

Assignment 1
Intro to Modern Analysis

1. For a subset A of a set X , let A^c denote the complement of A given by $A^c = X \setminus A$. Let I be an index set, and let $\{A_i\}_{i \in I}$ be a collection of subsets of X indexed by I . Prove or disprove.

(a) We have

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c.$$

(b) We have

$$\left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

2. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of nonempty subsets of \mathbb{R} which are “nested” in the sense that

$$A_0 \supset A_1 \supset A_2 \supset \cdots.$$

Is the intersection

$$\bigcap_{n \in \mathbb{N}} A_n$$

nonempty? Prove or disprove.

3. Let A denote the subset of \mathbb{R} given by

$$A = \left\{ \frac{|\cos(n)|}{n+1} : n \in \mathbb{N} \right\}.$$

Find with proof the values $\sup A$ and $\inf A$, whenever either exists.

4. Let A be a nonempty subset of \mathbb{R} bounded from above, and let x_0 be a point of \mathbb{R} . Show that x_0 is equal to $\sup A$ if and only if for each $\epsilon > 0$ there is an element $x \in A$ such that $x_0 - x < \epsilon$.

5. Let A be a nonempty set of *integers* that is bounded from above. Show that A has a largest element. (Hint: Let $x_0 = \sup A$. First show that x_0 is an integer by showing that if not, then there are integers $m, n \in A$ satisfying $x_0 - 1 < m < n < x_0$, which is an impossible statement [why?]. Then show that x_0 belongs to A .)

6. Let A, B be nonempty subsets of \mathbb{R} each bounded from above. Prove or disprove.

(a) $\sup(A \cup B) = \sup\{\sup A, \sup B\}.$

(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}.$

7. For a real number t , let $|t|$ denote the absolute value of t defined by

$$|t| = \begin{cases} t & t \geq 0 \\ -t & t \leq 0 \end{cases}.$$

(a) For a real number t and a nonnegative number a , show that $|t| \leq a$ if and only if $-a \leq t \leq a$.

(b) Let x and y be points of \mathbb{R}^n . Show that

$$|\|x\| - \|y\|| \leq \|x - y\|.$$