Problem Set 7
Slope Stability
Spring 2021

1. For a metric K, compute the gradient of the Donaldson functional M_K. (Here the term gradient means the vector dual to the derivative with respect to the L^2-inner product.)

2. Use the term heat flow to mean a flow of metrics $t \mapsto H_t$ satisfying

$$\dot{H}_t = -\frac{i}{2}(\Lambda F_t - \lambda I_E),$$

where λ is the unique scalar determined by the slope of E.

(a) Show that the right-hand side of the heat flow equation is a scalar multiple of the gradient of the Donaldson functional.

(b) For a solution to the heat flow, show that

$$\frac{\partial}{\partial t} M_K(H_t) = -\left\|\Lambda F_t - \lambda I_E\right\|^2_{L^2_K}.$$

Conclude that M_K is non-increasing along the heat flow.

3. For a metric K and an endomorphism s, let H_t denote the path of metrics

$$H_t = Ke^{ts}.$$

Find the first and second variations of the Donaldson functional along this path. (Hint: The second variation is a function of the quantity $\left\|\bar{\partial} s\right\|_{L^2_{H_t}}$.)

4. Conclude that the Donaldson functional is convex.

Putting the previous parts together, a pleasing picture arises: The Donaldson functional is non-increasing and convex along the heat flow $t \mapsto H_t$. In general, however, *the functional may not be bounded from below, and so the heat flow may fail to converge to a critical point. It turns out that an algebro-geometric criterion called slope stability is a necessary and sufficient condition to ensure that the functional has a critical point.*

5. The slope of a vector bundle was defined in a previous assignment. Compute the slopes of the following bundles over $\mathbb{CP}^n$.

(a) $\mathcal{O}(1)$

(b) $\mathcal{O}(k)$

(c) $\bigoplus_{j=1}^{r} \mathcal{O}(k_j)$

(d) $\bigotimes_{j=1}^{r} \mathcal{O}(k_j)$

There is a way of defining the rank, degree, and slope of a torsion-free coherent *sheaf* too, but that process is somewhat beyond the scope of this assignment. Assuming the possibility exists, however, we may define the following notion of stability: A bundle E is called **semi-stable** if for each proper coherent subhheaf $\mathcal{F}$, we have the inequality $\mu(\mathcal{F}) \leqslant \mu(E)$. If moreover the strict inequality $\mu(\mathcal{F}) < \mu(E)$ holds for each proper coherent subsheaf satisfying $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(E)$, then we say that E is **stable**.

6. Show that any line bundle is stable.

7. Is it true that $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ is semi-stable if and only if $k_1 \neq k_2$?

8. Suppose we are given an exact sequence of vector bundles

$$0 \to E \to F \to G \to 0.$$

(a) If two of the three bundles have the same slope, must the third also have the same slope?

(b) Show that $\mu(E) < \mu(F)$ if and only if

$$\mu(F) < \mu(G).$$

9. Let L be a line bundle and E a vector bundle. If E is stable, show that $E \otimes L$ is stable.

10. Assume E admits no holomorphic subbundles. If E admits an HE metric, show that E is stable.

The converse is harder to prove, but true.