Assignment 3
Intro to Modern Analysis

1. Let X be a set and consider the discrete metric d on X defined by

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}.$$

Show that X is compact if and only if X is finite.

2. Let x_n be the sequence of real numbers

$$x_n = \sqrt{1 + \frac{1}{n}}.$$

(a) Show that x_n converges to 1.

(b) Calculate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n.$$

3. Let s_n be a sequence of real numbers. Construct a new sequence σ_n by the averages

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

(a) If s_n converges to s, show that σ_n converges to s.

(b) Construct a sequence s_n which does not converge but which satisfies $\sigma_n \to 0$.

4. Construct a Cauchy sequence in $\mathbb{Q}$ that does not converge (to a point of $\mathbb{Q}$). (Hint: It might be useful to use the construction from Example 1.1 of Rudin or Proposition 2 from the notes.)

5. Let (M,d) be a metric space, and let p_n, q_n be two Cauchy sequences in X. Show that the sequence $d(p_n, q_n)$ is Cauchy in $\mathbb{R}$.

6. For a real number $p \geqslant 1$, let ℓ^p denote the vector space of sequences $x = (x_1, x_2, \ldots)$ of real numbers such that

$$\sum_{n=1}^{\infty} |x_n|^p$$

converges.

(a) Show that if $p \leqslant q$, then $\ell^q \subset \ell^p$.

(b) Suppose $p < q$. Find a sequence $x \in \ell^q$ but not in ℓ^p.

7. Let m denote the metric space whose elements are bounded infinite sequences of real numbers together with the metric

$$d(x,y) = \sup_{n=1,2,\ldots} |x_n - y_n|.$$

Show that m is complete.

8. Let a_n, b_n be sequences of real numbers. Suppose that

(i) $\sum_n a_n$ converges

(ii) b_n is bounded

(iii) b_n is monotonic.

Show that $\sum_n a_n b_n$ converges.

9. This problem has two parts.

(a) Show that if x, y are nonnegative real numbers, then

$$xy \leqslant \frac{1}{2}(x^2 + y^2).$$

(b) Let a_n be a sequence of nonnegative real numbers, and let $b_n = \sqrt{a_n}/n$. Show that if $\sum_n a_n$ converges, then $\sum_n b_n$ converges.

10. State and prove the convergence or divergence of $\sum_n a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(c) $a_n = (\sqrt[n]{n} - 1)^n$.

Extra. (Not to be graded) Let (M, d) be a metric space. Let $\mathcal{M}$ be the set of Cauchy sequences in M, that is, an element p of $\mathcal{M}$ consists of a sequence $P = (p_1, p_2, p_3, \ldots)$ of points of M.

(a) Problem 5 shows that we can associate to any two $P, Q \in \mathcal{M}$ a real number $\Delta(p, q)$ defined by

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

Show that the resulting function $\Delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is symmetric and nonnegative.

(b) Define a relation $\sim$ on $\mathcal{M}$ by $P \sim Q$ if and only if $\Delta(P, Q) = 0$. Show that the relation $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) Let M^* denote the set $\mathcal{M}/\sim$ of equivalence classes. Show that Δ induces a well-defined map on M^* which is a metric.

(d) Show that the resulting metric space M^* is complete.

(e) For any point $p \in M$, let P_p be the sequence whose terms are all p. Show that for any two points p, q in M, we have

$$\Delta(P_p, P_q) = d(p, q).$$

Conclude that there is a distance-preserving map $\varphi : M \to M^*$.

(f) Show that $\varphi(M)$ is dense in M^*.

(g) Show that $\varphi(M) = M^*$ if and only if M is complete.

As a result of this exercise, we may finally define $\mathbb{R}$: we may set $\mathbb{R} = M^*$ for $M = \mathbb{Q}$.