MATH 2610-01
Take-Home #3
Due on Friday, April 15

Chapter 6 (reprise)

1. Consider the constant-coefficient homogeneous equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1}$$

as in Chapter 6.

(a) Find a constant matrix A such that the equation becomes equivalent to a *system* of n equations

$$\mathbf{x}' = A\mathbf{x}$$

upon using the substitutions

$$x_k(t) = y^{(k-1)}(t) \qquad 1 \leqslant k \leqslant n.$$

(b) Show that the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

(Hint: Use induction on n.)

(c) If r is a root of $p_A(\lambda)$, show that the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^{n-1} \end{bmatrix}$$

is an eigenvector for A with eigenvalue r.

(d) Use the discussion at the beginning of Section 9.5 to deduce that, if r is a root of $p_A(\lambda)$, then $y(t) = e^{rt}$ solves equation (1).

2. Let D denote the linear operator $D = \frac{d}{dt}$. For a real number r, let V denote the vector space spanned by the list of functions

$$y_k(t) = t^k e^{rt} \qquad 0 \leqslant k < m.$$

Note that V has dimension m. We may restrict D to an operator on V.

(a) According to Exercise 25 of Section 6.2, the list $\{y_0, y_1, \ldots, y_{m-1}\}$ is a basis for V. Compute the matrix of D with respect to this basis.

(b) Find the characteristic polynomial of D on V.

(c) Show that r is the only eigenvalue of D on V.

(d) The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial. What is the algebraic multiplicity of r?

(e) The *geometric multiplicity* of an eigenvalue is the dimension of the eigenspace associated to it. What is the geometric multiplicity of r?

(f) Is D diagonalizable?

(g) The generalized eigenspace associated to r is the space

$$\ker(D - rI)^a$$

where a denote the algebraic multiplicity of r. In general, the dimension of this space is equal to a. Verify this for our situation. (Hint: The Cayley-Hamilton Theorem says that for an n-by-n matrix A with characteristic polynomial $p_A(\lambda)$, the matrix $p_A(A)$ is the zero matrix.)

3. Let $D = \frac{d}{dt}$. Set

$$a_1(t) = \cos t, \qquad a_2(t) = t\cos t$$
$$b_1(t) = \sin t, \qquad b_2(t) = t\sin t.$$

Let V denote the (real) span of a_1, b_1, a_2, b_2. We may restrict D to an operator on V.

(a) Show that V has (real) dimension 4.

(b) Compute the matrix of D with respect to the basis $\{a_1, b_1, a_2, b_2\}$.

(c) Compute the characteristic polynomial of D.

(d) Is D diagonalizable?

(e) Now let $V_{\mathbb{C}}$ denote vector space over $\mathbb{C}$ given by the complex span of a_1, b_1, a_2, b_2. Let $D_{\mathbb{C}}$ denote the restriction of $\frac{d}{dt}$ to $V_{\mathbb{C}}$. (The matrix of $D_{\mathbb{C}}$ is the same as the matrix of D.) Find all eigenvalues of $D_{\mathbb{C}}$ along with their algebraic and geometric multiplicities.

(f) Is $D_{\mathbb{C}}$ diagonalizable?

Chapter 9

4. Let A be a diagonalizable matrix with real entires, meaning that A is of the form

$$A = PDP^{-1}$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is diagonal and the columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of P are eigenvectors for A with eigenvalues $d_1, \ldots, d_n$.

(a) For a function $\mathbf{x}(t)$, write $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$. Show that $\mathbf{x}(t)$ solves $\mathbf{x}' = A\mathbf{x}$ if and only if $\mathbf{y}(t)$ solves $\mathbf{y}' = D\mathbf{y}$.

(b) Describe a general solution to $\mathbf{y}' = D\mathbf{y}$.

(c) Use (a) to show that a general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{d_1 t}\mathbf{u}_1 + \cdots + c_n e^{d_n t}\mathbf{u}_n.$$

(d) Use (c) to describe a general solution to $\mathbf{x}' = A\mathbf{x}$ when

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}.$$

5. Let A denote an n-by-n matrix with real entries. The goal of this problem is to describe a general solution to the system

$$t\mathbf{x}'(t) = A\mathbf{x}(t) \qquad t > 0, \tag{2}$$

if possible. (Note the similarity between this system and the Cauchy-Euler equations studied in Chapter 4.)

(a) If r is an eigenvalue of A with eigenvector $\mathbf{u}$, show that $\mathbf{x}(t) = t^r\mathbf{u}$ solves (2).

(b) Suppose A has n linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ with eigenvalues $r_1, \ldots, r_n$. Describe a general solution to (2) and prove that your description is indeed a general solution.

(c) We have seen how to define the matrix e^A when the base number is the number e. Construct a suitable definition of t^A when t is any positive number. (Hint: Recall that $t^a = e^{a \ln t}$.)

(d) With your definition from (c), show that

$$\frac{d}{dt}\left(t^A\right) = t^{-1}At^A$$

and also that

$$(t^A)^{-1} = t^{-A}.$$

(e) If $\mathbf{c} \in \mathbb{R}^n$ is any vector, use (d) to show that

$$\mathbf{x}(t) = t^A\mathbf{c}$$

solves (2).

(f) If $\mathbf{u}$ is a generalized eigenvector for A satisfying

$$(A - rI)^m\mathbf{u} = \mathbf{0},$$

show that

$$t^A\mathbf{u} = t^r\left[\mathbf{u} + \ln(t)(A - rI)\mathbf{u} + \cdots + \frac{\ln(t)^{m-1}}{(m-1)!}(A - rI)^{m-1}\mathbf{u}\right].$$

(g) Use (f) to describe a general solution to (2) when A is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(h) Part (e) says that t^A is a fundamental matrix for (2). Show that if $\mathfrak{X}(t)$ is *any* fundamental matrix for (2), then

$$t^A = \mathfrak{X}(t)\mathfrak{X}(1)^{-1}.$$

(i) Use the method of variation of parameters to describe an explicit *particular* solution to

$$\mathbf{x}'(t) = t^{-1}A\mathbf{x}(t) + \mathbf{f}(t), \qquad t > 0$$

in terms of matrix products involving t^A and indefinite integrals. (Hint: Section 9.7 of the textbook.)

(j) Use the method of variation of parameters to describe an explicit *general* solution to

$$\mathbf{x}'(t) = t^{-1}A\mathbf{x}(t) + \mathbf{f}(t), \qquad t > 0$$

in terms of matrix products involving t^A and indefinite integrals.

(k) Use the method of variation of parameters to describe an explicit solution to the initial value problem

$$\begin{cases} \mathbf{x}'(t) = t^{-1}A\mathbf{x}(t) + \mathbf{f}(t), & t > 0 \\ \mathbf{x}(1) = \mathbf{x}_1 \end{cases} \tag{3}$$

in terms of matrix products involving t^A and definite integrals. (Hint: Find an appropriate analogy of equation (16) from section 9.8.)

(l) Solve the initial value problem (3) when

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t^{-1} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

6. Let D denote the differential operator $D = d/dt$.

(a) Suppose that A, B are n-by-n real matrices that commute.

i. For any constant vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, show that the function

$$\mathbf{x}(t) = e^{At}\mathbf{a} + e^{Bt}\mathbf{b}$$

solves the system

$$(D - A)(D - B)\mathbf{x} = \mathbf{0}. \tag{4}$$

(Hint: Make sure you state clearly where you use that A, B commute. Also, everything here is matrix multiplication, so you cannot commute certain symbols past one another: For example, the product $B\mathbf{a}$ makes sense, but the product $\mathbf{a}B$ does not. As another example, the product DA is operator composition, so it may or may not be equal to AD; equality is something you would have to check carefully if you want to use it.)

ii. Now if $A = B$, show that the function

$$\mathbf{x}(t) = te^{At}\mathbf{a} + e^{At}\mathbf{b}$$

solves the system

$$(D - A)^2\mathbf{x} = \mathbf{0}. \tag{5}$$

(b) Suppose now that A, B are simultaneously diagonalizable, that is, suppose we can write

$$A = PD_AP^{-1} \quad \text{and} \quad B = PD_BP^{-1}$$

where D_A, D_B are diagonal.

i. Show that A and B commute.

ii. If we write $D_A = \operatorname{diag}(a_1, \ldots, a_n), D_B = \operatorname{diag}(b_1, \ldots, b_n)$, and the columns of P as $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then show that each function of the form

$$\mathbf{a}_k(t) = e^{a_kt}\mathbf{u}_k \quad \text{and} \quad \mathbf{b}_k(t) = e^{b_kt}\mathbf{u}_k \quad \text{for } k = 1, \ldots, n$$

solves the system (4).

iii. In the case that $A = B$, show that each function of the form

$$\mathbf{a}_k(t) = te^{a_kt}\mathbf{u}_k \quad \text{and} \quad \mathbf{b}_k(t) = e^{a_kt}\mathbf{u}_k \quad \text{for } k = 1, \ldots, n$$

solves the system (5).

Chapter 12

7. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -3x - y - 5 \\ \frac{dy}{dt} &= 13x + 3y + 23 \end{aligned}$$

(a) Find and classify all critical point(s).

(b) Sketch a phase plane diagram.

(c) Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that if any solution $(x(t), y(t))$ to the system satisfies

$$\sqrt{(x(0) + 2)^2 + (y(0) - 1)^2} < \delta$$

at $t = 0$, then it also satisfies

$$\sqrt{(x(t) + 2)^2 + (y(t) - 1)^2} < \epsilon$$

for all $t \geqslant 0$. (Hint: Think of ϵ as the semimajor axis of a certain ellipse.)