Problem Set 3: Laplace Equation on Riemannian manifolds

The purpose of this problem set is to solve the Laplace Equation $\Delta f = \rho$ on a Riemannian manifold M using a variational approach.

Preliminary concepts. We have fixed a Riemannian manifold (M, g). We often write the metric g as $\langle -, - \rangle$.

The gradient of a smooth function f on M is the vector field ∇f characterized by the rule

$$\langle \nabla f, X \rangle = X(f) \qquad \text{for each } X \in C^\infty(M, TM).$$

One can think of ∇f as the vector representing the one-form df using the duality $TM \simeq TM^*$ provided by the metric g. The vector field ∇f is unique and well-defined because g is non-degenerate and smooth because g and f are. Indeed, we can work out the local expression of ∇f as follows. Say that in coordinates $(x^1, \ldots, x^n)$ for M, we can write

$$\nabla f = f^i \frac{\partial}{\partial x^i},$$

where we are suppressing the summation when we have one lower and one raised index. Then taking $X = \partial/\partial x^j$ gives

$$f^i g_{ij} = \partial_j(f).$$

Multiplying both sides by the inverse g^{jk} and summing over j gives

$$f^k = g^{jk} \partial_j(f).$$

For a vector field X on M, the divergence of X is the real-valued smooth function $\operatorname{div}(X)$ defined by sending a point p to the trace of the linear endomorphism on T_pM given by $Y \mapsto \nabla_Y X(p)$ where ∇ is the Levi-Civita connection compatible with the metric g. Local coordinates $(x^1, \ldots, x^n)$ give rise to a basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ for T_pM. Say that X has local expression

$$X = X^j \frac{\partial}{\partial x^j}.$$

If Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection ∇, then

$$\nabla_i X = (\partial_i(X^k) + \Gamma_{ij}^k X^j) \frac{\partial}{\partial x^k}$$

It follows that $\operatorname{div}(X)$ has local expression

$$\operatorname{div}(X) = \partial_i(X^i) + \Gamma_{ij}^i X^j.$$

For a smooth function f on M, the Laplacian of f, denoted Δf, is the smooth function defined by

$$\Delta f = \operatorname{div}(\nabla f).$$

A local expression for Δf can be computed using the above local expressions for the gradient and the divergence.

1. Let X be a vector field on M. Show that we have the following equality of n-forms

$$(\mathrm{div}X)dV = d(i_X dV),$$

where i_X denotes the contraction of the n-form dV with the vector field X. In particular, the the following integral is zero

$$\int_M (\mathrm{div}X)\, dV = 0$$

by Stokes' theorem. (Hint: prove the equality at a single point using normal coordinates.)

2. (Integration by parts) For smooth functions f and h on M, check that

$$\int_M f(\Delta h)\, dV = -\int_M \langle \nabla f, \nabla h\rangle\, dV,$$

where ∇f denotes the gradient of f satisfying $\langle \nabla f, X\rangle = X(f)$ for any vector field X.

3. (Liouville's Theorem) Show that if f is harmonic on M, then f is constant.

4. For a vector field X with local expression

$$X = X^i \frac{\partial}{\partial x^i},$$

show that the divergence has local expression

$$\mathrm{div}(X) = \frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}X^i).$$

5. The Laplacian has local expression

$$\Delta f = \frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}g^{ij}\partial_j(f)).$$

6. A linear differential operator of second order of the form

$$L(f) = \sum_{j,k} a_{jk}\partial_{jk} f + \sum_{\ell} b_\ell \partial_\ell f + cf$$

is called **elliptic** if the matrix a_{jk} is positive definite. Check that Δ is elliptic.

Introduce the Sobolev space $L_1^2(M)$ in the following manner. Define an inner product on $C^\infty(M)$ by the rule

$$\langle f, h\rangle_{L_1^2} = \int_M (\langle \nabla f, \nabla h\rangle + fh)\, dV.$$

Let $L_1^2(M)$ denote the completion of $C^\infty(M)$ with respect to the norm $\|f\|_{L_1^2}^2 = \langle f, f\rangle_{L_1^2}$. The notation is meant to suggest that we are concerned with those functions where the L^2-norm of the function and the L^2-norm of the first derivative of the function are finite. Note that

$$\|f\|_{L_1^2}^2 = \|f\|_2^2 + \|\nabla f\|_2^2 .$$

7. Show that if f_j converges to f in L^2_1, then f_j converges to f in L^2 and ∇f_j converges to ∇f in L^2.

8. Define a functional $E : L^2_1(M) \to \mathbb{R}$ by the rule

$$E(f) = \int_M \left(\frac{1}{2} |\nabla f|^2 + f\rho \right) dV.$$

Show that E is well-defined and finite.

9. The **Poincaré Inequality** states that there is a positive constant $C > 0$ so that for each smooth function f on M satisfying $\int_M f\, dV = 0$, we have

$$\|f\|_2^2 \leqslant C \|\nabla f\|_2^2 .$$

Using the Poincaré inequality, show that there are constants $\epsilon, C > 0$ such that

$$E(f) \geqslant \epsilon \|f\|^2_{L^2_1} - C$$

for each $f \in L^2_1(M)$ with zero mean. (Hint: The geometric mean inequality of the form $|ab| \leqslant \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$ might be useful.)

The **Banach-Alaoglu Theorem** states that if B is a separable Banach space, then the closed unit ball in B is weakly compact. This means that if u_j is a sequence of vectors satisfying $\|u_j\| \leqslant 1$, then there is a vector u in B such that $\lim_{j\to\infty} \ell(u_j) = \ell(u)$ for each bounded linear functional $\ell \in B^*$.

10. (Lower semi-continuity of the norm) Let B be a Hilbert space with inner product $\langle -, - \rangle$. Show that if u_j converges weakly to u, then

$$\|u\| \leqslant \lim_{j\to\infty} \|u_j\| .$$

11. Show that there is a vector $F \in L^2_1(M)$ with zero mean such that F is achieves the infimum of E. (Hint: Take a minimizing sequence f_j for E where each f_j has zero mean. Argue that f_j converges weakly to a function F with zero mean. Show that F achieves the minimum of E.)

12. Show that if F achieves the minimum of E we have

$$\int_M \left(\langle \nabla F, \nabla \varphi \rangle + \varphi\rho \right)\, dV = 0 \qquad \text{for each } \varphi \in C^\infty(M).$$

13. Show that the operator Δ is self-adjoint with respect to the L^2-norm.

14. Show that if F achieves the minimum of E, then we have $\langle F, \Delta\varphi \rangle_{L^2} = \langle \rho, \varphi \rangle_{L^2}$ for each smooth function φ on M. This says that F is a weak solution to the equation $\Delta F = \rho$.

Standard regularity results concerning elliptic operators imply that if L is an elliptic operator with smooth coefficients and if ρ is smooth and f is a weak solution in $L^2(M)$ to $L(f) = \rho$, then f is smooth and in fact a bona fide solution to $L(f) = \rho$.

15. Let ρ be a smooth function on M such that

$$\int_M \rho \, dV = 0.$$

Show that there is a smooth function f on M, unique up to additive constant, such that $\Delta f = \rho$.