeneous polynomial in the quotient ring comes from a homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$. Then G acts on the images of such homogeneous polynomials by

$$(g \cdot i^* f) = i^*(g \cdot f)$$

where the action on the right-hand side is the one from a previous problem. Because the action of G restricts to one on $\tilde{X}$, this should be well-defined. I'm not sure if you want to check this...

4. Here is another viewpoint that might be useful. Recall that $\mathbb{P}^n$ can be regarded as the quotient of $\mathbb{C}^{n+1} \setminus 0$ by the diagonal action of $\mathbb{C}^*$.

(a) Let $\mathbb{C}^*$ act on the product $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}$ by the rule

$$\lambda \cdot (z, w) = (\lambda \cdot z, \lambda w) \qquad \lambda \in \mathbb{C}^*, (z, w) \in (\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}.$$

Show that the quotient

$$[(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}/\mathbb{C}^*]$$

may be identified with the total space of the line bundle $\mathcal{O}(1)$.

(b) Here is one way of thinking about the previous part. Let s denote a map

$$s : \mathbb{C}^{n+1} \setminus 0 \to (\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}.$$

Say that s is $\mathbb{C}^*$-equivariant if

$$s(\lambda \cdot z) = \lambda \cdot f(z) \qquad \lambda \in \mathbb{C}^*, z \in \mathbb{C}^{n+1} \setminus 0.$$

If s is $\mathbb{C}^*$ equivariant, show that s defines a map from $\mathbb{P}^n$ to the total space of $\mathcal{O}(1)$.

(c) Now suppose that the map s is a section of the projection onto the factor of $\mathbb{C}^{n+1} \setminus 0$. This means we can write

$$s(z) = (z, f(z))$$

for a function $f : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}$. If s is $\mathbb{C}^*$-equivariant, show that f is a homogeneous polynomial of degree 1.

(d) Now suppose that $\mathbb{C}^*$ acts on the product $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}$ by the rule

$$\lambda \cdot (z, w) = (\lambda \cdot z, \lambda^k w) \qquad \lambda \in \mathbb{C}^*, (z, w) \in (\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}$$

for a positive integer k. In the circumstances of the previous part, show that f is now a homogeneous polynomial of degree k.

5. Suppose that G acts on $\mathbb{C}^{n+1}$ via a group morphism $G \to SL(n+1, \mathbb{C})$ as in Problem 1. Then G determines an action on $\mathbb{C}^{n+1} \setminus 0 \times \mathbb{C}$ defined by

$$g \cdot (z, w) = (g \cdot z, w) \qquad z \in \mathbb{C}^{n+1} \setminus 0, w \in \mathbb{C}.$$

(a) Using the previous problem, show that this action determines one on the total space of $\mathcal{O}(1)$ over $\mathbb{P}^n$.

(b) Let s denote a map

$$s : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}^{n+1} \setminus 0 \times \mathbb{C}$$

which is $\mathbb{C}^*$-equivariant (with respect to the actions) and which is a section of the projection map onto the first factor. By the previous problem, s determines a section of $\mathcal{O}(1)$ and s can be written in the form

$$s(z) = (z, f(z))$$

where f is a homogeneous polynomial of degree 1. The group G acts on such sections by the rule

$$(g \cdot s)(z) = g \cdot s(g^{-1} \cdot z).$$

Show that

$$(g \cdot s)(z) = (z, f(g^{-1} \cdot z)),$$

and hence we recover the action described in Problem 1.

6. The linearization of the action from the previous problem is not unique. Indeed, for any group morphism $\chi : G \to \mathbb{C}^*$, that is, for any character of G, we have an associated action of G on the product space $\mathbb{C}^{n+1} \setminus 0 \times \mathbb{C}$ defined by

$$g \cdot (z, w) = (g \cdot z, \chi(g) w) \qquad z \in \mathbb{C}^{n+1} \setminus 0, w \in \mathbb{C}.$$

(a) Using a previous problem, show that this action determines one on the total space of $\mathcal{O}(1)$ over $\mathbb{P}^n$.

(b) In the notation of part (b) from the previous problem, show that now

$$(g \cdot s)(z) = (z, \chi(g) f(g^{-1} \cdot z)).$$