Problem Set 4
Curvature
Fall 2020

Throughout, let E denote a holomorphic vector bundle over a complex manifold X.

1. Show that a connection D on E induces, in a natural way, a connection D^* on the dual bundle E^* satisfying

$$d(\alpha(\sigma)) = (D^*\alpha)(\sigma) + \alpha(D\sigma)$$

for sections $\alpha \in A^0(E^*)$ and $\sigma \in A^0(E)$.

2. Let $\mathrm{End}(E)$ denote the corresponding endomorphism bundle.

(a) Show that a connection D on E also induces a connection on the bundle $\mathrm{End}(E)$ satisfying an identity similar to the one of the previous problem.

(b) The trace Tr describes a bundle map to the trivial bundle. Argue that

$$d \circ \mathrm{Tr} = \mathrm{Tr} \circ D,$$

where D denotes the induced connection on $\mathrm{End}(E)$.

3. If D is a connection, show that the composition D^2 is $C^\infty(X)$-linear.

4. Show that D^2 corresponds to an $\mathrm{End}(E)$-valued 2-form F_D.

5. Show that $DF_D = 0$, where D denotes the appropriate extension of the connection D. This formula is called the Bianchi identity.

6. Show that $\mathrm{Tr}(F_D)$ is closed.

7. For a hermitian metric H on E, show that there is a unique connection D_H satisfying

- $D_H^{0,1} = \bar{\partial}$
- $D_H H = 0$.

The connection D_H is called the Chern connection associated to H.

8. For a hermitian metric H, show that F_H belongs to $A^{1,1}(\mathrm{End}(E,H))$, where $\mathrm{End}(E,H)$ denotes the bundle of skew-hermitian endomorphisms of E.

9. If two metrics H, K satisfy

$$\langle \xi, \eta \rangle_H = \langle h\xi, \eta \rangle_K$$

for an endomorphism h of E, then write $H = Kh$.

(a) Show that h is a positive endomorphism of E.

(b) Show that h is self-adjoint with respect to both H and K.

(c) Show that

$$F_H = F_K + \bar{\partial}(h^{-1}\partial_K h).$$

(d) In particular, if $h = e^s$ for an endomorphism s, show that

$$F_H = F_K + \bar{\partial}\partial_K s.$$

10. Let $\mathcal{H}$ denote the space of hermitian metrics on E.

(a) Describe $\mathcal{H}$ as a subset of the space of sections of the bundle $E \otimes \overline{E}^*$, where $\overline{E}$ denotes the pullback of E under the conjugation involution.

(b) Describe the tangent space $T_H\mathcal{H}$ at a typical metric H.

11. The assignment $H \mapsto F_H$ describes a map $\mathcal{H} \to A^{1,1}(\mathrm{End}(E))$.

(a) Describe the derivative of this map at the point H.

(b) The trace determines a map from $A^{1,1}(\mathrm{End}(E)$ to $A^{1,1}(X)$. Show that the derivative of the composition

$$\mathcal{H} \to A^{1,1}(\mathrm{End}(E)) \to A^{1,1}(X)$$

has an image that is a subset of $\mathrm{im}\,\partial\bar{\partial}$.

12. Show that the cohomology class of

$\operatorname{Tr}(F_H)$ is independent of the metric H. (Hint: Since any two metrics are connected by a path, argue that it is enough to show that along a path H_t, the derivative of $\operatorname{Tr}(F_{H_t})$ lies in $\operatorname{im} \partial\bar{\partial}$.)