Problem Set 6
Slope
Fall 2020

**1.** For vector spaces $V, W$, show that there is a natural isomorphism

$$\mathrm{Hom}(V, W) \simeq W \otimes V^*.$$

Deduce that there is a natural isomorphism

$$\mathrm{Hom}(E, F) \simeq F \otimes E^*$$

for bundles too. (This justifies a portion of the hint from Problem 5 of Problem Set 4.)

**2.** Let $E$ be a complex vector bundle of rank $r$ over $X$.

(a) If $D_0, D_1$ are two connections on $E$, show that the difference $D_1 - D_0$ is $C^\infty(X)$-linear and deduce that it corresponds to an $\mathrm{End}(E)$-valued 1-form on $X$.

(b) If $\mathcal{C}$ denotes the space of connections on $E$, show that the tangent space $T_D\mathcal{C}$ to a connection $D$ can be identified with the space of $\mathrm{End}(E)$-valued 1-forms.

(c) Let $F : \mathcal{C} \to A^2(\mathrm{End}(E))$ denote the map taking a connection to its corresponding curvature form. For a connection $D$ and an $\mathrm{End}(E)$-valued 1-form $a$, show that

$$dF_D(a) = Da.$$

(d) The trace determines a map

$$A^2(\mathrm{End}(E)) \to A^2(X).$$

Show that the derivative of the composition

$$\mathcal{C} \to A^2(\mathrm{End}(E)) \to A^2(X)$$

has image contained in $\operatorname{im} d$, where $d$ is the de Rham differential $d : A^1(X) \to A^2(X)$.

**3.** Show that the cohomology class determined by the 2-form $\mathrm{Tr}(F_D)$ is independent of the choice of connection $D$. (Hint: Apply the fundamental theorem of Calculus to a path connecting two different connections.)

**4.** The degree of $E$ with respect to the Kähler form $\omega$ is defined by

$$\deg(E) = \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Show that

$$\deg(E) = \frac{i}{2\pi} \int_X \Lambda \mathrm{Tr}(F_D) \frac{\omega^n}{n!}$$

where $F_D$ is the curvature of a connection $D$.

**5.** A Hermitian metric $H$ on $E$ is called Hermitian–Einstein if there is a constant $\lambda$ such that

$$\Lambda F_H = \lambda I_E,$$

where $I_E$ denote the identity endomorphism. Show that the only possible option for $\lambda$ is the number

$$\lambda = \frac{-2\pi i \mu(E)}{\mathrm{vol}(X)}$$

where $\mu(E)$ denotes the *slope*

$$\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}.$$

**6.** Let $V$ denote a finite-dimensional vector space. The group $GL(V)$ acts on $\mathfrak{gl}(V)$ by conjugation

$$g \cdot A = gAg^{-1}.$$

The group also acts on $p$-copies of $\mathfrak{gl}(V)$ by the diagonal action consisting of conjugation in each slot.

(a) For a $g \in GL(V)$, the tangent space $T_g GL(V)$ can be identified with the vector space $\mathfrak{gl}(V)$ (because $GL(V)$ is the open subset of $\mathfrak{gl}(V)$ consisting of endomorphisms with nonzero determinant). Let $\xi$

be a tangent vector

$$\xi = \left.\frac{d}{dt}\right|_{t=0} g_t$$

realized as the derivative of a curve $g_t$ in $GL(V)$ passing through $I$ at time $t = 0$. Show that

$$\left.\frac{d}{dt}(g_t \cdot A)\right|_{t=0} = [\xi, A]$$

where

$$[\xi, A] = \xi A - A\xi$$

is the Lie bracket.

(b) Let $\varphi_2$ denote the bilinear form

$$\varphi_2(A, B) = -\mathrm{Tr}(AB)$$

on $\mathfrak{gl}(V)$. Show that $\varphi_2$ is invariant with respect to the diagonal action of $GL(V)$ on $\mathfrak{gl}(V) \times \mathfrak{gl}(V)$.

(c) Show that $\varphi_2$ enjoys the relation

$$\varphi_2([\xi, A], B) + \varphi_2(A, [\xi, B]) = 0.$$

(d) More generally, let $\varphi$ denote any multilinear function on $p$ copies of $\mathfrak{gl}(V)$. If $\varphi$ is invariant under the diagonal action of $GL(V)$, show that

$$\sum_{j=1}^{p} \varphi(A_1, \dots, [\xi, A_j], \dots, A_p) = 0$$

for any $\xi, A_j \in \mathfrak{gl}(V)$.

**7.** Let $\varphi$ denote a multi-$\mathbb{C}$-linear bundle map

$$\varphi : \mathrm{End}(E)^{\oplus p} \to \underline{\mathbb{C}}$$

which determines a multi-$C^\infty(X)$-linear map of sections

$$\varphi : A^0(\mathrm{End}(E))^{\oplus p} \to C^\infty(X).$$

More generally, $\varphi$ determines a multi-$C^\infty(X)$-linear map

$$\varphi : A(\mathrm{End}(E))^{\oplus p} \to A(X)$$

by the requirement

$$\varphi(\alpha_1 \otimes \sigma_1, \ldots, \alpha_p \otimes \sigma_p)$$
$$= \varphi(\sigma_1, \ldots, \sigma_p) \otimes \alpha_1 \wedge \cdots \wedge \alpha_p$$

for $\sigma_k \in A^0(\mathrm{End}(E))$ and $\alpha_k \in A(X)$.

(a) According to the previous problem, if $\varphi$ is invariant under the conjugation action of $GL(E)$, then $\varphi$ enjoys the relation

$$\sum_{k=1}^{p} \varphi(\sigma_1, \ldots, [\xi, \sigma_k], \ldots, \sigma_p) = 0$$

for $\sigma_k, \xi \in A^0(\mathrm{End}(E))$. More generally, define a bracket on $A(\mathrm{End}(E))$ by

$$[\omega, \eta] = \omega \wedge \eta - (-1)^{|\omega||\eta|} \eta \wedge \omega$$

for homogeneous $\omega, \eta \in A(\mathrm{End}(E))$. (Here the notation $\wedge$ is a combination of matrix multiplication and the wedge product.) Show that with this convention, if $\varphi$ is invariant, then $\varphi$ enjoys the relation

$$\sum_{k=1}^{p}(-1)^{|\eta|f(k)}\varphi(\omega_1,\ldots,[\eta,\omega_k],\ldots,\omega_p)=0$$

for homogeneous $\omega_k,\eta$, where

$$f(k)=\sum_{j>k}|\omega_j|.$$

(b) Show that the derivation property of $d$ implies the relation

$$\begin{aligned}&d(\varphi(\omega_1,\ldots,\omega_p))\\&=\sum_{k=1}^{p}(-1)^{g(k)}\varphi(\omega_1,\ldots,d\omega_k,\ldots,\omega_p),\end{aligned}$$

where $g(k)=\sum_{j<k}|\omega_j|$.

(c) Show that if $\varphi$ is invariant, then

$$\begin{aligned}&d(\varphi(\omega_1,\ldots,\omega_p))\\&=\sum_{k=1}^{p}(-1)^{g(k)}\varphi(\omega_1,\ldots,D\omega_k,\ldots,\omega_p).\end{aligned}$$

(Hint: Work locally. Let $\theta$ denote a connection matrix for a connection on $E$. For the induced connection on $\mathrm{End}(E)$, show that

$$D\omega_k = d\omega_k - [\theta, \omega_k].$$

Then use the previous two parts together.)

(d) For a connection $D$, its curvature $F_D$ determines via $\varphi$ a $2p$-form

$$\varphi(D) := \varphi(F_D, \ldots, F_D).$$

If $\varphi$ is invariant, show that $\varphi(D)$ is closed.

(e) Suppose $\varphi$ is invariant. If $D_1, D_0$ are two connections, show that the difference

$$\varphi(D_1) - \varphi(D_0)$$

is exact. (Hint: Let $D_t = D_0 + t\alpha$ where $\alpha = D_1 - D_0$ is an $\mathrm{End}(E)$-valued 1-form. Note that

$$\frac{d}{dt}\varphi(D_t) = \varphi(F_t; \dot{F}_t)$$

where

$$\varphi(\omega;\eta) = \sum_{k=1}^{p} \varphi(\omega, \dots, \overset{(k)}{\eta}, \dots, \omega).$$

Problem 2(c) implies that

$$\dot{F}_t = D_0\alpha + t[\alpha, \alpha] = D_t\alpha.$$

Part (c) implies

$$\frac{d}{dt}\varphi(D_t) = d(\varphi(F_t;\alpha)).$$

Set

$$\psi = \int_0^1 \varphi(F_t;\alpha)\,dt,$$

and conclude that

$$\varphi(D_1) - \varphi(D_0) = d\psi,$$

as desired.)

(f) Conclude that the cohomology class represented by $\varphi(D)$ is independent of the choice of connection $D$.

**8.** Let $\varphi$ be as before and suppose additionally that $\varphi$ is invariant under the action of the permutation group on the copies of $\mathfrak{gl}(V)$. For a Hermitian metric $H$ on $E$, set

$$\varphi(H) = \varphi(D_H)$$

where $D_H$ is the Chern connection corresponding to $H$. The $\partial\bar{\partial}$-lemma, together with the previous problem, implies that for each pair of metrics $H, K$, there is a $(p-1, p-1)$-form $R_\varphi(H, K)$ satisfying

$$i\bar{\partial}\partial R_\varphi(H, K) = \varphi(H) - \varphi(K).$$

Our goal is to construct $R_\varphi(H, K)$, modulo $\operatorname{im} \partial + \operatorname{im} \bar{\partial}$.

(a) Let $r(t, s)$ for $0 \leqslant t, s \leqslant 1$ denote a smooth surface of metrics satisfying the property that for each $s$, the map

$$t \mapsto r_s(t) := r(s, t)$$

is a smooth path of metrics starting at $K$ at $t = 0$ and ending at $H$ at $t = 1$. Let $\theta$

denote a one-form on the space of metrics $\mathcal{E}$. The pullback $r^*\theta$ is a one-form defined on the unit square $R = I^2 = [0,1] \times [0,1]$. Show that that

$$\iint_R d(r^*\theta) = \int_I r_1^*\theta - \int_I r_0^*\theta.$$

(b) Define a $A^{p-1,p-1}(X)$-valued one-form $\theta$ by the rule that at the metric $H$, the one form $\theta_H$ acts on the tangent space by

$$\theta_H(h) = -i\varphi(F_H; H^{-1}h).$$

Show that the exterior derivative of $\theta$ at $H$ evaluated on two tangent vectors $h, k$ is given by

$$\begin{aligned} &ip\varphi([\tau,\sigma], F, \ldots, F) \\ &+ ip\sum_{j=2}^{p} \varphi(\tau, F, \ldots, \overbrace{\bar\partial\partial\sigma}^{j}, \ldots, F) \\ &- ip\sum_{j=2}^{p} \varphi(\sigma, F, \ldots, \overbrace{\bar\partial\partial\tau}^{j}, \ldots, F). \end{aligned}$$

(Hint: See Donaldson's paper.)

(c) Show, using the Bianchi identity together with Problem 6(d), that the expression from the previous part lies in im $\partial$ + im $\bar{\partial}$.

(d) Set $R_\varphi(K, K) = 0$. For any other metric $H$, define

$$R_\varphi(H, K) = -i \int_0^1 \varphi(F_{H_t}; H_t^{-1}\dot{H})\, dt,$$

where $t \mapsto H_t$ is a path of metrics from $K$ to $H$. Use the previous parts to show that this definition is, modulo im $\partial$ + im $\bar{\partial}$, independent of the choice of path.

**9.** Using the notation of the previous problem, let $\varphi_1$ denote the invariant function associated to the trace

$$\varphi_1(A) = \mathrm{Tr}(A)$$

and let $\varphi_2$ denote the invariant function associated to the Killing form

$$\varphi_2(A, B) = -\operatorname{Tr}(AB).$$

(Here, the minus sign guarantees that the form becomes positive definite when restricted to the space of skew-hermitian endomorphisms.) Verify that indeed each $\varphi_k$ is invariant under the action of the general linear group. Then show that

$$R_1(H, K) = -i \int_0^1 \operatorname{Tr}(H_t^{-1}\dot{H})\, dt$$

$$R_2(H, K) = 2i \int_0^1 \operatorname{Tr}(H_t^{-1}\dot{H}F_t)\, dt.$$

**10.** The Donaldson functional can be defined by

$$M_K(H) = \int_X (R_2 + 2\lambda R_1 \omega) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

where $\lambda$ is as in Problem 5 and $R_1, R_2$ are as in Problem 9. For a path of metrics $H_t$

from $K$ to $H$, show that the variation of $M_K$ along this path is given by

$$\begin{aligned}&\frac{\partial}{\partial t} M_K(H_t)\\&= 2i \int_X \mathrm{Tr}(H_t^{-1} \dot{H}(F_t - \lambda\omega I_E)) \wedge \frac{\omega^{n-1}}{(n-1)!}.\end{aligned}$$

**11.** Show that if $H$ is a critical point of $M_K$, then $H$ is a Hermitian–Einstein metric.