Problem Set 11
Hilbert Mumford Criterion
Summer 2021

For simplicity, let's work over the field $\mathbb{C}$. We also simplify our lives by considering the following situation. Let G be a linear reductive group acting on a scheme X, where $X \subset \mathbb{C}^n$ is affine, G acts linearly (through a faithful embedding into GL_n), and the action is linearized using a character χ of G.[1] The last condition means the following. We view G as acting on the trivial line bundle $X \times \mathbb{C}$ by the rule

$$g \cdot (x, z) = (g \cdot x, \chi(g)z)$$

where $\chi : G \to GL_1 = \mathbb{C}^*$ is a fixed character. Let L_χ denote the trivial line bundle together with this choice of linearization. Note that with these conventions, we have a natural isomorphism

$$L_{\chi^{\otimes m}} \simeq (L_\chi)^{\otimes m}.$$

A section of the trivial bundle can be identified with a regular function $s : X \to \mathbb{C}$. I will call such a section *χ-equivariant* if

$$s(g \cdot x) = \chi(g)s(x),$$

that is, if s is an equivariant morphism with respect to the actions of G on X and $\mathbb{C}$. The space of χ-equivariant sections will be denoted $H^0(X, L_\chi)$. It is a subspace of the space of regular functions on X. A point $x \in X$ is called *semistable* if there is a positive integer m and a χ^m-equivariant section $s \in H^0(X, L_\chi^{\otimes m})$ such that $s(x) \neq 0$. The subset of semistable points will be denoted $X_{\mathrm{ss}}(L_\chi)$.

1. Let $X = \mathbb{C}^n$. Let $G = \mathbb{C}^*$ act on X through the usual diagonal action. A character χ of G takes the form $\chi(t) = t^d$ for some integer $d \in \mathbb{Z}$.

(a) Check that $H^0(X, L_\chi)$ is isomorphic to the vector space of homogeneous polynomials of degree d.

(b) Show the following

$$X_{\mathrm{ss}}(L_\chi) = \begin{cases} \mathbb{C}^n \setminus 0 & d > 0 \\ \mathbb{C}^n & d = 0 \\ \varnothing & d < 0 \end{cases}.$$

2. Let $X = \mathbb{C}^3$. Let $G = \{(t, t^{-1}, u) \in (\mathbb{C}^*)^3 : t \in \mathbb{C}^*, u = \pm 1\} \simeq \mathbb{C}^* \times \mu_2$. Let G act on X in the obvious way. Let $\chi : G \to \mathbb{C}^*$ denote the character

$$\chi(t, t^{-1}, u) = tu.$$

[1] This situation is very different from the one in MFK, where, in particular, X is projective and the action is linearized through any invertible sheaf L over X, not necessarily the trivial sheaf. Nevertheless, certain similarities persist.

(a) The regular functions on X can be identified with the polynomials in the variables x, y, z. Show that a monomial $x^a y^b z^c$ is χ-equivariant if and only if

$$a = b + 1, c \equiv 1 \bmod 2.$$

(b) Show that $H^0(X, L_\chi) \simeq xz\mathbb{C}[xy, z^2]$.

(c) Show that $X_{\mathrm{ss}}(L_\chi) = \mathbb{C}^* \times \mathbb{C}^2$.

3. For two positive integers $r \leqslant n$, let $X = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r) = M_{r\times n}$ be the set of r-by-n matrices. Let $G = GL_r = \mathrm{Aut}(\mathbb{C}^r)$ be the general linear group acting by post-composition:

$$g \cdot A = gA.$$

Let $\chi : GL_r \to \mathbb{C}^*$ be the determinant.

(a) Let $s : X \to \mathbb{C}$ be a regular function determined by a minor of maximal rank, that is, by an r-by-r minor. Show that s is χ-equivariant.

(b) Show that $X_{\mathrm{ss}}(L_\chi)$ contains the subset of full rank matrices.

A one-parameter subgroup of G is a morphism $\lambda : \mathbb{C}^* \to G$ of algebraic groups.