MATH 2610-01
Take-Home #2
Due March 18, 2022

1. **Inhomogeneous constant-coefficient equation.** Find a general solution to

$$y''' + y' = \csc x \cot x, \qquad 0 < x < \pi/2.$$

2. **Inhomogeneous "Shifted" Cauchy-Euler Equation.** Find a general solution to

$$(t+1)^2 y'' + 3(t+1)y' + y = (t+1)^{-1}, \qquad t > -1.$$

(Hint: First solve the corresponding homogeneous equation by devising a modification of the method for Cauchy-Euler equations.)

3. **Principal Symbol and Uniqueness.** Consider a linear homogeneous equation of the form

$$x(2x-3)y'' + p(x)y' + q(x)y = 0, \tag{1}$$

where p and q are differentiable on $\mathbb{R}$. The leading term $x(2x-3)$ is sometimes called the *principal symbol*, and it contains important information about the existence and uniqueness of solutions.

Suppose p and q are chosen such that

$$y_1(x) = x^3 \qquad \text{and} \qquad y_2(x) = x - 1$$

solve (1).

(a) Find $p(x), q(x)$.

(b) With p, q from part (a), consider the initial value problem

$$\begin{cases} x(2x-3)y'' + p(x)y' + q(x)y = 0 \\ y(1) = 1 \\ y'(1) = 3 \end{cases}.$$

For this IVP, prove the following: If b is any number less than zero, then the interval $(b, 3/2)$ admits at least two solutions.

(c*) With p, q as in the previous parts, the fact that $x = 3/2$ is also a root of the principal symbol suggests, in light of part (b), that there could be an IVP of the form

$$\begin{cases} x(2x-3)y'' + p(x)y' + q(x)y = 0 \\ y(1) = Y_0 \\ y'(1) = Y_1 \end{cases}$$

satisfying the following property: If b is any number greater than $3/2$, then the interval $(0, b)$ admits at least two solutions. Can you find such an IVP?

4. *Please note that this problem asks you to develop some aspects of the theory from class in a slightly more general setting than what we covered. Thus, while you cannot reference particular results from class during this problem, you can consult the strategies of proof we used and try to adapt them to this setting.* Let V be a vector space over $\mathbb{R}$. Let $L : V \to V$ denote a linear map, and let $F : V \to \mathbb{R}^n$ denote another linear map, where n is a positive integer. Let A denote the corresponding augmented linear map

$$\begin{aligned} A : V &\longrightarrow V \times \mathbb{R}^n \\ v &\longmapsto (L(v), F(v)). \end{aligned}$$

Suppose it is known that A is invertible.

(a) Show that V is infinite-dimensional. (Hint: Contradiction using rank-nullity theorem.)

(b) Show that $\dim \ker L \leqslant n$. (Hint: Consider what A does to a list of k linearly independent vectors in $\ker L$ where k is some positive number.)

(c) Show that in fact $\dim \ker L = n$. (Hint: Construct a basis.)

(d) Write $F = (f_1, \ldots, f_n)$ for the components of the linear map $F : V \to \mathbb{R}^n$ so that each f_k is a linear function $f_k : V \to \mathbb{R}$. For $v_1, \ldots, v_n \in V$, define

$$W(v_1, \ldots, v_n) = \det[f_i(v_j)].$$

If $v_1, \ldots, v_n$ belong to $\ker L$, show that the list $\{v_1, \ldots, v_n\}$ is linearly dependent if and only if $W(v_1, \ldots, v_n) = 0$. (Hint: For the reverse implication, you must use that $\ker A = \{0\}$.)

(e) Let $Y_1, \ldots, Y_n$ be given real numbers, and let g be a given vector in V. Suppose that we are given

- a vector $\varphi \in V$ satisfying $L(\varphi) = g$
- a basis $\{y_1, \ldots, y_n\}$ for $\ker L$.

Show that there are *unique* constants $c_1, \ldots, c_n \in \mathbb{R}$ such that if

$$y = \varphi + c_1 y_1 + \cdots + c_n y_n$$

then

$$A(y) = (g, Y_1, \ldots, Y_n).$$

(Hint: Don't forget to show both existence and uniqueness.)