Problem Set 8
GIT
Spring 2021

1. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1} \setminus 0$ by the rule

$$\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n).$$

Let $\mathbb{P}^n$ denote the topological quotient by this action. The goal of this problem is to understand the structure of $\mathbb{P}^n$ as a variety.

Let U_k denote the subset of $\mathbb{P}^n$ described by

$$U_k = \{[x_0, \ldots, x_n] : x_k \neq 0\}.$$

The map

$$\varphi_k : \mathbb{C}^n \to U_k$$

$$(y_1, \ldots, y_n) \mapsto [y_1, \ldots, \overbrace{1}^{k}, \ldots, y_n]$$

is a homeomorphism onto U_k.

(a) Show that the inverse of φ_k can be described by

$$[x_0, \ldots, x_n] \mapsto \left(\frac{x_0}{x_k}, \ldots, \widehat{\frac{x_k}{x_k}}, \ldots, \frac{x_n}{x_k}\right).$$

This suggests that we define the ring $\mathcal{O}(U_k)$ of holomorphic functions on $\mathcal{O}(U_k)$ to be the ring

$$\mathcal{O}(U_k) := \mathbb{C}\left[\frac{x_0}{x_k}, \ldots, \frac{\widehat{x_k}}{x_k}, \ldots, \frac{x_n}{x_k}\right].$$

(b) The overlap $U_k \cap U_j$ is a copy of $\mathbb{C}^* \times \mathbb{C}^{n-1}$. For simplicity, let's reorder indices so we are considering $U_0 \cap U_1$. The ring associated to U_0 via φ_0 is

$$\mathcal{O}(U_0) = \mathbb{C}\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right].$$

If V_1 denotes the variety inside

$$\operatorname{Spec} \mathbb{C}\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right]$$

determined by the vanishing of $\frac{x_1}{x_0}$, then the complement of V_1 corresponds to the intersection $U_0 \cap U_1$. This means that the ring of the intersection $U_0 \cap U_1$ corresponds, via φ_0, to the ring where we can invert $\frac{x_1}{x_0}$:

$$\mathbb{C}\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right].$$

On the other hand, the intersection also has a ring, via φ_1, that can be identified with

$$\mathbb{C}\left[\frac{x_0}{x_1}, \frac{x_1}{x_0}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right].$$

There is an obvious isomorphism from the first ring to the second described by

$$\begin{aligned} \frac{x_1}{x_0} &\mapsto \frac{x_0}{x_1} \\ \frac{x_k}{x_0} &\mapsto \frac{x_k}{x_1} \qquad\qquad 2 \leqslant k \leqslant n. \end{aligned}$$

Show that this is the isomorphism corresponding to the transition map

$$\varphi_0^{-1} \circ \varphi_1 : U_0 \cap U_1 \to U_0 \cap U_1.$$

2. Let $\tilde{L}$ denote the subset of $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}^{n+1}$ described by

$$\tilde{L} = \{(x, w) : x_i w_j - x_j w_i = 0\}.$$

(a) Show that the action of $\mathbb{C}^*$ on $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}^{n+1}$ described by

$$\lambda \cdot (x, w) = (\lambda \cdot x, w)$$

restricts to an action on $\tilde{L}$.

(b) Let L denote the topological quotient of $\tilde{L}$ by the action of $\mathbb{C}^*$. Show that L is the total space of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^n$.

(c) Let $p_2 : L \to \mathbb{C}^{n+1}$ denote the projection map onto the second factor. The morphism p_2 is often called the blow-up of $\mathbb{C}^{n+1}$ at the origin. Let E denote the preimage of the origin. Show that $E \simeq \mathbb{P}^n$.

(d) Show that E is exceptional in the sense that p_2 is an isomorphism away from E.

3. Let $\varphi : \mathbb{C}^{n+1} \dashrightarrow \mathbb{P}^n$ denote the rational map

$$(x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$$

which is defined on the dense open subset away from the origin. By the graph of φ, we mean the closure of the graph of φ when restricted to the dense open subset on which it is regular. Show that the graph of φ is L, that is, the blow-up of $\mathbb{C}^{n+1}$ at the origin.

4. Let X be the subvariety of $\mathbb{C}[x_1, \ldots, x_n]$ determined by k coprime polynomials $f_1, \ldots, f_k$. Assembling the polynomials together gives a map

$$F : \mathbb{C}^n \to \mathbb{C}^k$$

with components $F = (f_1, \dots, f_k)$. For a point x, the derivative is a linear map

$$dF_x : \mathbb{C}^n \to \mathbb{C}^k.$$

When the point x belongs to X and the map dF_x is surjective, then the tangent space T_xX can be identified with the kernel of dF_x so that we have the exact

$$0 \to T_xX \to \mathbb{C}^n \to \mathbb{C}^k \to 0.$$

In particular, when dF_x is surjective, the tangent space is $(n-k)$-dimensional. In such a case, we say that the point $x \in X$ is nonsingular. On the other hand, the point x is called singular if the linear map dF_x has less than full rank, which is equivalent to a determinantal condition expressed by the simultaneous vanishing of $\binom{n}{k}$ minors of size k-by-k. Writing $\Omega(dF)$ for this collection of $\binom{n}{k}$ new polynomials, the singular set of X can be expressed as the subvariety

$$V(f_1, \dots, f_k, \Omega(dF)).$$

(a) Suppose that $X = V(f)$ is determined by a single polynomial. Show that the set of singular points in X is the subariety

$$V\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

(b) Show that the singular set of the nodal cubic

$$X = \{(x, y) : x^3 + x^2 = y^2\}$$

is the origin.

(c) For the nodal cubic X, there is the usual rational map $\varphi : X \dashrightarrow \mathbb{P}^1$ defined away from the origin by $(x, y) \mapsto [x, y]$. Define the blow-up of X at the origin to be the graph Γ of φ, which is a closed subvariety of $X \times \mathbb{P}^1$. In this way, there is a natural projection map $\pi : \Gamma \to X$ back down to X. Show that π is an isomorphism away from the origin of X.

(d) Show that the preimage of the origin under the projection π consists of two points.

(e) Show that Γ is nonsingular. (Hint: The blow-up is covered by charts, so work with respect to an affine chart, such as $X \times U_0$. In this chart use the coordinate $[1, t]$ on U_0, where $t = x_1/x_0$. Write down all the equations involved, and express the portion of the blow-up Γ in this chart as the variety corresponding to two polynomials. Verify that the derivative of these two polynomials has full rank.)

(f) On the other hand, show that the blow up of the curve $y^2 = x^4 - x^5$ at the origin is still singular (and that, upon blowing up again, the singularity is resolved).

5. Let f denote the homogeneous polynomial

$$f(x_0, \ldots, x_n) = x_0,$$

and let $\tilde{X}$ denote the zero locus of f inside $\mathbb{C}^{n+1}$.

(a) Show that the origin belongs to $\tilde{X}$.

(b) Let $G = \mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by the rule

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda x_0, \cdots, \lambda x_n).$$

Show that this action restricts to one on $\tilde{X}$.

(c) Show that a point $\tilde{x}$ in $\tilde{X}$ determines an orbit that is dimension 1 if and only if $\tilde{x}$ is not the origin. (I think you can look at the stabilizer of $\tilde{x}$.)

(d) Let X denote the moduli space of 1-dimensional orbits of $\tilde{X}$. In other words, X is the topological quotient $(\tilde{X} \setminus 0)/\mathbb{C}^*$. There is a tautological bundle $\mathcal{O}_X(-1)$ over X whose fiber over an orbit is its corresponding closure (which is a line) in $\tilde{X} \subset \mathbb{C}^{n+1}$. Show that $\mathcal{O}_X(-1)$ is the blow-up of $\tilde{X}$ at the origin.

(e) Let x_k denote the usual linear function on $\mathbb{C}^{n+1}$. Show that each x_k determines a global section of the dual bundle $\mathcal{O}_X(1)$.

(f) In particular, $f = x_0$ is a global section of $\mathcal{O}_X(1)$. Show that f is equivalent to zero (along X).

(g) In fact, it is possible to show that we have the exact sequence (of $\mathbb{C}$-modules)

$$0 \to \mathbb{C}f \to \bigoplus_{k=0}^{n} \mathbb{C}x_k \to H^0(X, \mathcal{O}_X(1)) \to 0.$$

Show that this is equivalent to the exact sequence

$$0 \to \mathbb{C} \xrightarrow{e_1} \mathbb{C}^{n+1} \xrightarrow{A} \mathbb{C}^n \to 0$$

where e_1 is the first standard basis vector and A is the matrix

$$A = \begin{bmatrix} \mathbf{0} & I_n \end{bmatrix}.$$

(h) Any point $x \in X$ determines an evaluation map

$$\begin{aligned} \mathrm{ev}_x : H^0(X, \mathcal{O}_X(1)) &\to \mathbb{C} \\ s &\mapsto s(x), \end{aligned}$$

or equivalently, an element of the dual space $H^0(X, \mathcal{O}_X(1))^*$. It follows that we have a map

$$\begin{aligned} \varphi : X &\to H^0(X, \mathcal{O}_X(1))^* \\ x &\mapsto \mathrm{ev}_x. \end{aligned}$$

Provided the image of φ does not intersect zero, we in fact obtain a well-defined map

$$X \to \mathbb{P}(H^0(X, \mathcal{O}_X(1))^*).$$

From the previous part, provided the image of φ does not intersect zero, we in fact have a map

$$X \to \mathbb{P}^{n-1}.$$

Show that this map is equivalent to the map obtained from

$$\begin{aligned} \psi : \tilde{X} \setminus 0 &\to \mathbb{P}^{n-1} \\ (x_0, \ldots, x_n) &\mapsto [x_1, \ldots, x_n]. \end{aligned}$$

In fact, the map $X \to \mathbb{P}^{n-1}$ should be an isomorphism, but I don't really see how to prove this efficiently.

6. Let $G = \mathbb{C}^*$ act on $\mathbb{C}^3 \setminus 0$ by the rule

$$\lambda \cdot (x_0, x_1, x_2) = (\lambda x_0, \lambda^{-1} x_1, x_2).$$

(a) Show that the action descends to one on $\mathbb{P}^2$.

(b) Show that the action on $\mathbb{P}^2$ is not proper.

(c) Let L denote the total space of the line bundle $\mathcal{O}(-1)$, and let $\pi : L \to \mathbb{P}^2$ denote the projection map. Show that there is an action of G on L so that π is G-equivariant:

$$\pi(g \cdot x) = g \cdot \pi(x).$$

(d) Is the action of G on L unique?

(e) Show that an action of G on L determines an action of G on the total space of $\mathcal{O}(1)$.

(f) Show that an action of G on L determines an action of G on the total space of $\mathcal{O}(r)$.

(g) Show that an action of G on the total space of $\mathcal{O}(r)$ determines one on $H^0(\mathbb{P}^2, \mathcal{O}(r))$.

(h) The space $H^0(\mathbb{P}^2, \mathcal{O}(r))$ can be identified with the space of homogeneous polynomials of degree r in the variables x_0, x_1, x_2. So, when graded in the usual way, we have an identification

$$\bigoplus_{r=0}^{\infty} H^0(\mathbb{P}^2, \mathcal{O}(r)) \simeq \mathbb{C}[x_0, x_1, x_2].$$

Show that the induced action of G on this ring can be described by

$$\begin{aligned}\lambda \cdot x_0 &= \lambda x_0\\ \lambda \cdot x_1 &= \lambda^{-1} x_1\\ \lambda \cdot x_2 &= x_2.\end{aligned}$$

(i) Show that there is an identification of the G-invariant piece with

$$\mathbb{C}[x_0 x_1, x_2].$$