Take-home portion of final exam
Intro to Modern Analysis

1. Is the set of irrational numbers dense in $\mathbb{R}$?

2. For a real number $p > 1$, define a sequence a_n recursively by

$$\begin{aligned} a_0 &= 0 \\ a_{2n+1} &= a_{2n} + \frac{p-1}{p} && n \in \mathbb{N} \\ a_{2n+2} &= \frac{a_{2n+1}}{p} && n \in \mathbb{N}. \end{aligned}$$

Compute

$$\liminf_{n\to\infty} a_n \qquad \text{and} \qquad \limsup_{n\to\infty} a_n$$

whenever either exists.

3. Let a_n be a sequence of real numbers. Suppose that $\sum_n |a_{n+1} - a_n|$ converges. Must a_n converge?

4. Suppose E is a bounded subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. Must f be bounded? (I believe you might be able to Google this one.)

5. Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function. Suppose that

(a) f is bounded from above

(b) $f'(x) \geqslant 0$ for each $x \in [0, \infty)$.

Must there be a sequence of points $t_n \to \infty$ such that $f'(t_n) \to 0$?

6. For each positive integer n, let $\delta_n(x)$ be the function defined on $[-1, 1]$ by

$$\delta_n(x) = \begin{cases} n/2 & x \in [-1/n, 1/n] \\ 0 & \text{otherwise} \end{cases}.$$

Show that for any continuous function $f : [-1, 1] \to \mathbb{R}$, we have

$$f(0) = \lim_{n\to\infty} \int_{-1}^{1} \delta_n(x) f(x)\, dx.$$

Give an example of an integrable function $g : [-1, 1] \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{-1}^{1} \delta_n(x) g(x)\, dx$$

exists but is not equal to $g(0)$.