Assignment 2

Intro to Modern Analysis

1. Let A be a set, and let $P(A)$ be the collection of all subsets of A. Is there a bijection from A onto $P(A)$?

2. Let M be an infinite set, and let A be a countable set. Is there a bijection from M onto $M \cup A$?

3. Let $\{A_k\}_{k\geqslant 1}$ be a sequence of subsets of a metric space. Prove or disprove.

(a) For each integer $N > 0$, we have

$$\bigcup_{k=1}^{N} \overline{A_k} \subset \overline{\bigcup_{k=1}^{N} A_k}$$

(b) For each integer $N > 0$, we have

$$\bigcup_{k=1}^{N} \overline{A_k} = \overline{\bigcup_{k=1}^{N} A_k}.$$

(c) We have

$$\bigcup_{k=1}^{\infty} \overline{A_k} \subset \overline{\bigcup_{k=1}^{\infty} A_k}.$$

(d) We have

$$\bigcup_{k=1}^{\infty} \overline{A_k} = \overline{\bigcup_{k=1}^{\infty} A_k}$$

4. Let A° denote the set of interior points of A.

(a) Prove that A° is open.

(b) Prove that A is open if and only if $A = A^\circ$.

(c) If $B \subset A$ and B is open, prove that $B \subset A^\circ$.

(d) Prove that the complement of A° is the closure of the complement of A.

5. Let X be the interval $[0, 2) \subset \mathbb{R}$. The restriction of the usual metric on $\mathbb{R}$ to X is a metric on X.

(a) Is the set $[0, 1)$ open relative to X?

(b) Is the set $[1, 2)$ closed relative to X?

(c) Is the set $[1, 2)$ compact relative to X?

6. Give an example of an open cover of $(0, 1)$ which admits no finite subcover.

7. Let $f(x) = x^2$.

(a) Let $x_n = f(2 + \frac{1}{n})$. Does x_n converge? Prove your answer is correct.

(b) Let $y_n = f(n)$. Does y_n converge? Prove your answer is correct.

8. Let $\{A_n\}_{n\geqslant 1}$ be a sequence of open dense subsets of $\mathbb{R}$. Let A denote the intersection

$$A = \cap_{n\geqslant 1} A_n.$$

(a) Let U be an open subset of $\mathbb{R}$. Show that there is a sequence of points $x_n \in U \cap A_k$ together with a sequence of radii $0 < r_n < 1/n$ such that $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n) \cap A_n$.

(b) Show that the sequence x_n is Cauchy, and hence converges to a point x of $\mathbb{R}$.

(c) Show that $x \in U \cap A$.

(d) Conclude that A is dense in $\mathbb{R}$.

9. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p > 0$, write

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Fix a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and for each positive integer $k > 0$, let a_k denote the sequence of real numbers $a_k = \|x\|_k$. Show that

$$\lim_{k\to\infty} a_k = \max_{1\leqslant i\leqslant n} |x_i|.$$

10. In the notation of the previous question, define a function $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the rule

$$d_p(x, y) = \|x - y\|_p .$$

Is $(\mathbb{R}^n, d_p)$ a metric space for $p \in (0, 1)$ and $n > 1$?