Problem Set 10
Group schemes
Summer 2021

Remark. If you need hints, you can consult "Examples of group schemes" from the Stacks Project.

1. Let $\mathbb{G}_m$ denote the affine scheme

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}],$$

often called the "multiplicative group." Write down the morphism of rings

$$\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}[x, x^{-1}] \otimes \mathbb{Z}[x, x^{-1}]$$

determining the group structure

$$\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m.$$

2. For a positive integer n, let μ_n denote the affine scheme

$$\mu_n = \operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1)).$$

Write down the morphism of rings that determines the group structure.

3. Let $\mathbb{G}_a$ denote the "additive" group scheme

$$\mathbb{G}_a = \operatorname{Spec} \mathbb{Z}[x].$$

Write down the morphism of rings determining the group structure.

4. For a positive integer n, let GL_n denote the affine scheme

$$GL_n = \operatorname{Spec}(\mathbb{Z}[x_{ij}][1/d])$$

where d is the determinant of the n^2 variables x_{ij}. Write down the morphism of rings that determines the group structure.

5. Let $\mathbb{A}^n$ denote the affine space

$$\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n].$$

Describe the natural action of GL_n on $\mathbb{A}^n$ as a morphism of rings.

6. For any n, the group scheme $\mathbb{G}_m = GL_1$ embeds into GL_n via the "diagonal." Write down the ring map corresponding to this inclusion.

7. In light of the previous two exercises, there is a natural action of $\mathbb{G}_m$ on $\mathbb{A}^n$ through the diagonal. Describe it.

8. A $\mathbb{Z}$-grading of a ring R is a direct sum decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ such that $R_i \cdot R_j \subset R_{i+j}$. Show that an action of $\mathbb{G}_m$ on $\operatorname{Spec} R$ is the same data as a $\mathbb{Z}$-grading of R.

9. Determine the $\mathbb{Z}$-grading on $\mathbb{Z}[x, x^{-1}]$ coming from the action of $\mathbb{G}_m$ on itself (i.e. from the group structure).

10. Determine the $\mathbb{Z}$-grading on $\mathbb{Z}[x_1, \dots, x_n]$ coming from the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^n$.

11. Determine the $\mathbb{Z}$-grading on $\mathbb{Z}[x_{ij}][1/d]$ coming from the action of $\mathbb{G}_m$ on GL_n through the diagonal.

12. By a *graded ring* is typically meant an $\mathbb{N}$-graded ring $S = \bigoplus_{i\geqslant 0} S_i$. Show that in fact the $\mathbb{Z}$-grading on $\mathbb{Z}[x_0, \dots, x_n]$ coming from the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ makes the ring $\mathbb{Z}[x_0, \dots, x_n]$ into a graded ring.

Final comment: Projective n-space, denoted $\mathbb{P}^n$, usually means the locally ringed space associated to the graded ring $S = \mathbb{Z}[x_0, \dots, x_n]$ by taking the projective homogeneous spectrum (i.e. by applying Proj). There is a natural action of GL_{n+1} on the ring S, but the action may not preserve the grading. To remedy this, the standard approach, as far as I can tell, is to consider an associated scheme preserving the grading. From what I have seen, this scheme, denoted PGL_n, is the affine scheme associated to the degree zero piece

$$R_0 = \mathbb{Z}[\{x_{ij}\}_{0\leqslant i,j\leqslant n}][1/d]_{\deg=0},$$

where the grading comes from the diagonal action of $\mathbb{G}_m$ on GL_{n+1}. The inclusion of this degree zero piece corresponds to a surjective morphism of affine schemes

$$GL_{n+1} \to PGL_n.$$

Moreover, there is a natural action of PGL_n on $\mathbb{P}^n$ coming from the natural action of GL_{n+1} on $\mathbb{A}^{n+1}$.