MATH 2610-01
Final
Due in my office (SC 1408) between 11am and 12pm (noon) on Thursday, May 5

Directions: Select 6 (and only 6) out of 8 problems to complete entirely.

Chapter 2

1. **Exact Systems.** One of the ways we were able to deal with non-linear equations was to study those that were exact, that is, those of the form $dF = 0$ for a potential function $F(x, y)$. Solutions to such equations can be found implicitly as level sets of F.

When $F(x, y, t)$ is a function of three variables, its contours are surfaces, not curves. However, the intersection of two contours from two such functions, say F and G, is, under nice conditions, a curve, and therefore could describe a solution curve for a system of involving two variables $x(t), y(t)$.

More precisely, define the differential of $F(x, y, t)$ to be the formal expression

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt.$$

The equation $dF = 0$ then becomes equivalent to the equation

$$\frac{\partial F}{\partial x}x'(t) + \frac{\partial F}{\partial y}y'(t) + \frac{\partial F}{\partial t} = 0$$

on the two functions $x(t), y(t)$. Say that a system of equations on $x(t), y(t)$ is *exact* if it can be written in the form

$$\begin{cases} dF = 0 \\ dG = 0 \end{cases}$$

for two functions $F(x, y, t)$, $G(x, y, t)$. Such a system should then admit implict solutions of the form of intersections of contours from F and G, that is, a solution curve should be of the form

$$\begin{cases} F(x, y, t) = C_1 \\ G(x, y, t) = C_2. \end{cases}$$

(a) Show that the system

$$\begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2t \\ -1 \end{bmatrix}$$

is exact.

(b) Note that the system from part (a) is non-linear in the sense that the matrix depends on the variables x and y. As a result, the methods we developed in Chapter 9 don't apply. Nevertheless, use that the system is exact to show that solutions $(x(t), y(t))$ lie on ellipses in the xy-plane.

Chapter 9 (and Chapter 6)

2. **More General Linear Systems.** Let V, W be vector spaces. Let $L : V \to W$ be a linear map. Also let $F : V \to \mathbb{R}^n$ be a linear map, where n is a positive integer. Form the augmented linear map

$$\begin{aligned} A : V &\longrightarrow W \times \mathbb{R}^n \\ v &\longmapsto (L(v), F(v)). \end{aligned}$$

Suppose that A is *injective*.

(a) Show that $\dim \ker L \leqslant n$. (Hint: First show that $\ker L$ is finite-dimensional. Then consider the restriction of A to $\ker L$.)

(b) Given $v_1, \ldots, v_n \in V$, define

$$\omega(v_1, \ldots, v_n) = \det \begin{bmatrix} F(v_1) & \cdots & F(v_n) \end{bmatrix}$$

where the matrix on the right has columns given by $F(v_1), \ldots, F(v_n)$. For $v_1, \ldots, v_n$ in $\ker L$, show that the list $\{v_1, \ldots, v_n\}$ is linearly dependent if and only if $\omega(v_1, \ldots, v_n) = 0$.

(c) Now suppose in addition that A is surjective. Let φ denote a given vector in V. For brevity, set $g = L(\varphi)$. Let Y denote a given vector in $\mathbb{R}^n$. Show that there is a unique vector $v \in \ker L$ such that

$$A(\varphi + v) = (g, Y).$$

3. **Second-Order Linear Systems.** The goal of this problem is to study *second* order systems in normal form

$$\mathbf{x}''(t) = A(t)\mathbf{x}'(t) + B(t)\mathbf{x}(t) + \mathbf{f}(t). \tag{1}$$

We will try to follow the approach from Section 6.1 by adapting it to the vector-valued setting.

(a) Suppose that $A(t), B(t)$ and $\mathbf{f}(t)$ are continuous on an open interval (a, b) that contains the point t_0. Show that for any choices of vectors $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{R}^n$, there is a unique solution defined on (a, b) to the initial value problem

$$\begin{cases} \mathbf{x}''(t) = A(t)\mathbf{x}'(t) + B(t)\mathbf{x}(t) + \mathbf{f}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{x}'(t_0) = \mathbf{x}_1 \end{cases}$$

(Hint: Use the substitution $\mathbf{y}(t) = \mathbf{x}'(t)$ to transform the given second order system in n functions $\mathbf{x}$ into a first-order system in the $2n$ functions $(\mathbf{x}, \mathbf{y})$.)

(b) Given n functions $\mathbf{a}_1(t), \dots, \mathbf{a}_n(t)$, let $\mathbf{a}(t)$ denote the n-by-n matrix of functions

$$\mathbf{a}(t) = [\mathbf{a}_1(t) \cdots \mathbf{a}_n(t)]$$

whose columns are the given functions. Similarly, define another n-by-n matrix of functions $\mathbf{a}'(t)$ by

$$\mathbf{a}'(t) = [\mathbf{a}_1'(t) \cdots \mathbf{a}_n'(t)].$$

Given $2n$ functions $\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)$ that are differentiable, define their Wronskian to be

$$W(\mathbf{a}, \mathbf{b})(t) := \det \begin{bmatrix} \mathbf{a}(t) & \mathbf{b}(t) \\ \mathbf{a}'(t) & \mathbf{b}'(t) \end{bmatrix},$$

where the matrix on the right is the $2n$-by-$2n$ matrix with n-by-n blocks as indicated. If the list $\{\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)\}$ is linearly dependent, show that the Wronskian $W(\mathbf{a}, \mathbf{b})(t)$ is the zero function.

(c) Suppose that each member of the list $\{\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)\}$ solves the corresponding homogeneous system

$$\mathbf{x}''(t) = A(t)\mathbf{x}'(t) + B(t)\mathbf{x}(t) \tag{2}$$

on an interval I for which $A(t), B(t)$ are continuous. Show that the following are equivalent.

i. The list $\{\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)\}$ is linearly independent on I.

ii. There is a time $t_0 \in I$ such that $W(\mathbf{a}, \mathbf{b})(t_0) \neq 0$.

(d) Suppose that the list $\{\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)\}$ is a linearly independent set of solutions to (2) on an interval I for which $A(t), B(t)$ are continuous. Show that every solution $\mathbf{x}(t)$ to (2) can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{a}_1(t) + \cdots + c_n\mathbf{a}_n(t) + d_1\mathbf{b}_1(t) + \cdots + d_n\mathbf{b}_n(t)$$

for some constants $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$.

(e) Suppose the following is known.

i. The function $\mathbf{x}_p(t)$ is a particular solution to (1).

ii. The list $\{\mathbf{a}_1(t), \dots, \mathbf{a}_n(t), \mathbf{b}_1(t), \dots, \mathbf{b}_n(t)\}$ is a linearly independent set of solutions to the corresponding homogeneous system (2).

Show that any solution to (1) can be written in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{a}_1(t) + \cdots + c_n\mathbf{a}_n(t) + d_1\mathbf{b}_1(t) + \cdots + d_n\mathbf{b}_n(t)$$

for some constants $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$.

4. **A Special Second-Order System.** As a particular example of a second-order system, let's consider

$$\mathbf{x}''(t) + A^2\mathbf{x}(t) = \mathbf{0} \tag{3}$$

where A denotes a (constant) n-by-n matrix with real entries. (Note the similarity of this system with the equation $(D^2 + \beta^2)y = 0$ from Chapter 6.)

(a) Write appropriate definitions for $\cos A$ and $\sin A$.

(b) Use your definitions to show that

$$\frac{d}{dt}\sin(At) = A\cos(At) \qquad \text{and} \qquad \frac{d}{dt}\cos(At) = -A\sin(At).$$

(c) Show also that the "Pythagorean Theorem" holds

$$(\cos At)^2 + (\sin At)^2 = I.$$

(Hint: Show that both sides define solutions to a certain first-order matrix-valued initial value problem.)

(d) Show that for any constant vector $\mathbf{c} \in \mathbb{R}^n$, the functions

$$\mathbf{a}(t) = \cos(At)\mathbf{c} \qquad \text{and} \qquad \mathbf{b}(t) = \sin(At)\mathbf{c}$$

solve the system (3).

(e) Consider the first-order system on $2n$ functions $(\mathbf{x}, \mathbf{y})$ described by

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{y}(t) \\ \mathbf{y}'(t) &= -A^2\mathbf{x}(t). \end{aligned} \tag{4}$$

Show that $\mathbf{x}(t)$ solves (3) if and only if $(\mathbf{x}(t), \mathbf{x}'(t))$ solves (4).

(f) Deduce that a list of $2n$ linearly independent solutions to (3) forms a fundamental set of solutions to (3).

(g) If r is an eigenvalue of A with eigenvector $\mathbf{u}$, show that

$$\mathbf{a}(t) = \cos(rt)\mathbf{u} \qquad \text{and} \qquad \mathbf{b}(t) = \sin(rt)\mathbf{u}$$

solve the system (3).

(h) Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent eigenvectors for A with corresponding eigenvalues $r_1, \ldots, r_n$. For each $k = 1, \ldots, n$ define

$$\mathbf{a}_k(t) = \cos(r_k t)\mathbf{u}_k \qquad \text{and} \qquad \mathbf{b}_k(t) = \sin(r_k t)\mathbf{u}_k.$$

If none of the eigenvalues are zero, show that the list $\{\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_n, \mathbf{b}_n\}$ forms a fundamental set of solutions to (3).

(i) Describe a general solution to the system (3) when A is the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

(j) If we define $\mathfrak{X}(t)$ to be the block matrix

$$\mathfrak{X}(t) = \begin{bmatrix} \cos(At) & \sin(At) \\ -A\sin(At) & A\cos(At) \end{bmatrix},$$

then for any pair of constant vectors $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$, show that the function

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} = \mathfrak{X}(t) \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

is a solution of (4).

(k) If A is invertible, show that a particular solution to the non-homogeneous system

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{y}(t) \\ \mathbf{y}'(t) &= -A^2\mathbf{x}(t) + \mathbf{f}(t) \end{aligned}$$

is described by

$$\begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{y}_p(t) \end{bmatrix} = \mathfrak{X}(t)\mathbf{v}(t)$$

where

$$\mathbf{v}(t) = \int \begin{bmatrix} \cos(At) & -A^{-1}\sin(At) \\ \sin(At) & A^{-1}\cos(At) \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{bmatrix} dt.$$

(l) Describe a general solution to the non-homogeneous system

$$\mathbf{x}''(t) + A^2\mathbf{x}(t) = \mathbf{f}(t)$$

when

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} \cos(t) \\ 1 \end{bmatrix}.$$

5. **Another Second-Order System.** Let D denote the differential operator $D = d/dt$. Can you describe a general solution to

$$[(D - A)^2 + B^2]\mathbf{x} = \mathbf{0}$$

where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -7 & 9 \\ -3 & 5 \end{bmatrix}?$$

Chapter 12

6. **Linearization of the Van der Pol Oscillator.** In Section 12.3, we are able to study non-linear systems that are "almost linear" in the sense that they are approximated well by linear sytems. This technique of studying the corresponding linearization of a mapping is common in mathematics (and in fact, the motivation for much of Calculus).

For a scalar $\mu \geqslant 0$, let G^μ denote the operator

$$G^\mu[x] = x'' + \mu(x^2 - 1)x' + x.$$

The equation $G^\mu[x] = 0$ is equivalent to the second-order non-linear ODE

$$x'' + \mu(x^2 - 1)x' + x = 0,$$

which describes the *Van der Pol* oscillator, a (possibly) non-conservative oscillator with (possibly) non-linear damping.

For the purposes of this problem, let's regard G^μ as a map

$$G^\mu : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),$$

where $C^\infty(\mathbb{R})$ denotes the space of infinitely-differentiable functions defined on $\mathbb{R}$.

(a) Show that G^μ is a linear operator if and only if $\mu = 0$.

(b) By the previous part, when μ is positive, the operator G^μ fails to be linear, and we no longer have many of our main tools to study the equation $G^\mu[x] = 0$. One possible strategy is to try to study the corresponding *linearization* of G^μ at some input x, with the hope that this linearization retains enough of the original information about G^μ near x. Precisely, define the *linearization of G^μ at x* to be the new operator

$$L_x^\mu[v] := \lim_{h\to 0} \frac{G^\mu(x + hv) - G^\mu(x)}{h}.$$

Compute $L_x^\mu[v]$ for any $x, v \in C^\infty(\mathbb{R})$ and $\mu \in \mathbb{R}$.

(c) Show that L_x^μ is a linear operator.

(d) Show that $L_x^0 = G^0$. (In general, if a map is already linear, then its linearization at any point is equal to itself.)

(e) The constant function $x(t) = 0$ always solves $G^\mu[x] = 0$, and we can study the linearization L_0^μ near this solution. Using the variables (v, w), find a constant matrix A^μ such that the system

$$\begin{cases} w = v' \\ L_0^\mu[v] = 0 \end{cases}$$

is equivalent to

$$\begin{bmatrix} v' \\ w' \end{bmatrix} = A^\mu \begin{bmatrix} v \\ w \end{bmatrix}. \tag{5}$$

(f) Show the following for the system (5):

i. If $\mu = 0$, then the origin is a stable center.

ii. If $0 < \mu < 2$, then the origin is an unstable spiral.

iii. If $2 \leqslant \mu$, then the origin is an unstable node.

(Optional: Then take a look at the page for the Van der Pol equation on Wolfram MathWorld to see some solutions of $G^{\mu}[x] = 0$ for various μ and to be satisfied in your work.)

7. **More General Conservative Systems.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and consider the system

$$\begin{aligned} \mathbf{x}' &= \mathbf{v} \\ \mathbf{v}' &= -\nabla f(\mathbf{x}). \end{aligned} \tag{6}$$

(a) Find a function $E(\mathbf{x}, \mathbf{v})$ defined on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ such that trajectories of solutions to the system (6) lie on level sets of E. (Hint: Find a higher-dimensional analogue of the Energy Integral Lemma from Section 4.8 with the help of the observation $(y')^2 = \|y'\|^2$.)

(b) Suppose now that $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$.

i. Show that in this case the origin is the only critical point of (6). (Note that the origin is an absolute minimum of f, and so the philosophy of section 12.4 suggests that the origin should be a *stable center*, which we will now verify.)

ii. Show that the origin is stable in the following sense: for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that whenever $(\mathbf{x}(t), \mathbf{v}(t))$ is a solution to (6), the condition $\|(\mathbf{x}(0), \mathbf{v}(0))\| < \delta$ implies that we also have $\|(\mathbf{x}(t), \mathbf{v}(t))\| < \epsilon$ for each $t \geqslant 0$. Show that the origin, however, is not *asymptotically* stable by showing that every nonzero solution of (6) satisfies

$$\lim_{t\to\infty} (\mathbf{x}(t), \mathbf{v}(t)) \neq (\mathbf{0}, \mathbf{0}).$$

(c) Suppose now that $f(\mathbf{x}) = -\frac{1}{2}\|\mathbf{x}\|^2$.

i. Show that in this case, too, the origin is the only critical point of (6). (This time, however, the origin is an absolute maximum of f, and so we expect it to be an *unstable saddle*, which we will now verify.)

ii. Show that the origin is unstable by finding n linearly independent solutions $\{(\mathbf{x}_k^+(t), \mathbf{v}_k^+(t))\}_{k=1}^n$ to (6) such that

$$\lim_{t\to\infty} \left\|(\mathbf{x}_k^+(t), \mathbf{v}_k^+(t))\right\| = \infty.$$

iii. However, show that the origin is a saddle point by finding another n linearly independent solutions $\{(\mathbf{x}_k^-(t), \mathbf{v}_k^-(t))\}_{k=1}^n$ to (6) such that

$$\lim_{t\to\infty} (\mathbf{x}_k^-(t), \mathbf{v}_k^-(t)) = (\mathbf{0}, \mathbf{0}).$$

Chapter 13

8. **More General Fixed-Point Theorem.** The Banach Fixed-Point Theorem actually holds in a more general setting than the one of the textbook. Here is one small generalization. (There are even more general statements.)

(a) Let $(V, \|\cdot\|)$ be any vector space equipped with a norm, and let S denote a subset of V. Let G be a map from S to itself satisfying the following contraction condition: there is a constant $0 \leqslant K < 1$ such that

$$\|G(v) - G(w)\| \leqslant K\|v - w\| \qquad \text{for each } v, w \in S.$$

Show that G is continuous on S. (Hint: Show that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|v - w\| < \delta \implies \|G(v) - G(w)\| < \epsilon.$$

This actually shows that G is *uniformly* continuous.)

(b) A sequence x_n in S is called *Cauchy* if it satisfies the property that for each $\epsilon > 0$ there is an $N > 0$ such that

$$m, n \geqslant N \implies \|x_n - x_m\| \leqslant \epsilon.$$

Assume S is *complete* in the following sense: whenever a sequence x_n in S is Cauchy, then the sequence converges to a point of S. Show that G has a unique fixed point in S. (Hint: Follow the proof of the Banach Fixed Point Theorem in Chapter 13.)

(c) Let f be a real-valued continuously differentiable function mapping a closed interval $[a, b]$ to itself. Suppose there is a constant $0 \leqslant M < 1$ such that

$$|f'(x)| \leqslant M \qquad \text{for each } x \in [a, b].$$

Show that f is a contraction on $[a, b]$.

(d) Let f be a real-valued continuously differentiable function defined on a closed interval $[a, b]$. Suppose that $f(a) < 0$ and $f(b) > 0$. Suppose in addition that

$$0 < M_1 \leqslant |f'(x)| \leqslant M_2 \qquad \text{for each } x \in [a, b].$$

Find the unique root of f in $[a, b]$. (Hint: For a number λ, define the auxilary function $g_\lambda(x) = x - \lambda f(x)$. Find λ such that g_λ satisfies the hypothesis of part (c). You may describe the root as a limit of a sequence.)

(e) Find the unique solution to $e^x = -x + 2$. (Hint: You may describe your answer as a limit of a sequence.)

(f) It is not enough to assume the following weaker condition on G:

$$\|G(x) - G(y)\| \leqslant \|x - y\| \qquad \text{for each } x, y \in S. \tag{7}$$

Indeed, let $S = [1, \infty)$ and let $G : S \to S$ be defined by

$$G(x) = x + \frac{1}{x}.$$

It is a fact that S is complete. Show that, even though G satisfies the weaker condition (7), G has no fixed point on S.