Topic: The fundamental group

We explain how a point p in a topological space X gives rise to a group of homotopy classes of loops based at p, which is called the fundamental group $\pi_1(X, p)$.

0.1 Groups

A **binary operation** on a set S is simply a map $S \times S \to S$. For example, addition and multiplication are both binary operations on the integers $\mathbb{Z}$. Addition is not a binary operation on the set of odd integers because the sum of two odd numbers is not odd.

A **group** consists of the following data

- a set G
- a binary operation $G \times G \to G$ usually written $(g_1, g_2) \mapsto g_1 g_2$

satisfying the following requirements

(i) *Existence of identity*: There is an element $e \in G$ called an **identity element** such that $eg = ge = g$ for each $g \in G$.

(ii) *Existence of inverses*: For each element $g \in G$, there is an element $h \in G$ called an **inverse of** g satisfying $gh = hg = e$.

(iii) *Associativity:* For each $g, h, k \in G$, we have $g(hk) = (gh)k$.

We provide now several examples. We let the reader check through all the details. We encourage the reader to think carefully about how the axioms apply in each example, especially when indicated.

(a) The group $(\mathbb{Z}, +)$ of integers under the operation of addition.

(b) The group $(\mathbb{R}, +)$ of real numbers under the operation of addition.

(c) The group $(\mathbb{R}^\times, \cdot)$ of nonzero real numbers under the operation of multiplication.

(d) The group $(GL(n, \mathbb{R}), \cdot)$ of invertible n-by-n matrices under the operation of matrix multiplication.

(e) Let S_n denote the set of all bijections from $\underline{n} = \{1, \ldots, n\}$ to itself. Then S_n together with the operation of composition forms a group, called the **symmetric group on** n **letters**. What is the identity element?

(f) For a set X, the collection of all bijections $X \to X$ forms a group under the operator of composition called the **group of permutations of** X, denoted $\mathrm{Perm}(X)$. In particular, when $X = \underline{n} = \{1, \ldots, n\}$, we have $\mathrm{Perm}(X) = S_n$.

(g) A vector space V is a group under the operation of addition of vectors.

Exercise 0.1. For a group G, the following hold.

(i) The identity element $e \in G$ is unique. This means that if e, e' are elements of G such that $eg = ge = g$ and $e'g = ge' = g$ for each $g \in G$, then $e = e'$.

(ii) For each $g \in G$, the inverse of g in G is unique. This means that if h, h' are elements of G such that $gh = hg = e$ and $gh' = h'g = e$, then $h = h'$.

Hence, by the proposition, we are justified in saying *the* identity of the group G, and *the* inverse to an element $g \in G$. We denote the inverse of $g \in G$ by g^{-1}.

For groups G, H, a **homomorphism from** G **into** H consists of a map $\varphi : G \to H$ satisfying the following properties.

(a) $\varphi(gh) = \varphi(g)\varphi(h)$ for each $g, h \in G$

(b) $\varphi(e_G) = e_H$

(c) $\varphi(g)^{-1} = \varphi(g^{-1})$ for each $g \in G$.

That is, φ is a map from G into H preserving all of the group structure. In fact, for φ to satisfy all these requirements, it is sufficient only for (a) to hold.

Exercise 0.2. If $\varphi : G \to H$ is a map such that $\varphi(gh) = \varphi(g)\varphi(h)$ for each $g, h \in G$, then φ is a group homomorphism.

We say that a homomorphism $\varphi : G \to H$ of groups is an **isomorphism** if there is a homomorphism of groups $\psi : H \to G$ such that ψ is the inverse of φ, meaning that $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$.

Exercise 0.3. (i) Check that if $f : X \to Y$ is an injective map, then f admits a left-inverse, that is a map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Verify that the converse is true as well.

(ii) Check that if $f : X \to Y$ is a surjective map, then f admits a right-inverse, that is a map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. Verify that the converse is true as well.

(iii) Conclude that $f : X \to Y$ is bijective if and only if f admits an inverse.

Exercise 0.4. Show that a homomorphism of groups $\varphi : G \to H$ is an isomorphism if and only if φ is bijective.

Hint. The forward direction is clear by the previous exercise. Conversely, if φ is bijective, then φ admits an inverse $\psi : H \to G$ by the previous exercise. It remains to show that ψ is a homomorphism of groups, which is routine.

A **subgroup** H of a group G consists of a subset $H \subset G$ satisfying the following conditions.

(i)' The identity e belongs to H.

(ii)' *Closed under the group operation:* For each $h_1, h_2 \in H$, the product $h_1 h_2$ belongs to H.

(iii)' *Closed under taking inverses:* For each $h \in H$, the inverse h^{-1} belongs to H.

Equivalently, we say that a subset H of a group G is a subgroup of G if H is a group in its own right when endowed with the group operation coming from G.

Exercise 0.5. (i) For a group G, check that $\{e\}$ is a subgroup, called the **trivial subgroup**.

(ii) Check that the set of even integers forms a subgroup of the set of integers under addition.

(iii) Check that the set of invertible n-by-n matrices with determinant 1 forms a subgroup of the group of all invertible n-by-n matrices. This group is called the **special linear group**, denoted $SL(n, \mathbb{R})$.

(iv) Check that the

Exercise 0.6. Check that the intersection $\bigcap_\alpha H_\alpha$ of any collection $\{H_\alpha\}$ of subgroups of G is a subgroup of G.

By the previous exercise, for a subset $S \subset G$, there is a smallest subgroup of G containing S, which we denote by $\langle S \rangle$, and call **the subgroup generated by** S. It can be defined by the rule

$$\langle S \rangle = \bigcap_{\substack{H \leq G \\ H \supset S}} H,$$

that is, the intersection of all subgroups containing S. This is indeed the smallest subgroup of G containing S for the following reason. If H is any subgroup containing S, then $\langle S \rangle$ is contained in H by the definition of $\langle S \rangle$.

A group homomorphism $\varphi : G \to H$ determines a subgroup of G called the **kernel of** φ, denoted $\ker \varphi$, given by $\ker \varphi = \{g \in G : \varphi(g) = e\}$. This is indeed a subgroup of G, as we let the reader verify.

Exercise 0.7. Verify that the kernel of a group homomorphism $\varphi : G \to H$ is a subgroup of G.

A group homomorphism $\varphi : G \to H$ also determines a subgroup of H called the **image of** φ, denoted $\mathrm{im}\varphi$ or $\varphi(G)$, given by $\mathrm{im}\varphi = \{\varphi(g) : g \in G\}$. We let the reader check that this is a subgroup of H.

Exercise 0.8. Verify that the image of a group homomorphism $\varphi : G \to H$ is a subgroup of H.

Exercise 0.9. Verify that if $\varphi : G \to H$ is an injective homomorphism of groups, then the image of φ is isomorphic to G.

Hint. An injective map has a left-inverse, that is, a map $\psi : H \to G$ satisfying $\psi \circ \varphi = \mathrm{id}_G$. We may restrict ψ to the image of φ to get a map $\psi|_{\varphi(G)} : \varphi(G) \to G$. Check that this restricted $\psi|_{\varphi(G)}$ defines a homomorphism of groups and that this restricted $\psi|_{\varphi(G)}$ is also a right-inverse for φ, meaning that $\varphi \circ \psi|_{\varphi(G)} = \mathrm{id}_{\varphi(G)}$. It follows that G is isomorphic to $\varphi(G)$.

We say that a group G is **abelian** (or **commutative**) if $gh = hg$ for each $g, h \in G$. For example, the group of integers under addition is abelian. As another example, a vector space is an abelian group. However, for $n \geq 3$, the group S_n is not abelian, as we let the reader check now.

Exercise 0.10. Verify that S_n is not abelian if $n \geq 3$.

Hint: First prove the case $n = 3$. Assume the claim is true for n. Construct an injective group homomorphism $\varphi_n : S_n \to S_{n+1}$. Show that if S_n is not abelian, then neither is S_{n+1}. Conclude the proof by induction on n.

For a group G, we use the term **order** to mean the cardinality $|G|$ of the set G. By a **finite group of order** n we mean a group G with finite order $|G| = n$.

Every group of finite order can be realized as a subgroup of the symmetric group S_n for n large enough. In particular, we have the following theorem.

Theorem 0.11. *Every group G of order n is isomorphic to a subgroup of S_n.*

Proof. Consider the map $\varphi : G \to \mathrm{Perm}(G)$ defined by assigning to each $g \in G$, the map $\varphi_g(h) = gh$ given by left-multiplication by g. By the above exercise, we see that φ is a group homomorphism. We in fact claim that φ is injective, which will prove the theorem, because $\mathrm{Perm}(G)$ is clearly isomorphic to S_n.

Suppose that g_1 and g_2 map to the same element of $\mathrm{Perm}(G)$. Then for each $h \in G$, we have the equality $g_1 h = g_2 h$. It follows that we have the chain of equalities

$$\begin{aligned} g_1 &= g_1 e \\ &= g_1(hh^{-1}) \\ &= (g_1 h)h^{-1} \\ &= (g_2 h)h^{-1} \\ &= g_2(hh^{-1}) \\ &= g_2 e \\ &= g_2. \end{aligned}$$

Therefore, the map φ is injective. □

Exercise 0.12. Verify that that is only one group of order two up to isomorphism.

0.2 The fundamental group

For two points $p, q \in X$, a **path** from p to q is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. We say that X is **path-connected** if any two points in X may be connected by a path in X. If $\gamma : [0,1] \to X$ is a path from p to q, then there is a path $\bar{\gamma} : [0,1] \to X$ from q to p called the **reverse** of γ defined by $\bar{\gamma}(t) = \gamma(1-t)$. (Why is $\bar{\gamma}$ continuous?)

Let γ and τ be two paths from p to q in X. A **homotopy from γ to τ with endpoints fixed** is a continuous map $H : [0,1] \times [0,1] \to X$ such that

For a point $p \in X$, a **loop based at** p is a path from p to p, that is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = p$. For a loop γ based at p, we let $[\gamma]$ denote its homotopy class. We let $\pi_1(X, p)$ denote the collection of all equivalence classes of loops based at p.

We claim that $\pi_1(X, p)$ can be endowed with the structure of a group.