Lemma. Let X be any set. Let $f_n : X \to \mathbb{R}$ be a sequence of functions. If f_n is uniformly Cauchy, then there is a function $f : X \to \mathbb{R}$ such that $f_n \to f$ uniformly.

Proof. Let $\epsilon > 0$. There is a positive integer $N > 0$ such that

$$m, n \geqslant N \implies |f_m(x) - f_n(x)| < \epsilon/2 \quad \text{for each } x \in X.$$

As a result, for each $x \in X$, the sequence of real numbers $f_n(x)$ is Cauchy, and by the completeness of $\mathbb{R}$, we may define the function

$$f(x) = \lim_{n \to \infty} f_n(x).$$

I will now show the following:

$$n \geqslant N \implies |f(x) - f_n(x)| < \epsilon \quad \text{for each } x \in X.$$

Showing this will complete the proof.

Let n be a positive integer satisfying $n \geqslant N$. Let x be any point of X. Because $f(x) = \lim_{m \to \infty} f_m(x)$, there is a positive integer $m_x \geqslant N$ such that

$$|f(x) - f_{m_x}(x)| < \epsilon/2.$$

Now using all of our work, the triangle inequality gives

$$|f(x) - f_n(x)| \leqslant |f(x) - f_{m_x}(x)| + |f_{m_x}(x) - f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This is as desired. $\square$