Let $\mathrm{Cl}(S)$ denote the closure of S and $L(S)$ the set of limit points of S. We will show that $\mathrm{Cl}(S) = S \cup L(S)$.

We first show that $\mathrm{Cl}(S) \subset S \cup L(S)$. Say that x is a point of the closure of S. If x belongs to S, then we are done. If not, then we claim that every neighborhood U of x intersects S in some point (which must be different from x), and hence x is a limit point. Indeed, if there were a neighborhood U of x disjoint from S, then the complement $X \setminus U$ would be a closed set containing S which does not contain $x \in \mathrm{Cl}(S)$. This contradicts the minimality of $\mathrm{Cl}(S)$.

We next show that $S \cup L(S) \subset \mathrm{Cl}(S)$. Certainly we have $S \subset \mathrm{Cl}(S)$. Moreover, if x is a limit point of S, then we claim that x belongs to any closed subset containing S, and hence x belongs to $\mathrm{Cl}(S)$. Indeed, suppose that there were a closed subset C containing S such that $x \notin C$. Then the complement $X \setminus C$ would be an open neighborhood of x which is disjoint from S, which is a contradiction to the fact that $x \in L(S)$.

Next we show that $\mathrm{Cl}(S) = X \setminus \mathrm{Int}(X \setminus S)$.

Suppose that x is not in the closure $\mathrm{Cl}(S)$. Then there is a closed set C containing S such that $x \notin C$. Hence the open set $X \setminus C$ is an open neighborhood of x satisfying $X \setminus C \subset (X \setminus S)$. It follows that x belongs to $\mathrm{Int}(X \setminus S)$ and hence x does not belong to $X \setminus \mathrm{Int}(X \setminus S)$.

Conversely, suppose that x does not belong to $X \setminus \mathrm{Int}(X \setminus S)$, meaning that x belongs to $\mathrm{Int}(X \setminus S)$. Then there is an open neighborhood U of x satisfying $x \in U \subset X \setminus S$. Then the complement $C = X \setminus U$ is a closed subset containing S such that $x \notin C$. By the minimality of the closure, we conclude that x is not in the closure $\mathrm{Cl}(S)$.

Let $\{U_\alpha\}$ be an open cover of X. We show that C is closed in X if and only if $C \cap U_\alpha$ is closed in U_α for each α.

One direction is by definition of the subspace topology. Conversely, suppose that $C \cap U_\alpha$ is closed in U_α for each α. Write $V = X \setminus C$. Note that $V \cap U_\alpha = U_\alpha \setminus (U_\alpha \cap C)$ for each α. Since $U_\alpha \cap C$ is closed in U_α for each α, we see that $V \cap U_\alpha$ is open in U_α for each α. Hence V is open in X by part (i). Hence C is closed in X by definition.