MATH S1202: Calculus IV
Quiz 3
June 7, 2018

1. Compute the gradient $\nabla f(3,4)$ where

$$f(x,y) = \sqrt{x^2+y^2}.$$

Solution. We compute that

$$\nabla f(x,y) = \frac{1}{f}\langle x, y\rangle.$$

We have $f(3,4) = 5$ and hence

$$\nabla f(3,4) = \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle.$$

2. Compute the area enclosed by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

using ideas from Chapter 15. (Hint: There are several ways of doing this problem, but perhaps the easiest is to use the change of variables $x = 2r\cos(\theta)$ and $y = 5r\sin(\theta)$.)

Solution. Using the coordinates

$$\begin{aligned} x &= 2r\cos\theta \\ y &= 5r\sin\theta, \end{aligned}$$

the region R in the xy-plane enclosed by the ellipse is mapped to by the region S in the (r,θ)-plane given by

$$S = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 1. \end{cases}$$

The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = 10r.$$

It follows that the area of R is given by

$$\begin{aligned} \text{Area}(R) &= \iint_R dA \\ &= \int_0^{2\pi}\int_0^1 10r\,dr\,d\theta \\ &= 10\pi. \end{aligned}$$

3. Let C be the line segment from $(0,0)$ to $(1,1)$ in the xy-plane. Compute the line integral $\int_C xy\,ds$.

Solution. A parametrization of C is given by $\mathbf{r}(t) = (t,t)$ for $0 \leqslant t \leqslant 1$. We compute that

$$|\mathbf{r}'(t)| = \sqrt{2}.$$

It follows that

$$\begin{aligned} \int_C xy\,ds &= \int_0^1 t\cdot t\cdot\sqrt{2}\,dt \\ &= \sqrt{2}\int_0^1 t^2\,dt \\ &= \frac{\sqrt{2}}{3}. \end{aligned}$$