Problem Set 1: Kähler preliminaries

The purpose of this problem set is to become acquainted with the statement of Calabi's conjecture.

**1.** Show that $\mathbb{CP}^n$ is a complex manifold.

**2.** Show that any complex manifold $M$ enjoys a natural almost complex structure $J$.

**3.** Show that if $g$ is a Hermitian metric and if $\omega(X,Y) = g(JX,Y)$, then $\omega$ is antisymmetric and is a real 2-form of type $(1,1)$.

**4.** Show that if $g$ is a Hermitian metric and if

$$g_{j\bar{k}} = g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right),$$

then $g_{\bar{k}j} = g_{j\bar{k}}$ and $g_{jk} = g_{\bar{j}\bar{k}} = 0$. Hence $g$ enjoys the local expression

$$g = g_{j\bar{k}}(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j).$$

**5.** Show that $g$ is Kähler if and only if $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$ for each $i, j, k$.

**6.** Find a local expression for the Fubini-Study metric $\omega_{FS}$ on the chart $U_0$ of $\mathbb{CP}^n$. Here $U_0$ is the subset $U_0 = \{[Z_0 : \ldots : Z_n] : Z_0 \neq 0\}$. Verify that at the point $[1 : 0 : \ldots : 0]$, the Fubini-Study metric is positive definite.

**7.** Understand the statement of the $\partial\bar{\partial}$-lemma and its proof. Do also Exercise 1.15 in Gabor's book.

**8.** Exercise 1.23 of Gabor's book.

**9.** Understand the statement of the Calabi-Yau Theorem (Theorem 1.24 of Gabor's book).