Problem Set 5
Chern–Weil
Fall 2020

1. Let V be a finite-dimensional vector space.

(a) Show that there is a natural isomorphism

$$\mathrm{End}(V) \simeq V \otimes V^*$$

by showing that $\mathrm{End}(V)$ satisfies the universal property of the tensor product for the spaces V and V^*.

(b) Also by the universal property of the tensor product, the bilinear map

$$\begin{aligned} V \times V^* &\to \mathbb{C} \\ (v, \alpha) &\mapsto \alpha(v) \end{aligned}$$

determines a linear functional Tr on $V \otimes V^*$. If an endomorphism T of V is represented by a matrix A, check that $\mathrm{Tr}(T) = \mathrm{Tr}(A)$ under the identification of $\mathrm{End}(V)$ with $V \otimes V^*$ described in the previous part.

2. Let $\Lambda^k(V)$ denote the space of alternating k-linear forms on V^*.

(a) Show that there is a natural isomorphism of $\Lambda^1(V)$ with V.

(b) More generally, show that there is a map

$$\begin{aligned} V^k &\to \Lambda^k(V) \\ (v_1, \ldots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k \end{aligned}$$

that is alternating and k-linear and whose kernel consists of those k-tuples that are linearly dependent.

(c) If V has dimension r, show that $\Lambda^k(V)$ has dimension $\binom{r}{k}$. (Hint: Work with a basis e_i.)

(d) If T is an endomorphism of V, show that, for each k, T induces an endomorphism $\Lambda^k T : \Lambda^k(V) \to \Lambda^k(V)$ satisfying

$$\Lambda^k T(v_1 \wedge \cdots \wedge v_k) = Tv_1 \wedge \cdots \wedge Tv_k.$$

(e) In particular, since $\Lambda^r(V)$ has dimension 1, the map $\Lambda^r T$ must correspond to multiplication by a scalar. Call that scalar $\det T$. Show that if T is represented by a matrix A, then $\det T = \det A$.

(f) Show that any pair of endomorphisms S, T of $\Lambda^\ell(V), \Lambda^k(V)$ respectively determine an endomorphism $S \wedge T$ of $\Lambda^\ell(V) \wedge \Lambda^k(V)$ in such a way that $T \wedge T$ corresponds to $\Lambda^2 T$ when T is an endomorphism of $V = \Lambda^1(V)$.

3. With the notation from the previous problems, define a polynomial in t by the rule

$$f^T(t) = \det(I + tT)$$

for an endomorphism T of V.

(a) Show that the degree of f^T is r.

(b) If we write the coefficients of f^T as

$$f^T(t) = \sum_{k=0}^{r} f_k(T) t^k,$$

show that $f_k(T)$ is invariant under the conjugation action of $GL(V)$ on $\mathrm{End}(V)$.

(c) In fact, show that

$$f_k(T) = \mathrm{Tr}\Lambda^k T.$$

(d) If T is hermitian with respect to a metric on V, show that each $f_k(T)$ is real.

4. Let E be a complex vector bundle of rank r over X. For a connection D on E, define the total chern form

$$f(E, D) = \det\left(I_E + \frac{i}{2\pi} F_D\right),$$

an element of the ring $A(X)$.

(a) Show that we may write $p(E, D)$ in the form

$$1 + f_1(E, D) + \cdots + f_r(E, D)$$

for $f_k(E, D) \in A^{2k}(X)$.

(b) In particular, show that

$$f_1(E, D) = \mathrm{Tr}\left(\frac{i}{2\pi} F_D\right).$$

(c) Show that each $f_k(E, D)$ is closed. (Hint: First show the generalized Bianchi identity $D\Lambda^k F_D = 0$.)

(d) Moreover, if D_1, D_0 are two connections, show that the difference

$$f_k(E, D_1) - f_k(E, D_0)$$

is exact. (Hint: If $\theta = D_1 - D_0$, define a path $D_t = D_0 + t\theta$. Show that $\dot{F}_t = D_t\theta$, and use this to show

$$\frac{d}{dt} f_k(E, D_t) = k\left(\frac{i}{2\pi}\right)^k d\mathrm{Tr}(\theta \wedge \Lambda^{k-1} F_t).$$

Upon setting

$$\Phi = k\left(\frac{i}{2\pi}\right)^k \int_0^1 \mathrm{Tr}(\theta \wedge \Lambda^{k-1} F_t)\, dt,$$

verify that

$$f_k(E, D_1) - f_k(E, D_0) = d\Phi,$$

as desired.)

(e) Conclude that the cohomology class of $f_k(E, D)$ is independent of the choice of connection. This class, denoted $c_k(E)$, is called the kth Chern class of E.

5. Show that $c_1(E \oplus E') = c_1(E) + c_1(E')$.

6. Show that $c_k(E^*) = (-1)^k c_k(E)$.

7. Show that $c_1(E) = c_1(\det E)$.

8. In Problem Set 2, it was claimed that for a line bundle L with metric H, its first Chern class is locally represented by

$$-\frac{i}{2\pi}\partial\bar{\partial}\log\|s\|_H^2$$

where s is a section. Check that this is true.

9. For line bundles L, L', show that

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

10. Now suppose that E is holomorphic and equipped with a hermitian metric H.

(a) If D is unitary with respect to H, show that each $f_k(E, D)$ is real.

(b) If D_H is the Chern connection associated with H, show that in addition each $f_k(E, D_H)$ is of type (k, k).

11. Find

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(\ell)),$$

if $\mathcal{O}(-1)$ is the tautological bundle.

12. For a complex manifold X, define its Chern classes $c_k(X)$ to be the classes associated to the (complexified) tangent bundle.

(a) Show that $c_1(X) = -c_1(K_X)$, where K_X is the canonical bundle.

(b) Find $\int_{\mathbb{CP}^1} c_1(\mathbb{CP}^1)$.