Picard's Existence and Uniqueness Theorem

A SELF-CONTAINED PROOF

Preface

These notes were written during the spring of 2021 and edited during the spring of 2022. They arose out of teaching a course on ordinary differential equations at Vanderbilt University.

The goal of these notes is to provide a (fairly) complete proof of an existence and uniqueness result for first order equations often called *Picard's Theorem*. The approach is closely modeled on the approach presented in the final chapter of the book by Nagle, Saff, and Snider, but more concepts/techniques from analysis are supplied in order to make the approach as self-contained as possible.

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1 Introduction

The goal of these notes is to provide a (fairly) complete proof of an existence and uniqueness result for first order equations often called *Picard's Theorem.*

Precisely, we will consider an initial value problem (IVP) of the form

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

where f is a given function and (x_0, y_0) is a given initial condition. We would like to understand what conditions on f guarantee the existence (and uniqueness) of a solution $y(x)$ defined near x_0.

Recall that, at the beginning of the semester, we described such conditions on f. In fact, we stated (and often exploited) the following result, which is actually one version of *Picard's Theorem.*

Theorem 1 (Picard's Theorem)**.** *Suppose f and its partial derivative f_y are continuous on a closed rectangle containing the initial point (x_0, y_0). Then there is a $\delta > 0$ such that the IVP* (1) *admits a solution $\varphi(x)$ defined on $(x_0 - \delta, x_0 + \delta)$ and moreover any solution to the IVP defined on that same interval agrees with φ.*

Our specific goal, then, is to provide a detailed proof of this.

Remark 2. It is possible to weaken the hypotheses on f to require only that f is "uniformly Lipschitz continuous in y" and continuous in x. Our primary goal is not to prove such a general statement. However, if time permits, we will discuss how to weaken the hypotheses in this way.

Remark 3. It is important to note that this result is an implication; it says nothing about a converse. We have seen that, in certain situations, it is reasonable to expect a type of converse when f or its partial derivative fail to be continuous, that is, it is reasonable to expect the existence of multiple solutions (or no solution at all!) when the hypotheses on f are not met. However, because we have no general result of this type, such "converses" must be dealt with on a case-to-case basis with explicit counterexamples and reasoning particular to the IVP in question.

2 Restatement as a "fixed point" problem

The first step in the proof is restating the initial value problem.

Precisely, define an operator T in the following way. Given a function $y(x)$, the operator T determines a new function $T[y](x)$ defined by the rule

$$T[y](x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt, \tag{2}$$

where f and (x_0, y_0) are given in the IVP (1). Unlike other operators defined on spaces of functions we've seen in this course, the operator T is, in general, *not linear.*

Example 4. As an example, suppose the IVP is

$$\begin{cases} y'(x) = 2y(x) \\ y(0) = 1 \end{cases}.$$

In this situation, we have

$$f(x, y) = 2y$$

and $(x_0, y_0) = (0, 1)$. The operator T is defined by the rule

$$T[y](x) = 1 + \int_0^x 2y(t)\, dt.$$

For example, if $y(x) = 1 + 2x$, then

$$T[y](x) = 1 + \int_0^x 2(1 + 2t)\, dt = 1 + 2x + \frac{(2x)^2}{2}.$$

As another example, if $y(x) = e^{2x}$, then

$$T[y](x) = 1 + \int_0^x 2e^{2t} dt = 1 + (e^{2x} - 1) = e^{2x} = y(x).$$

So the function $y(x) = e^{2x}$ is fixed by T (because $T[y] = y$).

The utility of T is that it characterizes solutions to the IVP (1) because such solutions are exactly the "fixed points" of T, as the following result asserts.

Lemma 5. *A function $y(x)$ solves the IVP (1) if and only if $T[y] = y$.*

Proof. Suppose $y(x)$ solves the IVP. Then compute

$$\begin{aligned} T[y](x) &= y_0 + \int_{x_0}^x f(t, y(t))\, dt && \text{definition of } T \\ &= y(x_0) + \int_{x_0}^x y'(t)\, dt && y \text{ solves the IVP} \\ &= y(x) && \text{fundamental theorem of Calculus.} \end{aligned}$$

This means that $T[y] = y$.

Conversely, suppose that $T[y] = y$. Then compute

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(T[y](x)\right) && T[y] = y \\ &= \frac{d}{dx}\left(y_0 + \int_{x_0}^x f(t, y(t))\, dt\right) && \text{definition of } T \\ &= f(x, y(x)) && \text{fundamental theorem of Calculus.} \end{aligned}$$

In addition, we have

$$\begin{aligned} y(x_0) &= T[y](x_0) \\ &= y_0 + \int_{x_0}^{x_0} f(t, y(t))\, dt \\ &= y_0. \end{aligned}$$

Therefore, y solves the IVP (1). $\square$

Therefore, to solve the IVP (1), it suffices to find the "fixed points" of T.

To gain insight into this procedure of finding fixed points, let us consider a simpler example of a fixed point problem.

Example 6. Find a real number x that solves the equation

$$x = \frac{x}{2} + \frac{1}{x}.$$

Of course, elementary algebraic techniques (such as the quadratic formula) can be used to find directly all of the solutions to this equation, but let us describe another approach, which goes by the name of *successive approximations.*

The idea behind this alternate approach is that solving the equation is equivalent to finding a fixed point of the function

$$g(x) = \frac{x}{2} + \frac{1}{x}.$$

To be precise, by a *fixed point of g* we mean a point x such that $g(x) = x$.

The method of successive approximations describes a procedure that can, under suitable conditions, lead to finding a fixed point.

This method starts by selecting an initial guess x_0 for a fixed point. To be concrete, let us suppose we guess $x_0 = 1$ is a fixed point of g. We can check that in fact $x_0 = 1$ is not fixed by g because

$$g(x_0) = \frac{x_0}{2} + \frac{1}{x_0} = \frac{3}{2}.$$

Nevertheless, the guess is a good starting point, because $g(x_0) = 3/2$ only differs from $x_0 = 1$ by 0.5 units.

To get a better guess (provided the map g satisfies certain hypotheses), the method of successive approximations asserts that our next guess should be the output of our previous guess. Namely, our next guess x_1 should be $x_1 = g(x_0)$, or, $x_1 = 3/2$. This next guess is still not a fixed point of g because

$$g(x_1) = \frac{x_1}{2} + \frac{1}{x_1} = \frac{17}{12},$$

but it is closer to a fixed point than x_0 because $g(x_1) = 17/12$ only differs from $x_1 = 3/2$ by $1/12$.

The method of successive approximations is therefore an iterative or recursive procedure. Starting with an initial guess x_0, the method says that the next guess should always be obtained by applying g. In other words, starting with an initial guess x_0, we obtain a sequence

$$x_{n+1} := g(x_n) \qquad n = 0, 1, \ldots$$

When suitable conditions are met on g, this sequence will converge to a limit

$$L = \lim_{n\to\infty} x_n,$$

which turns out to be a fixed point of g. Indeed, for such a limit L, we have, provided g is continuous, that

$$\begin{aligned} g(L) &= g\left(\lim_{n\to\infty} x_n\right) && \text{definition of } L \\ &= \lim_{n\to\infty} g(x_n) && g \text{ is continuous} \\ &= \lim_{n\to\infty} x_{n+1} && \text{definition of } x_{n+1} \\ &= L. \end{aligned}$$

Let us investigate what the limit L is for

$$g(x) = \frac{x}{2} + \frac{1}{x}$$

and our initial guess $x_0 = 1$. Because L is a fixed point, we must have

$$L = g(L) = \frac{L}{2} + \frac{1}{L}.$$

Then using elementary algebraic techniques as described earlier, we find that L satisfies

$$2L^2 = L^2 + 2$$

or L must be $\pm\sqrt{2}$. Because we started with an initial guess that is positive ($x_0 = 1$) and g maps positive numbers to positive numbers, we conclude that for such an initial guess, the fixed point that our sequence x_n approaches must be $\sqrt{2}$.

The point of the method of successive approximations is that, provided suitable hypotheses are met on g, we know the sequence x_n will converge to a fixed point L, without having to find or determine L explicitly by any formula.

3 Limits and convergence

Throughout this section, V will denote a vector space over $\mathbb{R}$, which need not be finite-dimensional.

Definition 7. A **norm** on V is a function

$$\|\cdot\| : V \to \mathbb{R}$$

satisfying the following properties

1. $\|v\| \geqslant 0$ for each $v \in V$, with equality if and only if $v = 0$
2. $\|cv\| = |c| \, \|v\|$ for each $c \in \mathbb{R}, v \in V$
3. $\|v + w\| \leqslant \|v\| + \|w\|$ for each $v, w \in V$.

The last property is called the **triangle inequality** and can be interpreted as saying that the length of the diagonal of the parallelogram formed by the vectors v and w is at most the sum of the lengths of the sides of the parallelogram.

Example 8. On $\mathbb{R}^n$, we have the standard Euclidean norm

$$\|v\| = \sqrt{\sum_{k=1}^{n} v_k^2}$$

for $v = (v_1, \ldots, v_n) \in \mathbb{R}$. When $n = 1$, this norm specializes to the absolute value.

Example 9. Let $C[a, b]$ denote the vector space of continuous real-valued functions defined on a closed interval $[a, b]$. In other words, a vector in $C[a, b]$ is a continuous function $y : [a, b] \to \mathbb{R}$. This is indeed a vector space when equipped with the usual operations of function addition and scalar multiplication for functions.

Define a norm on $C[a, b]$ by declaring

$$\|y\| = \max_{x \in [a,b]} |y(x)|.$$

In other words, the number $\|y\|$ is the maximum vertical distance between the graph of y and the x-axis. This norm on $C[a, b]$ is often called the **supremum norm** or the **sup-norm** for short.

To check that this indeed defines a norm, we will verify all three properties in the definition.

1. Because $|y(x)|$ is nonnegative, the norm is nonnegative. Moreover, if $\|y\| = 0$, then this implies that

   $$|y(x)| = 0 \qquad \text{for each } x \in [a, b]$$

   and so y is the zero function.

2. If c is a real number, then $|cy(x)| = |c||y(x)|$ and so

$$\|cy\| = |c| \, \|y\| \, .$$

3. For any two functions y, z and any point $x \in [a, b]$, the triangle inequality on $\mathbb{R}$ implies

$$|y(x) + z(x)| \leqslant |y(x)| + |z(x)|.$$

By definition of the sup-norm, we know $|y(x)| \leqslant \|y\|$. As a result, we obtain the inequality

$$|y(x) + z(x)| \leqslant \|y\| + \|z\| \, .$$

Taking the maximum of both sides gives

$$\|y + z\| \leqslant \|y\| + \|z\| \, .$$

Example 10. More generally, let V be any vector space equipped with a norm $\|\cdot\|_V$, and let X be any set. Let W denote the set of maps $y : X \to V$ such that the number

$$\sup_{x \in X} \|y(x)\|_V$$

is finite. Then W "inherits" a sup-norm from V defined by

$$\|y\|_W = \sup_{x \in X} \|y(x)\|_V \qquad y \in W.$$

As a special case, when $V = \mathbb{R}$ with the Euclidean norm and $X = [a, b]$, then the normed space W contains the space $C[a, b]$ as a subset:

$$C[a, b] \subset W.$$

This is a consequence of the extreme value theorem.

Definition 11. Let v_n be a sequence of vectors in a normed space V, and let v be another vector. Say that v_n **converges to** v to mean for each $\epsilon > 0$, there is an $N > 0$ such that

$$n \geqslant N \implies \|v_n - v\| < \epsilon.$$

Lemma 12. *The limit of a convergent sequence is unique.*

Proof. Suppose that v_n converges to v and $\tilde{v}$. Let $\epsilon > 0$ be arbitrary. There is an $N > 0$ such that

$$n \geqslant N \implies \|v_n - v\| < \epsilon/2 \quad \text{and} \quad \|v_n - \tilde{v}\| < \epsilon/2.$$

Then by the triangle inequality, we have for such $n \geqslant N$ that

$$\|v - \tilde{v}\| = \|v - v_n + v_n - \tilde{v}\| \leqslant \|v - v_n\| + \|v_n - \tilde{v}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

In other words, we have

$$\|v - \tilde{v}\| < \epsilon.$$

Since ϵ is arbitrary, we find that $\|v - \tilde{v}\| = 0$. By the third property of a norm, we conclude that $v = \tilde{v}$. □

Example 13. Let x_n be the sequence of points $x_n = 1/n$ in $\mathbb{R}$. Then we claim that x_n converges to the point $x = 0$.

Indeed, let $\epsilon > 0$ be given. Choose a positive integer N satisfying $N > 1/\epsilon$. Then for $n \geqslant N$, we have

$$\|x_n - x\| = |\frac{1}{n} - 0| = \frac{1}{n} \leqslant \frac{1}{N} < \epsilon.$$

Example 14. Let $y_n \in C[0,1]$ be the sequence of functions defined by

$$y_n(x) = x^2 + \frac{x}{n}.$$

Then we claim that the sequence converges to the function y defined by

$$y(x) = x^2.$$

Indeed, let $\epsilon > 0$ be given. Choose a positive integer N satisfying $N > 1/\epsilon$. Then for $n \geqslant N$, we have

$$\begin{aligned} \|y_n - y\| &= \max_{x \in [0,1]} |y_n(x) - y(x)| \\ &= \max_{x \in [0,1]} \left| \frac{x}{n} \right| \\ &= \frac{1}{n} \leqslant \frac{1}{N} < \epsilon. \end{aligned}$$

Convergence in the space $C[a,b]$ with respect to the sup-norm is often called "uniform" convergence because for a given ϵ, the choice of N does not depend on x or is *uniform* for all $x \in [a,b]$. There is a weaker version of convergence, called *pointwise* convergence, which is sometimes useful, so we mention it as well. These two notions of convergence, pointwise and uniform, can be defined for any sequences of functions defined on $[a,b]$, not just continuous ones.

Definition 15. Let y_n be a sequence of functions defined on $[a,b]$, not necessarily continuous. Let y denote another function defined on $[a,b]$.

- Say that y_n **converges pointwise to** y to mean for each point $x \in [a,b]$ and each $\epsilon > 0$, there is a number $N_{x,\epsilon} > 0$ such that

$$n \geqslant N_{x,\epsilon} \implies |y_n(x) - y(x)| < \epsilon.$$

- Say that y_n **converges uniformly to** y to mean for each $\epsilon > 0$ there is a number $N_\epsilon > 0$ such that

$$n \geqslant N_\epsilon \implies |y_n(x) - y(x)| < \epsilon \qquad \text{for each } x \in [a, b].$$

Remark 16. The fundamental difference between the two notions of convergence is that the choice of $N_{x,\epsilon}$ is allowed to depend on $x \in [a, b]$ in the pointwise definition, but it is not allowed to depend on x in the uniform one.

Remark 17. Convergence with respect to the sup-norm on $C[a, b]$ is equivalent to uniform convergence.

Example 18. Consider the sequence y_n of functions defined on $[0, 1]$ by

$$y_n(x) = x^n.$$

Then for the pointwise limit y, we may compute it explicitly as

$$y(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}.$$

In particular the pointwise limit is no longer continuous, even though each term y_n in the sequence is.

In addition, we claim that the convergence $y_n \to y$ is not uniform, even though it is pointwise. To see this, choose $\epsilon = 1/4$. Let N be arbitrarily large. Select $n \geqslant N$. Because $y_n(x) = x^n$ is continuous at $x = 1$, there is a $\delta > 0$ such that

$$1 - \delta < x < 1 \implies x^n > 1 - \epsilon = 3/4$$

But then for such x we have $y(x) = 0$ and so

$$|y(x) - y_n(x)| = x^n > 3/4 > \epsilon.$$

In fact, more generally, as soon as we know that the pointwise limit is not continuous even though each term in the sequence is, we can be certain the convergence is not uniform, according to the following result.

Lemma 19. *Let y_n be a sequence of functions defined on $[a, b]$. Suppose that*

- *each y_n is continuous and*
- *the sequence y_n converges uniformly to a function y.*

Then y is continuous.

Proof. Let $\epsilon > 0$ be given. By uniform convergence, there is an $N > 0$ such that

$$n \geqslant N \implies |y_n(x) - y(x)| < \epsilon/3 \qquad \text{for each } x \in [a,b].$$

Choose $n \geqslant N$. Because y_n is continuous on the compact interval $[a,b]$, it is uniformly continuous. This means there is a $\delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |y_n(x_1) - y_n(x_2)| < \epsilon/3.$$

Then for $|x_1 - x_2| < \delta$, using the triangle inequality and all of the previous sentences we have

$$\begin{aligned} |y(x_1) - y(x_2)| &= |y(x_1) - y_n(x_1) + y_n(x_1) - y_n(x_2) + y_n(x_2) - y(x_2)| \\ &\leqslant |y(x_1) - y_n(x_1)| + |y_n(x_1) - y_n(x_2)| + |y_n(x_2) - y(x_2)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

This shows that y is (uniformly) continuous. □

Corollary 20. *If a sequence $y_n \in C[a,b]$ converges in $C[a,b]$ to a function y, then the limit function y must also belong to $C[a,b]$.*

Lemma 21. *Let y_n be a sequence in $C[a,b]$. Suppose*

- *$y_n \to y$ in $C[a,b]$ (or uniformly)*
- *there is a constant $M > 0$ independent of n such that $\|y_n\| \leqslant M$.*

Then $\|y\| \leqslant M$.

Proof. Let $\epsilon > 0$ be arbitrary. There is an $N > 0$ such that

$$n \geqslant N \implies \|y - y_n\| < \epsilon.$$

By the triangle inequality, we have

$$\|y\| \leqslant \|y - y_n\| + \|y_n\| < \epsilon + M.$$

Because ϵ is arbitrary, we conclude that

$$\|y\| \leqslant M,$$

as desired. □

4 Cauchy sequences and complete subsets

Definition 22. Let v_n be a sequence in a normed space V. We say that v_n is **Cauchy** to mean for each $\epsilon > 0$, there is an $N > 0$ such that

$$m, n \geqslant N \implies \|v_n - v_m\| < \epsilon.$$

Remark 23. Roughly, the term Cauchy means that the points in the sequence become close to one another. In contrast, a convergent sequence is one in which the points become close to another given point, namely, the limit of the sequence.

Example 24. Consider the sequence $x_n = 1/n$ in $\mathbb{R}$. We claim that this sequence is Cauchy. Indeed, let $\epsilon > 0$ be given. Choose $N > 2/\epsilon$. Then for $m, n \geqslant N$, we have

$$\|x_n - x_m\| \leqslant \frac{1}{n} + \frac{1}{m} \leqslant \frac{1}{N} + \frac{1}{N} < \epsilon.$$

In fact, the sequence in the previous example converges (to the point 0). More generally, when a sequence converges, then it is automatically Cauchy. The converse, however, is more subtle, and may not be true.

Lemma 25. *If v_n is convergent, then v_n is Cauchy.*

Proof. Let $\epsilon > 0$. Because v_n converges, say, to v, there is an $N > 0$ such that

$$n \geqslant N \implies \|v_n - v\| < \epsilon/2.$$

Then for $m, n \geqslant N$, we have by the triangle inequality that

$$\|v_n - v_m\| = \|v_n - v + v - v_m\| \leqslant \|v_n - v\| + \|v - v_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Definition 26. Let S be a subset of a normed space V. We say S is **complete** to mean every Cauchy sequence of points in S converges to a point of S.

Example 27. It is a standard result in real analysis to show that the space $S = V = \mathbb{R}$ is complete with respect to the usual Euclidean norm.

Similarly, the space $S = V = \mathbb{R}^n$ is complete. (This can actually be deduced from the fact that $\mathbb{R}$ is complete.)

Example 28. Let $S = (0, 1]$ be the half-open interval in $\mathbb{R}$. Then S is *not complete*. Indeed, consider the sequence $x_n = 1/n$ in S. This sequence is Cauchy. However it converges to the point 0, which is not in S. Since the limit of a convergent sequence is unique, the sequence does not converge to any point in S.

In fact, the previous example is just an illustration of the general fact that if S is complete, then it must contain its limit points.

Lemma 29. *If S is complete, then S is closed (that is, S contains all of its limit points).*

Proof. If S did not contain the limit point x_∞, then choose a sequence x_n in S converging to x_∞. This sequence converges, and so it is Cauchy. However, it does not converge to any point of S. $\square$

Lemma 30. *Any closed interval $S = [a, b]$ is complete.*

Proof. Let x_n be a Cauchy sequence in $[a, b]$. Then x_n is also a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence x_n converges to a point, say x_∞, in $\mathbb{R}$. Since each x_n belongs to $[a, b]$, the point x_∞ is by definition a limit point of $[a, b]$. Since $[a, b]$ is closed, it contains all its limit points, and so x_∞ belongs to $[a, b]$. $\square$

Lemma 31. *The space $C[a, b]$ is complete.*

Proof. Let y_n be a Cauchy sequence in $C[a, b]$. For a given $\epsilon > 0$, there is an $N > 0$ such that

$$m, n \geqslant N \implies |y_m(x) - y_n(x)| < \epsilon \qquad \text{for each } x \in [a, b]. \tag{3}$$

This means that for each $x \in [a, b]$, the sequence of real numbers $y_n(x)$ is Cauchy. Since $\mathbb{R}$ is complete, the sequence of real numbers $y_n(x)$, for x fixed, converges to a real number we denote by $y(x)$. In other words, we may define a real-valued function y on $[a, b]$ by the rule

$$y(x) = \lim_{n\to\infty} y_n(x).$$

If we could show that the sequence y_n converges to y uniformly, then we would know y belongs to $C[a, b]$ from Corollary 20 and the proof would be complete.

To see that the convergence is uniform, let $\epsilon > 0$ be given. Taking $m \to \infty$ in (3), we find that

$$n \geqslant N \implies |y(x) - y_n(x)| < \epsilon \qquad \text{for each } x \in [a, b].$$

This means that $y_n \to y$ uniformly. $\square$

5 Contraction mappings and fixed points

The notion of continuity is supposed to express that nearby points are mapped to nearby points. With that in mind, we have the following general definition.

Definition 32. Let V, W be normed vector spaces, let S be a subset of V, and let $T : S \to W$ be a map. We say T is **continuous at** $v_0 \in S$ to mean the following: for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $v \in S$, we have

$$\|v - v_0\|_V < \delta \implies \|T(v) - T(v_0)\|_W < \epsilon.$$

We say that T is **continuous on** S to mean that T continuous at each point of S.

Example 33. For example, if $V = W = \mathbb{R}$ and S is any subset of V, then the notion of continuous expressed above is just the usual notion of continuous from single variable Calculus.

More generally, if $V = \mathbb{R}^n$ and $W = \mathbb{R}$, then the notion of continuous coincides with the one from multivariable Calculus.

Lemma 34. *In the setup of Definition 32, suppose that*

- $v_n \in S$ *converges to* $v_\infty \in S$
- T *is continuous on* S.

Then $T(v_n)$ *converges to* $T(v_\infty)$.

Proof. Let $\epsilon > 0$. Because T is continuous at v_∞, there is a $\delta > 0$ such that for each $v \in S$, we have

$$\|v - v_\infty\|_V < \delta \implies \|T(v) - T(v_\infty)\|_W < \epsilon.$$

Because $v_n \to v_\infty$, there is an $N > 0$ such that

$$n \geqslant N \implies \|v_n - v_\infty\|_V < \delta.$$

Putting everything together, we find that for $n \geqslant N$, we have

$$\|T(v_n) - T(v_\infty)\|_W < \epsilon.$$

□

Remark 35. Another way of formulating the previous result is by saying that for continuous T, we have

$$T\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} T(v_n).$$

In other words, a continuous map commutes with the limit symbol.

Definition 36. Let S be a subset of V, and let $T : S \to V$. We say that T is a **contraction on** S to mean

- $T(S) \subset S$

- there is a constant $0 \leqslant K < 1$ such that

$$\|T(v) - T(w)\| \leqslant K \|v - w\| \qquad \text{for each } v, w \in S.$$

Lemma 37. *If $T : S \to S$ is a contraction on S, then T is continuous on S.*

Proof. This is an exercise for the reader (or, an exericise on the final exam, for those readers in my course). □

Example 38. For example, let $V = \mathbb{R}$ and $S = [0, 1]$. Given K satisfying $0 \leqslant K < 1$, define $T : [0, 1] \to \mathbb{R}$ by the rule

$$T(x) = Kx.$$

Then T is a contraction on $S = [0, 1]$.

Lemma 39. *Let f be a map from a closed interval $[a, b]$ to itself, and suppose that there is a constant $0 \leqslant K < 1$ such that*

$$|f'(x)| \leqslant K \qquad \textit{for each } x \in [a, b].$$

Then f is a contraction on $[a, b]$.

Proof. This is another exercise for the interested reader. My hint is to use the Mean Value Theorem. □

Example 40. Return to Example 6 where we had a function

$$g(x) = \frac{x}{2} + \frac{1}{x}.$$

It is possible to regard g as a map from the closed interval $[1, 2]$ to itself. Then we compute that for this g, we have

$$g'(x) = \frac{1}{2} - \frac{1}{x^2}.$$

On the closed interval $[1, 2]$, we have

$$-\frac{1}{2} \leqslant g'(x) \leqslant \frac{1}{4}.$$

So we deduce that

$$|g'(x)| \leqslant \frac{1}{2}.$$

By the previous lemma, g is a contraction.

We saw that the function g from Example 6 has a fixed point in the interval $[1, 2]$. This is really a consequence of the fact that g is a contraction on that interval.

Theorem 41. *Suppose g is a contraction on a closed interval $[a, b]$. Then g has a unique fixed point. Moreover, for any initial point $x_0 \in [a, b]$, the sequence of successive approximations*

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, \ldots$$

converges to the unique fixed point.

Proof. The statement of the result gives a recipe for finding a fixed point. We proceed in steps to verify that the recipe works. Let x_0 be any initial point in $[a, b]$.

Assertion 1. The sequence of successive approximations is well-defined.
Proof of Assertion 1. Because g maps the closed interval $[a, b]$ to itself, we can define $x_1 = g(x_0)$ and x_1 belongs to $[a, b]$. Similarly, because x_1 belongs to $[a, b]$, we can define $x_2 = g(x_1)$ and x_2 belongs to $[a, b]$. Proceed in this way.

Assertion 2. The sequence x_n is Cauchy.
Proof of Assertion 2. Think of x_n as $g^n(x_0)$. Then, for example, we have

$$|x_2 - x_1| = |g(g(x_0)) - g(x_0)| \leqslant K|g(x_0) - x_0| = K|x_1 - x_0|.$$

More generally, for $m \leqslant n$, we have

$$\begin{aligned}
|x_m - x_n| &= |g^m(x_0) - g^n(x_0)| \\
&\leqslant K^m|x_0 - g^{n-m}(x_0)| \\
&\leqslant K^m(|x_0 - x_1| + |x_1 - x_2| + \cdots + |x_{n-m-1} - x_{n-m}|) \\
&\leqslant K^m(|x_0 - x_1| + K|x_0 - x_1| + \cdots + K^{n-m-1}|x_0 - x_1|) \\
&\leqslant K^m|x_0 - x_1|(1 + K + K^2 + \cdots + K^{n-m-1}) \\
&< K^m|x_0 - x_1|\frac{1}{1-K}.
\end{aligned}$$

Because $0 \leqslant K < 1$, once m is a chosen suitably large, we can make the quantity on the right as small as we need.

Assertion 3. The sequence x_n converges to a point x_∞ in $[a, b]$.
Proof of Assertion 3. This is because $[a, b]$ is complete.

Assertion 4. The limit x_∞ is a fixed point of g.
Proof of Assertion 4. As a contraction, g is continuous by Lemma 37. So we have

$$g(x_\infty) = g\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_{n+1} = x_\infty.$$

Assertion 5. A fixed point of g is unique.
Proof of Assertion 5. Let z denote a fixed point of g. Then we have

$$|z - x_\infty| = |g(z) - g(x_\infty)| \leqslant K|z - x_\infty|.$$

In other words,

$$|z - x_\infty|(1 - K) \leqslant 0.$$

Because $0 \leqslant K < 1$, we find that $|z - x_\infty| = 0$, or, equivalently, $z = x_\infty$. □

In fact, this argument extends more generally to any contraction on a complete subset of a normed space. The details are left as an exercise on the final exam.

Theorem 42. *Let S be a complete subset of a normed space V. If T is a contraction on S, then T has a unique fixed point.*

We would really like to apply this result to the operator T defined on $C[a, b]$ by the rule

$$T[y](x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Indeed, we know that the fixed points of this operator correspond to solutions to the IVP (1), and obtaining results about such solutions (and their uniqueness) is our primary goal.

With that in mind, introduce the following notation. For a function $y_0 \in C[a, b]$ and a positive number α, let $S_\alpha(y_0)$ denote the subset of $C[a, b]$ defined by

$$S_\alpha(y_0) = \{y \in C[a, b] : \|y - y_0\| \leqslant \alpha\}.$$

Sometimes the set $S_\alpha(y_0)$ is called the **closed α-corridor around** y_0.

Lemma 43. *For any $y_0 \in C[a, b]$ and any $\alpha > 0$, the subset $S_\alpha(y_0)$ is complete.*

Proof. Let y_n be a Cauchy sequence in $S_\alpha(y_0)$. Because $C[a, b]$ is complete, we know that y_n converges to a point y_∞ in $C[a, b]$. It follows that $(y_n - y_0)$ converges to $(y_\infty - y_0)$. In addition, because we have the uniform bound

$$\|y_n - y_0\| \leqslant \alpha$$

which is independent of n, Lemma 21 implies that we have the bound

$$\|y_\infty - y_0\| \leqslant \alpha.$$

In other words, y_∞ belongs to $S_\alpha(y_0)$ too. □

Corollary 44. *If T is a contraction on $S_\alpha(y_0)$, then T has a unique fixed point.*

As a result, it remains only to determine a suitable subset $S_\alpha(y_0)$ for which T is a contraction. Then we can get exactly the type of existence and uniqueness result we seek.

6 Proof of Picard's Theorem

Theorem 45. *Suppose f, f_y are continuous on a closed rectangle containing the point (x_0, y_0). Then the IVP*

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

admits a unique solution on a small interval $[x_0 - h, x_0 + h]$ containing x_0.

Proof. The basic idea is to find an interval $I = [x_0 - h, x_0 + h]$ and a subset $S \subset C(I)$ so that S is complete and T is a contraction on S.

Step 1. Determining a suitable $I = [x_0 - h, x_0 + h]$.

First, if necessary, select a smaller rectangle R_1 centered about (x_0, y_0) which is inside R. Because R_1 is centered about (x_0, y_0), we may write R_1 in the form

$$R_1 = \{(x, y) : |x - x_0| \leqslant h_1, |y - y_0| \leqslant \alpha_1\}$$

for some positive real numbers h_1, α_1. Because R_1 is closed and bounded, the extreme value theorem says there are positive real numbers M, L such that

$$|f(x, y)| \leqslant M \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(x, y)\right| \leqslant L \qquad \text{for each } (x, y) \in R_1.$$

Choose h satisfying

$$0 < h < \min\left\{h_1, \frac{\alpha_1}{M}, \frac{1}{L}\right\}.$$

Then let I denote the interval $I = [x_0 - h, x_0 + h]$.

Step 2. Determining a suitable $S \subset C(I)$.

Using α_1 from the rectangle R_1 and y_0 from the initial condition, set

$$S = S_{\alpha_1}(y_0) = \{g \in C(I) : \|g - y_0\|_I \leqslant \alpha_1\}$$

where y_0 is the constant function with value y_0.

Step 3. Showing that T maps S to S.

Let g be a function in S. Then $T[g]$ is continuous, so $T[g]$ belongs to $C(I)$. Moreover, note that when x belongs to I, we have $|x - x_0| \leqslant h$ so

$$\begin{aligned} |T[g](x) - y_0| = \left|\int_{x_0}^{x} f(t, g(t))\, dt\right| &\leqslant \int_{x_0}^{x} |f(t, g(t))|\, dt \\ &\leqslant M|x_0 - x| \\ &\leqslant Mh \\ &\leqslant \alpha_1. \end{aligned}$$

This shows that $T[g]$ belongs to S.

Step 4. There is a shrinking factor $0 \leqslant K < 1$.

Choose $K = Lh$. By definition of h, we have $0 \leqslant K < 1$. In addition, for functions $u, v \in S$, the mean value theorem implies the existence of a function z defined on I such that

$$f(t, u(t)) - f(t, v(t)) = \frac{\partial f}{\partial y}(t, z(t))(u(t) - v(t)).$$

Using this, we have for $x \in I$ that

$$\begin{aligned}
|T[u](x) - T[v](x)| &= \left| \int_{x_0}^{x} [f(t, u(t)) - f(t, v(t))] \, dt \right| \\
&= \left| \int_{x_0}^{x} \frac{\partial f}{\partial y}(t, z(t))[u(t) - v(t)] \, dt \right| \\
&\leqslant L \int_{x_0}^{x} (u(t) - v(t)) \, dt \\
&\leqslant L|x - x_0| \, \|u - v\|_I \\
&\leqslant Lh \, \|u - v\|_I \\
&\leqslant K \, \|u - v\|_I \, .
\end{aligned}$$

As a result, we have

$$\|T[u] - T[v]\|_I \leqslant K \, \|u - v\|_I \, ,$$

as desired.

Step 5. Concluding that T has a fixed point in S.

This is because T is a contraction on S and S is complete.

Step 6. Showing that any solution to the IVP must lie in S.

Let u be any solution to the IVP defined on I. Let x_1 denote a point of I. By the mean value theorem, there is a point x between x_0 and x_1 such that

$$u(x_1) - u(x_0) = u'(x)(x_1 - x_0).$$

Because u solves the IVP, we have $u(x_0) = y_0$ and so

$$u(x_1) - y_0 = u'(x)(x_1 - x_0).$$

Taking the absolute value, we find

$$\begin{aligned}
|u(x_1) - y_0| &= |u'(x)||x_1 - x_0| \\
&= |f(t, u(t))||x_1 - x_0| \\
&\leqslant Mh \\
&\leqslant \alpha_1.
\end{aligned}$$

This shows that u belongs to S. □

Example 46. As an example, let's compute the Picard iterations for the IVP

$$\begin{cases} y'(x) = 2y(x) \\ y(0) = 1 \end{cases}$$

from Example 4. Here the operator T is

$$T[y](x) = 1 + \int_0^x 2y(t)\,dt.$$

For the initial choice $y_0(x) = y_0 = 1$ given by the constant function at $y_0 = 1$, we compute

$$y_1(x) = T[y_0](x) = 1 + \int_0^x 2 \cdot 1\,dt = 1 + 2x.$$

We then compute

$$y_2(x) = T[y_1](x) = 1 + \int_0^x 2(1+2t)\,dt = 1 + 2x + \frac{(2x)^2}{2}.$$

It is possible to demonstrate, by induction, that

$$y_n(x) = 1 + 2x + \frac{(2x)^2}{2} + \cdots + \frac{(2x)^n}{n!}.$$

We recognize y_n as the nth partial sum of the function

$$y(x) = e^{2x},$$

and so the sequence y_n tends to this function (uniformly on any closed neighborhood containing $x = 0$). We also recognize that $y(x) = e^{2x}$ is the unique solution to the IVP (defined in any closed neighborhood containing $x = 0$).

It is possible to obtain a similar existence/uniqueness theorem IVPs modeled on first-order systems, namely, IVPs of the form

$$\begin{cases} \mathbf{x}'(t) = F(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

where $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a map of $n+1$ variables and $(t_0, \mathbf{x}_0)$ is an initial condition in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$.

Theorem 47. *Suppose F and each $\partial F/\partial x_i$ $(i = 1, \ldots, n)$ are continuous on a closed rectangular box in $\mathbb{R}^{n+1}$ containing the initial point $(t_0, \mathbf{x}_0)$. Then the IVP*

$$\begin{cases} \mathbf{x}'(t) = F(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

admits a unique solution defined on an interval of the form $[t_0 - h, t_0 + h]$ for some $h > 0$.

Proof Sketch. For a closed interval $I \subset \mathbb{R}$, let $C(I, \mathbb{R}^n)$ denote the vector space of continuous maps $\mathbf{x} : I \to \mathbb{R}^n$. Equip this space with the sup-norm it inherits from $\mathbb{R}^n$:

$$\|\mathbf{x}\| = \max_{t \in I} \|\mathbf{x}(t)\|_{\mathbb{R}^n} \qquad \text{for } \mathbf{x} \in C(I, \mathbb{R}^n).$$

Define an operator

$$T : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$$

by the rule

$$T[\mathbf{x}](t) = \mathbf{x}_0 + \int_{t_0}^{t} F(s, \mathbf{x}(s))\, ds.$$

As in the proof of Theorem 45, the idea is to show that T is a contraction when restricted to a suitable complete subset $S \subset C(I, \mathbb{R}^n)$ for a suitable interval of the form $I = [x_0 - h, x_0 + h]$. The attentive reader can fill in the details.