MATH S1202: Calculus IV
Quiz 1
May 24, 2018

1. Compute the integral of the constant function $f(x,y) = \pi$ over the rectangle $R = [-1,1] \times [0,1]$.
Solution. By properties of the integral, we know that

$$\begin{aligned}\iint_R \pi \, dA &= \pi \cdot \iint_R 1 \, dA \\ &= \pi \cdot \text{Area}(R) \\ &= \pi \cdot 2 \\ &= 2\pi.\end{aligned}$$

2. Compute the integral of the function $f(x,y) = \sqrt{1-y^2}$ over the rectangle $R = [-1,1] \times [0,1]$.
Solution. The graph of $f(x,y)$ is the portion of the surface $z = \sqrt{1-y^2}$ lying over R, which is a portion of a cylinder along the x-axis with radius 1 and height 2. The integral of $f(x,y)$ over R is by definition the volume of this portion of the cylinder over R, which is, one-quarter of the volume of the total cylinder (it is helpful to draw a picture). We therefore have

$$\begin{aligned}\iint_R \sqrt{1-y^2} \, dA &= \frac{1}{4}\text{Volume(Cylinder)} \\ &= \frac{1}{4} \cdot \pi \cdot 1^2 \cdot 2 \\ &= \frac{\pi}{2}.\end{aligned}$$

3. Compute the integral of $f(x,y) = \sin(y^2)$ over the triangle bounded by the lines $y = x, y = \sqrt{\pi}$, and $x = 0$.
Solution. The region D of integration admits a description as a type I region as

$$D = \begin{cases} 0 \leqslant x \leqslant \sqrt{\pi} \\ x \leqslant y \leqslant \sqrt{\pi} \end{cases}$$

and a type II region as

$$D = \begin{cases} 0 \leqslant y \leqslant \sqrt{\pi} \\ 0 \leqslant x \leqslant y \end{cases} \quad .$$

Using the description of D as a type I region gives the iterated integral

$$\int_D \sin(y)^2 \, dA = \int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \sin(y^2) \, dy \, dx,$$

but this is difficult, because we don't know an (easy) anti-derivative for $\sin(y^2)$. On the other hand, if we use the description of D as a type II region, then we obtain the iterated integral

$$\begin{aligned}\int_D \sin(y)^2 \, dA &= \int_0^{\sqrt{\pi}} \int_0^y \sin(y^2) \, dx \, dy \\ &= \int_0^{\sqrt{\pi}} y \sin(y^2) \, dy \\ &= -\frac{1}{2} \cos(y^2) \Big|_{y=0}^{y=\sqrt{\pi}} \\ &= -\frac{1}{2}(\cos(\pi) - \cos(0)) \\ &= -\frac{1}{2}(-1-1) \\ &= 1.\end{aligned}$$

4. Find the volume of the solid below the surface $z = (x^2 + y^2)^2$, inside the cylinder $x^2 + y^2 = 1$, and above the xy-plane.

Solution. If we use the polar coordinates

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta), \end{aligned}$$

then the solid lies above the disc D in the xy-plane with description

$$D = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 1 \end{cases} .$$

The volume is therefore

$$\begin{aligned} \text{Volume} &= \iint_D (x^2 + y^2)^2 \, dA \\ &= \iint_D (r^2)^2 \, dA \\ &= \int_0^{2\pi} \int_0^1 r^4 \cdot r \, dr \, d\theta \\ &= 2\pi \cdot \int_0^1 r^5 \, dr \\ &= 2\pi \cdot \frac{r^6}{6}\bigg|_{r=0}^{r=1} \\ &= 2\pi \cdot \left(\frac{1}{6} - 0\right) \\ &= \frac{\pi}{3}. \end{aligned}$$