MATH S1202: Calculus IV
Quiz 4
June 16, 2016

Define curves in the following way.

- Let D be the region *in the upper half xy-plane* between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
- Let C denote the boundary of D with the positive orientation. (Note that C consists of 4 arcs.)
- Let C_1 denote the piece of C lying along the circle $x^2 + y^2 = 4$.

1. Determine whether the vector field is conservative *and* if it is conservative, find a potential function.

(a) $\vec{F}(x, y) = (x^3, y^3)$

(b) $\vec{G}(x, y) = (-y^3, x^3)$

Solution. (a) The vector field $\vec{F}$ is conservative and a potential for $\vec{F}$ is given by $f(x, y) = \frac{1}{4}(x^4 + y^4)$.

(b) If we denote the components of $\vec{G}$ by $P = -y^3$ and $Q = x^3$, then we compute that $Q_x = 3x^2 \neq -3y^2 = P_y$. Since the domain of $\vec{G}$ is all of $\mathbb{R}^3$, which is simply connected, we conclude that $\vec{G}$ is not conservative.

2. Compute the line integral of the vector field over the curve C_1.

(a) $\vec{F}(x, y) = (x^3, y^3)$

(b) $\vec{G}(x, y) = (-y^3, x^3)$

Hint: For (b), you may use the trig identity

$$\sin^4 t + \cos^4 t = \frac{1}{4}(\cos(4t) + 3)$$

Solution. (a) The fundamental theorem of line integrals implies that

$$\int_{C_1} \vec{F} \cdot d\vec{r} = f(-2, 0) - f(2, 0) = 0,$$

where $f(x, y) = \frac{1}{4}(x^4 + y^4)$ from 1(a).

(b) We parametrize C_1 by $\vec{r}(t) = (2\cos(t), 2\sin(t))$ for $0 \leqslant t \leqslant \pi$. The derivative is given by $\vec{r}'(t) = (-2\sin(t), 2\cos(t))$. We compute that

$$\begin{aligned}
\int_C \vec{G} \cdot d\vec{r} &= \int_0^\pi \vec{G}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\
&= \int_0^\pi \langle -(2\sin(t))^3, (2\cos(t))^3 \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle dt \\
&= \int_0^\pi 16 \int_0^\pi (\sin^4(t) + \cos^4(t))\, dt \\
&= 4 \int_0^\pi (\cos(4t) + 3) dt \\
&= 12\pi.
\end{aligned}$$

3. Compute the line integral of the vector field over the closed curve C.

(a) $\vec{F}(x, y) = (x^3, y^3)$

(b) $\vec{G}(x,y) = (-y^3, x^3)$

Solution. (a) Since C is closed and $\vec{F}$ is conservative, we conclude that $\int_C \vec{F} \cdot d\vec{r} = 0$.

(b) The curve C is the boundary of the region D described in polar coordinates

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

by

$$D = \begin{cases} 0 \leqslant \theta \leqslant \pi \\ 1 \leqslant r \leqslant 2 \end{cases}.$$

We may apply Green's theorem to find that

$$\begin{aligned} \int_C \vec{G} \cdot d\vec{r} &= \iint_D (3x^2 + 3y^2) dA \\ &= \int_0^\pi \int_1^2 3r^2 r \, dr \, d\theta \\ &= \frac{3\pi}{4} \cdot r^4 \Big|_{r=1}^{r=2} \\ &= \frac{3\pi}{4} \cdot (16 - 1) \\ &= \frac{45\pi}{4}. \end{aligned}$$