MATH S1202: Calculus IV
Final take-home portion
Due: June 28
50 points

1. Let a, b be two positive numbers, and let m, n be any real numbers satisfying $m \neq n$. Give an expression of the area enclosed by the ellipse

$$\frac{(x+my)^2}{a^2} + \frac{(x+ny)^2}{b^2} = 1$$

in terms of the variables a, b, m, n.

2. Compute the double integral

$$\iint_T \left|y^2 - x^2\right| \, dA$$

where T is triangle with vertices $(-1,1), (1,1)$, and $(1,-1)$.

3. Let S be a sphere of radius a and let D be a spherical rectangle on S, meaning that D is of the form

$$D = \{(\rho, \varphi, \theta) : \rho = a, \varphi_1 \leqslant \varphi \leqslant \varphi_2, \theta_1 \leqslant \theta \leqslant \theta_2\}$$

for some fixed angles φ_1, φ_2 and θ_1, θ_2.

(i) Give an expression for the area of D in terms of $a, \varphi_1, \varphi_2, \theta_1, \theta_2$.

(ii) The boundary of D consists of four arcs. Let C_u be the upper (northernmost) arc and C_ℓ be the lower (southernmost) arc. Give expressions for C_u and C_ℓ in terms of spherical coordinates.

(iii) Compute the lengths of C_u and C_ℓ in terms of $a, \varphi_1, \varphi_2, \theta_1, \theta_2$.

(iv) Assume that Colorado is such a spherical rectangle D. Look up values for $a, \varphi_1, \varphi_2, \theta_1, \theta_2$ based on actual physical data (identifying the physical meaning of each constant), and compute the difference in the lengths of C_u and C_ℓ for Colorado. Write your answer in meters. (On a map, it is easy to forget that the lengths of C_u and C_ℓ are different.)

4. Let a, b, c be fixed positive real numbers. Find the minimum volume bounded by the planes $x = 0, y = 0, z = 0$, and a plane which is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the first octant $x > 0, y > 0, z > 0$.

5. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $x^2 + y^2 = ay$, where $a > 0$.